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# The Method of Transport for mixed hyperbolic-parabolic systems

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## Abstract

Starting from a numerical scheme for solving systems of hyperbolic partial differential equations the transition to parabolic equations of the type of advection-diffusion equations needs a different treatment of the viscous part. Since we are using a genuine multi-dimensional scheme also the fact that the diffusion acts in infinitely many directions shall be captured properly. Therefore, to be able to use this scheme we have developed a decomposition of the scalar advection-diffusion equation into a special system of advection equations. In particular the interaction of the advection and diffusion part will be taken into account. The extension to the Navier-Stokes equations which are a system of mixed hyperbolic-parabolic type is possible and will be pointed out.

**key words:** Finite Volume Schemes, Hyperbolic Conservation Laws, Numerical Viscosity, Euler Equations, Advection-Diffusion Equation, Navier-Stokes Equations

**classification:** 35Q30 65M12 76M25 76N10 76R99

# 1 Introduction

The Method of Transport is a genuine multi-dimensional finite volume scheme which was developed originally for the Euler equations and was extended for several systems of hyperbolic conservation laws. The idea of the scheme is build up out of two parts. In the first step the system of partial differential equations is decomposed into a finite number of advection equations with variable coefficients. This technique will be presented in section 2 for the Euler equations. In the second step these advection equations will be solved with the multi-dimensional scheme which will be introduced in section 3.

For simulating viscous flows the Navier-Stokes equations are a proper model. This system is still a conservation law but no longer of hyperbolic type. The equations for the momentum and energy include viscosity and heat flux which is described by second order derivatives. This needs a different numerical treatment of these equations. Since the diffusion is a process which does not act in a specific direction, similar to the acoustic waves in the Euler equations, we want to take these infinitely many directions into account. In the decomposition of the system a linearisation error is introduced and the resulting scheme is only of first order accuracy. In order to obtain second order, correction terms are introduced which are added to the scalar equations. These terms look like a viscosity but with a negative sign because they damp the numerical viscosity. However they do not change the character of the equations.

Having this fact in mind we will show here that it is possible to model parabolic equations with a finite number of advection equations with variable coefficients. To get a first insight how to handle such scalar parabolic equations we start our investigation with the advection-diffusion equation in section 4 and show in section 5 our results for the system of Navier-Stokes equations.

## 2 The two-dimensional Euler equations

### 2.1 Decomposition in finitely many advection equations

We consider the two-dimensional Euler equations

$$\frac{\partial}{\partial t}\mathbf{U} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{U}) + \frac{\partial}{\partial x}\mathbf{G}(\mathbf{U}) = 0, \quad (1)$$

with

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho E + p) \end{pmatrix}, \quad \mathbf{G}(\mathbf{U}) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix}.$$

Here,  $\rho$  is the density,  $u$  and  $v$  are the velocities in  $x$ - and  $y$ -direction,  $E$  is the total energy per unit mass and  $p$  is the pressure. Since the system is not closed we use the equation of state

$$p = (\gamma - 1)\rho\left(E - \frac{u^2 + v^2}{2}\right)$$

to get a relation between the pressure and the quantities in  $\mathbf{U}$ . Here  $\gamma$  is the ratio of the specific heat capacities with a value of 1.4 for air.

In the one-dimensional case we can write the fluxes as

$$\mathbf{F}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \mathbf{U} = \mathbf{A} \mathbf{U} = \mathbf{R} \Lambda \mathbf{R}^{-1} \mathbf{U} = \sum_{i=1}^3 (\alpha_i \mathbf{r}_i \lambda_i) \quad (2)$$

with  $\mathbf{R}$  the matrix of the right eigenvectors and  $\lambda_i$  the eigenvalues of the Jacobian matrix  $\mathbf{A}$  and  $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \text{diag}(u + c, u, u - c)$ . In the two-dimensional case the hyperbolicity still allows the diagonalisation of each Jacobian matrix

$$\mathbf{F}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \mathbf{U} \quad \text{and} \quad \mathbf{G}(\mathbf{U}) = \frac{\partial \mathbf{G}}{\partial \mathbf{U}} \mathbf{U}$$

