# Knot removal with radial function interpolation

Eidgenössische
Technische Hochschule
Zürich

Seminar für
Angewandte
Mathematik

# Knot removal with radial function interpolation

M. Buhmann and A. Le Méhauté[1]

Research Report No. 94-09
September 1994

Seminar für Angewandte mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

[1] Département I T I, ENST Bretagne, BP 832, 29285 Brest, France

# Knot removal with radial function interpolation

M. Buhmann and A. Le Méhauté [1]

Seminar für Angewandte mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 94-09 September 1994

**Abstract**

In this note we study interpolants to $n$-variate, real valued functions from radial function spaces, *i.e.*, spaces that are spanned by radially symmetric functions $\varphi(\|\cdot - x_j\|_2)$ defined on $\mathbb{R}^n$. Here $\|\cdot\|_2$ denotes the Euclidean norm, $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a given "radial (basis) function"which we take here to be $\varphi(r) = (r^2 + c^2)^{\beta/2}$, $-n \leq \beta < 0$, and the $\{x_j\} \subset \mathbb{R}^n$ are prescribed "centres", or knots. We analyse the effect of removing a knot from a given interpolant, in order that in applications one can see how many knots can be eliminated from an interpolant so that the interpolant remains within a given tolerance from the original one.

**AMS(MOS) Subject Classification:** 41A15, 41A30, 42C15, 65D15

[1] Département I T I, ENST Bretagne, BP 832, 29285 Brest, France

# Knot removal with radial function interpolation

Martin D. Buhmann and Alain Le Méhauté

**Abstract.** In this note we study interpolants to $n$-variate, real valued functions from radial function spaces, *i.e.*, spaces that are spanned by radially symmetric functions $\varphi(\|\cdot - x_j\|_2)$ defined on $\mathbb{R}^n$. Here $\|\cdot\|_2$ denotes the Euclidean norm, $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a given "radial (basis) function"which we take here to be $\varphi(r) = (r^2 + c^2)^{\beta/2}$, $-n \le \beta < 0$, and the $\{x_j\} \subset \mathbb{R}^n$ are prescribed "centres", or knots. We analyse the effect of removing a knot from a given interpolant, in order that in applications one can see how many knots can be eliminated from an interpolant so that the interpolant remains within a given tolerance from the original one.

## Enlèvement de nœuds et interpolation par fonctions radiales

**Résumé.** Nous analysons le résultat produit sur un interpolant par fonctions radiales en dimension quelconque lorsque l'on retire ou ajoute un nouveau point d'interpolation (nœud). Les fonctions radiales considérées ici sont de la forme $\varphi(\|\cdot - x_j\|_2)$, où $\|\cdot\|_2$ désigne la norme euclidienne de $\mathbb{R}^n$ avec $\varphi(r) = (r^2 + c^2)^{\beta/2}$, $-n \le \beta < 0$. Le théorème présenté peut être utilisé dans une méthode de réduction des bases de données par enlèvement des nœuds dont la contribution à l'interpolant global est inférieure à une tolérance donnée. Il est une contribution théorique au problème de la réduction des bases de données. De telles méthodes n'existent jusqu'à présent que pour des représentations polynomiales par morceaux (Bézier, B-splines).

**Version abrégée.**

Etant donné un ensemble $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ de points quelconques (non régulièrement répartis), on considère un interpolant par fonctions radiales de la forme

$$s(x) = \sum_{j=1}^{m} c_j \varphi(\|x - x_j\|_2), \quad \forall\, x \in \mathbb{R}^n, \tag{1}$$

