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Uniqueness of Piecewise Lipschitz Continuous Solutions of the Cauchy-Problem for $2 \times 2$ Conservation Laws

R. Jeltsch and X. Wang
Abstract

We prove uniqueness theorems in the class of piecewise Lipschitz continuous solutions of the Cauchy-Problem for $2 \times 2$ conservation laws and improve the results of DiPerna [Di] in this class.

Keywords: uniqueness, conservation laws, $2 \times 2$ system

Subject Classification: 35L65, 35A05, 35L67, 35L60
1. Introduction.

We consider strictly hyperbolic systems of two conservation laws in one dimension:

\[(1.1,a)\quad \partial_t U + \partial_x f(U) = 0, \quad (x,t) \in R \times R^+;\]

\[(1.1,b)\quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad f(U) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix}\]

with the initial data

\[(1.2)\quad U(x,0) = U_0(x),\]

where \( R = (-\infty, +\infty) \) and \( R^+ = [0, +\infty) \).

Here the condition of strict hyperbolicity means that the Jacobian \( \nabla f \) of \( f \) has two real and distinct eigenvalues:

\[\sigma_1(U) < \sigma_2(U)\]

and the function \( f \) is in \( C^3(\Omega) \) for some open set \( \Omega \) in \( R^2 \).

The system (1.1) is genuinely nonlinear, if for each \( i \) with \( i = 1, 2 \)

\[r_i : \nabla \sigma_i \neq 0,\]

where \( r_i \) denotes the right eigenvector of \( \nabla f \) corresponding to \( \sigma_i \).

A bounded integrable function \( U(\cdot, \cdot) \) is called a weak solution of the Cauchy-Problem: (1.1) and (1.2), if \( U \) satisfies (1.1) and (1.2) in the sense of distributions, i.e.,

\[\int_{R \times R^+} (U \partial_t \phi + f(U) \partial_x \phi) \, dx \, dt + \int_R U_0(x) \phi(x,0) \, dx = 0\]

for any smooth function \( \phi \) with compact support in \( t \geq 0 \).

It is well known that uniqueness is lost in the class of weak solutions. Thus we consider uniqueness in the following subclasses of the weak solutions. \( |\cdot| \) denotes the Euclidean norm in \( R^2 \).

**Definition 1.1** A weak solution \( U(\cdot, \cdot) \) is called a solution in the class \( i \), if it satisfies the following conditions:

\[(1.4)\quad U \text{ is bounded in the sense of Lebesgue's measure, i.e.,}\]

\[\text{esssup}_{(x,t) \in R \times R^+} |U(x,t)| < +\infty.\]
(1.5) $U$ has a finite number of curves of discontinuity in each compact set in $R \times R^+$. 
(1.6) Each curve of discontinuity is smooth. 
(1.7) $U$ is Lipschitz continuous in the region between any two curves of discontinuity. 
(1.8) Across any discontinuity $x = \gamma(t)$ the weak solution $U$ satisfies Lax's entropy condition:

\[(1.8.a)\quad \sigma_1(U^+) < \gamma' < \sigma_1(U^-)\]

or

\[(1.8.b)\quad \sigma_2(U^+) < \gamma' < \sigma_2(U^-),\]

where $U^\pm = U(x \pm 0, t)$ and $\gamma' = \frac{\partial \gamma}{\partial t}$.

**Remark** The condition (1.5) refers to the count of the number of discontinuities. We consider the discontinuities after the intersection of two discontinuities as new discontinuities.

**Definition 1.2** A weak solution $U(\cdot, \cdot)$ is called a solution in the class $2$, if it satisfies the conditions: (1.4)-(1.7), and

(1.9) For any curve of the discontinuity $x = \gamma(t)$ there is a number $\sigma > 0$ independent of $t$, such that across $x = \gamma(t)$ the weak solution $U$ satisfies the strong entropy condition:

\[
\begin{cases} 
\sigma_1(U^+) < \gamma'; \\
0 < \sigma < \sigma_1(U^-) - \gamma'
\end{cases}
\]

or

\[
\begin{cases} 
\gamma' < \sigma_2(U^-); \\
0 < \sigma < \gamma' - \sigma_2(U^+).
\end{cases}
\]

We show in this paper the following two theorems.

**Theorem 1.3:** Suppose that the system is genuinely nonlinear. For every positive number $C$ there exists a number $\theta > 0$ depending only on $f$ and $C$ with the following property.

If $U$ and $\bar{U}$ are two solutions in the class 1 and satisfy

\[
\text{esssup}_{(x,t) \in R \times R^+} |U(x,t)| < C; \\
\text{esssup}_{(x,t) \in R \times R^+} |\bar{U}(x,t)| < C
\]

and for any discontinuity

\[|U^+ - U^-| < \theta;\]
\[ |\bar{U}^+ - \bar{U}^-| < \theta, \]

then \( \bar{U} = U \) in \( R \times R^+ \).

For a general system we have

**Theorem 1.4:** For every weak solution \( U \) in the class 2 there exists a number \( \theta > 0 \) depending on \( f \) and \( U \) with the following property.

If \( \bar{U} \) is a solution in the class 1 with

\[ \max_{(x,t) \in R \times R^+} |U(x,t) - \bar{U}(x,t)| < \theta, \]

then \( \bar{U} = U \) in \( R \times R^+ \).

DiPerna [1] has shown

**Theorem D:** Suppose that the system (1.1) is genuinely nonlinear and admits an additional conservation law:

(1.10) \[ \eta(U)_t + q(U)_x = 0. \]

For every \( \bar{U} \in R^2 \) there exists a number \( \theta > 0 \) depending on \( f \) and \( \bar{U} \) with the following property.

If \( U \) is a solution in the class 1 and \( \bar{U} \) another weak solution with

\[ \eta(U)_t + q(U)_x \leq 0 \]

in the sense of distribution and \( U \) and \( \bar{U} \) satisfy

\[ \max_{(x,t) \in R \times R^+} |(U(\bar{U})(x,t))| \leq \theta \]

and

\[ \max_{(x,t) \in R \times R^+} |(\bar{U} - \bar{U})(x,t)| \leq \theta, \]

then \( \bar{U} = U \) in \( R \times R^+ \).

Here DiPerna requires: (i) Since \( \bar{U} \) is a fix vector in \( R^2 \), \( U(x,t) \) and \( \bar{U}(x,t) \) are almost constant. They have to stay in a ball with the sufficiently small radius \( \theta \) and the center \( \bar{U} \). (ii) The distance between \( U \) and \( \bar{U} \) is sufficient small.

We do not require the assumption of existence of an additional conservation law (1.10). In the theorem 1.3 both solutions can have a large variation and there is no restriction for
the distance between $U$ and $\bar{U}$. In the theorem 1.4 both solutions can have an arbitrary large, but bounded variation and the system must not be genuinely nonlinear.

In our view the technique applied here to show the identity of two weak solutions is more important than the results. With this technique we can consider furthermore the rarefaction waves and the $n \times n$ systems. This will be confirmed in our next papers.

For more information on uniqueness we would like to refer to the papers: [Liu], [FX], [DG] and the references cited there.

