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Eidgenössische Technische Hochschule
CH-8092 Zürich
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Abstract

Most commonly used schemes for unsteady multidimensional systems of hyperbolic conservation laws use dimensional splitting. In each coordinate direction a scheme for a one dimensional system is used. Such an approach does not take in account the infinitely many propagation directions which are present in a system in several space dimensions. In 1992 M. Fey introduced what he called the **M**ethod of **T**ransport, MoT, for the Euler equations of gas dynamics. It is a finite volume method which uses the transport along characteristics. It does not compute fluxes across cellsides but from one cell to another. These type of schemes can be developed by first rewriting the Euler equation as a sum with integrals of infinitely many transport equations. One of these terms is related to the transport by the velocity while the integrals reflect the acoustic waves. In the numerical scheme the integrals are replaced by finite sums. The method can be modified such as to become a second order scheme. The technique can be applied to the magneto-hydrodynamic equations and the shallow water equation. Numerical examples for the shallow water equation are given.

Keywords: nonlinear hyperbolic conservation laws, multi-dimensional schemes, method of transport, second order, Euler equations of gas dynamics, shallow water equation, magneto-hydrodynamic equations

Subject Classification: 65M06, 65M25, 76N99, 76N20, 76W05

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1 Introduction

The traditional approach for the development of schemes to solve numerically systems of m hyperbolic conservation laws in several space dimensions was the following. First one develops a scheme for the one dimensional problem. For a good introduction to a large class of one dimensional schemes see [15]. All these schemes use in one way or another that, all the propagation speeds are finite. This fact is due to the hyperbolicity of the differential equations, i.e. the Jacobian of the flux function can be diagonalized. There are m real eigenvalues and a full set of eigenvectors. In the large class of Godunov-type schemes propagation speeds are computed implicitly by solving a Riemann problem exactly or approximately across a cell boundary. This approach is then extended to several space dimensions using the one-dimensional Godunov-type methods at each cell intervall.

One has to compute fluxes across cell sides and this is done as in the one-dimensional approach. The numerical approximation on each side of the cell is considered to be constant and the one-dimensional scheme is applied. Hence if one has a triangular mesh, information from one cell travels in the three space directions given by the normal vectors of the sides of the triangles, see Figure 1.

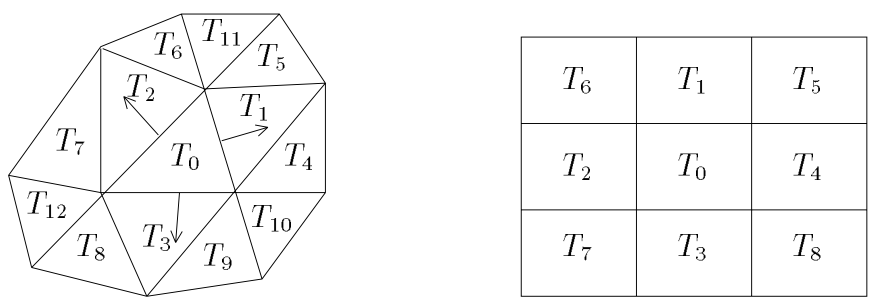

Figure 1: Discretization of space with triangles and rectangles

The triangle T_0 has only three neighboring triangles T_1, T_2, T_3 into which information travels in the next time step while the other nine neighbors don't get any information in one time step. In fact it needs three time steps to move information from T_0 to its neighbors T_{10}, T_{11}, T_{12}. One has a similar effect in a mesh with quadrilaterals. If the control volumes are of hexagonal shape each neighbor has one cell side in common with T_0. Thus, there is no neighbor which does not get information in one time step. Our approach here is different. Since most multidimensional systems propagate their information in infinitely many directions we try to model this situation. Fey considered in particular the Euler equations of gas dynamics. The sound waves propagating in all directions was the guide for his approach. He decomposed the state vector $\boldsymbol{U}$ in what he called three waves in each computational cell. Then he propagated them as sound waves in all directions $\boldsymbol{u} + \boldsymbol{n}c$, where $\boldsymbol{u}$ is the velocity of the fluid, c the speed of sound and $\boldsymbol{n}$ runs over all unit vectors. This lead to rather complicated and thus computationally complex integrals, for details see [4], [9]. For these reasons, in [8] a simplification was introduced. It turns out that this simplification can be interpreted as a "linearization" or better decomposition of the Euler equations into a finite number of transport equations. This new approach has the advantage that it shows the freedom to select certain directions and that this selection can be done independently of the mesh. Moreover any scheme for the linear advection equation can be applied to these transport equations. In addition one can add correction terms such that the overall scheme becomes second order accurate.

In Section 2 we derive the decomposition of the Euler equations into infinitely many transport equations and give ways how to reduce these infinitely many equations to a finite number of equations in Section 3. In Section 4 we show in one space dimension how correction terms can be incorporated such as to create a second order scheme. In Section 5 we describe the overall scheme and then give numerical examples in the last section.

2 Linearization of the Euler equations

We consider the Euler equations in two space dimensions which have the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial}{\partial x}\,\boldsymbol{F}(\boldsymbol{U}) + \frac{\partial}{\partial y}\,\boldsymbol{G}(\boldsymbol{U}) = 0\ , \tag{2.1}$$

where

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}\ ,\quad \boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}\ ,\quad \boldsymbol{G} = \begin{pmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}\ . \tag{2.2}$$

Here, ρ is the density, u, v the velocities in x and y directions, E is the total energy and p is the pressure. The system is closed with the equation of state

$$p = (\gamma - 1)\Big(E - \rho\,\frac{u^2+v^2}{2}\Big)\ . \tag{2.3}$$

By introducing the matrix

$$\mathcal{F}(\boldsymbol{U}) = (\boldsymbol{F}, \boldsymbol{G}) \tag{2.4}$$

we can write the Euler equation in the more compact form

$$\boldsymbol{U}_t + \operatorname{div}(\mathcal{F}) = 0\ , \tag{2.5a}$$

where the divergence acts on each row of $\mathcal{F}$. Using the velocity vector $\boldsymbol{u}^T = (u, v)$ one obtains the formula

$$\mathcal{F}(\boldsymbol{U}) = \boldsymbol{U}\boldsymbol{u}^T + c\boldsymbol{L}(\boldsymbol{U}) \ , \tag{2.5b}$$

where

$$\boldsymbol{L}(\boldsymbol{U}) = \frac{p}{c}\begin{pmatrix} \boldsymbol{0}^T \\ \boldsymbol{I} \\ \boldsymbol{u}^T \end{pmatrix} . \tag{2.5c}$$

