Doctoral Thesis

On representations of the elliptic quantum group $E_{\gamma, \tau}(glN)$

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Abstract

We study the representation theory of the elliptic quantum group $E_{\gamma,T}(gl_N)$ for irrational values of the deformation parameter $\gamma$.

In Chapter 1, we introduce the basic notions and concepts. Chapter 2 is devoted to the study of the main properties of irreducible highest weight modules of finite type. We prove, in particular, that there is a unique (up to isomorphism) irreducible representation with given highest weight. In Chapters 3 and 4, we study the symmetric ($S^nV(z)$), resp. exterior powers ($\wedge^nV(z)$) of the fundamental vector representation. Finite dimensional modules are the topic of Chapter 5. Here we derive a set of necessary and sufficient conditions for the finite dimensionality of a highest weight module and show that any finite dimensional module can be realized (up to taking the tensor product with a suitable 1-dimensional module) as a sub-quotient of a tensor product of fundamental vector representations. In the last two Chapters, we study the tensor product of symmetric powers of the fundamental representation and give an explicit construction of $E_{\gamma,T}(gl_N)$-modules which are associated, in a natural way, to generalized Young Tableaus.
Riassunto

In questo lavoro studiamo la teoria delle rappresentazioni del gruppo quantistico ellittico $E_{\gamma,\tau}(gl_N)$ per valori irrazionali del parametro di deformazione $\gamma$.
Il primo capitolo è dedicato alle definizioni fondamentali, mentre nel capitolo 2 studiamo le proprietà principali dei moduli irreducibili di peso più alto. Dimostriamo, in particolare, che esiste un’unica (a meno di isomorfismo) rappresentazione irreducibile di peso più alto. Nei capitoli 3 e 4 studiamo le rappresentazioni simmetriche ($S^nV(z)$), risp. i prodotti esterni ($\wedge^nV(z)$), della rappresentazione fondamentale. I moduli di dimensione finita sono l’argomento del capitolo 5. Qui deriviamo un’insieme di condizioni necessarie e sufficienti sul vettore di peso ellittico affinché un modulo irreducibile sia di dimensione finita e dimostriamo che che ogni modulo di dimensione finita può essere realizzato come sotto quoziente di un prodotto tensoriale di rappresentazioni fondamentali. Infine, negli ultimi due capitoli diamo una costruzione esplicita degli $E_{\gamma,\tau}(gl_N)$-moduli associati a diagrammi di Young generalizzati.
Chapter 1

Introduction

In this introductory chapter we recall the definition of the elliptic quantum group $E_{\tau,\tau}(gl_N)$ [3] and introduce some basic notations.

1.1 Basic Notations

Let $\mathfrak{h}$ be the Cartan sub-algebra of $gl_N$. It is the Abelian Lie algebra of diagonal complex $N \times N$ matrices. We identify $\mathfrak{h}$ with its dual space via the non-degenerate bilinear form $(x, y) = \text{tr}(xy)$, and with $\mathbb{C}^N$ via the orthonormal basis $e_j = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $j$-th position ($j = 1, \ldots, N$).

Diagonalizable $\mathfrak{h}$-Modules

A diagonalizable $\mathfrak{h}$-module of finite type (over $\mathbb{C}$) is a complex vector space $W$ which admits a weight decomposition $W = \bigoplus_{\mu \in \mathfrak{h}^*} W[\mu]$ such that:

1. $\mathfrak{h}$ acts on $W[\mu]$ by $h\omega = \mu(h)\omega$, ($h \in \mathfrak{h}$, $\omega \in W[\mu]$).
2. All spaces of fixed weight $W[\mu]$ are finite-dimensional.
3. The set $\{ \mu | W[\mu] \neq 0 \}$ is at most countable.

Let $W_1 \otimes \cdots \otimes W_k$ be $\mathfrak{h}$-modules of finite type. For any function $f : \mathfrak{h}^* \to \text{End}(W_1 \otimes \cdots \otimes W_k)$ and any $i = 1, \ldots, k$ we define an operator $f(h^{(i)}) \in \text{End}(W_1 \otimes \cdots \otimes W_k)$ by the rule:

$$f(h^{(i)}) v_1 \otimes \cdots \otimes v_k = f(\mu) v_1 \otimes \cdots \otimes v_k, \quad (1.1)$$

for any $v \in W_1 \otimes \cdots \otimes W_k$.

For a finite-dimensional vector space $W$ over $\mathbb{C}$ denote by $\text{Fun}_N(W)$ the space of meromorphic functions on $\mathfrak{h}^*$ with values in $W$, such that:

$$F(\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_N) = F(\lambda_1, \ldots, \lambda_j, \ldots, \lambda_N) \quad (1.2)$$
$$F(\lambda + \mathbb{C}(1, \ldots, 1)) = F(\lambda). \quad (1.3)$$

If $W$ is a diagonalizable $\mathfrak{h}$-module of finite type, set

$$\text{Fun}_N(W) = \bigoplus_{\mu \in \mathfrak{h}^*} \text{Fun}_N(W[\mu]). \quad (1.4)$$

The space $\text{Fun}_N(W)$ is a vector space over $\text{Fun}_N(\mathbb{C})$. The space $W$ is naturally embedded in $\text{Fun}_N(W)$ as the subspace of constant functions. If $W$ is an $\mathfrak{h}$-module of
finite type, then $\text{Fun}_N(W)$ is an $\mathfrak{h}$-module of finite type with the natural pointwise action of $\mathfrak{h}$ and $\text{Fun}_N(W)[\mu] = \text{Fun}_N(W[\mu])$.

Let $U$ be a diagonalizable $\mathfrak{h}$-module of finite type and a vector space over $\text{Fun}_N(\mathbb{C})$. Suppose that the action of $\mathfrak{h}$ commutes with multiplication by functions. Then each weight subspace $U[\mu]$ is a vector space over $\text{Fun}_N(\mathbb{C})$. Assume that all the weight subspaces are finite-dimensional over $\text{Fun}_N(\mathbb{C})$. Then one can define a diagonalizable $\mathfrak{h}$-module of finite type $W$, such that $U = \text{Fun}_N(W)$ as $\mathfrak{h}$-modules of finite type, in the following way. For any $\mu$ such that $U[\mu] \neq 0$, pick up a basis $f_1, \ldots, f_k$ of $U[\mu]$ over $\text{Fun}_N(\mathbb{C})$ and set $W[\mu] = \bigoplus_{i=1}^k \mathbb{C}f_i$, otherwise, set $W[\mu] = 0$. Then define $W = \bigoplus_{\mu \in \mathfrak{h}^*} W[\mu]$ to be the diagonalizable $\mathfrak{h}$-module of finite type such that $W[\mu]$ is a weight-subspace of weight $\mu$.

Operator Algebra

We denote by $D(W)$ the space of difference operators acting in $\text{Fun}_N(W)$. It is spanned over $\mathbb{C}$ by operators of the form

$$f(\lambda) \mapsto \varphi(\lambda)f(\lambda + \mu),$$

where $\varphi(\lambda) \in \text{Fun}_N(\text{End}(\mathbb{C}))$ and $\mu \in \mathfrak{h}^*$.

1.2 The Elliptic Quantum Group $E_{\gamma,\tau}(gl_N)$

The R-Matrix of $E_{\gamma,\tau}(gl_N)$

Let us fix a point $\tau$ in the upper half plane and a generic complex number $\gamma$, then

$$\Theta(z) = -\sum_{j \in \mathbb{Z} + \frac{1}{2}} e^{i\pi j^2 + 2\pi iz(j + \frac{1}{2})}$$

(1.6)

is Jacobi's first theta function. The function $\Theta(z)$ has multipliers $-1$ as $z \rightarrow z + 1$ and $-\exp(-2\pi iz - \pi i\tau)$ as $z \rightarrow z + \tau$. It is entire with only simple zeros lying on the lattice $\mathbb{Z} + \mathbb{Z}\tau$.

The $R$-matrix of the elliptic quantum group $E_{\gamma,\tau}(gl_N)$ is the meromorphic function from $\mathbb{C} \otimes \mathfrak{h}^*$ to $\text{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$

$$R(z, \lambda) = \sum_{i=1}^N E_{i,i} \otimes E_{i,i} + \sum_{i \neq j}^{N} \alpha(z, \lambda_i - \lambda_j)E_{i,i} \otimes E_{j,j} + \sum_{i \neq j}^{N} \beta(z, \lambda_i - \lambda_j)E_{i,j} \otimes E_{j,i},$$

(1.7)

where the functions $\alpha(z, \lambda), \beta(z, \lambda)$ are ratios of theta functions:

$$\alpha(z, \lambda) = \frac{\Theta(z)\Theta(\lambda)}{\Theta(z - \lambda)}, \quad \beta(z, \lambda) = -\frac{\Theta(z + \lambda)\Theta(\gamma)}{\Theta(z - \gamma)\Theta(\lambda)}$$

(1.8)

and $E_{i,j}$ is a matrix such that $E_{i,j}e_k = \delta_{j,k}e_i$. Note that $R(z, \lambda)$ is invariant under the symmetric group $S_N$ (the Weyl group of $gl_N$), in the sense that for any permutation $\sigma$

$$R(z, \sigma \lambda) = \sigma \otimes \sigma R(z, \lambda)\sigma^{-1} \otimes \sigma^{-1},$$

(1.9)
1.2. The Elliptic Quantum Group $E_{\gamma,T}(gl_N)$

where $S_N$ acts linearly on $h$ and on $C^N$ by permutation of the coordinates.

The $R$-matrix of the elliptic quantum group $E_{\gamma,T}(gl_N)$ is a solution of the dynamical Yang-Baxter equation with spectral parameter:

\[
R(z_1 - z_2, \lambda - \gamma h^{(3)}(12))R(z_1 - z_3, \lambda)^{(13)}R(z_2 - z_3, \lambda - \gamma h^{(11)}(23)) = R(z_2 - z_3, \lambda)^{(23)}R(z_1 - z_3, \lambda - \gamma h^{(2)}(13))R(z_1 - z_2, \lambda)^{(12)}. \tag{1.10}
\]

We adopt the standard notation: for instance, $R(z, \lambda - \gamma h^{(3)}(12))$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R(z, \lambda - \gamma \mu_3) \otimes 1$ if $v_3$ has weight $\mu_3$.

The $R$-matrix of $E_{\gamma,T}(gl_N)$ has zero weight:

\[
[R(z, \lambda), h^{(1)} + h^{(2)}] = 0, \tag{1.11}
\]

and satisfies the “unitarity” relation:

\[
R(z, \lambda)R(-z, \lambda)^{(21)} = 1_{C^N \times C^N}. \tag{1.12}
\]

Modules Over The Elliptic Quantum Group $E_{\gamma,T}(gl_N)$

A module over the elliptic quantum group $E_{\gamma,T}(gl_N)$ is a diagonalizable $h$-module of finite type $W$ together with $D(W)$-valued meromorphic functions $L_{ij}(z)$, $i, j = 1, \ldots, N$, in a complex variable $z$, subject to relations (1.14)-(1.16). We combine the functions $L_{ij}(z)$ into a matrix $L(u)$ with noncommuting entries:

\[
L(z) = \sum_{i,j} E_{ij} \otimes L_{ij}(z). \tag{1.13}
\]

The defining relations are:

\[
L_{ij}(z) w(\lambda) = w(\lambda - \gamma \epsilon^j) L_{ij}(z) \tag{1.14}
\]

for any $w(\lambda) \in \text{Fun}_N(C)$,

\[
[L(z), h^{(1)} + h^{(2)}] = 0, \tag{1.15}
\]

\[
R^{(12)}(z_1 - z_2, \lambda - \gamma h^{(3)}(12))L^{(13)}(z_1)L^{(23)}(z_2) = L(z_2)^{(23)}L^{(13)}(z_1)R^{(12)}(z_1 - z_2, \lambda). \tag{1.16}
\]

The last equality holds in $\text{End}(C^N \otimes C^N \otimes \text{Fun}_N(W))$. As before $L^{(13)}(z) = \sum_{i,j} E_{ij} \otimes 1 \otimes L_{ij}(z)$ and $L^{(23)}(z) = \sum_{i,j} 1 \otimes E_{ij} \otimes L_{ij}(z)$.

Relations (1.14) can be rewritten as $L(z)\varphi(\lambda + \gamma h^{(1)}) = \varphi(\lambda)L(z)$. Formula (1.15) means that for any $\mu \in h^*$

\[
L_{ij}(z)\text{Fun}_N(W[\mu]) \subset \text{Fun}_N(W[\mu - \epsilon^i + \epsilon^j]). \tag{1.17}
\]

According to (1.14) $L_{ij}(z)$ is a difference operator which acts on $\text{Fun}_N(W)$ by

\[
L_{ij}(z)w(\lambda) = L_{ij}(z, \lambda)w(\lambda - \gamma \epsilon^j). \tag{1.18}
\]

Set $L(z, \lambda) = \sum_{i,j} E_{ij} \otimes L_{ij}(z, \lambda)$. 

Example 1.1. For any \( w \in \mathbb{C} \) the assignment
\[ L(z, \lambda) = R(z - w, \lambda), \]
(1.19)
makes \( \mathbb{C}^N \) into an \( E_{\gamma,\tau}(gl_N) \)-module. This module is called the (fundamental) vector representation of \( E_{\gamma,\tau}(gl_N) \) and is denoted by \( V(w) \).

Let \( V, W \) be two \( E_{\gamma,\tau}(gl_N) \)-modules. Then the \( \mathfrak{h} \)-module \( V \otimes W \) is made into an \( E_{\gamma,\tau}(gl_N) \)-module by the co-product of \( E_{\gamma,\tau}(gl_N) \):
\[ L_{ij}(z, \lambda) = \sum_{k=1}^{N} L_{ik}(z, \lambda - \gamma) \otimes L_{kj}(z, \lambda). \]
(1.20)

Sometimes we will also make use of the opposite co-product defined by:
\[ L_{ij}(z, \lambda) = \sum_{k=1}^{N} L_{kj}(z, \lambda) \otimes L_{ik}(z, \lambda - \gamma \lambda^{(1)}). \]
(1.21)

For an \( E_{\gamma,\tau}(gl_N) \)-module of finite type \( W \), a vector subspace \( \text{Fun}_N(V) \) of \( \text{Fun}_N(W) \) is an \( E_{\gamma,\tau}(gl_N) \)-submodule of \( W \), if
\[ L_{ij}(z)\text{Fun}_N(V) \subset \text{Fun}_N(V), \quad \forall i, j \in \{1, \ldots, N\}. \]
(1.22)

Let \( W \) be an \( E_{\gamma,\tau}(gl_N) \)-module of finite type and \( V \) an \( E_{\gamma,\tau}(gl_N) \)-submodule of \( W \), then we define an \( E_{\gamma,\tau}(gl_N) \)-module structure on \( \text{Fun}_N(W)/\text{Fun}_N(U) \) by:
\[ L_{ij}(z)[w] = L_{ij}(z)w \mod \text{Fun}_N(U), \]
(1.23)
where \([w] = w \mod \text{Fun}_N(U)\).

Explicit Relations

In the next Chapters we will need a more "explicit" form of the defining relations (1.14)-(1.16) of the elliptic quantum group. Let \( \alpha_{ij}(z, \lambda) \), resp. \( \beta_{ij}(z, \lambda) \), be the functions:
\[ \alpha_{ij}(z, \lambda) = \alpha(z, \lambda_i - \lambda_j), \]
\[ \beta_{ij}(z, \lambda) = \beta(z, \lambda_i - \lambda_j). \]
(1.24)

Using (1.13) it is easy to see that:
\[ \alpha_{kl}(z_1 - z_2, \lambda - \gamma) L_{kl}(z_1) L_{ij}(z_2) + \beta_{ik}(z_1 - z_2, \lambda - \gamma) L_{li}(z_1) L_{kj}(z_2) = \alpha_{ij}(z_1 - z_2, \lambda) L_{ij}(z_2) L_{kl}(z_1) + \beta_{ij}(z_1 - z_2, \lambda) L_{li}(z_2) L_{kj}(z_1) \]
(1.25)
for \( k \neq l, \quad i \neq j \),
\[ L_{ki}(z_1) L_{kj}(z_2) = \alpha_{ij}(z_1 - z_2, \lambda) L_{ki}(z_2) L_{kj}(z_1) + \beta_{ij}(z_1 - z_2, \lambda) L_{ki}(z_2) L_{kj}(z_1) \]
(1.26)
for \( i \neq j \),
\[ L_{li}(z_2) L_{kl}(z_1) = \alpha_{ik}(z_1 - z_2) L_{ki}(z_1) L_{li}(z_2) + \beta_{ik}(z_1 - z_2, \lambda - \gamma) L_{li}(z_1) L_{ki}(z_2) \]
(1.27)
1.3 Morphisms

Let $V, W$ be two $E_{\gamma, \tau}(gl_N)$-modules of finite type, then any function $\phi(\lambda)$ with values in the space $\text{Hom}(V, W)$ induces a homomorphism $\text{Fun}_N(V) \to \text{Fun}_N(W)$. A morphism of an $E_{\gamma, \tau}(gl_N)$-module of finite type $V$ to an $E_{\gamma, \tau}(gl_N)$-module of finite type $W$ is a meromorphic function with values in $\text{Hom}(V, W)$ such that the induced morphism $\text{Fun}_N(V) \to \text{Fun}_N(W)$ commutes with the action of the operators $L_{ij}(z)$ for all indices $i, j \in \{1, \ldots, N\}$ and all generic values of $z \in \mathbb{C}$:

$$L_{ij}(z) \phi(\lambda) = \phi(\lambda) L_{ij}(z).$$

A morphism is an isomorphism, if the homomorphism $\phi(\lambda)$ is non-degenerate for generic $\lambda$.

If $\phi_1(\lambda) \in \text{Hom}(V_1, W_1)$ and $\phi_2(\lambda) \in \text{Hom}(V_2, W_2)$ are morphisms, then

$$(\phi_1 \otimes \phi_2)(\lambda) = \phi^{(1)}_1(\lambda - \gamma \phi^{(2)}_2(\lambda))$$

is a morphism from $V_1 \otimes V_2$ to $W_1 \otimes W_2$.

An $E_{\gamma, \tau}(gl_N)$-module $W$ is irreducible if for all non-trivial morphisms $\phi(\lambda) \in \text{Hom}(V, W)$ the map $\phi(\lambda)$ is surjective for generic values of the parameter $\lambda$. An $E_{\gamma, \tau}(gl_N)$-module is reducible, if it is not irreducible.

**Example 1.2.** Consider the two $E_{\gamma, \tau}(gl_N)$-modules $W_1 = V(z_1) \otimes V(z_2)$ and $W_2 = V(z_2) \otimes V(z_1)$, then one can see, using equation (1.10), that $R(\lambda) = R(z_1 - z_2, \lambda)P$, where $P$ is the flip $Pv_1 \otimes v_2 = v_2 \otimes v_1$, is a surjective morphism

$$R(\lambda) : \text{Fun}_N(V(z_1) \otimes V(z_2)) \to \text{Fun}_N(V(z_2) \otimes V(z_1))$$

for generic values of $z_1, z_2$. In fact, it can be easily shown that if $z_1 - z_2 \not\equiv \gamma \ (\gamma \text{ generic})$, the $E_{\gamma, \tau}(gl_N)$-modules $W_1$ and $W_2$ are irreducible (see Section 2.6).
1.4 Graphical Interpretation of the \( R \)-Matrices

It is well known that the Yang-Baxter equation has a graphical interpretation. In fact, let us consider a graph generated by \( n \) "crossing" lines. Let us number this lines from 1 to \( n \) from left to right at the bottom of the drawing. Reading the drawing from the bottom to the top we assign to each crossing between lines \( j \) and \( k \) an \( R \)-matrix

\[
R^{jk}(z_j - z_k, \lambda - \sum h^{(l)})
\]  

(1.35)

with the convention that the line \( j \) is at the left of the line \( k \) bellow the crossing and that the sum is over all the numbers \( l \) assigned to the lines at the left of the crossing.

Definition 1.3. Let \( D \) be any graph generated by \( n \) "crossing" lines. We denote by \( W_D \) the operator obtained multiplying together the operators assigned to each crossing of \( D \) by the previous procedure (reading the graph from the bottom to the top).

With this rules we can represent the "unitarity" and the Yang-Baxter equation diagrammatically (see Figure 1.2 and Figure 1.3). In the next section we will need the following Lemma:

Lemma 1.4. Let \( z_j \in \mathbb{C} \) be generic, then

If two diagrams \( D \) and \( D' \) induce the same permutation the associated operators \( W_D \) and \( W_{D'} \) are equal.

Proof. We use the fact that the symmetric group \( S_n \) is generated by adjacent transposition \( s_1, \ldots, s_{n-1} \) with relations

\[
(a) \ s_j^2 = 1, \quad (b) \ s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}, \quad (c) \ s_j s_k = s_k s_j \text{ if } |j - k| \geq 2.
\]  

(1.36)
To a diagram we associate in an obvious way a word in the generators $s_j$, so that its image in $S_n$ is the induced permutation. For example, we associate the word $s_3s_2s_1s_2s_3s_2$ to the diagram on the left in Fig.1.4. If two diagrams induce the same permutation, the corresponding words can be obtained from each other by applying a sequence of relations. To each relation there corresponds a property of $R$-matrices that implies that the corresponding endomorphism $W_D(z,\lambda)$ coincide: namely, we have (a) the "unitarity", (b) the dynamical Yang–Baxter equation, and that (c) $R^{jk}(z,\lambda - \sum_{l \in L} h(l))$ commutes with $R^{rs}(z,\lambda - \sum_{l \in K} h(l))$ if the sets $\{j, k\}$ and $\{r, s\}$ are disjoint, and are either contained in or have empty intersection with $L$ and $K$.

Remark 1. It is important that the parameters $z_j$ are generic, in fact for special values of this ones the R-matrix becomes singular and the product is meaningless if one does not consider a suitable limit.
Chapter 2

Highest Weight Modules

2.1 Introduction and Definitions

Definition 2.1. A singular vector in an $E_{\gamma}(gl_N)$-module of finite type $W$ is a non-zero element $w \in \text{Fun}_N(W)$ such that

$$L_{ji}(z)w = 0 \quad \forall i < j, \quad \forall z \in \mathbb{C}.$$ (2.1)

An element $w \in \text{Fun}_N(W)$ is of $h^{(i)}$-weight $(i = 1, \ldots, N) \mu^i$ if $h^i w = \mu^i w$.

An element $w \in \text{Fun}_N(W)$ is of elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\}_{i=1}^{N})$, where $\Omega \in \mathbb{C}^N$, if it is of $h^i$-weight $\Omega^i$ and

$$L_{ii}(z)w = \Lambda_{ii}(z, \lambda)w.$$ (2.2)

An $E_{\gamma}(gl_N)$-module of finite type is a highest weight $E_{\gamma}(gl_N)$-module of finite type with elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\}_{i=1}^{N})$ and highest weight cyclic vector (highest weight vector, for short) $w \in \text{Fun}_N(W)$, if $w$ is a singular vector of elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\}_{i=1}^{N})$ and $\text{Fun}_N(W)$ is generated over $\text{Fun}_N(\mathbb{C})$ by elements of the form:

$$L_{i_1,j_1}(z_1) \cdots L_{i_k,j_k}(z_k)w,$$ (2.3)

where $z_1, \ldots, z_k$ are generic (by generic we mean that $z_j - z_i \neq 0, \gamma \mod \mathbb{Z} + \mathbb{Z} \gamma$, $j \neq i$).

Remark 2. If $W_1$ and $W_2$ are two highest weight $E_{\gamma}(gl_N)$-modules, then there is an $E_{\gamma}(gl_N)$-module structure on $W_1 \otimes W_2$ but in general this space isn't a highest weight $E_{\gamma}(gl_N)$-module.

Remark 3. In this thesis we consider only highest weight $E_{\gamma}(gl_N)$-modules of elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\}_{i=1}^{N})$ such that the functions $\Lambda_{ii}(z, \lambda)$ are not identically zero.

The representation theory of $E_{\gamma}(gl_N)$ depends strongly on the actual value of the deformation parameter $\gamma \in \mathbb{C}$. In what follows we will almost always assume that $\gamma$ is a (real) irrational number.

2.2 Normal Form

Definition 2.2. Let $\text{Fun}_N(W)$ be a highest weight $E_{\gamma}(gl_N)$-module of finite type with elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\})$ and highest weight vector $\omega$.  

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An element $w \in \text{Fun}_N(W)$ is a word if it is of the form
\[ w = L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega \] (2.4)
for some $i_1, j_1 \in \{1, \ldots, N\}, z_i \in \mathbb{C}$. An element $w \in \text{Fun}_N(W)$ is in weak normal form if it is a word and
\[ i_1 \leq i_2 \leq \cdots \leq i_n \quad j_1 \geq j_i, \quad \forall i \] (2.5)
and it is in normal form if
\[ i_1 \leq i_2 \leq \cdots \leq i_n \quad j_1 > j_i, \quad \forall i. \] (2.6)

**Lemma 2.3.** Let $\text{Fun}_N(W)$ be a highest weight $E_{\gamma,(gl(N))}$-module of finite type with elliptic weight $(\Omega, \{A_{ij}(\lambda, \lambda)\})$ and highest weight vector $\omega$, then $\text{Fun}_N(W)$ is generated by words in weak normal form.

**Proof.** $\text{Fun}_N(W)$ is, by definition, generated by words of the form
\[ L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega. \] (2.7)
This means that we only have to prove that such words can be expressed as linear combination (over $\text{Fun}_N(\mathbb{C})$) of words in weak normal form.

Let $w \in \text{Fun}_N(W)$ be the word
\[ w = L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega. \] (2.8)
Using that
\[ \alpha_{i_1,i_2}(z_1 - z_2, \lambda) L_{i_1,j_2}(z_1) L_{i_2,j_2}(z_2) + \beta_{i_1,i_2}(z_1 - z_2, \lambda) L_{i_1,j_2}(z_1) L_{i_2,j_2}(z_2) = \alpha_{i_2,i_1}(z_1 - z_2, \lambda) L_{i_2,j_1}(z_1) L_{i_1,j_1}(z_2) + \beta_{i_2,i_1}(z_1 - z_2, \lambda) L_{i_2,j_1}(z_1) L_{i_1,j_2}(z_2) \] (2.9)
we can rewrite $w$ as a linear combination of words
\[ L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega \] (2.10) with $i_1 \leq i_2 \leq \cdots \leq i_n$.

Now we claim that after this operation the only non-zero terms are automatically in weak normal form. Or to be more precise, all terms of the form
\[ L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega \] (2.11)
with $i_1 \leq i_2 \leq \cdots \leq i_n$ are identically zero if $j_k < i_k$ for some $k$. We prove this by induction.
1. $L_{i,j}(z)\omega \neq 0$ only if $i \leq j$.
2. Let $w$ be of the form $L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega$ with $i_1 \leq i_2 \leq \cdots \leq i_n$ and $j_k \geq i_k, \quad k = 2, \ldots, n$. If $j_1 \geq i_1$ we are done. So we only have to consider the following situation:
\[ j_1 < i_1 \leq i_2 \leq j_2. \] (2.12)
2.2. Normal Form

In this case we have:

\[
\begin{align*}
\alpha_{j_2 j_1}(z_1 - z_2, \lambda) & L_{i_1 j_1}(z_1) L_{i_2 j_2}(z_2) \quad + \beta_{j_2 j_1}(z_1 - z_2, \lambda) L_{i_1 j_2}(z_2) L_{i_2 j_1}(z_1) = \\
& \alpha_{i_2 i_1}(z_1 - z_2, \lambda - \gamma h) L_{i_1 j_2}(z_1) L_{i_2 j_1}(z_2) \\
& + \beta_{i_2 i_1}(z_1 - z_2, \lambda) L_{i_1 j_2}(z_1) L_{i_2 j_1}(z_2) \\
(2.13)
\end{align*}
\]

This implies,

\[
\begin{align*}
L_{i_1 j_1}(z_1) L_{i_2 j_2}(z_2) L_{i_3 j_3}(z_3) \ldots L_{i_n j_n}(z_n) \omega = \\
br(z_1 - z_2, \lambda) L_{i_1 j_2}(z_1) L_{i_2 j_3}(z_2) L_{i_3 j_3}(z_3) \ldots L_{i_n j_n}(z_n) \omega \\
+ s(z_1 - z_2, \lambda) L_{i_1 j_2}(z_2) L_{i_2 j_3}(z_2) L_{i_3 j_3}(z_3) \ldots L_{i_n j_n}(z_n) \omega \\
+ t(z_1 - z_2, \lambda) L_{i_1 j_2}(z_2) L_{i_2 j_2}(z_2) L_{i_3 j_3}(z_3) \ldots L_{i_n j_n}(z_n) \omega, \\
(2.14)
\end{align*}
\]

for some functions \(r(z_1 - z_2, \lambda), s(z_1 - z_2, \lambda), t(z_1 - z_2, \lambda)\) which can be explicitly computed from (2.13). Now, one can see that the right hand side of equation (2.14) is identically zero by the induction hypothesis. In fact

\[
\begin{align*}
L_{i_1 j_2}(z_1) L_{i_2 j_2}(z_2) L_{i_3 j_3}(z_3) \ldots L_{i_n j_n}(z_n) \omega. \\
(2.15)
\end{align*}
\]

Lemma 2.4. Let \(\text{Fun}_N(W)\) be a highest weight \(E_{\gamma,N}(gl_N)\)-module of finite type with elliptic weight \((\Omega, \{A_{ii}(z, \lambda)\})\) and highest weight vector \(\omega\), then \(\text{Fun}_N(W)\) is generated over \(\text{Fun}_N(\mathbb{C})\) by words in normal form.

Proof. Using Lemma 2.3, it is clear that we only have to show that one can eliminate the diagonal terms in the "surviving" word in weak normal form:

\[
L_{i_1 j_2}(z_1) L_{i_2 j_3}(z_2) \ldots L_{i_n j_n}(z_n) \omega, \\
(2.16)
\]

where \(i_1 \leq i_k\) if \(i < k\) and \(j_1 \geq j_i\).

We prove the statement by induction on the length of the words.

1. \(L_{ij}(z) \omega\) is in normal form if \(i < j\) and equal to \(A_{ii}(z, \lambda)\) otherwise.
2. Consider a word if the form (2.16) where \(i_1 \leq \ldots i_n\) and \(i_k < j_k, k = 2, \ldots, n,\) then if \(i_1 < j_1\) there is nothing to prove. So we only have to consider the case:

\[
L_{i_1 i_2}(z_1) L_{i_2 j_2}(z_2) \ldots L_{i_n j_n}(z_n) \omega. \\
(2.17)
\]

Now, we notice that if \(i_1 = i_2\) we can "commute" \(L_{i_1 j_1}(z_1)\) and \(L_{i_2 j_2}(z_2)\) using the relation:

\[
L_{ij}(z)L_{ik}(w) = \alpha_{jk}(z-w, \lambda)L_{jk}(w)L_{ij}(z) + \beta_{jk}(z-w, \lambda)L_{kj}(w)L_{ij}(z). \\
(2.18)
\]

This implies that we can assume without loss of generality that:

\[
i_1 < i_2 \leq i_3 \leq \ldots \leq i_n. \\
(2.19)
\]

Using the relation (2.13) we can rewrite (2.17) as:

\[
\begin{align*}
L_{i_1 i_2}(z_1) L_{i_2 j_2}(z_2) \ldots L_{i_n j_n}(z_n) \omega = \\
r(z_1 - z_2, \lambda) L_{i_1 j_2}(z_1) L_{i_2 j_3}(z_2) \ldots L_{i_n j_n}(z_n) \omega \\
s(z_1 - z_2, \lambda) L_{i_1 j_2}(z_2) L_{i_2 j_3}(z_2) \ldots L_{i_n j_n}(z_n) \omega \\
t(z_1 - z_2, \lambda) L_{i_1 j_2}(z_2) L_{i_2 j_2}(z_2) \ldots L_{i_n j_n}(z_n) \omega. \\
(2.20)
\end{align*}
\]

\[
\Box
\]
Now, the first two terms in the right hand side of (2.20) are identically zero, while the last is in normal form by the induction hypothesis.

**Corollary 2.5.** Let $\text{Fun}_N(W)$ be a highest weight $E_{\gamma, \tau}(gl_N)$-module of finite type with elliptic weight $(\Omega, \{\Lambda_i(z, \lambda)\})$ and highest weight vector $\omega$, then the $E_{\gamma, \tau}(gl_N)$-module $\text{Fun}_N(W)$ has a basis in normal form.

### 2.3 Partial Order

Let $W$ be a highest weight $E_{\gamma, \tau}(gl_N)$-module of finite type with highest weight vector $\omega$ and elliptic weight $(\Omega, \{\Lambda_i(z, \lambda)\})$.

The vectors $e_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0), i = 1, \ldots, N - 1$ are a basis in the weight space, this means that any weight $\mu$ is of the form

$$\mu = \Omega - \sum_{i=1}^{N-1} n_i e_i, \quad n_i \in \mathbb{Z}. \quad (2.21)$$

**Definition 2.6.**

1. Let $\text{Fun}_N(W[\mu_1])$ and $\text{Fun}_N(W[\mu_2])$ be two spaces of weights $\mu_j = \Omega - \sum_{i=1}^{N-1} n_i^{(j)} e_i$ in a highest weight $E_{\gamma, \tau}(gl_N)$-module of finite type $W$, then we say that $\text{Fun}_N(W[\mu_1]) < \text{Fun}_N(W[\mu_2])$ if $n_i^{(1)} \leq n_i^{(2)}, \forall i = 1, \ldots, N - 1$ and $n_k^{(1)} \neq n_k^{(2)}$ for some $k$.

2. Let $w_1 \in \text{Fun}_N(W[\mu_1])$ and $w_2 \in \text{Fun}_N(W[\mu_2])$ be two vectors. We say that $w_1 < w_2$ if $\text{Fun}_N(W[\mu_1]) < \text{Fun}_N(W[\mu_2])$.

### 2.4 Structure Theorems

In the rest of this Chapter we will prove the Main Structure Theorems about $E_{\gamma, \tau}(gl_N)$-modules for an irrational deformation parameter $\gamma$.

**Theorem 2.7.** Let $W$ be a highest weight $E_{\gamma, \tau}(gl_N)$-module with elliptic weight $(\Omega, \{\Lambda_i(z, \lambda)\})$ and highest weight vector $\omega$, then the following two statements are equivalent:

1. $W$ is reducible.
2. There is a non-zero vector $\eta \in \text{Fun}_N(W)$ not proportional to $\omega$, such that $L_{z_i}(\eta) = 0, \forall j > i, z \in \mathbb{C}$.

**Theorem 2.8.** Let $U$ and $W$ be two irreducible highest weight $E_{\gamma, \tau}(gl_N)$-modules with elliptic weight $(\Omega, \{\Lambda_i(z, \lambda)\})$ and highest weight vector $\nu$ and $\omega$ respectively. Then there is an $E_{\gamma, \tau}(gl_N)$-module isomorphism $\phi$ between $U$ and $W$.

**Theorem 2.9.** Let $\gamma$ be an irrational number, $W$ a highest weight $E_{\gamma, \tau}(gl_N)$-module with elliptic weight $(\Omega, \{\Lambda_i(z, \lambda)\})$ and highest weight vector $\omega$, then

1. The functions $\Lambda_i(z, \lambda)$ factorize, that is, they are of the form:

$$\Lambda_i(z, \lambda) = G_{ii}(z)W_i(\lambda), \quad i = 1, \ldots, N. \quad (2.22)$$

2. If the $E_{\gamma, \tau}(gl_N)$-module $W$ is of finite type then there is a 1-dimensional $E_{\gamma, \tau}(gl_N)$-module such that $\tilde{W} = W \otimes U$ has an elliptic weight of the form

$$\Lambda_i(z, \lambda) = G_{ii}(z)W_i(\lambda), \quad i = 1, \ldots, N \quad (2.23)$$
with functions $G_{ii}(z)$ and $W_{ii}(\lambda)$ of the form

$$G_{ii}(z) = \prod_{k=1}^{n_i} \frac{\Theta(z - i_{k}^i)}{\Theta(z - s_{k}^i)}, \quad \sum_{k} i_{k}^i - \sum_{k} s_{k}^i = \gamma(\Omega^i - \Omega^i) \quad (2.24)$$

and

$$W_{ii}(\lambda) = \prod_{m=1}^{i-1} \frac{\Theta(\lambda_m - \lambda_i - \gamma(\Omega^m - \Omega^f))}{\Theta(\lambda_m - \lambda_i)}, \quad i \neq 1, \quad W_{11} = 1. \quad (2.25)$$

### 2.4.1 Proof of Theorem 2.7

**Definition 2.10.** Let $W$ be a highest weight $E_{\gamma,\tau}(gl_N)$-module and $\omega$ a singular vector in $W$, then we define $W(\omega)$ to be the $E_{\gamma,\tau}(gl_N)$-module over $\text{Fun}_N(\mathbb{C})$ generated by words of the form

$$L_{i_1,j_1}(z_1) \ldots L_{i_n,j_n}(z_n)\omega, \quad (2.26)$$

for some generic $z_i \in \mathbb{C}$.

**Lemma 2.11.** Let $\text{Fun}_N(W)$ be a highest weight $E_{\gamma,\tau}(gl_N)$-module of finite type with elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\})$ and highest weight vector $\omega$ and $V$ any $E_{\gamma,\tau}(gl_N)$-module. Moreover let $\phi(\lambda)$ be a morphism of $E_{\gamma,\tau}(gl_N)$-modules

$$\phi(\lambda) : \text{Fun}_N(V) \to \text{Fun}_N(W), \quad (2.27)$$

then:

i) $\text{im} \phi(\lambda) \subset \text{Fun}_N(W)$ is a sub-module.

ii) If the highest weight vector $\omega$ is in the image of $\phi(\lambda)$, then $\text{im} \phi(\lambda) = \text{Fun}_N(W)$.