2. Localization.

We show in this section that it is sufficient to prove the identity of two weak solutions in a small trapezoid, in order to prove the identity in the halfplane: $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. In this trapezoid the discontinuities can be easily described.

Let $M > 0$ be a given constant,

$$K(a, b, c, d) := \left\{ (x,t) \in \mathbb{R} \times [0, T] \left| M(t-a) + b \leq x \leq -M(t-a) + d; \atop a \leq t \leq c \right. \right\}$$

with $0 \leq a < c, b < d$ and $M(c-a) < \frac{d-b}{2}$ a trapezoid and $U$ and $\bar{U}$ two weak solutions. Then we introduce a class of trapezoids depending on $f$, $U$ and $\bar{U}$.

**Definition 2.1:** A trapezoid $K(a, b, c, d)$ is called a trapezoid in the class $T(M)$, if $K(a, b, c, d)$ satisfies the following conditions.

(2.1) The weak solutions $U$ and $\bar{U}$ are almost everywhere identical in the ground line:

$$\left\{ (x,t) \in K(a, b, c, d) \left| b \leq x \leq d, t = a \right. \right\}.$$

(2.2) There is no or a finite number of discontinuities that run from one point in the ground line and don’t intersect with the side lines:

$$x = M(t-a) + b \text{ and } x = -M(t-a) + d, \quad a < t \leq c.$$

They don’t intersect each other in the trapezoid $K(a, b, c, d)$ either and each of them intersects with the upper boundary

$$\left\{ (x,t) \in K(a, b, c, d) \left| b + Mc \leq x \leq d - Mc, t = c \right. \right\}$$

at only one point.
**Definition 2.2:** A trapezoid $K(a, b, c, d)$ in the class $T(M)$ has the property $T_0$, if for each $(x, a) \in K(a, b, c, d)$ with $b < x < d$ there is $\varepsilon > 0$ with $b < x - \varepsilon < x + \varepsilon \leq d$, so that $U$ and $\bar{U}$ are almost everywhere identical in the trapezoid $K(a, x - \varepsilon, a + \varepsilon, x + \varepsilon) \subset K(a, b, c, d)$.

**Lemma 2.3:** Suppose that $U$ and $\bar{U}$ are two weak solutions satisfying the conditions (1.4)-(1.7).

If there is a number $M > 0$, so that any trapezoid in $T(M)$ has the property $T_0$, then $U = \bar{U}$ in $R \times R^+$. 

**Proof:** Suppose that $U \neq \bar{U}$ in $R \times R^+$. Let $W := U - \bar{U}$ and 

$$D(0, b, c, d) := \iint_{K(0,b,c,d)} |W|^2 \, dx \, dt.$$ 

Then there are $b_0, c_0$ and $d_0$, so that 

$$D(0, b_0, c_0, d_0) > 0.$$ 

Let 

$$t_0 := \sup \{ t \in [0, c_0] \mid D(0, b_0, t, d_0) = 0 \}.$$ 

It follows that 

$$t_0 < c_0;$$

(2.3) 

$$D(0, b_0, t, d_0) > 0, \quad t \in (t_0, c_0);$$

$$D(0, b_0, t_0, d_0) = 0$$

and $U$ and $\bar{U}$ are identical in the segment 

$$\left\{(x, t) \in K(a, b, c, d) \mid b_0 + M t_0 \leq x \leq d_0 - M t_0, \, t = t_0 \right\}.$$
Since $U$ and $\overline{U}$ satisfy the conditions (1.4)-(1.7), it is obvious in the geometric views, that there is in the class $T(M)$ a finite number of trapezoids \( \{ K(t_0, b_i, t_0 + \varepsilon, d_i) \}_{i=1}^n \) with $\varepsilon > 0$, so that
\[
\bigcup_{i=1}^n K(t_0, b_i, t_0 + \varepsilon, d_i) = K(t_0, b_0, t_0 + \varepsilon, d_0).
\]
It follows from the property $T_0$ and compactness that
\[
D(t_0, b_0, t_0 + \varepsilon, d_0) = 0.
\]
Thus
\[
D(0, b_0, t_0 + \varepsilon, d_0) = D(0, b_0, t_0, d_0) + D(t_0, b_0, t_0 + \varepsilon, d_0) = 0.
\]
This is a contradiction to (2.3).

Without loss of generality we assume that the trapezoid
\[
(2.4) \quad K := \left\{ (x, t) \mid Mt - 1 \leq x \leq Mt + 1, 0 \leq t \leq c \right\}
\]
is in the class $T(M)$ and any discontinuities in $K$ start from $(0,0)$. Then we shall show the identity of $U$ and $\overline{U}$ in $K$.

With the help of the entropy condition we can reduce the number of discontinuities in the trapezoid $K$.

**Lemma 2.4:** Suppose that $U$ and $\overline{U}$ satisfy the conditions (1.4)-(1.7). Then $U$ (resp. $\overline{U}$) has in $K$ at most two discontinuities. One of them $\alpha(\cdot)$ (resp. $\overline{\alpha}(\cdot)$) satisfies (1.8.a) and the another $\beta(\cdot)$ (resp. $\overline{\beta}(\cdot)$) satisfies (1.8.b).

**Lemma 2.5:** Suppose that $U$ and $\overline{U}$ satisfy the conditions (1.4)-(1.7). Then the discontinuities satisfy
\[
\alpha(t) < \beta(t)
\]
\[
\overline{\alpha}(t) < \overline{\beta}(t)
\]
for any $t \in (0, c)$.

The proof is trivial and we omit it.

3. **Hyperbolicity and Symmetrizer.**

**Lemma 3.1:** For every $C > 0$ there exists $\theta > 0$ depending on $C$ and $f$, so that for each $V$ with $|V| \leq C/2$ there is a neighborhood of $V$
\[
\Omega_\theta(V) = \{ U \in \mathbb{R}^2 \mid |U - V| \leq \theta, |U| \leq C \},
\]
in which there is a smooth, regular and real matrix $L$ with the following properties:

(3.1) It holds that
\[ \nabla f = L^{-1} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} L \] for all $U \in \Omega(V)$.

(3.2) Let $A := L^\top L$. There is a positive constant depending on $f$ and $C$ with
\[ \langle W, A(U)W \rangle \geq \text{const} \langle W, W \rangle \]
for all $U \in \Omega(V)$ and all $W \in \mathbb{R}^2$, where the notation $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^2$.

(3.3) the Matrixes $L$ and $A$ are uniformly bounded in $U \in \Omega(V)$, i.e., there is a positive constant depending on $f$ and $C$ with
\[ \max \{ |L|, |A| \} \leq \text{const} \]
for all $U \in \Omega(V)$ and all $W \in \mathbb{R}^2$.

(3.4) The derivatives of $L$ and $A$ uniformly bounded in $U \in \Omega(V)$, i.e., there is a positive constant depending on $f$ and $C$ with
\[ \max \{ |\partial_{u_1} L|, |\partial_{u_2} L| \} \leq \text{const} \]
and
\[ \max \{ |\partial_{u_1} A|, |\partial_{u_2} A| \} \leq \text{const} \]
for all $U \in \Omega(V)$ and all $W \in \mathbb{R}^2$.