Here, c is the speed of sound given by $c^2 = p\gamma/\rho$, $\boldsymbol{0}$ is the zero vector and $\boldsymbol{I}$ is the identity matrix. Observe that (2.5) represents a formulation of the Euler equations which is independent of the number N of space dimensions. $\boldsymbol{U}$ and $\boldsymbol{L}$ have to be modified in the obvious way, e.g.. $\boldsymbol{u}^T = (u, v)$ is replaced by $\boldsymbol{u}^T = (u, v, w)$ in three space dimensions. Then $\mathcal{F}$ becomes a 5×3 matrix. In what follows we shall use the dimension independent notation (2.5) and only if we want to exemplify formulas we shall do this using the case of two space dimensions.

In [4] Fey used physical arguments to motivate his decomposition of $\boldsymbol{U}$ in three parts. Each portion was then moved according its own propagation velocities leading to what he termed $\mathcal{U}$, $\mathcal{C}^+$ and $\mathcal{C}^-$-waves. While the $\mathcal{U}$-wave is related to the advection by the fluid velocity $\boldsymbol{u}$ the $\mathcal{C}^+$- and $\mathcal{C}^-$-waves are related to the acoustic waves. It is the aim of this section to decompose the Euler equation (2.5) directly in a similar fashion. One can write

$$\boldsymbol{U} = \boldsymbol{R}_1 + \boldsymbol{R}_2 \ , \tag{2.6}$$

where $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are functions of $\boldsymbol{U}$ given by

$$\boldsymbol{R}_2(\boldsymbol{U}) := \frac{\gamma - 1}{\gamma}\begin{pmatrix} \rho \\ \rho\boldsymbol{u} \\ \rho\boldsymbol{u}^T\boldsymbol{u}/2 \end{pmatrix} , \quad \boldsymbol{R}_1(\boldsymbol{U}) := \frac{1}{\gamma}\begin{pmatrix} \rho \\ \rho\boldsymbol{u} \\ \rho H \end{pmatrix} . \tag{2.7}$$

Here, H is the total enthalpy given by $H = (E + p)/\rho$. The functions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are natural extensions of the right eigenvectors of the Jacobian of the flux function in one space dimension, see [5], [8]. In this case the eigenvector $\boldsymbol{R}_2$ is transported by the speed of the fluid and hence in several space dimensions it is advected by the fluid velocity $\boldsymbol{u}$. In the original scheme by Fey [4] the transport associated with $\boldsymbol{R}_2$ from a computational cell Ω_0 to any point $\boldsymbol{x}$ in space was given by the $\mathcal{U}$-wave

$$\mathcal{U}_{\Omega_0}(\boldsymbol{x}, t + \Delta t) = \int_{\Omega_0} \boldsymbol{R}_2(\boldsymbol{U}(\boldsymbol{y}, t))\, \delta(\boldsymbol{x} - (\boldsymbol{y} + \Delta t\, \boldsymbol{u}(\boldsymbol{y}, t))\, d\boldsymbol{y}$$

where δ is the usual delta-function. Here we observe that the transport by $\boldsymbol{u}$ can be written by the transport equation

$$\phi_2 := \frac{\partial \boldsymbol{R}_2}{\partial t} + \operatorname{div}(\boldsymbol{R}_2\, \boldsymbol{u}^T) = 0 \ . \tag{2.8}$$

In one space dimension $\boldsymbol{R}_1$ is associated with the acoustic waves. In N space dimension the acoustic signal is propagated by $\boldsymbol{u}+c\boldsymbol{n}$ where $\boldsymbol{n}$ is a unit vector in $\mathbb{R}^N$. This leads to the transport equation

$$\phi_1(\boldsymbol{n}) := \frac{\partial \boldsymbol{R}_1}{\partial t} + \operatorname{div}(\boldsymbol{R}_1(\boldsymbol{u}^T + c\boldsymbol{n}^T)) = 0 \ . \tag{2.9}$$

Since the acoustic waves move in all directions we have to split $\boldsymbol{R}_1$ in (2.6) in all directions and integrate these portions over the unit sphere. With the identities

$$\frac{1}{|S|}\int_S ds = 1 \ \text{ and } \ \frac{1}{|S|}\int_S \boldsymbol{n}\, ds = 0 \ , \tag{2.10}$$

where S is the unit sphere in $\mathbb{R}^N$ and $|S|$ its surface, we can rewrite (2.6) as

$$\boldsymbol{U} = \boldsymbol{R}_2 + \frac{1}{|S|}\int_S \boldsymbol{R}_1\, ds \tag{2.11}$$

and combine equations (2.8) and (2.9) in the same way, i.e.

$$\phi_2 + \frac{1}{|S|}\int_S \phi_1\, ds = \boldsymbol{U}_t + \operatorname{div}(\boldsymbol{U}\boldsymbol{u}^T) = 0 \ . \tag{2.12}$$

Here we have used the identities in (2.10) and (2.11). Comparing (2.12) with the Euler equation (2.5) we observe that they differ by the term $c\boldsymbol{L}(\boldsymbol{U})$ in the flux function. This missing term is associated with the $\mathcal{C}^-$-wave of Fey. We make the Ansatz $\alpha\boldsymbol{L}\boldsymbol{n}$ for the vector to be transported where α is a scalar to be determined. The corresponding transport equation is

$$(\alpha\boldsymbol{L}\boldsymbol{n})_t + \operatorname{div}(\alpha\boldsymbol{L}\boldsymbol{n}(\boldsymbol{u}^T + c\boldsymbol{n}^T)) = 0 \ . \tag{2.13}$$