**Proof.** i) Let $\omega_i$ be any vector in the image of $\phi(\lambda)$, the by definition there is a vector $\nu_i \in \text{Fun}_N(V)$ such that $\omega_i = \phi(\lambda)\nu_i$. This implies that

$$L_{ij}(z)\omega_i = L_{ij}(z)\phi(\lambda)\nu_i = \phi(\lambda)L_{ij}(z)\nu_i \in \text{im} \phi(\lambda) \quad \forall i, j = 1, \ldots, N. \quad (2.28)$$

ii) If the highest weight vector $\omega \in \text{Fun}_N(W)$ is in the image of $\phi(\lambda)$, then the image has to be identically equal to $\text{Fun}_N(W)$ because, by definition, any vector of $\text{Fun}_N(W)$ is a linear combination of vectors of the form

$$L_{i_1,j_1}(z_1)L_{i_2,j_2}(z_2) \ldots L_{i_n,j_n}(z_n)\omega, \quad (2.29)$$

which are in $\text{im} \phi(\lambda)$ by i).

**Lemma 2.12.** Let $W$ be highest weight $E_{\gamma,\tau}(gl_N)$-module with elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\})$ and highest weight vector $\omega$, then the following two statements are equivalent:

i) $W$ is reducible.

ii) There is a non zero vector $\eta \in \text{Fun}_N(W)$ not proportional to $\omega$, such that $L_{ij}(z)\eta = 0, \quad \forall i > j, \quad z \in \mathbb{C}$.
Proof. We first show that i) implies ii). By definition $W$ is reducible only if there is an $E_{\gamma,\tau}(gl_N)$-module $V$ and a non-zero morphism $\phi(\lambda)$ such that the image of $\phi(\lambda)$ isn’t equal to $\text{Fun}_N(W)$.

Now, let $\eta_i$ be a vector of highest weight in the image of $\phi(\lambda)$ and $\nu_i$ one of its pre-images, then

$$L_{ji}(z)\eta_i = L_{ji}(z)\phi(\lambda)\nu_i = \phi(\lambda)L_{ji}(z)\nu_i = 0, \quad \forall j > i,$$

(2.30)

because, by construction, all vectors of $\text{Fun}_N(V)$ of weight higher than the weight of $\eta_i$ are mapped to zero. Now, using Lemma 2.11 it follows that the weight of $\eta_i$ is lower than that of $\omega$, and this proves the claim.

Now we show that ii) implies i).

Let $\eta$ be a singular vector of $W$ not proportional to $\omega$, then from Lemma 2.3 (see Remark 4) it follows that $W(\eta)$ contains only vectors of weight lower that the weight of $\omega$. Using this, we can construct a non-surjective morphism

$$\phi(\lambda) : W(\eta) \rightarrow W$$

(2.31)

by defining $\phi(\lambda)$ to be the natural inclusion. □

Remark 4. Strictly speaking, Lemma 2.11 can be applied only to highest weight modules. It is however easy to see that it also remains true in this case.

**Definition 2.13.** Let $\text{Fun}_N(W)$ be a highest weight $E_{\gamma,\tau}(gl_N)$-module of finite type with elliptic weight $(\Omega, \{\Lambda_{\alpha}(z, \lambda)\})$ and highest weight vector $\omega$, then we define the sequence of modules $\text{Fun}_N(U^i), i \geq 0$ inductively by:

$$\text{Fun}_N(U^0) = 0,$$

$$\text{Fun}_N(U^i) \supset \text{Fun}_N(U^{i-1}),$$

(2.32)

where $\text{Fun}_N(U^i)$ is the sub-module generated by all vectors which generate the module $\text{Fun}_N(U^{i-1})$ plus all vectors of $\text{Fun}_N(W)$, not proportional to $\omega$, such that

$$L_{ji}(z)\eta \in \text{Fun}_N(U^{i-1}), \forall j > i, \eta \in \mathbb{C}.$$  

(2.33)

Moreover, we set $\text{Fun}_N(\bar{U}) = \bigcup_{i \in \mathbb{N}} \text{Fun}_N(U^i)$.

**Lemma 2.14.** Let $\text{Fun}_N(W)$ be a highest weight $E_{\gamma,\tau}(gl_N)$-module of finite type with elliptic weight $(\Omega, \{\Lambda_{\alpha}(z, \lambda)\})$ and highest weight vector $\omega$, then the $E_{\gamma,\tau}(gl_N)$-submodule of $\text{Fun}_N(W)$, $\text{Fun}_N(\bar{U})$ defined in 2.13, is well defined and unique.

Proof. The fact that $\text{Fun}_N(\bar{U})$ is well defined follows from the fact that all spaces of fixed weight $\text{Fun}_N(W[\mu])$ are finite-dimensional. In fact this implies that for each $\mu \in \mathfrak{h}^*$ there is an $i_0 \geq 0$ such that

$$\text{Fun}_N(U^j)[\mu] = \text{Fun}_N(U^{i_0})[\mu], \quad \forall j \geq i_0.$$  

(2.34)

Let us now prove that it is unique. In order to do this we assume that there is an other $E_{\gamma,\tau}(gl_N)$-submodule of $\text{Fun}_N(W)$, $\text{Fun}\bar{Q}$ (and an other sequence $\text{Fun}_N(Q^i), i \geq 0$) which satisfies Definition 2.13.

We prove that $\text{Fun}_N(\bar{Q})$ is equal to $\text{Fun}_N(\bar{U})$ by induction. $\text{Fun}_N(Q^0)$ is clearly equal to $\text{Fun}_N(U^0)$.

Assume now that $\text{Fun}_N(Q^{i-1}) = \text{Fun}_N(U^{i-1})$, then it follows from Definition 2.13.
that \( \text{Fun}_N(Q') \) is generated by all vectors which generate \( \text{Fun}_N(Q'^{-1}) \) plus all vectors of \( \text{Fun}_N(W) \), not proportional to \( \omega \), such that

\[
L_{ji}(z)\eta \in \text{Fun}_N(Q'^{-1}), \forall j > i, z \in \mathbb{C}
\]  

(2.35)

But, this vectors are the same vectors which are "added" to \( \text{Fun}_N(Q') \), because by the induction hypothesis \( \text{Fun}_N(Q'^{-1}) = \text{Fun}_N(U'^{-1}) \).

**Definition 2.15.** We call the maximal sub-module of Corollary 2.16 the maximal sub-module generated by all singular vectors of \( W \).

**Corollary 2.16.** Let \( \text{Fun}_N(W) \) be a highest weight \( E_{\gamma, r}(gl_N) \)-module of finite type with elliptic weight \((\Omega, \{\lambda_i(z, \lambda)\})\) and highest weight vector \( \omega \), then there is an irreducible \( E_{\gamma, r}(gl_N) \)-module \( \bar{W} \) with the same elliptic weight.

**Proof.** If \( W \) is irreducible there is nothing to prove.

It suffices to show that

\[
\text{Fun}_N(\bar{W}) = \text{Fun}_N(W)/\text{Fun}_N(U),
\]

(2.36)

where \( \text{Fun}_N(U) \) is the maximal proper \( E_{\gamma, r}(gl_N) \)-submodule of the space \( \text{Fun}_N(W) \), is an irreducible highest weight module.

That \( \text{Fun}_N(\bar{W}) \) is a highest weight module follows directly from the fact that \( \text{Fun}_N(W) \) is a highest weight module. So, let us show that it is irreducible.

Assume that it is not the case, then there is a non-zero vector \( \eta \in \text{Fun}_N(\bar{W}) \), not proportional to the highest weight vector of \( \text{Fun}_N(W) \), such that

\[
L_{ji}(z)\eta = 0, \forall j > i, z \in \mathbb{C}.
\]

(2.37)

Now, let \( \eta \in \text{Fun}_N(W) \) be any lift of \( \bar{\eta} \) to \( \text{Fun}_N(W) \), then (2.37) implies

\[
L_{ji}(z)\eta \in \text{Fun}_N(U), \forall j > i, z \in \mathbb{C}.
\]

(2.38)

But this is impossible by construction. In fact, let \( \mu \) be the weight of \( \eta \), then there is an \( i_0 > 0 \) such that

\[
\eta \in \text{Fun}_N(U'[^0\mu]), \forall i > i_0.
\]

(2.39)

\[\square\]

### 2.4.2 Proof of Theorem 2.8

In order to prove this Theorem we first define a linear map \( \phi(\lambda) \) from \( \text{Fun}_N(W) \) to \( \text{Fun}_N(V) \), then we prove that this map is actually a morphism, and in the last step that it is an isomorphism.

1. Let \( e_{ij}(t_1, t_2, \ldots, t_n) = L_{i_1,j_1}(t_1)L_{i_2,j_2}(t_2) \ldots L_{i_n,j_n}(t_n)\omega \) be a basis of \( W \) such that the words \( e_{ij} \) are in normal form, the we define \( \phi(\lambda) \) by the rules

\[
1. \ \phi(\lambda)\omega = \nu.
\]

2. \( \phi(\lambda)e_{ij}(t_1, t_2, \ldots, t_n) = L_{i_1,j_1}(t_1)L_{i_2,j_2}(t_2) \ldots L_{i_n,j_n}(t_n)\phi(\lambda)\omega.\)

3. Extend \( \phi(\lambda) \) \( \text{Fun}_N(\mathbb{C}) \)-linearly to \( \text{Fun}_N(W) \) and \( \text{Fun}_N(V) \).

Until to now \( \phi(\lambda) \) is "only" a linear map between vector spaces over the field \( \text{Fun}_N(\mathbb{C}) \), in order to show that \( \phi(\lambda) \) is a morphism, we have to prove that

\[
\phi(\lambda)L_{ij}(z) = \phi(\lambda)L_{ij}(z), \ \forall i, j = 1, \ldots, N, z \in \mathbb{C}.
\]

(2.40)
We prove (2.40) by induction on the weight of the spaces on which \( L_t(z) \) acts.

a) \( L_t(z) \phi(\lambda) \omega = 0 = \phi(\lambda) L_t(z) \omega, \forall j > i. \)

b) \( L_t(z) \phi(\lambda) \omega = \phi(\lambda) L_t(z) \omega \) because by definition the two \( E_{\gamma,t}(gl_N) \)-modules \( W, V \) have the same elliptic weight.

c) Consider the vector

\[
\eta = L_{i,i+n}(z) \phi(\lambda) \omega - \phi(\lambda) L_{i,i+n}(z) \omega, \quad n \leq N - i, \tag{2.41}
\]

and let \( \eta_{k,l} \) be the elements of \( \text{Fun}_N(V) \):

\[
\eta_{k,l} = L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) \eta, \quad k_i > l_i \quad \forall i = 1, \ldots, n. \tag{2.42}
\]

We first prove that \( \eta_{k,l} \) is identically zero for all choices of \( k_i > l_i \) and \( z_i \in \mathbb{C} \) generic.

In order to do this, we notice that \( \eta_{k,l} \) is identically zero unless:

\[
k_i \neq k_j, \quad \forall i \neq j \in \{1, \ldots, n\}
\]

\[
k_s = l_s + 1, \quad \forall s = 1, \ldots, n \tag{2.43}
\]

In fact, in all other cases the weight \( \mu(\eta_{k,l}) \) would not be of the form

\[
\tilde{\mu} = \mu(\nu) - \sum_{i=1}^{N-1} r_i e_i, \quad r_i \in \mathbb{N}, \tag{2.44}
\]

for some \( r_i \in \mathbb{N} \), and would therefore vanish because these are the weights with a space \( \text{Fun}_N(V[\tilde{\mu}]) \) of non-zero dimension.

Assume now that (2.43) holds and let us rewrite \( \eta_{k,l} \) as:

\[
\eta_{k,l} = \eta_{k,l}^{(1)} + \eta_{k,l}^{(2)}, \tag{2.45}
\]

where

\[
\eta_{k,l}^{(1)} = L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) \eta = \phi(\lambda) L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) \phi(\lambda) \omega - \phi(\lambda) L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) L_{i,i+n}(z) \omega, \tag{2.46}
\]

\[
\eta_{k,l}^{(2)} = \phi(\lambda) L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) L_{i,i+n}(z) \omega - L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) \phi(\lambda) L_{i,i+n}(z) \omega. \tag{2.47}
\]

We show that \( \eta_{k,l}^{(1)} = \eta_{k,l}^{(2)} = 0. \)

Let us begin with the proof that \( \eta_{k,l}^{(1)} = 0. \) The weight of the vectors

\[
L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) L_{i,i+n}(z) \phi(\lambda) \omega, \phi(\lambda) L_{k_1,i_1}(z_1) L_{k_2,i_2}(z_2) \ldots L_{k_n,i_n}(z_n) L_{i,i+n}(z) \omega \tag{2.48}
\]

is equal to the weight of the highest weight vector of \( V \). This implies that they are proportional to it, because \( V \) is irreducible. Moreover, the proportionality factors, which are functions of \( \lambda, z_1, \ldots, z_n, z \), can be computed putting the two products in normal form, using the relations of \( E_{\gamma,t}(gl_N) \). This means that they will depend only on the universal functions \( \alpha_{ij}(\lambda), \beta_{ij}(\lambda) \) and the elliptic weight functions...
2.4. Structure Theorems

\[ \Lambda_i(w, \lambda), i = 1, \ldots, N. \] Now, because we can compute them using the same sequence of "commutations" and because the weight functions \( \Lambda_i(w, \lambda), i = 1, \ldots, N \) are the same for both spaces, they will coincide. This means that

\[ \eta_{k,i}^{(1)} = F(\lambda, z_1, \ldots, z_n, z)\phi(\lambda)\omega - \phi(\lambda)F(\lambda, z_1, \ldots, z_n, z)\omega = 0, \quad (2.49) \]

because \( \phi(\lambda) \) is \( \text{Fun}_N(\mathbb{C}) \)-linear.

Let us now show that \( \eta_{k,i}^{(1)} = 0 \). This can be achieved by writing \( L_{i,i+n}(z)\omega \) in the chosen basis:

\[ L_{i,i+n}(z)\omega = \sum_{\alpha, \beta} c_{\alpha, \beta} e_{\alpha, \beta}. \quad (2.50) \]

Let us now focus on a single element of the sum (2.50). From the definition of \( \phi(\lambda) \) it follows that

\[ \phi(\lambda)e_{\alpha, \beta} = \phi(\lambda)L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2)\cdots L_{\alpha_m, \beta_m}(t_m)\omega = \]

\[ L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2)\cdots L_{\alpha_m, \beta_m}(t_m)\phi(\lambda)\omega. \quad (2.51) \]

This implies that \( \eta_{k,i}^{(2)} \) can be rewritten as a sum of terms of the form:

\[ \phi(\lambda)L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2)\cdots L_{\alpha_m, \beta_m}(t_m)\omega - \]

\[ L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2)\cdots L_{\alpha_m, \beta_m}(t_m)\phi(\lambda)\omega. \quad (2.52) \]

Using the same argument as before it follows then that

\[ \eta_{k,i}^{(2)} = \sum_{\alpha, \beta} c_{\alpha, \beta} \left( \phi(\lambda)F_{\alpha, \beta}(\lambda, z_1, \ldots, z_n, z)\omega - \right. \]

\[ \left. F_{\alpha, \beta}(\lambda, z_1, \ldots, z_n, z)\phi(\lambda)\omega \right), \quad (2.53) \]

which is identically zero because of the \( \text{Fun}_N(\mathbb{C}) \)-linearity of \( \phi(\lambda) \).

Up to now, we have only proved that \( \eta_{k,i} = 0, \quad \forall k_1 > l_i \). But our goal is to show that

\[ \eta = L_{i,i+n}(z)\phi(\lambda)\omega - \phi(\lambda)L_{i,i+n}(z)\omega = 0. \quad (2.54) \]

In order to fill this gap, we rewrite \( \eta_{k,i} \) as

\[ \eta_{k,i} = L_{k_1,l_1}(z_1)\eta_{k,i}. \quad (2.55) \]

What we have proved up to now is equivalent to

\[ L_{k_1,l_1}(z_1)\eta_{k,i} = 0, \quad \forall k_1 > l_1, z \in \mathbb{C}. \quad (2.56) \]

But because \( V \) is irreducible, it means that

\[ \eta_{k,i} = 0 \text{ or } \eta_{k,i} = \omega, \quad c \in \text{Fun}_N(\mathbb{C}). \quad (2.57) \]

Using the same argument as before (2.45)-(2.53), one can show that the factor \( c \in \text{Fun}_N(\mathbb{C}) \) is actually equal to zero, and this means that we can apply the same procedure to \( \eta_{k,i}^2 \) eliminating one after the other, all \( L_{k_2,l_2}(z_2) \) in front of \( \eta \). What
remains is the claim.

Let us now show, by induction on the weight of the vectors on which $\phi(\lambda)$ acts, that it is a morphism. In order to do this, we assume that

$$L_{i,j}(z)\phi(\lambda)\tilde{\omega} = \phi(\lambda)L_{i,j}(z)\tilde{\omega}, \quad \forall i, j, z \in \mathbb{C}, \quad (2.58)$$

holds for all vectors $\tilde{\omega}$ of weight of the form

$$\mu(\tilde{\omega}) = \mu(\omega) - \sum_{i=1}^{N-1} k_i e_i, \quad k_i \geq 0,$$

$$\sum_{i=1}^{N-1} k_i \leq n - 1, \quad (2.59)$$

and show that this implies that

$$L_{i,j}(z)\phi(\lambda)\tilde{\omega} = \phi(\lambda)L_{i,j}(z)\tilde{\omega}, \quad \forall i, j, z \in \mathbb{C}, \quad (2.60)$$

for all vector of weight of the form

$$\mu(\tilde{\omega}) = \mu(\nu) - \sum_{i=1}^{N-1} k_i e_i, \quad k_i \geq 0,$$

$$\sum_{i=1}^{N-1} k_i = n. \quad (2.61)$$

Let $\tilde{\omega}$ be any vector with property (2.61) we show that

1. $L_{ji}(z)\phi(\lambda)\tilde{\omega} = \phi(\lambda)L_{ji}(z)\tilde{\omega}, \quad \forall j > i$
2. $L_{ii}(z)\phi(\lambda)\tilde{\omega} = \phi(\lambda)L_{ii}(z)\tilde{\omega}, \quad \forall i$
3. $L_{i,i+n}(z)\phi(\lambda)\tilde{\omega} = \phi(\lambda)L_{i,i+n}(z)\tilde{\omega}, \quad \forall i, i + n \leq N$

The proof of the first statement is relatively simple. In fact, using

$$L_{i,j}(z)\phi(\lambda)\tilde{\omega} - \phi(\lambda)L_{i,j}(z)\tilde{\omega} = \sum_{\alpha, \beta} c_{\alpha, \beta} L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2) \ldots L_{\alpha_m, \beta_m}(t_m)\omega,$$  

yields

$$L_{i,j}(z)\phi(\lambda)\tilde{\omega} - \phi(\lambda)L_{i,j}(z)\tilde{\omega} = \sum_{\alpha, \beta} c_{\alpha, \beta} L_{i,j}(z)L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2) \ldots L_{\alpha_m, \beta_m}(t_m)\phi(\lambda)\tilde{\omega} - \sum_{\alpha, \beta} c_{\alpha, \beta} \phi(\lambda)L_{i,j}(z)L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2) \ldots L_{\alpha_m, \beta_m}(t_m)\tilde{\omega}. \quad (2.62)$$

Let us now focus on one term of the previous sum over the indices $\alpha, \beta$

$$L_{i,j}(z)L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2) \ldots L_{\alpha_m, \beta_m}(t_m)\phi(\lambda)\tilde{\omega} - \phi(\lambda)L_{i,j}(z)L_{\alpha_1, \beta_1}(t_1)L_{\alpha_2, \beta_2}(t_2) \ldots L_{\alpha_m, \beta_m}(t_m)\tilde{\omega}. \quad (2.63)$$

Both term in (2.64) can be rewritten as linear combinations of words in normal form with the coefficients depending only on the universal functions $\alpha_{ij}, \beta_{ij}$ and the
weight functions $\Lambda_{kl}(w, \lambda)$ that are equal for both modules. Moreover the weight of each term in the remaining sum is in the set (2.59), so the $L$'s can be freely commuted with $\phi(\lambda)$ by the induction hypothesis. And this means that (2.64) is equal to zero.

The proof of the second statement is also rather simple. To see that it holds we show that

$$L_{kl}(w)(L_{ii}(z)\phi(\lambda) - \phi(\lambda)L_{ii}(z))\tilde{\omega} = 0, \quad \forall k > l, w \in \mathbb{C}. \quad (2.65)$$

From the defining relations of $E_{\gamma,r}(gl_N)$ we have the two relations

$$L_{kl}(w)L_{ii}(z) = r_1(w - z, \lambda)L_{ii}(z)L_{kl}(w) + r_2(w - z, \lambda)L_{ii}(w)L_{kl}(z), \quad r_3(w - z, \lambda)L_{ki}(w)L_{li}(z) + r_4(w - z, \lambda)L_{kl}(z)L_{ii}(w), \quad (2.66)$$

for some functions $r_i(w - z, \lambda), i = 1, \ldots, 5$. Now, the relative order of the indices $i, k, l$ can be

$$k > l \geq i, \quad i \geq k > l, \quad k > i > l. \quad (2.67)$$

Let us consider explicitly only the first case $k > l \geq i$, then

$$L_{kl}(w)L_{ii}(z)\phi(\lambda)\tilde{\omega} =$$

$$= r_1(w - z, \lambda)L_{ii}(z)L_{kl}(w) + r_2(w - z, \lambda)L_{ii}(w)L_{kl}(z) + r_3(w - z, \lambda)L_{ii}(z)L_{ki}(w)L_{li}(z) + r_4(w - z, \lambda)L_{ii}(w)L_{kl}(z)L_{li}(w), \quad (2.68)$$

because of 1. Now the weight of the vectors on the right of $\phi(\lambda)$ are in the set (2.59). This implies that we can commute also the other $L$'s, by the induction hypothesis. Using then the relation in the "opposite" direction, we get the desired result.

Let us now show that 3. holds. In order to do this we use a trick similar to the one used in the first induction step. We show that

$$\eta_{k,l} = L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n)(L_{i_1,i_{n+1}}(z_{n+1})\phi(\lambda)\tilde{\omega} - \phi(\lambda)L_{i_{1+n},i_{n+1}}(z_{n+1})\tilde{\omega}) \quad (2.69)$$

is identically zero for all $k_s > l_s, s = 1, \ldots, n + 1$ by splitting (2.69) in

$$\eta_{k,l} = \eta_{k,l}^{(1)} + \eta_{k,l}^{(2)}, \quad (2.70)$$

where

$$\eta_{k,l}^{(1)} = L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n)(\phi(\lambda)L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n)\phi(\lambda)\tilde{\omega} - \phi(\lambda)L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n)\tilde{\omega}) \quad \eta_{k,l}^{(2)} = \phi(\lambda)L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n+1)\phi(\lambda)L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n+1)\tilde{\omega} \quad (2.71)$$

As usual we start by showing that $\eta_{k,l}^{(1)}$ is identically zero. This can be done by rewriting

$$L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n+1)\phi(\lambda)L_{k_1,i_1}(z_1)L_{k_2,i_2}(z_2) \cdots L_{k_n,i_n}(z_n+1)\tilde{\omega}. \quad (2.72)$$
as a linear combination of words of the form
\[ L_{i_1,j_1} \cdots L_{i_s,j_s} L_{j_{s+1},i_{s+1}} \cdots L_{j_n,i_n+1} \cdots \quad s \leq n, \] (2.73)
where
\[ i_k \leq j_k, \quad \forall k = 1, \ldots, s, \quad i_k < j_k, \quad k = s + 1, \ldots, n. \] (2.74)

Then, by the first statement we get
\[ L_{i_1,j_1} \cdots L_{i_s,j_s} L_{j_{s+1},i_{s+1}} \cdots L_{j_n,i_n+1} \phi(\lambda) \omega = L_{i_1,j_1} \cdots L_{i_s,j_s} L_{j_{s+1},i_{s+1}} \cdots L_{j_n,i_n+1} \phi(\lambda) L_{j_{s+1},i_{s+1}} \omega. \] (2.75)

It follows from the fact that the weight of the vector on the right of \( \phi(\lambda) \) is always in the set (2.59) that we can commute also the remaining \( L' \)'s. To prove that \( \eta^{(2)}_{k,d} \) is zero we write \( \omega \) in the basis we chose and use a similar argument. The last step is to show that
\[ \eta_{k,d} = 0, \quad \forall k > i \Rightarrow L_{i+i+n}(z) \phi(\lambda) \omega = \phi(\lambda) L_{i+i+n}(z). \] (2.76)
This can be achieved by eliminating all \( L_{k,s}(z) \) in the same way as we did it in the first induction step.

Let us now show, that the morphism \( \phi(\lambda) \) is bijective. From the definition of irreducibility given in Section 1.3, it follows directly that \( \phi(\lambda) \) is surjective because it is, by construction, non-zero.

It remains to be shown, that \( \phi(\lambda) \) is injective. In order to do this, it suffices to prove that:
\[ \ker \phi(\lambda) = 0. \] (2.77)

Let \( w_1 \) be an element of \( \ker \phi(\lambda) \). Then, from
\[ 0 = L_1(w) \phi(\lambda) w_1 = \phi(\lambda) L_1(w) w_1, \quad \forall 1 \leq i, j \leq N, z \in \mathbb{C}, \] (2.78)
it follows that \( \ker \phi(\lambda) \) is a sub-module of \( W \). This implies that, if \( \ker \phi(\lambda) \neq 0 \), we can construct a non-surjective, non-zero morphism
\[ \bar{\phi}(\lambda) : \ker \phi(\lambda) \to \text{Fun}_N(W), \] (2.79)
with \( \bar{\phi}(\lambda) \) given by the natural inclusion. But this would be in contradiction with the irreducibility of \( W \).

**Remark 5.** It is possible to relax a little bit the hypothesis of Theorem 2.8, and require only that \( V \) is irreducible. In this case the map \( \phi(\lambda) \), defined in Theorem 2.8 is still a morphism but, of course, not necessarily an injective one.

### 2.4.3 Proof of Theorem 2.9

**Lemma 2.17.** Let \( \gamma \) be an irrational number, \( W \) a highest weight \( \mathfrak{gl}_N \)-module with weight vector \((\Omega, \{ \Lambda_i(z, \lambda) \}_{i=1}^{N} \})\) and highest weight vector \( \omega \), then the functions \( \Lambda_i(z, \lambda) \) \( i = 1, \ldots, N \) are of the form:
\[ \Lambda_i(z, \lambda) = G_i(z) W_i(\lambda). \]
Proof. From the defining relations \((i = k, l = j)\) we have:

\[
\{\alpha_{ji}(z_1 - z_2, \lambda - \gamma h) \mathcal{L}_{ji}(z_1) \mathcal{L}_{ji}(z_2) + \beta_{ji}(z_1 - z_2, \lambda - \gamma h) \mathcal{L}_{ij}(z_1) \mathcal{L}_{ji}(z_2)\} \omega
\]

\[
= \{\alpha_{ji}(z_1 - z_2) \mathcal{L}_{ii}(z_2) \mathcal{L}_{ji}(z_1) + \beta_{ji}(z_1 - z_2) \mathcal{L}_{ij}(z_2) \mathcal{L}_{ji}(z_1)\} \omega
\]

(2.80)

and

\[
\mathcal{L}_{ii}(z_1) \mathcal{L}_{ii}(z_2) \omega = \mathcal{L}_{ii}(z_2) \mathcal{L}_{ii}(z_1) \omega.
\]

(2.81)

So for \(j > i\) we get the system of equations:

\[
\alpha_{ji}(z_1 - z_2, \lambda - \gamma h) \mathcal{L}_{ji}(z_1, \lambda) \mathcal{L}_{ii}(z_2, \lambda - \gamma h^j) = \alpha_{ji}(z_1 - z_2, \lambda) \mathcal{L}_{ii}(z_2, \lambda) \mathcal{L}_{ji}(z_1, \lambda - \gamma h^j)
\]

(2.82)

and

\[
\mathcal{L}_{ii}(z_1, \lambda) \mathcal{L}_{ii}(z_2, \lambda - \gamma h^j) = \mathcal{L}_{ii}(z_2, \lambda) \mathcal{L}_{ii}(z_1, \lambda - \gamma h^j).
\]

(2.83)

Using the explicit form of the functions \(\alpha\) and \(\beta\) we get

\[
\frac{\alpha_{ji}(z_1 - z_2, \lambda - \gamma (\Omega^i - \Omega^j))}{\alpha_{ji}(z_1 - z_2, \lambda - \gamma h^j)} = \frac{\Theta(\lambda_i - \lambda_j - \gamma(\Omega^i - \Omega^j) + \gamma) \Theta(\lambda_i - \lambda_j)}{\Theta(\lambda_i - \lambda_j - \gamma(\Omega^i - \Omega^j)) \Theta(\lambda_i - \lambda_j + \gamma)}
\]

(2.84)

This implies that:

\[
\frac{\mathcal{L}_{ii}(z_1, \lambda - \gamma h^j) \mathcal{L}_{ji}(z_2, \lambda)}{\mathcal{L}_{ii}(z_1, \lambda) \mathcal{L}_{jj}(z_2, \lambda - \gamma h^j)} = \frac{\Theta(\lambda_i - \lambda_j - \gamma(\Omega^i - \Omega^j) + \gamma) \Theta(\lambda_i - \lambda_j)}{\Theta(\lambda_i - \lambda_j - \gamma(\Omega^i - \Omega^j)) \Theta(\lambda_i - \lambda_j + \gamma)}
\]

(2.85)

The right hand side of equation (2.85) is constant as a function of \(z_1\) and \(z_2\), so the right hand side has to be \(z_1, z_2\) independent too. This implies that

\[
\frac{\mathcal{L}_{ii}(z, \lambda - \gamma h^j)}{\mathcal{L}_{ii}(z, \lambda)} = F_i^j(\lambda), \quad \forall i, j \in \{1, \ldots, n\}.
\]

(2.86)

Now, let \(H_i(z, \lambda)\) be the functions

\[
H_i(z, \lambda) = \frac{\mathcal{L}_{ii}(z, \lambda)}{\mathcal{L}_{ii}(z_0, \lambda)}
\]

(2.87)

These functions are by definition 1-periodic in \(\lambda\) and using equation (2.86) one can see that \(H_i(z, \lambda)\) is also \(\gamma\) periodic in \(\lambda\). In fact:

\[
\frac{H_i(z, \lambda)}{H_i(z, \lambda - \gamma h^j)} = \frac{\mathcal{L}_{ii}(z, \lambda) \mathcal{L}_{ii}(z_0, \lambda - \gamma h^j)}{\mathcal{L}_{ii}(z, \lambda - \gamma h^j) \mathcal{L}_{ii}(z_0, \lambda)} = \frac{F_i^j(\lambda)}{F_i^j(\lambda)} = 1.
\]

(2.88)

This implies that the functions \(H_i(z, \lambda)\) are constant in \(\lambda\) because \(\gamma\) is an irrational number and there are no \((1, \gamma)\)-periodic functions other than the constant ones. \(\Box\)
Lemma 2.18. Let \( \gamma \) be an irrational number and \( W \) a 1-dimensional highest weight \( E_{\gamma,\tau}(gl_N) \)-module elliptic weight \( (\Omega, \{\Lambda_{ii}(z, \lambda)\}_{i=1}^{n}) \) and highest weight vector \( \omega \), then:

1. \( L_{ij}(z), \quad i \neq j \) act by zero.
2. \( \gamma(\Omega_i - \Omega_j) = q_1 + q_2 \tau, \quad q_1, q_2 \in \mathbb{Z} \).

Proof. 1. \( L_{ij}(z), \quad i \neq j \) act by zero because all the space of weight different from \( \Omega \) are 0-dimensional.

2. In addition to the equations for \( A_{M} \) of the previous Lemma, we have:

\[
\beta_{ij}(z_1 - z_2, \lambda - \gamma \Omega)\Lambda_{ii}(z_1, \lambda)\Lambda_{jj}(z_2, \lambda - \gamma h^i) = \beta_{ij}(z_1 - z_2, \lambda)\Lambda_{ii}(z_2, \lambda)\Lambda_{jj}(z_1, \lambda - \gamma h^i) \quad (2.91)
\]

From Lemma 2.17 we know that the weight functions \( \Lambda_{ii}(z, \lambda) \) are of the form \( \Lambda_{ii}(z, \lambda) = G_{ii}(z)W_{ii}(\lambda) \). This implies:

\[
\frac{G_{ii}(z_1)G_{jj}(z_2)}{G_{ii}(z_2)G_{jj}(z_1)} = \frac{\beta_{ij}(z_1 - z_2, \lambda)}{\beta_{ij}(z_1 - z_2, \lambda - \gamma \Omega)} = \frac{\Theta(z_1 - z_2 + \lambda_i - \lambda_j)}{\Theta(z_1 - z_2 + \lambda_i - \lambda_j - \gamma(\Omega^i - \Omega^j))} \quad (2.92)
\]

\[
\Rightarrow \gamma(\Omega^i - \Omega^j) = q_1 + q_2 \tau, \quad q_1, q_2 \in \mathbb{Z} \quad (2.93)
\]

Lemma 2.19. Let \( \gamma \) be an irrational number, \( W \) a highest weight module with elliptic weight \( (\Omega, \{\Lambda_{ii}\}) \), then there is a 1-dimensional \( E_{\gamma,\tau}(gl_N) \)-module \( U \) such that the weight functions of \( \tilde{W} = U \otimes W \) are of the form \( \tilde{\Lambda}_{ii}(z, \lambda) = G_{ii}(z)W_{ii}(\lambda) \)

\[
W_{11}(\lambda) = 1, \quad W_{kk}(\lambda) = \frac{\prod_{m=1}^{k-1} \Theta(\lambda_m - \lambda_k - \gamma(\Omega^m - \Omega^k))}{\Theta(\lambda_m - \lambda_k)}. \quad (2.94)
\]

Proof. Let \( w \) be the highest weight vector of \( W \), then by definition

\[
L_{ii}(z)w = 0, \quad \forall j > i. \quad (2.95)
\]

Using the defining relations of \( E_{\gamma,\tau}(gl_N) \) we get the system of equations:

\[
\alpha_{ji}(z_1 - z_2, \lambda - \gamma h)\Lambda_{jj}(z_1, \lambda)\Lambda_{ii}(z_2, \lambda - \gamma h^i) = \alpha_{ji}(z_1 - z_2, \lambda)\Lambda_{jj}(z_2, \lambda)\Lambda_{ii}(z_1, \lambda - \gamma h^i) \quad (2.96)
\]

In order to find the general solution of (2.96) we notice:

1. \( \Lambda_{ii}(z, \lambda) = G_{ii}(z)W_{ii}(\lambda) \) from Lemma 2.17

2. The functions \( W_{ii}(\lambda) \) are solution of

\[
\Theta(\lambda_i - \lambda_j + \gamma(\Omega^i - \Omega^j) + \gamma)W_{ii}(\lambda)W_{jj}(\lambda - \gamma h^i) = \Theta(\lambda_i - \lambda_j + \gamma)W_{jj}(\lambda)W_{ii}(\lambda - \gamma h^i) \quad (2.97)
\]
3. Let $W_{ii}(\lambda)$ be a solution of (2.97) and $F_{ii}(\lambda)$ a solution of,

$$F_{ii}(\lambda)F_{jj}(\lambda - \gamma h^i) = F_{jj}(\lambda)F_{ii}(\lambda - \gamma h^i), \forall i, j,$$  

then $W_{ii}(\lambda) = W_{ii}(\lambda)F_{ii}(\lambda)$ is a new solution of (2.97). The equations (2.98) are the equations for the $\lambda$-part of a 1-dimensional $E_{\gamma, r}(gl_N)$-module with $\Omega = 0$.

4. Let $W_{ii}(\lambda)$ and $\tilde{W}_{ii}(\lambda)$ be two solutions of (2.97), then $F_{i} = \frac{W_{ii}(\lambda)}{\tilde{W}_{ii}(\lambda)}$ is a solution of (2.98).

5. The functions

$$W_{ii}(\lambda) = 1, \quad W_{kk}(\lambda) = \prod_{m=1}^{k-1} \frac{\Theta(\lambda_m - \lambda_k - \gamma(\Omega^m - \Omega^k))}{\Theta(\lambda_m - \lambda_k)}$$  

are a solution of (2.97).

\[ \square \]

**Definition 2.20.** An $E_{\gamma, r}(gl_N)$-module $W$ with elliptic weight $(\Omega, \{\Lambda_{ii}(z, \lambda)\}_{i=1}^N)$ is in canonical form if:

$$\Lambda_{ii} = G_{ii}(z) \prod_{m=1}^{k-1} \frac{\Theta(\lambda_m - \lambda_k - \gamma(\Omega^m - \Omega^k))}{\Theta(\lambda_m - \lambda_k)}, \quad i \neq 1, \quad \Lambda_{11} = 1.$$  

**Lemma 2.21.** Let $\gamma$ be an irrational number, $W$ a highest weight $E_{\gamma, r}(gl_N)$-module of finite type in canonical form, then

$$\Lambda_{ii}(z, \lambda) = c_t \prod_{k=1}^{n_i} \frac{\Theta(z - t_k)}{\Theta(z - s_k)} W_{ii}(\lambda), \quad c_t \in \mathbb{C},$$  

with $\sum t_k - \sum s_k = \gamma(\Omega^i - \Omega^j)$.