où $\varphi$ est de la forme $\varphi(r) = (r^2 + c^2)^{\beta/2}$, $-n \le \beta < 0$, ce qui couvre en particulier les cas des multiquadriques inverses ($\beta = -1$) [1,2,10]. Les coefficients $c_j$ sont obtenus par résolution du système linéaire $s(x_j) = f(x_j), j = 1, 2, \ldots, m$. Ce problème a toujours une solution unique pourvu que les points soient distincts parce qu'alors la matrice d'interpolation $A' = \{\varphi(\|x_j - x_k\|_2)\}_{j,k=1}^m$ est toujours définie positive, et ceci quel que soit $n$ [8]. Les fonctions radiales ayant des supports non bornés, l'ajout d'un nouveau point d'interpolation $x_{new}$ se traduit par une modification globale du résultat. Désignant par $s_{old}$ l'interpolant sur les points distincts $x_1, x_2, \ldots, x_m$ et $s_{new}$ celui sur $x_1, x_2, \ldots, x_m, x_{new}$, ($x_{new} \notin \{x_i\}_{i=1}^m$) on obtient le résultat suivant:

**Théorème.** *Soit* $h = \max_{1\le j\le m} \min_{1\le k\le m} \|x_j - x_k\|_2$, $A = \{\varphi(\|x_j - x_k\|_2)\}_{j,k=0}^m$, $x_0$ *désignant* $x_{new}$*; soit* $\lambda(A)$ *la plus petite valeur propre de la matrice* $A$*. Il existe une constante* $C$ *dependant de* $M$ *et indépendante de* $h$*, de* $x_{new}$ *et de* $f_{new} := f(x_{new})$ *telle que*

$$\|s_{old} - s_{new}\|_\infty \le C\lambda(A)^{-1/2} |f_{new} - s_{old}(x_{new})| h^M,$$

*où* $M$ *est un entier arbitraire mais vérifiant* $\dim \mathbb{P}_n^M < m$, $\|\cdot\|_\infty$ *étant évalué sur un ensemble* $\Omega \subset \mathbb{R}^n$ *compact tel que* $\{x_j\}_{j=0}^m \subset \Omega$ *et* $\mathbb{P}_n^M$ *l'espace des polynomes en* $\mathbb{R}^n$ *d'ordre* $\le M$.

Même si la constante $C$ apparaît inconnue, elle sera explicite dans la preuve du théorème.

Les fonctions radiales étant invariantes par translation, on peut supposer $x_{new} = 0$. On montre alors que

$$s_{old}(x) - s_{new}(x) = (\alpha^T a' - \varphi(0))^{-1}(s_{old}(0) - f_{new})(\alpha^T a'(x) - \varphi(\|x\|_2)),$$

où $\alpha = (A')^{-1}a'$, $a' = \{\varphi(\|x_j\|_2)\}_{j=1}^m$ et $a'(x) = \{\varphi(\|x - x_j\|_2)\}_{j=1}^m$. On remarque d'abord que $(\alpha^T a' - \varphi(0))^{-1}$ est borné par $\left(\varphi(0)\sqrt{\lambda(A)}\right)^{-1} = c^\beta \lambda(A)^{-1/2}$, cf. [11]. Puis, on désigne par $E$ la fonction définie par $E(x) = \alpha^T a'(x) - \varphi(\|x\|_2)$, $\alpha^T a'(x)$ étant l'interpolant radial qui interpole $\varphi(\|x\|_2)$ en $\{x_j\}_{j=1}^m$. Cet interpolant minimise l'erreur d'approximation de $\varphi(\|x\|_2)$ par une combinaison linéaire quelconque de la forme

$$\tilde{s}(x) = \sum_{j=1}^m \varphi(\|x_j\|_2) c_j(x).$$

Par passage aux transformées de Fourier, on obtient que pour tout $\{c_j(x)\}_{j=1}^m$ :

$$|E(x)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\|y\|_2) \Big| 1 - \sum_{j=1}^m c_j(x) \exp(i(x - x_j) \cdot y) \Big| \, dy.$$

Ensuite, par un choix particulier de $\{c_j(x)\}_{j=1}^m$ et par application de la formule de Taylor, on aboutit à $\|E\|_\infty \leq C c^{-\beta} h^M$, ce qui donne l'inégalité desirée.