Note that (3.1) implies
\[ A \nabla f = L^\top \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} L \]
and
\[ (A \nabla f)^\top = A \nabla f. \]

The row vectors of $L$ are the left eigenvectors of $\nabla f$. We can prove this lemma by directly calculating $L$. In order to show the properties (3.1)–(3.4) we must apply the strict hyperbolicity of $\nabla f$. We omit here the details of proof.

A matrix $A$ is called a symmetrizer of the Matrix $\nabla f$, if $A$ is positive definite and $(A \nabla f)^\top = A \nabla f$ holds. With its help one can show the existence and the uniqueness of classical solution of conservation laws (1.1). We refer to [KL]. We apply in this paper the symmetrizer to show the uniqueness of solution with discontinuities and avoid the assumption of existence of the entropy pair $(\eta, q)$ [Dip].
4. Local Estimation.

Let \( t_1, t_2 \) with \( 0 \leq t_1 < t_2 < c \)

\[
\begin{align*}
\Gamma_1 : & \quad x = \gamma_1(t), \quad t \in [t_1, t_2] \\
\Gamma_r : & \quad x = \gamma_r(t), \quad t \in [t_1, t_2]
\end{align*}
\]

be two smooth curves in \( K \) and

\[
G := \{(x, t) \in K \mid \gamma_1(t) \leq x \leq \gamma_r(t), \quad t_1 \leq t \leq t_2\}
\]

be a subset of \( K \).

In this section we assume, that there is a symmetrizer satisfying (3.1)-(3.4), there is no discontinuity in \( G \) and \( U \) and \( \mathcal{U} \) satisfy the condition (1.7) and

\[
\text{esssup}_{(x,t) \in \mathbb{R} \times \mathbb{R}^+} |U(x,t)| < C
\]

and

\[
\text{esssup}_{(x,t) \in \mathbb{R} \times \mathbb{R}^+} |\mathcal{C}(x,t)| < C.
\]

We measure the distance between \( U \) and \( \mathcal{U} \) in \( G \) by the integration

\[
J(t) := \int_{t_1}^{t_2} <W, A(U)W > \phi \, dx, \quad t \in [t_1, t_2]
\]

with a strictly positive smooth function \( \phi \). Let

\[
I(\tau, c, \phi) := \iint_{G(\tau)} \left[ <W, A(U)W > (\partial_t \phi) \\
+ <W, A(U) \nabla f(U)W > (\partial_x \phi) \\
+ c <W, W > \phi \right] dx \, dt,
\]

where \( \tau \in [t_1, t_2] \), \( G(\tau) := \{(x, t) \in G \mid t_1 \leq t \leq \tau\} \), \( W = \mathcal{U} - U \) and \( c > 0 \) is a real number.

Then we have

**Lemma 4.1**: There is \( c > 0 \) depending on \( f \), \( U \) and \( \mathcal{U} \), so that for any \( \tau \in [t_1, t_2] \)

\[
J(\tau) \leq I(\tau, c, \phi) + J(t_1) - J(\Gamma_1, t_1, \tau) + J(\Gamma_r, t_1, \tau)
\]

holds, where

\[
J(\Gamma_r, t_1, \tau) := \int_{t_1}^{\tau} <W, A[\gamma'_r(t) \cdot E - \nabla f]W > \phi \bigg|_{x=\gamma_r(t)} \, dt,
\]

\[
J(\Gamma_1, t_1, \tau) := \int_{t_1}^{\tau} <W, A[\gamma'_1(t) \cdot E - \nabla f]W > \phi \bigg|_{x=\gamma_1(t)} \, dt
\]
and $E$ is the unit matrix.

**Proof:** Let $U_n := \Psi_n \ast U$ and $\mathcal{U}_n := \Psi_n \ast U$, where $\Psi_n$ is a $\delta$-function. $U_n$ (resp. $\mathcal{U}_n$) is smooth, if $U$ (resp. $\mathcal{U}$) is Lipschitz continuous. We estimate the integral:

$$
\int \int_{G_n} \partial_t [\langle W_n, A_n W_n \rangle \phi] + \partial_x [\langle W_n, A_n (\nabla f) W_n \rangle \phi] \, dx \, dt,
$$

where $W_n = \mathcal{U}_n - U_n$, $A_n := A(U_n)$ and $(\nabla f)_n := f(U_n)$. By use of Green’s theorem and the definition of weak solutions we can take apart this integration by the integrations along the boundaries of $G(\tau)$, i.e., $J(\tau)$, $J(t_1)$, $J(\Gamma_1, t_1, \tau)$ and $J(\Gamma_r, t_1, \tau)$. The remaining term can be estimated by

$$
\int \int_{G(\tau)} c < W, W > \phi \, dx \, dt.
$$

Then the conclusion follows, when $n \to \infty$.

Let

$$
I_\Delta(\tau, c, \phi) := \int \int_{G(\tau)} [\langle W, A(U)W \rangle \phi] + \langle W, A(U)[f(\mathcal{U}) - f(U)] \rangle \phi + c < W, W > \phi \, dx \, dt,
$$

$$
J_\Delta(\Gamma_r, t_1, \tau) := \int_{t_1}^\tau < W, A[\gamma'_r(t) \cdot W - (f(\mathcal{U}) - f(U))] \rangle \phi \bigg|_{x = \gamma_r(t)} \, dt,
$$

and

$$
J_\Delta(\Gamma_1, t_1, \tau) := \int_{t_1}^\tau < W, A[\gamma'_1(t) \cdot W - (f(\mathcal{U}) - f(U))] \rangle \phi \bigg|_{x = \gamma_1(t)} \, dt.
$$

Similarly we have

**Lemma 4.2:** There is $c > 0$ depending on $f$, $U$ and $\mathcal{U}$, so that for any $\tau \in [t_1, t_2]$

$$
J(\tau) \leq I_\Delta(\tau, c, \phi) + J(t_1) - J_\Delta(\Gamma_1, t_1, \tau) + J_\Delta(\Gamma_r, t_1, \tau)
$$

holds.

We omit here its proof.

**Lemma 4.3:** Let $c$ be the positive number determined in the Lemma 4.1 and the Lemma 4.2. Then there exists a positive constant $M_1$, so that for the testfunction

$$
\phi_{\bar{c}}(x, t) := e^{-\frac{x^2}{4\bar{c}^2}}
$$

with $\bar{c} > c$ the estimates:

$$
I(\tau, \bar{c}, \phi_{\bar{c}}) \leq 0;
$$

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\[ I_{\Delta}(\tau, \varepsilon, \phi_{\varepsilon}) \leq 0; \]
\[ J(\tau) \leq J(t_1) - J(\Gamma_l, t_1, \tau) + J(\Gamma_r, t_1, \tau) \]
and
\[ J(\tau) \leq J(t_1) - J_\Delta(\Gamma_l, t_1, \tau) + J_\Delta(\Gamma_r, t_1, \tau). \]
hold.

The conclusion follows from using \( \phi_{\varepsilon} \) as a testfunction in the Lemma 4.1 and the Lemma 4.2.