**Proof.** Let $w$ be the highest weight vector of $W$, $j \in \{2, \ldots, N\}$, $i = j - 1$ and $d_j$ the dimension of the space of weight $\mu = \Omega - h^i + h^j$. Then there are, by definition, $z_1, \ldots, z_{d_j} \in \mathbb{C}$ such that the vectors

$$L_{ij}(z_l)w, \quad l = 1, \ldots, d_j$$  

are a basis of $\text{Fun}_{\mathbb{C}}(W[\Omega - h^i + h^j])$. This means that for all $z_0 \in \mathbb{C}$ there are functions $c_t(\lambda, z_0) \in \text{Fun}_{\mathbb{C}}(\mathbb{C})$ such that

$$L_{ij}(z_0)w + \sum_{l=1}^{d_j} c_t(\lambda, z_0)L_{ij}(z_l)w = 0.$$  

Using that

$$L_{ji}(x)L_{ij}(z) = \frac{\beta_{ij}(x - z, \lambda - \gamma\Omega)}{\alpha_{ij}(x - z, \lambda - \gamma\Omega)} L_{ij}(x)L_{ji}(z) + \frac{\alpha_{ij}(x - z, \lambda)}{\alpha_{ji}(x - z, \lambda - \gamma\Omega)} L_{ii}(z)L_{ji}(x) + \frac{\beta_{ij}(x - z, \lambda)}{\alpha_{ji}(x - z, \lambda - \gamma\Omega)} L_{ii}(z)L_{jj}(x)$$

(2.104)
we get:

\[ 0 = L_{ji}(x) \sum_{l=0}^{d_j} c_l(\lambda, z_0) L_{ij}(z_l) w = \sum_{l=0}^{d_j} c_l(\lambda - \gamma h^i, z_0) \left\{ \frac{\beta_{ij}(x - z_l, \lambda)}{\alpha_{ij}(x - z_l, \lambda - \gamma \Omega)} L_{ii}(z_l) L_{jj}(x) \right\} \]

This implies:

\[ 0 = \sum_{l=0}^{d_j} c_l(\lambda - \gamma h^i, z_0) \left\{ \frac{\beta_{ij}(x - z_l, \lambda)}{\alpha_{ij}(x - z_l, \lambda - \gamma \Omega)} G_{ii}(z_l) G_{jj}(x) \right\} \]

Using the explicit expressions for \( \alpha \) and \( \beta \), we get:

\[ 0 = \sum_{l=0}^{d_j} c_l(\lambda - \gamma h^i, z_0) \left\{ \frac{\Theta(x - z_l + \lambda_j - \lambda_i)}{\Theta(x - z_l) \Theta(\lambda_j - \lambda_i)} G_{ii}(z_l) G_{jj}(x) \right\} \]

\[ - \frac{\Theta(x - z_l + \lambda_j - \lambda_i - \gamma(\Omega^j - \Omega^i))}{\Theta(x - z_l) \Theta(\lambda_j - \lambda_i - \gamma(\Omega^j - \Omega^i))} G_{ii}(x) G_{jj}(z_l) \]  

From the last equation it follows that the functions

\[ F_{jj}(x) = \frac{G_{jj}(x)}{G_{ii}(x)} \]

satisfy the functional equations:

\[ F_{jj}(x + 1) = F_{jj}(x), \quad F_{jj}(x + \tau) = e^{2 \pi i (\Omega^j - \Omega^i)} F_{jj}(x). \]  

This means that the functions \( F_{jj}(x) \) are products of ratios of theta functions. Using that \( F_{22}(x) = G_{22}(x) \) we get the claimed result.

\[ \square \]

### 2.5 Morphisms, Intertwiners and \( R \)-matrices

**Definition 2.22.** Let \( W_1 \) and \( W_2 \) be two highest weight \( E_{\gamma, \tau}(gl_N) \)-modules. A morphism

\[ \phi_{1,2}(\lambda) : W_2 \otimes W_1 \rightarrow W_1 \otimes W_2, \]  

is called an intertwiner between \( W_1 \otimes W_2 \) and \( W_2 \otimes W_1 \).

**Lemma 2.23.** Let \( W_1, W_2 \) and \( W_3 \) be \( E_{\gamma, \tau}(gl_N) \)-modules, then the maps

\[ \phi_{1,23}(\lambda) : W_2 \otimes W_3 \otimes W_1 \rightarrow W_1 \otimes W_2 \otimes W_3 \]

\[ \phi_{12,3}(\lambda) : W_3 \otimes W_1 \otimes W_2 \rightarrow W_1 \otimes W_2 \otimes W_3 \]  

(2.110)
given by

$$\phi_{1,23}(\lambda) = \phi_{1,23}^{12}(\lambda - \gamma h^{(3)})\phi_{1,3}^{23}(\lambda)$$

$$\phi_{12,3}(\lambda) = \phi_{12,3}^{23}(\lambda)\phi_{1,3}^{23}(\lambda - \gamma h^{(2)}).$$  \hspace{1cm} (2.111)

where $\phi_{ij}(\lambda)$ is an intertwiner between $W_j \otimes W_i$ and $W_i \otimes W_j$, are intertwiners between $W_2 \otimes W_3 \otimes W_1$ and $W_1 \otimes W_2 \otimes W_3$, resp. $W_3 \otimes W_1 \otimes W_2$ and $W_1 \otimes W_2 \otimes W_3$.

Proof. We prove only the first statement, because the second is completely analogous.

The $L$-matrix on $W_2 \otimes W_3 \otimes W_1$ is, by definition, equal to:

$$L_{231}(z, \lambda) = L_{231}^{01}(z, \lambda - \gamma(h^{(1)} + h^{(3)}))L_{33}^{02}(z, \lambda - \gamma h^{(1)})L_{13}^{03}(z, \lambda),$$  \hspace{1cm} (2.112)

while the $L$-matrix on $W_1 \otimes W_2 \otimes W_2$ is

$$L_{123}(z, \lambda) = L_{123}^{01}(z, \lambda - \gamma(h^{(2)} + h^{(3)}))L_{22}^{02}(z, \lambda - \gamma h^{(2)})L_{33}^{03}(z, \lambda).$$  \hspace{1cm} (2.113)

Now, using the definition of $\phi_{ij}(\lambda)$ we get

$$L_{123}(z, \lambda)\phi_{1,23}(\lambda) =$$

$$L_{123}^{01}(z, \lambda - \gamma(h^{(2)} + h^{(3)}))L_{22}^{02}(z, \lambda - \gamma h^{(2)})L_{33}^{03}(z, \lambda)$$

$$\times \phi_{12,3}^{12}(\lambda - \gamma(h^{(0)} + h^{(3)}))\phi_{1,3}^{23}(\lambda - \gamma h^{(0)}) =$$

$$L_{123}^{01}(z, \lambda - \gamma(h^{(2)} + h^{(3)}))L_{22}^{02}(z, \lambda - \gamma h^{(2)})L_{33}^{03}(z, \lambda)$$

$$\times \phi_{12,3}^{12}(\lambda - \gamma h^{(3)})\phi_{1,3}^{23}(\lambda - \gamma h^{(3)}) =$$

$$L_{123}^{01}(z, \lambda - \gamma(h^{(1)} + h^{(3)}))L_{11}^{02}(z, \lambda - \gamma h^{(1)})L_{22}^{03}(z, \lambda - \gamma h^{(3)}).$$  \hspace{1cm} (2.114)

which is the desired result.

Definition 2.24. Let $W_1$ and $W_2$ be two $E_{\gamma}(gl_N)$-modules and let $\phi_{1,2}(\lambda)$ be an intertwiner between $W_2 \otimes W_1$ and $W_1 \otimes W_2$, then we define the $R$-matrix on $W_1 \otimes W_2$ associated to $\phi_{1,2}(\lambda)$ to be the linear map

$$R_{W_1, W_2}(\lambda) = \phi_{1,2}(\lambda)P^{12},$$  \hspace{1cm} (2.115)

where $P^{12}$ is the flip on $W_1 \otimes W_2$.

Corollary 2.25. Let $W_1$, $W_2$, and $W_3$ be $E_{\gamma}(gl_N)$-modules, then the maps on $W_1 \otimes (W_2 \otimes W_3)$ and $(W_1 \otimes W_2) \otimes W_3$ are given by:

$$R_{W_1, W_2 \otimes W_3}(\lambda) = \phi_{12}^{12}(\lambda - \gamma h^{(3)})\phi_{1,3}^{23}(\lambda - \gamma h^{(2)}),$$

$$R_{W_1 \otimes W_2, W_3}(\lambda) = \phi_{12}^{23}(\lambda - \gamma h^{(2)}).$$  \hspace{1cm} (2.116)

are $R$-matrices on $W_1 \otimes (W_2 \otimes W_3)$ and $(W_1 \otimes W_2) \otimes W_3$, respectively.

In Theorem 2.8, we proved that two irreducible $E_{\gamma}(gl_N)$-modules with the same elliptic weight are isomorphic. A consequence of this fact is that if the tensor products $W_i \otimes W_j$ are irreducible, then there is a unique intertwiner (up to a trivial factor) between $W_i \otimes W_j$ and $W_j \otimes W_i$ and a unique $R$-matrix.
Theorem 2.26. Let $W_1, W_2$ and $W_3$ be highest weight $E_{\gamma,\tau}(gl_N)$-modules of finite type such that $W_1 \otimes W_2 \otimes W_3$ is irreducible, then the $R$-matrices satisfy the (generalized) Yang-Baxter Equation.

*Proof.* This Theorem follows by looking at the morphism

$$\phi_{123}(\lambda) : W_3 \otimes W_2 \otimes W_1 \rightarrow W_1 \otimes W_2 \otimes W_3,$$  \hspace{1cm} (2.117)

along two different paths and by noticing that, because $W_1 \otimes W_2 \otimes W_3$ is irreducible, $\phi_{123}(\lambda)$ is unique. In fact, let $\phi_{123}^L(\lambda)$ and $\phi_{123}^R(\lambda)$ be the morphism along the left, resp. right, path. Then:

$$(\phi_{123}^L(\lambda) - \phi_{123}^R(\lambda))w_3 \otimes w_2 \otimes w_1 = 0,$$  \hspace{1cm} (2.118)

by definition. Which is a contradiction, unless $\phi_{123}^L(\lambda) = \phi_{123}^R(\lambda)$.

Another class of modules for which we can define $R$-matrices (and morphisms) are modules of the form

$$W_1 \otimes W_2,$$  \hspace{1cm} (2.119)

where $W_1$ is the tensor product of a finite number of fundamental representations evaluated at generic points

$$W_1 = V(z_1) \otimes V(z_2) \otimes \ldots \otimes V(z_n), \quad z_i - z_j \neq \pm \gamma \forall i \neq j$$  \hspace{1cm} (2.120)

and $W_2$ is an arbitrary $E_{\gamma,\tau}(gl_N)$-module.

In fact, because $W_2$ is an $E_{\gamma,\tau}(gl_N)$-module there is an $L$-matrix on $W_2$. It is then easy to see that

$$R_{W_1, W_2}(\lambda) = L^{n,n+1}(z_n, \lambda) \ldots L^{1,n+1}(z_1, \lambda - \gamma \sum_{i=2}^{n} h^{(i)})$$  \hspace{1cm} (2.121)

is an $R$-matrix on $W_1 \otimes W_2$. 

\hfill \Box
2.6 An example

We will conclude this chapter by showing that the tensor product of a finite number of fundamental vector representations evaluated at generic points

\[ V(z_1) \otimes V(z_2) \otimes \ldots V(z_n), \quad z_i - z_j \neq \pm \gamma \quad \forall i \neq j, \quad (2.122) \]

is irreducible.

In order to do this consider the \( E_{\gamma, \tau}(gl_N) \)-module

\[ W = W_1 \otimes W_2, \quad (2.123) \]

where \( W_1 = V(z_1) \otimes V(z_2) \) and \( W_2 = V(t_1) \otimes V(t_2) \). On \( W \) the \( R \)-matrix takes the form

\[
R_{W_1, W_2}(\lambda) = R_{23}(z_2 - t_1, \lambda - \gamma^1)R_{13}(z_1 - t_1, \lambda) \times \\
R_{24}(z_2 - t_2, \lambda - \gamma(\gamma^1 + \gamma^3))R_{14}(z_1 - t_2, \lambda - \gamma^2),
\]

which is well defined if \( z_i - t_k \neq \pm \gamma \). In particular if \( z_1 = t_1 \) and \( z_2 = t_2 \), (2.124) becomes

\[
R_{W_1, W_2}(\lambda) = \\
R_{23}(z_2 - z_1, \lambda - \gamma^1)P_{24}R_{14}(z_1 - z_2, \lambda - \gamma(\gamma^1 + \gamma^3)) = P_{23}P_{24},
\]

where \( P_{ij} \) is the flip operator \( P_v \otimes w = w \otimes v \). Moreover, using (2.121) we get

\[
R_{W_1, W_2}(\lambda)e_{j_1} \otimes j_{j_2} \otimes k_1 \otimes k_2 = e_{j_1} \otimes e_{j_2} \otimes L_{i_2 j_2}(z_2)L_{i_1 j_1}(z_1)e_{k_1} \otimes e_{k_2}.\quad (2.126)
\]

Comparing (2.126) with (2.125) one gets

\[
L_{i_2 j_2}(z_2)L_{i_1 j_1}(z_1)e_{k_1} \otimes e_{k_2} = \\
\delta_{i_1 k_1} \delta_{i_2 k_2} e_{j_1} \otimes e_{j_2}.
\]

(2.127)

And this proves that \( V(z_1) \otimes V(z_2) \) is a highest weight \( E_{\gamma, \tau}(gl_N) \)-module.

It is now easy to conclude that \( V(z_1) \otimes V(z_2) \) is irreducible. In fact let \( \eta = \sum c_{ij}(\lambda)e_i \otimes e_j \) be a singular vector

\[ L_{ji}(z)\eta = 0, \quad \forall j > i, z \in \mathbb{C}. \quad (2.128) \]

Using (2.126), we get

\[ L_{j_2 i_2}(z_2)L_{j_1 i_1}(z_1)\eta = c_{j_1 j_2}(\lambda - 2\gamma^1)e_1 \otimes e_1 = 0, \quad \forall j_2, j_1 > 1. \quad (2.129) \]

And this implies that \( c_{j_1 j_2} = 0, \quad \forall j_2, j_1 \neq 1 \). Which is the claimed result. This result can be generalized to an arbitrary number of factors with the formula

\[
L_{i_1 j_1}(z_1)L_{i_2 j_2}(z_{n-1})\ldots L_{i_n j_n}(z_1)e_{k_1} \otimes e_{k_2} \otimes \ldots e_{k_n} = \\
\delta_{i_1 k_1} \delta_{i_2 k_2} \ldots \delta_{i_n k_n} e_{j_1} \otimes e_{j_2} \otimes \ldots e_{j_n},
\]

(2.130)
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Chapter 3

Symmetric Modules

In this chapter we study symmetric powers of the fundamental representations. Such modules arise as sub-modules of tensor products of fundamental vector representations evaluated at special points. We will, in particular, give an explicit construction, compute the elliptic weight and the matrix elements for such representations. Moreover we will prove that symmetric modules are irreducible and study a suitable infinite dimensional limit.

3.1 Symmetric Powers of the Fundamental Vector Representation

Definition 3.1. For any $n = 1, 2, \ldots$, the symmetric group $S_n$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n = \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ by permuting the factors. A tensor in $\mathbb{C}^n \otimes \mathbb{C}^n$ is called symmetric if it is invariant under the symmetric group and the vector space of such vectors is denoted by $S^n \mathbb{C}^N$.

Lemma 3.2. Let $\gamma$ and $\lambda \in \mathfrak{h}^*$ be generic. Then $R(z, \lambda)$ is a nonsingular matrix for all $z \neq \pm \gamma$ (modulo $\mathbb{Z} + \tau \mathbb{Z}$).

(i) The image of $R(-\gamma, \lambda)$ is $S^2 \mathbb{C}^N$.

(ii) The kernel of $R^{\text{reg}}(\gamma, \lambda) \equiv \text{res}_{z = \gamma} R(z, \lambda)$ is $J_2(\mathbb{C}^N) = S^2 \mathbb{C}^N$.

Proof. We have the "unitarity property" $R(z, \lambda)R(-z, \lambda)^{(21)} = 1$ which implies that $R(z, \lambda)$ is nonsingular unless $z$ or $-z$ is a pole. This occurs only if $z = \pm \gamma$. The other claims follow easily from the definition of $R$. \qed

Definition 3.3. Let us define operators $W_n(z, \lambda) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ \((z \in \mathbb{C}^n, \lambda \in \mathfrak{h}^*)\) recursively by the conditions:

\begin{equation}
W_1(z, \lambda) = 1
\end{equation}

\begin{equation}
W_{n+1}(z, \lambda) = R(z_1 - z_2, \lambda - \gamma \sum_{j=3}^{n+1} h^{(j)})^{(12)} \ldots
\end{equation}

\begin{equation}
R(z_1 - z_n, \lambda - \gamma h^{(n+1)}) (1 \otimes W_n(z_2, \ldots, z_{n+1}, \lambda - \gamma h^{(1)})) \quad (1 \otimes W_n(z_2, \ldots, z_{n+1}, \lambda - \gamma h^{(1)})).
\end{equation}

Definition 3.4. Let $z^S = (0, \gamma, 2\gamma, \ldots, (n-1)\gamma)$. We set

\begin{equation}
W_n^S(\lambda) = W_n(z^S, \lambda)
\end{equation}

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Remark 6. Note that the spectral parameter in any of the $R$-matrices of the product defining $W^S_n(\lambda)$ is always a negative multiple of $\gamma$, so that there are no divergent $R$-matrices in this product.

**Theorem 3.5.** Let $\lambda$ be generic. Then the image of $W^S_n(\lambda)$ is equal to $S^n(\mathbb{C}^N)$.

**Proof.** Let $P$ denote the flip $u \otimes v \rightarrow v \otimes u$ on $\mathbb{C}^N \otimes \mathbb{C}^N$. Since every permutation is a product of adjacent transpositions, we have

$$S^n(\mathbb{C}^N) = \{v \in (\mathbb{C}^N)^{\otimes n} \mid P^{(j,j+1)} v = v, \quad j = 1, \ldots, n-1\}. \quad (3.4)$$

Let us first show that the image of $W^S_n(\lambda)$ is contained in $S^n(\mathbb{C}^N)$. It suffices to show that $P^{(j,j+1)} W^S_n(\lambda) = W^S_n(\lambda)$ for all $j = 1, \ldots, n-1$. This follows from Lemma 3.2 and the fact that we can always find a representation of $W^S_n(\lambda)$ such that the highest crossing is the one between line $j$ and line $j+1$.

Let us now show the converse, by considering the limit $\gamma \rightarrow 0$. The operator $W^S_n(\lambda)$ is a product of $R$-matrices whose spectral parameter is a negative integer multiple of $\gamma$. Let us write $R_\gamma(z, \lambda)$ for our $R$-matrix to make the $\gamma$-dependence apparent. The $R$-matrix at $z = -k\gamma$ ($k = 1, 2, \ldots$) is then regular as a function of $\gamma$ at $\gamma = 0$ and its limit

$$\lim_{\gamma \rightarrow 0} R_\gamma(-k\gamma, \lambda) = \frac{k}{k+1} \text{Id} + \frac{1}{k+1} P; \quad (3.5)$$

acts as the identity on $S^2(\mathbb{C}^N)$. Hence, if $\gamma = 0$, $W^S_n(\lambda)$ acts on symmetric tensors as the identity and thus the image contains all symmetric tensors. It follows that, for generic $\gamma$, the dimension of the image is at least the dimension of $S^n(\mathbb{C}^N)$. But since the image is contained in $S^n(\mathbb{C}^N)$, they must coincide.

**Theorem 3.6.** Let $V^{\otimes n}(z)$, for $n = 0, 1, 2, \ldots$, denote the $E_{\gamma,\tau}(gl_N)$-module $V(z) \otimes \cdots \otimes V(\tau + \cdots + \tau V(z+\gamma(n-1)))$. Then the subspace $S^n(\mathbb{C}^N)$ is an $E_{\gamma,\tau}(gl_N)$-submodule of $V^{\otimes n}(z)$.

**Definition 3.7.** The $E_{\gamma,\tau}(gl_N)$-module $S^n(\mathbb{C}^N)$ of the previous definition is called the $n$-th symmetric power of the vector representation with evaluation point $z$ and is denoted by $S^nV(z)$.

What the theorem means is that the $L$-operator on $\mathbb{C}^N \otimes (\mathbb{C}^N)^{\otimes n}$

$$L(w, \lambda) = R(w-z, \lambda - \gamma \sum_{j=2}^n h^{(j)})^{(01)}$$

$$R(w-z-\gamma, \lambda - \gamma \sum_{j=3}^n h^{(j)})^{(02)} \cdots R(w-z-\gamma(n-1), \lambda)^{(0n)}, \quad (3.6)$$

(the factors in the tensor products are numbered from 0 to $n$) preserves the subspaces $\mathbb{C}^N \otimes S^n(\mathbb{C}^N)$.

In order to prove this theorem, we introduce another operator on $\mathbb{C}^N \otimes (\mathbb{C}^N)^{\otimes n}$. It is the $L$-operator corresponding to the "opposite co-product":

$$L^{op}(w, \lambda) = R(w-z-\gamma(n-1), \lambda - \gamma \sum_{j=1}^{n-1} h^{(j)})^{(0n)} \cdots R(w-z-\gamma, \lambda - \gamma h^{(1)})^{(02)} R(w-z, \lambda)^{(01)}. $$
3.2 Elliptic Weight of Symmetric Modules

Lemma 3.8. Let \( L(w, \lambda) \) be the \( L \)-matrix on \( \mathbb{C}^N \otimes (\mathbb{C}^N)^\otimes n \) and \( W_n^S(\lambda) \) the operator of Definition 3.7, then

\[
L(w, \lambda) \left( 1 \otimes W_n^S(\lambda - \gamma h(0)) \right) = \left( 1 \otimes W_n^S(\lambda) \right) L^p(w, \lambda).
\] (3.7)

Proof. The left-hand side is, by definition, \( W_{n+1}(Z, \lambda) \), with

\[
Z = (z, w, w + \gamma, \ldots, w + (n-1)\gamma),
\]

with the notational convention that the factors are numbered from 0 to \( n \). The right-hand side is another representation of \( W_{n+1}(Z, \lambda) \) as a product of \( R \)-matrices. \( \square \)

3.2 Elliptic Weight of Symmetric Modules

In the last section we constructed the \( n \)-th symmetric power, \( S^nV(z) \), of the fundamental vector representation. In this section we will compute the elliptic weight of such modules.

Theorem 3.9. Let \( S^nV(z) \) be the \( n \)-th symmetric power of the fundamental vector representation of Definition 3.7 then the elliptic weight of this \( E_{\gamma, \tau}(gl_N) \)-module is

\[
\Omega = (n, 0, \ldots, 0) \quad \Lambda_{11}(w, \lambda) = 1
\]

\[
\Lambda_{ii}(z, \lambda) = \frac{\Theta(w - z)}{\Theta(w - z - n\gamma)} \frac{\Theta(\lambda_i - \lambda_i - n\gamma)}{\Theta(\lambda_i - \lambda_i)} \quad \text{if } i = 2, \ldots, N.
\] (3.9)

Proof. We identify the \( E_{\gamma, \tau}(gl_N) \)-module \( S^nV(z) \) with the symmetric subspace \( S^n(C^N) \) in the tensor product \( (C^*)^\otimes n \). From this one can see that the vector

\[
\omega = e_1 \otimes \cdots \otimes e_1
\] (3.10)

is the symmetric vector of highest weight in \( (C^*)^\otimes n \), therefore it is the highest weight vector of the symmetric power \( S^nV(z) \).

The computation of the elliptic weight is straightforward, in fact using the co-product formula, we get:

\[
L_{ii}(w, \lambda)e_1 \otimes \cdots \otimes e_1 =
L_{ii}(w - z, \lambda - \gamma(n-1)\ell^i)e_1 \otimes \cdots \otimes L_{ii}(w - z - \gamma(n-2)\ell^i)e_1 \otimes \cdots \otimes L_{ii}(w - z - (n-1)\gamma)e_1.
\] (3.11)

Using that

\[
L_{ii}(w, \lambda)e_1 = \begin{cases} 
1, & \text{if } i = 1 \\
\frac{\Theta(w-z)}{\Theta(w-z-\gamma)} \frac{\Theta(\lambda_i-\lambda_i+\gamma)}{\Theta(\lambda_i-\lambda_i)}, & \text{if } i \neq 1.
\end{cases}
\] (3.12)

we get the desired result. In fact

\[
L_{ii}(w, \lambda)e_1 \otimes \cdots \otimes e_1 =
\frac{\Theta(w-z)}{\Theta(w-z-\gamma)} \frac{\Theta(w-z-(n-1)\gamma)}{\Theta(w-z-n\gamma)} \cdots \frac{\Theta(\lambda_i-\lambda_i+\gamma)}{\Theta(\lambda_i-\lambda_i+(n-1)\gamma)} \cdots \frac{\Theta(\lambda_i-\lambda_i+\gamma(n-1))}{\Theta(\lambda_i-\lambda_i+\gamma(n-2))} \cdots \frac{\Theta(\lambda_i-\lambda_i+\gamma(n-1))}{\Theta(\lambda_i-\lambda_i+(n-1)\gamma)}
\]

\[
\frac{\Theta(\lambda_i-\lambda_i+\gamma(n-1))}{\Theta(\lambda_i-\lambda_i)} \frac{\Theta(w-z)}{\Theta(w-z-n\gamma)} \frac{\Theta(\lambda_i-\lambda_i-n\gamma)}{\Theta(\lambda_i-\lambda_i)} e_1 \otimes \cdots \otimes e_1
\] (3.13)

\( \square \)
3.3 Matrix Elements for Symmetric Modules

In this section we will compute the matrix elements \( L_{ij}(w, \lambda), \forall i, j \in \{1, \ldots, N\} \). The computation of the matrix elements is more complicated than the computation of the elliptic weight and we need some preliminary work.

The main result of this section is summarized in the next theorem.

**Theorem 3.10.** Let \( S^nV(z) \) be the \( n \)-th symmetric power of the fundamental representation defined in Section 3.7. Moreover let \( e_n \) be the symmetric vector

\[
e_n = \sum_{\sigma \in S^n} \sigma(e_1^{\otimes n_1} \otimes \cdots \otimes e_N^{\otimes n_N}), \quad n = \sum_{i=1}^N n_i,
\]

then the diagonal matrix elements are equal to

\[
L_{ii}(w, \lambda)e_n = \frac{\Theta(w - z - \gamma n_i) \prod_{k=1, k \neq i}^N \Theta(\lambda_i - \lambda_k + \gamma n_k)}{\Theta(w - z - \gamma n) \Theta(\lambda_i - \lambda_j)} e_n, \tag{3.15}
\]

and the non-diagonal ones are:

\[
L_{ij}(w, \lambda)e_n = -\frac{\Theta(w - z - \gamma n_i + \lambda_i - \lambda_j + \gamma)\Theta(\gamma(n_j + 1))}{\Theta(w - z - \gamma n) \Theta(\lambda_i - \lambda_j)} \prod_{\substack{l \geq 1, i \neq l, j \neq l \neq i \neq j}}^{l \neq i, j} \Theta(\lambda_j - \lambda_l + \gamma n_l) e_{\bar{n}_k}, \tag{3.16}
\]

\[
\bar{n}_k = \begin{cases} 
  n_k, & k \neq i, j \\
  n_i - 1, & k = i \\
  n_j + 1, & k = j
\end{cases}
\]

if \( n_i \neq 0 \) and identically zero otherwise.

**Definition 3.11.** Let \( z \) be a number in \( \mathbb{C} \), then the elliptic quantum number \([z]\) is defined as

\[
[z] = \frac{\Theta(\gamma z)}{\Theta(\gamma)}.
\]

**Definition 3.12.** With \([e_1 \otimes \cdots \otimes e_n]\) we denote the vector

\[
[e_1 \otimes \cdots \otimes e_n] = \sum_{\sigma \in S^n} \sigma(e_1 \otimes \cdots \otimes e_n). \tag{3.18}
\]

**Lemma 3.13.** Let \( \overline{e}_n \) be the vector in \((\mathbb{C}^N)^{\otimes n}\) of the form

\[
\overline{e}_n = e_1^{\otimes n_1} \otimes e_2^{\otimes n_2} \otimes \cdots \otimes e_N^{\otimes n_N}, \quad \sum n_i = n,
\]

and \( e_n = [\overline{e}_n] \) its image in \( S^nV(z) \). Moreover, let \( g_n(\lambda) \) be the function

\[
g_n(\lambda) = \prod_{j<k} \prod_{r+s=0}^{n_j-1} \Theta(\lambda_j - \lambda_k - \gamma(r - s) + \gamma) \Theta(\lambda_j - \lambda_k - \gamma(r - s)), \tag{3.20}
\]

then there is a non-zero constant \( C_n \) such that

\[
W_{\lambda}[\overline{e}] = C_n g_n(\lambda) e, \tag{3.21}
\]

where

\[
C_n = \prod_{l=1}^N \frac{|n_l|!}{n_l!} \quad (\text{see Lemma 3.15}). \tag{3.22}
\]
3.3. Matrix Elements for Symmetric Modules

Proof. We know from Theorem 3.5 that the right hand side of (3.21) must be proportional to the symmetric vector $e_n$. It suffices, therefore, to compute the coefficient of $e_n$ in $W^S_n(\lambda)e_n$. In order to do this, we notice that only the diagonal elements in the $R$-matrices give a non trivial contribution to this coefficient. This implies that $g_n(\lambda)$ is a product of functions

$$\alpha(z, \lambda) = \frac{\Theta(z)\Theta(\lambda + \gamma)}{\Theta(z - \gamma)\Theta(\lambda)}.$$  \hspace{1cm} (3.23)

The values of $z$ occurring here are of the form $-m\gamma$ for some positive integers $m$, so that no zeros or divergences appear. This implies that the $z$-dependent part of $\alpha(z, \lambda)$ contribute only to the constant $C_n$.

Let us now compute the $\lambda$-dependent part, using a suitable representation of $W^S_n(\lambda)$ by a diagram. As above we number the lines from 1 to $n$ from left to right at the bottom of the diagram. Let us say that the first $n_1$ lines have weight 1, the next $n_2$ lines have weight 2 and so on. Now, we compute the contribution to the product of a crossing between a line of weight $j$ and a line of weight $k \neq j$. Suppose that there are $n_i$ lines of weight $i$ to the left of the crossing, then the crossing contributes with

$$\alpha(z, \lambda_j - \lambda_k - (n_j - n_k)\gamma),$$  \hspace{1cm} (3.24)

for some $z \in \mathbb{C}$, to the product.

Taking together all crossings between lines of weight $j$ and $k$, and taking the product over all $j < k$ yields:

$$g_n(\lambda) = \prod_{j < k} \prod_{r=0}^{n_j-1} \prod_{s=0}^{n_k-1} \frac{\Theta(\lambda_j - \lambda_k - \gamma(r - s) + \gamma)}{\Theta(\lambda_j - \lambda_k - \gamma(r - s))},$$  \hspace{1cm} (3.25)

which is the desired result. \hfill $\square$

Lemma 3.14. Let $\bar{e}_\sigma$ be the vector in $(\mathbb{C}^N)^\otimes n$ of the form

$$\bar{e}_\sigma = e_{\sigma(1)}^{\otimes n_{\sigma(1)}} \otimes e_{\sigma(2)}^{\otimes n_{\sigma(2)}} \otimes \ldots \otimes e_{\sigma(N)}^{\otimes n_{\sigma(N)}}, \quad \sum_{i=1}^N n_i = n,$$  \hspace{1cm} (3.26)
and \( e_\sigma = [e_\sigma] \) its image in \( S^n V(z) \). Moreover, let \( g_{n,\sigma}(\lambda) \) be the function

\[
g_{n,\sigma}(\lambda) = \prod_{j<k} \prod_{t=0}^{n_{\sigma(j)}-1} \prod_{s=0}^{n_{\sigma(k)}-1} \frac{\Theta(\lambda_{\sigma(j)} - \lambda_{\sigma(k)} - \gamma(r-s) + \gamma)}{\Theta(\lambda_{\sigma(j)} - \lambda_{\sigma(k)} - \gamma(r-s))},
\]

then there is a non-zero constant \( C_n \) such that

\[
W_n^\lambda(\lambda)^\sigma = C_n g_{n}(\lambda)^e, 
\]

where

\[
C_n = \frac{\prod_{i=1}^{N} [n_i]!}{[n]!} \quad (\text{see Lemma 3.15}).
\]

Proof. The proof is similar to Lemma 3.13.

Lemma 3.15. Let \( \overline{e}_n \) be the vector in \((\mathbb{C}^N)^{\otimes n}\) of the form

\[
\overline{e}_n = e_1^{\otimes n_1} \otimes e_2^{\otimes n_2} \otimes \ldots \otimes e_N^{\otimes n_N}, \quad \sum n_i = n,
\]

and \( e_n = [\overline{e}_n] \) its image in \( S^n V(z) \), then the constant \( C_n \) of Lemma 3.13 and 3.14 is

\[
\frac{\prod_{i=1}^{N} [n_i]!}{[n]!}.
\]

Proof. The computation of \( C_n \) is straightforward. In fact, let us number the lines of the diagram that permutes the factors from 1 to \( n \) from left to right. Let us say that the first \( n_1 \) lines have weight 1, the next \( n_2 \) lines have weight 2 and so on. Now, the contribution to the product of the crossing between the lines of weight \( i \) and the lines of weight \( j \) is:

\[
\prod_{s=1}^{n_i} \prod_{r=1}^{n_j} \frac{\Theta(\gamma \sum_{i=1}^{j} n_l - \gamma s - \gamma \sum_{i=1}^{j} n_l + \gamma r)}{\Theta(\gamma \sum_{i=1}^{j} n_l - \gamma s - \gamma \sum_{i=1}^{j} n_l + \gamma r)}.
\]

which is equivalent to

\[
\prod_{r=1}^{n_j} \frac{\Theta(\gamma \sum_{i=1}^{j} n_l - \gamma s + \gamma)}{\Theta(\gamma \sum_{i=1}^{j} n_l - \gamma s + \gamma)} = \prod_{r=1}^{n_j} \frac{\sum_{i=1}^{j} n_l - r + 1}{\sum_{i=1}^{j} n_l - r + 1},
\]

where in (3.33) we have introduced the “quantum numbers” notation.

Taking the product over all \( i < j \), yields:

\[
\prod_{i<j} \prod_{r=1}^{n_j} \frac{\sum_{i=1}^{j} n_l - r + 1}{\sum_{i=1}^{j} n_l - r + 1}.
\]

Expression (3.34) can be simplified, noticing that:

\[
\prod_{i<j} \prod_{r=1}^{n_j} \frac{\sum_{i=1}^{j} n_l - r + 1}{\sum_{i=1}^{j} n_l - r + 1} \quad (3.35)
\]
and that
\[ \prod_{j=r}^{N} \prod_{i=1}^{n_i} \left( \sum_{i=1}^{n_i} r - 1 \right) = \frac{\prod_{i=1}^{n_i} [n_i]}{[n_i]!}. \] (3.36)

Putting everything together one can see that (3.34) is equal to
\[ \prod_{i=1}^{N-1} \prod_{j=1}^{n_i} \left( n - \sum_{i=1}^{j} n_i \right)! \frac{[n_i]!}{[n - \sum_{i=1}^{j} n_i]!} = \prod_{i=1}^{N-1} \prod_{j=1}^{n_i} \left( n - \sum_{i=1}^{j-1} n_i \right)! \frac{[n_i]!}{[n - \sum_{i=1}^{j-1} n_i]!} = \prod_{i=1}^{N} [n_i]! \frac{[n_i]!}{[n_i]!}, \] (3.37)

which is the desired result. \( \square \)

In the next Lemma we compute the diagonal terms \( L_{ii}(w, \lambda) \)

**Lemma 3.16.** Let \( \vec{e}_\sigma \) be the vector
\[ \vec{e}_\sigma = e_1^{n_1} \otimes e_2^{n_2} \otimes \cdots \otimes e_n^{n_N} \] (3.38)

and \( e = [e_\sigma] \) the associated vector in \( S^n V(z) \), then
\[ L_{ii}(w, \lambda)e = \Lambda_{ii}(w, \lambda, n)e, \] (3.39)

with \( \Lambda_{ii}(w, \lambda, n) \) equal to
\[ \Lambda_{ii}(w, \lambda, n) = \frac{\Theta(w - z - \gamma n_i)}{\Theta(w - z - \gamma n)} \prod_{1 \leq k \leq i} \Theta(\lambda_i - \lambda_k + \gamma n_k) \prod_{i \neq k} \Theta(\lambda_i - \lambda_k) \] (3.40)

**Proof.** In order to prove this Lemma, we use that the functions \( \Lambda_{ii}(w, \lambda, n) \), defined in (3.39), factorize in a \( \lambda \) and a \( w \)-dependent part. This can be verified using that, up to a function of \( \lambda \),
\[ \Lambda_{ii}(w, \lambda, n)e_\sigma \otimes [e_1^{n_1} \otimes e_2^{n_2} \otimes \cdots \otimes e_n^{n_N}] \] (3.41)
is equal to
\[ e_\sigma \otimes I L(w, \lambda) W_n^S(\lambda - h^0) e_1^{n_1} \otimes e_2^{n_2} \otimes \cdots \otimes e_N^{n_N} \]
\[ = e_\sigma \otimes I W_n^S(\lambda) L(w, \lambda) = e_\sigma \otimes e_1^{n_1} \otimes e_2^{n_2} \otimes \cdots \otimes e_N^{n_N} \] (3.42)

and that (3.42) is a symmetric vector. It suffices, therefore, to compute the coefficient in front of
\[ e_1^{n_1} \otimes e_2^{n_2} \otimes \cdots \otimes e_N^{n_N} \] (3.43)

which is a product of functions
\[ \alpha(x, \lambda) = \frac{\Theta(x) \Theta(\lambda - \gamma)}{\Theta(x - \gamma) \Theta(\lambda)} \] (3.44)
with values of \( x \) of the form \( x = w - z - \gamma k, k = 0, \ldots, n - 1 \).

