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Author(s):
Jeltsch, Rolf

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Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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Abstract

We shall review stability requirements for time discretizations of ordinary and partial differential equations. If a constant time step is used and the method involves more than two time levels stability is always related to the location of roots of a polynomial in circular or half plane regions. In several cases the coefficients of the polynomial depend on a real or complex parameter. Hurwitz determinants allow to create a fraction free Routh array to test the stability of time discretizations. A completely different technique, called order stars, is used to relate accuracy of the schemes with their stability.

Keywords: stability of time discretizations, ordinary differential equations, partial differential equations, Von Neumann analysis, Routh algorithm, fraction free, order stars

Subject Classification: 12D10, 65L07, 65M12

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1 Introduction

When Hurwitz wrote the famous paper on the conditions which guarantee that all roots of a polynomial lie in the left half plane he was motivated by a control problem [10]. At that time numerical multi-step schemes for time discretizations such as Adams-Bashforth had already been published [2]. However since these schemes are stable by their design and they had been used for hand calculation only the problem of stiffness did not arise and the concept of stability regions was not created. To the authors knowledge most applications of the Hurwitz criterion to numerical time discretizations have been done in the second half of this century. We shall describe in Section 2 several stability problems where the Hurwitz criterion has been or could be used. However most researchers employed the Schur criterion [22] or after the transformation (2-1) the Routh algorithm. When the coefficients of the polynomial depend on parameters the Routh algorithm has the drawback that it involves divisions. Therefore scaled fraction free Routh algorithms had been introduced. It turns out that the Hurwitz determinants form a scaled fraction free Routh array with the slowest growth of the normalized degrees. This development is briefly treated in Section 3 where a Routh-type algorithm to compute the Hurwitz determinants in a fraction free way is given too.

In Section 4 we present the order star technique which relates stability to accuracy of time discretizations and thus the highly non linear Hurwitz criterion can be avoided.

2 Stability of time discretizations

Discretizations of time dependent problems involving differential equations lead always to recurrence relations. Usually the discretization is linear in the sense that it either involves the unknown function or the differential equation in a linear fashion. If the time step is constant and the differential equation is not a nonlinear partial differential equation then the growth of the solution of the recurrence relation can be analyzed by looking at the roots of polynomials. There is a large variety of examples and applications in this general frame work. Here we outline typical examples only. In the first subsection we deal with ordinary differential equations while in the second subsection we treat two examples involving partial differential equations.

2.1 Stability of discretizations of ordinary differential equations

For simplicity we start with a scalar differential equation. The initial value problem is

$$\begin{aligned} y' &= f(t,y), \quad 0 \le t \le T \\ y(0) &= y_0 \quad \text{given} . \end{aligned}$$

Let Δt be the time step, $t_n = n\Delta t$ and y_n is a numerical approximation to $y(t_n)$. A linear k-step scheme computes recursively y_{n+k} from the relation

$$\sum_{i=0}^{k} \alpha_i \; y_{n+i} = \Delta t \sum_{i=0}^{k} \beta_i \; f(t_{n+i}, y_{n+i}) \;\; n = 0, 1, 2, \dots \;\; .$$

α_i, β_i are fixed real numbers independent on t, Δt and y. A necessary condition for such a scheme to be **convergence**, i.e. $y_n \to y(t)$ as $\Delta t = t/n \to 0$, is that the scheme is **stable**. A linear multi-step method is **stable** if all roots of the characteristic polynomial

$$\rho(\zeta) = \sum_{i=0}^{k} \alpha_i \, \zeta^i$$

are inside the unit disk and the roots of modules 1 are simple. The transformation

$$(2-1) \qquad z = \frac{\zeta + 1}{\zeta - 1}, \quad \zeta = \frac{z+1}{z-1}$$

maps the unit disk of the ζ-plane into the left half plane. Hence a method is stable if and only if all roots of

$$r(z) = (z-1)^k \; \rho\Big(\frac{z+1}{z-1}\Big)$$

satisfy $Re\, z \leq 0$ and if one root has $Re\, z = 0$ then it is a simple root. Hence $r(z)$ is the first example where the Hurwitz criteria could be applied. For more details see [7], [9].

While the above stability is necessary for convergence, i.e. if $\Delta t \to 0$, for positive Δt an additional concept called **stability region** has to be introduced. This concept is best explained when applying a general linear method to the system

$$(2-2) \qquad y' = f(t, y), \qquad t \geq 0, \qquad y \in \mathbb{R}^m \; .$$

The class of general linear methods is very large. It contains among others the most popular methods such as linear multi-step and Runge-Kutta methods, see [8], p 313. For all these methods it makes sense when studying the growth of errors to consider the behavior of the scheme when applied to the leading term of the variational equation of (2-2). Let $y(t)$, $\varphi(t)$ be two exact solutions of (2-2). Then the difference $e(t) = y(t) - \varphi(t)$ satisfies

$$e' = J(t)\; e + \dots$$

where the Jacobian is

$$J(t) = \frac{\partial f}{\partial y}\; (t, \varphi(t)) \; .$$

Hence one studies the application of a method to

$$y' = J(t)\, y \qquad y \in \mathbb{R}^m, \; J \in \mathbb{R}^m \times \mathbb{R}^m \; .$$