Des résultats analogues existent pour les fonctions radiales appelées splines polyharmoniques, de la forme

$$\varphi(r) = \begin{cases} r^{2\mathrm{N}-n} \log r, & n \text{ pair}, \mathrm{N} > \frac{n}{2}, \mathrm{N} \text{ entier}, \\ r^{2\mathrm{N}-n}, & \text{sinon, avec } \mathrm{N} > \frac{n}{2}, \end{cases}$$

et de leur version translatée du type $\varphi(\sqrt{r^2 + c^2})$, pour lesquelles on peut avoir $\mathrm{N} \leq \frac{1}{2} n$.

---

Radial function methods are multivariate interpolation and approximation methods which currently enjoy a large amount of interest by researchers and in applications [2]. Their aim is to interpolate a given $f \in C(\mathbb{R}^n), n > 0$, by a linear combination of the form

(1) $$s(x) = \sum_{j=1}^m c_j \varphi(\|x - x_j\|_2), \quad \forall\, x \in \mathbb{R}^n,$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is the prescribed radial (basis) function, $\{x_j\}_{j=1}^m \subset \mathbb{R}^n$ are given distinct points ("centres" or knots), and the coefficients $c_j$ are chosen such that

(2) $$f(x_j) = s(x_j), \qquad j = 1, 2, \ldots, m.$$

Such interpolants exists uniquely, e.g., for $\varphi(r) = \sqrt{r^2 + c^2}, c \geq 0$ (the multiquadric function) or $\varphi(r) = (r^2 + c^2)^{-1/2}$, $c > 0$ (inverse multiquadric), both of which have very similar approximation properties [1,2,10]. Sometimes a polynomial of degree $\mathrm{N}-1$, say, in $n$ variables $p \in \mathbb{P}_n^{\mathrm{N}-1}$, has to be added to (1) to make the system uniquely solvable [2–4,10], *i.e.*,

$$(3) \qquad s(x) = \sum_{j=1}^{m} c_j \varphi(\|x - x_j\|_2) + p(x), \quad \forall\, x \in \mathbb{R}^n,$$

the extra degrees of freedom being taken up by the conditions

$$(4) \qquad \sum_{j=1}^{m} c_j q(x_j) = 0, \quad \forall\, q \in \mathbb{P}_n^{\mathrm{N}-1},$$

where we require the knots to be such that they satisfy the non-degeneracy conditions

$$(5) \qquad \dim \mathbb{P}_n^{\mathrm{N}-1}|_{\{x_j\}_{j=1}^m} = \dim \mathbb{P}_n^{\mathrm{N}-1}.$$

An example is $\varphi(r) = r^2 \log r$, $\mathrm{N} = 2$, $n = 2$, the so-called thin-plate spline [3,4]. It is well-known that these approximations are highly useful, in particular thin-plate splines and multiquadric-type radial functions [2–4,10]. However, often when using an interpolation process to approximate one wants to decrease the number of the $\{x_j\}_{j=1}^m$ that are initially used, in order to facilitate the evaluation of (1) or (3) without changing the interpolant $s$ beyond a given uniform tolerance. More generally, one wants to eliminate knots from general approximants (not necessarily interpolants) in order to facilitate their computation. One reason for this may be that $s$ has to be evaluated often on a computer, once an initial interpolant or approximant has been found. We therefore study in this paper the effect of removing one centre at a time from a given interpolant or, equivalently, inserting one centre to a given interpolant. Such methods are only known so far for piecewise-polynomial approximants (Bézier, B-spline) [6,7]. For a further discussion of applications and another approach to make evaluations of radial function interpolants faster, see [9].