Let us first compute explicitly the \( \lambda \)-dependent part. This can be done by noticing that

\[
L(w, \lambda)\big|_{w=z-\gamma} W_n^S(\lambda - \gamma h^0) = W_{n+1}^S(\lambda).
\]  

Equation (3.45) implies that

\[
L(z-\gamma, \lambda) W_n^S(\lambda - \gamma h^0) e_i \otimes e_i^{\otimes n_1} \otimes e_i^{\otimes n_1} \otimes \cdots \otimes e_i^{\otimes n_1} \otimes \cdots \otimes e_i^{\otimes n_1}
\]

is a symmetric vector. Using Lemma 3.14 we get that the \( \lambda \)-dependent part is equal to

\[
g_{n, \sigma}(\lambda - \gamma e^t)^{-1} g_{n, \sigma}(\lambda),
\]

where the permutation \( \sigma \) is defined by

\[
\sigma(1 \ 2 \ \ldots \ N) = (i \ \ldots \ \hat{i} \ \ldots \ N)
\]

and

\[
\tilde{n}_j = \begin{cases} 
  n_j & j \neq i \\
  n_j + 1 & j = i.
\end{cases}
\]

The explicit computation yields:

\[
g_{n, \sigma}(\lambda - \gamma e^t)^{-1} g_{n, \sigma}(\lambda) = 
\]

\[
C \prod_{1 \leq k \neq \hat{k}} n_i \prod_{s=0}^{n_i-1} \frac{\Theta(\lambda_i - \lambda_k - \gamma(r-s) + \gamma)}{\Theta(\lambda_i - \lambda_k - \gamma(r-s))} \times 
\]

\[
\prod_{1 \leq k \neq \hat{k}} n_j \prod_{s=0}^{n_j-1} \frac{\Theta(\lambda_i - \lambda_j - \gamma(r-s) - \gamma)}{\Theta(\lambda_i - \lambda_j - \gamma(r-s))}.
\]

The first term in (3.50) is \( g_{n, \sigma}(\lambda) \) while the second is \( g_{n, \sigma}(\lambda - \gamma e^t)^{-1} \). The constant \( C \) is the \( w \)-dependent part of \( \Lambda_{\mu}(w, \lambda, n) \) evaluated in \( w = z - \gamma \), so it will be ignored for the rest of this computation.

Now, using that

\[
\prod_{r=0}^{n_i-1} \frac{\Theta(\lambda_i - \lambda_k - \gamma(r-s) + \gamma)}{\Theta(\lambda_i - \lambda_k - \gamma(r-s))} = \frac{\Theta(\lambda_i - \lambda_j + \gamma s + \gamma)}{\Theta(\lambda_i - \lambda_k + \gamma s)} \prod_{r=0}^{n_i-1} \frac{\Theta(\lambda_i - \lambda_k - \gamma(r-s))}{\Theta(\lambda_i - \lambda_k - \gamma(r-s) + \gamma)},
\]

we can simplify (3.50) to:

\[
\prod_{1 \leq k \neq \hat{k}} \frac{\Theta(\lambda_i - \lambda_k + \gamma s + \gamma)}{\Theta(\lambda_i - \lambda_k + \gamma s)} = \prod_{1 \leq k \neq \hat{k}} \frac{\Theta(\lambda_i - \lambda_k + \gamma) \Theta(\lambda_i - \lambda_k + \gamma n_k)}{\Theta(\lambda_i - \lambda_k) \Theta(\lambda_i - \lambda_k + \gamma)}. 
\]

(3.53)

\[
= \prod_{1 \leq k \neq \hat{k}} \frac{\Theta(\lambda_i - \lambda_k + \gamma n_k)}{\Theta(\lambda_i - \lambda_k)},
\]

(3.54)
which is the desired result.

Now we compute the \( w \)-dependent part. This can be done using that, by definition,
\[ L_{\alpha}(w, \lambda) e = e^i \otimes \mathbb{1} L(w, \lambda) e_i \otimes e, \]  
(3.55)
which is equivalent to
\[ L_{\alpha}(w, \lambda) e = g_n(\lambda - \gamma^{\ell^i})^{-1} e^i \otimes \mathbb{1} L(w, \lambda) W_n^S(\lambda - \gamma^{\ell^i}) e_i \otimes e, \]  
(3.56)
\[ = g_n(\lambda - \gamma^{\ell^i})^{-1} e^i \otimes \mathbb{1} W_n^S(\lambda) L^{np}(w, \lambda) e_i \otimes e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(n)}. \]  
(3.57)

Now we chose \( \sigma \) as in the previous computation of the \( \lambda \)-dependent part of \( \Lambda_{\alpha}(w, \lambda, n) \). With this choice of \( \sigma \) one can see that the first \( n_i \) \( R \)-matrices in the product \( L^{np}(w, \lambda) \) give a trivial contribution. The other contribute with
\[ \prod_{j=1}^{N} \prod_{s=0}^{n_i-1} \alpha(w - z - \sum_{k=1}^{i-1} n_k - \gamma s), \]  
(3.58)
which can be simplified to
\[ \frac{\Theta(w - z - \gamma n_i)}{\Theta(w - z - \gamma n)}, \quad n = \sum_{i=1}^{N} n_i. \]  
(3.59)

Lemma 3.17. Let \( \bar{e}_\sigma \) be the vector
\[ \bar{e}_\sigma = e_i^{\otimes n_i} \otimes e_i^{\otimes n_i} \otimes \ldots \otimes e_1^{\otimes n_1} \]  
(3.60)
and \( e = [e_i] \) the associated vector in \( S^n \mathbb{V}(z) \), then
\[ L_{ij}(w, \lambda)e = \Lambda_{ij}(w, \lambda, n)e, \]  
(3.61)
where \( \Lambda_{ij}(w, \lambda, n) \) is equal to
\[ \Lambda_{ij}(w, \lambda, n) = \frac{\Theta(w - z - \gamma n_i + \lambda_i - \lambda_j + \gamma) \Theta(\gamma(n_j + 1))}{\Theta(w - z - \gamma n) \Theta(\lambda_i - \lambda_j)} \prod_{i \neq j, s} \frac{\Theta(\lambda_j - \lambda_i + \gamma n_i)}{\Theta(\lambda_j - \lambda_i)}. \]  
(3.62)

and \( e \) equal to
\[ e = [e_1^{\otimes n_1} \otimes \ldots \otimes e_i^{\otimes (n_i-1)} \otimes \ldots \otimes e_j^{\otimes (n_j+1)} \otimes \ldots \otimes e_N^{\otimes n_N}]. \]  
(3.63)

Proof. One could, in principle, compute the matrix elements using the same procedure used for the diagonal terms. But a closer look shows that the computations involved would be too complicated. Instead of this, we use the following trick. One of the defining relations of the elliptic quantum group states:
\[ L_{ij}(w, \lambda)L_{ii}(z_1, \lambda - \gamma^{\ell^i}) = \alpha_{ij}(w - z_1, \lambda)L_{ii}(w, \lambda)L_{ij}(z_1, \lambda - \gamma^{\ell^i}) + \beta_{ij}(w - z_1, \lambda)L_{ij}(z_1, \lambda)L_{ii}(z_1, \lambda - \gamma^{\ell^i}) \]  
(3.64)
Let $e$ be the vector of $S^N V(z)$ of the form
\[ e = [e_i^{\otimes n_i} \otimes \ldots \otimes e_N^{\otimes n_N}] = [e_z], \quad (3.65) \]
where $e_\sigma$ is equal to
\[ e_\sigma = e_j^{\otimes n_j} \otimes e_i^{\otimes n_i} \otimes e_1^{\otimes n_1} \otimes \ldots \otimes e_N^{\otimes n_N}, \quad \sigma(12 \ldots N) = (ji1 \ldots N), \quad (3.66) \]
then
\[ L_{ij}(z, \lambda) L_{ij}(w, \lambda - \gamma \ell^j) e \quad (3.67) \]
is identically zero for $z_1 = \gamma(n_i - 1) + z$. This implies that the matrix element $L_{ij}(w, \lambda)$ is of the form:
\[ L_{ij}(w, \lambda)e = \beta_{ij}(w - z_1, \lambda) L_{ij}(z_1, \lambda) \Lambda_{ij}(w) \Lambda_{ij}(z_1)^{-1} e, \quad (3.68) \]
where $\Lambda_{ii}(w)$, $\Lambda_{ii}(z_1)$ are the $\lambda$-independent part of $\Lambda(w, \lambda)$.
So, it remains to compute the function $L_{ij}(z_1, \lambda)$. In order to do this we evaluate both sides of (3.68) at a special value of $w$, using that
\[ L_{ij}(w, \lambda)e = e^i \otimes 11 L(w, \lambda)e_j \otimes e \quad (3.69) \]
and that
\[ L(w, \lambda)e_j \otimes e|_{w=z-\gamma} \quad (3.70) \]
is a symmetric vector. Equation (3.70) implies that
\[ L(z - \gamma, \lambda)e_j \otimes e = G(\lambda)[e_j \otimes e], \quad (3.71) \]
with
\[ G(\lambda) = g_{n,\sigma}(\lambda - \gamma \ell^j)^{-1} g_{\tilde{n},\sigma}(\lambda) C_{\tilde{n}}^{-1} C_n, \quad (3.72) \]
where where $\tilde{n}$ is:
\[ \tilde{n}_i = \begin{cases} n_i & \text{if } i \neq j \\ n_j + 1 & \text{if } i = j \end{cases} \quad (3.73) \]
In order to simplify expression (3.72), we rewrite it as
\[ g_{n,\sigma}(\lambda) = \prod_{\lambda_j > \lambda_i,j}^{n_j} g_{ji}(n, \lambda) \prod_{r \neq i,j} g_{is}(n, \lambda) \times g_{ij}(n, \lambda) H(\lambda), \quad (3.74) \]
where
\[ g_{kl}(n, \lambda) = \prod_{r=0}^{n_k-1} \prod_{s=1}^{n_s-1} \frac{\Theta(\lambda_k - \lambda_l - \gamma(r - s + 1))}{\Theta(\lambda_k - \lambda_l - \gamma(r - s))}, \quad (3.75) \]
and $H(\lambda)$ is $\lambda_i$ and $\lambda_j$ independent. Equation (3.72) is equivalent to
\[ g_{n,\sigma}(\lambda - \ell^j)^{-1} g_{\tilde{n},\sigma}(\lambda) = \prod_{\lambda_j \neq i,j}^{n_j} g_{ji}(n, \lambda - \gamma \ell^j)^{-1} g_{ji}(\tilde{n}, \lambda) \]
\[ \prod_{r > 1 \neq i,j}^{s} g_{sr}(n, \lambda - \gamma \ell^j)^{-1} g_{sr}(\tilde{n}, \lambda) \]
\[ g_{ji}(n, \lambda - \gamma \ell^j)^{-1} g_{ji}(\tilde{n}, \lambda). \quad (3.76) \]
3.4. Infinite Dimensional Limit

Let us now compute the single terms appearing in equation (3.72). The first yields:

\[
\prod_{r=0}^{n_1-1} \prod_{s=0}^{n_2-1} \left[ \frac{\Theta(\lambda_j - \lambda_i - \gamma(r - s))}{\Theta(\lambda_j - \lambda_i - \gamma(r - s + 1))} \right]^{-1} \times \\
\prod_{r=0}^{n_1-1} \prod_{s=0}^{n_2-1} \frac{\Theta(\lambda_j - \lambda_i - \gamma(r - s - 1))}{\Theta(\lambda_j - \lambda_i - \gamma(r - s))} \\
= \prod_{s=0}^{n_1-1} \frac{\Theta(\lambda_j - \lambda_i + \gamma(s + 1))}{\Theta(\lambda_j - \lambda_i + \gamma s)} = \frac{\Theta(\lambda_j - \lambda_i + \gamma n_i)}{\Theta(\lambda_j - \lambda_i)}.
\]

(3.77)

The second is equal to 1. And the last one is

\[
\prod_{r=0}^{n_1-1} \prod_{s=0}^{n_2-1} \left[ \frac{\Theta(\lambda_j - \lambda_i - \gamma(r - s))}{\Theta(\lambda_j - \lambda_i - \gamma(r - s + 1))} \right]^{-1} \times \\
\prod_{r=0}^{n_1-1} \prod_{s=0}^{n_2-1} \frac{\Theta(\lambda_j - \lambda_i - \gamma(r - s - 1))}{\Theta(\lambda_j - \lambda_i - \gamma(r - s))} \\
= \prod_{s=0}^{n_1-1} \frac{\Theta(\lambda_j - \lambda_i + \gamma(s + 1))}{\Theta(\lambda_j - \lambda_i + \gamma s)} = \frac{\Theta(\lambda_j - \lambda_i + \gamma n_i)}{\Theta(\lambda_j - \lambda_i)}.
\]

(3.78)

Putting every thing together we get that \(G(\lambda)\) is equal to

\[
\prod_{i \geq 1 \atop i \neq j} \frac{\Theta(\lambda_j - \lambda_i + \gamma n_i)}{\Theta(\lambda_j - \lambda_i)} \Theta(\gamma(n_j + 1))
\]

(3.79)

and a short computation yields the final expression for \(\Lambda_{ij}(w, \lambda, n)\)

\[
\Lambda_{ij}(w, \lambda, n) = \frac{\Theta(w - z - \gamma n_i + \lambda_i - \lambda_j + \gamma) \Theta(\gamma(n_j + 1))}{\Theta(w - z - \gamma n_i) \Theta(\lambda_i - \lambda_j)} \times \\
\prod_{i \geq 1 \atop i \neq i, j} \frac{\Theta(\lambda_j - \lambda_i + \gamma n_i)}{\Theta(\lambda_j - \lambda_i)},
\]

(3.80)

which is the desired result. \(\square\)

3.4 Infinite Dimensional Limit

One of the biggest problem in the representation theory of \(E_{r, r}(gl_N)\) is to construct (and classify) infinite dimensional \(E_{r, r}(gl_N)\)-modules. In fact, by the fusion procedure outlined in the previous sections, one only obtains finite dimensional \(E_{r, r}(gl_N)\)-modules. In what follows, we will construct a (family) of infinite dimensional \(E_{r, r}(gl_N)\)-modules which are a suitable “limit” of the symmetric powers \(S^n V(z)\).

**Theorem 3.18.** Let \(\gamma\) be an irrational number and \(\Lambda \in \mathbb{C}\), generic, then there is a highest weight \(E_{r, r}(gl_N)\)-module \(V(z, \Lambda)\) with elliptic weight:

\[
\Omega = (\Lambda, 0, \ldots, 0), \quad \Lambda_{11}(w, \lambda) = 1 \\
\Lambda_{ij}(w, \lambda) = \frac{\Theta(w - z) \Theta(\lambda^1 - \lambda^i - \gamma \Lambda)}{\Theta(w - z - \Lambda \gamma) \Theta(\lambda^j - \lambda^j)}.
\]

(3.81)
The idea behind our construction is the following. By the Uniqueness Theorem 2.8 for irreducible $E_{\gamma,T}(gl_N)$-modules, we know that an $E_{\gamma,T}(gl_N)$-module $S^\lambda V(z)$ is uniquely determinated by its elliptic weight

$$\Omega = (n,0,\ldots,0), \quad L_i(w,\lambda)\omega = \Lambda_{ii}(w,\lambda)\omega, \quad i = 1,\ldots,N,$$

(3.82)

where

$$\Lambda_{ii}(w,\lambda) = \frac{\Theta(w-z)\Theta(\lambda_1 - \lambda_i - n\gamma)}{\Theta(w-z-n\gamma)\Theta(\lambda_1 - \lambda_i)}. \quad (3.83)$$

Now, we notice that $n\gamma$ (and so $\Omega$) is actually only defined up to an integer. In fact if we change $n\gamma$ to $n\gamma + 1$ we get the same $E_{\gamma,T}(gl_N)$-module (up to isomorphism).

Let now $\gamma$ be an irrational number, then there is a sequence $\{n^{(i)}\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty} n^{(i)}\gamma \mod \mathbb{Z} + \mathbb{Z}\tau = \gamma A \mod \mathbb{Z} + \mathbb{Z}\tau, \quad (3.84)$$

for each $A \in \mathbb{R}$.

Using this, one could define the "limit" $E_{\gamma,T}(gl_N)$-module as the unique $E_{\gamma,T}(gl_N)$-module with elliptic weight

$$\Omega = (A,0,\ldots,0), \quad L_i(w,\lambda)\omega_\infty = \Lambda_{ii}(w,\lambda)\omega_\infty, \quad i = 1,\ldots,N,$$

(3.85)

where the functions $\Lambda_{ii}(w,\lambda)$ are defined as:

$$\Lambda_{ii}(w,\lambda) = \lim_{i\to\infty} \frac{\Theta(w-z)\Theta(\lambda_1 - \lambda_i - n^{(i)}\gamma)}{\Theta(w-z-n^{(i)}\gamma)\Theta(\lambda_1 - \lambda_i)}. \quad (3.86)$$

This isn’t, however, the whole story. In fact, we also need to prove that an $E_{\gamma,T}(gl_N)$-module with weight (3.85) exists, and in order to do this we need to know the action of the matrix elements on arbitrary elements of the $E_{\gamma,T}(gl_N)$-module. This problem can be solved using the explicit form of the matrix elements derived in Section 3.16.

The results are summarized in the next Lemma.

**Lemma 3.19.** Let $\gamma$ be an irrational number, $\lambda \in \mathbb{R}$ generic, then there is an infinite dimensional $E_{\gamma,T}(gl_N)$-module $S^\lambda V(z)$ with highest weight vector $\omega_\infty$ and elliptic weight

$$\Omega = (A,0,\ldots,0), \quad \Lambda_{ii}(w,\lambda) = 1$$

$$\Lambda_{ii}(w,\lambda) = \frac{\Theta(w-z)\Theta(\lambda_1 - \lambda_i - \Lambda\gamma)}{\Theta(w-z-\Lambda\gamma)\Theta(\lambda_1 - \lambda_i)}, \quad (3.87)$$

spanned by

$$\omega_\infty = \omega[0,\ldots,0], \quad \omega[n_2,\ldots,n_N], \quad n_l \in \mathbb{N}, \sum_l n_l > 0, \quad l = 2,\ldots,N. \quad (3.88)$$
Moreover, the action of the matrix elements is:

\[
L_{\lambda}(w, \lambda)\omega[n_2, \ldots, n_N] = \frac{\Theta(w - z - \gamma \Lambda + \sum_{k=2}^{N} n_k)}{\Theta(w - z - \gamma \Lambda)} \times 
\prod_{k=2}^{N} \frac{\Theta(\lambda^i - \lambda^k + \gamma n_k)}{\Theta(\lambda^i - \lambda^k)} \omega[n_2, \ldots, n_N] \tag{3.89}
\]

\[
L_{ij}(w, \lambda)\omega[n_2, \ldots, n_N] = -\frac{\Theta(w - z - \gamma \Lambda + \gamma \sum_{k=2}^{N} n_k + \gamma + \lambda_1 - \lambda_i)}{\Theta(w - z - \gamma \Lambda)} \times 
\prod_{k=2}^{N} \frac{\Theta(\lambda^i - \lambda^k + \gamma n_k)}{\Theta(\lambda^i - \lambda^k)} \omega[n_2, \ldots, n_j + 1, \ldots, n_N], \quad n_j > 0, j \neq 1 \tag{3.91}
\]

\[
L_{ij}(w, \lambda)\omega[n_2, \ldots, n_N] = -\frac{\Theta(w - z - \gamma n_1 + \lambda_j - \lambda_1 + \gamma)}{\Theta(w - z - \gamma \Lambda)} \times 
\Theta(\gamma(n_j + 1)) \prod_{k=2}^{N} \frac{\Theta(\lambda^i - \lambda^k + \gamma n_k)}{\Theta(\lambda^i - \lambda^k)} \omega[n_2, \ldots, n_j + 1, \ldots, n_N], \quad n_j > 0, j \neq 1 \tag{3.92}
\]

Proof. To prove this Lemma it suffices to perform the substitutions

\[
\gamma n \rightarrow \gamma n^{(i)} \tag{3.94}
\]

\[
\gamma n_i \rightarrow \gamma n_i, \quad i \neq 1 \tag{3.95}
\]

\[
\gamma n_1 \rightarrow \gamma n^{(i)} = \sum_{k=2}^{N} \gamma n_k^{(i)} \tag{3.96}
\]

and to take the limit \(\gamma n^i \rightarrow \gamma \Lambda\).

\[
\text{Lemma 3.20. The } E_{\gamma, r}(gl_N)\text{-module of Lemma 3.19 exists for generic } \Lambda \in \mathbb{C}.
\]
Proof. Lemma 3.19 can be stretched to hold for generic complex number \( \Lambda \). It is in fact easy to see that, as a consequence of the fact the operators \( L_{ij}(w, \lambda) \) are meromorphic functions of \( \Lambda \in \mathbb{C} \), the defining relations of the elliptic quantum group \( E_{\gamma, \tau}(gl_N) \) hold in a neighborhood of any irrational number \( \Lambda_0 \). And this impels the claim. \( \square \)

The \( E_{\gamma, \tau}(gl_N) \)-module defined in Theorem 3.18, is the generalization to \( E_{\gamma, \tau}(gl_N) \) of the Verma \( E_{\gamma, \tau}(gl_N) \)-module introduced in [5], with some minor twists in the conventions. In order to facilitate a comparison with [5] we reproduce here the a “translation table” between the two conventions. Rescaling, for later convenience, the basis vectors of \( S^\infty_N V(z) \) by

\[
\omega[n_2, \ldots, n_N] = \frac{1}{\prod_{i=2}^{N} [n_i]} e[n_2, \ldots, n_N], \quad (3.97)
\]

we get the Verma \( E_{\gamma, \tau}(gl_N) \)-module.

**Definition 3.21.** Let \( \gamma \) an irrational number and \( \Lambda \in \mathbb{C} \) generic, then the Verma \( E_{\gamma, \tau}(gl_N) \)-module \( V(z, \Lambda) \) is the vector space spanned by

\[
\omega = e[0, \ldots, 0], \quad e[n_2, \ldots, n_N], \quad n_i \in \mathbb{N}, i = 2, \ldots, N. \quad (3.98)
\]

<table>
<thead>
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<th>Here</th>
</tr>
</thead>
<tbody>
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<td>( z )</td>
<td>( z + \gamma - (\Lambda + 1)\eta )</td>
</tr>
<tr>
<td>( k )</td>
<td>( n_2 )</td>
</tr>
<tr>
<td>( 2\eta )</td>
<td>( \gamma )</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>( n )</td>
</tr>
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<td>( \lambda )</td>
<td>( \lambda_1 - \lambda_2 )</td>
</tr>
<tr>
<td>( e_k )</td>
<td>( [k] ! e_k )</td>
</tr>
</tbody>
</table>

**Table 3.1: Translation Table**
3.4. Infinite Dimensional Limit

With the following $E_{\gamma,r}(gl_N)$-action:

$$L_{11}(w,\lambda)e[n_2,\ldots,n_N] = \frac{\Theta(w - z - \gamma \Lambda + \sum_{k=2}^{N} n_k)}{\Theta(w - z - \gamma \Lambda)} \prod_{k=2}^{N} \frac{\Theta(\lambda^1 - \lambda^k + \gamma n_k)}{\Theta(\lambda^1 - \lambda^k)} e[n_2,\ldots,n_N]$$  
(3.99)

$$L_{ii}(w,\lambda)e[n_2,\ldots,n_N] = \frac{\Theta(w - z - \gamma n_i)}{\Theta(w - z - \gamma \Lambda)} \times \prod_{k=2}^{N} \frac{\Theta(\lambda^i - \lambda^k + \gamma n_k)}{\Theta(\lambda^i - \lambda^k)} e[n_2,\ldots,n_N], \quad i \neq 1$$  
(3.100)

$$L_{1j}(w,\lambda)e[n_2,\ldots,n_N] = -\frac{\Theta(w - z - \gamma \Lambda + \gamma \sum_{k=2}^{N} n_k + \gamma + \lambda_1 - \lambda_i)}{\Theta(w - z - \gamma \Lambda)} \prod_{k=2}^{N} \frac{\Theta(\lambda^i - \lambda^k + \gamma)}{\Theta(\lambda^i - \lambda^k)} e[n_2,\ldots,n_j + 1,\ldots,n_N]$$  
(3.101)

$$L_{ij}(w,\lambda)e[n_2,\ldots,n_N] = -\frac{\Theta(w - z - \gamma n_i + \lambda_i - \lambda_j + \gamma)}{\Theta(w - z - \gamma \Lambda)} \prod_{k=2}^{N} \frac{\Theta(\lambda^j - \lambda^k + \gamma n_k)}{\Theta(\lambda^j - \lambda^k)} e[n_2,\ldots,n_j - 1,\ldots,n_N], \quad n_j > 0, j \neq 1$$  
(3.102)

3.4.1 Reducibility of Verma Modules $V(z,\Lambda)$

In the last section we introduced the Verma $E_{\gamma,r}(gl_N)$-module $V(z,\Lambda)$. This $E_{\gamma,r}(gl_N)$-modules are irreducible for generic $\Lambda$. In this section we derive a necessary and sufficient condition for the reducibility of $V(z,\Lambda)$.

Theorem 3.22. Let $V(z,\Lambda)$ be the Verma $E_{\gamma,r}(gl_N)$-module of Definition 3.21, then $V(z,\Lambda)$ is reducible if and only if $\Lambda \in \mathbb{N}$. In this case it contains a sub-module $W(z,\Lambda)$ with highest weight vector

$$\omega^1 = e[n,0,\ldots,0].$$  
(3.103)

Moreover the quotient $V(z,\Lambda)/W(z,\Lambda)$ is isomorphic to $S^nV(z)$.

Proof. The first step in the prove is to find all (proper) cyclic vectors of $V(z,\Lambda)$. From the very definition of a cyclic vector this amounts to find all solutions of the system of equations

$$L_{ij}(w,\lambda)\omega = 0 \quad \forall j > i \geq 1.$$  
(3.104)
Now, because the spaces of fixed weight in $V(z, \Lambda)$ have all dimension $\leq 1$, this is equivalent to find vectors $e[n_2, \ldots, n_N]$ such that

$$L_{j,i}(w, \lambda) e[n_2, \ldots, n_N] = 0 \quad \forall j > i \geq 1.$$  \hfill (3.105)

Equation (3.105) implies that

$$n_j = 0 \quad \forall j, i \neq 1,$$

$$\Lambda + 1 = \sum n_k \quad j \neq 1.$$  \hfill (3.106) \hfill (3.107)

So, a (proper) cyclic vector exists only if $\Lambda \in \mathbb{N}$ and in this case it is equal to

$$\omega^1 = e[\Lambda + 1, 0, \ldots, 0].$$  \hfill (3.108)

Let now $W(z, \Lambda)$ be the $E_{\gamma, \tau}(gl_N)$-module generated $\omega^1$, then it's clear, by construction, that the $E_{\gamma, \tau}(gl_N)$-module

$$V(z, \Lambda)/W(z, \Lambda)$$  \hfill (3.109)

is irreducible and, by the uniqueness Theorem, isomorphic to $S^n V(z)$. The elliptic weight of $W(z, \Lambda)$ is obtained from the definition of $V(z, \Lambda)$ using that $\Lambda = n \in \mathbb{C}$.
Chapter 4

Exterior Powers

In this chapter we study the exterior powers of the fundamental representation. Such representation will play a pivotal role in Chapter 5, where we will show that any finite dimensional $E_{\gamma,\tau}(gl_N)$-module is a sub-quotient of a suitable tensor product of fundamental vector representations.

4.1 Exterior Power of the Fundamental Vector Representations

Let $W$ be the $E_{\gamma,\tau}(gl_N)$-module

$$W(z, \epsilon) = V(z) \otimes V(z + \gamma + \epsilon) \otimes \ldots \otimes V(z + (n-1)\gamma + (n-1)\epsilon) \quad (4.1)$$

and $W_n^\wedge(\lambda, \epsilon)$ the operator in $End((\mathbb{C}^N)^{\otimes n})$ defined by:

$$W_n^\wedge(\lambda, \epsilon) = W_n(z^\wedge, \lambda)^{(n-\ldots-1)} \quad (4.2)$$

where $z^\wedge = (z + (n-1)\gamma + (n-1)\epsilon, \ldots, z)$ and $W_n$ is the operator associated to the diagram that permutes the factors. This operator is singular in the limit $\epsilon \to 0$, in fact it is a product of $R-$matrices with spectral parameters $k\gamma + k\epsilon, k = 0, 1, 2, \ldots$, so we have to “regularize” it.

Figure 4.1: Diagram that permutes four lines

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**Definition 4.1.** The regularized operator $W_{\lambda, \text{reg}}^\Lambda(z)$ is the operator defined by

$$W_{\lambda, \text{reg}}^\Lambda(z) = \lim_{\epsilon \to 0} \Theta(\epsilon)^{-1} W_{n, \text{reg}}^\Lambda(\lambda, \epsilon). \quad (4.3)$$

**Theorem 4.2.** Let $\lambda$ be generic, then the kernel of $W_{\lambda, \text{reg}}^\Lambda(z)$ is equal to $J_n(\mathbb{C}^N)$, where

$$J_n(\mathbb{C}^N) = \sum_{j=1}^{n-1} \{ P^{(j+1)} v + v \mid v \in \mathbb{C}^N \} \quad (4.4)$$

**Proof.** Let us first show that the vector space $J(\mathbb{C}^N)$ is contained in the kernel of $W_{\lambda, \text{reg}}^\Lambda(z, \lambda)$. If a vector is of the form $u_k = p(k+1)^{p_{ij}} v + v$ then $W_{\lambda, \text{reg}}^\Lambda(z, \lambda)$ vanishes on it (see Lemma 1.4) there is a representation of $W_{\lambda, \text{reg}}^\Lambda(z, \lambda)$ such that the crossing between the lines $j$ and $j+1$ is at the bottom.

Let us now show the converse. Let $A(\mathbb{C}^N)$ be the space of antisymmetric tensors. We have the direct sum decomposition

$$(\mathbb{C}^N)^{\otimes n} = J_n(\mathbb{C}^N) \oplus A_n(\mathbb{C}^N) \quad (4.5)$$

In fact $A_n(\mathbb{C}^N)$ is the orthogonal complement of $J_n(\mathbb{C}^N)$ with respect to the standard inner product. As $\gamma \to 0$, we have

$$\lim_{\gamma \to 0} R_\gamma(k, \lambda) = \frac{k}{k-1} \text{Id} - \frac{1}{k-1} P, \quad k = 2, 3, \ldots$$

$$\lim_{\gamma \to 0} \frac{1}{\gamma} R_\gamma(\gamma, \lambda) = \text{Id} - P.$$ 

Thus, for $\gamma = 0$, $\gamma^{-n+1} W_{\lambda, \text{reg}}^\Lambda(z)$ acts as a non-zero multiple of the identity on $A_n((\mathbb{C})^N)$. The dimension of the kernel is therefore at most the dimension of $J_n((\mathbb{C})^N)$, and the claim follows. \qed

**Definition 4.3.** Let $n \leq N$. We define the $n$-th exterior power of the fundamental vector representation to be the $E_{\gamma, \tau}(gl_N)$-module

$$\wedge^n V(z) = V(z) \otimes \ldots \otimes V(z + (n-1)\gamma) \big/ J_n(\mathbb{C}^N) \cong \wedge^n \mathbb{C}^N. \quad (4.6)$$

**4.2 Elliptic weight of Exterior Powers**

The computation of the elliptic weight of an exterior power of the fundamental representation is more complicated than in the case of a symmetric sub-module, because the $E_{\gamma, \tau}(gl_N)$-modules $\wedge^n V(z)$ are defined as quotients between $V(z) \otimes V(z + \gamma) \otimes \ldots \otimes V(z + (n-1)\gamma)$ and $J_n(\mathbb{C}^N)$.

**Theorem 4.4.** Let $n \leq N$ and $\wedge^n V(z)$ be the $E_{\gamma, \tau}(gl_N)$-module

$$\wedge^n V(z) = V(z) \otimes \ldots \otimes V(z + (n-1)\gamma) \big/ J_n(\mathbb{C}^N) \cong \wedge^n \mathbb{C}^N, \quad (4.7)$$

then $\wedge^n V(z)$ is an irreducible highest weight $E_{\gamma, \tau}(gl_N)$-module. Moreover the vector

$$\omega = \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma(e_1 \otimes \ldots \otimes e_n) = e_1 \wedge e_2 \wedge \ldots \wedge e_n \quad (4.8)$$
is the highest weight vector of $\wedge^n V(z)$ of elliptic weight

$$L_{ij}(z,\lambda)\omega = \Lambda_{ij}(z,\lambda)\omega,$$

where

$$\Lambda_{ij}(w,\lambda) = \frac{\Theta(w-z-\gamma)}{\Theta(w-z-(n-1)\gamma)} \prod_{k=1 \atop k \neq i}^{n} \frac{\Theta(\lambda^i - \lambda^k)}{\Theta(\lambda^i - \lambda^k - \gamma)}, \quad 1 \leq i \leq n$$

and

$$\Lambda_{il}(w,\lambda) = \frac{\Theta(w-z-\gamma)}{\Theta(w-z-n\gamma)} \prod_{k=1}^{n} \frac{\Theta(\lambda^i - \lambda^k + \gamma)}{\Theta(\lambda^i - \lambda^k)}, \quad n < i \leq N$$

Proof. Let us identify the equivalence classes $[v]$ of $v$ in $V(z) \otimes \ldots \otimes V(z+(n-1)\gamma)$ modulo $J_n(C^N)$ with $\wedge^n C^N$.

The vector

$$\omega = \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma(e_1 \otimes \ldots \otimes e_n),$$

is a vector of highest weight in $\wedge^n V(z)$. Because all spaces of fixed weight have dimension 1 (or 0) it has also to be the highest weight vector (if $\wedge^n V(z)$ is a highest weight module).

Let us show that $\wedge^n V(z)$ is an irreducible highest weight $E_{\gamma,\tau}(gl_N)$-module. In order to do this, we compute the matrix elements on each vector of $\wedge^n V(z)$.

Let $i$ be different from $j$, then

$$L_{ij}(w,\lambda)e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_n}, k_1 < k_2 < \ldots < k_n$$

is identically zero unless $i = k_s$ for some $s \in \{1, \ldots, n\}$. And, in this case, it is equal to zero if $j = k_l$ for some $l \in \{1, \ldots, n\}$. This means that we only have to compute the matrix elements in the case

$$L_{ij}(w,\lambda)e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_{n-1}} \wedge e_i.$$

Now using that the tensors in $\wedge^n V(z)$ are totally anti-symmetric we get up to a a factor $(-1)^m$, for some $m \in N$,

$$L_{ij}(w,\lambda)(-1)^m e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_{n-1}} \wedge e_i \wedge e_{e_{k_n}}.$$ 

Applying the co-product formula to (4.15), we get the result

$$L_{ij}(w,\lambda)e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_{n-1}} \wedge e_i =$$

$$L_{ij}(w,\lambda - \gamma \sum_{l=2}^{n} h^{(l)}) e_{k_1} \wedge \ldots \wedge L_{ij}(w,\lambda - \gamma h^{(n)}) e_{k_{n-1}} \wedge L_{ij}(w,\lambda)e_i =$$

$$\prod_{l=1}^{n} F_l(w) \prod_{l=1}^{n} G_l(\lambda - \gamma \sum_{k=l+1}^{n} h^{(k)}) \frac{\Theta(w-z-\gamma(n-1) + \lambda_i - \lambda_j)}{\Theta(w-z-n\gamma)\Theta(\lambda_i - \lambda_j)} e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_{n-1}} \wedge e_i.$$
The product over the functions $F_i(w)$ can be easily computed and is equal to
\[
\prod_{i=1}^{n} F_i(w) = \frac{\Theta(w - z)}{\Theta(w - z - (n-1)\gamma)}, \tag{4.17}
\]
while the functions $G_i(\lambda)$ are equal to
\[
G_i(\lambda) = \frac{\Theta(\lambda_i - \lambda_{k_i} - \gamma)}{\Theta(\lambda_i - \lambda_{k_i})}. \tag{4.18}
\]

Let us now compute the diagonal terms.
If $i \neq k_l \quad \forall l = 1, \ldots, n$, then
\[
L_{ii}(w, \lambda)e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_n} = \\
L_{ii}(w, \lambda - \gamma \sum_{l=2}^{n} h^{(l)})e_{k_1} \wedge L_{ii}(w, \lambda - \gamma \sum_{l=2}^{n} h^{(l)}) \wedge L_{ii}(w, \lambda)e_{k_n} = \\
\prod_{l=1}^{n} F_l(w) \prod_{i=1}^{n} G_l(\lambda - \gamma \sum_{k=l+1}^{n} h^{(k)})e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_n}, \tag{4.19}
\]
and
\[
\prod_{i=1}^{n} F_i(w) = \frac{\Theta(w - z)}{\Theta(w - z - n\gamma)}, \tag{4.20}
\]
and
\[
G_i(\lambda) = \frac{\Theta(\lambda_i - \lambda_{k_i} - \gamma)}{\Theta(\lambda_i - \lambda_{k_i})}. \tag{4.21}
\]
If $i = k_l$ for some $l \in 1, \ldots, n$, then we use the same trick as before and get
\[
L_{ii}(w, \lambda)e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_{n-1}} \wedge e_i = \\
\prod_{l=1}^{n-1} F_l(w) \prod_{i=1}^{n-1} G_l(\lambda - \gamma \sum_{k=l+1}^{n} h^{(k)})e_{k_1} \wedge e_{k_2} \wedge \ldots \wedge e_{k_{n-1}} \wedge e_i, \tag{4.22}
\]
because
\[
L_{ii}(w, \lambda)e_i = e_i \tag{4.23}
\]
The product over the functions $F_i$ is equal to
\[
\prod_{i=1}^{n} F_i(w) = \frac{\Theta(w - z)}{\Theta(w - z - (n-1)\gamma)}, \tag{4.24}
\]
while the functions $G_i$ are equal to
\[
G_i(\lambda) = \frac{\Theta(\lambda_i - \lambda_{k_i} - \gamma)}{\Theta(\lambda_i - \lambda_{k_i})}. \tag{4.25}
\]
In the case $k_i = i$, it is an easy exercise to verify that one gets the claimed result. The irreducibility and the fact that the $E_{7,1}(gl_N)$-module we are considering is a highest weight module follow also easily from this explicit formulas.
4.2. Elliptic weight of Exterior Powers

The \( E_{\tau,\tau}(gl_N) \)-module of Theorem 4.4 is not in canonical form. The next two lemmas show how it can be "normalized".