Clearly

$$\frac{1}{|S|}\int_S \boldsymbol{L}\boldsymbol{n}\, ds = 0 \tag{2.14}$$

and

$$\frac{1}{|S|}\int_S \boldsymbol{n}\boldsymbol{n}^T\, ds = \frac{1}{N}\boldsymbol{I} \ . \tag{2.15}$$

Hence, we rewrite (2.11) as

$$\boldsymbol{U} = \boldsymbol{R}_2 + \frac{1}{|S|}\int_S \boldsymbol{R}_1\, ds + \frac{\alpha}{|S|}\int_S \boldsymbol{L}\boldsymbol{n}\, ds \tag{2.16}$$

and combine the equations (2.8), (2.9) and (2.13) in the same way, i.e.

$$\boldsymbol{\phi}_2 + \frac{1}{|S|}\int_S \phi_1\, ds + \frac{\alpha}{|S|}\int_S \boldsymbol{L}\boldsymbol{n}\, ds = \boldsymbol{U}_t + \operatorname{div}(\boldsymbol{U}\boldsymbol{u}^T + \frac{\alpha}{N}c\boldsymbol{L}) = 0 \ . \tag{2.17}$$

Comparing this equation with the Euler equation (2.5) we find that they are identical for $\alpha = N$. Observe that this factor N for the $\mathcal{C}^-$-wave had already been derived by Fey [4] to make his scheme consistent with the Euler flux.

In conclusion we collect the result using the decomposition

$$\boldsymbol{U} = \boldsymbol{R}_2 + \frac{1}{|S|} \int_S \boldsymbol{R}_1 \, ds + \frac{N}{|S|} \int_S \boldsymbol{L}\boldsymbol{n} \, ds \tag{2.18}$$

and the linear advection equations

$$\boldsymbol{\phi}_2 = \frac{\partial \boldsymbol{R}_2}{\partial t} + \operatorname{div}(\boldsymbol{R}_2 \, \boldsymbol{u}^T) = 0 \tag{2.19a}$$

$$\boldsymbol{\phi}_1(\boldsymbol{n}) = \frac{\partial \boldsymbol{R}_1}{\partial t} + \operatorname{div}(\boldsymbol{R}_1(\boldsymbol{u}^T + c\boldsymbol{n}^T)) = 0 \quad \text{for all } \boldsymbol{n} \text{ with } \boldsymbol{n}^T \boldsymbol{n} = 1 \tag{2.19b}$$

$$\boldsymbol{\phi}_3(\boldsymbol{n}) = \frac{\partial (N\boldsymbol{L}\boldsymbol{n})}{\partial t} + \operatorname{div}(N\boldsymbol{L}\boldsymbol{n}(\boldsymbol{u}^T + c\boldsymbol{n}^T)) = 0 \quad \text{for all } \boldsymbol{n} \text{ with } \boldsymbol{n}^T \boldsymbol{n} = 1 \tag{2.19c}$$

we obtain the Euler equations back by composing the equation (2.19) in the same way as $\boldsymbol{U}$ was decomposed, i.e.

$$\boldsymbol{\phi}_2 + \frac{1}{|S|} \int_S \boldsymbol{\phi}_1(\boldsymbol{n}) \, ds + \frac{1}{|S|} \int_S \boldsymbol{\phi}_3(\boldsymbol{n}) \, ds = \boldsymbol{U}_t + \operatorname{div}(\mathcal{F}) = 0 \, . \tag{2.20}$$

Since the equations (2.19) are all linear transport equations we say that the equations (2.18), (2.19) represent a certain linearization of the Euler equations. In the next section we shall make use of this representation to create a numerical scheme. Finally observe that we could have combined (2.19 b) and (2.19 c) in one acoustic wave by defining

$$\boldsymbol{R}_a(\boldsymbol{n}) := \boldsymbol{R}_1 + N\boldsymbol{L}\boldsymbol{n} \tag{2.21}$$

which is transported by the advection equation

$$\boldsymbol{\phi}_a(\boldsymbol{n}) := \frac{\partial \boldsymbol{R}_a(n)}{\partial t} + \operatorname{div}(\boldsymbol{R}_a(\boldsymbol{u}^T + c\boldsymbol{n}^T)) = 0 \quad \text{for all } \boldsymbol{n} \text{ with } \boldsymbol{n}^T \boldsymbol{n} = 1 \, . \tag{2.22}$$

Hence

$$\boldsymbol{U} = \boldsymbol{R}_2 + \frac{1}{|S|} \int_S \boldsymbol{R}_a(\boldsymbol{n}) \, ds \tag{2.23}$$

and

$$\boldsymbol{\phi}_2 + \frac{1}{|S|} \int_S \boldsymbol{\phi}_a(\boldsymbol{n}) \, ds = \boldsymbol{U}_t + \operatorname{div}(\mathcal{F}) = 0 \, . \tag{2.24}$$

Observe that in one space dimension the integrals have to be replaced by sums with two terms since the unit sphere has only the two elements 1 and -1. We leave the details in this situation to the reader.

3 Decomposition in finitely many advection equations

If we would want to solve the Euler equations the disadvantage of the formulation (2.18), (2.19) or of (2.22), (2.23) and (2.24) is that we have to represent $\boldsymbol{U}$ by an integral and we have to solve infinitely many advection equations. We shall now try to replace the integrals by a finite sum of k terms and replace (2.22) by k advection equations. Hence, let us choose k unit vectors $\boldsymbol{n}_i \in \mathbb{R}^N$. We replace (2.22) and (2.23) by

$$\boldsymbol{U} = \boldsymbol{R}_2 + \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{R}_a(\boldsymbol{n}_i) \quad \text{and} \tag{3.1}$$

$$\boldsymbol{\phi}_a(\boldsymbol{n}_i) = \frac{\partial \boldsymbol{R}_a(\boldsymbol{n}_i)}{\partial t} + \operatorname{div}(\boldsymbol{R}_a(\boldsymbol{n}_i)(\boldsymbol{u}^T + c\boldsymbol{n}_i^T)) = 0, \quad i = 1, 2, \ldots, k \ . \tag{3.2}$$