5. Proof of the theorems.

At first we show the Theorem 1.3. According to the conclusions in the section 2 we shall show, that the trapezoid \( K \) determined in (2.4) has the property \( T_0 \). We set the constant \( M \) in the section 2 by

\[ M := \max_{(x, t) \in \mathbb{R} \times [0, t]} \left\{ \sum_{i=1}^{2} |\sigma_i(U)| + |\sigma_i(\bar{U})| \right\}. \]

**Lemma 5.1:** For every point \((x, 0)\) with \(-1 \leq x < 0\) or \(0 < x \leq 1\) there exists \( \epsilon > 0\), so that \( U = \bar{U} \) in the trapezoid \( K(0, x - \epsilon, x + \epsilon) \).

**Proof:** We choose \( \epsilon \) so small, that there is in the trapezoid \( K(0, x - \epsilon, x + \epsilon) \) a symmetrizer \( A \) having all properties in the Lemma 3.1. Then we use the Lemma 4.3 and obtain

\[ J(\tau) \leq J(0) - J(\Gamma_l, 0, \tau) + J(\Gamma_r, 0, \tau) \]

for any \( \tau \in [0, \epsilon] \), where \( \Gamma_l : x = Mt + x - \epsilon \) and \( \Gamma_r : x = -Mt + x + \epsilon \) are the two side line of the trapezoid \( K(0, x - \epsilon, x + \epsilon) \). Since \( U = \bar{U} \) on the ground line \( t = 0 \), we have

\[ J(\tau) \leq -J(\Gamma_l, 0, \tau) + J(\Gamma_r, 0, \tau). \]

We investigate the sign of the integrand of \( J(\Gamma_l, 0, \tau) \) and find

\[ < W, A[\gamma_l(t) \cdot E - (2f)]W > \phi \bigg|_{x = Mt + x - \epsilon} \]
\[ = W^T L^T \left( \begin{array}{cc}
M - \sigma_1 & 0 \\
0 & M - \sigma_2
\end{array} \right) LW \bigg|_{x = Mt + x - \epsilon} \geq 0. \]

It follows that

\[ -J(\Gamma_l, 0, \tau) \leq 0. \]
Similarly
\[ J(\Gamma_r, 0, \tau) \leq 0. \]

Thus \( J(\tau) \leq 0. \) This implies that \( U = \tilde{U} \) in the trapezoid \( K(0, x - \epsilon, x + \epsilon) \).

In order to show that \( K \) has the property \( T_0 \), we have to show furthermore that \( U \) and \( \tilde{U} \) are identical in a neighborhood of \((0, 0)\). Before we do this we consider two small trapezoides on the right and left side of \((0, 0)\).

Let \( \gamma_r(t) := \min \{ \alpha(t), \bar{\alpha}(t) \} \), \( \gamma_l(t) := \max \{ \beta(t), \bar{\beta}(t) \} \),

\[
G^1_\epsilon := \{(x, t) \in K \mid \gamma_r - \epsilon \leq x \leq \gamma_r, \quad 0 \leq t \leq \epsilon \}
\]

and

\[
G^2_\epsilon := \{(x, t) \in K \mid \gamma_l \leq x \leq \gamma_l + \epsilon, \quad 0 \leq t \leq \epsilon \}. \]

Remark: From the Lemma 2.5 it follows that \( \gamma_r(t) \leq \gamma_l(t) \) for \( t \in [0, \epsilon] \).

**Lemma 5.2:** There exists \( \epsilon > 0 \), so that \( U = \tilde{U} \) holds in \( G^1_\epsilon \) and \( G^2_\epsilon \).

**Proof:** We show here only that \( U = \tilde{U} \) holds in the trapezoid \( G^1_\epsilon \) and choose \( \epsilon \) so small, that there is in the trapezoid \( G^1_\epsilon \) a symmetrizer \( A \) having all properties in the Lemma 3.1.

Then we use the Lemma 4.3 and obtain

\[ J(\tau) \leq J(0) - J(\gamma_r - \epsilon, 0, \tau) + J(\gamma_r, 0, \tau) \]

for any \( \tau \in [0, \epsilon] \). Similarly as in the proof of the last lemma we have \( J(0) = 0, -J(\gamma_r - \epsilon, 0, \tau) \leq 0. \) Then

\[ J(\tau) \leq J(\gamma_r, 0, \tau). \]

It remains to show that

\[
(5.2.1) \quad J(\gamma_r, 0, \tau) \leq 0. \]

As in the last lemma we investigate the sign of the integrand of \( J(\gamma_r, 0, \tau) \) and find

\[
< W, A[\gamma'_r(t) \cdot E - (\nabla f)]W > \phi \bigg|_{x=\gamma_r(t)}^{x=\gamma_r(t)} \]

\[
= W^r L^r \begin{pmatrix}
\gamma'_r(t) - \sigma_1 & 0 \\
0 & \gamma'_r(t) - \sigma_2
\end{pmatrix} LW \bigg|_{x=\gamma_r(t)}. \]
The curve $\gamma_r(\cdot)$ consists of two curves of discontinuities: $\alpha(\cdot)$ and $\tilde{\alpha}(\cdot)$. In the case that $(\gamma_r(t), t) = (\alpha(t), t)$ it follows from the entropy condition (1.8) that

$$0 > [\gamma_r^t(t) - \sigma_1(U)]_{x = \gamma_r(t) - 0}$$

$$> [\gamma_r^t(t) - \sigma_2(U)]_{x = \gamma_r(t) - 0}.$$  

and

$$J(\gamma_r, 0, \tau) \leq 0.$$  

In the case that $(\gamma_r(t), t) = (\tilde{\alpha}(t), t)$ we have also according to the entropy condition (1.8)

$$0 > [\gamma_r^t(0) - \sigma_1(U)]_{x = \gamma_r(0) - 0}$$

$$> [\gamma_r^t(0) - \sigma_2(U)]_{x = \gamma_r(0) - 0}.$$  

Since $\overline{U}(\tilde{\alpha}(0) - 0, 0) = \overline{U}(\alpha(0) - 0, 0) = \overline{U}(-\alpha(0) - 0, 0)$, we have

$$0 > [\gamma_r^t(0) - \sigma_1(U)]_{x = \gamma_r(0) - 0}$$

$$> [\gamma_r^t(0) - \sigma_2(U)]_{x = \gamma_r(0) - 0}.$$  

Then we can find $\epsilon > 0$, so that for $t \in (0, \epsilon)$

$$0 > [\gamma_r^t(t) - \sigma_1(U)]_{x = \gamma_r(t) - 0}$$

$$> [\gamma_r^t(t) - \sigma_2(U)]_{x = \gamma_r(t) - 0}.$$  

This implies also (5.2.1).  

Let $\gamma_1(t) := \max\{\alpha(t), \tilde{\alpha}(t)\}$ and $\gamma_2(t) := \min\{\beta(t), \tilde{\beta}(t)\}$.  

**Lemma 5.3:** For every positive number $C$ there exists a number $\theta > 0$ depending only on $f$ and $C$ with the following property.

If $U$ and $\overline{U}$ are two solutions in the class 1 and satisfy

$$\text{esssup}_{(x, t) \in \mathbb{R} \times \mathbb{R}^+} |U(x, t)| < C,$$
esssup \((x, t) \in \mathbb{R} \times \mathbb{R}^+\) \(|\vec{U}(x, t)| < C\)

and for any discontinuity

\[|U^+ - U^-| < \theta,\]

\[|\vec{C}^+ - \vec{C}^-| < \theta,\]

then there exists \(\epsilon > 0\), so that \(\gamma_1(t) < \gamma_2(t)\) hold for \(t \in (0, \epsilon)\).