Now, given points $\{x_j\}_{j=1}^m$ and $x_{new} \notin \{x_j\}_{j=1}^m$, all in $\mathbb{R}^n$ and distinct, we compare in the uniform norm the radial function interpolant $s_{old}$ to $\{f(x_j)\}_{j=1}^m$ at $\{x_j\}_{j=1}^m$, with the interpolant $s_{new}$ that interpolates $f_{new} = f(x_{new})$ at $x_{new}$ as well.

We find an upper bound on the uniform distance between $s_{old}$ and $s_{new}$ that depends in a simple way on $x_{new}$ and $f_{new}$. Of course it depends on properties of the interpolation matrices $A' = \{\varphi(\|x_j - x_k\|_2)\}_{j,k=1}^m$ and $A = \{\varphi(\|x_j - x_k\|_2)\}_{j,k=0}^m$, $x_0$ standing for $x_{new}$, and on the spacing $h$ of the centres too. We make no claim as to the optimality of this bound, since the work on the subject is currently in progress.

We also restrict ourselves in this note to radial functions of the form

$$(6) \qquad \varphi(r) = (r^2 + c^2)^{\beta/2}, \quad -n \le \beta < 0, \quad c > 0,$$

for which interpolants (1) with no polynomials added always exist uniquely whenever the $x_j$ are pairwise distinct, for arbitrary $n$ [8]. It is also shown there that the interpolation matrices $A$, $A'$ and their inverses are positive definite. In the sequel, $\|\cdot\|_\infty$ always denotes the uniform norm restricted to a compactum $\Omega \in \mathbb{R}^n$ which contains all the centres including the new one, *i.e.*, $\|g\|_\infty$ denotes $\sup_{x \in \Omega} |g(x)|$ for all $g : \mathbb{R}^n \to \mathbb{R}$.

**Theorem.** *Under the stated conditions, we have the estimate*

$$\|s_{old} - s_{new}\|_\infty \le C\lambda(A)^{-1/2}|f_{new} - s_{old}(x_{new})|h^M, \tag{7}$$

*where* $h = \max_{1\le j\le m}\min_{1\le k\le m}\|x_j - x_k\|_2$, $A$ *is the matrix* $\{\varphi(\|x_j - x_k\|_2)\}_{j,k=0}^m$, $x_0$ *standing for* $x_{new}$, $\lambda(A)$ *is its smallest eigenvalue,* $M$ *is arbitrary but integral and so that* $\dim \mathbb{P}_n^M < m$, *and, finally,* $C$ *is a positive constant independent of* $h$, $x_{new}$ *and* $f_{new}$, *but dependent on* $M$.

This theorem gives a way to estimate the effect of inserting a knot $x_{new}$ to $s_{old}$ which of course also allows to estimate the effect of removing a knot from $s_{new}$.

We remark further that the constant $C$ will be made explicit in the proof of the theorem, so that it is, strictly speaking, no longer an unknown.

We remark finally that the condition $\dim \mathbb{P}_n^M < m$ is no real restriction because usually $m$ will be reasonably large in comparison to $M$.

**Proof:** We claim first that the left-hand side of (7) can be bounded by

$$\|s_{old} - s_{new}\|_\infty \le \lambda(A)^{-1/2}|f_{new} - s_{old}(x_{new})| \times \|\varphi(\|\cdot - x_{new}\|_2) - S\|_\infty, \tag{8}$$

where $S(x)$ is the radial function interpolant to $\varphi(\|x - x_{new}\|_2)$ at $\{x_j\}_{j=1}^m$. Without loss of generality we take $x_{new} = 0$. We let $E(x)$ denote $\varphi(\|x\|_2) - S(x)$. Then (8) becomes

$$\|s_{old} - s_{new}\|_\infty \le \lambda(A)^{-1/2}|f_{new} - s_{old}(0)| \times \|E\|_\infty. \tag{9}$$