To simplify the investigation one assumes J to be constant. If J is diagonizable then it is enough to study first the scalar equation

$$(2-3) \qquad y' = \lambda y\,, \quad \lambda \in \mathbb{C}\,.$$

Applying a general linear method leads to a recurrence relation where the growth of the solutions is governed by the roots ζ_i, $i = 1, 2, \ldots, k$ of a characteristic polynomial

$$\Phi(\zeta, \mu) = \sum_{i=0}^{k} \sum_{j=0}^{s} \alpha_{ij} \mu^j \zeta^i$$

where $\mu = \Delta t\, \lambda$ and $\alpha_{i,j}$ are method dependent real numbers. One defines the **stability region** S to be the following set

$$S := \left\{ \mu \in \overline{\mathbb{C}} \,\middle|\, \Phi(\zeta, \mu) = 0 \Longrightarrow \left\{ \begin{array}{ll} |\zeta| \le 1 & \\ |\zeta| = 1 & \text{then this is a simple root} \end{array} \right. \right\}.$$

With this definition the numerical solution of the scheme applied to (2-3) stays bounded for a fixed $\Delta t > 0$ as $n \to \infty$ if and only if $\mu \in S$. Again one can map the unit disk of the ζ-plane in the left half plane by the transformation (2-1) and introduces

$$\Psi(z, \mu) = (z-1)^k\, \Phi\Big(\frac{z+1}{z-1}\,,\ \mu\Big) = \sum_{i=0}^{k} a_{k-i}(\mu)\, z^i\,.$$

Hence

$$S = \left\{ \mu \in \overline{\mathbb{C}} \,\middle|\, \Psi(z, \mu) = 0 \Longrightarrow \left\{ \begin{array}{ll} Re\, z \le 0 & \\ Re\, z = 0\,, & \text{then this is a simple root} \end{array} \right. \right\}.$$

We therefore need to test for all $\mu \in \overline{\mathbb{C}}$ whether $\mu \in S$ or not, i.e. we need to show that all roots of $\Psi(\cdot, \mu)$ are in the left half plane. Again one can apply the Hurwitz criteria. In order to determine S one can plot candidates for boundary points ∂S by solving $\Psi(iy, \mu) = 0$ for μ for all $y \in \mathbb{R}$. This is a good way if one wants to obtain S numerically but it is impractical if one should prove results on S. in Section 4 we shall present the order star technique which can be used to prove properties of S. In many cases one has information on the spectrum of J and therefore it is often enough to show that a certain subset of $\overline{\mathbb{C}}$ belongs to S. This leads to different stability definitions such as

$$A_0 - \text{stability} \iff (-\infty, 0] \subset S, \ \text{ see Section 3}$$

$$A(\alpha) - \text{stability} \iff \Big\{ \mu \in \mathbb{C} \Big|\, |\arg(-\mu)| \le \alpha \Big\} \subset S$$

$$A - \text{stability} \iff \Big\{ \mu \in \mathbb{C} \Big|\, Re\, \mu \le 0 \Big\} \subset S\,.$$

Also one might be interested to have $D_r \subset S$ where D_r is the generalized circle

$$D_r = \left\{ \begin{array}{ll} \{\mu \in \mathbb{C} \,|\, |\mu + r| \le r\} & \text{if } r > 0 \\ \{\mu \in \overline{\mathbb{C}} \,|\, |\mu + r| \ge -r\} & \text{if } r < 0 \end{array} \right. .$$

In Section 2.2 we shall discuss stability of semi discretizations of convection-diffusion equations. There one might be interested in the following sets included in S:

$$\begin{aligned}
&\text{ellipses} && \Big\{\mu = \xi + i\eta \,\Big|\, \Big(\frac{\xi}{g}+1\Big)^2 + \Big(\frac{\eta}{c}\Big)^2 \le 1\Big\} \\
&\text{ovals} && \Big\{\mu = \xi + i\eta \,\Big|\, \Big(\frac{\xi}{g}+1\Big)^2 + \Big(\frac{\eta}{c}\Big)^4 \le 1\Big\} \\
&\text{parabolas} && \Big\{\mu = \xi + i\eta \,\Big|\, \xi + \Big(\frac{\xi}{c}\Big)^2 \le 0\Big\} .
\end{aligned}$$

In general if a set $\Omega \subset \overline{\mathbb{C}}$ is simply connected one has by the maximum principle that $\Omega \subset S$ if and only if $\partial\Omega \subset S$ and $\zeta(\mu)$ has no poles in Ω. Here $\zeta(\mu)$ is the algebraic function defined by

$$\Phi(\zeta(\mu), \mu) \equiv 0 .$$

Using again the transformation (2-1) we find for a simply connected Ω that $\Omega \subset S$ if and only if

i) $\partial\Omega \subset S$

and

ii) $\sum_{i=0}^{k} a_i(\mu) \neq 0$ for all $\mu \in \Omega$.