We want the equation

$$\boldsymbol{\phi}_2 + \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{\phi}_a(\boldsymbol{n}_i) = \boldsymbol{U}_t + \operatorname{div}(\mathcal{F}) = 0 \tag{3.3}$$

to hold exactly. This sets certain requirements on the choice of the $\boldsymbol{n}_i$. Using (2.6) we find that (3.1) only holds if

$$\boldsymbol{R}_1 = \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{R}_a(\boldsymbol{n}_i) = \boldsymbol{R}_1 + \frac{N}{k}\, \boldsymbol{L} \sum_{i=1}^{k} \boldsymbol{n}_i \ . \tag{3.4}$$

Hence we need that

$$\sum_{i=1}^{k} \boldsymbol{n}_i = 0 \ . \tag{3.5}$$

In order that (3.3) holds we need that

$$\begin{aligned} \boldsymbol{U}\boldsymbol{u}^T + c\boldsymbol{L} &= \boldsymbol{R}_2\, \boldsymbol{u}^T + \frac{1}{k} \sum_{i=1}^{k} \boldsymbol{R}_a(\boldsymbol{n}_i)(\boldsymbol{u}^T + c\boldsymbol{n}_i^T) \\ &= \boldsymbol{U}\boldsymbol{u}^T + c\boldsymbol{L}\, \frac{N}{k} \sum_{i=1}^{k} \boldsymbol{n}_i\, \boldsymbol{n}_i^T \ . \end{aligned}$$

Therefore we obtain the additional equation

$$\frac{N}{k} \sum_{i=1}^{k} \boldsymbol{n}_i\, \boldsymbol{n}_i^T = \boldsymbol{I} \tag{3.6}$$

for the $\boldsymbol{n}_i$. Collecting these results we can represent the Euler equations (3.3) by a combination of the $k+1$ advection equations (2.8) and (3.2) if (3.5) and (3.6) hold. Let us look a different choices for the $\boldsymbol{n}_i$'s.

a) **One dimensional case: $N = 1$.** Since the unit sphere has two elements we can choose $k = 2$ and $\boldsymbol{n}_1 = (1)$ and $\boldsymbol{n}_2 = (-1)$. One obtains three advection equations which transport the three eigenvectors of the Jacobian of the flux function, $\boldsymbol{R}_2$, $\boldsymbol{R}_a(\boldsymbol{n}_1)$, $\boldsymbol{R}_a(\boldsymbol{n}_2)$ with the characteristic speeds, u, $u + c$ and $u - c$. Applying Fey's scheme to this choice of $\boldsymbol{n}_i$'s leads to the scheme of Steger and Warming.

b) **Two dimensional case: $N = 2$.** A very natural choice for the $\boldsymbol{n}_i$'s would be to choose $\boldsymbol{n}_i$ in direction of the coordinate axis, i.e. $k = 4$ and

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{n}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \boldsymbol{n}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{3.7a}$$

Note, that this choice is **not** related to the dimensional splitting approach. The final propagation directions $(\boldsymbol{u} + \boldsymbol{n}_i c)$ are in general not alligned with the coordinate axes. The advantage of the linearization presented here is that we can also choose another set of $\boldsymbol{n}_i$'s, e.g. $k = 4$ and

$$\boldsymbol{n}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{n}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \boldsymbol{n}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \boldsymbol{n}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{3.7b}$$

If we apply Fey's approach using the $\boldsymbol{n}_i$'s given in (3.7b) we are lead to the so called simplified method of transport by Fey. Observe that due to equation (3.5) we can write the acoustic transport equations as

$$\frac{\partial \widetilde{\boldsymbol{R}}_a(\tilde{\boldsymbol{n}}_i)}{\partial t} + \operatorname{div}(\widetilde{\boldsymbol{R}}_a(\tilde{\boldsymbol{n}}_i)(\boldsymbol{u}^T + c\tilde{\boldsymbol{n}}_i^T)) = 0 \qquad i = 1, 2, 3, 4 \tag{3.8}$$

where

$$\widetilde{\boldsymbol{R}}_a(\tilde{\boldsymbol{n}}) = \boldsymbol{R}_1 + \boldsymbol{L}\tilde{\boldsymbol{n}} \tag{3.9}$$

and

$$\tilde{\boldsymbol{n}}_i = \sqrt{N}\,\boldsymbol{n}_i \qquad i = 1, 2, 3, 4\ . \tag{3.10}$$

c) **Three dimensional case: $N = 3$.** Again we could choose the $\boldsymbol{n}_i$'s in direction of the coordinate axis which would lead to $k = 6$. For a Cartesian grid it would however be better to choose $\boldsymbol{n}_i = \frac{1}{\sqrt{N}} \cdot \tilde{\boldsymbol{n}}_i$ with

$$\tilde{\boldsymbol{n}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tilde{\boldsymbol{n}}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \tilde{\boldsymbol{n}}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \tilde{\boldsymbol{n}}_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \tilde{\boldsymbol{n}}_5 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$
$$\tilde{\boldsymbol{n}}_6 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \tilde{\boldsymbol{n}}_7 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \tilde{\boldsymbol{n}}_8 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Hence $k = 8$. One obtains eight transport equations (3.8) with (3.9).

Before we end this section let us generalize the above approach slightly. The choice of $\boldsymbol{R}_a(\boldsymbol{n})$ in (2.21) was motivated by the exact Euler equations. For the numerical modelling of the acoustic waves we make the Ansatz

$$\boldsymbol{R}_{an}(\alpha, \boldsymbol{\nu}) = \boldsymbol{R}_1 + \alpha \boldsymbol{L}\boldsymbol{\nu}, , \quad \alpha \in \mathbb{R}, \quad \boldsymbol{\nu} \in \mathbb{R}^N \tag{3.11}$$

and combine these waves such that

$$\boldsymbol{R}_1 = \sum_{i=1}^{k} \omega_i \; \boldsymbol{R}_{an}(\alpha_i, \boldsymbol{\nu}_i) \; . \tag{3.12}$$

