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Concentration-cancellation and Hardy spaces

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Concentration-cancellation and Hardy spaces

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Abstract

Let $v^\varepsilon$ a sequence of DiPerna-Majda approximate solutions to the 2-D incompressible Euler equations. We prove that if the vorticity sequence is weakly compact in the Hardy space $H^1(\mathbb{R}^2)$ then a subsequence of $v^\varepsilon$ converges strongly in $L^2(\mathbb{R}^2)$ to a solution of the Euler equations. This phenomenon is directly related to the cancellation effects exhibited by “phantom vortices”.

Keywords: Riesz transform, equibounded, Dunford-Pettis theorem

Subject Classification: 35Q10 (76D05)
In their fundamental paper [4] DiPerna and Majda study the convergence of approximate solutions \( v^\varepsilon \) of the 2-D inviscid Euler equations as the regularization parameter \( \varepsilon \) goes to zero. They give several examples of sequences of compactly supported approximate solutions \( v^\varepsilon \) (as defined in Defintion 1.1, [4]) whose vorticity \( \omega^\varepsilon \) is bounded in \( L^1 \) which fail to be compact in \( L^2 \) so that in the limit concentration phenomena occur. Moreover in Th. 1.3 of [4] a criterion which rules concentrations out is proposed: it is shown that a uniform bound on a logarithmic Morrey norm of \( \omega^\varepsilon \) yields strong \( L^2 \)-convergence of the velocity field. In this note another criterion for compactness is introduced: we show that strong \( L^2 \)-compactness of \( v^\varepsilon \) follows from weak compactness of \( \omega^\varepsilon \) in the Hardy space \( H^1(R^2) \). Since \( H^1(R^2) \) is not rearrangement invariant the fine structure of the vorticity plays a crucial role in getting strong \( L^2 \)-convergence. We recall that by Dunford-Pettis theorem (see [5], VIII, Th.1.3) a necessary and sufficient condition for a subset \( \Lambda \) of \( L^1(R^2) \) to be weakly pre-compact in \( L^1(R^2) \) is that there exist a positive function \( G(s) : R^+ \to R^+ \) such that

\[
\lim_{s \to +\infty} \frac{G(s)}{s} = +\infty
\]

and

\[
\sup_{f \in \Lambda} \int_{R^2} G(|f|) \, dx < +\infty
\]

Let \( R_i \), \( i = 1, 2 \), denote the Riesz transforms:

\[
R_j f(x) = \int_{R^2} \frac{x_j - y_j}{|x - y|^3} f(y) \, dx
\]

We formulate our result as follows. **Theorem 1.** Let \( v^\varepsilon \) be a sequence of approximate solutions such that for every \( t \geq 0 \)

\[
\|\omega^\varepsilon(\cdot, t)\|_{H^1} < C \quad 0 < \varepsilon \leq \varepsilon_0
\]

and \( \omega^\varepsilon \) satisfies weak uniform control at infinity (cfr.[4],(3.5)). Moreover let there be a function \( G(s) : R^+ \to R^+ \) such that (1) and (2) hold for \( \Lambda = \{ \omega^\varepsilon \}, \{ R_i \omega^\varepsilon \}, i = 1, 2 \). Then there is a subsequence of \( v^\varepsilon \) which converges strongly in \( L^2 \) to a weak classical solution \( v \) of the Euler equations. Moreover \( v \in W^{1,1}(R^2) \). We recall (see [6]) that a function \( f \) belongs to the Hardy space \( H^1(R^2) \) iff there is a sequence of numbers \( \lambda_j \) satisfying \( \sum_j \lambda_j < \infty \) and a series of functions (atoms) \( a_j \) such that

\[
f = \sum_{j=1}^{\infty} \lambda_j a_j
\]