**Proof:** Let \(V_m := \lim_{t \to 0} U(\alpha(t) + 0, t) = \lim_{t \to 0} U(\beta(t) - 0, t)\) and \(\vec{V}_m := \lim_{t \to 0} \vec{U}(\vec{\alpha}(t) + 0, t) = \lim_{t \to 0} \vec{U}(\vec{\beta}(t) - 0, t)\). We shall show here that

\[(5.3.1) \quad V_m = \vec{V}_m.\]

Then it follows from the Rankine-Hugoniot jump condition that \(\alpha'(0) = \vec{\alpha}'(0)\) and \(\beta'(0) = \vec{\beta}'(0)\). This implies the conclusion because of the Lemma 2.5.

We show now (5.3.1). Let

\[U_l := U(\alpha(0) - 0, 0) = \vec{U}(\vec{\alpha}(0) - 0, 0)\]

and

\[U_r := U(\beta(0) + 0, 0) = \vec{U}(\vec{\beta}(0) + 0, 0).\]

We obtain from the Rankine-Hugoniot jump condition that \(V_m\) and \(\vec{V}_m\) satisfy the equation with the unknown \(V\):

\[
\begin{cases}
< L_l(V), V - U_l > = 0 \\
< L_r(V), V - U_r > = 0,
\end{cases}
\]

where \(L_l(V)\) is the left eigenvector corresponding to the second eigenvalue of the matrix

\[
\int_0^1 \nabla f(\delta(V - U_l) + U_l) \, d\delta
\]

and \(L_r(V)\) the left eigenvector corresponding to the first eigenvalue of the matrix

\[
\int_0^1 \nabla f(\delta(V - U_r) + U_r) \, d\delta.
\]

Then we consider a mapping

\[
\Phi(V) := -\left( \frac{L_l(U_l)}{L_r(U_l)} \right)^{-1} \begin{pmatrix} < L_l(V), V - U_l > \\ < L_r(V), V - U_r > \end{pmatrix} + U
\]

and estimate \(|d\Phi|\) by use of the estimations in the Lemma 3.1. If \(|V - U_l| < \theta, |V - U_r| < \theta\) and

\[
\text{esssup}_{(x, t) \in \mathbb{R} \times \mathbb{R}^+} |V| < C
\]
then $|d\Phi| \leq \text{konst} \, \theta$ holds, where the constant depends on $f$ and $C$. We choose $\theta$ so small that $|d\Phi| \leq 1/2$ holds. It follows that

$$|\Phi(V_m) - \Phi(\bar{V}_m)| < \frac{1}{2}|V_m - \bar{V}_m|.$$ 

Then (5.3.1) follows from $\Phi(V_m) = V_m$ and $\Phi(\bar{V}_m) = \bar{V}_m$. 

In order to continue the proof we require the following lemma.

**Lemma 5.4:** Suppose that the system is genuinely nonlinear. For every positive number $C$ there exists a number $\theta > 0$ depending only on $f$ and $C$ with the following property.

If $U$ and $U'$ are two solutions in the class 1 and satisfy

$$\text{esssup}_{(x,t) \in R \times R_+} |U(x,t)| < C,$$

$$\text{esssup}_{(x,t) \in R \times R_+} |U'(x,t)| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|U'^+ - U'^-| < \theta,$$

then there exists $\epsilon > 0$, so that $J(\gamma_1 + 0, 0, t) \geq 0$ and $J(\gamma_2 - 0, 0, t) \leq 0$ hold for $t \in (0, \epsilon)$.

We shall show this lemma in the next section.

**Lemma 5.5:** Suppose that the system is genuinely nonlinear. For every positive number $C$ there exists a number $\theta > 0$ depending only on $f$ and $C$ with the following property.

If $U$ and $U'$ are two solutions in the class 1 and satisfy

$$\text{esssup}_{(x,t) \in R \times R_+} |U(x,t)| < C,$$

$$\text{esssup}_{(x,t) \in R \times R_+} |U'(x,t)| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|U'^+ - U'^-| < \theta,$$

then there exists $\epsilon > 0$, so that $U = \bar{U}$ in

$$G^3_\epsilon := \left\{ (x,t) \in K \left| \gamma_1 \leq x \leq \gamma_2, 0 \leq t \leq \epsilon \right. \right\}$$
This conclusion is a direct corollary of the last lemma and the Lemma 4.3.

It remains to show the identity of $U$ and $\bar{U}$ in the regions between $\alpha$ and $\bar{\alpha}$ and between $\beta$ and $\bar{\beta}$. In the following lemma we prove that $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$.

**Lemma 5.6:** Suppose that the system is genuinely nonlinear. For every positive number $C$ there exists a number $\theta > 0$ depending only on $f$ and $C$ with the following property.

If $U$ and $\bar{U}$ are two solutions in the class 1 and satisfy

$$\operatorname{esssup}_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |U(x,t)| < C,$$

$$\operatorname{esssup}_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |\bar{U}(x,t)| < C$$

and for any discontinuity

$$|U^+ - U^-| < \theta,$$

$$|\bar{U}^+ - \bar{U}^-| < \theta,$$

then there exists $\epsilon > 0$, so that $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ hold for $t \in [0, \epsilon]$.

**Proof:** If $\alpha \neq \bar{\alpha}$, there is a region

$$G := \left\{ (x,t) \in K \left| \begin{array}{l}
\gamma_l(t) \leq x \leq \gamma_r(t) ; \\
\gamma_l(t) \leq t \leq \gamma_r(t)
\end{array} \right. \right\}$$

with the following properties:

(5.6.1) The two side lines $\gamma_l$ und $\gamma_r$ are either $\alpha$ or $\bar{\alpha}$.

(5.6.2) The weak solutions $U$ and $\bar{U}$ are identical on the left side of $\gamma_l$ and on the right side of $\gamma_r$, i.e., for $t \in [t_1, t_2]$

$$U(\gamma_l(t) - 0, t) = \bar{U}(\gamma_l(t) - 0, t)$$

and

$$U(\gamma_r(t) + 0, t) = \bar{U}(\gamma_r(t) + 0, t)$$

hold.

(5.6.3) The left and right side lines don't intersect for $t \in (t_1, t_2)$, i.e., for $t \in (t_1, t_2)$

$$\gamma_l(t) < \gamma_r(t)$$

holds.

(5.6.4) $U$ and $\bar{U}$ are Lipschitz continuous in $G$. 

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\[(5.6.5) \quad \gamma_l(t_1) = \gamma_r(t_1).\]

According to the Lemma 4.3 we have in \(G\)
\[
J(\tau) \leq J(t_1) - J_{\Delta}^l(\Gamma_l, t_1, \tau) + J_{\Delta}^r(\Gamma_r, t_1, \tau).
\]
It follows from \((5.6.5)\) that \(J(t_1) = 0\). By using \((5.6.1), (5.6.2)\) and the Rankine-Hugoniot jump condition we have
\[
J_{\Delta}^l(\Gamma_l, t_1, \tau) = J_{\Delta}^r(\Gamma_r, t_1, \tau) = 0.
\]
This implies that \(U = \bar{U}\) in \(G\). Then it follows from \((5.6.2)\) that \(\gamma_l\) and \(\gamma_r\) are not the discontinuities of \(U\) or \(\bar{U}\). This is a contradiction to \((5.6.1)\).