In order to establish (9) we note that

$$s_{old}(x) - s_{new}(x) = (\alpha^T a' - \varphi(0))^{-1}(s_{old}(0) - f_{new})(\alpha^T a'(x) - \varphi(\|x\|_2)), \tag{10}$$

where $a'$ is the vector $\{\varphi(\|x_j\|_2)\}_{j=1}^m$, $\alpha = (A')^{-1}a'$, $a'(x) = \{\varphi(\|x - x_j\|_2)\}_{j=1}^m$, so that specifically $\alpha^T a'(x) = S(x)$. It is straightforward to verify (10) because of the linear independence of distinct translates of $\varphi(\|x\|_2)$, which is implied by the aforementioned nonsingularity results of [8], by checking that the right-hand side of (10) vanishes at $\{x_j\}_{j=1}^m$ and is $s_{old}(0) - f_{new}$ at $x = 0$, where one uses in particular that $\alpha^T a'(x_j) = S(x_j) = \varphi(\|x_j\|_2)$ for all $j = 1, 2, \ldots, m$. That the right-hand side of (10) is $s_{old}(0) - f_{new}$ at $x = 0$ is even easier to verify. Hence (10) is a consequence of the fact that both left- and right-hand side are linear combinations of translates of $\varphi(\|x\|_2)$ by $x_1, x_2, \ldots, x_m$. We note further that, according to [11], Theorem 2.1 (c),

$$(\alpha^T a' - \varphi(0))^2 = E(0)^2 \ge \left(\int_{\mathbb{R}^n} \widehat{\varphi}(\|y\|_2)\, dy\right)^2 \lambda(A) = \varphi(0)^2\lambda(A) = c^{2\beta}\lambda(A).$$

Here, $\widehat{\varphi}(\|y\|_2)$ is the distributional Fourier transform of $\varphi(\|x\|_2)$. The integral in the expression above is finite because $\widehat{\varphi}$ decays exponentially and $\widehat{\varphi}(\|y\|_2)$ is integrable over all regions of small $\|y\|_2$, $\|y\|_2 < 1$, say. Specifically, $\widehat{\varphi}(\|y\|_2)$ is

$$\frac{2\pi^{n/2}}{\Gamma\left(-\frac{1}{2}\beta\right)} K_{(n+\beta)/2}(c\|y\|_2)/\left(\tfrac{1}{2}\|y\|_2/c\right)^{(n+\beta)/2}$$

([2], for instance), where $K_{(n+\beta)/2}$ denotes a modified Besselfunction which decays exponentially and is positive on the positive axis. Precisely, we have $K_{(n+\beta)/2}(z) \sim z^{-(n+\beta)/2}$, $z \to 0$, (but $K_0(z) \sim -\log z$ for $z \to 0$) and $K_{(n+\beta)/2}(z) = O(\exp(-z)/\sqrt{z})$, $z \to \infty$. We thus just have to bound $|E(x)|$ from above by a constant multiple of $h^M$, uniformly in $x$, in order to obtain (7).

A suitable pointwise and uniform estimate for $E$ can be derived in the following fashion. According to [12],

$$|E(x)| \le |\tilde{s}(x) - \varphi(\|x\|_2)|,$$

where $\tilde{s}$ is *any* function of the form

$$\tilde{s}(x) = \sum_{j=1}^{m} \varphi(\|x_j\|_2) c_j(x).$$

The result in [12] states specifically that replacing $\tilde{s}$ by $S$ minimizes the expression

$$\frac{|\tilde{s}(x) - \varphi(\|x\|_2)|}{\int\limits_{\mathbb{R}^n} \widehat{\varphi}(\|y\|_2)\, dy},$$

so that the theory in [12] applies because $\varphi$ satisfies

$$0 < \int_{\mathbb{R}^n} \widehat{\varphi}(\|y\|_2)\, dy < \infty.$$

We get by taking Fourier transforms

$$|E(x)| \le \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\|y\|_2) \Big| 1 - \sum_{j=1}^{m} c_j(x) \exp(i(x - x_j)\cdot y) \Big|\, dy. \tag{11}$$

