Concentration-cancellation and Hardy spaces

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Abstract

Let $v^\varepsilon$ a sequence of DiPerna-Majda approximate solutions to the 2-D incompressible Euler equations. We prove that if the vorticity sequence is weakly compact in the Hardy space $H^1(R^2)$ then a subsequence of $v^\varepsilon$ converges strongly in $L^2(R^2)$ to a solution of the Euler equations. This phenomenon is directly related to the cancellation effects exhibited by “phantom vortices”.

Keywords: Riesz transform, equibounded, Dunford-Pettis theorem

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In their fundamental paper [4] DiPerna and Majda study the convergence of approximate solutions \( v^\epsilon \) of the 2-D inviscid Euler equations as the regularization parameter \( \epsilon \) goes to zero. They give several examples of sequences of compactly supported approximate solutions \( v^\epsilon \) (as defined in Definition 1.1, [4]) whose vorticity \( \omega^\epsilon \) is bounded in \( L^1 \) which fail to be compact in \( L^2 \) so that in the limit concentration phenomena occur. Moreover in Th. 1.3 of [4] a criterion which rules concentrations out is proposed: it is shown that a uniform bound on a logarithmic Morrey norm of \( \omega^\epsilon \) yields strong \( L^2 \)-convergence of the velocity field. In this note another criterion for compactness is introduced: we show that strong \( L^2 \)-compactness of \( v^\epsilon \) follows from weak compactness of \( \omega^\epsilon \) in the Hardy space \( H^1(R^2) \). Since \( H^1(R^2) \) is not rearrangement invariant the fine structure of the vorticity plays a crucial role in getting strong \( L^2 \)-convergence. We recall that by Dunford-Pettis theorem (see [5], VIII, Th.1.3) a necessary and sufficient condition for a subset \( \Lambda \) of \( L^1(R^2) \) to be weakly pre-compact in \( L^1(R^2) \) is that there exist a positive function \( G(s) : R^+ \to R^+ \) such that

\[
\lim_{s \to +\infty} \frac{G(s)}{s} = +\infty
\]

and

\[
\sup_{f \in \Lambda} \int_{R^2} G(|f|) \, dx < +\infty
\]

Let \( R_i \) \( i = 1, 2 \), denote the Riesz transforms:

\[
R_j f(x) = \int_{R^2} \frac{x_j - y_j}{|x - y|^3} f(y) \, dx
\]

We formulate our result as follows. **Theorem 1.** Let \( v^\epsilon \) be a sequence of approximate solutions such that for every \( t \geq 0 \)

\[
\|\omega^\epsilon(\cdot, t)\|_{H^1} < C \quad 0 < \epsilon \leq \epsilon_0
\]

and \( \omega^\epsilon \) satisfies weak uniform control at infinity (cfr.[4],(3.5)). Moreover let there be a function \( G(s) : R^+ \to R^+ \) such that (1) and (2) hold for \( \Lambda = \{\omega^\epsilon\}, \{R_i \omega^\epsilon\}, i = 1, 2 \). Then there is a subsequence of \( v^\epsilon \) which converges strongly in \( L^2 \) to a weak classical solution \( v \) of the Euler equations. Moreover \( v \in W^{1,1}(R^2) \). We recall (see [6]) that a function \( f \) belongs to the Hardy space \( H^1(R^2) \) if there is a sequence of numbers \( \lambda_j \) satisfying \( \sum_j |\lambda_j| < \infty \) and a series of functions (atoms) \( a_j \) such that

\[
f = \sum_{j=1}^{\infty} \lambda_j a_j
\]

where the \( a_j \)'s have the following properties a) \( a_j \) is supported on a ball \( B_j \) and \( \|a_j\|_\infty < \frac{1}{|B_j|^{1/2}} \int_{B_j} a_j(x) \, dx = 0 \) The \( H^1 \)-norm of \( f \) can be defined as the infimum of the expressions \( \sum_{j=1}^{\infty} |\lambda_j| \) on all possible representations of \( f \) as in (4). If condition b) were dropped the resulting space would be \( L^1(R^2) \). It is the subtle cancellation
effect due to b) (cfr. "phantom vortices" in [4], 1.A) together with (2) which yields strong $L^2$-compactness. Proof of Theorem 1. To prove the theorem we introduce the stream function $\psi^\varepsilon$ such that