**Proof of Theorem 1.3:** By putting the Lemma 5.2, the Lemma 5.5 and the Lemma 5.6 together we show that there is a neighborhood of \((0,0)\), in which \(U = \bar{U}\). This means that the trapezoid \(K\) has the property \(T_0\). Then the theorem follows from Lemma 2.3.

**Proof of Theorem 1.4:** We show here the identity of the discontinuities, i.e., \(\alpha = \bar{\alpha}\) and \(\beta = \bar{\beta}\). The rest is similar with the proof of the last theorem.

**Proposition.** For every weak solution \(U\) in the class \(2\) there exists a number \(\theta > 0\) depending on \(f\) and \(U\) with the following property.

If \(\bar{U}\) is a solution in the class \(1\) with
\[
\max_{(x, t) \in R \times R^+} \left| U(x, t) - \bar{U}(x, t) \right| < \theta,
\]
then the discontinuity of \(U\) is also the discontinuity of \(\bar{U}\).

Let \(U^+ := U(\alpha(t) + 0, t)\) and \(U^- := U(\alpha(t) - 0, t)\). The strong entropy condition \((1.9)\) implies that
\[
\sigma_1(U^-) - \sigma_1(U^+) > \sigma > 0.
\]
We assume that \(\bar{U}\) is continuous across the curve \(\alpha(\cdot)\).

Then \(\bar{U}(\alpha(t), t) = \bar{U}(\alpha(t) + 0, t) = \bar{U}(\alpha(t) - 0) = U(\alpha(t) - 0)\) and
\[
\left| \sigma_1(U^-) - \sigma_1(\bar{U}) \right|_{(x, t) = (\alpha(0), 0)} = 0.
\]
Since \(\sigma_1(\cdot)\) is Lipschitz continuous, there is a constant depending on \(f\) and \(U\), such that
\[
\left| \sigma_1(U^+) - \sigma_1(\bar{U}) \right|_{(x, t) = (\alpha(0), 0)} \leq \text{const} \left| U^+ - \bar{U} \right|_{(x, t) = (\alpha(0), 0)} \leq \text{const} \theta
\]

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holds. Then we have
\[
0 < \sigma < [\sigma_1(U^-) - \sigma_1(U^+)]
\]
\[
= [\sigma_1(U^+) - \sigma_1(U^-)]
\]
\[
\leq \text{const} \theta.
\]
Then there exists \( \epsilon > 0 \), so that for \( t \in [0, \epsilon] \)
\[
\sigma < \text{const} \theta
\]
holds. We choose \( \theta < \frac{\sigma}{\text{const}} \) and obtain a contradiction. Thus the proposition is shown.

We continue showing that \( \alpha = \bar{\alpha} \) and \( \beta = \bar{\beta} \). According to this proposition \( \bar{\alpha} \) is identical either with \( \alpha \) or with \( \beta \).

If \( \bar{\alpha} \) and \( \beta \) are identical, the curve \( \bar{\beta} \) is on the right side of \( \beta \) because of the Lemma 2.5. Applying the Lemma 2.5 again we obtain that \( \bar{\beta} \) is on the right side of \( \alpha \). This is a contradiction to the proposition, that \( \bar{\beta} \) is identical either with \( \alpha \) or with \( \beta \). This implies that \( \alpha \) and \( \bar{\alpha} \) must be identical.

Similarly one show that \( \beta \) and \( \bar{\beta} \) are identical.

We show here that \( J(\gamma_1 + 0, 0, t) \geq 0 \) holds for \( t \in (0, \epsilon) \) and one can obtain the other estimate in the Lemma 5.4: \( J(\gamma_2 - 0, 0, t) \leq 0 \), in the same way. We investigate the sign of the integrand of \( J(\gamma_1 + 0, 0, t) \):

\[
< W, A(U)[\gamma_1 I E - \nabla f(U)]W > \bigg|_{x = \gamma_1 + 0}.
\]

The curve \( \gamma_1(\cdot) \) consists of two discontinuities: \( \alpha(\cdot) \) and \( \bar{\alpha}(\cdot) \). From now on we assume that \( \gamma_1(t) = \bar{\alpha}(t) \), \( \alpha(t) < \bar{\alpha}(t) \) hold for \( t \in [t_1, t_2] \) and \( \alpha(t_1) = \bar{\alpha}(t_1) \). Then we shall show that for \( t \in [t_1, t_2] \)

\[
\int_{t_1}^{t} < W, A(U)[\gamma_1 I E - \nabla f(U)]W > \bigg|_{x = \gamma_1 + 0} \geq 0
\]

(6.1)

holds, if \( t_1 \) and \( t_2 \) are sufficiently small. In the other case where \( \gamma_1(t) = \alpha(t) \) we can obtain

\[
\int_{t_1}^{t} < W, A(U)[\gamma_1 I E - \nabla f(U)]W > \bigg|_{x = \alpha + 0} \geq 0
\]

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in the same way.

We introduce the following notations:

\[
U_{\tilde{a}} := U(\tilde{a}(t) + 0, t)
\]
\[
U_{\alpha} := U(\alpha(t) + 0, t)
\]
\[
\tilde{U}_{\tilde{a}} := \tilde{U}(\tilde{a}(t) + 0, t)
\]
\[
\tilde{U}_{\alpha} := \tilde{U}(\alpha(t) + 0, t)
\]
\[
W_{\tilde{a}} := \tilde{U}_{\tilde{a}} - U_{\tilde{a}}
\]

and

\[
W_{\alpha} := \tilde{U}_{\alpha} - U_{\alpha}.
\]

Obviously \( U_{\tilde{a}} := U_{\tilde{a}+} = U_{\tilde{a}+} \) and \( \tilde{U}_{\alpha} := \tilde{U}_{\alpha+} = \tilde{U}_{\alpha+} \) hold. According to the Lemma 5.2 there exists \( \epsilon > 0 \), so that \( U_{\alpha+} = \tilde{U}_{\alpha+} = U_{\alpha-} = \tilde{U}_{\alpha-} \) and \( W_{\alpha+} = W_{\alpha-} \) hold, if \( 0 \leq t_1 \leq t \leq t_2 \leq \epsilon \).

We introduce a new function \( \psi : R^2 \times R^2 \rightarrow R \)

\[
\psi(V_1, V_2) := \langle V_1 - V_2, A(U_{\tilde{a}+})[\tilde{a}' \cdot E - \nabla f(U_{\tilde{a}+})](V_1 - V_2) \rangle.
\]

Instead of the proving (6.1) we prove that for \( t \in [t_1, t_2] \)

\[
(6.2) \quad \int_{t_1}^{t} \Psi(U_{\tilde{a}+}, \tilde{U}_{\tilde{a}+}) d\tau \geq 0
\]

holds, if \( t_1 \) and \( t_2 \) are sufficiently small.