Specializing further, the left-hand term in (11) is at most

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\|y\|_2) \Big| 1 - \sum_{\ell=1}^{\tilde{m}} \tilde{c}_{j_\ell}(x) \exp(i(x - x_{j_\ell})\cdot y) \Big|\, dy \tag{12}$$

for any $\tilde{m}, \tilde{c}_{j_\ell}(x)$ which we choose *independent of* $m$ and in such a fashion that $\tilde{m} > \dim \mathbb{P}_n^M$, so we take $c_{j_\ell}(x) = \tilde{c}_{j_\ell}(x)$, $\ell = 1, 2, \ldots, \tilde{m}$, and $c_j(x) = 0$ otherwise. We choose $\tilde{m}$ and the $\tilde{c}_{j_\ell}(x)$ to be such that the term in modulus signs in (12) is bounded above by a fixed constant multiple of $h^M \|y\|_2^M$, where we also assume that $h \le \|x_{j_\ell} - x\|_2 \le 2\tilde{m}h$, $j = 1, 2, \ldots, \tilde{m}$, which can be satisfied for a suitable selection of $j_\ell \in \{1, 2, \ldots, m\}$, $\ell = 1, 2, \ldots, \tilde{m}$. The constant in front of $h^M \|y\|_2^M$ in this bound will be made explicit. The condition on the aforementioned term in modulus signs in (12) can be met by requiring that the $\tilde{c}_{j_\ell}(x)$ sum to 1 and

$$\sum_{\ell=1}^{\tilde{m}} \tilde{c}_{j_\ell}(x) \big(h^{-1}(x - x_{j_\ell})\big)^{\alpha} = 0, \quad \forall\, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n,\ 0 < |\alpha| < M, \tag{13}$$

where we are using standard multi-index notation for the multivariate power $(x - x_{j_\ell})^\alpha$ and for $|\alpha|$. Such a system of equations is always solvable for $\tilde{m} > \dim \mathbb{P}_n^M$, because of the linear independence of distinct polynomials. A bound $C_0$ on the sum of absolute values of the coefficients exists and can be computed explicitly if desired by forming the coefficients according to Cramer's rule as a fraction of two multivariate Vandermonde determinants (see, e.g., [5] for an algorithm for the fast computation of Vandermonde determinants). The form of the requirements (13), the upper and lower bounds on the distance between $x$ and $x_{j_\ell}$ and the continuity of the Vandermonde determinant show that the bound $C_0$ may be made independent of $h$, although it may still depend on $x$; a dependence that may be eliminated by taking the supremum over all $x$ within our compact domain $\Omega$.

Hence (12) is at most a multiple $C_0 C_1 h^M$ of $h^M$, where

$$\begin{aligned} C_1 &:= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}(\|y\|_2) \frac{2\cdot(2\tilde{m})^M}{M!} \|y\|_2^M \, dy \\ &= \frac{2^{M+2-n/2+\beta/2}\tilde{m}^M}{M!\pi^{n/2}\Gamma\big(-\frac{1}{2}\beta\big)} \int_{\mathbb{R}^n} K_{(n+\beta)/2}(c\|y\|_2)\|y\|_2^{M-n/2-\beta/2}\, dy \\ &= \frac{2^{M+3-n/2+\beta/2}\tilde{m}^M}{\sqrt{\pi}M!\Gamma\big(\frac{1}{2}(n-1)\big)\Gamma\big(-\frac{1}{2}\beta\big)} \int_0^\infty K_{(n+\beta)/2}(cr)r^{n/2-1+M-\beta/2}\, dr \\ &= \frac{2^{2M+1}c^{\beta/2-n/2-M}\tilde{m}^M\Gamma\big(\frac{1}{2}(n+M)\big)\Gamma\big(\frac{1}{2}(M-\beta)\big)}{\sqrt{\pi}M!\Gamma\big(\frac{1}{2}(n-1)\big)\Gamma\big(-\frac{1}{2}\beta\big)}, \end{aligned}$$

using the expansion of the exponential and (13). Letting $C := c^{-\beta} C_0 \cdot C_1$, this provides a *uniform upper bound* of $Cc^{-\beta}h^M$ for (11), as required in conjunction with our upper bound $c^\beta/\sqrt{\lambda(A)}$ on $|E(0)|^{-1}$, and supplying, finally, the desired bound on the right-hand side of (7) in the statement of the theorem. ■