There seem to be some redundant parameters in (3.12) since $\boldsymbol{\nu}_i$ needs not to be a unit vector any more. However we want to be able to have an advection equation of the form

$$\boldsymbol{\phi}_{an}(\boldsymbol{\nu}_i) = \frac{\partial \boldsymbol{R}_{an}(\alpha_i, \boldsymbol{\nu}_i)}{\partial t} + \operatorname{div}(\boldsymbol{R}_{an}(\alpha_i, \boldsymbol{\nu}_i)(\boldsymbol{u}^T + c\boldsymbol{\nu}_i^T) = 0, \quad i = 1, 2, \ldots, k \; , \tag{3.13}$$

i.e. we want to be able to advect by $\boldsymbol{u}^T + c\boldsymbol{\nu}_i^T$. Let us derive the conditions on ω_i, α_i and $\boldsymbol{\nu}_i$ such that

$$\boldsymbol{U} = \boldsymbol{R}_2 + \sum_{i=1}^{k} \omega_i \; \boldsymbol{R}_{an}(\alpha_i, \boldsymbol{\nu}_i) \tag{3.14}$$

and

$$\boldsymbol{\phi}_2 + \sum_{i=1}^{k} \omega_i \; \boldsymbol{\phi}_{an}(\alpha_i, \boldsymbol{\nu}_i) = \boldsymbol{U}_t + \operatorname{div}(\mathcal{F}) = 0 \; . \tag{3.15}$$

Clearly, (3.14) directly implies that

$$\sum_{i=1}^{k} \omega_i = 1 \tag{3.16}$$

and

$$\sum_{i=1}^{k} \omega_i \, \alpha_i \, \boldsymbol{\nu}_i = 0 \; . \tag{3.17}$$

(3.15) holds if

$$\sum_{i=1}^{k} \omega_i \, \alpha_i \, \boldsymbol{\nu}_i \, \boldsymbol{\nu}_i^T = I \tag{3.18}$$

and

$$\sum_{i=1}^{N} \omega_i \, \boldsymbol{\nu}_i = 0 \; . \tag{3.19}$$

Since the Ansatz (3.11), (3.13), (3.14) is more general than (2.21), (3.1), (3.2) all representations of the Euler equations discussed so far are included in the approach (3.11), (3.13), (3.14). For example with the choice

$$\omega_i = \frac{1}{k} \; , \quad \boldsymbol{\nu}_i = \boldsymbol{n}_i \; , \quad \alpha = N \tag{3.20}$$

we get the formulas (3.1), (3.2), (3.3), (3.4). With the choice

$$\omega_i = \frac{1}{k}\,, \quad \boldsymbol{\nu}_i = \tilde{\boldsymbol{n}}_i\,, \quad \alpha = 1 \tag{3.21}$$

we obtain the formulas (3.8), (3.9), (3.10). However the new approach gives more freedom to counteract the geometric imbalance introduced by the mesh. For example, in two space dimensions we could choose $k = 8$,

$$\begin{aligned} &\boldsymbol{\nu}_1 = (1,0)^T,\ \boldsymbol{\nu}_2 = (0,1)^T,\ \boldsymbol{\nu}_3 = (-1,0)^T,\ \boldsymbol{\nu}_4 = (0,-1)^T, \\ &\boldsymbol{\nu}_5 = (1,1)^T,\ \boldsymbol{\nu}_6 = (-1,1)^T,\ \boldsymbol{\nu}_7 = (-1,-1)^T,\ \boldsymbol{\nu}_8 = (1,-1)^T, \end{aligned} \tag{3.22}$$

and the weights

$$\omega_i = \begin{cases} \omega & i \le 4 \\ \frac{1}{4} - \omega & i > 4 \end{cases} \tag{3.23}$$

and

$$\alpha_i = \begin{cases} 2 & i \le 4 \\ 1 & i > 4 \end{cases}\ . \tag{3.24}$$

With this, choice (3.16), (3.17), (3.19) are satisfied. (3.18) reduces to

$$2\omega \cdot 2\boldsymbol{I} + (\frac{1}{4} - \omega) \cdot 4\boldsymbol{I} = \boldsymbol{I}\ .$$

Hence, the formulas based on (3.22), (3.23), (3.24) are valid for all real ω. If $\omega = 0$ one obtains the case (3.7a) while for $\omega = \frac{1}{4}$ one obtains (3.7b). We get therefore a class of representations for the Euler equations with a free parameter ω. This freedom is even more important in view of the high order extension. Special choices of α_i and ω_i will reduce the number and amplitude of the correction terms introduced in the next section.

4 Second order extension of the decomposition

In Section 2 we have decomposed the Euler equations in infinitely many transport equations and in Section 3 we have approximated the Euler equation by finitely many transport equations. Numerically we shall use this approximation to find a numerical solution at time $t + \Delta t$ given that the solution at time t was exact. The error committed by this process is of the order Δt^2. We shall illustrate this statement in this section and then show how one can modify the finite number of advection equations to get an error which is of order Δt^3 only. To do this we shall restrict ourselves to the one-dimensional case and to the density variable. We shall expand the exact solution $\rho(x, t+\Delta t)$ in a Taylor series and omit the argument if it is x, t. For the exact solution we have

$$\rho(x, t+\Delta t) = \rho + \rho_t\,\Delta t + \rho_{tt}\,\frac{\Delta t^2}{2!} + O(\Delta t^2)\,. \tag{4.1}$$

Using the conservation of mass and momentum we obtain

$$\begin{aligned} \rho_t &= -(\rho u)_x \\ \rho_{tt} &= -(\rho u)_{tx} = (\rho u^2 + p)_{xx} = (\rho(u^2 + \frac{c^2}{\gamma}))_{xx}\,. \end{aligned} \tag{4.2}$$

To transport the solution by the finite number of transport equations (3.2) we have to choose first the $\boldsymbol{n}_i$. In the one-dimensional case there is only one choice namely $\boldsymbol{n}_1 = (1)$ and $\boldsymbol{n}_2 = (-1)$. The first component ρ of $\boldsymbol{U}$ has to be decomposed according to (3.1) in

$$\rho = \rho_1 + \rho_2 + \rho_3 \tag{4.3}$$

where $\rho_1 = \rho_3 = \frac{1}{2\gamma}\,\rho$ and $\rho_2 = \frac{\gamma-1}{\gamma}\,\rho$. For each part we have the transport equation

$$(\rho_{1/3})_t + ((u \pm c)\,\rho_{1/3})_x = 0 \quad \text{and} \quad (\rho_2)_t + (u\rho_2)_x = 0\,. \tag{4.4}$$