where the \( a_j \)'s have the following properties a) \( a_j \) is supported on a ball \( B_j \) and \( \|a_j\|_\infty < \frac{1}{|B_j|} \int_{B_j} a_j(x) \, dx = 0 \) The \( H^1 \)-norm of \( f \) can be defined as the infimum of the expressions \( \sum_{j=1}^{\infty} \lambda_j \) on all possible representations of \( f \) as in (4). If condition b) were dropped the resulting space would be \( L^1(R^2) \). It is the subtle cancellation
effect due to b) (cfr."phantom vortices"in [4], 1.A) together with (2) which yields strong $L^2$-compactness. \textit{Proof of Theorem 1.} To prove the theorem we introduce the stream function $\psi^\varepsilon$ such that

\begin{equation}
\triangle \psi^\varepsilon = \omega^\varepsilon
\end{equation}

and we proceed as in the proof of Th. 3.1 in [4]. It is known that for every $f$ in $BMO(R^2)$ there are $g_i$ in $L^\infty(R^2)$, $i = 0, 1, 2$, such that

$$f = g_0 + \sum_{i=1,2} R_i g_i$$

Hence

$$\int_{R^2} f \omega^\varepsilon \, dx = \int_{R^2} \omega^\varepsilon (g_0 + \sum_{i=1,2} R_i g_i) \, dx = \int_{R^2} \omega^\varepsilon g_0 - \sum_{i=1,2} g_i R_i \omega^\varepsilon \, dx$$

By our assumption (2) the sequence $\{\omega^\varepsilon\}$ and its Riesz transforms admit a weakly convergent subsequence in $L^1(R^2)$. Therefore there is a subsequence such that

\begin{equation}
\omega^\varepsilon \to \omega \quad \text{weakly in } H^1(R^2)
\end{equation}

The statement of Th.1 is guaranteed by showing that for all $\rho \in C_\infty(R^2)$

\begin{equation}
\lim_{\varepsilon \to 0} \int_{R^2} \rho |v^\varepsilon|^2 \, dx = \int_{R^2} \rho |v|^2 \, dx
\end{equation}

Indeed after integrating by parts (7) is seen to hold iff (see [4], (3.7)-(3.10))

\begin{equation}
\lim_{\varepsilon \to 0} \int_{R^2} \rho \psi^\varepsilon \omega^\varepsilon \, dx = \int_{R^2} \rho \psi \, d\omega
\end{equation}

where $\psi$ is the stream function corresponding to $\omega$ in (5). We recall that

$$\frac{\partial^2}{\partial x_j \partial x_k} f = -R_j R_k \triangle f$$

Hence

$$\frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon = -R_j R_k \omega^\varepsilon$$

Since the Riesz transform maps $H^1(R^2)$ continuously into itself we get that

\begin{equation}
\left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{L^1} \leq \left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{H^1} \leq C \| \omega^\varepsilon \|_{H^1}
\end{equation}

and $\psi^\varepsilon$ stays bounded in $W^{2,1}(R^2)$. We recall that for any bounded domain $\Omega$ in $R^2$ by the Gagliardo-Sobolev imbedding theorem $W^{2,1}(R^2)$ is continuously imbedded in $C(\overline{\Omega})$. Therefore

\begin{equation}
\| \psi^\varepsilon \|_{C(\overline{\Omega})} \leq C \| \omega^\varepsilon \|_{H^1}
\end{equation}
Moreover (see [1], Lemma 5.8) if \( u \in W^{2,1}(R^2) \) for any \( P_o \in R^2 \) we have that for \( \delta > 0 \) if \(|\Delta P| < \frac{\delta}{2}\)

\[
(11) \quad |u(P_o + \Delta P) - u(P_o)| \leq C \left( \frac{1}{\delta^2} \|u(P + \Delta P) - u(P)\|_{L^1(B_{\delta}(P_o))} \right) + \frac{1}{\delta} \sum_i \left\| \frac{\partial}{\partial x_i} u(P + \Delta P) - \frac{\partial}{\partial x_i} u(P) \right\|_{L^1(B_{\delta}(P_o))} + \sum_{i,j} \left\| \frac{\partial^2}{\partial x_j \partial x_i} u(P + \Delta P) - \frac{\partial^2}{\partial x_j \partial x_i} u(P) \right\|_{L^1(B_{\delta}(P_o))}
\]