Again i) can be checked by an extension of the Hurwitz criteria to polynomials with complex coefficients, see e.g. [23], p. 179.

2.2 Stability of discretizations of partial differential equations

Here we shall treat two examples. The first is concerned with a general convection-diffusion equation in several space dimensions while the second one will be needed in the discussion, Section 4.

Example 1: Semi discretizations of convection-diffusion equations.

In this example we follow the article by Wesseling [26]. The convection-diffusion equation is given by

$$u_t + Lu = 0, \quad u = u(x,t),\ x(x_1, x_2 \ldots, x_m) \in \mathbb{R}^m,\ t \ge 0$$

where L is the linear operator

$$Lu = \sum_{\alpha=1}^{m} \Big(c_\alpha \frac{\partial}{\partial x_\alpha} - d\,\frac{\partial^2}{\partial x_\alpha^2}\Big)\, u .$$

Let us discretize (2-4) first in space-direction. Δx_α is the step-size and $e_\alpha = (0, 0, \ldots, 1, 0, \ldots, 0)$ the unit vector in direction x_α. Using for example central differences one obtains a system of infinitely many ordinary differential equations

$$(2-4) \qquad \frac{du_j}{dt} = -L_{\Delta x}\, u_j, \quad j = (j_1, j_2, \ldots, j_m) \in \mathbb{Z}^m$$

where

$$L_{\Delta x}\, u_j = \frac{1}{\Delta t}\left[\sum_{\alpha=1}^{m} \frac{\nu_\alpha}{2}\,(u_{j+e_\alpha} - u_{j-e_\alpha}) - \delta_\alpha(u_{j+e_\alpha} - 2u_j + u_{j-e_\alpha})\right],$$

the Courant number in direction x_α is $\nu_\alpha = c_\alpha\, \Delta t/\Delta x_\alpha$ and the diffusion number is $\delta_\alpha = d\, \Delta t/\Delta x_\alpha^2$. If δ_α is chosen differently the central difference for the advection part $c_\alpha\, \partial/\partial x_\alpha$ is replaced by the upwind differences. To make a Neumann stability analysis one defines the symbol $\hat{L}_{\Delta x}(\theta) = e^{-ij\theta}\, L_{\Delta x}\, e^{i<j,\theta>}$ where $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ and $<,>$ is the usual scalar product. Substitution of each mode $u_j = y(t)\, e^{i<j,\theta>}$ in (2-4) gives

$$\frac{dy}{dt} = -\hat{L}_{\Delta x}(\theta)\, y\ .$$

One introduces the set

$$S_L = \{-\Delta t\, \hat{L}_{\Delta x}(\theta) \in \mathbb{C} \mid \text{for all } \theta \in \mathbb{R}^m\}\ .$$

Theorem 1 *If one applies a time discretization with a stability region S to (2-4) then*

$$S_L \subset S$$

is sufficient for Neumann stability of the overall scheme.

This result is applied in the following way. For simple set Ω e.g. ellipses, ovals, parabolas, one shows

a) $S_L \subset \Omega$

b) $\Omega \subset S$.

This is one reason why we claimed in Section 2.1 that one is interested to show $\Omega \subset S$. In [26] Wesseling shows a) for many different sets Ω.

Example 2: Full discretization of the advection equation.

When analyzing stability of difference schemes for solving initial boundary value problems of systems of partial differential equations

$$u_t = A(x,t)\, \frac{\partial u}{\partial x} + B(x,t)\, u + f(x,t) \quad x \in [0,1],\ t \geq 0,\ u \in \mathbb{R}^N$$

it was shown in a series of papers [6], [20], [21] that it is enough to study the initial value problem

$$(2-5) \qquad \begin{aligned} &\frac{\partial u}{\partial t} = c \, \frac{\partial u}{\partial x}, \quad x \in \mathbb{R}, \; t \geq 0, \; u \in \mathbb{R} \\ &u(x,0) \text{ given .} \end{aligned}$$

We discretize simultaneously space and time. Let Δx, Δt be the step-sizes and $x_m = m \cdot \Delta x$, $t_n = n \cdot \Delta t$ the grid points. u_{nm} is a numerical approximation to the exact solution $u(x_m, t_n)$. A general k-step difference scheme has the form

$$(2-6) \qquad \sum_{\ell=-r_k}^{S_k} a_{k\ell} \, u_{n+k,m+\ell} + \sum_{j=0}^{k-1} \sum_{\ell=-r_j}^{s_j} a_{j\ell} \, u_{n+j,m+\ell} = 0$$

where $a_{j\ell}$ are method dependent coefficients which usually depend on the Courant number $\nu = c\Delta t/\Delta x$. A scheme is called explicit if $a_{k0} \neq 0$, $a_{k\ell=0}$ if $\ell \neq 0$. Let $\tilde{u}_n$ be the Fourier transform of u_n where