If we sum up the Taylor expansions

$$\rho_i(x, t+\Delta t) = \rho_i + (\rho_i)_t\,\Delta t + (\rho_i)_{tt}\,\frac{(\Delta t)^2}{2!} + O(\Delta t^3)$$

and use

$$\begin{aligned} (\rho_{1/3})_{tt} &= \Big(((u \pm c)((u \pm c)\,\rho_{1/3})_x\Big)_x \\ (\rho_2)_{tt} &= (u(\rho_2)_x)_x \end{aligned}$$

we obtain for the solution $\tilde{\rho}$ constructed using (3.1), (3.2)

$$\tilde{\rho}(x, t+\Delta t) = \rho + \rho_t\,\Delta t + \Big(\rho_{tt} + \frac{\rho}{2}\,(\gamma u u_x + c c_x)\Big)\,\frac{\Delta t^2}{2!} + O(\Delta t^3)\,. \tag{4.5}$$

Comparing (4.1) and (4.5) shows that the solution of the Euler equation and the one by the "linearized" system (3.1), (3.2) differ by $O(\Delta t^2)$ after time Δt has passed. The idea is now that we modify the equation (3.1), (3.2) such that the solution of these modified equations approximate $\rho(x, t+\Delta t)$ up to a term of size $O(\Delta t^3)$. Let us replace $\boldsymbol{R}_a(\boldsymbol{n}_i)$ by

$$\boldsymbol{R}_{a,2}(\boldsymbol{n}_i) = \boldsymbol{R}_1 + (\boldsymbol{L} + \boldsymbol{K})\,\boldsymbol{n}_i\,. \tag{4.6}$$

Hence (3.1) is transformed into

$$\boldsymbol{U} = \boldsymbol{R}_2 + \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{R}_{a,2}(\boldsymbol{n}_i) \tag{4.7}$$

and (3.2) into

$$\frac{\partial \boldsymbol{R}_{a,2}(\boldsymbol{n}_i)}{\partial t} + \operatorname{div}(\boldsymbol{R}_{a,2}(\boldsymbol{n}_i)(\boldsymbol{u}^T + c\boldsymbol{n}_i^T)) = 0, \quad i = 1, 2, \ldots, k \tag{4.8}$$

It turns out that in the 1−D case with $\boldsymbol{n}_1 = (1)$, $\boldsymbol{n}_2 = (-1)$ the solution constructed by (4.7), (2.19 a) and (4.8) gives an error term of size $O(\Delta t^3)$ if $\boldsymbol{K}$ is chosen as follows

$$\boldsymbol{K} = \begin{pmatrix} k^\rho \\ k^m \\ k^E \end{pmatrix} = \begin{pmatrix} -\frac{\Delta t\,\rho}{2\,c}(\gamma\,u\,u_x + c\,c_x) \\ -\frac{\Delta t}{2}\rho((\gamma-2)c\,u_x + u\,c_x) + u\,k^\rho \\ -\frac{\Delta t\,\rho\,c}{2\gamma(\gamma-1)}(u\,u_x - c\,c_x) + u\,k^m - \frac{u^2}{2}\,k^\rho \end{pmatrix} \tag{4.9}$$

The advantage of this approach is that the structure of the advection equations is not changed. Hence, the implementations in a numerical scheme is the same as without the correction term. Clearly in the multidimensional case the Taylor series expansions corresponding to (4.5) depend on the choices of the ω_i, α_i, $\boldsymbol{\nu}_i$. However correction terms $\boldsymbol{K}$ can always be found. For example in the Euler case in two space dimensions see [6] and for the shallow water equation see [10].

5 The numerical realization of the transport

So far we have replaced the nonlinear Euler equations by a finite, $k+1$, set of linear advection equations such that if one starts with the same solution at t the solutions differ by $O(\Delta t^3)$ at time $t + \Delta t$. Hence if the linear advection equations are solved exactly we obtain a second order scheme in time and space. Clearly it is enough to have a scheme to solve the scalar equation

$$u_t + \operatorname{div}(u\boldsymbol{a}^T) = 0 \tag{5.1}$$

where $\boldsymbol{a}$ is the local advection velocity which is a function depending on the space variable only. Schemes to solve this problem have been around for a long time, [2], [14], [3]. Here, we follow the approach of [11]. For simplicity we restrict ourselves to the two dimensional case. Assume we have a Cartesian grid with the step size Δx

and Δy in x and y direction. Let $x_i = i\Delta x$ and $y_j = j\Delta y$ and (x_i, y_j) is the center of the finite volume $V_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$. Let

$$u_{ij}^n = \frac{1}{|V_{ij}|} \int_{V_{ij}} u(x, y, n\Delta t)\, dxdy \tag{5.2}$$

be the average value of u over the cell V_{ij}. We assume now that in each control volume the solution is constant and has the value u_{ij}^n. If we assume that $\boldsymbol{a}(x, y)$ is also constant in this volume, e.g. has the value $\boldsymbol{a}(x_i, y_j)$, and we ignore effects from neighboring cell then we can say that the quantity u in V_{ij} is transported by $\Delta t\, \boldsymbol{a}(x_i, y_j)$, see Fig. 2

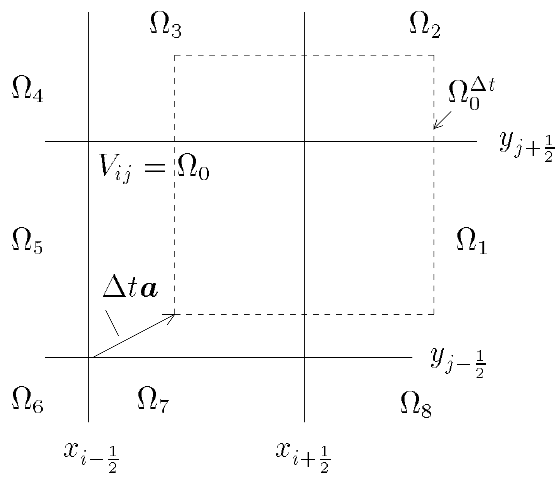

Figure 2: Movement of all points by $\Delta t\, \boldsymbol{a}(x_i, y_j)$

Let

$$\begin{aligned} \Omega_0^{\Delta t} &= \Omega_0 + \Delta t\, \boldsymbol{a}(x_i, y_j) \\ &= \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, (x, y) - \Delta t\, \boldsymbol{a}(x_i, y_j) \in \Omega_0 \right\}. \end{aligned} \tag{5.3}$$