It remains to add that more general results than the theorem which give error estimates of the type (7) are available. They apply to all polyharmonic splines

$$\varphi(r) = \begin{cases} r^{2\mathrm{N}-n}\log r, & n \text{ even, } \mathrm{N} > \frac{n}{2}, \mathrm{N} \text{ an integer,} \\ r^{2\mathrm{N}-n}, & \text{otherwise with } \mathrm{N} > \frac{n}{2}, \end{cases} \tag{14}$$

and their shifted versions $\varphi(\sqrt{r^2+c^2})$, of which (6) is a special case, but negative exponents are admitted in (6) because the shift avoids the singularity of (14) at the origin. Those results will be published elsewhere, as will computational experiments regarding the theoretical results. It should be noted that (14) and their shifted versions cover all of the most commonly studied and used radial functions [2,10].

## Réferences


1. Buhmann, M.D., Multivariate interpolation in odd-dimensional Euclidean spaces using multiquadrics, Constr. Approx. 6 (1990), 21–34.
2. Buhmann, M.D., New developments in radial basis function interpolation, in *Multivariate Approximation: From CAGD to Wavelets*, K. Jetter and F. Utreras, eds, World Scientific, Singapore, 1993, 35–75.

3. Duchon, J., Splines minimizing rotation-invariant semi-norms in Sobolev spaces, in *Multivariate Approximation Theory*, W. Schempp and K. Zeller, eds, Birkhäuser-Verlag, Basel, 1979, 85–100.
4. Duchon, J., Sur l'erreur d'interpolation des fonctions de plusieurs variables par les $D^m$-splines, RAIRO Anal. Num. 12, 4 (1978), 325–334.
5. Gohberg, I and V. Olshevsky, Fast inversion of Chebyshev–Vandermonde matrices, Numer. Math. 67 (1994), 71–92.
6. Le Méhauté, A., and Y. Lafranche, A knot removal strategy for scattered data in $\mathbb{R}^2$, in *Mathematical Methods in CAGD*, T. Lyche and L.L. Schumaker, eds, Academic Press, 1989, 419–425.
7. Lyche, T., and K. Mørken, Knot removal for parametric B-splines curves and surfaces, Computer Aided Geom. Design 4 (1987), 217–230.
8. Micchelli, C.A., Interpolation with scattered data: distance matrices and conditionally positive definite functions, Constr. Approx. 6 (1986), 11–22.
9. Powell, M.J.D., Tabulation of thin plate splines interpolants on a very fine two-dimensional grid, in *Numerical Methods in Approximation Theory, vol 9*, D. Braess and L.L. Schumaker, eds, Birkhäuser-Verlag, Basel, 1992, 221–244.
10. Powell, M.J.D., The theory of radial basis functions approximation in 1990, in *Advances in Numerical Analysis II*, W.A. Light, ed., Oxford University Press, 1992, 105–210.
11. Schaback, R., Error estimates and condition numbers for radial basis function interpolation, ms., 1993.
12. Schaback, R., Comparison of radial basis function interpolants, in *Multivariate Approximation: From CAGD to Wavelets*, K. Jetter and F. Utreras, eds, World Scientific, Singapore, 1993, 293–305.

Martin D. Buhmann, Mathematik Departement, ETH–Zentrum, CH–8092 Zürich.

Alain Le Méhauté, Département I T I, ENST Bretagne, BP 832, 29285 Brest.