but they cannot be diagonalised simultaneously. To get a more compact formulation we introduce the matrix

$$\mathcal{F}(\mathbf{U}) = (\mathbf{F}(\mathbf{U}), \mathbf{G}(\mathbf{U})) = \mathbf{U} \mathbf{u}^T + \begin{pmatrix} \mathbf{0}^T \\ \mathbf{I} \\ \mathbf{u}^T \end{pmatrix} p,$$

where  $\mathbf{0}$  is the zero vector,  $\mathbf{I}$  the identity and  $\mathbf{u}^T = (u, v)$  the velocity vector. The Euler equations now become

$$\mathbf{U}_t + \text{div}(\mathcal{F}) = \mathbf{U}_t + \nabla \cdot \mathcal{F} = 0, \quad (3)$$

where the divergence acts on the rows of  $\mathcal{F}$ .

Considering the last term of (2) we have vectors  $\alpha_i \mathbf{r}_i$  which propagate with the characteristic speeds  $\lambda_i$ . We want to decompose  $\mathbf{U}$  in such a way that the multidimensional flux is obtained by propagating these quantities along the characteristic surfaces. In order to get a consistent numerical flux it is sufficient to define the two vectors and the matrix respectively (see [2], [3])

$$\mathbf{R}_1(\mathbf{U}) = \frac{1}{\gamma} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ E + p \end{pmatrix}, \quad \mathbf{R}_2(\mathbf{U}) = \frac{\gamma - 1}{\gamma} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho \mathbf{u}^2 / 2 \end{pmatrix}, \quad \mathbf{L}(\mathbf{U}) = \frac{\rho c}{\gamma} \begin{pmatrix} \mathbf{0}^T \\ \mathbf{I} \\ \mathbf{u}^T \end{pmatrix}$$

where  $\mathbf{R}_2$  will be convected by the fluid velocity  $\mathbf{u}$  and  $\mathbf{R}_1$  and  $\mathbf{L}$  propagate along the characteristic surfaces the so called Monge cone (see Fig. 1). By this approach infinitely many propagation directions have to be considered. However, it turns out

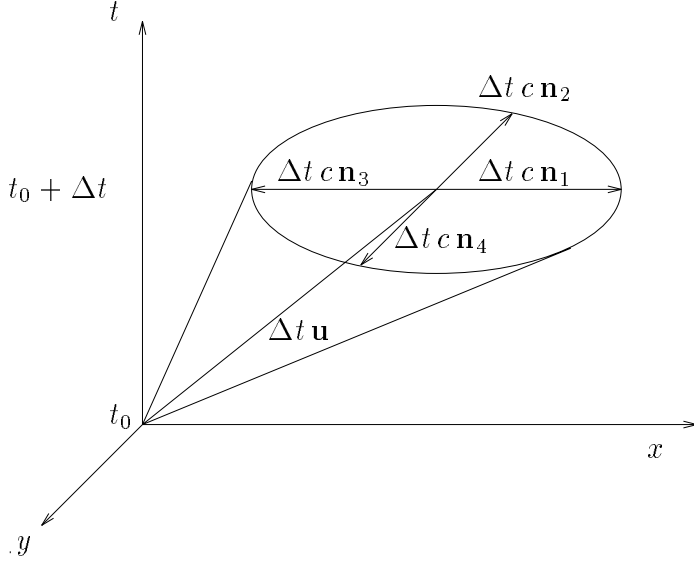


Figure 1: Characteristic cone with four unit vectors  $\mathbf{n}_i$

that it is sufficient for consistency of the scheme when we approximate the Monge cone by a finite number of vectors (see [5]). Then  $\mathbf{U}$  can be written as

$$\mathbf{U} = \mathbf{R}_1 + \mathbf{R}_2 + \frac{\alpha}{k} \sum_{i=1}^k \mathbf{L} \mathbf{n}_i, \quad k \in \mathbb{N}, \quad (4)$$