$$u_n(x) = u_{nm} \quad \text{if } \; |x - x_m| < \Delta \, x/2 \; .$$

With a similar interpretation (2-6) is an equation for all x and we can Fourier transform it. This gives

$$(2-7) \qquad \alpha_k(e^{i\Delta x\theta}) \, \tilde{u}_{n+k} + \sum_{j=0}^{k-1} \alpha_j(e^{i\Delta x\theta}) \, \tilde{u}_{n+j} = 0$$

where

$$\alpha_j(\mu) = \sum_{\ell=-r_j}^{s_j} a_{j\ell} \, \mu^\ell \; .$$

In order that one can solve (2-7) for $\tilde{u}_{n+k}$ it is necessary and sufficient that $\alpha_k(\mu)$ $\neq 0$ for $|\mu| = 1$. This makes it possible that we can always request the following **normalization conditions**:

a) $\alpha_k(1) = 1$

b) r_k, s_k are such that

$$\begin{aligned} r_k &= \text{the number of roots of } \alpha_k(\mu) \text{ with } |\mu| < 1 \\ s_k &= \text{the number of roots of } \alpha_k(\mu) \text{ with } |\mu| > 1 \; . \end{aligned}$$

(2-7) is a linear recurrence relation. Hence to study the growth of its solution one has to introduce the characteristic function

$$\Phi(\zeta, \mu) = \sum_{j=0}^{k} \alpha_j(\mu) \, \zeta^j \; .$$

Definition 1 *The method is* **stable** *if*

$$\left.\begin{array}{rcl}\Phi(\zeta,\mu) & = & 0\\ |\mu| & = & 1\end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} |\zeta| & \le & 1\\ \textit{if } |\zeta| & = & 1, \ \textit{then } \zeta \textit{ is a simple root}\,.\end{array}\right.$$

As in Section 2.1 we map the unit disk of the ζ-plane into the left half plane using the transformation (2-1). Let

$$\Psi(z) = (z-1)^k\, \Phi\Big(\frac{z+1}{z-1},\ \mu\Big) = \sum_{i=0}^{k} a_{k-i}\, z^i\ ,$$

then the method is stable if

$$\left.\begin{array}{rcl}\Psi(z) & = & 0\\ |\mu| & = & 1\end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} Re\, z & \le & 0\\ Re\, z & = & 0, \ \textit{then } z \textit{ is a simple root}\,.\end{array}\right.$$

a_i are polynomials in $\mu \in \mathbb{C}$ with $|\mu| = 1$ and are functions of ν with $\nu \in (0, \nu_0]$.

2.3 General stability problem

In the two previous subsections we have given several examples of discretizations of time dependent problems. In all cases the stability analysis leads to a polynomial

$$\Psi(z) = \sum_{i=0}^{k} a_{k-i}\, z^i = a_0\, z^k + a_1\, z^{k-1} + \ldots + a_k$$

where a_i is a polynomial in $\mu \in \mathbb{C}$ and possibly also a function of a real parameter ν. One needs to show that the roots of $\Psi(z)$ are in the left half plane for all μ on a curve $C \subset \mathbb{C}$. Such curves can be straight lines e.g. $\{\mu \in \mathbb{C}\,|\, Im\, \mu = 0,\ Re\, \mu \le 0\}$, circles, boundaries of ellipses, parabolas and ovals. In the next section we shall explain in a simple example how the Hurwitz determinants can be used to test for stability.

3 Hurwitz determinants and fraction free Routh algorithms

In this section we show how the Hurwitz criterion can be used to test stability of schemes. We take the particularly simple example of testing A_0-stability for a general linear method. This was also a part of criterias to test for stiff stability the author had devised in [12], [13], [14]. We therefore consider the special case.

$$(3-1) \qquad \Psi(z) = a_0\, z^n + a_1\, z^{n-1} + \ldots + a_n$$

where all a_i are real. The Routh-algorithm is then defined as follows:

Routh Algorithm Let $a_j = 0$ if $j > n$.

The first two rows of the Routh array are defined as follows

$$\begin{aligned} r_{0j} &= a_{2j-2} \quad \text{for} \quad j = 1,2,3,\ldots \\ r_{1j} &= a_{2j-1} \quad \text{for} \quad j = 1,2,3,\ldots \ . \end{aligned}$$

For $i = 2, 3, \ldots$ *one computes*

$$r_{ij} = -\frac{1}{r_{i-1,1}} \begin{vmatrix} r_{i-2,1} & r_{i-2,j+1} \\ r_{i-1,1} & r_{i-1,j+1} \end{vmatrix} . \tag{3-2}$$

□

Let us assume that the Routh array is regular, i.e. $r_{i1} \neq 0$ for $i = 0, 1, 2, \ldots$. The following Routh criterion is an easy way to test stability.