Let \( V_0 \in R^2 \) and \( S(V_0) \) be the shock set [Liu]

\[
S(V_0) := \left\{ V \in R^2 \bigg| \frac{f_1(V) - f_1(V_0)}{v_1 - v_{01}} = \frac{f_2(V) - f_2(V_0)}{v_2 - v_{02}} \right\}
\]

where \( V = (v_1, v_2)^T \) and \( V_0 = (v_{01}, v_{02})^T \). Let

\[
S(V, V_0) := \frac{f_1(V) - f_1(V_0)}{v_1 - v_{01}} = \frac{f_2(V) - f_2(V_0)}{v_2 - v_{02}}.
\]

It is not difficult to show that \( S(V, V_0) \) is either the first or the second eigenvalue of the matrix

\[
\int_{0}^{1} \nabla f(\delta(V - V_0) + V_0) \, d\delta,
\]

if \( V \neq V_0 \). Let

\[
S_i(V_0) := \left\{ V \in S(V_0) | S(V, V_0) \text{ is the } i \text{-th eigenvalue} \right\}
\]

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with \( i = 1, 2 \). Then \( S(V_0) = S_1(V_0) \cup S_2(V_0) \). By using of the standard method [Smo] we can show that \( S_1(V_0) \) and \( S_2(V_0) \) are two one-parameter families and for any \( t \in [t_1, t_2] \) there is \( \tilde{U} \in S_1(U_{\hat{\omega}_-}) \), so that

\[
\alpha' - \sigma_1(U_{\hat{\omega}_-}) = S(\tilde{U}, \tilde{U}_{\hat{\omega}_-}) - \sigma_1(\tilde{U}_{\hat{\omega}_-})
\]

holds, if \(|U^+ - U^-|\) and \(|C^+ - C^-|\) are sufficiently small.

In the following lemmas we show the estimate (6.2) with the help of the intermediate state \( \tilde{U} \). Without special explanation the positive constants in this section depend only on \( C, \theta \) and \( f \).

**Lemma 6.1.** Suppose that the system is genuinely nonlinear. For every positive number \( C \) there exists a number \( \theta > 0 \) depending only on \( f \) and \( C \) with the following property.

If \( U \) and \( \tilde{U} \) are two solutions in the class 1 and satisfy

\[
\begin{align*}
\text{esssup}_{(x,t) \in \mathbb{R} \times \mathbb{R}^+} |U(x,t)| < C; \\
\text{esssup}_{(x,t) \in \mathbb{R} \times \mathbb{R}^+} |\tilde{U}(x,t)| < C
\end{align*}
\]

and for any discontinuity

\[
|U^+ - U^-| < \theta; \\
|\tilde{U}^+ - \tilde{U}^-| < \theta,
\]

then there is a constant, so that

\[
\psi(\tilde{U}, \tilde{U}_{\hat{\omega}_+}) \geq \text{const} |\tilde{U} - \tilde{U}_{\hat{\omega}_+}|^2.
\]

**Proof:** Let \( R := \tilde{U} - U_{\hat{\omega}_+} \) and \( L = (L_1, L_2)^T \) be the left eigenvectors of \( \nabla f(U_{\hat{\omega}_+}) \) with \(|L_i| = 1, i = 1, 2\).

From the Lemma 3.1 follows that

\[
\psi(\tilde{U}, U_{\hat{\omega}_+})
\]

\[
= \langle R, [L(U_{\hat{\omega}_+})]^T \begin{pmatrix} \alpha' - \sigma_1(U_{\hat{\omega}_+}) & 0 \\ 0 & \alpha' - \sigma_2(U_{\hat{\omega}_+}) \end{pmatrix} L(U_{\hat{\omega}_+}) R \rangle,
\]

\[
= |R|^2 \sum_{i=1}^2 [\alpha' - \sigma_i(U_{\hat{\omega}_+})](\cos \rho_i)^2,
\]

where \( \rho_i \) with \( i = 1, 2 \) is the angle between \( L_i \) and \( R \).

In order to show the conclusion we show that

\[
\sum_{i=1}^2 [\alpha' - \sigma_i(U_{\hat{\omega}_+})](\cos \rho_i)^2 > \text{const} > 0
\]

(6.1.1)
holds.

Since the system is genuinely nonlinear,

\[ \tilde{a}' - \sigma_1(U_{\tilde{a}}) \bigg|_{t=0} > \text{const} \left| U_{\tilde{a}+} - U_{\tilde{a}-} \right|_{t=0}. \]

Then it follows from the entropy condition that

\[ \tilde{a}' - \sigma_1(U_{\tilde{a}}) \bigg|_{t=0} > \text{const} \left| U_{\tilde{a}+} - U_{\tilde{a}-} \right|_{t=0} > 0 \]

holds. The equality (5.3.1) means that \( U_{\tilde{a}} \bigg|_{t=0} = U_{\tilde{a}} \bigg|_{t=0} \) holds. It follows that

\[ \tilde{a}' - \sigma_1(U_{\tilde{a}}) \bigg|_{t=0} > \text{const} \left| U_{\tilde{a}+} - U_{\tilde{a}-} \right|_{t=0} > 0. \]

A short estimate shows that there is a constant, so that

\[ \left| \cos \rho_1 \right|_{t=0} > \text{const} > 0. \]

Then we have

\[ \left| \tilde{a}' - \sigma_1(U_{\tilde{a}}) \right| (\cos \rho_1)^2 \bigg|_{t=0} \geq \text{const} \left| U_{\tilde{a}+} - U_{\tilde{a}-} \right|_{t=0} > 0. \]

By using the Rankine-Hugoniot jump condition we obtain

\[ \left| \cos \rho_2 \right|_{t=0} > \text{const} \left| U_{\tilde{a}+} - U_{\tilde{a}-} \right|_{t=0}. \]

Then we have

\[ \sum_{i=1}^{2} \left| \tilde{a}' - \sigma_i(U_{\tilde{a}}) \right| (\cos \rho_i)^2 \bigg|_{t=0} \geq \text{const} \left| U_{\tilde{a}+} - U_{\tilde{a}-} \right|_{t=0} \geq \text{const} \left| \mathcal{U}_{\tilde{a}+} - 2 \mathcal{U}_{\tilde{a}} \right|_{t=0} \geq \text{const} \left| \mathcal{U}_{\tilde{a}+} - \mathcal{U}_{\tilde{a}-} \right|_{t=0}. \]

We choose \( \theta \) so small, that

\[ \sum_{i=1}^{2} \left| \tilde{a}' - \sigma_i(U_{\tilde{a}}) \right| (\cos \rho_i)^2 \bigg|_{t=0} > \text{const} \left| \mathcal{U}_{\tilde{a}+} - \mathcal{U}_{\tilde{a}-} \right|_{t=0} \]

holds, if \( \left| \mathcal{U}_{\tilde{a}+} - \mathcal{U}_{\tilde{a}-} \right| \leq \theta. \)

Thus there is \( \epsilon > 0 \), so that for \( 0 \leq t_1 \leq t_2 \leq \epsilon \) and \( t \in [t_1, t_2] \) the estimate (6.1.1) holds, since the right is a continuous function of \( t. \)
Lemma 6.2. Suppose that the system is genuinely nonlinear. For every positive number $C$ there exists a number $\theta > 0$ depending only on $f$ and $C$ with the following property.