where the  $\mathbf{n}_i$  have to be chosen as

$$\sum_{i=1}^k \mathbf{n}_i = 0, \quad (5)$$

so that the last term drops out. If we furthermore demand that

$$\frac{\alpha}{k} \sum_{i=1}^k \mathbf{n}_i \mathbf{n}_i^T = \mathbf{I}$$

we can decompose  $\mathcal{F}$  corresponding to  $\mathbf{U}$  as

$$\mathbf{R}_2 \mathbf{u}^T + \frac{1}{k} \left\{ \sum_{i=1}^k [(\mathbf{R}_1 + \alpha \mathbf{L} \cdot \mathbf{n}_i)(\mathbf{u} + \mathbf{n}_i c)^T] \right\}. \quad (6)$$

Assemble (4) and (6) we obtain the Euler equations rewritten as a sum of linear advection equations

$$\mathbf{U}_t + \nabla \cdot \mathcal{F} = (\mathbf{R}_2)_t + \nabla \cdot (\mathbf{R}_2 \mathbf{u}^T) + \frac{1}{k} \sum_{i=1}^k [(\mathbf{R}_n)_t + \nabla \cdot (\mathbf{R}_n (\mathbf{u} + \mathbf{n}_i c)^T)]$$

with

$$\mathbf{R}_n = \mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i$$

Hence the  $(k + 1)$  advection equations

$$\Phi_2 = (\mathbf{R}_2)_t + \nabla \cdot (\mathbf{R}_2 \mathbf{u}^T) = 0 \quad (7)$$

$$\Phi_n(\mathbf{n}_i) = \frac{1}{k} \left\{ (\mathbf{R}_n)_t + \nabla \cdot ((\mathbf{R}_n)(\mathbf{u} + \mathbf{n}_i c)^T) \right\} = 0 ; \quad i = 1, \dots, k \quad (8)$$

are a linearisation of the Euler equations (3).

## 2.2 Error analysis and second order approximation

To solve equations (7) and (8) we have to discretise them. We know that for the discretisation in space a approximation order of two can be achieved but not for the time discretisation. Therefore, consider here a semi-discretisation in time only and assume we know an exact solution at a given time  $t$ . We eliminate the time dependency of  $\mathbf{u}$  and  $c$  by freezing the time so that the advection velocities become functions of space only. The error at time  $t + \Delta t$  between the discretised and exact solution is of the order  $\Delta t^2$ . We will show this statement in this section and derive a modification of the linearised equations so that we obtain an error which is of order  $\Delta t^3$  only.

We will restrict ourselves to the study of the one-dimensional equation of mass conservation. A two-dimensional example will be given in section 4. We expand the exact solution  $\rho(x, t + \Delta t)$  in a Taylor series and get (omitting the arguments  $x$  and  $t$ )

$$\rho(x, t + \Delta t) = \rho + \rho_t \Delta t + \rho_{tt} \frac{\Delta t^2}{2} + O(\Delta t^3) .$$

The time derivatives can be replaced by spatial ones using the Euler equations

$$\begin{aligned} \rho_t &= -(\rho u)_x \\ \rho_{tt} &= -(\rho u)_{tx} = (\rho u^2 + p)_{xx} \end{aligned}$$

and we obtain

$$\rho(x, t + \Delta t) = \rho - (\rho u)_x \Delta t + (\rho u^2 + p)_{xx} \frac{\Delta t^2}{2} + O(\Delta t^3) . \quad (9)$$

Since for the one-dimensional case there are only two vectors  $\mathbf{n}_i$ , namely  $\mathbf{n}_1 = 1$  and  $\mathbf{n}_2 = -1$ , the decomposition of the density leads to

$$\rho = \rho_1 + \rho_2 + \rho_3$$

with

$$\rho_1 = \rho_3 = \frac{1}{2\gamma} \rho \text{ and } \rho_2 = \frac{\gamma - 1}{\gamma} \rho .$$