Theorem 2 *Routh-Criterion*
$\Psi(z)$ *has all zeros in the left half plane if and only if* $r_{01}, r_{11}, r_{21}, r_{31}, \ldots$ *have the same sign.* □

The disadvantage of this algorithm is that r_{ij} with $i > 1$ become rational functions of a_i, $i = 0, 1, 2, \ldots$. Hence if a_i are elements of a polynomial ring $J[\mu]$ then r_{ij} become rational functions. If one implements the algorithm with a symbol manipulation language computations become rather involved. Hence fraction free algorithms are introduced. Let $P[a]$ be the polynomial ring in the $n+1$ variables a_0, $a_1, \ldots, a_n$ and $R[a]$ the rational functions in the same variables.

Definition 2 $\{m_{ij}\}$ *is called a* **scaled fraction free Routh array** *if there exist scaling factors* $K_i \in R[a]$ *such that*

$$m_{ij} = K_i \, r_{ij} \in P[a] \ .$$

Example 3: Barnett in [1] suggested to multiply (3-2) through by the numerator. Hence one obtains

$$m_{ij} = -\begin{vmatrix} m_{i-2,1} & m_{1-2,j+1} \\ m_{i-1,1} & m_{i-1,j+1} \end{vmatrix} \quad \begin{aligned} i &= 2,3,\ldots \\ j &= 1,2,\ldots \ . \end{aligned}$$

Clearly $m_{ij} \in P[a]$. Let $\nu_i = \max_j\{\text{degree of } m_{ij} \in P[a]\}$. It is easy to see that ν_i are the Fibonacci numbers which grow exponentially.

Example 4. The **reduced *PRS* algorithm and the sub-resultant *PRS* algorithm** are fraction free versions of Euclid's algorithm [3], [4]. Hence they yield a fraction free Routh algorithm, see [15]. Contrary to Example 3 the normalized degrees ν_i grow only linearly. In fact $\nu_i = 2i - 1$, $i = 1, 2, \ldots$. See [15] for details.

The question is which scaled fraction free Routh array shows the slowest growth in ν_i. Here comes the relation to the famous paper by Hurwitz which is celebrated in this volume. Using Hurwitz's original notation the so called Hurwitz matrix is

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & \ldots & a_{2n-1} \\ a_0 & a_2 & a_4 & & a_{2n-2} \\ 0 & a_1 & a_3 & & \\ 0 & a_0 & a_2 & & \\ 0 & 0 & a_1 & & \\ \vdots & \vdots & \vdots & & \end{bmatrix} .$$

Let H_{ij} be the minor formed from the first i rows, the first $i-1$ columns and the $i-1+j$th column of H. It is well known that

$$\begin{aligned} H_{ij} &= H_{i-1,1}\, r_{ij} && \text{for } i = 2, 3, \ldots,\ j = 1, 2, 3, \\ H_{1j} &= r_{1j} = a_{2j-1} && j = 1, 2, 3, \ldots \\ H_{0j} &= r_{0j} = a_{2j-2} && j = 1, 2, 3, \ldots \ . \end{aligned} \tag{3-3}$$

Note that $H_{i-1,1}$ are called Hurwitz determinants. Due to (3-3) H_{ij} is a scaled fraction free Routh array. From the definition of H_{ij} if is obvious that $\nu_i = i$, $i = 1, 2, \ldots$. Hence one has linear growth and slower growth than in Example 4. Since Sylvester has shown that if n is odd and i even then $H_{i1}, H_{i2}, H_{i3}, \ldots$ have no common factor in $P[a]$ one cannot have a slower growth than $\nu_i = i$. Hence H_{ij} is the scaled fraction free Routh array where the growth of the normalized degrees ν_i is minimal.

Let us consider the following algorithm, see [14]

$$\begin{aligned} n_{0j} &= r_{0j} = a_{2j-2} && j = 1, 2, 3 \ldots \\ n_{1j} &= r_{1j} = a_{2j-1} && j = 1, 2, 3 \ldots \end{aligned} \tag{3-4}$$

$$\begin{aligned} n_{ij} &= -\frac{1}{d_i} \begin{vmatrix} n_{i-2,1} & n_{i-2,j+1} \\ n_{i-1,1} & n_{i-1,j+1} \end{vmatrix} \quad i = 2, 3, \ldots \ \ j = 1, 2, 3 \ldots \\ d_i &= \begin{cases} 1 & \text{for } i = 2, 3 \\ n_{i-3,1} & \text{for } i = 4, 5, 6, \ldots \end{cases} . \end{aligned} \tag{3-5}$$

This algorithm can be used to compute H_{ij} for one has the following

Theorem 3 *[15], [5]. Assume $n_{i1} \neq 0$ and the n_{ij} are computed by (3-4), (3-5). Then $n_{ij} = H_{ij}$.*

During the conference J. Garloff pointed out to the author that the algorithm (3-4), (3-5) and the theorem has been proved much earlier than in [15] by G. Fichera [5].