Hence, the contribution of cell Ω_0 to the flux between cell Ω_0 and Ω_j is

$$\begin{aligned} f_{\Omega_0 \Omega_j} &= u_{ij}^n \int_{\Omega_0^{\Delta t} \cap \Omega_j} dxdy \\ &= u_j^n \left| \Omega_0^{\Delta t} \cap \Omega_j \right|. \end{aligned} \tag{5.4}$$

Collecting all these fluxes gives the final formula

$$\begin{aligned} u_{ij}^{n+1} &= u_{ij}^n - \frac{1}{|V_{ij}|} \sum_{j=1}^{8} (f_{\Omega_0 \Omega_j} - f_{\Omega_j \Omega_0}) \\ &= \frac{1}{|V_{ij}|} \sum_{j=0}^{8} f_{\Omega_0 \Omega_j}. \end{aligned} \tag{5.5}$$

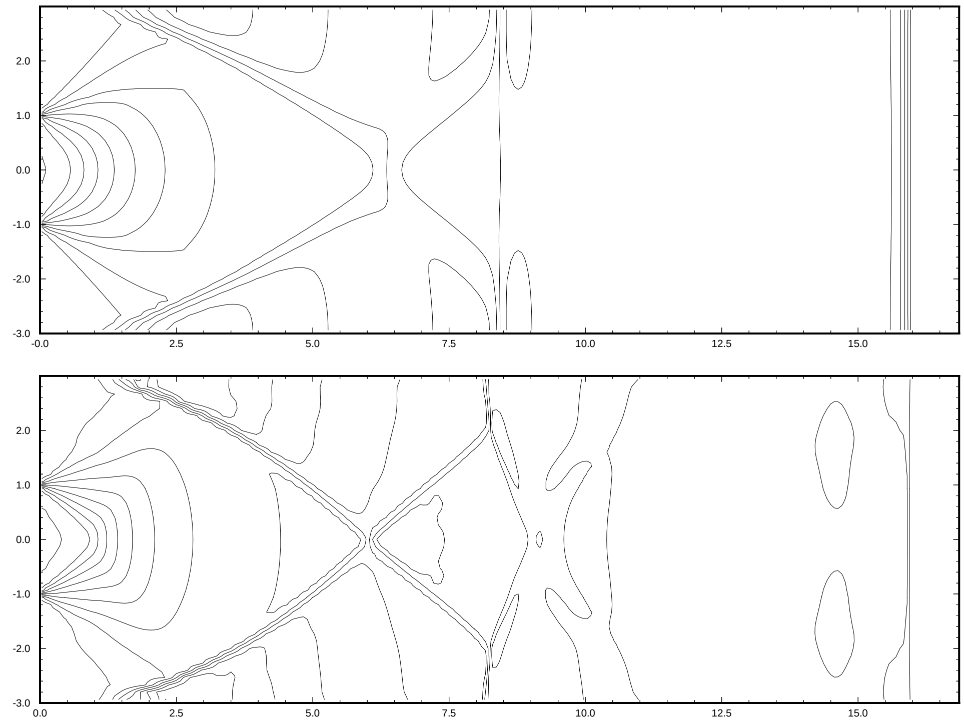

Figure 3: Abrupt expansion in a channel. Plotted are 10 contours lines of the geopotential h.

Here Ω_0 denotes the control volume V_{ij} and Ω_j are the eight neighboring cells. Moreover we have assumed that Δt is restricted such that $\Omega_0^{\Delta t} \subset \bigcup_{j=0}^{8} \Omega_j$. Clearly, replacing u locally in space by a piecewise constant leads to a first order scheme. Hence to get a second order scheme one replaces u by a piecewise linear function. This does not affect formula (5.5). However $\boldsymbol{a}(x, y)$ should locally be replaced by piecewise linear functions. Hence $\Omega_0^{\Delta t}$ is no longer just a shifted rectangle but becomes in general a quadrangle. Moreover formula (5.4) for $f_{\Omega_0 \Omega_j}$ has to be replaced by a more complicated integral. This can be done and for details see [10], [5]. If we apply this scheme to each of the transport equations (2.19a), (4.8) we obtain an overall second order scheme. At this point we have omitted to discuss the piecewise linear reconstruction. See [7] for details.

What has been done here for the Euler equations one can equally well do for the shallow water equations. In the next section we give some numerical examples.

6 Numerical results

6.1 Expansion in a channel.

We consider the supercritical flow of an abrupt expansions in a channel of lenght 8 m and width 1.5 m. The opening is one third of the total width and the flow arrives with a height of 96 mm and a Froude number of 2. Figure 3 shows solutions at time $t = 5\,\text{sec}$. The space discretisation uses a cartesian grid with 240×45 points and a time step $\Delta t = 10^{-3}$. The upper picture shows the solution with the first order method using the formulas (5.4), (5.5). The lower picture shows the solution of the second order extension using the four vectors $\mathbf{n}_i \in \{(1,1)^T, (-1,1)^T, (1,-1)^T, (-1,-1)^T\}$ with coefficients $\omega_i = 1/4$ and $\alpha_i = 1$ in (3.13). The curved shock structure in the lower picture is well captured and coincides with measuments by Hager and Mazumder [12].