Each of this quantity will be expanded in a Taylor series

$$\rho_i(x, t + \Delta t) = \rho_i + (\rho_i)_t \Delta t + (\rho_i)_{tt} \frac{\Delta t^2}{2} + O(\Delta t^3) .$$

Again the time derivatives will be replaced by spatial derivatives now using the corresponding advection equation (7), (8)

$$\begin{aligned}(\rho_2)_t &= -(u\rho_2)_x \\(\rho_{1/3})_t &= ((u \pm c)\rho_{1/3})_x \\(\rho_2)_{tt} &= (u(\rho_2)_x)_x \\(\rho_{1/3})_{tt} &= (((u \pm c)((u \pm c)\rho_{1/3})_x)_x).\end{aligned}$$

Summing up this expansions the comparison with (9) shows that the numerical solution differs from the exact solution by  $O(\Delta t)^2$ :

$$\sum_{i=1}^3 \rho_i(x, t + \Delta t) = \rho + \rho_t \Delta t + \left( \rho_{tt} + \left( \frac{\rho}{2} (\gamma u u_x + c c_x) \right)_x \right) \frac{\Delta t^2}{2} + O(\Delta t^3).$$

If we modify the equations (7) and (8) by replacing  $\mathbf{L}$  by  $(\mathbf{L} + \mathbf{K})$  we can achieve an approximation of the exact solution up to a term of  $O(\Delta t^3)$ .  $\mathbf{K}$  is called a correction term and if we choose for the one-dimensional Euler equations

$$\mathbf{K} = \begin{pmatrix} k^\rho \\ k^m \\ k^{\rho E} \end{pmatrix} = \begin{pmatrix} -\frac{\Delta t \rho}{2c} (\gamma u u_x + c c_x) \\ -\frac{\Delta t}{2} \rho ((\gamma - 2)c u_x + u c_x) + u k^\rho \\ -\frac{\Delta t \rho c}{2\gamma(\gamma - 1)} (u u_x - c c_x) + u k^m - \frac{u^2}{2} k^\rho \end{pmatrix}$$

we obtain an error term of size  $O(\Delta t^3)$ . These correction terms can always be found, like for the two-dimensional Euler equations [4] or the Shallow-water-equations [6]. Note that the decomposition is independent of the numerical scheme and that any scheme which solves an advection equation with variable coefficients can be used.

### 3 The Method of Transport

In the previous section we have replaced the system of Euler equations by a finite number of linear advection equations with variable coefficients, so it will be sufficient to have a scheme for solving the scalar equation

$$u_t + \nabla \cdot (u \mathbf{a}^T) = 0$$

where  $\mathbf{a} = \mathbf{a}(\mathbf{x})$  is the local advection velocity. We consider the Method of Transport which is a multidimensional finite volume scheme. In this method fluxes will be computed not across cell sides but from one cell to all adjacent cells which have at least one point of contact. Furthermore the method uses an explicit time integration so that the scheme is described by

$$u_{\Omega_i}^{n+1} = u_{\Omega_i}^n - \frac{1}{|\Omega_i|} \sum_{j \in V} (F_{\Omega_i \Omega_j} - F_{\Omega_j \Omega_i})$$

where  $\Omega_i$  is a control volume with volume  $|\Omega_i|$  and  $V$  is the set of indices of all adjacent cells of  $\Omega_i$ . The contributions  $F_{\Omega_i, \Omega_j}$  are an approximation of the multidimensional flux from  $\Omega_i$  into  $\Omega_j$

$$F_{\Omega_i, \Omega_j} = \int_{S^{\Delta t}(\Omega_i, \Omega_j)} u_i^n d\mathbf{x} \quad (10)$$

with

$$S^{\Delta t}(\Omega_i, \Omega_j) = T(\Omega_i) \cap \Omega_j$$

and

$$T(\Omega_i) = [ \mathbf{y} \mid \mathbf{y} = \mathbf{x} + \Delta t \mathbf{a}(\mathbf{x}); x \in \Omega_i ] .$$