**Lemma 4.5.** There exists a 1-dimensional \( E_{\tau,\tau}(gl_N) \)-module \( W_n(z) \) with elliptic weight

\[
\Omega = 0
\]

\[
\Lambda_{ii}(w, \lambda) = \frac{\Theta(w - z)}{\Theta(w - z - (n - 1)\gamma)} \prod_{j=1}^{n} \frac{\Theta(\lambda^j - \lambda^i)}{\Theta(\lambda^j - \lambda^i - \gamma)}, \quad 0 \leq i \leq n
\]

\[
\Lambda_{ii}(w, \lambda) = \frac{\Theta(w - z)}{\Theta(w - z - (n - 1)\gamma)}, \quad n < i \leq N. \quad (4.26)
\]

**Proof.** In order to prove this Lemma it suffices to show that

\[
\Lambda_{ij}(w_1, \lambda)\Lambda_{ii}(w_2, \lambda - \gamma\ell) = \Lambda_{ii}(w_2, \lambda)\Lambda_{ij}(w_2, \lambda - \gamma\ell) \quad (4.27)
\]

\[
\Lambda_{ij}(w_1, \lambda)\Lambda_{ij}(w_2, \lambda - \gamma\ell) = \Lambda_{ij}(w_2, \lambda)\Lambda_{ii}(w_2, \lambda - \gamma\ell) \quad (4.28)
\]

In fact it follows from the definition of \( E_{\tau,\tau}(gl_N) \), that the only non trivial relations for a one dimensional \( E_{\tau,\tau}(gl_N) \)-module are

\[
\alpha_{ki}(z_1 - z_2, \lambda - \gamma h) = \beta_{ki}(z_1 - z_2, \lambda - \gamma h)
\]

\[
\beta_{ki}(z_1 - z_2, \lambda - \gamma h) = \alpha_{kj}(z_1 - z_2, \lambda - \gamma h) + \gamma
\]

\[
\alpha_{ij}(z_1 - z_2, \lambda - \gamma h) = \beta_{ij}(z_1 - z_2, \lambda - \gamma h) + \gamma
\]

\[
\beta_{ij}(z_1 - z_2, \lambda - \gamma h) = \alpha_{ij}(z_1 - z_2, \lambda - \gamma h) + \gamma
\]

Using that \( \Omega = 0 \) and that \( L_{ij}(w, \lambda) = 0 \) for \( i \neq j \), one can derive the previous equations.

The second set of equations is trivial because the \( \Lambda \)-functions all have the same \( w \)-dependence.

To prove that the first set of equation is also satisfied, we consider three cases.

\((0 \leq i, j \leq n)\)

We rewrite \( L_{ii}(w, \lambda) \) and \( L_{jj}(w, \lambda) \) as:

\[
\Lambda_{ii}(w, \lambda) = \frac{\Theta(w - z)}{\Theta(w - z - (n - 1)\gamma)} \prod_{k=1}^{n} \frac{\Theta(\lambda^i - \lambda^k)}{\Theta(\lambda^i - \lambda^k - \gamma)} \frac{\Theta(\lambda^i - \lambda^j)}{\Theta(\lambda^i - \lambda^j - \gamma)}
\]

\[
\Lambda_{jj}(w, \lambda) = \frac{\Theta(w - z)}{\Theta(w - z - (n - 1)\gamma)} \prod_{k=1}^{n} \frac{\Theta(\lambda^j - \lambda^k)}{\Theta(\lambda^j - \lambda^k - \gamma)} \frac{\Theta(\lambda^j - \lambda^i)}{\Theta(\lambda^j - \lambda^i - \gamma)}
\]

Now, one can see that the equations of the first set are satisfied, because they imply

\[
\frac{\Theta(\lambda^j - \lambda^i)}{\Theta(\lambda^j - \lambda^i - \gamma)} \frac{\Theta(\lambda^i - \lambda^j + \gamma)}{\Theta(\lambda^i - \lambda^j)} = \frac{\Theta(\lambda^j - \lambda^i)}{\Theta(\lambda^j - \lambda^i - \gamma)} \frac{\Theta(\lambda^j - \lambda^i + \gamma)}{\Theta(\lambda^j - \lambda^i)}, \quad (4.30)
\]

which are, of course, correct.

Suppose, now that \( 1 \leq n \) and \( n < j \leq N \). In this case the equations are trivial because \( \Lambda_{ii}(w, \lambda) \) does not depend from \( \lambda^j \).

The case \( n < i, j \leq N \) is even simpler. \( \Box \)
Lemma 4.6. There exists a 1-dimensional $E_{\gamma,T}(gl_N)$-module $W_n^*(z)$ with elliptic weight
\begin{equation}
\Omega = 0
\end{equation}
\begin{equation}
\Lambda_{ii}(w,\lambda) = \left(\frac{\Theta(w-z)}{\Theta(w-z-(n-1)\gamma)}\prod_{j=1}^{n} \frac{\Theta(\lambda^i - \lambda^j)}{\Theta(\lambda^i - \lambda^j - \gamma)}\right)^{-1}, \quad 0 \leq i \leq n
\end{equation}
\begin{equation}
\Lambda_{ii}(w,\lambda) = \left(\frac{\Theta(w-z)}{\Theta(w-z-(n-1)\gamma)}\right)^{-1}, \quad n < i \leq N.
\end{equation}

Corollary 4.7. The $E_{\gamma,T}(gl_N)$-module $\wedge^n V(z)$ factorize in a tensor product of two $E_{\gamma,T}(gl_N)$-modules $W$ and $\wedge^N_N V(z)$, where $W$ is the $E_{\gamma,T}(gl_N)$-module of the previous lemma and $\wedge^N_N V(z)$ is given by $\wedge^N_N V(z) = W_n^*(z) \otimes \wedge^n V(z)$.

Moreover, the $E_{\gamma,T}(gl_N)$-module $\wedge^N_N V(z)$ has elliptic weight
\begin{equation}
\Omega = (1,1,\ldots,1,0,\ldots,0), \quad \Lambda_{ii}(w,\lambda) = 1 \quad \text{for} \quad i = 1,\ldots,n
\end{equation}
\begin{equation}
\Lambda_{ii}(w,\lambda) = \frac{\Theta(w-z-(n-1)\gamma)}{\Theta(w-z-n\gamma)} \prod_{k=1}^{n} \frac{\Theta(\lambda^i - \lambda^k - \gamma)}{\Theta(\lambda^i - \lambda^k)}, \quad \text{for} \quad i = n + 1,\ldots,N.
\end{equation}

Proof. The co-product formula implies that
\begin{equation}
\Lambda_{ii}^{1,2}(w,\lambda) = \Lambda_{ii}(w,\lambda - \gamma\Omega_1)\Lambda_{ii}(w,\lambda).
\end{equation}
A simple computation shows that the elliptic weight of $\wedge^N_N V(z)$ has the desired form. □

Definition 4.8. The $i$-dimensional $E_{\gamma,T}(gl_N)$-module $W_n^*(z)$ of Corollary 4.7 is called the $n$-th normalization $E_{\gamma,T}(gl_N)$-module.

Definition 4.9. Let $\wedge^n V(z)$ the $n$-th exterior power of the fundamental representation and $W_n^*$ the $n$-th normalization $E_{\gamma,T}(gl_N)$-module, then we define the $n$-th canonical exterior power of the fundamental representation $\wedge^N_N V(z)$ to be the $E_{\gamma,T}(gl_N)$-module
\begin{equation}
\wedge^N_N V(z) = W_n^*(z-(n-1)\gamma) \otimes \wedge^n V(z-(n-1)\gamma).
\end{equation}
Chapter 5

Finite Dimensional Modules

In this section we will derive a set of conditions on the weight vector of an irreducible $E_{\gamma},(gl_N)$-module, which are equivalent to its finite-dimensionality. Using this criterion, we will moreover prove that any finite dimensional $E_{\gamma},(gl_N)$-module is isomorphic to the tensor product of sub-quotients of powers of the fundamental vector representation (up to taking the tensor product with a 1-dimensional $E_{\gamma},(gl_N)$-module).

**Theorem 5.1.** Let $\gamma$ be an irrational number and $W$ an irreducible highest weight $E_{\gamma},(gl_N)$-module in canonical form with highest weight vector $\omega$ and elliptic weight $(\Xi, \{A_{hi}(w, \lambda)\}_{i=1}^{N})$, where

\[
A_{hi}(w, \lambda) = \frac{\Theta(\lambda^i - \lambda^j - \gamma(\lambda^i - \lambda^j))}{\Theta(\lambda^i - \lambda^j)}.
\]

Then, $W$ is of finite dimension if and only if there are $N - 1$ functions of the form

\[
P_i(w) = \prod_{j=1}^{n_i} \Theta(w - s_j^i), \quad i = 2, \ldots, N,
\]

such that

\[
G_{hi}(w) = \prod_{i=2}^{N} \frac{P_i(w)}{P_i(w - \gamma)}.
\]

**Definition 5.2.** The "polynomials" $P_i(w), \quad i = 2, \ldots, N$ associated to a finite dimensional $E_{\gamma},(gl_N)$-module $W$ are called the Elliptic Drinfeld Polynomials of $W$.

**Corollary 5.3.** Let $\gamma$ be an irrational number and $W$ a finite dimensional irreducible $E_{\gamma},(gl_N)$-module, then $W$ isomorphic to the tensor product of sub-quotients of powers of the fundamental vector representation.

**Proof.** Let $s_j^i, i = 2, \ldots, N, i = j, \ldots, n_i$ be the zeros of the Drinfeld Polynomials associated to $W$, than using the definition of the co-product and the results of Chapter 4, one can see that the $E_{\gamma},(gl_N)$-module

\[
\widehat{W} = \bigotimes N^{i=1} V(s_j^i)
\]

has the same elliptic weight as $W$.

Let, now, $U$ be the maximal sub-module of $\widehat{W}$ generated by all singular vectors of
\( \mathcal{W} \) not proportional to the highest weight vector of \( \mathcal{W} \), then
\[
\text{Fun}_N(\mathcal{W})/\text{Fun}_N(U)
\]
is irreducible and by Theorem 2.8 isomorphic to \( W \).

**Definition 5.4.** Let \( G(w) \) be a quotient of products of theta functions:
\[
G(w) = \prod_{i=1}^{n} \frac{\Theta(w - z_i)}{\Theta(w - p_i)} \mod \mathbb{Z} + \mathbb{Z} \tau, \quad \forall i, j,
\]
then we define \( D_z(G) \), resp. \( D_p(G) \), to be the divisor of zeros \( \mod \mathbb{Z} + \mathbb{Z} \tau \), resp. of poles, of \( G(w) \).
Moreover, we define the divisor of \( G(w) \) to be \( D(G) = D_z(G) - D_p(G) \).

**Lemma 5.5.** Let \( \gamma \) be an irrational number and \( W \) an irreducible highest weight \( E_{\gamma, \tau}(gl_N) \)-module in normal form with elliptic weight \( \langle \Omega, \{ A_{ii}(w, \lambda) \}_{i=1}^{N} \rangle \) and highest weight vector \( \omega \). Moreover, let \( D_j = D_z(G_{jj}/G_{ii}), j = i + 1, \ j = 2, \ldots, N \) be the divisor of zero \( \mod \mathbb{Z} + \mathbb{Z} \tau \) of the \( w \)-dependent part of
\[
\Lambda_{jj}(w, \lambda)/\Lambda_{ii}(w, \lambda), \quad j = i + 1,
\]
where
\[
\Lambda_{ii}(w, \lambda) = G_{ii}(w) \prod_{i=1}^{i-1} \frac{\Theta(\lambda^i - \lambda^j - \gamma(\Omega^i - \Omega^j))}{\Theta(\lambda^i - \lambda^j)}.
\]
Then the dimension, over \( \text{Fun}_N(\mathbb{C}) \) of the space \( W[\mu] \) of weight \( \mu = \Omega - e_i + e_j \), \( j = i + 1 \) is equal to the degree of the divisor \( D_j \).

**Proof.** It’s easy to see that we have only to prove the statement of this Lemma for \( i = 1 \) for a normalized \( E_{\gamma, \tau}(gl_N) \)-module with elliptic weight
\[
\Lambda_{11}(w, \lambda) = 1, \quad \Lambda_{22}(w, \lambda) = \prod_{i=1}^{n} \frac{\Theta(w - t_i) \Theta(\lambda^1 - \lambda^2 - \gamma(\Omega^1 - \Omega^2))}{\Theta(w - q_i) \Theta(\lambda^1 - \lambda^2)}.
\]
From the fact that \( W \) is a highest weight \( E_{\gamma, \tau}(gl_N) \)-module, it follows that there are \( z_i \in \mathbb{C} \mod \mathbb{Z} + \mathbb{Z} \tau, i = 1, \ldots, p \) such that
\[
L_{12}(z)\omega = \sum_{i} c_i(z, \lambda)L_{12}(z_i)\omega
\]
for some, minimal, \( p \in \mathbb{N} \).
Let us now define the functions \( F(w, z, \lambda) \) and \( F_1(w, \lambda) \) by
\[
F(w, z, \lambda)\omega = L_{21}(w)L_{12}(z)\omega, \quad F_1(w, \lambda)\omega = L_{21}(w)L_{12}(z_i)\omega
\]
then the dimension (over \( \text{Fun}_N(\mathbb{C}) \)) of \( W[\mu], \ \mu = \Omega - e_1 + e_2 \) is equal to the dimension of the space of functions \( \mathcal{F} \) spanned by \( F_1(w, \lambda) \). In fact, assume that there are \( c_i(\lambda) \in \text{Fun}_N(\mathbb{C}) \) such that
\[
\sum_{i} c_i(\lambda)F_1(w, \lambda) = 0,
\]
then the vector

$$\eta = \sum_i c_i (\lambda + \gamma h^i) L_{12}(z_i) \omega$$

(5.14)

is identically zero, because by construction

$$L_{21}(w) \eta = \sum_i c_i (\lambda) F_i(w, \lambda) \omega.$$  

(5.15)

And this implies that \( \eta = 0 \) because \( W \) is irreducible.

Let us now compute the dimension of \( F \). From the defining relations of the quantum group \( \hat{E}_{\gamma+\theta}(gl_N) \) we can compute the functions \( F_i(w, \lambda) \) explicitly. In fact, they are equal to

$$F_i(w, \lambda) = r(w - z_i, \lambda) \lambda_{22}(z_i, \lambda) - s(w - z_i, \lambda) \lambda_{22}(w, \lambda),$$

(5.16)

where \( r(w - z_i, \lambda) \) and \( s(w - z_i, \lambda) \) are given by:

$$r(w - z_i, \lambda) = \frac{\Theta(\gamma) \Theta(w - z - \lambda_{12} + \gamma \Omega_{12})}{\Theta(\lambda_{12} - \gamma (\Omega_{12} + 1)) \Theta(w - z)}$$

(5.17)

$$s(w - z_i, \lambda) = \frac{\Theta(\gamma) \Theta(w - z - \lambda_{12}) \Theta(\lambda - \gamma \Omega_{12})}{\Theta(\lambda_{12}) \Theta(w - z) \Theta(\lambda_{12} - \gamma (\Omega_{12} + 1))}$$

(5.18)

where \( \lambda_{12} = \lambda_1 - \lambda_2 \), \( \Omega_{12} = \Omega_1 - \Omega_2 \).

The last two equations imply that the functions \( F_i(w, \lambda) \) have poles (at most) in the points \( q_i \mod \mathbb{Z} + \mathbb{Z} \tau, i = 1, \ldots, n \) and that they satisfy, for \( \lambda \) generic, the functional equations

$$F_i(w + 1, \lambda) = F_i(w, \lambda),$$

(5.19)

$$F_i(w + \tau, \lambda) = e^{2\pi i (\lambda - \gamma \Omega_{12})} F_i(w, \lambda).$$

(5.20)

It is well known, that the dimension of the space of functions with property (5.19) and (5.20) is equal to the degree of the divisor of poles. Which implies that \( \dim(F) \leq n \). It remains to prove that it is exactly \( n \). This is equivalent to find \( z_1, \ldots, z_n \in \mathbb{C} \mod \mathbb{Z} + \mathbb{Z} \tau \) such that the corresponding \( F_i(w, \lambda) \) are a basis of \( F \). If all the zeros of \( \Lambda_{22}(z, \lambda) \) are different, then one can easily see that the functions \( F(w, t_i, \lambda), i = 1, \ldots, n \) are linearly independent. In fact, in this case they are equal to

$$F_i(w, \lambda) = -\frac{\Theta(\gamma) \Theta(w - t_i - \lambda_{12}) \Theta(\lambda_{12} - \gamma \Omega_{12})}{\Theta(\lambda_{12}) \Theta(w - t_i) \Theta(\lambda_{12} - \gamma (\Omega_{12} + 1))} \Lambda_{22}(w, \lambda).$$

(5.21)

Then the linear independence follows from:

$$F_i(t_j, \lambda) = \delta_{ij} \frac{\Theta(\lambda_{12} - \gamma \Omega_{12})}{\Theta(\lambda_{12} - \gamma (\Omega_{12} + 1))} \Theta(0) \frac{d}{dw} \Lambda_{22}(w, \lambda)|_{w=t_j}.$$  

(5.22)

We will show now that there are \( z_1, \ldots, z_n \in \mathbb{C} \) such that the functions \( F(w, z_i, \lambda) \) are a basis of \( F \) in the case that \( G_{22}(w) \) has zeros of higher multiplicity.

Let \( d \in \mathbb{R} \) the number

$$d = \min_{i,j=1,\ldots,n, i \neq j} |t_i - t_j|,$$  

(5.23)

where \( t_i \in \mathbb{C} \) are the zeros of \( G_{22}(w) \) in the fundamental parallelogram. Then for
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Figure 5.1: Fundamental Parallelogram of $\Theta(w)$.

$\epsilon \in \mathbb{R}$ small enough, there is a complex number $c$ such that $|c| < \epsilon$ and such that the equation

$$G_{22}(w) = c$$

(5.24)

has exactly $n$ simple solutions (in the fundamental parallelogram) $z_i, i = 1, \ldots, n$ which satisfy the conditions

$$|z_i - z_j| = \begin{cases} < \epsilon, \\ > \frac{d}{2}, \end{cases}$$

(5.25)

Where the first inequality holds for solutions of (5.24) which are close to the same zero of $G_{22}(w)$, while the second holds for zeros which are close to different zeros of $G_{22}(w)$ (see figure 5.2). We will show that the functions

$$F_i(w, \lambda) = F(w, z_i, \lambda), \quad i = 1, \ldots, n,$$

(5.26)

with this choice of $z_1, \ldots, z_n$ are a basis of $\mathcal{F}$.

In order to do this, we notice that if the functions $F_i(w, \lambda)$ are not a basis of $\mathcal{F}$, then there are $c_i(\lambda) \in \operatorname{Fun}_N(\mathbb{C})$ such that

$$\sum_{i=1}^{n} c_i(\lambda) F_i(w, \lambda) = 0.$$  

(5.27)

This implies that for any choice of $w_j \in \mathbb{C}, j = 1, \ldots, n$, different from the poles of the functions $F_i(w, \lambda)$, the matrix

$$(F_{ij})_{i,j=1}^{n} = (F_i(w_j, \lambda))_{i,j=1}^{n}$$

(5.28)

has a vanishing determinant. We will, therefore, prove that:

$$\det(F_i(z_j, \lambda)) \neq 0.$$  

(5.29)
Let us first rewrite the functions $F_t(w, \lambda)$ as

$$F_t(w, \lambda) = \frac{\Theta(\gamma)\Theta(\lambda - \gamma\Omega_{12})}{\Theta(\lambda_{12} - \gamma(\Omega_{12} + 1))} \frac{\Lambda_{22}(w, \lambda) - \Lambda_{22}(z_i, \lambda)}{\Theta(w - z_i)} + \Delta F_t(w, \lambda), \quad (5.30)$$

where

$$\Delta F_t(w, \lambda) = F_t(w, \lambda) - \frac{\Theta(\gamma)\Theta(\lambda - \gamma\Omega_{12})}{\Theta(\lambda_{12} - \gamma(\Omega_{12} + 1))} \frac{\Lambda_{22}(w, \lambda) - \Lambda_{22}(z_i, \lambda)}{\Theta(w - z_i)}. \quad (5.31)$$

It is now easy to see that the functions $F_t(w, \lambda)$ are a basis of $\mathcal{F}$. In fact the determinant of the matrix

$$\tilde{F} = (F_t(z_j, \lambda))_{i,j=1}^n = \left( \frac{\Theta(\gamma)\Theta(\lambda - \gamma\Omega_{12})}{\Theta(\lambda_{12} - \gamma(\Omega_{12} + 1))} \frac{\Lambda_{22}'(z_j, \lambda)}{\Theta'(0)} \delta_{ij}, \right)_{i,j=1}^n, \quad (5.32)$$

where

$$\Lambda_{22}'(z_j, \lambda) = \frac{d}{dw} \Lambda(w, \lambda)|_{w=z_i}, \quad \Theta'(0)$$

is equal to

$$\frac{\Theta(\gamma)\Theta(\lambda - \gamma\Omega_{12})}{\Theta(\lambda_{12} - \gamma(\Omega_{12} + 1))} \prod_{i=1}^n \Lambda_{22}'(z_i, \lambda). \quad (5.34)$$

Now, from the representation (5.30) of $F_t(w, \lambda)$, it follows that the matrix

$$F = \tilde{F} + \Delta F,$$ \quad (5.35)

where

$$\Delta F_{ij}(\lambda) = \Delta F_t(z_j, \lambda), \quad (5.36)$$

is invertible if $\|\tilde{F}\| > \|\Delta F\|$, where the norm of the matrix involved is, by definition, $\|A\| = \max_{i,j} |A_{ij}|$.

The norm of $\tilde{F}$ can be computed from its explicit form:

$$\|\tilde{F}\| = \left| \frac{\Theta(\gamma)\Theta(\lambda - \gamma\Omega_{12})}{\Theta(\lambda_{12} - \gamma(\Omega_{12} + 1))} \max_{1 \leq i \leq n} |\Lambda_{22}'(z_i, \lambda)| \right|. \quad (5.37)$$

Let us now compute the norm of the second matrix in (5.35).

In order to do this, we have to distinguish two cases. If $z_i$ and $z_j$ are close, that is if $|z_i - z_j| < \epsilon$, then there is a function $H(\lambda) \in \text{Fun}_N(\mathbb{C})$ such that

$$|\Delta F_{ij}| \leq C|H(\lambda)||G_{22}(z_i)|. \quad (5.38)$$

This follows from:

$$\lim_{w \to z_j} \Delta F_j = H(\lambda)G_{22}(z_j) = H(\lambda)G_{22}(z_i), \quad (5.39)$$
because, by construction, $G_{22}(z_i) = G_{22}(z_j)$.

If $z_i$ and $z_j$ are "distant", that is, if they are close to different zeros of $\Delta_{22}(w, \lambda)$, then

$$|\Delta F_j| \leq \frac{K}{d}|H(\lambda)||G_{22}(z_i)|,$$  \hspace{1cm} (5.40)

because, by construction, $G_{22}(z_i) = G_{22}(z_j)$.

This implies, that

$$\|\Delta F\| \leq C|H(\lambda)|\max_i|G_{22}(z_i)|.$$  \hspace{1cm} (5.41)

Now, using that

$$\frac{G_{22}(z_j)}{G_{22}(z_j)}$$  \hspace{1cm} (5.42)

can be made as small as desired, by choosing an $\epsilon > 0$ small enough, we get the desired result. $\square$

**Definition 5.6.** Two complex numbers $z_1, z_2$ are called generic if the equation

$$z_1 - z_2 + \gamma k = 0 \mod \mathbb{Z} + \mathbb{Z} \tau,$$  \hspace{1cm} (5.43)

has no solution $k \in \mathbb{Z}$.

**Lemma 5.7.** Let $z_1, z_2$ be non generic complex numbers, then Equation (5.43) has a unique solution.

**Proof.** The existence of a solution, follows from the definition of generic pairs. Assume now that there are two solutions to (5.43), then

$$z_1 - z_2 + \gamma k = z_1 - z_2 + \gamma k_1 = 0 \mod \mathbb{Z} + \mathbb{Z} \tau,$$  \hspace{1cm} (5.44)

implies that $\gamma(k - k_1) = 0 \mod \mathbb{Z} + \mathbb{Z} \tau$ which is impossible because $\gamma$ is a (real) irrational number. $\square$

This trivial Lemma give us the possibility to introduce an order relation between non generic complex numbers.

**Definition 5.8.** Let $z_1, z_2$ be two non generic complex numbers, then $z_1 < z_2$ if there is a $k \in \mathbb{N}$ such that $z_1 - z_2 + \gamma k = 0 \mod \mathbb{Z} + \mathbb{Z} \tau$.

**Definition 5.9.** Let $W$ be a finite dimensional $E_{\gamma, r}(gl_N)$-module in normal form with highest weight vector $\omega$ and elliptic weight $(\omega, \{\Lambda_{ui}(w, \lambda)\}_{i=1}^N)$, where

$$\Lambda_{ui}(w, \lambda) = G_{ui}(w) \prod_{i=1}^{t-1} \frac{\Theta(\lambda^i - \lambda^i - \gamma(\Omega^i - \Omega^i))}{\Theta(\lambda^i - \lambda^i)},$$  \hspace{1cm} (5.45)

then we define the functions $\Gamma_j(w), j = 2, \ldots, N$ to be:

$$\Gamma_j(w) = \frac{G\beta_j(w)}{G\alpha_i(w)}, i = j - 1, j = 2, \ldots, N.$$  \hspace{1cm} (5.46)
Lemma 5.10. Let $W$ be a finite dimensional irreducible $E_{\gamma, \tau}(gl_N)$-module in normal form with highest weight vector $\omega$ and elliptic weight $(\Omega, \{\Lambda_i(w, \lambda)\}_{i=1}^N)$. Moreover, let $\Gamma_j(w)$ be the functions of Definition 5.9. Then, for each function $j = 2, \ldots, n$ there is a pair of numbers $(z, p) \in \mathbb{C}^2$, where $z \in \mathcal{D}_Z(\Gamma_j)$ and $p \in \mathcal{D}_P(\Gamma_j)$, such that $p - z = n \gamma \mod \mathbb{Z} + \mathbb{Z} \tau$ for some number $n \in \mathbb{N}$.

Proof. We first notice, that it suffices to prove the Lemma for $j = 2$. In fact the other cases can be reduced to this by taking the tensor product with 1-dimensional $E_{\gamma, \tau}(gl_N)$-module with elliptic weight:

$$\Omega = 0$$

$$\Lambda_i(w, \lambda) = G_{j-1,j-1}(w)^{-1} F_i(\lambda), \; i = 1, \ldots, N \quad (5.47)$$

for suitably chosen functions $F_i(\lambda) \in \text{Fun}_{\mathbb{N}}(\mathbb{C})$.

Let $z_1, \ldots, z_n$ be the zeros of $G_{22}(w)$, then using that "non genericity" is an equivalence relation, we split the set $\{z_1, \ldots, z_n\}$ in equivalence classes, where each class contains all zeros which are non generic between them selves.

Let now $s_1$ be the biggest zero in some equivalence class, then if all poles of $G_{22}(w)$ are generic with respect to $s_1$, the $E_{\gamma, \tau}(gl_N)$-module $W$ is infinite dimensional. In fact, we will eventually prove, that all vectors

$$\omega_1 = L_{12}(s_1) \omega$$

$$\omega_{n+1} = L_{12}(s_1 + \gamma n) \omega_n, \quad (5.48)$$

are non-zero.

Remark 7. We will assume that the vectors (5.48) are regular. Or to be more precise that

$$\lim_{\varepsilon \to 0} L_{12}(z + \gamma n + \varepsilon) \omega_n \quad (5.49)$$

is finite. This because in the event that

$$L_{12}(z + \gamma n + \varepsilon) \omega_n \quad (5.50)$$

has a pole of order $m$ in $\varepsilon = 0$, it follows that

$$\lim_{\varepsilon \to 0} (z + \gamma n + \varepsilon)^m L_{12}(z + \gamma n + \varepsilon) \omega_n \neq 0. \quad (5.51)$$

And this is exactly what we need to prove.

In order to show that the vectors (5.48) are all different from zero, we first need to show that

$$L_{21}(w) \omega_n |_{w = s_1 + \gamma m} \neq \infty, \; \forall m \geq n. \quad (5.52)$$

We prove this by induction. $L_{21}(w) \omega$ is clearly regular in $w = s_1 + \gamma m, \; m \geq 0$. Assume now that (5.52) holds for $n \in \mathbb{N}$, then using the defining relations of $E_{\gamma, \tau}(gl_N)$ we get that:

$$L_{21}(w)L_{12}(s_1 + \gamma n) \omega_n =$$

$$r_1(w - (s_1 + \gamma n), \lambda)L_{12}(s_1 + \gamma n) L_{21}(w) \omega_n +$$

$$r_2(w - (s_1 + \gamma n), \lambda)L_{11}(w)L_{22}(s_1 + \gamma n) \omega_n +$$

$$r_3(w - (s_1 + \gamma n), \lambda)L_{11}(s_1 + \gamma n) L_{22}(w) \omega_n, \quad (5.53)$$
for some functions $r_i(w - (s_1 + \gamma m), \lambda)$ regular in $w = s_1 + \gamma m, m \geq n + 1$. Now, the first term on the right hand side of (5.53) is regular by the induction hypothesis. The second one is zero by (5.60). That the last term on the left hand side of (5.53) is regular in $w = s_1 + \gamma m, m \geq n + 1$ is also a consequence of equation (5.60).

Let, now, $D_{12}(w)$ be the element in $E_{\gamma,r}(gl_N)$:

$$D_{12}(w) = \frac{\Theta(\lambda^1 - \lambda^2)}{\Theta(\lambda^1 - \lambda^2 - \gamma(h^1 - h^2))} \{L_{22}(w + \gamma)L_{11}(w) - L_{12}(w + \gamma)L_{21}(w)\},$$

then it has been proven in [6],[5] that

$$D_{12}(w) = \frac{\Theta(\lambda^1 - \lambda^2)}{\Theta(\lambda^1 - \lambda^2 - \gamma(h^1 - h^2))} \{L_{11}(w + \gamma)L_{22}(w) - L_{21}(w + \gamma)L_{12}(w)\},$$

and that

$$D_{12}(w)L_{kl}(z) = L_{kl}(z)D_{12}(w), \forall z \in \mathbb{C}, \ k,l \in \{1,2\}.$$ (5.56)

Moreover, consider the relation:

$$a_{12}(z_1 - z_2, \lambda - \gamma h)L_{22}(z_1)L_{12}(z_2) + b_{12}(z_1 - z_2, \lambda - \gamma h)L_{12}(z_1)L_{22}(z_2) = L_{12}(z_2)L_{22}(z_1)$$

It follows, then from equation (5.57) that

$$L_{22}(w)L_{12}(s_1)\omega = \frac{1}{a_{12}(z_1 - z_2, \lambda - \gamma h)}L_{12}(s_1)L_{22}(w)\omega.$$ (5.58)

Which is equivalent to:

$$L_{22}(w)L_{12}(s_1)\omega = F(\lambda)\frac{\Theta(w - s_1 - \gamma)}{\Theta(w - s_1)} G_{22}(w)L_{12}(s_1)\omega,$$ (5.59)

for some function $F(\lambda) \in \text{Fun}_N(\mathbb{C})$.

So we get that:

a) $L_{12}(s_1)\omega$ is an eigenvector of $L_{22}(w)$,

b) $L_{22}(w)L_{12}(s_1)\omega$ has a zero in $w = s_1 + \gamma$.

Applying, (5.59) $n$ times, we get that:

$$L_{22}(w)\omega_n = F_n(\lambda)\frac{\Theta(w - s_1 - n\gamma)}{\Theta(w - s_1)} G_{22}(w)L_{12}(s_1)\omega_n.$$ (5.60)

We are now ready to show that the vectors $\omega_n$ are all different from zero, if all poles of $G_{22}(w)$ are generic with respect to $s_1$. In fact, assume that $\omega_n = 0$ for some $n \in \mathbb{N}$, then:

$$0 = L_{21}(s_1 + n\gamma)\omega_n = L_{21}(s_1 + n\gamma)L_{12}(s_1 + (n - 1)\gamma)\omega_{n-1} = -\frac{\Theta(\lambda^1 - \lambda^2 - \gamma(h^1 - h^2))}{\Theta(\lambda^1 - \lambda^2)} D_{12}(s_1 + (n - 1)\gamma)\omega_{n-1} + L_{11}(s_1 + n\gamma)L_{22}(s_1 + (n - 1)\gamma)\omega_{n-1}. $$ (5.61)
But the last term in the right hand side of equation (5.61) is identically zero because of equation (5.60). So, using (5.56), we get

\[ 0 = D_{12}(s_1 + (n - 1)\gamma)\omega_{n-1} = L_{12}(s_1 + (n - 2)\gamma) \ldots L_{12}(s_1)D_{12}(s_1 + (n - 1)\gamma)\omega. \]

(5.62)

And using (5.54)

\[ 0 = G_{22}(s_1 + n\gamma)G_{11}(s_1 + (n - 1)\gamma) \times \]

\[ L_{12}(s_1 + (n - 2)\gamma) \ldots L_{12}(s_1)\omega \]

(5.63)

The last equation implies that \( \omega_{n-1} = 0 \) because \( G_{11}(w) = 1 \) and by assumption \( G_{22}(w + n\gamma) \neq 0, \forall n > 0 \). And this leads to the contradiction \( \omega_n = \omega_{n-1} = \cdots = \omega = 0 \).

Assume now, that there is a pole, say \( t_1 \) that is non-generic, or in other words, that the equation

\[ s_1 - t_1 - \gamma k = 0 \mod \mathbb{Z} + \mathbb{Z}r, \]

(5.64)

has a solution. Then, if \( k < 0 \) it follows, with a similar argument, that the \( E_{\gamma, r}(gl_N) \)-module is infinite dimensional, because all vectors of (5.48) are non-zero.

Lemma 5.11. Let \( W \) be an \( E_{\gamma, r}(gl_N) \)-module in normal form with cyclic vector \( \omega \) and elliptic weight \( (\Omega, \{\Lambda_{ii}(w, \lambda)\}_{i=1}^N) \). Moreover, let \( z \in D_{x}(\Gamma_1), 1 < i \leq N \).

Then:

a) the \( E_{\gamma, r}(gl_N) \)-module \( \Lambda^l_{N} V(z - \gamma) \otimes W \) is reducible.

b) the \( E_{\gamma, r}(gl_N) \)-module \( \hat{U} \) generated by the cyclic vector \( \omega_1 \otimes \omega_2 \in \Lambda^l_{N} V(z - \gamma) \otimes W, \omega_2 = w \), is a proper sub-module.

c) there is a 1-dimensional \( E_{\gamma, r}(gl_N) \)-module \( \hat{U} \) such that

\[ \hat{W} = \hat{U} \otimes (\Lambda^l_{N} V(z - \gamma) \otimes W/\hat{U}), \]

(5.65)

has elliptic weight

\[ \tilde{\Lambda}_{11}(w, \lambda) = 1, \]

\[ \tilde{\Lambda}_{ii}(w, \lambda) = G_{ii}^2(w, \lambda)\hat{W}(\lambda), \quad i < l - 1 \]

\[ \tilde{\Lambda}_{l-1,l-1}(w, \lambda) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} G_{l-1,l-1}^2(w)\hat{W}_{l-1,l-1}(\lambda) \]

\[ \tilde{\Lambda}_{l}(w, \lambda) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} G_{l}^2(w)\hat{W}_{l}(\lambda), \quad \tilde{G}_{l}^2(w) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} G_{l}^2(w) \]

\[ \tilde{\Lambda}_{ii}(w, \lambda) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} G_{ii}^2(w)\hat{W}_{ii}(\lambda), \quad i > l. \]

(5.66)

which means that \( \mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma_1) - (z) + (z + \gamma) \).

Proof. From the definition of the co-product it follows that the elliptic weight of \( \Lambda^l_{N} V(z - \gamma) \otimes W \) is:

\[ \tilde{\Lambda}_{ii}(w, \lambda) = \Lambda_{ii}^l(w, \lambda - \gamma \Omega)\Lambda_{ii}^2(w, \lambda), \quad l < i \leq N, \]

(5.67)

where \( \Lambda_{ii}^l(w, \lambda) \), resp. \( \Lambda_{ii}^2(w, \lambda) \), are the elliptic weights of \( \Lambda^l_{N} V(z - \gamma) \) and \( W \).

Using Lemma 5.5, one can see that \( \Lambda^l_{N} V(z - \gamma) \otimes W \) is reducible. In fact let \( n \) be
the degree of the divisor of zeros of $\Gamma_{1}^{T}(w)$, then the dimension of the sub-space of $\wedge_{\mathcal{X}}V(z - \gamma) \otimes W$ of weight 

$$\mu = \Omega^{1} + \Omega^{2} - e^{l-1} + e^l$$

(5.68)

is $n + 1$, which is bigger than the degree of the divisor of zeros of $\Gamma_{1}^{T}(w) = \Gamma_{1}^{T}(w)\Gamma_{2}^{T}(w)$. Which is equal to $n$ because there is a cancellation between the pole of $G_{1}^{T}(w)$ and a zero of $G_{2}^{T}(w)$. And this proves a).