If $U$ and $\mathcal{U}$ are two solutions in the class 1 and satisfy

\[ \operatorname{esssup}_{(x,t)\in \mathbb{R} \times \mathbb{R}^+} |U(x,t)| < C, \]
\[ \operatorname{esssup}_{(x,t)\in \mathbb{R} \times \mathbb{R}^+} |\mathcal{U}(x,t)| < C \]

and for any discontinuity

\[ |U^+ - U^-| < \theta, \]
\[ |\mathcal{U}^+ - \mathcal{U}^-| < \theta, \]

then there is a constant, so that

\[ \psi(U_{\Delta^+}, \mathcal{U}_{\Delta^+}) \geq \text{const} |\mathcal{U} - \mathcal{U}_{\Delta^+}|^2 - \text{const} \Delta^2, \]

where $\Delta(\cdot) := \bar{a}(\cdot) - a(\cdot)$.

**Proof:** We consider $\psi(\mathcal{U}, \mathcal{U}_{\Delta^+})$ as a perturbation of $\psi(U_{\Delta^+}, \mathcal{U}_{\Delta^+})$ and have the estimate

\[ |\psi(U_{\Delta^+}, \mathcal{U}_{\Delta^+}) - \psi(\mathcal{U}, \mathcal{U}_{\Delta^+})| \]
\[ \leq \text{const} |\mathcal{U} - U_{\Delta^+}| |\mathcal{U} - \mathcal{U}_{\Delta^+}| + \text{const} |\mathcal{U} - U_{\Delta^+}|^2 \]

It holds that

\[ |\mathcal{U} - U_{\Delta^+}| \leq |\mathcal{U}_{\Delta^-} - U_{\Delta^-}| \]
\[ = |\mathcal{U}_{\Delta^-} - \mathcal{U}_{\Delta^-}| \]
\[ \leq \text{const} \Delta \]

and

\[ |\mathcal{U} - U_{\Delta^+}|^2 \]
\[ \leq 2(|\mathcal{U} - U_{\Delta^+}|^2 + |U_{\Delta^+} - U_{\Delta^+}|^2) \]
\[ \leq \text{const} \Delta^2. \]

Then we have by use of the last lemma that

\[
\psi(U_{\Delta^+}, \mathcal{U}_{\Delta^+}) = \psi(\mathcal{U}, \mathcal{U}_{\Delta^+}) + [\psi(U_{\Delta^+}, \mathcal{U}_{\Delta^+}) - \psi(\mathcal{U}, \mathcal{U}_{\Delta^+})]
\geq \text{const} |\mathcal{U} - \mathcal{U}_{\Delta^+}|^2 - \text{const} \Delta |\mathcal{U} - \mathcal{U}_{\Delta^+}| - \text{const} \Delta^2.
\]

(6.2.1)

For any positive number $\mu$ it holds that

\[ \Delta |\mathcal{U} - \mathcal{U}_{\Delta^+}| \]
\[ \leq \mu^{-1} \Delta^2 + \mu |\mathcal{U} - \mathcal{U}_{\Delta^+}|^2. \]
We insert this estimation in (6.2.1). Obviously we can find a number \( \mu \), so that the conclusion holds.

As a corollary of this lemma we have

**Lemma 6.3.** Suppose that the system is genuinely nonlinear. For every positive number \( C \) there exists a number \( \theta > 0 \) depending only on \( f \) and \( C \) with the following property.

If \( U \) and \( \bar{U} \) are two solutions in the class 1 and satisfy

\[
\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |U(x,t)| < C, \\
\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |\bar{U}(x,t)| < C
\]

and for any discontinuity

\[
|U^+ - U^-| < \theta, \\
|\bar{U}^+ - \bar{U}^-| < \theta,
\]

then there is a constant, so that

\[
\psi(U_{\bar{U}}^+, \bar{U}_{\bar{U}}^+) \geq \text{const} |U_{\bar{U}}^+ - \bar{U}_{\bar{U}}^+|^2 - \text{const} \Delta^2.
\]

**Lemma 6.4.** Suppose that the system is genuinely nonlinear. For every positive number \( C \) there exists a number \( \theta > 0 \) depending only on \( f \) and \( C \) with the following property.

If \( U \) and \( \bar{U} \) are two solutions in the class 1 and satisfy

\[
\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |U(x,t)| < C, \\
\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |\bar{U}(x,t)| < C
\]

and for any discontinuity

\[
|U^+ - U^-| < \theta, \\
|\bar{U}^+ - \bar{U}^-| < \theta,
\]

then there is a constant, so that for any \( t \in [t_1, t_2] \)

\[
\int_{t_1}^t \Delta^2(\tau) \, d\tau \leq \text{const} \cdot (t - t_1)^2 \int_{t_1}^t |U_{\bar{U}}^+ - \bar{U}_{\bar{U}}^+|^2 \, d\tau.
\]
**Proof:** Taylor’s expansion leads to

\[ \Delta(t) \leq \int_{t_1}^{t} |\alpha'(\tau) - \bar{\alpha}'(\tau)| \, d\tau. \]

Since \( \alpha' \) is the eigenvalue of the matrix

\[ \int_{0}^{1} \nabla f(\delta(U_{\bar{\alpha}+} - U_{\bar{\alpha}-}) + U_{\alpha-}) \, d\delta \]

and \( \bar{\alpha}' \) the eigenvalue of the matrix

\[ \int_{0}^{1} \nabla f(\delta(U_{\bar{\bar{\alpha}+}} - \bar{U}_{\bar{\alpha}-}) + \bar{U}_{\alpha-}) \, d\delta, \]

it isn’t difficult to show that there is a constant, so that

\[ |\alpha'(\tau) - \bar{\alpha}'(\tau)| \leq \text{const}(|U_{\alpha+} - \bar{U}_{\alpha+}| + |U_{\alpha-} - \bar{U}_{\alpha-}|) \]
\[ \leq \text{const}(|U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}| + \Delta(t)). \]

Then we have

\[ \Delta(t) \leq \text{const} \left( \int_{t_1}^{t} \Delta(\tau) \, d\tau + \int_{t_1}^{t} |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}| \, d\tau \right). \]

By use of Schwartz’s inequality we obtain that

\[ \Delta^2(t) \leq \text{const} \cdot (t - t_1) \int_{t_1}^{t} [\Delta^2(\tau) + |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|^2] \, d\tau. \]

Then the conclusion follows, if \( t_1 \) and \( t_2 \) are sufficiently small and \( t \in [t_1, t_2] \). □

**Proof of the estimation (6.2):**

Putting the last two lemmas together we obtain that

\[ \int_{t_1}^{t} \psi(U_{\bar{\alpha}+}, \bar{U}_{\bar{\alpha}+}) \]
\[ \geq \left[ \text{const} - \text{const} \cdot (t - t_1)^2 \right] \int_{t_1}^{t} |U_{\bar{\alpha}+} - \bar{U}_{\bar{\alpha}+}|^2 \, d\tau. \]

If \( t_1 \) and \( t_2 \) are sufficiently small and \( t \in [t_1, t_2] \), we have the estimation (6.2). □

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