4 Order star technique

The Routh-Hurwitz criterion is very good to test stability of a particular scheme if no parameter is involved in the scheme. However if the scheme depends on parameters such as μ and ν discussed in Section 2 the high level of nonlinearity involved in the Routh-Hurwitz criterion makes it extremely difficult, if not impossible to prove general results. One disadvantage is that the technique does not take in account that the numerical schemes approximate the exact solutions. For example, when solving ordinary differential equations $y' = f(t, y)$ we have seen in Section 2.1 that it makes sense to consider the equation $y' = \lambda y$. The exact solution is $y(t) = y_0\, e^{\lambda t}$. Hence the exact solution satisfies the simple recurrence relation

$$y(t_n + \Delta t) = e^{\mu} y(t_n), \quad \text{where} \quad \mu = \lambda\, \Delta t \;.$$

Thus the dominant root ζ_1 of $\Phi(\zeta, \mu)$ should approximate e^{μ} for μ close to zero. In a similar fashion in Example 2 the dominant root of $\Phi(\zeta, \mu)$ should approximate μ^{ν} for μ close to 1. In 1978 Wanner, Hairer, Nørsett [25] developed the so called order star technique which relates this approximation property to the stability requirements. The basic idea is that for a special method or the exact solution stability and accuracy is formulated in a function theoretical frame work. If it is done for a special method this involves the algebraic function defined by $\Phi(\zeta, \mu) \equiv 0$. Then one compares the **general** scheme to this special method or the exact solution. This comparison usually calls for an application of the argument principle. The technique was very successful and many open problems concerning time discretization of ordinary and partial differential equations could be solved, [11], [8]. To explain the idea in more detail we restrict ourselves to the full discretization of the advection equation which has already be treated in Example 2.

4.1 Order stars of full discretizations of the advection equation

We have already formulated the stability of a method given by (2-6) in terms of Φ in Definition 1. The accuracy is measured by the error order p which is defined as follows. One substitutes any sufficiently differentiable function $u(x, t)$ into the left hand side of (2-6) and expand this in powers of Δx with the requirement that $\nu = c\Delta t/\Delta x$ is kept fixed, i.e.

$$(4-1) \qquad \sum_{j=0}^{k} \sum_{\ell=-r_j}^{s_j} a_{j\ell}\, u(x_{m+\ell}, t_{n+j}) = O\Big((\Delta x)^{p+1}\Big) \ .$$

Theorem 4 *[16], [24]. Let the method be stable and satisfy $\Phi(1,1) = 0$. Then the following three conditions are equivalent:*

i) The scheme has order p.

ii) $\Phi(\mu^\nu, \mu) = O\Big((\mu-1)^{p+1}\Big)$ as $\mu \to 1$.

iii) The algebraic function ζ given by $\Phi(\zeta(\mu), \mu) \equiv 0$ has exactly one branch $\zeta_1(\mu)$ which is analytic in a neighborhood of $\mu = 1$ and satisfies

$$(4-2) \qquad \mu^\nu - \zeta_1(\mu) = O\Big((\mu-1)^{p+1}\Big) \ \ \text{as } \mu \to 1 \ .$$

□

Hence we have expressed stability and accuracy both in terms of the algebraic function defined by $\Phi(\zeta, \mu) \equiv 0$.

To motivate the results one would like to prove we observe that the exact solution of $u_t = cu_x$ is constant along the characteristic lines $x + ct = \text{const}$. We introduce the stencil of a scheme by the set of indices:

$$(4-3) \qquad I = \{(j,\ell) \in \mathbb{Z} \times \mathbb{Z} : 0 \le j \le k,\ -r_j \le \ell \le s_j\} \ .$$

A difference stencil is called regular if the characteristic line through (x_m, t_{n+k}) does not pass through any other point $(x_{m+\ell}, t_{n+j})$, $(j,\ell) \in I$ [16]. The advection equation is of hyperbolic type and hence we know that information does not travel across characteristic lines. It turns out that due to the normalization conditions the characteristic line through the point (x_m, t_{n+k}) plays an important role. Let R, S be the number of stencil points to the left and right, respectively of this crucial characteristic line. With this definition the following conjecture has been made:

Conjecture: *Let a method of form (2-6) for the linear advection equation (2-5) be normalized and have a convex stencil. Then if the method is stable it implies that*

$$(4-4) \qquad p \le 2 \ \min\{R, S\} \ .$$

□

A method is **convex** if the convex hull of the set of indices does not contain a point $(j,\ell) \in \mathbb{Z} \times \mathbb{Z}$ which is not a element of I. For $k = 1$ this result has been proved using the order star technique, [18]. For $k = 2$ [19], [17] have proved for certain subclasses of schemes this conjecture.