By (weak) continuity of the Riesz transforms from \( H^1(R^2) \) into itself there is a subsequence of \( \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \), that converges weakly in \( H^1(R^2) \) to a \( \phi_{i,j} \in H^1(R^2) \). On the other hand weak convergence in \( H^1(R^2) \) implies weak convergence in \( L^1(R^2) \) (indeed \( L^\infty \subset BMO \)) so that we have

\[
\frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \rightarrow \phi_{i,j} \text{ weakly in } L^1(R^2)
\]

By the full version of Dunford-Pettis theorem for every \( \kappa > 0 \) there is a \( \delta > 0 \) such that for any \( P \in \Omega \)

\[
\left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{L^1(B_{\delta}(P))} < \kappa
\]

uniformly in \( \varepsilon \). We observe that if \(|\Delta P| < \delta\)

\[
\left\| \frac{\partial^2}{\partial x_j \partial x_k} [\psi^\varepsilon(P + \Delta P) - \psi^\varepsilon(P)] \right\|_{L^1(B_{\delta}(P_o))} < C \left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{L^1(B_{2\delta}(P_o))}
\]

Therefore for every \( P_o \) given \( \kappa_o > 0 \) we can find a \( \delta_o > 0 \) such that if \(|\Delta P| < \frac{\delta_o}{2}\)

\[
\sum_{i,j} \left\| \frac{\partial^2}{\partial x_j \partial x_k} [\psi^\varepsilon(P + \Delta P) - \psi^\varepsilon(P)] \right\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}
\]

uniformly in \( \varepsilon \). Moreover since \( W^{1,1}(\Omega) \) is compactly imbedded in \( L^p \) for any \( p < 2 \) both \( \{\psi^\varepsilon\} \) and \( \{\frac{\partial}{\partial x_k} \psi^\varepsilon\} \) are compact in \( L^1 \). Hence by Kondratchev compactness criterion (see [1]) there is a \( \delta_1 > 0 \) such that if \(|\Delta P| < \delta_1\)

\[
\frac{1}{\delta_o} \|u(P + \Delta P) - u(P)\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}
\]

\[
\frac{1}{\delta_o} \sum_i \left\| \frac{\partial}{\partial x_i} u(P + \Delta P) - \frac{\partial}{\partial x_i} u(P) \right\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}
\]

and by (11)

\[
(12) \quad |\psi^\varepsilon(P_o + \Delta P) - u(P_o)| < \kappa_o
\]
uniformly in \( \epsilon \). The sequence \( \psi^\epsilon \) is equibounded by (10) and equicontinuous by (12) and by Ascoli theorem we can extract a subsequence such that

(13) \[ \psi^\epsilon \rightarrow \psi \quad \text{strongly in } C(\Omega) \]

By (6) we have that \( \omega^\epsilon \rightharpoonup \omega \) weakly in \( M(\Omega) \) so that (8) holds and the same argument as in Th. 1.3 of [4] yields the statement of the theorem.

Remark. The first example in (1, §4) in [4] (phantom vortices) shows a sequence of vorticities which stays bounded in \( H^1(R^2) \) whose velocity field fails to converge strongly in \( L^2 \); in the second example one has strong \( L^1(R^2) \) convergence of the vorticity but the sequence does not lie in \( H^1(R^2) \) and again concentrations occur. By looking at the proof of Delort’s recent deep result ([3]), weak convergence of \( \omega^\epsilon \) in \( L^1(R^2) \) is sufficient to pass to the limit in the quadratic terms of the Euler equations, due to their special structure. It is interesting that every bounded sequence in \( H^1(R^2) \) admits a weakly(*) convergent subsequence whose limit stays in \( H^1(R^2) \) (see [2], Lemma (4.2)). However, since \( (VMO)^* = H^1(R^2) \) and \( L^\infty \not\subset VMO \), this does not yield weak \( L^1 \)-convergence. It is worth observing that condition (2) for \( \omega^\epsilon \) is rearrangement invariant and so in the time dependent case it is conserved by the particle trajectory map. On the other hand, as for the bounds (3,4) of Th. 3.1 in [4], it is not clear what happens to the \( H^1 \)-norm as time goes by, since \( H^1(R^2) \) is not rearrangement invariant.

Bibliography

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