$$\Delta \psi^\varepsilon = \omega^\varepsilon$$

and we proceed as in the proof of Th. 3.1 in [4]. It is known that for every $f$ in $BMO(R^2)$ there are $g_i$ in $L^\infty(R^2)$, $i = 0, 1, 2$, such that

$$f = g_0 + \sum_{i=1,2} R_i g_i$$

Hence

$$\int_{R^2} f \omega^\varepsilon \, dx = \int_{R^2} \omega^\varepsilon (g_0 + \sum_{i=1,2} R_i g_i) \, dx = \int_{R^2} \omega^\varepsilon g_0 - \sum_{i=1,2} g_i R_i \omega^\varepsilon \, dx$$

By our assumption (2) the sequence $\{\omega^\varepsilon\}$ and its Riesz transforms admit a weakly convergent subsequence in $L^1(R^2)$. Therefore there is a subsequence such that

$$\omega^\varepsilon \rightharpoonup \omega \quad \text{weakly in } H^1(R^2)$$

The statement of Th.1 is guaranteed by showing that for all $\rho \in C_c^\infty(R^2)$

$$\lim_{\varepsilon \to 0} \int_{R^2} \rho |\psi^\varepsilon|^2 \, dx = \int_{R^2} \rho |\psi|^2 \, dx$$

Indeed after integrating by parts (7) is seen to hold iff (see [4], (3.7)-(3.10))

$$\lim_{\varepsilon \to 0} \int_{R^2} \rho \psi^\varepsilon \omega^\varepsilon \, dx = \int_{R^2} \rho \psi \omega \, dx$$

where $\psi$ is the stream function corresponding to $\omega$ in (5). We recall that

$$\frac{\partial^2}{\partial x_j \partial x_k} f = -R_j R_k \Delta f$$

Hence

$$\frac{\partial^2 \psi^\varepsilon}{\partial x_j \partial x_k} = -R_j R_k \omega^\varepsilon$$

Since the Riesz transform maps $H^1(R^2)$ continuously into itself we get that

$$\left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{L^2} \leq \left\| \frac{\partial^2}{\partial x_j \partial x_k} \omega^\varepsilon \right\|_{H^1} \leq C \|\omega^\varepsilon\|_{H^1}$$

and $\psi^\varepsilon$ stays bounded in $W^{2,1}(R^2)$. We recall that for any bounded domain $\Omega$ in $R^2$ by the Gagliardo-Sobolev imbedding theorem $W^{2,1}(R^2)$ is continuously imbedded in $C(\overline{\Omega})$. Therefore

$$\|\psi^\varepsilon\|_{C(\overline{\Omega})} \leq C \|\omega^\varepsilon\|_{H^1}$$
Moreover (see [1], Lemma 5.8) if \( u \in W^{2,1}(\mathbb{R}^2) \) for any \( P_o \in \mathbb{R}^2 \) we have that for \( \delta > 0 \) if \( |\Delta P| < \frac{\delta}{2} \)

\[
(11) \quad |u(P_o + \Delta P) - u(P_o)| \leq C \left( \frac{1}{\delta^2} \|u(P + \Delta P) - u(P)\|_{L^1(B_o(P_o))} \right)
+ \frac{1}{\delta} \sum_i \left\| \frac{\partial}{\partial x_i} u(P + \Delta P) - \frac{\partial}{\partial x_i} u(P) \right\|_{L^1(B_o(P_o))}
+ \sum_{i,j} \left\| \frac{\partial^2}{\partial x_j \partial x_i} u(P + \Delta P) - \frac{\partial^2}{\partial x_j \partial x_i} u(P) \right\|_{L^1(B_o(P_o))}
\]

By (weak) continuity of the Riesz transforms from \( H^1(\mathbb{R}^2) \) into itself there is a subsequence of \( \frac{\partial^2}{\partial x_j \partial x_i} \psi^\varepsilon \), that converges weakly in \( H^1(\mathbb{R}^2) \) to a \( \phi_{i,j} \in H^1(\mathbb{R}^2) \). On the other hand weak convergence in \( H^1(\mathbb{R}^2) \) implies weak convergence in \( L^1(\mathbb{R}^2) \) (indeed \( L^\infty \subset BMO \)) so that we have

\[
\frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \rightharpoonup \phi_{i,j} \quad \text{weakly in } L^1(\mathbb{R}^2)
\]