Let us show now that the $E_{\nu, \tau}(gl_{N})$-module generated by the highest weight vector $\omega = \omega_{1} \otimes \omega_{2}$ is a proper sub-module of $\wedge_{\mathcal{X}}V(z - \gamma) \otimes W$.

We assume here that $z \notin \mathcal{Z}(G_{\nu})$ and not a pole of $G_{l-1}(w)$. The other case can be treated in a symmetric way. In order to do this, it suffices to show that there is a vector $\eta$, in the space of weight (5.68), which is not of the form

$$\sum_{i=1}^{n} c_{i}(\lambda)L_{ij}(z_{i})\omega_{1} \otimes \omega_{2}, \; j = l - 1.$$  

(5.69)

Let now $v_{1} \in \wedge_{\mathcal{X}}V(z - \gamma)$ be a vector of weight $\mu_{1} = \Omega^{1} - e^{l-1} + e^l$. We claim that $u = v_{1} \otimes \omega_{2}$ is not of the form (5.69).

In fact, let $c_{i}(z_{0}, \ldots, z_{n}, \lambda) \in \text{Fun}_{N}(\mathbb{C}), i = 1, \ldots, n$ be $n$ functions which solve the equation in $W$: 

$$[L_{ij}(z_{0}) - \sum_{i=1}^{n} c_{i}(z_{0}, \ldots, z_{n}, \lambda)L_{ij}(z_{i})]\omega_{2} = 0, \quad j = l - 1,$$

(5.70)

where the vectors

$$L_{ij}(z_{i})\omega_{2}, \quad i = 1, \ldots, n, j = l - 1,$$

(5.71)

are a basis of $W[\Omega^{2} + e^{l-1} - e^l]$. And, let $\eta$ be the vector

$$\eta = [L_{ij}(z_{0}) - \sum_{i=1}^{n} c_{i}(z_{0}, \ldots, z_{n}, \lambda)L_{ij}(z_{i})]\omega_{1} \otimes \omega_{2}, \quad j = l - 1.$$  

(5.72)

Then, using that

$$L_{ij}(w)\omega_{1} \otimes \omega_{2} = \omega_{1} \otimes L_{ij}(w, \lambda)\omega_{2} + L_{ij}(w, \lambda - \gamma\Omega^{2})\omega_{1} \otimes L_{H}(w, \lambda)\omega_{2},$$

(5.73)

one can see that $u = v_{1} \otimes \omega_{2} \notin U$ if and only if $\eta = 0$. In fact, the following holds:

$$\eta = [L_{ij}(z_{0}, \lambda) - \sum_{i=1}^{n} c_{i}(z_{0}, \ldots, z_{n}, \lambda)L_{ij}(z_{i})]\omega_{1} \otimes \omega_{2}$$

$$= \omega_{1} \otimes [L_{ij}(z_{0}, \lambda) - \sum_{i=1}^{n} c_{i}(z_{0}, \ldots, z_{n}, \lambda)L_{ij}(z_{i}, \lambda)]\omega_{2} + L_{ij}(z_{0}, \lambda - \gamma\Omega^{2})\omega_{1} \otimes L_{H}(z_{0}, \lambda)\omega_{2} - \sum_{i=1}^{n} c_{i}(z_{0}, \ldots, z_{n}, \lambda)L_{ij}(z_{i}, \lambda - \gamma\Omega^{2})\omega_{1} \otimes L_{H}(z_{i}, \lambda)\omega_{2}.$$  

(5.74)
The first term on the right hand side of (5.74) is identically zero, by construction. It holds therefore that:

\[ \eta = F(z_0 \ldots, z_n, \lambda)v_1 \otimes \omega_2, \]  

(5.75)

for some function \( F(z_0 \ldots, z_n, \lambda) \in \text{Fun}_N(\mathbb{C}) \).

We show now, that \( F(z_0 \ldots, z_n, \lambda) \) is identically zero. This can be done, by noticing that the functions \( c_i(z_0 \ldots, z_n, \lambda) \in \text{Fun}_N(\mathbb{C}), i = 0, \ldots, n \) are solution of the equation:

\[
0 = \sum_{i=0}^{n} c_i(z_0, \ldots, z_n, \lambda) \left\{ \frac{\Theta(z - \gamma - z_i + \lambda_i - \lambda_j)}{\Theta(z - z_i) \Theta(\lambda_i - \lambda_j)} \times \right. \\
G_{jj}(z_i) G_H(z) \\
- \frac{\Theta(z - \gamma - z_i + \lambda_i - \lambda_j - \gamma(\Omega_i - \Omega_j))}{\Theta(z - z_i) \Theta(\lambda_i - \lambda_j - \gamma(\Omega_i - \Omega_j))} \times \\
G_{jj}(z_i) G_H(z) \left\} 
\]  

(5.76)

(see Lemma 2.21 for a derivation). Moreover from Section 4, we know that

\[
L_{jj}(w) \omega_1 = H(\lambda) \frac{\Theta(w - z + \gamma + \lambda_j - \lambda_i)}{\Theta(w - z)} \omega_1, 
\]  

(5.77)

for some function \( H(\lambda) \in \text{Fun}_N(\mathbb{C}) \). Putting everything together we get that:

\[
F(z_0 \ldots, z_n, \lambda) = \tilde{H}(\lambda) G_{jj}(z)^{-1} \sum_{i=0}^{n} c_i(z_0, \ldots, z_n, \lambda) \times \\
\frac{\Theta(z_i - z + \gamma + \lambda_j - \lambda_i - \gamma(\Omega_j - \Omega_i))}{\Theta(z_i - z) \Theta(\lambda_j - \lambda_i - \gamma(\Omega_j - \Omega_i))} G_H(z_i) G_{jj}(z), 
\]  

(5.78)

for some function \( H(\lambda) \in \text{Fun}_N(\mathbb{C}) \). But (5.78) is the right hand side of (5.76), so that

\[
F(z_0 \ldots, z_n, \lambda) = \\
\tilde{H}(\lambda) G_{jj}(z)^{-1} \sum_{i=0}^{n} c_i(z_0, \ldots, z_n, \lambda) \left\{ \frac{\Theta(z_i - z + \gamma + \lambda_j - \lambda_i)}{\Theta(z_i - z) \Theta(\lambda_j - \lambda_i)} \times \\
G_{jj}(z_i) G_H(z) \right\} = \\
\tilde{H}(\lambda) \sum_{i=0}^{n} c_i(z_0, \ldots, z_n, \lambda) \left\{ \frac{\Theta(z_i - z + \gamma + \lambda_j - \lambda_i)}{\Theta(z_i - z) \Theta(\lambda_j - \lambda_i)} \times \\
G_{jj}(z_i) \right\} \Gamma(z) = 0, 
\]  

(5.79)

because, by the hypothesis of the Lemma, \( z \) is a zero of \( \Gamma(z) \). And this proves b). Let us now prove c). In order to do this we have to compute

\[
L_{ii}(w)[u], \forall i = 1, \ldots, N, 
\]  

(5.80)
where \([u] = v_1 \otimes \omega_2 \mod U\).

Let \(i \neq l\), then it follows from the definition of the co-product it follows that:

\[
L_{ii}(w)[u] = [L_{ii}(w, \lambda - \gamma \Omega)v_1 \otimes L_{ii}(w, \lambda)\omega_2 = \]

\[
G_{ii}^1(w)G_{ii}^2 (w)H_i(\lambda)[u] = \]

\[
\left\{ \begin{array}{ll}
H_i(\lambda)[u]G_{ii}^{(2)}(w), & i < l - 1 \\
\frac{\Theta(w - z + \gamma)}{\Theta(w - z)}H_i(\lambda)G_{ii}^{(2)} (w)[u], & i = l - 1 \\
\frac{\Theta(w - z + \gamma)}{\Theta(w - z)}H_i(\lambda)G_{ii}^{(2)} (w)[u], & i > l
\end{array} \right. 
\]

(5.81)

for some function \(H(\lambda) \in \text{Fun}_N(\mathbb{C})\).

If \(i = l\) the computation is a little bit more complicated. In order to carry it out we notice that, by definition of the quotient:

\[
[L_{jl}(w)\omega_1 \otimes \omega_2] = 0, \forall z \in \mathbb{C}, j = l - 1. 
\]

(5.82)

This implies that there is a function \(H(\lambda) \in \text{Fun}_N(\mathbb{C})\) such that

\[
[v_1 \otimes \omega_2] = H(\lambda)[\omega_1 \otimes L_{jl}(z, \lambda)\omega_2].
\]

(5.83)

From (5.83) one can then see, that:

\[
L_{ll}(w)[u] = [L_{ll}(w, \lambda - \gamma \Omega)v_1 \otimes L_{ll}(w, \lambda)\omega_2 = \]

\[
H(\lambda - \gamma \lambda^l)[L_{ll}(w, \lambda - \gamma \lambda^l)\omega_1 \otimes L_{ll}(w, \lambda)L_{ll}(z, \lambda)\omega_2 = \]

\[
\tilde{H}(\lambda) \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} \tilde{G}_{ll}^2 (w),
\]

(5.84)

where \(\tilde{G}_{ll}^2 (w) = \frac{\Theta(w - z - \gamma)}{\Theta(w - z)}\) and \(\tilde{H}(\lambda)\) is an, unimportant, function in \(\text{Fun}_N(\mathbb{C})\). Now, from Theorem 2.9, it follow that there is a 1-dimensional \(E_{\gamma, \tau}(\mathfrak{gl}_N)\)-module \(\tilde{U}\) such that

\[
\tilde{U} \otimes (\Lambda_{\lambda^l} V(z - \gamma) \otimes W/U)
\]

(5.85)

has elliptic weight:

\[
\Lambda_{11}(w, \lambda) = 1,
\]

\[
\Lambda_{ii}(w, \lambda) = G_{ii}^2 (w, \lambda)W_i(\lambda), \quad i < l - 1
\]

\[
\Lambda_{l-1,l-1}(w, \lambda) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} G_{l-1,l-1}^2 (w)W_{l-1,l-1}(\lambda)
\]

\[
\Lambda_{ll}(w, \lambda) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} \tilde{G}_{ll}^2 (w)W_{ll}(\lambda)
\]

\[
\Lambda_{ii}(w, \lambda) = \frac{\Theta(w - z + \gamma)}{\Theta(w - z)} G_{ii}^2 (w)W_{ii}(\lambda), \quad i > l.
\]

(5.86)

which is the desired result.

\(\Box\)

**Corollary 5.12.** Let \(W\) be irreducible \(E_{\gamma, \tau}(\mathfrak{gl}_N)\)-module in normal form with highest weight vector \(\omega\) and elliptic weight \((\Omega, \{\Lambda_{ii}(w, \lambda)\}_{i=1}^N)\), where

\[
\Lambda_{ii}(w, \lambda) = G_{ii}^2 (w) \prod_{i=1}^{n-1} \frac{\Theta(\lambda' - \lambda' - \gamma(\Omega' - \Omega'))}{\Theta(\lambda' - \lambda')}
\]

(5.87)
Then, there is a function \( P_j(w) \) of the form

\[
P_j(w) = \prod_{j=1}^{n_j} \Theta(w - s^j), \quad 2 \leq j \leq N,
\]

such that

\[
\frac{G_{jj}(w)}{G_{ii}(w)} = \frac{P_j(w + \gamma)}{P_j(w)}, \quad i = j - 1.
\]

**Proof.** From Lemma 5.10, we now that if \( W \) is finite dimensional, that there exists a zero \( z \in \mathcal{D}_z(\Gamma_j) \) and a pole \( p \in \mathcal{D}_p(\Gamma_j) \) such that

\[
p - z = \gamma n \mod \mathbb{Z} + \mathbb{Z} \tau.
\]

Moreover, from Lemma 5.11, we know that there is series of \( E_{\gamma, \tau}(gl_N) \)-modules

\[
W^0 \to W^1 \to \cdots \to W^n,
\]

such that

\[
\dim_{\text{Fun}_{\gamma, \tau}(\mathbb{C})} W^{i+1} < \dim_{\text{Fun}_{\gamma, \tau}(\mathbb{C})} W^i \dim_{\text{Fun}_{\gamma, \tau}(\mathbb{C})} \Lambda_{\mathbb{C}}^j V(z - (i + 1)\gamma),
\]

and such that

\[
\mathcal{D}(\Gamma_{j+1}^i) = \mathcal{D}(\Gamma_j^i) - (z + \gamma i) + (z + \gamma(i + 1)).
\]

So putting everything together, we get:

a) \( W^n \) is finite dimensional.

b) \( \Gamma_j^{(n)}(w) = \frac{\Theta(w - z - \gamma n)}{\Theta(w - z)} \Gamma_j(w) \).

But, because \( W^n \) is finite dimensional, we can start from the beginning with a new zero and a new pole until all the zeros (and poles) of \( \Gamma_j(w) \) are eliminated. This implies that

\[
\Gamma_j(w) = \prod_{i=1}^{n} \frac{\Theta(w - z_i)}{\Theta(w - z_i - \gamma n_i)},
\]

where \( z_i, i = 1, \ldots, n \) are the zeros of \( \Gamma_j(w) \) (with their multiplicity).

Now, rewriting each factor of (5.94) as

\[
\frac{\Theta(w - z_i)}{\Theta(w - z_i - \gamma n_i)} = \prod_{l=1}^{n_i} \frac{\Theta(w - z_i - (l - 1)\gamma)}{\Theta(w - z_i - \gamma l)} =
\]

\[
\prod_{l=1}^{n_i} \Theta(w - z_i - (l - 1)\gamma) \quad \prod_{l=1}^{n_i} \Theta(w - z_i - \gamma l) =
\]

\[
= \frac{P_j(w)}{P_j(w - \gamma)}, \quad P_j(w) = \prod_{l=0}^{n_i-1} \Theta(w - z_i - \gamma l),
\]

we get the desired result. \( \square \)
Chapter 6

Tensor Products of Symmetric Modules

In this Chapter we study the tensor product of symmetric modules $S^n V(z) \otimes S^m V(w)$ and show that it is irreducible if

$$z - w \notin \{ -\gamma n, -\gamma(n - 1), \ldots, \gamma m \}.$$  \hspace{1cm} (6.1)

Moreover, we will show that if $n = m$:

1. $R_{S^n V(z), S^n V(z)}(\lambda) = P$, where $P$ is the flip operator $Pv \otimes w = w \otimes v$.
2. $S^n V(z) \otimes S^n V(z)$ is irreducible.

6.1 R-Matrix on $S^n V(z) \otimes W$

In this section we study the $R$-matrix between symmetric $E_{\gamma, r}(gl_N)$-modules $S^n V(z)$ and arbitrary $E_{\gamma, r}(gl_N)$-modules $W$.

**Definition 6.1.** Let $W$ be an $E_{\gamma, r}(gl_N)$-module and $i, j$ two vectors in $N^k$ such that

$$i = (i_1, i_2, \ldots, i_k),$$  \hspace{1cm} (6.2)

$$j = (j_1, j_2, \ldots, j_k),$$  \hspace{1cm} (6.3)

then by $S_{ij}(z, \lambda)$ we denote the operator

$$S_{ij}(z, \lambda) = \sum_{\sigma \in S^k} L_{i_1 j_{\sigma(1)}}(z + (k - 1)\gamma, \lambda) L_{i_2 j_{\sigma(2)}}(z + (k - 2)\gamma, \lambda - \gamma^2) \ldots L_{i_k j_{\sigma(k)}}(z, \lambda - \sum_{l=1}^{k-1} \ell^{\sigma(l)}).$$  \hspace{1cm} (6.4)

**Theorem 6.2.** Let $S^n V(z)$ be the $n$-th symmetric power of the fundamental representation and $W$ any $E_{\gamma, r}(gl_N)$-module, then the matrix on $S^n V(z) \otimes W$

$$R_{S^n V(z) \otimes W}(\lambda) \text{sym}(e_{i_1} \otimes e_{j_1} \otimes \ldots \otimes e_{j_k}) \otimes w = \sum_i \text{sym}(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}) \otimes S_{ij}(z, \lambda)w,$$

$$i = (i_1, i_2, \ldots, i_k), \quad j = (j_1, j_2, \ldots, j_k),$$  \hspace{1cm} (6.5)

is an $R$-matrix.

Moreover if $W = S^m V(z)$, then the $R$-matrix depends only on the difference $x = z - w$, and its poles are located in $x = z - w = \gamma(m - k)$, $k = 0, \ldots, n - 1$. 

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The rest of this section is devoted to the proof of this theorem.

**Definition 6.3.** We denote the right-inverse of $W^n(\lambda)$ by $Q^n(\lambda)$:

$$W^n(\lambda)Q^n(\lambda) = P_{S_n \mathbb{C}^N},$$

where $W^n(\lambda)$ is the operator that permutes the first $n$ factors and $P_{S_n \mathbb{C}^N}$ is the projector on $S^n \mathbb{C}^N$:

$$P_{S_n \mathbb{C}^N} = \frac{1}{n!} \sum_{\sigma \in S^n} \sigma.$$

**Lemma 6.4.** The matrix

$$L^{n,n+1}(z + \gamma(n - 1), \lambda)L^{n-1,n+1}(z + \gamma(n - 2), \lambda - \gamma h^1)$$

$$\ldots L^{1,n+1}(z, \lambda - \gamma \sum_{l=1}^{n-1} h^l) \in \text{End}(\mathbb{C}^N \otimes W),$$

leaves the space $S^n \mathbb{C}^N \otimes W$ invariant.

**Proof.** Let us number, as usual, the single factors from 1 to $n + 1$ from left to right, where $n + 1$ is associated with $W$.

\[
\begin{array}{cccc}
\mathbb{C}^N & \mathbb{C}^N & \mathbb{C}^N & W \\
1 & 2 & \ldots & n & n+1
\end{array}
\]

Then, from the graphical representation of

$$L^{n,n+1}(z + \gamma(n - 1), \lambda)L^{n-1,n+1}(z + \gamma(n - 2), \lambda - \gamma h^1)$$

$$\ldots L^{1,n+1}(z, \lambda - \gamma \sum_{l=1}^{n-1} h^l) \in \text{End}(\mathbb{C}^N \otimes W),$$

one can see that

\[\text{Figure 6.1. Left Side} \quad \text{Figure 6.2. Right Side}\]

where the boxes represent the permutation of the first $n$ factors.

Moreover, we proved in Theorem 3.5 that the image of $W^n(\lambda)$ is $S^n \mathbb{C}^N$, and this completes the proof.

**Lemma 6.5.** Let $S^n V(z) \subseteq \mathbb{C}^N \otimes^n$ and $S^m V(w) \subseteq \mathbb{C}^N \otimes^m$ be two symmetric powers of the fundamental vector representation of $E_{\gamma,\tau}(gl_N)$ and $W$ any $E_{\gamma,\tau}(gl_N)$-
6.1. R-Matrix on $S^n V(z) \otimes W$

module, then the matrices

$$R_{S^n V(z), W}(\lambda) = L^{n,n+m+1}(z + \gamma(n-1), \lambda) \ldots$$

$$L^{1,n+m+1}(z, \lambda - \gamma \sum_{t=1}^{n-1} h^t) \otimes S^n V(z) \otimes W,$$

$$R_{S^n V(w), W}(\lambda) = L^{n+m,n+m+1}(w + \gamma(m-1), \lambda) \ldots L^{n+1,n+m+1}(w, \lambda - \gamma \sum_{t=1}^{m-1} h^t) \otimes S^n V(w) \otimes W,$$

where $\lambda_{ij} = \lambda - \gamma \sum_{t=1}^{n} h^t - \gamma \sum_{t=j-n}^{m} h^t$, satisfy the Yang-Baxter equation

$$R_{S^n V(z), S^n V(w)}(\lambda - \lambda_{i,j-n}) R_{S^n V(z), W}(\lambda)$$

$$= R_{S^n V(w), W}(\lambda - \lambda_{i,j-n}) R_{S^n V(n), W}(\lambda - \sum_{t=1}^{m} h^t)$$

where the factors in $C^{N \otimes n}$ are numbered from 1 to $n$ and those of $C^{N \otimes m}$ are numbered from $n+1$ to $n+m$ from left to right. This implies that the matrices defined in (6.5), (6.5), (6.5) are R-matrices on $S^n V(z) \otimes W$ and $S^n V(w) \otimes W$ and $S^n V(z) \otimes S^n V(w)$.

**Proof.** The best way to see that this lemma holds, is to consider the graphical representation of the R-matrix.

The associated equation on $C^{N \otimes n} \otimes C^{N \otimes m} \otimes W$ is

$$\Rightarrow R_{23}(\lambda) R_{13}(\lambda - \gamma \sum_{t=n+1}^{m+n} h^t) R_{12}(\lambda) \times$$

$$W^1(\lambda) W^2(\lambda - \sum_{t=1}^{n} h^t)$$

**Figure 6.3. Left Side**
where the boxes \((W^i)\) represent the permutation of \(1, \ldots, n\) and \(n + 1, \ldots, n + m\), respectively.

Now, multiplying both sides with the right-inverses \(Q_1, Q_2\) of \(W_1, W_2\), we get:

\[
R_{23}R_{13}R_{12}W_1^* Q_2 Q_1 = R_{23} R_{13} R_{12} |S^2 V_{(z)} \otimes S^n V_{(w)} \otimes W |,
\]

\[
(6.18)
\]

\[
R_{12} R_{13} R_{23} W_2^* Q_1 Q_2 = R_{23} R_{13} R_{12} |S^2 V_{(z)} \otimes S^n V_{(w)} \otimes W |,
\]

\[
(6.19)
\]

which is the desired result.

\hspace{1cm}

\[\square\]

**Corollary 6.6.** Let \(S^n V(z)\) be the \(n\)-th symmetric power of the fundamental vector representation of \(E_{\gamma, \tau}(gl_N)\) and \(W\) any \(E_{\gamma, \tau}(gl_N)\)-module, then the \(R\)-matrix on \(S^n V(z) \otimes W\) has the form

\[
R_{S^n V(z) \otimes W} (\lambda) \sum (e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n}) \otimes w =
\sum_{i} \sum (e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}) \otimes S_{ij}(z, \lambda) w.
\]

\[
(6.20)
\]

**Proof.** It suffices to compute

\[
L^{n,n+1}(z + \gamma(n-1), \lambda) L^{n-1,n+1}(z + \gamma(n-2), \lambda - \gamma h^1) \ldots L^{1,n+1}(z, \lambda - \gamma \sum_{i=1}^{n-1} h^i) \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^n \otimes W),
\]

\[
(6.21)
\]

\[
(6.22)
\]

on symmetric vectors. The explicit computation yields:

\[
L^{n,n+1}(z + \gamma(n-1), \lambda) L^{n-1,n+1}(z + \gamma(n-2), \lambda - \gamma h^1) \ldots L^{1,n+1}(z, \lambda - \gamma \sum_{i=1}^{n-1} h^i) \sum_{i} \sum_{\sigma \in S^n} e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes L_{i_j j_{(1)}}(z + (n-1)\gamma, \lambda)
\]

\[
L_{i_j j_{(2)}}(z + (n-2)\gamma, \lambda - \gamma \ell_{j_{(1)}}) \ldots L_{i_m j_{(n)}}(z, \lambda - \sum_{i=1}^{n-1} \ell_{j_{(i)}}) w.
\]

\[
(6.23)
\]

From lemma 6.4 we know that right-hand side is symmetric in the first \(n\) factors, this implies that the right-hand side of (6.1) can be simplified to

\[
\sum_{i} \sum_{\sigma \in S^n} \sum_{\sigma \in S^n} L_{i_j j_{(1)}}(z + (n-1)\gamma, \lambda)
\]

\[
L_{i_j j_{(2)}}(z + (n-2)\gamma, \lambda - \gamma \ell_{j_{(1)}}) \ldots L_{i_m j_{(n)}}(z, \lambda - \sum_{i=1}^{n-1} \ell_{j_{(i)}}) w,
\]

\[
(6.24)
\]

which is the desired result.

\hspace{1cm}

\[\square\]
Corollary 6.7. Let $S_{ij}(z, \lambda)$ be the operators of definition (6.1), then
$$S_{t(i)j}(z, \lambda) = S_{ij}(z, \lambda), \quad \forall t \in S^n. \quad (6.25)$$

Definition 6.8. By $S_j(z, \lambda)$ we denote the operator
$$S_j(z, \lambda) = S_{ij}(z, \lambda), \quad i = (1, 1, \ldots, 1). \quad (6.26)$$

Lemma 6.9. Let $S_j(z, \lambda)$ be the operator defined in 6.8, then
$$S_j(z, \lambda) = g(\lambda)^{-1} L_{1j_1}(z, \lambda) L_{1j_2}(z + \gamma, \lambda - \gamma \ell_1) \ldots L_{1j_n}(z + \gamma(n - 1), \lambda - \gamma \sum_{l=1}^{n-1} \ell_l), \quad (6.27)$$

where the function $g_j(\lambda)$ is
$$g(\lambda) = \prod_{i=1}^N \frac{|n_i|!}{[n_i]!} \prod_{j=k} \prod_{r=0}^{n_j-1} \prod_{s=0}^{n_k-1} \frac{\Theta(\lambda^j - \lambda^k - \gamma(r - s - 1))}{\Theta(\lambda^j - \lambda^k - \gamma(r - s))} \quad (6.28)$$

Proof. This lemma is a simple consequence of the Yang-Baxter equation. The explicit form of the function $g(\lambda)$ is computed in 3.14. □

Lemma 6.10. Let $W$ be an $E_{\gamma, \tau}(gl_N)$-module and $\omega \in W$, then
$$S_i^{(m)}(z, \lambda) S_j^{(m)}(w, \lambda - \gamma \sum_{q=1}^n \ell_q) \omega = \sum_{i,j} R_{ij}^{k}(\lambda) S_k^{(m)}(w, \lambda) S_j^{(m)}(z, \lambda - \gamma \sum_{q=1}^m \ell_q) \omega. \quad (6.29)$$

where $R_{ij}^{k}(\lambda)$ is the $R$-matrix on $S^n V(z) \otimes S^m V(w)$ ( $S_i^{(n)}$, resp. $S_j^{(m)}$ are the operators of Definition 6.9 of length $n$, resp. $m$).

Proof. Applying the Yang-Baxter equation on $S^n V(z) \otimes S^m V(w) \otimes W$
$$R_{S^n V(z), S^m V(w)}(\lambda - \gamma h^{n+m+1}) R_{S^n V(z), W}(\lambda)$$
$$R_{S^m V(w), W}(\lambda - \gamma \sum_{l=1}^n h_l)$$
$$= R_{S^m V(w), W}(\lambda) R_{S^n V(z), W}(\lambda - \gamma \sum_{l=1}^m h_l)$$
$$= R_{S^n V(z), S^m V(w)}(\lambda), \quad (6.30)$$
on both side to $e_i \otimes e_j \otimes \omega$, on gets
$$e_i \otimes e_i \otimes S_i^{(n)}(z) S_j^{(m)}(w) \omega + \ldots$$
$$e_i \otimes e_i \otimes \sum_{i,j} R_{ij}^{k}(\lambda) S_k^{(m)}(w) S_j^{(m)}(z) \omega + \ldots. \quad (6.32)$$

where the dots are linear independent terms. And this is the desired result. □

6.2 $R$-matrix on $S^n V(z) \otimes S^m V(w)$

Lemma 6.11. The $R$-matrix on $S^n V(z) \otimes S^m V(w)$ is well defined and satisfies the unitarity condition if
$$z - w \notin \{ -\gamma n, -\gamma(n - 1), \ldots, \gamma m \}. \quad (6.33)$$
Proof. From the definition of the \( R \)-matrix we have that
\[
R^{n,n+1}(z - w + \gamma(n-1), \lambda_{n,n+1})R^{n,n+2}(z - w + \gamma(n-2), \lambda_{n,n+2}) \times \\
\ldots R^{n,n+m}(z - w - \gamma(n-m), \lambda_{n,n+m}) \times \\
\ldots R^{1,n+1}(z - w, \lambda_{1,n+1}) \ldots R^{1,n+m}(z - w - \gamma(m-1), \lambda_{1,n+m}),
\]
where \( \lambda_{ij} = \lambda - \gamma \sum_{k=1}^{h(0)} \gamma \sum_{m \geq i+n} h(m) \), is an \( R \)-matrix on \( V(z) \otimes V(z + \gamma) \otimes \ldots \otimes V(z + \gamma(n-1)) \otimes V(z) \otimes \ldots \otimes V(w + \gamma) \otimes \ldots \otimes V(w + \gamma(m-1)) \). Using that (6.34) leaves the sub-space \( S^n V(z) \otimes S^m V(w) \) invariant, it follows that it is also an \( R \)-matrix for \( S^n V(z) \otimes S^m V(w) \). Now, if \( z \) and \( w \) satisfies the condition (6.33) one can see that the \( R \)-matrix (6.34) is a product of invertible matrices, and therefore invertible. The unitarity of (6.34) can be verified using the unitarity of the \( R \)-matrix on \( V(z_1 \otimes V(z_2) \).

Lemma 6.12. The \( R \)-matrix on \( S^n V(z) \otimes S^n V(z) \) is well defined and invertible if \( z = w \) and equal to the flip operator
\[
P v \otimes w = w \otimes v. \tag{6.35}
\]
Proof. Using the representation (6.20) of \( R \)-matrix on \( S^n V(z) \otimes S^n V(w) \) and the explicit form of the matrix elements, it follows that \( RS^n V(z), S^n V(w) (\lambda) \) is well defined (has no poles) if
\[
z - w \neq \gamma(k - n), \quad k = 0, 1, 2, \ldots, n - 1. \tag{6.36}
\]
In particular it is well defined in \( z = w \). In this case, using the symmetries of the operators \( S_{ij}(z, \lambda) \), it follows that
\[
S_{ij}(z, \lambda)|_{z=w}e_k = \delta_{ik}F_{ij}e_j. \tag{6.37}
\]
In fact, if all \( j_l \neq i_k, l = 1, \ldots, n \), then (6.37) holds because
\[
L_{ij}(z, \lambda)e_k = 0, \quad \text{for } i \neq j, \tag{6.38}
\]
unless at least one \( k_l \) is equal to \( i \).

Let us now consider the case where some \( j_k = j_{k+1} = \ldots = j_{k+n} \) are equal to \( i_k \).
Using the symmetries of \( S_{ij}(w) \) on can see that we can, without loss of generality, assume that this happens for \( j_{n-1} = \ldots = j_n = i_n \). Then we can rewrite the left hand side of (6.37) as
\[
S_{ij}(z)e_k = \left\{ \sum_{\sigma \in S^{n-1}} L_{i_1,j_{a(1)}}(z + \gamma(n-1))L_{i_2,j_{a(2)}}(z + \gamma(n-2)) + \right. \\
\ldots L_{i_{n-1},j_{a(n-1)}}(z + \gamma(n-1))L_{i{n-1},j_{a(n-2)}}(z + \gamma(n-2)) + \\
\left. \ldots L_{i_{n-1},j_{a(n-1)}}(z + \gamma(n-1))L_{i{n-1},j_{a(n-2)}}(z + \gamma(n-2)) + \ldots \right. \\
\ldots L_{i_{n-1},j_{a(n-1)}}(z + \gamma(n-1))L_{i{n-1},j_{a(n-2)}}(z + \gamma(n-2))
\]
\[
\ldots L_{i_n,j_{a(n-1)}}(z + \gamma(n-1))L_{i_n,j_{a(n-2)}}(z + \gamma(n-2)) \}
\]
\[
e_k. \tag{6.39}
\]
It is then clear that if all \( k_l \neq i_n \), that (6.39) vanishes identically. Because of (6.38), if \( i_n \neq j_l \), and because of

\[
L_{ii}(z)_{z=w}e_k = \frac{\Theta(z-w-\gamma n_i)}{\Theta(z-w-\gamma n)}g(\lambda)e_k, \tag{6.40}
\]

where \( n_i \) is the number of \( k_l = i \) and \( g(\lambda) \) is some function of \( \lambda \), if \( i_n = j_n \). (The other cases are treated in a similar way).

It remains to show that the coefficients \( F_{ij} \) are equal to 1. The fact that they are non-zero follows by taking the limit \( \gamma \to 0 \) of the matrix elements and noticing that in this limit \( F_{ij} \) are a sum of positive numbers. In fact

\[
\lim_{\gamma \to 0} \Lambda_{ij}(z,\lambda, n) = \lim_{\gamma \to 0} \frac{\Theta(\gamma n_i)}{\Theta(\gamma n)} \prod_{k \geq 1, k \neq i} \frac{\Theta(\lambda_i - \lambda_k + \gamma n_k)}{\Theta(\lambda_i - \lambda_k)} = \frac{n_i}{n} \tag{6.41}
\]

and

\[
\lim_{\gamma \to 0} \Lambda_{ii}(z,\lambda, n) = \lim_{\gamma \to 0} \frac{\Theta(\lambda_i - \lambda_j - \gamma(n_i - 1))\Theta(\gamma(n_j + 1))}{\Theta(\gamma n)\Theta(\lambda_i - \lambda_j)} \times \prod_{k \geq 1, k \neq i,j} \frac{\Theta(\lambda_i - \lambda_k + \gamma n_k)}{\Theta(\lambda_i - \lambda_k)} = \frac{n_j + 1}{n}, \tag{6.42}
\]

where \( n_i \) is the number of \( k_s \) equal to \( l \) and \( n = \sum n_i \). The last step is a consequence of (6.29). \( \square \)

**Corollary 6.13.** The tensor product of two symmetric powers of the fundamental representation \( S^n V(z) \otimes S^m V(w) \) is irreducible for values of \( z, w \) in \( \mathbb{C} \) such that

\[
z - w \notin \{-\gamma n, -\gamma(n-1), \ldots, \gamma m\}. \tag{6.43}
\]

**Proof.** In order to prove this Corollary we first consider the module \( W_1 \otimes W_2 \), where

\[
W_1 = S^n V(t_1) \otimes S^m V(t_2) \tag{6.44}
\]

for \( z, w \) generic and show and show that the \( R \)-matrix on \( W_1 \otimes W_2 \) is equal to the flip if

\[
t_1 = z, \quad t_2 = w. \tag{6.45}
\]

On \( W = W_1 \otimes W_2 \) the \( R \)-matrix takes the form

\[
R_{W_1,W_2}(\lambda) = R_{23}(t_2 - z, \lambda - \gamma h^{(1)})R_{13}(t_1 - z, \lambda) \times \\
R_{24}(t_2 - w, \lambda - \gamma(h^{(1)} + h^{(3)}))R_{14}(t_1 - w, \lambda - \gamma h^{(2)}). \tag{6.46}
\]

Using the previous results one can see that the point \( t_1 = z, t_2 = w \) is not a singular value of \( R_{W_1,W_2}(\lambda) \). In this case the \( R \)-matrix becomes

\[
R_{W_1,W_2}(\lambda) = R_{23}(w - z, \lambda - \gamma h^{(1)})P_{13}P_{24} \times \\
R_{14}(z - w, \lambda - \gamma h^{(2)}) \tag{6.47}
\]
and the claim follows from the unitarity of the \( R \)-matrix on \( S^n V(z) \otimes S^m V(w) \).

Now, using Theorem 6.2 in combination with (6.47), it follows that

\[
\delta_{i_k j_k}^{(m)}(w, \lambda - \gamma h^{(1)}) \delta_{i_{j_1} j_{j_1}}^{(m)}(z, \lambda) e_{k_1} \otimes e_{k_2} = \delta_{i_{k_1} k_1} \delta_{i_{j_1} j_1} e_{j_1} \otimes e_{j_2}, \tag{6.48}
\]

where \( e_{i_1} \otimes e_{j_2} \) are the standard basis vectors of \( S^n V(z) \otimes S^m V(w) \). The irreducibility is then a simple consequence of (6.48).

Corollary 6.13 can be easily generalized to the tensor product of an arbitrary number of symmetric powers of the fundamental vector representation.

**Theorem 6.14.** Let \( W \) be the \( E_{\gamma, \tau}(gl_N) \)-module

\[
W = S^{n_1} V(z_1) \otimes S^{n_2} V(z_2) \otimes \ldots \otimes S^{n_k} V(z_k), \tag{6.49}
\]

then \( W \) is irreducible if

\[
z_l - z_k \notin \{-\gamma n_1, -\gamma(n_1 - 1), \ldots, \gamma n_k\}, \quad \forall l > k. \tag{6.50}
\]
Chapter 7

Fusion of Symmetric Modules

7.1 Introduction

In the Chapter 5 we showed that any irreducible \( E_{\gamma,\tau}(gl_N) \)-module can be realized as a sub-quotient of a suitable tensor product of fundamental representations. In this section we will study such \( E_{\gamma,\tau}(gl_N) \)-modules more in detail. In particular we will give an explicit construction of the \( E_{\gamma,\tau}(gl_N) \)-modules associated to (generalized) Young Tableaus.

In order to explain the ideas involved we start with a simple example. Consider the pair of \( E_{\gamma,\tau}(gl_N) \)-modules

\[
W_1 = S^2V(z) \otimes V(w), \quad W_2 = V(w) \otimes S^2V(z).
\]

From Chapter 6 we know that the \( R \)-matrices on \( W_1 \) (resp. \( W_2 \)) have poles only if

\[
\begin{align*}
&z - w = \gamma - \gamma k, k = 0, 1 \mod Z + Z\tau \quad \text{on } W_1, \\
&w - z = 2\gamma \mod Z + Z\tau \quad \text{on } W_2
\end{align*}
\]

Moreover, we notice that the two equations (7.2) can not be satisfied simultaneously if \( \gamma \) generic.

Using the "unitarity" of the \( R \)-matrices it follows that if one of the two equations (7.2) is satisfied, the \( E_{\gamma,\tau}(gl_N) \)-modules \( W_1, W_2 \) become reducible.