6.2 Explosion test

Solving a sperical symmetric problem on a Cartesian mesh causes a lot of problems for any kind of numerical method as shown in [16] for the Euler equations. In the case of the shallow water equations, we compute the circular explosion problem as defined in [1]. The initial values are given as

$$h(\mathbf{x},0) = \begin{cases} 1/g & \text{if} \quad |\mathbf{x}| \le 0.35 \\ 0.1/g & \quad \text{else} \end{cases} ,$$

where g is the gravition constant, h is the geopotential and $\mathbf{u}(\mathbf{x},0) = 0$ where $\mathbf{u}$ is the usual velocity vector. The two-dimensional calculations were done on the square domain $[-1.5, 1.5]^2$ with 500 points in each direction. The CFL-number was 0.8. Figures 4 and 5 show the solutions along the line $y = 0$. The dotted line in Figure 4 shows the geopotential h (left) and the velocity $|\mathbf{u}|$ (right) for the first order method and Figure 5 shows the results for the second order approximation. For comparison, the solid line indicates the solution of the one-dimensional radial symmetric problem using a spacial discretisation of 20000 points on the same interval.

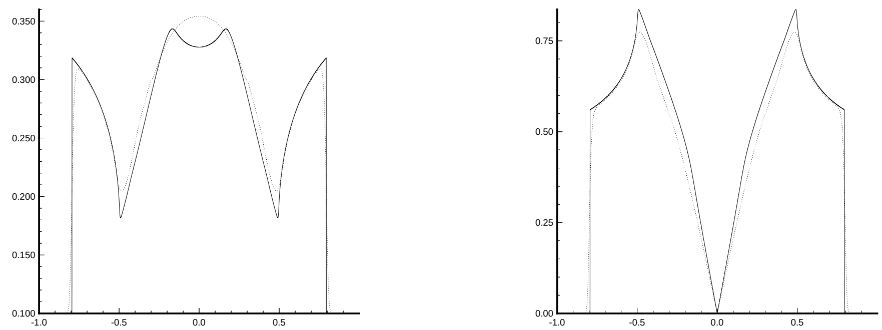

Figure 4: Explosion test problem at the time $t = 0.5$, $y = 0$. First order solution for geopotential (left) and velocity (right).

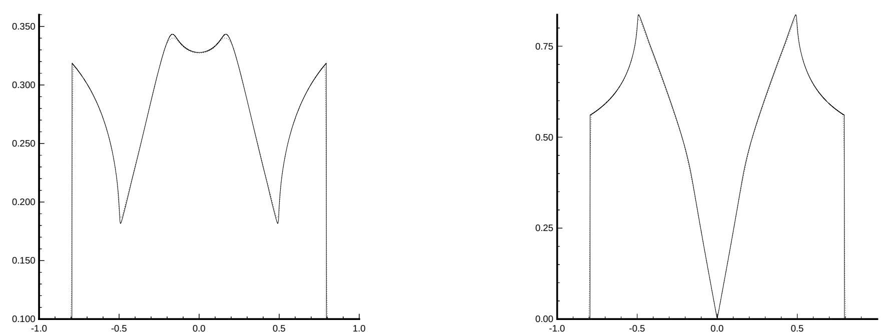

Figure 5: Explosion test problem at the time $t = 0.5$, $y = 0$. Second order solution for geopotential (left) and velocity (right).

References

[1] S. J. Billett. A Class Upwind Methods for Conservation Laws. Dissertation, Cranfield University, 1994.

[2] P. N. Childs and K. W. Morton. Characteristic galerkin methods for scalar conservation laws in one dimension. *SIAM J. Numer. Anal.*, 27(3):533-594, 1990.

[3] P. Collela. Multidimensional upwind methods for hyperbolic conservation laws. *J. of Comp. Phys.*, 87:171-200, 1990.

[4] M. Fey. Ein echt mehrdimensionales Verfahren zur Lösung der Eulergleichungen, Dissertation, ETH Zürich, 1993.

[5] M. Fey. Method of transport for solving the Euler equations. Submitted to *J. Comp. Phys.*, 1994.

[6] M. Fey. Decomposition of the multidimensional Euler equations into advection equations, Research Report 95-14, Seminar für Angewandte Mathematik, ETH Zürich, 1995.

[7] M. Fey. Approximation and Stability Analysis for the High Order Method of Transport for the linear Advection Problem, in preparation.

[8] M. Fey and R. Jeltsch. A simple multidimensional Euler scheme, *Proceedings of the First European Computational Fluid Dynamics Conference, Brussels, 7-11 September 1992*, Elsevier Science Publishers,1992.

[9] M. Fey and R. Jeltsch, A new multidimensional Euler-scheme, in A. Donato and F. Oliveri (Editors), Nonlinear Hyperbolic Problems: Theoretical, Applied and Computational Aspects, Notes on Numerical Fluid Mechanics, **43**, 227-234 Vieweg Verlag, 1993.

[10] M. Fey, A.-T. Morel. Multidimensional method of transport for the shallow water equations, Research Report 95-04, Seminar für Angewandte Mathematik, ETH Zürich, 1995.

[11] M. Fey and H. J. Schroll. Monoton split- and unsplit methods for single conservation law in two space dimensions. Technical Report 57, Inst. für Geometrie und Praktische Mathematik, RWTH Aachen, 1989.

[12] W. H. Hager and S. K. Mazumder. Supercritical flow at abrupt expansions. *Proc. Instn Civ. Engrs Wat., Marit. and Energy*, **96**, Sept., 153-166, 1992.

[13] W. N. G. Hitchon, D. J. Koch, and J. B. Adams. An efficient scheme for convection-dominated transport. *J. Comp. Phys.*, 83:79-95, 1989.

[14] R.J. LeVeque. High resolution finit volume methods on arbitrary grids via wave propagation. *J. Comp. Phys.*, 78:36-63, 1988.

[15] R.J. LeVeque. Numerical Methods for Conservation Laws, Lecture Notes in Mathematics, Birkhäuser, 1992.

[16] R. J. LeVeque and R. Walder, Grid Alignment Effects and Rotated Methods for Computing Complex Flows in Astrophysics, Proceedings of the 9th GAMM Conference on Numerical Methods in Fluid Dynamics, Notes on Numerical Fluid Mechanics,Vieweg Verlag, 1991

[17] P. L. Roe, Beyond the Riemann Problem, ICASE-NASA, Langley Workshop, September 1991