If we assume that the quantity  $u$  and the velocity  $\mathbf{a}$  are constant in each volume we obtain a first order scheme and the contributions read

$$F_{\Omega_i, \Omega_j} = u_{\Omega_i}^n |S^{\Delta t}(\Omega_i, \Omega_j)| .$$

The transport of the quantity  $u$  is described for the two dimensional case by a shift of  $\Omega_i$  by  $\Delta t \mathbf{a}$ , cf. Fig. 2. However, if we choose piecewise linear functions for  $u$

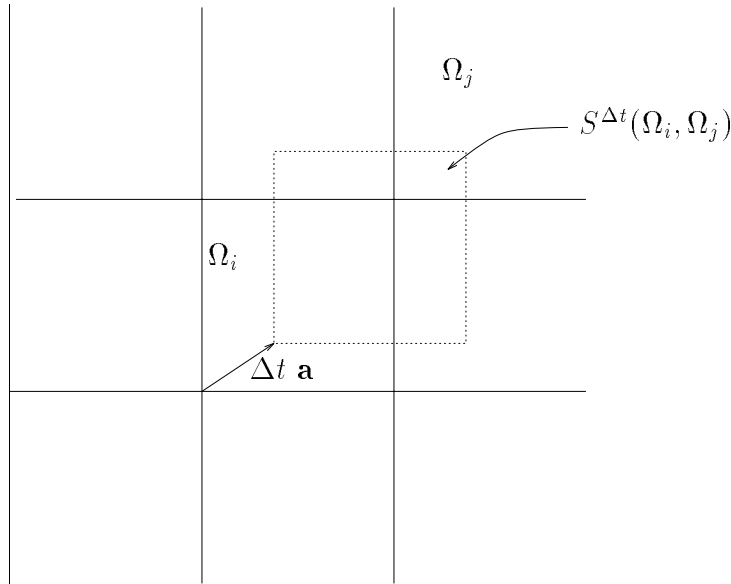


Figure 2: Transport of  $\Omega_i$  as a shift by  $\Delta t \mathbf{a}$

and  $\mathbf{a}$  the integration of (10) is more complicated since the transport is no longer a shift. Considering the two dimensional case a rectangle will now be carried over into a parallelogram. For more details see [8].



## 4 The advection-diffusion equation

### 4.1 Decomposition of the equation

We consider the two-dimensional advection-diffusion equation

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(au) + \frac{\partial}{\partial y}(bu) = \frac{\partial}{\partial x}\left(\epsilon\frac{\partial}{\partial x}u\right) + \frac{\partial}{\partial y}\left(\epsilon\frac{\partial}{\partial y}u\right) \quad (11)$$

with constant coefficients  $a$ ,  $b$  and  $\epsilon$ . This equation can also be written in conservation form

$$u_t + (au - \epsilon u_x)_x + (bu - \epsilon u_y)_y = u_t + \nabla \cdot \mathcal{F} = 0$$

with

$$\mathcal{F} = \mathcal{F}(u, u_x, u_y) .$$

The flux function now depends on  $u$  as well as on the derivatives  $u_x$  and  $u_y$ . The advection equation is of hyperbolic type but the advection-diffusion equation is parabolic. By changing the character of the equation also the character of the solution changes. The main difference between the solutions of hyperbolic and parabolic equations is that the analytical solution is always smooth even if the initial solution is non-smooth. This is also true for the numerical solution provided that the resolution of the grid is fine enough.

First we consider the advection-diffusion equation in one space dimension. We introduce the transformation

$$\tau = t , \quad z = x - at .$$

This leads to

$$\begin{aligned} u_t &= u_\tau \frac{\partial \tau}{\partial t} + u_z \frac{\partial z}{\partial t} &= u_\tau - au_z \\ u_x &= u_\tau \frac{\partial \tau}{\partial x} + u_z \frac{\partial z}{\partial x} &= u_z \\ u_{xx} &= u_\tau \frac{\partial^2 \tau}{\partial x^2} + u_{\tau\tau} \left(\frac{\partial \tau}{\partial x}\right)^2 + u_z \frac{\partial^2 z}{\partial x^2} + u_{zz} \left(\frac{\partial z}{\partial x}\right)^2 &= u_{zz} . \end{aligned}$$

Hence, it follows

$$u_\tau - \epsilon u_{zz} = 0 .$$

The advection part drops out and a diffusion equation is left with unchanged diffusion part. There is a superposition of the diffusion and the advection that means the quantity  $u$  will be advected with the velocity  $a$  and simultaneously smeared by the diffusion. This result is from a physical point of view not very surprising but it is ignored in most of the splitting methods, [7], [1].