But let us be more explicit and consider the special case:

\[
\begin{align*}
&z - w = \gamma \quad (z - w = \gamma - \epsilon).
\end{align*}
\]

In the limit \( \epsilon \to 0 \) the \( R \)-matrix on \( W_1 \) will develop a pole. So, in analogy to Chapter 4, we define a "regularized" \( R \)-matrix on \( W_1 \) by:

\[
R_{S^2V(z) \otimes V(w)}^{reg}(\lambda) = \res \mid_{\epsilon = 0} R_{S^2V(z), V(w)}(\lambda).
\]

It is then possible, using the results of Section 7.2, to show that for generic values of \( \gamma \) the kernel of \( R_{S^2V(z), V(w)}^{reg}(\lambda) \) is equal to

\[
\ker R_{S^2V(z), V(w)}^{reg}(\lambda) \approx S^3(C^N).
\]

Moreover using the Yang-Baxter equation it follow that the vector space

\[
\ker R_{S^2V(z), V(w)}^{reg}(\lambda)P^{1,2} \approx S^3(C^N)
\]

where
is an invariant sub-space of $W_2$. In fact the operator $R_{S^2V(z),V(w)}^{reg}(\lambda)P^{1,2}$ "commutes" with the $L$-matrix on $W_2$:

$$R_{S^2V(z),V(w)}^{reg}(\lambda)P^{12}L^{(01)}(\lambda - \gamma h^1)L^{(02)}(\lambda) = L^{(02)}(\lambda - \gamma h^1)L^{(01)}(\lambda)R_{S^2V(z),V(w)}^{reg}(\lambda - \gamma h^0).$$ (7.7)

So, using equation (7.7) we can define an $E_{\gamma,\tau}(gl_N)$-module structure on

$$W_3 = V(z - \gamma) \otimes S^2V(z)/R_{S^2V(z),V(w)}^{reg}(\lambda)P^{12} \cong V(z - \gamma) \otimes S^2V(z)/S^3(C_N).$$ (7.8)

The vectors in $W_3$ have the symmetries of the tensors associated to the Young-

\[
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\]

Figure 7.1: Young Tableau associated to $W_3$

diagram 7.1, it is therefore natural to associate the same Young Tableau to the $E_{\gamma,\tau}(gl_N)$-module $W_3$.

### 7.2 Representations Associated to $YT$

The main result of this section is the explicit construction of $E_{\gamma,\tau}(gl_N)$-modules associated to (standard) Young Tableaus. Such $E_{\gamma,\tau}(gl_N)$-modules are realized as sub-quotients of a suitable tensor product of symmetric $E_{\gamma,\tau}(gl_N)$-modules.

**Definition 7.1.** Let $Y$ be a Young Tableau with $k \leq N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_k$, then we associate a positive integer and a pair of natural coordinates $(i, j) \in \mathbb{N}^* \times \mathbb{N}^*$ to each box of $Y$ starting from the upper left corner (see figure 7.2).

\[
\begin{array}{|c|c|}
\hline
i & j \\
\hline
\end{array}
\]

Figure 7.2: Standard Young Tableau

**Definition 7.2.** Let $Y$ be a Young Tableau with $k \leq N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_k$, then for each $1 \leq i, j \leq k$, $Y^{ij}$ is the sub-Tableau of $Y$ which contains only the boxes of rows $i$ and $j$. 
7.2. Representations Associated to YT

Definition 7.3. 1. The order of a Young Tableau is a positive number which counts the number of pairs of boxes in $Y$ which are on a diagonal (from left to right) minus the number of pair of boxes which satisfy the condition $(i_1 - i_2) - (j_1 - j_2) = -1$, where $(i_1, j_1)$ are the natural coordinates of the $l$-th box. As we will see later the order of a Young Tableau is related to the order of the pole of an operator associated to $Y$.

$$\mathcal{O}(Y) = \sum_{i < j} \mathcal{O}(Y_{ij}).$$

(7.10)

Proof. From the definition of the order of a Young Tableau it is clear that each pair of boxes that contribute to $\mathcal{O}(Y)$ is in one and only one sub-Tableau $Y_{ij}$ for some $1 < i < j < k$. And this implies the claim. □

Definition 7.4. Let $Y$ be a Young Tableau with $k \leq N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_k$ and $\mathcal{O}(Y)$ its order, then

$$\mathcal{O}(Y) = \# \left\{\{(i_1, j_1), (i_2, j_2)\} | (i_1, j_1) \in Y, l = 1, 2, \right.$$ 

$$\text{and } i_2 = i_1 + d, j_2 = j_1 + d, d \in \mathbb{N}^* \}$$

$$- \# \left\{\{(i_1, j_1), (i_2, j_2)\} | j_1 < j_2 | (i_1, j_1) \in Y, l = 1, 2, \right.$$ 

$$\text{and } (i_1 - i_2) - (j_1 - j_2) = -1\right\}.$$  (7.9)

2. Analogously, the order of a sub-Tableau $Y_{ij}$ in the the number of pairs of boxes in $Y_{ij}$ which are on a diagonal minus the number of pair of boxes which satisfy the condition $(i_1 - i_2) - (j_1 - j_2) = -1$

Lemma 7.4. Let $Y$ be a Young Tableau with $k \leq N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_k$ and $\mathcal{O}(Y)$ its order, then

Let $Y$ be a Young Tableau with $k \leq N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_k$, then we define $\mathcal{S}_Y(z)$ and $\mathcal{S}_Y(z, \epsilon)$ to be the $E_{\gamma, \tau}(\mathfrak{gl}_N)$-modules

$$\mathcal{S}_Y(z) = S^{n_1}V(z_1) \otimes S^{n_2}V(z_2) \cdots S^{n_k}V(z_k), z_i = z - \gamma(i - 1),$$

$$\mathcal{S}_Y(z, \epsilon) = S^{n_1}V(z_1) \otimes S^{n_2}V(z_2) \cdots S^{n_k}V(z_k),$$

$$z_i = z - \gamma(i - 1) - (i - 1)\epsilon.$$  (7.11)

The last object we need, is the operator that permutes the factors in a tensor product of symmetric $E_{\gamma, \tau}(\mathfrak{gl}_N)$-modules.

Definition 7.5. Let $Y$ be a Young Tableau with $k \leq N$ rows of length $n_1 \geq n_2 \geq \cdots \geq n_k$, then we define $W_{n_1, \ldots, n_k}(z_1, \ldots, z_k; \lambda)$ recursively by the rules:

a) $W_{n_1, n_2}(z_1, z_2; \lambda) = \mathcal{R}_{S^{n_1}V(z_1), S^{n_2}V(z_2)}(\lambda).$

b) $W_{n_1, \ldots, n_k}(z_1, \ldots, z_k; \lambda) = \mathcal{R}_{S^{n_1}V(z_1), S^{n_2}V(z_2)}(\lambda) \mathcal{R}_{S^{n_1}V(z_1), S^{n_2}V(z_2)}(\lambda) \cdots$

$$\cdots \mathcal{R}_{S^{n_1}V(z_1), S^{n_2}V(z_2)}(\lambda - \gamma \sum_{i=1}^k h(i)) W_{n_2, \ldots, n_k}(z_2, \ldots, z_k; \lambda - \gamma h(1))$$  (7.12)

Lemma 7.7. Let $\mathcal{S}_Y(z, \epsilon)$ be the $E_{\gamma, \tau}(\mathfrak{gl}_N)$-module $S^{n_1}V(z) \otimes \cdots \otimes S^{n_k}V(z - (k - 1)\gamma + \gamma(k - 1)\epsilon)$ associated to the Young Tableau $Y$, then the operator $W_{n_1, \ldots, n_k}(z_1, \ldots, z_k; \lambda), z_i = z - \gamma(i - 1) - \epsilon(i - 1)$ that permutes the factors of $\mathcal{S}_Y(z, \epsilon)$ has a pole of order $\mathcal{O}(Y)$ in $\epsilon = 0 \mod \mathbb{Z} + \mathbb{Z}\tau$. 
Definition 7.8. Let $\tilde{S}_Y(z, \epsilon)$ be the $E_{\gamma,r}(gl_N)$-module $S^{n_1}V(z) \otimes \ldots \otimes S^{n_k}V(z) - (k-1)\gamma + \gamma(k-1)\epsilon$ associated to the Young Tableau $Y$, then we define $W_{n_1, \ldots, n_k}^{reg}(\lambda)$ to be the operator on $\tilde{S}_Y(z)$:

$$W_{n_1, \ldots, n_k}^{reg}(\lambda) = \lim_{\epsilon \to 0} \Theta(\epsilon)^{O(Y)}W_{n_1, \ldots, n_k}(\epsilon, \lambda).$$

(7.13)

Proof. (of Lemma 7.7) We first prove that $\lim_{\epsilon \to 0} \Theta(\epsilon)^{O(Y)}W_{n_1, \ldots, n_k}(\lambda)$ is well-defined.

In Section 6.1 we showed that the $R$-matrix on $S^{n_1}V(z_1) \otimes S^{n_2}V(z_2)$ has a pole only if $z_1 - z_2 = (n - k)$ for some $k \in \{0, 1, \ldots, n-1\}$. It's easy to see that this happens exactly $O(Y)$ times.

It remains to show that $W_{n_1, \ldots, n_k}^{reg}(\lambda)$ is not identically zero. In order to do this it suffices to show that

$$W_{n_1, \ldots, n_k}^{reg}(\lambda)\epsilon_1^{\otimes n_1} \otimes \cdots \otimes \epsilon_k^{\otimes n_k} \neq 0.$$  

(7.14)

Using the explicit form of $W_{n_1, \ldots, n_k}(z_1, \ldots, z_k; \lambda)$ one can see that (7.14) is equal to

$$W_{n_1, \ldots, n_k}(z_1, \ldots, z_k; \lambda)\epsilon_1^{\otimes n_1} \otimes \cdots \otimes \epsilon_k^{\otimes n_k} =$$

$$w(z_1, \ldots, z_k; \lambda)\epsilon_1^{\otimes n_1} \otimes \cdots \otimes \epsilon_k^{\otimes n_k} + \text{linear independent term},$$

(7.15)

(7.16)

where the function $w(z_1, \ldots, z_k; \lambda)$ is given by

$$w(z_1, \ldots, z_k; \lambda) = \prod_{i<j} w_{ij}(z_1, \ldots, z_k; \lambda),$$

(7.17)

$$w_{ij}(z_1, \ldots, z_k; \lambda) =$$

$$\prod_{l=1}^{n_i} \prod_{m=1}^{n_j} \alpha((j - i)\gamma + (l - m)\gamma + (i - m)\epsilon, \lambda_i - \lambda_j + \sigma_{ij}),$$

(7.18)

for some, unimportant, $\sigma_{ij}$.

Now, using the functional form of $\alpha(z, \lambda)$ it is easy to see that in the limit $\epsilon \to 0$ there is

1. a pole for each crossing associated to spaces with coordinates such that $(i_1 - i_2) - (j_1 - j_2) = -1$, $j_1 < j_2$,
2. a zero for each crossing associated to spaces with coordinates such that \( i_2 = i_1 + d, j_2 = j_2 + d, d \in \mathbb{N}^* \), where \((i_1, j_1)\) are the natural coordinates of the boxes.

And this ends the proof. \( \square \)

**Lemma 7.9.** Let \( Y \) be a Young Tableau with \( k \) rows of length \( n_1 \geq n_2 \cdots \geq n_k \) and \( S_Y(z) \) the (opposite) \( E_{\gamma,\tau}(gl_N) \)-module associated to \( Y \) (see Definition 7.5). Moreover, let \( P^{1,2,\ldots,k} \) be the flip on \( S^{n_1}CN \otimes \ldots \otimes S^{n_k}CN \):

\[
P^{1,2,\ldots,k} v_1 \otimes v_2 \otimes \ldots \otimes v_k = v_k \otimes \ldots \otimes v_2 \otimes v_1, \quad v_i \in S^{n_i}CN,
\]

then

\[
W^{reg}_{n_1,\ldots,n_k}(\lambda) P^{1,2,\ldots,k} \mathcal{L}^{0,1}(z,\lambda) = \mathcal{L}^{0,1}_{op}(z,\lambda) W^{reg}_{n_1,\ldots,n_k}(\lambda - \gamma h(0)),
\]

where \( \mathcal{L}^{0,1}(z,\lambda) \), resp. \( \mathcal{L}^{0,1}_{op}(z,\lambda) \) is the \( L \)-matrix, resp. the opposite \( L \)-matrix, on \( S_Y(z) \).

**Proof.** Equation (7.20) follows from the previous definitions and the Yang-Baxter Equation. \( \square \)

It follows from Lemma 7.9 that the kernel of \( W^{reg}_{n_1,\ldots,n_k}(\lambda) P^{1,2,\ldots,k} \) is an invariant sub-module of \( S_Y(z) \). This motivates the following definition.

**Definition 7.10.** Let \( Y \) be a Young Tableau with \( k \) rows of length \( n_1 \geq n_2 \cdots \geq n_k \) and \( S_Y(z) \) the (opposite) \( E_{\gamma,\tau}(gl_N) \)-module associated to \( Y \), then the Young \( E_{\gamma,\tau}(gl_N) \)-module \( W_Y(z) \) is:

\[
F_{n_1,\ldots,n_k}(W_Y(z)) = \frac{F_{n_1,\ldots,n_k}(S_Y(z))}{\ker W^{reg}_{n_1,\ldots,n_k} P^{1,2,\ldots,k}}.
\]

We will see in the following sections that the \( E_{\gamma,\tau}(gl_N) \)-module \( W_Y(z) \) has the same dimension, over \( \text{Fun}_N(\mathbb{C}) \), as the classical \( GL_n \)-modules associated to the same Young Tableau. But in order to do this we need to study the kernel of \( W^{reg}_{n_1,\ldots,n_k} \) more in details.

### 7.2.1 The Kernel of \( W^{reg}_{n_1,\ldots,n_k} \)

In this Section we will compute the kernel of the operator \( W^{reg}_{n_1,\ldots,n_k}(\lambda) \). This will be done in two steps. In the first one we will consider the case of a tensor product of two symmetric spaces. In the second one, we will generalize the results to a arbitrary tensor product of symmetric \( E_{\gamma,\tau}(gl_N) \)-modules associated to standard Young Tableaus.

#### The kernel of \( W^{reg}_{n_1,n_2} \)

**Definition 7.11.** Let \( W \) be the tensor product of \( n \) copies of \( \mathbb{C}^N \)

\[
W = \mathbb{C}^N \otimes \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N.
\]

Let us number the single factors of \( W \) from 1 to \( n \) from left to right. Moreover let \( l_1 \) and \( l_2 \) be two integers such that:

\[
1 \leq l_1 \leq l_2 \leq n.
\]

Then by \( S^{l_1,l_2}(\lambda) \) we denote the operator on \( W \) defined recursively by the rules:
Chapter 7. Fusion of Symmetric Modules

1. $S^{l_1,l_1}(\lambda) = 1$

2. $S^{l_1,l_1+1}(\lambda) = R^{l_1,l_1+1}(\lambda, \lambda - \gamma \sum_{i=1}^{l_1-1} h^{(i)})$

3. 

$$S^{l_1,l_1+k}(\lambda) = S^{l_1,l_1+k}(\lambda - \gamma h^{(l_1+k)}) \times$$

$$R^{l_1,l_1+k}(-k, \lambda - \gamma \sum_{i=1}^{l_1-1} h^{(i)}) \ldots R^{l_1,(k-1),l_1+k}(-\gamma, \lambda - \gamma \sum_{i=1}^{l_1+(k-2)} h^{(k)})$$

Remark 8. $S^{l_1,n_1}(\lambda)$ is the R-matrix on $S^{n_1}V(z)$.

Definition 7.12. Let $S^{l_1,l_2}(\lambda)$ be the operator of Definition 7.11 and $w = e_{i_1} \otimes \cdots \otimes e_{i_n}$ a standard basis vector of $CN \otimes CN$, then we define the function $g^{l_1,l_2}_{i_1,...,i_n}(\lambda) \in \text{Fun}_N(C)$ by:

$$S^{l_1,l_2}(\lambda)e_{i_1} \otimes \cdots \otimes e_{i_{l_1}}, e_{i_1} \otimes \cdots \otimes e_{i_{l_2}} \otimes \cdots \otimes e_{i_n} =$$

$$g^{l_1,l_2}_{i_1,...,i_n}(\lambda)e_{i_1} \otimes \cdots \otimes e_{i_{l_1}}, \otimes [e_{i_1} \otimes \cdots \otimes e_{i_{l_2}}] \otimes \cdots \otimes e_{i_n}.$$

(7.25)

where the braces "][]" mean, as usual, symmetrization.

Moreover, let $\sigma \in S^{l_1-l_2}$ be a permutation of the indices $i_1, \ldots, i_2$ and $P^\sigma$ the associated operator on $W$:

$$P^\sigma e_{i_1} \otimes \cdots \otimes e_{i_{l_1-1}} \otimes e_{i_{l_1}} \otimes \cdots \otimes e_{i_{l_2}} \otimes \cdots \otimes e_{i_n} =$$

$$e_{i_1} \otimes \cdots \otimes e_{i_{l_1-1}} \otimes e_{i_{l_1}} \otimes \cdots \otimes e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(l_1-1)} \otimes \cdots \otimes e_{i_n}.$$  

(7.26)

then we define the action of $\sigma \in S^{l_1-l_2}$ on the functions $g^{l_1,l_2}_{i_1,...,i_n}(\lambda)$ by:

$$S^{l_1,l_2}(\lambda)P^\sigma e_{i_1} \otimes \cdots \otimes e_{i_{l_1-1}} \otimes e_{i_{l_1}} \otimes \cdots \otimes e_{i_{l_2}} \otimes \cdots \otimes e_{i_n} =$$

$$\sigma(g^{l_1,l_2}_{i_1,...,i_n}(\lambda))e_{i_1} \otimes \cdots \otimes e_{i_{l_1-1}} \otimes [e_{i_1} \otimes \cdots \otimes e_{i_{l_2}}] \otimes \cdots \otimes e_{i_n}.$$  

(7.27)

Definition 7.13. 1. Let $\tilde{S}_Y(z)$ and $\tilde{S}_Y(z,\epsilon)$ be the $E_{\gamma,\tau}(gl_N)$-modules

$$S^{n_1}V(z) \otimes S^{n_2}V(z - \gamma) \subset CN \otimes CN \otimes \cdots \otimes CN$$

and

$$S^{n_1}V(z - \gamma) \otimes S^{n_2}V(z - \gamma - \epsilon) \subset CN \otimes CN \otimes \cdots \otimes CN.$$  

Moreover let $w$ be an arbitrary element of $\text{Fun}_N(\tilde{S}_Y(z))$, resp. $\text{Fun}_N(\tilde{S}_Y(z,\epsilon))$ then a vector $\tilde{w} \in \text{Fun}_N(CN \otimes CN \otimes \cdots \otimes CN)$ is a lift of $w$ to $CN \otimes CN \otimes \cdots \otimes CN$ if

$$w = S^{n_1,n_1}(\lambda)S^{n_1+1,n_1+1}(\lambda - \gamma \sum_{i=1}^{n_1} h^{(i)})\tilde{w}.$$  

(7.28)

2. Let $w$ be a canonical basis element of $S^{n_1}V(z) \otimes S^{n_2}V(z - \gamma)$, resp. $S^{n_1}V(z) \otimes S^{n_2}V(z - \gamma - \epsilon)$, then the canonical lift of $w = [e_{i_1}, \ldots, e_{i_{n_1}}] \otimes [e_{j_1}, \ldots, e_{j_{n_2}}]$ to $CN \otimes CN \otimes \cdots \otimes CN$ we mean the vector of $\text{Fun}_N(CN \otimes CN \otimes \cdots \otimes CN)$:

$$\tilde{w} = g^{l_1,l_2}_{i_1,...,i_n}(\lambda)^{-1}g^{l_1+1,l_2}_{j_1,...,j_{n_2}}(\lambda - \gamma \sum_{m=1}^{n_1} h^{(m)})^{-1}e_{i_1} \otimes \cdots \otimes e_{i_{n_1}}$$

$$\otimes e_{j_1} \otimes \cdots \otimes e_{j_{n_2}}.$$  

(7.29)

where the indices $i_l, 1 \leq l \leq n_1$ and $j_k, 1 \leq k \leq n_2$ are in increasing order from left to right.
Remark 9. The functions "g" of the previous definition can be computed explicitly by using Lemma 3.13.

The last objects we need, are some short hand notations for special products of $R$-matrices, which will appear often in what follows.

Remark 10. To simplify the notation we will omit the shifts in the "$\lambda$", because they do not play any role and they can be reconstructed from the context. Moreover the spectral parameter will be only specified when it takes a special value ($z = \pm \gamma$).

Definition 7.14. Let $W$ be the $E_{\gamma, \tau}(gl_N)$-module $V(z_1) \otimes \ldots \otimes V(z_n)$ and let $l_1, l_2$ be two positive integers, then:

\[
\begin{align*}
\tilde{R}_{l_1, l_2}^{l_1} &= \begin{cases} 
R_{l_1, l_2} \ldots R_{l_1+1, l_1} & \text{if } l_2 > l_1 \\
R_{l_1, l_2} \ldots R_{l_1-1, l_1} & \text{if } l_2 < l_1 \\
1 & \text{if } l_1 = l_2.
\end{cases} \\
\tilde{R}_{l_1, l_2}^{l_2} &= \begin{cases} 
R_{l_1, l_2} \ldots R_{l_1+1, l_1} & \text{if } l_2 > l_1 \\
R_{l_1, l_2} \ldots R_{l_1-1, l_1} & \text{if } l_2 < l_1 \\
1 & \text{if } l_1 = l_2.
\end{cases}
\end{align*}
\]

Lemma 7.15. Let $\hat{S}_T(z, \epsilon)$ be the $E_{\gamma, \tau}(gl_N)$-module associated to the Young Tableau with two rows of length $n_1$ and $n_2$ and let $l_1, l_2$ be two integers such that:

\[
l_1 < l_2 < n_1 + n_2, \quad l_2 = l_1 + n_1,
\]

then there is an operator

\[
\tilde{W}(\lambda, \epsilon) : \mathbb{C}^{n_1 \otimes n_2} \otimes \mathbb{C}^{n_1 \otimes n_2} \rightarrow \hat{S}_T(z, \epsilon),
\]

such that

\[
W_{n_1, n_2}(\lambda, \epsilon)w = W_{n_1, n_2}(\lambda, \epsilon)S_{n_1, n_1 + 1, n_1 + n_2}(\lambda) \tilde{w}
\]

\[
= \tilde{W}(\lambda, \epsilon)R_{l_1, l_2}(\gamma + \epsilon, \lambda)S_{l_1, n_1}(\lambda)S_{n_1 + 1, l_2}(\lambda) \tilde{w},
\]

where $\tilde{w}$ is a (canonical) lift of $w$. Moreover, in the limit $\epsilon \to 0$ mod $\mathbb{Z} + \mathbb{Z}T$:

\[
\tilde{W}_{n_1, n_2}(\lambda)w = \lim_{\epsilon \to 0} \Theta(\epsilon)W_{n_1, n_2}(\lambda, \epsilon)
\]

\[
= \tilde{W}(\lambda, 0)A_{l_1, l_2}(\lambda)S_{l_1, n_1}(\lambda)S_{n_1 + 1, l_2}(\lambda) \tilde{w}(\lambda),
\]

where

\[
A_{l_1, l_2}(\lambda) = \lim_{\gamma \to 0} \Theta(\epsilon)R_{l_1, l_2}(\gamma + \epsilon, \lambda).
\]

Remark 11. The name $A_{l_1, l_2}(\lambda)$ has been chosen so to remind the reader that in Chapter 4 it has been shown that $\ker A_{l_1, l_2}(\lambda) = S^2 \mathbb{C}^N$.

Proof. The first statement is a simple consequence of the Yang-Baxter equation. The equations (7.34) and (7.35) are, in fact, two representations of the operator that permutes the factors of $\mathbb{C}^{n_1 \otimes (n_2 + n_2)}$ associated to the two different diagrams.
of figure 7.4. It remains to show that (7.35) has the claimed limit. This isn’t a priori clear because the Yang-Baxter equation does not hold, in general, for the single coefficients of the Laurent expansion, in \(\epsilon\), of \(W_{n_1,n_2}(\lambda,\epsilon)\).

In order to do this let us first rewrite (7.35) in a more explicit way. One has:

\[
W_{n_1,n_2}(\lambda,\epsilon)S^{1,n_1}(\lambda)S^{n_1+1,n_1+n_2}(\lambda)\tilde{w} = \\
D^{n_1+n_2}(\lambda,\epsilon)\ldots D^{1+1}(\lambda,\epsilon)\times \\
C^{1}(\lambda,\epsilon)\ldots C^{l-1}(\lambda,\epsilon)\times \\
\frac{R^{n_1}_{1,1,n_1+1}R^{n_2}_{1,1,n_2+1}R^{1,1}_{1,1}(\gamma + \epsilon,\lambda)S^{1,n_2}S^{2,n_2}\tilde{w}}{W^{1,\ldots,n_1,n_1+1,\ldots,n_2}(\lambda)},
\]

where \(C^k(\lambda,\epsilon) = R^k_{1,1}(\lambda,\epsilon), \quad k = 1,\ldots,l-1,\) and \(D^j(\lambda,\epsilon) = R^j_{1,1}(\lambda,\epsilon).\) Now, we notice that the only operator in the third line of (7.38) which is singular in the limit \(\epsilon \to 0\) is \(R^j_{1,1}(\gamma + \epsilon,\lambda),\) and that each of the operators \(C^k\) and \(D^j\) contains exactly one \(R\)-matrix which is singular in the same limit.

In order to show that the singularities of the operators \(C^k\) and \(D^j\) are only apparent we use that:

\[
\lim_{\epsilon \to 0} C^{l-1}(\epsilon,\lambda)v = \frac{R^{l-1}(0,\lambda)}{P^{l-1}(0,\lambda)},
\]

for all \(\epsilon \neq 0.\) Then, it follows that

\[
\lim_{\epsilon \to 0} C^{l-1}(\epsilon,\lambda)v,
\]

is regular on vectors in \(\text{im}W^{l_1,\ldots,n_2;1,2}(\lambda,\epsilon)\). In fact, by definition:

\[
C^{l-1}(\epsilon,\lambda) = R^{l-1,l_1}\ldots R^{l-1,1,2}.
\]

Now, using that

\[
R^{l-1,0}(0,\lambda) = P^{l-1},
\]

and the definition of \(A^{l-1,1,2} = A^{l-1,1,2}^{-1}\)

\[
\lim_{\epsilon \to 0} A^{l-1,1,2}(\epsilon,\lambda) = A^{l-1,1,2}^{-1},
\]

one can see that the Laurent expansion of \(C^{l-1}\) has the form

\[
C^{l-1}(\epsilon,\lambda)v = \frac{1}{\epsilon} R^{l-1,l_1}\ldots R^{l-1,1,2}A^{l-1,1,2}P^{l-1,1,2}v \\
+ C^{l-1}(0,\lambda)v + O(\epsilon)v.
\]
But, the first term of the right hand side of (7.44) vanishes identically on vectors \( w \) in the image of \( W^{l_1,\ldots,l_n,l_1+1,\ldots,l_2}(\lambda) \).

This means that the order of the singularity in \( \epsilon = 0 \) of (7.38) is not increased by the singularities of \( C^{l_1-1} \).

Now, noticing that

\[
\text{im } C^{l_1-1}W^{l_1,\ldots,l_n,l_1+1,\ldots,l_2}(\lambda) \\
\subset \mathbb{C}^N \otimes (l_1-2) \otimes S^{(l_1-1+2)}(\mathbb{C}^N) \otimes \mathbb{C}^N \otimes \mathbb{C}^{N \otimes (l_1+n_2-1) + 2},
\]

for all values of \( \epsilon \neq 0 \) one can see, by induction, that the claimed result holds. (The operators \( D_\alpha \) are treated in a symmetric way.)

**Definition 7.16.**

1. Let \( \tilde{S}_Y(z) \) be that \( \mathbb{E}^{lT}(g_{N}) \)-module \( S^{n_1}V(z) \otimes S^{n_2}V(z - \gamma) \)
associated to the Young Tableau with two rows of length \( n_1 \geq n_2 \). Then by \( Q^{l_1,l_2}_{n_1,n_2}(\lambda) \) we denote the operator:

\[
Q^{l_1,l_2}_{n_1,n_2}(\lambda) = A^{l_1,l_2}(\lambda)S^{n_1}(\lambda)S^{n_2+1,l_2}(\lambda),
\]

where \( l_1 \) and \( l_2 \) are two integers such that \( 1 \leq l_1 \leq n_1, \ l_2 = l_1 + n_1 \).

2. Let \( \tilde{S}_Y(z) \) be the same \( \mathbb{E}^{lT}(g_{N}) \)-module, then we define \( J^{l_1,l_2}(\tilde{S}_Y(z)) \) to be the vector space over \( \text{Fun}_N(\mathbb{C}) \):

\[
J^{l_1,l_2}(\tilde{S}_Y(z)) = \ker Q^{l_1,l_2}_{n_1,n_2}(\lambda) \cap \text{Fun}_N(\tilde{S}_Y(z)).
\]

3. Let \( \tilde{S}_Y(z) \) be that \( \mathbb{E}^{lT}(g_{N}) \)-module \( S^{n_1}V(z) \otimes S^{n_2}V(z - \gamma) \)
associated to the Young Tableau with two rows of length \( n_1 \geq n_2 \), then we define \( J(\tilde{S}_Y(z)) \) to be the vector space:

\[
J(\tilde{S}_Y(z)) = \sum_{l=1}^{n_2} J^{l_1+n_1}(\tilde{S}_Y(z)).
\]

**Remark 12.** We will eventually prove that

\[
J(\tilde{S}_Y(z)) = \sum_{l=1}^{n_2} J^{l_1+n_1}(\tilde{S}_Y(z)),
\]

is equal to the kernel of \( W^{n_2}_{n_1,n_2}(\lambda) \).

**Lemma 7.17.** Let \( w \) be a vector in \( J^{l_1,l_2}(\tilde{S}_Y(z)) \), then \( w \) is a linear combination
(over \( \text{Fun}_N(\mathbb{C}) \)) of vectors of the form (7.54) and (7.55).

**Proof.** Let \( \tilde{w} \in \text{Fun}_N(\mathbb{C}^{N \otimes n_1} \otimes \mathbb{C}^{N \otimes n_2}) \) be the canonical lift of a standard basis vector of \( \text{Fun}_N(\tilde{S}_Y(z)) \):

\[
w = [e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{n_1}}] \otimes [e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{n_2}}],
\]

then by definition

\[
Q^{l_1,l_2}_{n_1,n_2}(\lambda)w = Q^{l_1,l_2}_{n_1,n_2}(\lambda)S^{n_1}(\lambda)S^{n_2+1,n_1+n_2}(\lambda)\tilde{w} =
Q^{l_1,l_2}_{n_1,n_2}(\lambda)g_{i_1,i_2,...,i_{n_1}}(\lambda)^{-1}g_{j_1,j_2,...,j_{n_2}}(\lambda) - \gamma \sum_{m=1}^{n_1} h_m^{(m)} x_e
\]

\[
e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_{n_1}} \otimes e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_{n_2}}.
\]
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This implies that

\[
g_{t_1,n_1,\ldots,t_n,n_1}^{l_1,l_2} (\lambda) g_{j_1,j_2,\ldots,n_2}^{l_1,l_2} (\lambda - \gamma \sum_{m=1}^{n_1} h_{(m)}) Q_{n_1,n_2}^{l_1,l_2} (\lambda) w
\]

\[
= Q_{n_1,n_2}^{l_1,l_2} (\lambda) e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{m_1}} \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_{m_2}}
\]

\[
= g_{t_1,n_1,\ldots,t_n,n_1}^{l_1,l_2} (\lambda) g_{j_1,j_2,\ldots,n_2}^{l_1,l_2} (\lambda - \gamma \sum_{m=1}^{n_1} h_{(m)}) A_{l_1,l_2}^{t_1,t_2} (\lambda) w^{l_1,l_2},
\]

(7.52)

where the vector \( w^{l_1,l_2} \) is given by

\[
w^{l_1,l_2} = e_{i_1} \otimes \cdots \otimes [e_{i_1} \otimes \cdots \otimes e_{i_{m_1}} \otimes [e_{i_{m_1}+1} \otimes \cdots \otimes e_{i_{m_2}}] \otimes \cdots \otimes e_{i_{m_2}+n_2}.
\]

(7.53)

From equation (7.52) it's then easy to see that the vectors in \( J^{l_1,l_2}(S_Y(z)) \) have the form:

\[
\tilde{w} = \sum_{\sigma \in S^{(l_1,l_2)}} \sigma (G_{t_1,n_1,\ldots,t_n,n_1}^{n_1,n_2} (\lambda)) P_{l_1,l_2} w^{l_1,l_2},
\]

(7.54)

where \( P_{l_1,l_2} \) is a permutation which acts in the \( l_1, \ldots, l_2 \) positions and the functions \( G \) are given by:

\[
G_{t_1,n_1,\ldots,t_n,n_1}^{n_1,n_2} (\lambda) = \frac{g_{t_1,n_1,\ldots,t_n,n_1}^{l_1,l_2} (\lambda) g_{j_1,j_2,\ldots,n_2}^{l_1,l_2} (\lambda - \gamma \sum_{m=1}^{n_1} h_{(m)})}{g_{l_1,l_2}^{n_1,n_2} (\lambda) g_{j_1,j_2,\ldots,n_2}^{l_1,l_2} (\lambda - \gamma \sum_{m=1}^{n_1} h_{(m)})}.
\]

(7.55)

Definition 7.18. Let \( z \in \mathbb{C} \) be a generic complex number and \( R(z, \lambda) \) the \( R \)-matrix, then we define the limit \( \gamma \to 0 \) of \( R(z, \lambda) \) as:

\[
\lim_{\gamma \to 0} R(\gamma x, \lambda) = \frac{x}{x-1} - \frac{1}{x-1} P, \text{ for } z \neq \gamma,
\]

(7.56)

\[
A = 1 - P, \text{ for } z = \gamma.
\]

(7.57)

We call this limit the rational limit.

Lemma 7.19. Let \( S_Y(z, \epsilon) \) be the \( E_{\gamma,\epsilon}(\mathfrak{g}_L) \)-module associated to the Young Tableau with two rows of length \( n_1 \geq n_2 \). Then:

\[
\lim_{\gamma \to 0} \lim_{\eta \to 0} \frac{\Theta(\gamma \eta)}{\gamma} W_{n_1,n_2}(\lambda) = B \prod_{i=1}^{n_2} A^{t_1,n_1+i}, \quad z = \gamma k, \epsilon = \gamma \eta,
\]

(7.58)

for some invertible operator \( B \).

Proof. We will prove here only that

\[
\lim_{\eta \to 0} \lim_{\gamma \to 0} \frac{\Theta(\gamma \eta)}{\gamma} W_{n_1,n_2}(\lambda) = B \prod_{i=1}^{n_2} A^{t_1,n_1+i},
\]

(7.59)
for some matrix $B$. The proof that $B$ is invertible can be found in [7]. It has been shown in Lemma 7.7, that $W_{n_1,n_2}(\epsilon, \lambda)$ has a pole of order one in $\epsilon = 0 \mod \mathbb{Z} + \mathbb{Z}\tau$, it acts on vectors of $S^{n_1}C^N \otimes S^{n_2}C^N$. This means that $W_{n_1,n_2}(\gamma\eta, \lambda)$ has a pole of the same order in $\gamma\eta = 0 \mod \mathbb{Z} + \mathbb{Z}\tau$ and that $\Theta(\gamma\eta)W_{n_1,n_2}(\gamma\eta, \lambda)$ is analytic in $\gamma, \eta$, for generic values of $\lambda$. So, the equality of the two limit follows from the continuity of $\Theta(\gamma\eta)W_{n_1,n_2}(\gamma\eta, \lambda)$.

In order to prove the second claim, we first take the limit $\epsilon \to 0$. Then it is easy to see that:

$$\lim_{\gamma \to 0} \frac{\Theta(\gamma\eta)}{\gamma} W_{n_1,n_2}(\lambda) = \Theta'(0)\eta \lim_{\gamma \to 0} \widetilde{W}_{n_1,n_2},$$

where in $\widetilde{W}_{n_1,n_2}$ all $R$-matrices are replaced by the corresponding $R$-matrices by the rule:

$$R(\gamma\eta, \lambda) \to R(\eta) = \frac{\eta}{\eta - 1} - \frac{1}{\eta - 1}P.$$  \hspace{1cm} (7.61)

Let us now compute the limit $\eta \to 0$. From the definition of $W_{n_1,n_2}(\lambda)$, and its graphical representation, one can see that $\widetilde{W}_{n_1,n_2}$ can be rewritten as:

$$\widetilde{W}_{n_1,n_2} = B \times Z^{n_2} Z^{n_2-1} \cdots Z^1,$$

where

$$Z^k = R^{1,n-k+1} \cdots R^{k-1,n-1} R^{k,n_1+n_2},$$

and $B$ contains only $R$-matrixes which are regular in the limit $\eta \to 0$. Now noticing that all the $R$-matrices in $Z^k$, for some fixed $k$, have the same spectral parameter, that $Z^{n_2}$ contains $n_2$ singular matrices and that all the other terms are regular in the point $\eta = 0$, we get:

$$\lim_{\eta \to 0} \eta \widetilde{W}_{n_1,n_2} =$$

$$= B \times \prod_{i=1}^{n_2} A^{i,n_1+i} \times \eta \frac{d^{n_2-1}}{d\eta^{n_2-1}} (Z^{n_2-1} \cdots Z^1),$$

where $A^{i,j} = \frac{1}{2}(I - P^{i,j})$. \hspace{1cm} \Box

**Corollary 7.20.** Let $S_Y(z)$ the associated to the Young Tableau with two rows of length $n_1 \geq n_2$ and $\widetilde{W}_{n_1,n_2}$ the operator of the previous Lemma, then the image of $\widetilde{W}_{n_1,n_2}$ is:

$$\text{im } W = \text{im } S^{1,n_1} S^{n_1+1,n_1+n_2} \prod_{i=1}^{n_1} A^{i,n_1+i},$$

or in other words, the image of $\widetilde{W}_{n_1,n_2}$ is identical to the space of classical symmetric tensors associated to a Young Tableau $Y$. 
Proof. This Lemma is simple consequence of the Yang Baxter Equation. In fact
using that \( W_{n_1,n_2} \) acts on vectors in \( S^{n_1}\mathbb{C}^N \otimes S^{n_2}\mathbb{C}^N \) it follows that:

\[
W_{n_1,n_2} w = B \prod_{i=1}^{n_1} A^i,^{n_1+i} w = B \prod_{i=1}^{n_1} A^i,^{n_1+i} S^{1,n_1} S^{n_1+1,n_1+n_2} w \quad (7.66)
\]

But (7.66) is equal to

\[
S^{1,n_1} S^{n_1+1,n_1+n_2} \prod_{i=1}^{n_1} A^i,^{n_1+i} B \tilde{w},
\]

because of the Yang Baxter Equation. \( \square \)

The basis vectors of \( J(\tilde{S}_Y(z)) \) are complicated linear combinations, over
\( \text{Fun}_N(\mathbb{C}) \), of tensors in \( S^{n_1}\mathbb{C}^N \otimes S^{n_2}\mathbb{C}^N \). For later use, we will compute the limit
\( \gamma \to 0 \) of those vectors.