4.2 Outline of the order star technique

We have already expressed stability and accuracy of the "general" scheme in terms of the algebraic function $\zeta(\mu)$ which is the solution of $\Phi(\zeta,\mu) \equiv 0$. We do the same with the "good" scheme $\hat{\zeta}(\mu)$ or the exact solution. Now one considers

$$\varphi(\mu) = \frac{\text{"general" scheme}}{\text{"good" or exact}} = \begin{cases} \dfrac{\zeta(\mu)}{\hat{\zeta}(\mu)} & \text{if } k = 1 \\[2ex] \dfrac{\zeta(\mu)}{\mu^{\nu}} & \text{if } k > 1 \end{cases}$$

and define the order star Ω by

$$\Omega = \{\mu \mid |\varphi(\mu)| > 1\} \; .$$

If $k = 1$ then Ω is a set of $\mathbb{C}$ and for $k > 1$ it has to be defined on the Riemann surface of the algebraic function $\zeta(\mu)$. Let Δ be the "unit" disk. Then stability can be formulated as follows.

Lemma 5 *The scheme is stable if and only if* $\Omega \cap \partial\Delta = \emptyset$. □

The accuracy can also be expressed in terms of the order star Ω using the principle branch $\zeta_1(\mu)$ and (4-2).

Lemma 6 *Assume the scheme is stable and has* $\Phi(1,1) = 0$. *Then it is of order* p *if and only if the order star* Ω *consists as the principle branch close to* $\mu = 1$ *of* $p+1$ *sectors with angle* $\pi/(p+1)$ *separated by* $p+1$ *sectors of the complement of* Ω *with the same angle.* □

As an illustration we give such an order star in the case $k = 2$ with $p = 8$ in Fig. 1.

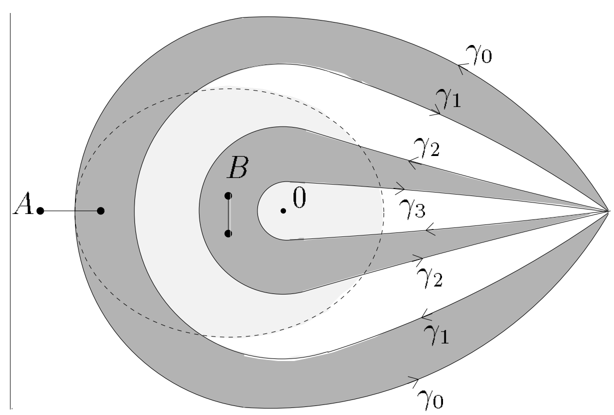

Fig. 1: Order star in the the case $k = 2$, $p = 8$. A and B denote cuts where the two sheets of the Riemann surface are connected

Let m be the number of sectors of Ω inside Δ. Hence Ω has at most $m+1$ sectors outside Δ. By Lemma 4.3 one obtains $p+1 \leq 2m+1$. Hence if we obtain a bound for m we obtain a bound for p. The argument principle is used to relate m to the location of poles of φ in Ω. To show this we do this considering the component Ω_1 given in Fig. 2.

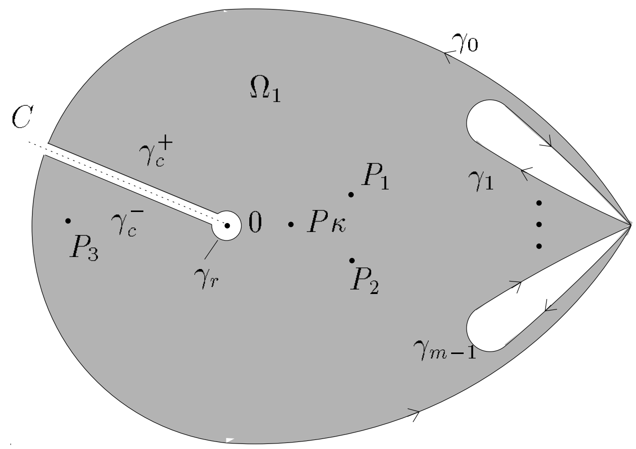

Fig. 2: Simple component Ω_1.

The important part of the order star theory is that by the Cauchy-Riemann differential equations it follows from the fact that $|\varphi|$ = constant along $\partial\Omega$ the argument of φ decreases along $\partial\Omega$. Hence

$$\frac{1}{2\pi i} \int_{\gamma_j} \frac{\varphi'}{\varphi}\, d\varphi \leq -1, \quad j = 1, 2, \ldots, m-1 \; .$$

Since $\varphi(\mu) = \zeta(\mu)/\mu^\nu$ has an essential singularity at $\mu = 0$ we have to make the cut c indicated in Fig. 2. Let α_1 be the leading exponent of φ at $\mu = 0$, i.e.

$$\varphi(\mu) = \frac{\zeta(\mu)}{\mu^\nu} = \mu^{\alpha_1} g(\mu), \quad g(0) \neq 0 \; .$$

We now apply the argument principle to the set $\Omega_1 \backslash \{cut\}$.