Let us consider some special diffusion equations. In the heat equation  $u$  represents the temperature and  $\epsilon$  the thermal diffusivity

$$\epsilon = \chi = \frac{\kappa}{\rho c_p}$$

with  $\kappa$  and  $c_p$  as thermal conductivity and specific heat respectively. When  $u$  represent a velocity then  $\epsilon$  will be the kinematic viscosity

$$\epsilon = \nu = \frac{\mu}{\rho}$$

where  $\mu$  is the dynamic viscosity. We notice that the thermal diffusivity and the kinematic viscosity are both positive values with the unit  $m^2/sec$ . Our aim is to write the flux in a form similar to (6) as a quantity times a propagation velocity. If we divide the coefficient  $\epsilon$  or one of its equivalent by a length we achieve a velocity which will be used as propagation speed

$$u_t - \frac{\epsilon}{l}(lu_x)_x = 0 .$$

Since the equation is of parabolic type the propagation speed is infinite. Thus it seems unreasonable to define a finite propagation velocity. However, we can define such a  $l$  which will for that reason depend on the spatial discretisation. We will see this in section 4.3.

To take into consideration the superposition mentioned above we introduce as new propagation velocity  $(\mathbf{a} - \epsilon \mathbf{H}^{-1} \mathbf{n}_i)$  with

$$\mathbf{a}^T = (a, b) \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} l^x & 0 \\ 0 & l^y \end{pmatrix},$$

where  $\mathbf{H}$  represents  $l$  in two dimensions. Another property of the diffusion is that it acts in all directions similar to the propagation of the quantities  $\mathbf{R}_1$  and  $\mathbf{L}$ . To take both facts into account we rewrite equation (11) as

$$u_t + \nabla \cdot (u \mathbf{a}^T) + \frac{1}{k} \left\{ \sum_{i=1}^k [(\mathbf{H}(\nabla u)^T \mathbf{n}_i)_t + \nabla \cdot ((\mathbf{H}(\nabla u)^T \mathbf{n}_i)(\mathbf{a} - \epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T)] \right\} = 0 .$$

Note that the terms  $(\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i)_t$  and  $(\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i) \mathbf{a}^T$  drop out because of (5). To get a representation similar to the Euler equations we define

$$\mathbf{R}_1 = 0, \mathbf{R}_2 = u, \text{ and } \mathbf{L} = (\mathbf{H}(\nabla u)^T) .$$

The quantity  $u$  is given by

$$u = R_1 + R_2 + \frac{\alpha}{k} \sum_{i=1}^k \mathbf{L} \mathbf{n}_i$$

and the  $(k + 1)$  advection equations

$$(\mathbf{R}_2)_t + \nabla \cdot (\mathbf{R}_2 \mathbf{a}^T) = 0 \quad (12)$$

$$\frac{1}{k} \left\{ (\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i)_t + \nabla \cdot ((\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i)(\mathbf{a} - \epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T) \right\} = 0 \quad i = 1, \dots, k \quad (13)$$

are a linearisation of (11), where equation (12) contains the advection part and equations (13) contain the diffusion part. Note that there appear no problems in the limit  $\epsilon \rightarrow 0$  because the propagation velocity of equations (13) tend to zero and hence the influence of this equations can be neglected against (12).

## 4.2 Error analysis and second order approximation

A semi-discretisation in time of (12) and (13) yields that this system of advection equations is only a first order approximation of (11).

In an analogous proceeding to the Euler equations, an error analysis of the discretised solution can be performed to get a second order approximation of (11) by modifying the linearised equations.