By the full version of Dunford-Pettis theorem for every \( \kappa > 0 \) there is a \( \delta > 0 \) such that for any \( P \in \Omega \)

\[
\left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{L^1(B_o(P))} < \kappa
\]

uniformly in \( \varepsilon \). We observe that if \( |\Delta P| < \delta \)

\[
\left\| \frac{\partial^2}{\partial x_j \partial x_k} [\psi^\varepsilon(P + \Delta P) - \psi^\varepsilon(P)] \right\|_{L^1(B_o(P))} < C \left\| \frac{\partial^2}{\partial x_j \partial x_k} \psi^\varepsilon \right\|_{L^1(B_{2\delta}(P_o))}
\]

Therefore for every \( P_o \) given \( \kappa_o > 0 \) we can find a \( \delta_o > 0 \) such that if \( |\Delta P| < \frac{\delta_o}{2} \)

\[
\sum_{i,j} \left\| \frac{\partial^2}{\partial x_j \partial x_k} [\psi^\varepsilon(P + \Delta P) - \psi^\varepsilon(P)] \right\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}
\]

uniformly in \( \varepsilon \). Moreover since \( W^{1,1}(\Omega) \) is compactly imbedded in \( L^p \) for any \( p < 2 \) both \( \{\psi^\varepsilon\} \) and \( \{\frac{\partial}{\partial x_i} \psi^\varepsilon\} \) are compact in \( L^1 \). Hence by Kondratchev compactness criterion (see [1]) there is a \( \delta_1 > 0 \) such that if \( |\Delta P| < \delta_1 \)

\[
\frac{1}{\delta_o} \left\| u(P + \Delta P) - u(P) \right\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}
\]

\[
\frac{1}{\delta_o} \sum_i \left\| \frac{\partial}{\partial x_i} u(P + \Delta P) - \frac{\partial}{\partial x_i} u(P) \right\|_{L^1(B_{\delta_o}(P_o))} < \frac{\kappa_o}{3}
\]

and by (11)

\[
(12) \quad |\psi^\varepsilon(P_o + \Delta P) - u(P_o)| < \kappa_o
\]
uniformly in $\epsilon$. The sequence $\psi^\epsilon$ is equibounded by (10) and equicontinuous by (12) and by Ascoli theorem we can extract a subsequence such that

$$
\psi^\epsilon \to \psi \quad \text{strongly in } C(\Omega)
$$

By (6) we have that $\omega^\epsilon \to \omega$ weakly in $M(\Omega)$ so that (8) holds and the same argument as in Th. 1.3 of [4] yields the statement of the theorem.

Remark. The first example in (1, §4) in [4] (phantom vortices) shows a sequence of vorticities which stays bounded in $H^1(R^2)$ whose velocity field fails to converge strongly in $L^2$; in the second example one has strong $L^1(R^2)$ convergence of the vorticity but the sequence does not lie in $H^1(R^2)$ and again concentrations occur. By looking at the proof of Delort’s recent deep result ([3]), weak convergence of $\omega^\epsilon$ in $L^1(R^2)$ is sufficient to pass to the limit in the quadratic terms of the Euler equations, due to their special structure. It is interesting that every bounded sequence in $H^1(R^2)$ admits a weakly(*) convergent subsequence whose limit stays in $H^1(R^2)$ (see [2], Lemma (4.2)). However, since $(VMO)^* = H^1(R^2)$ and $L^\infty \not\subset VMO$, this does not yield weak $L^1$-convergence. It is worth observing that condition (2) for $\omega^\epsilon$ is rearrangement invariant and so in the time dependent case it is conserved by the particle trajectory map. On the other hand, as for the bounds (3,4) of Th. 3.1 in [4], it is not clear what happens to the $H^1$-norm as time goes by, since $H^1(R^2)$ is not rearrangement invariant.

Bibliography

## Research Reports

<table>
<thead>
<tr>
<th>Report No.</th>
<th>Authors</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>91-03</td>
<td>I. Vecchi</td>
<td>Concentration-cancellation and Hardy spaces</td>
</tr>
<tr>
<td>91-02</td>
<td>R. Jeltsch, B. Pohl</td>
<td>Waveform Relaxation with Overlapping Splittings</td>
</tr>
</tbody>
</table>