$$\begin{aligned} -\kappa &= \frac{1}{2\pi i} \int_{\partial(\Omega_1\backslash\{cut\})} \frac{\varphi'}{\varphi} \;, \\ &= \underbrace{\frac{1}{2\pi i}\int_{\gamma_0}}_{<0} + \underbrace{\frac{1}{2\pi i}\Big(\int_{\gamma_c^+} + \int_{\gamma_c^-}\Big)}_{\substack{=0 \\ \text{as } r \to 0}} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_r}}_{\leq \alpha_1} + \underbrace{\frac{1}{2\pi i}\int_{\gamma_1+\ldots\gamma_{m-1}}}_{\leq -(m-1)} \; . \end{aligned}$$

Hence

$$m \leq \lceil \alpha_1 \rceil + 1 + \kappa \; .$$

It remains to determine α_1 and κ. However due to the normalization we know that exactly r_k poles one in Δ. α_1 can be determined easily from the stencil of the scheme using the Newton-Puiseux diagram. Hence one gets a bound for m and thus for p. Unfortunately components of Ω can become rather complicated and thus one is able to prove the conjecture in certain cases only.

References

[1] Barnett S., *New reductions of Hurwitz determinants.* Int. J. Control, 18, 1973, 977-991.

[2] Bashforth F., *An attempt to test the theory of capillary action by comparing the theoretical and measured forms of drops of fluid. With an explanation of the method of integration employed in constructing the tables which give the theoretical form of such drops.* By J.C. Adams, Cambridge University Press 1883.

[3] Brown W.S., Traub J.F., *On Euclid's Algorithm and the Theory of Subresultants.* J. Assoc. Comput. Mach., **18**, 1971, 505-514.

[4] Collins G.E., *Subresultants and Reduced Polynomial Remainder Sequences.* J. Assoc. Comput. Mach., **14**, 1967, 128-142.

[5] Fichera G., *Alcune osservazioni sulle condizioni di stabilità per le equazioni algebriche a coefficienti reali.* Bolletino della Unione Matematica Italia, Ser. III, **2**, 1947, 103-109.

[6] Gustafsson B., Kreiss H.-O., Sundström A., *Stability theory of difference approximations for mixed initial boundary value problems II.* Math. Comp. **26**, 649-686, 1972.

[7] Hairer E., Nørsett S.P., Wanner G., *Solving ordinary differential equations I.* Springer, 1987.

[8] Hairer E., Wanner G., *Solving ordinary differential equations II.* Springer, 1991.

[9] Henrici P., *Discrete variable methods in ordinary differential equations.* J. Wiley & Sons, 1992.

[10] Hurwitz A., *Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt.* Mathematische Annalen XLVI, 273-284, 1895.

[11] Iserles A., Nørsett S.P., *Order stars.* Chapman & Hall, 1991.

[12] Jeltsch R., *Stiff stability and its relation to A_0- and $A(0)$-stability.* SIAM J. Numer. Anal. **13**, 1976, 8-17.

[13] Jeltsch R., *Stiff stability of multi-step multi-derivative methods.* SIAM J. Numer. Anal, **14**, 1977, 760-772.

[14] Jeltsch R., *Corrigendum to "Stiff stability of multi-step multi-derivative methods".* SIAM J. Numer. Anal, **16**, 1979, 339.

[15] Jeltsch R., *An optimal fraction free Routh array.* Int. J. Control, **30**, 1979, 653-660.

[16] Jeltsch R., Kiani P., Raczek K., *Counterexamples to a stability barrier.* Numer. Math., **52**, 301-316, 1988.

[17] Jeltsch R., Renaut R.A., Smit J.H., *An accuracy barrier for stable three-time-level difference schemes for hyperbolic equations.* Research Report No 95-01, 1995, Seminar für Angewandte Mathematik, ETH Zürich.

[18] Jeltsch R., Smit J.H., *Accuracy barriers of difference schemes for hyperbolic equations.* SIAM J. Numer. Anal. **24**, 1-11, (1987).

[19] Jeltsch R., Smit J.H., *Accuracy barriers of three-time-level difference schemes for hyperbolic equations.* Ann. University of Stellenbosch, 1992/2, 1-34, 1992.

[20] Kreiss H.-O., *Difference approximations for the initial-boundary value problem for hyperbolic differential equations, in: Numerical Solutions of Nonlinear Differential equations.* Proc. Adv. Sympos., Madison, Wisconsin, 141-166, 1966.

[21] Kreiss H.-O., *Stability theory for difference approximations of mixed initial-boundary value problem I.* Math. Comput. **22**, 703-714, 1968.

[22] Lambert J.D., *Computational methods in ordinary differential equations,* J. Wiley & Sons, 1973.

[23] Marden M., *Geometry of polynomials.* American Mathematical Society, 1966.

[24] Strang G., Iserles A., *Barriers to stability.* SIAM J. Numer. Anal. **20**, 1251-1257, 1983.

[25] Wanner G., Hairer E., Nørsett S.P., *Order stars and stability theorems.* BIT 18, 475-489, 1978.

[26] Wesseling P., *A method to obtain Neumann stability conditions for the convection-diffusion equation.* In: Numerical Methods for Fluid Dynamics V, K W Morton and M J Baines, editors, 211-224, Clarendon Press, 1995.