**Definition 7.21.** Let \( J(\tilde{S}_Y(z)) \) be the vector space of Definition 7.16, then we define
\( \tilde{J}(\tilde{S}_Y(z)) \) to be the vector space over \( \text{Fun}_N(\mathbb{C}) \) generated by elements of \( J(\tilde{S}_Y(z)) \)
in the limit \( \gamma \to 0 \).

**Lemma 7.22.** Let \( \tilde{J}(\tilde{S}_Y(z)) \) be the vector space of Definition 7.21, then

\[
\ker \tilde{W}_{n_1,n_2} = \tilde{J}(\tilde{S}_Y(z)), \quad (7.68)
\]

and

\[
\dim_{\text{Fun}_N(\mathbb{C})} \tilde{J}(\tilde{S}_Y(z)) \leq \dim_{\text{Fun}_N(\mathbb{C})} J(\tilde{S}_Y(z)). \quad (7.69)
\]

Proof. The first statement follows from the explicit form of the \( g \)-functions given
in Section 3.3.

The second one, from the fact that each basis vector of \( \tilde{J}(\tilde{S}_Y(z)) \) is the limit of at
least one basis vector of \( J(\tilde{S}_Y(z)) \). \( \square \)

We have now all the ingredients to prove the main Theorem on the kernel of
\( W^\text{reg}_{n_1,n_2} \).

**Theorem 7.23.** Let \( \tilde{S}_Y(z) \) be the \( E_{n_1,n_2}(gl_N) \)-module associated to the Young
Tableau \( Y \) with two rows of length \( n_1 \geq n_2 \). Moreover let \( \gamma \) be a generic number,
then the kernel of \( W^\text{reg}_{n_1,n_2} \) is equal to:

\[
J_Y(\tilde{S}_Y(z)) = \sum_{j=1}^{n_2} J_Y^n+1n+1j(\tilde{S}_Y(z)). \quad (7.70)
\]

Proof. From the previous discussion it should be clear that \( J_Y(\tilde{S}_Y(z)) \) is contained
in the kernel of \( W^\text{reg}_{n_1,n_2} \). It remains, therefore, to show the converse.

In order to do this we consider the limit \( \gamma \to 0 \) described previously. From Lemma
7.22 we have that:

\[
\dim_{\text{Fun}_N(\mathbb{C})} \tilde{J}(\tilde{S}_Y(z)) \leq \dim_{\text{Fun}_N(\mathbb{C})} J(\tilde{S}_Y(z)). \quad (7.71)
\]

On the other side:

\[
\dim_{\text{Fun}_N(\mathbb{C})} J(\tilde{S}_Y(z)) \leq \dim_{\text{Fun}_N(\mathbb{C})} \ker W^\text{reg}_{n_1,n_2} \leq \dim_{\text{Fun}_N(\mathbb{C})} \ker \tilde{W}_{n_1,n_2} = \dim_{\text{Fun}_N(\mathbb{C})} \tilde{J}(\tilde{S}_Y(z)). \quad (7.72)
\]
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where the second inequality holds because the dimension of the kernel of a family of operators evaluated in a special point (\( \gamma = 0 \)) can only be greater than the dimension of the kernel in a generic point (\( \gamma \) generic). But if \( J(\tilde{Y}(z)) \) has the same dimension of \( \ker W_{n_1,n_2}^{\text{reg}} \), they coincide with it.

The kernel of \( W_{n_1,\ldots,n_k}^{\text{reg}}(\lambda) \)

In what follows, we will generalize the results about the kernel of \( W_{n_1,\ldots,n_k}^{\text{reg}}(\lambda) \) to arbitrary tensor products of symmetric spaces associated to Standard Young Tableau.

**Lemma 7.24.** Let \( \tilde{Y}(z,\epsilon) \) be the \( E_{\gamma,T}(gl_N) \)-module associated to Young Tableau with \( k \) rows of length \( n_1 \geq n_2 \cdots \geq n_k \) and \( R_{S^{n_1},S^{n_k}} \) the R-matrix associated to the crossing of \( S^{n_i}V(z_I) \) and \( S^{n_k}V(z_k) \), then

\[
W_{S^{n_1},S^{n_k}}^{\text{reg}}(\lambda) := \lim_{\epsilon \to 0} \Theta(\epsilon)^{O(Y)} R_{S^{n_1},S^{n_k}}, \tag{7.73}
\]

is well defined and not identical to zero.

**Proof.** The proof of this Lemma is analogous to the proof of Lemma 7.6. \( \square \)

**Corollary 7.25.** Let \( \tilde{Y}(z,\epsilon) \) be the \( E_{\gamma,T}(gl_N) \)-module associated to Young Tableau with \( k \) rows of length \( n_1 \geq n_2 \cdots \geq n_k \), then in the limit \( \epsilon \to 0 \) the regularized R-matrix associated to the crossing of two spaces \( S^{n_i}V(z_i), S^{n_k}V(z_k) \), satisfies the (generalized) Yang Baxter Equation.

**Proof.** This Lemma follows from the fact that \( \Theta(Y) = \sum \Theta(Y^{ij}) \), where the sum runs over all \( i \leq j \), as showed in Lemma 7.4. \( \square \)

**Definition 7.26.** Let \( \tilde{Y}(z) \) be the vector space associated to a Young Tableau with \( k \) rows of length \( n_1 \geq n_2 \cdots \geq n_k \), then we define \( J_{i,i+1}(\tilde{Y}(z)) \) to be the vector space over \( \text{Fun}_N(\mathbb{C}) \):

\[
J_{i,i+1}(\tilde{Y}(z)) = \text{Fun}_N(S^{n_i}V(z) \otimes \ldots \otimes S^{n_k}V(z - \gamma(i - 2))) \otimes \ker W_{n_1,\ldots,n_k}^{\text{reg}} \otimes \ldots S^{n_k}V(z - \gamma(k - 1))), \tag{7.74}
\]

and \( J_Y(\tilde{Y}(z)) \):

\[
J_Y(\tilde{Y}(z)) = \sum_{i=1}^{k_1} J_{i,i+1}(\tilde{Y}(z)) \tag{7.75}
\]

**Theorem 7.27.** Let \( \tilde{Y}(z) \) be the vector space associated to a Young Tableau with \( k \) rows of length \( n_1 \geq n_2 \cdots \geq n_k \), and let \( \gamma \) be a generic number, then

\[
\ker W_{n_1,\ldots,n_k}^{\text{reg}}(\lambda) = J_Y(\tilde{Y}(z)) \tag{7.76}
\]

**Proof.** Let us show that the space \( J(\tilde{Y}(z)) \) is contained in the kernel of the operator \( \ker W_{n_1,\ldots,n_k}^{\text{reg}}(\lambda) \). If a vector is of \( J_{i,i+1}(\tilde{Y}(z)) \) for some \( i < k \), then \( W_{n_1,\ldots,n_k}^{\text{reg}}(\lambda) \) vanishes on it because there is a representation of \( W_{n_1,\ldots,n_k}^{\text{reg}}(\lambda) \) which has the crossing between spaces \( i \) and \( i + 1 \) on the bottom of the diagram.

The converse, from the rational limit \( \gamma \to 0 \). \( \square \)

**Corollary 7.28.** Let \( W_Y(\lambda) \) be the \( E_{\gamma,T}(gl_N) \)-module associated to a standard Young Tableau, then the dimension over \( \text{Fun}_N(\mathbb{C}) \) of \( W_Y(z) \) is equal to the (classical) dimension of a \( GL_N \)-module associated to the same Young Tableau.
7.3 Elliptic Weight of $W_Y(z)$

**Theorem 7.29.** Let $W_Y(z)$ be the $E_{\gamma,\tau}(gl_N)$-module associated to the Young Tableau $Y$ with $k \leq N$ rows of length $n_1, n_2, \ldots, n_k$, then the space of highest weight is 1-dimensional. Moreover, there are functions $g_i(\lambda) \in \text{Fun}_N(\mathbb{C})$, $i = 1, \ldots, N$ such that the elliptic weight of the vector of highest weight $\omega$ of $W_Y(z)$, where $\omega \in \text{Fun}_N(W_Y(z))$ is image in $W_Y(z)$ of $e_k^{\otimes n_k} \otimes \ldots \otimes e_1^{\otimes n_1}$, is equal to:

$$
\Lambda_i(\omega, \lambda) = \prod_{l=2}^{k} \frac{\Theta(w - z + \gamma(l - 1))}{\Theta(w - z + \gamma(l - 1) - \gamma n_l)} g_i(\lambda),
$$

Moreover, there is a 1-dimensional $E_{\gamma,\tau}(gl_N)$-module $W^*$ such that the elliptic weight of $W_{Y,\lambda'}(z) = W^* \otimes W_Y(z)$ takes the form:

$$
\Lambda(\omega, \lambda) = \prod_{l=1}^{k} \frac{\Theta(w - z + \gamma(l - 1))}{\Theta(w - z + \gamma(l - 1) - \gamma n_l)} g_i(\lambda)
$$

\hspace{1cm} (7.77)

**Proof.** Let us first show that the space of highest weight is 1-dimensional. In the last Section we showed that the dimension of the kernel of the operator $W_{n_1, \ldots, n_k}(\lambda)P^{1 \ldots n}$ is equal to the dimension of the kernel of the operator $W_{n_1, \ldots, n_k}(\lambda)P^{1 \ldots n}$ obtained by taking the limit $\tau \rightarrow 0$ of $W_{n_1, \ldots, n_k}(\lambda)P^{1 \ldots n}$. In this limit $W_{n_1, \ldots, n_k}(\lambda)P^{1 \ldots n}$ is equal, up to an invertible matrix, to the classical Young projectors associated to the Young Tableau $Y$. So, the claim follows from the analogous classical result. Let us now compute the elliptic weight of $\omega$. We consider two different cases.

Let $i$ be greater that $k$, then using the co-product formula one gets, after a straightforward computation, that

$$
\Lambda_i(\omega, \lambda) = \prod_{l=2}^{k} \frac{\Theta(w - z + \gamma(l - 1))}{\Theta(w - z + \gamma(l - 1) - \gamma n_l)} g_i(\lambda),
$$

\hspace{1cm} (7.80)

where the right hand side of (7.80) is computed in $\omega \in S^{n_k}V(z_k) \otimes \ldots \otimes S^{n_l}V(z_l)$, $z_i = z - \gamma(i - 1)$. Now, using the explicit expressions for the matrix elements derived in Section 3.3 Theorem 3.10, equation (7.80) takes the form

$$
\Lambda_i(\omega, \lambda) = \prod_{l=1}^{k} \frac{\Theta(w - z + \gamma(l - 1))}{\Theta(w - z + \gamma(l - 1) - \gamma n_l)} g_i(\lambda)|\omega|,
$$

\hspace{1cm} (7.81)
for some function \( g_1(\lambda) \in \text{Fun}_N(\mathbb{C}) \).

The case \( 1 \leq k \) is a little bit more involved. In order to perform the calculations in this case too, we notice that the image, in \( W_\gamma(z) \), of

\[
e_{nk} \otimes \cdots \otimes e_{n1} \quad \text{and} \quad e_{nk} \otimes \cdots \otimes e_{n1+1} \otimes e_{k} \otimes e_{k-1} \otimes \cdots \otimes (e_{1}^{(n_1-n_k)} \otimes e_{1}^{(n_1-n_k+1)}),
\]

are equal (up to a factor in \( \text{Fun}_N(\mathbb{C}) \)). So, applying the co-product formula to (7.83) and with the explicit form of the action of the matrix elements on \( S_{nk} V(z_k) \otimes \cdots \otimes S_{n1} V(z_1), \ z_i = z - \gamma(i - 1) \), we get after a short computation:

\[
L_{ii}(w, \lambda)[\omega] = \prod_{i=2}^{k} \frac{\Theta(w - z + \gamma(l - 1))}{\Theta(w - z + \gamma(l - 1) - \gamma n_1)} \times \frac{\Theta(w - z - \gamma n_i)}{\Theta(w - z - \gamma n_1)} g_i(\lambda)[\omega],
\]

which is the claimed result.

Let us now prove the last statement.

From Theorem 2.9 we know that there is a 1-dimensional \( \mathfrak{g} \ell_N \)-module \( W^* \) such that \( W^* \otimes W_\gamma(z) \) is in canonical form. Dividing the \( z \)-dependent part of \( \Lambda_{ii}(z, \lambda) \) by the \( z \)-dependent part of \( \Lambda_{11}(z, \lambda) \) we get the desired result.

**Definition 7.30.** The \( \mathfrak{g} \ell_N \)-module \( W_{\gamma_N}(z) \) of Theorem 7.29 is called the canonical \( \mathfrak{g} \ell_N \)-module associated to the Young Tableau \( Y \) with Drinfeld Polynomials

\[
P_k(w) = \prod_{i=0}^{\Delta_k-1} \Theta(w - z - \gamma(n_k + l)), \quad k = 2, \ldots, N,
\]

where \( \Delta_k = n_{k-1} - n_k \).

**Theorem 7.31.** The \( \mathfrak{g} \ell_N \)-modules \( W_\gamma(z) \) are irreducible highest weight modules.

**Proof.** This Theorem can be proved by taking the rational limit \( \gamma \to 0 \) and using an analogous result on Yangians (see [9],[10]).

**7.4 Representation Associated To GYT**

The \( \mathfrak{g} \ell_N \)-modules \( W_\gamma(z) \) studied in the last section, e.g. the \( \mathfrak{g} \ell_N \)-modules which are associated to (standard) Young Tableaux, are not the unique way to fuse symmetric \( \mathfrak{g} \ell_N \)-modules. In this section we will introduce an other class of \( \mathfrak{g} \ell_N \)-modules, which are naturally associated to generalized Young Tableaux.

The proofs of the results of this section are almost equal to the proof of the results of Section 7.2 and will therefore be omitted.

**Definition 7.32.** 1. A generalized Young Tableau \( Y_g \) is a pair \( (Y_1, Y_2) \), where \( Y_1 \) and \( Y_2 \) are two Young Tableau such that \( Y_1 \subset Y_2 \).

In other words \( Y_2 \) has \( k \) rows of length \( n_1 \geq n_2 \geq \cdots \geq n_k \) and \( Y_1 \) has \( i \) rows of length \( l_1 \geq l_2 \geq \cdots \geq l_i \), with the additional condition that \( l_m < n_m \).

2. Let \( Y_g = (Y_1, Y_2) \) be a generalized Young Tableau, then to each box of \( Y_2 \setminus Y_1 \) we associate a positive integer and a pair of natural coordinates \( (i, j) \in \mathbb{N}^* \times \mathbb{N}^* \) to each box in \( Y_2 \setminus Y_1 \) starting from the upper left corner. (see figure 7.5)

3. Let \( Y_g = (Y_1, Y_2) \) be a generalized Young Tableau, then \( Y_g^{ij} \) is the sub-Tableau if \( Y_g \) which contains the boxes of \( Y_2 \setminus Y_1 \) which are in row \( i \) and \( j \).
Definition 7.33. 1. The order of a generalized Young Tableau is a positive number which counts the number of pairs of boxes in \( Y_g = (Y_1, Y_2) \) which are on a diagonal (from left to right) minus the number of pairs of boxes which satisfy the condition \((i_1 - i_2) - (j_1 - j_2) = -1\). As we will see later the order of a Young Tableau is related to the order of a pole of an operator associated to \( Y \).

\[
\mathcal{O}(Y_g) = \# \{((i_1, j_1), (i_2, j_2)) \mid (i_1, j_1) \in Y_2 \setminus Y_1, i = 1, 2, \text{ and } i_2 = i_1 + d, j_2 = j_1 + d, d \in \mathbb{N}^* \} - \\
\# \{((i_1, j_1), (i_2, j_2)) \mid j_1 < j_2, (i_1, j_1) \in Y_2 \setminus Y_1, i = 1, 2, \text{ and } (i_1 - i_2) - (j_1 - j_2) = -1 \}
\]

(7.86)

2. Analogously the order if a sub-Tableau \( Y_{gI} \) in the the number of pairs of boxes in \( Y_{gI} \) which are on a diagonal minus the number of pairs of boxes which satisfy the condition \((i_1 - i_2) - (j_1 - j_2) = -1\).

Lemma 7.34. Let \( Y_g = (Y_1, Y_2) \) be a generalized Young Tableau, then

\[
\mathcal{O}(Y_g) = \sum_{i \leq j} \mathcal{O}(Y_{gij}).
\]

(7.87)

Proof. From the definition of the order of a Young Tableau it is clear that each pair of boxes that contribute to \( \mathcal{O}(Y_g) \) is in one and only one sub-Tableau \( Y_{gij} \) for some \( 1 \leq i < j \leq k \). And this implies the claim.

Definition 7.35. 1. Let \( Y_g = (Y_1, Y_2) \) be generalized Young Tableau, then we define \( \hat{S}_Y(z) \) and \( \hat{S}_Y(z, \epsilon) \) to be the \( E_{r, \tau}(gl_N) \)-modules

\[
\hat{S}_Y(z) = S^{(n_1 - l_1)}V(z_1) \otimes S^{(n_2 - l_2)}V(z_2) \cdots S^{(n_k - l_k)}V(z_k),
\]

\[
z_i = z + \gamma l - \gamma (k - 1),
\]

\[
\hat{S}_Y(z, \epsilon) = S^{(n_1 - l_1)}V(z_1) \otimes S^{(n_2 - l_2)}V(z_2) \cdots S^{(n_k - l_k)}V(z_k),
\]

\[
z_i = z + \epsilon (k - 1) + \gamma l - (k - 1)\epsilon,
\]

(7.88)

where \( n_1 \geq n_2 \geq \cdots \geq n_1 \), resp. \( l_1 \geq l_2 \geq \cdots \geq l_1 \), are the lengths of the rows of \( Y_1 \), resp. \( Y_2 \).

And \( S_Y(z), S_Y(z, \epsilon) \) in the same way, but with the factors in the opposite order.

The last object we need, is the operator that permutes the factors in our tensor product of symmetric \( E_{r, \tau}(gl_N) \)-module.

Definition 7.36. Let \( z_1, z_2, \ldots, z_n \in \mathbb{C} \) be generic complex numbers, then we denote by \( W^{l_1, \ldots, l_n}_{a_1, \ldots, a_n}(z_1, \ldots, z_k; \lambda) \) the operator that permutes the factors in the tensor
7.5. Elliptic Weight of $W_{Y_g}$

product associated to a generalized Young Tableau. Its recursive definition can be derived from Definition 7.6.

**Lemma 7.37.** Let $\tilde{S}_Y(z, \epsilon)$ be the $E_{\gamma, \tau}(gl_N)$-module associated to the generalized Young Tableau $Y_g = (Y_1, Y_2)$, then $W^{l_{1}, \ldots, l_k}_{n_1, \ldots, n_k}(z_1, \ldots, z_k; \lambda)$ that permutes the factors of $\tilde{S}_Y(z, \epsilon)$ has a pole of order $O(Y_g)$ in $\epsilon = 0$ mod $Z + Z\tau$.

**Definition 7.38.** Let $\tilde{S}_Y(z, \epsilon)$ be the $E_{\gamma, \tau}(gl_N)$-module associated to the generalized Young Tableau $Y_g = (Y_1, Y_2)$, then we define the operator $W^{l_{1}, \ldots, l_k, \text{reg}}_{\alpha}(\lambda)$ on $\tilde{S}_Y(z)$ to be:

$$W^{l_{1}, \ldots, l_k, \text{reg}}_{\alpha}(\lambda) = \lim_{\epsilon \to 0} \Theta(\epsilon)^{O(Y_g)} W^{l_{1}, \ldots, l_k}_{n_1, \ldots, n_k}(\lambda).$$

(7.89)

**Definition 7.39.** Let $Y_g = (Y_1, Y_2)$ be a generalized Young Tableau with two rows, then we define $J_{Y_g}(\tilde{S}_Y(z))$ to be the vector space over $\text{Fun}_N(\mathbb{C})$

$$J_{Y_g}(\tilde{S}_Y(z)) = \sum_{k=i_1+1}^{n_2} J^{k, n_1+k}(\tilde{S}_Y(z)), \quad (7.90)$$

where

$$J^{k, n_1+k}(\tilde{S}_Y(z)) = \ker A^{l_1, l_2}(\lambda) S^{l_1, n_1, l_2(n_2+1, l_2)} \cap \tilde{S}_Y(z). \quad (7.91)$$

2. Let $Y_g = (Y_1, Y_2)$ be a generalized Young Tableau, then we define $J_{Y_g}(\tilde{S}_Y(z))$ to be the vector space over $\text{Fun}_N(\mathbb{C})$

$$J_{i, i+1}(\tilde{S}_Y(z)) = \text{Fun}_N(S^{(n_1-i_1)}(z_1) \otimes \ldots \otimes S^{(n_i-i_{i+1})}(z_{i-1}) \otimes J_{Y_g}(\tilde{S}_Y(z)) \otimes \ldots \otimes S^{(n_{i+1}-i_{i+1})}(z_1)), \quad (7.92)$$

$$J_{Y_g}(\tilde{S}_Y(z)) = \sum_{i_1}^{k} J_{i, i+1}(\tilde{S}_Y(z)).$$

**Theorem 7.40.** Let $\tilde{S}_Y(z)$ be the vector space associated to a generalized Young Tableau with $Y_g = (Y_1, Y_2)$. And let $\gamma$ be a generic number, then

$$\ker W^{l_{1}, \ldots, l_k, \text{reg}}_{\alpha}(\lambda) = J_{Y_g}(\tilde{S}_Y(z)). \quad (7.93)$$

**Definition 7.41.** Let $Y_g = (Y_1, Y_2)$ be a generalized Young Tableau and $S_Y(z)$ the (opposite) $E_{\gamma, \tau}(gl_N)$-module associated to $Y_g$, then the Young $E_{\gamma, \tau}(gl_N)$-module $W_{Y_g}(z)$ is:

$$\text{Fun}_N(W_Y(z)) = \text{Fun}_N(S_Y(z))/\ker W^{l_{1}, \ldots, l_k, \text{reg}}_{\alpha}(\lambda) P^{1, 2, \ldots, k}. \quad (7.94)$$

7.5 Elliptic Weight of $W_{Y_g}$

In this section we will compute the elliptic weight of the $E_{\gamma, \tau}(gl_N)$-modules $W_{Y_g}(z)$ associated to generalized Young Tableau, or to be more precise, the elliptic weight of a suitably normalized $E_{\gamma, \tau}(gl_N)$-module $W_{Y_{g,N}}(z)$. We showed, in fact in Theorem 2.9, that any $E_{\gamma, \tau}(gl_N)$-module can be "normalized" to an $E_{\gamma, \tau}(gl_N)$-module such that the first component of its elliptic weight is identically equal to one.
Definition 7.42. 1. Let \( Y_g \) be a generalized Young Tableau, then the highest weight “filling” of \( Y_g \) is obtained associating 1 to the first box of each column, 2 to the second and so on.

2. Let \( Y_g \) be a generalized Young Tableau \( Y_g = (Y_1, Y_2) \), then the Tableau \( Y_g^{(i)} \) are obtained from \( Y_g \) by removing the columns of \( Y_2 \setminus Y_1 \) of length \( i \geq i \), \( i = 1, \ldots, N \) and all boxes with labels different from 1 and \( i \).

3. Let \( Y_g^{(i)}, i = 1, \ldots, N \) be the Tableau of 1., then we define \( G_{ih,Y_g^{(i)}}(w) \) to be the function:

\[
G_{ih,Y_g^{(i)}}(w) = \prod_{b \in Y_g^{(i)}} \frac{\Theta(w - z - \gamma i_b + \gamma j_b)}{\Theta(w - z - \gamma i_b + \gamma j_b - \gamma)}. 
\]  
(7.95)

Theorem 7.43. Let \( W_{Y_g}(z) \) be the \( \mathfrak{g} \)-module associated to the generalized Young Tableau \( Y_g \), then:

1. \( W_{Y_g}(z) \) is an irreducible highest weight \( \mathfrak{g} \)-module,

2. there is a 1-dimensional \( \mathfrak{g} \)-module \( W^* \) such that the tensor product \( W_{Y_g,N}(z) = W^* \otimes W_{Y_g}(z) \) has the elliptic weight

\[
\Lambda_{ih}(w, \lambda) = G_{ih,Y_g^{(i)}}(w)W_{ih}(\lambda), 
\]  
(7.96)

where the functions \( G_{ih,Y_g^{(i)}}(w) \) are defined in 7.42.

Proof. The computation of the elliptic weight is straightforward. The irreducibility can be proved by taking the rational limit. \( \square \)

Definition 7.44. The \( \mathfrak{g} \)-module \( W_{Y_g,N}(z) \) of Theorem 7.43 is called the canonical \( \mathfrak{g} \)-modules associated to the generalized Young Tableau \( Y_g \).

7.6 A final Conjecture

In this section we study the \( \mathbb{R} \)-matrix on \( W_{Y_g}(z) \otimes W \), where \( W_{Y_g}(z) \) is an \( \mathfrak{g} \)-module associated to a (generalized) Young Tableau \( Y_g \) and \( W \) is any irreducible \( \mathfrak{g} \)-module and formulate a Conjecture.

Definition 7.45. 1. Let \( Y_g \) be a generalized Young Tableau, then a labeling \( \ell \) of \( Y_g \) with integers in \( \{1, 2, \ldots, N\} \) is a map from the boxes of \( Y_g \) to \( \{1, 2, \ldots, N\} \).

2. Let \( \ell \) be a labeling of \( Y_g \), then \( \ell \) is admissible if the labels associated to the boxes on a same row do not decrease from left to right.

3. Let \( \ell \) an admissible labeling of \( Y_g \), then \( \ell \) is a basis labeling of \( Y_g \) if the labels associated to the boxes on a same column increase from the top to the bottom of the Tableau.

In the next Lemma we will show that there is a bijection between basis labelings of \( Y_g \) and certain vectors in \( W_{Y_g} \).
Definition 7.46. 1. Let \( Y_g \) be a generalized Young Tableau with \( k \) rows of length \( n_i - l_i, i = 1, \ldots, k \), and \( \ell \) an admissible labeling of \( Y_g \), then we define the vector 

\[
\omega_\ell = v_k \otimes v_{k-1} \otimes \cdots \otimes v_1 \in S^{n_k-l_k}V(z - \gamma(k - 1)) \otimes \cdots \otimes S^{n_1-l_1}V(z),
\]

(7.97)

associated to the labeling \( \ell \), by the rule:

let \( i_1 \leq i_2 \leq \cdots i_s, s = n_j - l_j \) be the labels of the boxes in the \( j \)-th row of \( Y_g \), then

\[
\omega_\ell = [e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_s}] \in S^{n_j-l_j}V(z - \gamma(j - 1)).
\]

(7.98)

2. Let \( \omega_\ell \in S^{n_k-l_k}V(z - \gamma(k - 1)) \otimes \cdots \otimes S^{n_1-l_1}V(z) \) be the vector associated to an admissible labeling \( \ell \), then we define \( \tilde{\omega}_\ell \) to be the image of \( \omega_\ell \) in \( W_{Y_g} \).

Lemma 7.47. The family of vectors \( \tilde{\omega}_\ell \) of Definition 7.46 associated to basis labelings is a basis of \( W_{Y_g} \).

Proof. We show first that the vectors in \( W_{Y_g} \) which are associated to admissible labelings which are not basis labelings can be rewritten as a linear combination of the latter.

In order to do this, let us choose two arbitrary adjacent rows in \( Y_g \), say row \( r \) and \( r + 1 \). Moreover, let us choose and mark to boxes, one on row \( r(B_1) \) and one on row \( r + 1(B_2) \), which are on the same column. Finally, let us mark all boxes of row \( r \) on the right of \( B_1 \) and all boxes of row \( r + 1 \) on the left of \( B_2 \).

Then there are functions \( g_\sigma \in \text{Fun}_N(\mathbb{C}) \) such that

\[
\sum_\sigma g_\sigma(\lambda) \tilde{\omega}_\sigma(\ell) = 0, \quad (\tilde{\omega} = 0)
\]

(7.99)

where the sum runs over all admissible labelings of \( Y_g \) which are obtained by permuting the labels in the marked boxes of an arbitrary but fixed starting labeling \( \ell \). The correctness of (7.99) follows directly from the definition of \( W_{Y_g} \) and the properties of the vectors in the kernel of \( W_{Y_g}^{n_1, n_2, \ldots, n_k}(\lambda) \).

Let us now show that vectors associated to non-basis labelings are linear combinations, over \( \text{Fun}_N(\mathbb{C}) \), of vectors associated to basis labelings. We do this by induction on the number of rows of \( Y_g \).

If \( Y_g \) has only one row there is nothing to prove.

Assume, now, that any vector associated to a non-basis labeling is a linear combination of vectors which satisfies the "column condition", i.e. labels increase in columns, from row \( s \) to the bottom of \( Y_g \). If the "column condition" is satisfied also for the rows \( s \) and \( s - 1 \) there is nothing to prove. So we only consider the case where there is a violation of this rule for a pair of boxes \( B_1 \) in \( s - 1 \) and \( B_2 \) in \( s \).
Let us now mark the boxes from $B_1$ to the end of row $s-1$ and form the beginning of the row $s$ to $B_2$, as described previously. Moreover let $i_1 \leq i_2 \leq \ldots \leq i_m$, resp. $j_1 \leq j_2 \leq \ldots \leq j_n$ be the labels of the marked boxes in row $s-1$, resp. $s$. Then by construction:

$$j_1 \leq j_2 \leq \ldots \leq j_n \leq i_1 \leq i_2 \leq \ldots \leq i_m$$

(7.100)

where at least two labels are different. (Otherwise the vector $\tilde{w}_\ell$ would be identically zero.) Using (7.99) it follows now, that the vectors we are considering can be rewritten as a linear combination of vectors associated to admissible labelings obtained by permuting the labels in the marked boxes. Such vectors have the following properties:

- The violation of the "column rule" between boxes $B_1$ and $B_2$ is removed.
- At least one label in row $s-1$ is smaller.

After this operation the new vectors may violate the "column rule" in rows form $s$ to the bottom of $Y_\gamma$, but this "violations" can be removed by the induction hypothesis (without changing the labels in the row $s-1$). Now using that the labels are bounded from below, it follow that after a finite number of steps the column rule must be satisfied also for the rows $s-1$ and $s$.

It remains to show that the vectors $\tilde{w}_\ell$ associated to basis labelings are linear independent. This follows from the previous considerations on the kernel of $W_{n_1,n_2,\ldots,n_k}(\lambda)$ in the limit $\gamma \to 0$. □

**Definition 7.48.** The basis of vectors of $W_{Y_\gamma}$ given by basis labelings is called the canonical basis of $W_{Y_\gamma}$.

**Definition 7.49.** Let $\tilde{w}_\ell \in W_{Y_\gamma}(z)$ be the image in $W_{Y_\gamma}(z)$ of a vector associated to an admissible labeling $\ell$, then we denote by $C_f^\ell(\lambda) \in \text{Fun}_N(\mathbb{C})$ the coordinates of $\tilde{w}_\ell$ in the canonical basis of $W_{Y_\gamma}(z)$.

**Definition 7.50.** Let $Y_\gamma$ be a generalized Young Tableau. Moreover let $(i,j)$ be a ordered pair of basis labelings of $Y_\gamma$, then the operator $S^Y_{i,j}(z)$ associated to the pair $(i,j)$ is defined as:

$$S^Y_{i,j}(z) = \sum_\ell C_f^\ell(\lambda)S_{t_1,j_1}(z)\ldots S_{t_n,j_n}(z) - \gamma(k-1),$$

(7.101)

where the sum runs thought all admissible labelings which have a non-zero $i$ component and $\ell_m$, resp. $j_m$ are the vectors of labels associated to the $m$-th row of $Y_\gamma$.

**Conjecture 7.1.** Let $W_{Y_\gamma}(z)$ be the $E_{\gamma,\tau}(gl_N)$-module associated to a generalized Young Tableau $Y_\gamma$ and let $W$ be any $E_{\gamma,\tau}(gl_N)$-module, then an R-matrix on $W_{Y_\gamma}(z) \otimes W$ is given by:

$$R_{W_{Y_\gamma}(z),W}(\lambda) = \sum_i w_i \otimes S^Y_{i,j}(z)w, \quad \forall w \in W, \quad \forall \lambda \in W_{Y_\gamma}(z) \otimes W_{Y_\gamma}(z).$$

(7.102)

where the vectors $w_i$ are the basis vectors associated to the corresponding labelings of $Y_\gamma$ and the sum runs over all basis vectors of $W_{Y_\gamma}(z)$.

Moreover, if $W = W_{Y_\gamma}(z)$, then the R-matrix is $\pm P$, where $P$ is the flip on $W_{Y_\gamma}(z) \otimes W_{Y_\gamma}(z)$.
The statement (7.102) will be proved in what follows. What remains a conjecture is that the $R$-matrix on $W_{Y_g}(z) \otimes W_{Y_g}(z)$ is (proportional to) the flip on $W_{Y_g}(z) \otimes W_{Y_g}(z)$. The importance of the last statement is that it implies the irreducibility of tensor products of modules $W_{Y_{z_1}}(z) \otimes W_{Y_{z_2}}(w)$ for generic values of $z, w$ and the irreducibility of the single factors.

**Definition 7.51.** Let $i$ be the labeling of $Y_g$ associated to the highest weight vector of $W_{Y_g}(z)$, then we denote by $S_{Y_g}^{1/2}$ the operator, of Definition 7.50, $S_{Y_g}^{1/2}$.

**Corollary 7.52.** Let $W_{Y_g}(z)$ be the $E_{\gamma,N}(gl_N)$-module associated to a generalized Young Tableau $Y_g$ and let $\omega \in W_{Y_g}(z)$ be its highest weight vector, then the vectors

$$\omega_i = S_{Y_g}^{1/2}(z)\omega,$$

are well defined and form a basis of $W_{Y_g}(z)$.

We will prove this Theorems in two steps. In the first one we will show that the $R$-matrix on $W_{Y_g}(z) \otimes W$ has the form of Theorem 7.50. In the second that it is equal to the flip if $W = W_{Y_g}(z)$.

**Lemma 7.53.** The $R$-matrix on $W_{Y_g}(z) \otimes W$ has the form of Theorem 7.37 7.1.

**Proof.** In order to compute the $R$-matrix $W_{Y_g}(z) \otimes W$ we first consider the "lifted" $R$-matrix on

$$S^{n_k-l_k} V(z_k) \otimes \ldots \otimes S^{n_1-l_1} V(z_1) \otimes W.$$  

This is by definition

$$R_{S^{n_k-l_k} V(z_k), W(\lambda)} \ldots R_{S^{n_1-l_1} V(z_1), W(\lambda - \sum_{i=1}^{k-1} h_i)}.$$  

(7.105)

Let now $P^{(1, \ldots, n)}$ be the flip on the first $n$ factors and $W_{n_1, \ldots, n_k}^{l_1, \ldots, l_k, reg}(\lambda)$ the regularized operator of Theorem 7.40, then

$$W_{n_1, \ldots, n_k}^{l_1, \ldots, l_k, reg}(\lambda - \gamma h^{(n+1)}) P^{(1, \ldots, n)} R_{S^{n_k-l_k} V(z_k), W(\lambda)}$$

$$\ldots R_{S^{n_1-l_1} V(z_1), W(\lambda - \gamma \sum_{i=1}^{k-1} h_i)} =$$

$$R_{S^{n_k-l_k} V(z_k), W} \ldots R_{S^{n_1-l_1} V(z_1), W} W_{n_1, \ldots, n_k}^{l_1, \ldots, l_k, reg}(\lambda) P^{(1, \ldots, n)}.$$  

(7.106)

The correctness of the last identity follows from its graphical interpretation. (see Figure 7.8).

This implies that the $R$-matrix on (7.104) leaves the kernel of the operator $W_{n_1, \ldots, n_k}^{l_1, \ldots, l_k, reg}(\lambda) P^{(1, \ldots, n)}$ invariant, or to be more precise the space

$$\text{ker} \{W_{n_1, \ldots, n_k}^{l_1, \ldots, l_k, reg}(\lambda) P^{(1, \ldots, n)}\} \otimes W.$$  

(7.107)

In other words it follows that the $R$-matrix $W_{Y_g}(z) \otimes W$ is given by the quotient of (7.105) with (7.107). Which is the desired result. ∎
Figure 7.8: Graphical Representation of Equation (7.106)
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