First we consider the one-dimensional case. Assume that the exact solution at time  $t$  is known and expand the solution  $u(x, t + \Delta t)$  in a Taylor series

$$u(x, t + \Delta t) = u + u_t \Delta t + u_{tt} \frac{\Delta t^2}{2} + O(\Delta t^2) . \quad (14)$$

By using the advection-diffusion equation the time derivatives can be replaced by spatial ones

$$u_t = -a u_x + \epsilon u_{xx} \quad (15)$$

$$u_{tt} = a^2 u_{xx} - 2a \epsilon u_{xxx} + \epsilon^2 u_{xxxx} . \quad (16)$$

For the numerical solution we consider the time derivatives of each advection equation. For the first derivatives we get

$$\begin{aligned} u_t &= -a u_x \\ \frac{1}{2} (l u_x)_t &= -\frac{1}{2} \left( a - \frac{\epsilon}{l} \right) (l u_x)_x \\ \frac{1}{2} (-l u_x)_t &= -\frac{1}{2} \left( a + \frac{\epsilon}{l} \right) (-l u_x)_x . \end{aligned}$$

Summing up this equations we obtain

$$u_t = -a u_x + \frac{\epsilon}{l} (l u_x)_x = -a u_x + \epsilon u_{xx} . \quad (17)$$

Inserting this equation into the Taylor expansion we see that the linearised equations are at least of first order. For the second derivatives we get

$$u_{tt} = a^2 u_{xx}$$

$$\begin{aligned}\frac{1}{2}(lu_x)_{tt} &= \frac{1}{2}\left(a - \frac{\epsilon}{l}\right)^2(lu_x)_{xx} \\ \frac{1}{2}(-lu_x)_{tt} &= \frac{1}{2}\left(a + \frac{\epsilon}{l}\right)^2(-lu_x)_{xx} .\end{aligned}$$

Again summing up we obtain

$$u_{tt} = a^2 u_{xx} - 2a\epsilon u_{xxx} . \quad (18)$$

Thus the linearised equations are not of order  $\Delta t^3$  because the term  $\epsilon^2 u_{xxxx}$  in equation (16) is missing. Since the advection part is still correct it will be sufficient to introduce a correction term in equation (13)

$$\frac{1}{k} \left\{ (\mathbf{R}_1 + \alpha(\mathbf{L} + \mathbf{K})\mathbf{n}_i)_t + \nabla \cdot ((\mathbf{R}_1 + \alpha(\mathbf{L} + \mathbf{K})\mathbf{n}_i)(\mathbf{a} - \epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T) \right\} = 0 .$$

Now the time derivatives read

$$\begin{aligned}u_t^k &= -au_x + \epsilon u_{xx} + \frac{\epsilon}{l} k_x \\ u_{tt}^k &= a^2 u_{xx} - 2a\epsilon u_{xxx} - 2\frac{\epsilon}{l} k_{xx} .\end{aligned}$$

The term  $k_{xx}$  is of order  $\Delta t^3$  and can be neglected. To determine  $k_x$  we demand that the Taylor series of the exact and the numerical solution shall be equal up to terms of order  $\Delta t^3$

$$u + u_t \Delta t + u_{tt} \frac{\Delta t^2}{2} = u + u_t^k \Delta t + u_{tt}^k \frac{\Delta t^2}{2} .$$

Hence it follows

$$\Delta t \frac{\epsilon}{l} k_x = \frac{\Delta t^2}{2} \epsilon^2 u_{xxxx}$$

so the correction term

$$\mathbf{K} = \frac{\Delta t l}{2} \epsilon u_{xxx}$$

achieves that the linearised equations (12) and (13) approximate equation (11) up to  $O(\Delta t^3)$ . Note that the correction terms are not due to the interaction of the advection and diffusion part (there is no dependency on  $a$  since the advection part will be solved exactly in time) but they are necessary to get a second order approximation in time of the viscous part.

In the two-dimensional case the time derivatives of the exact solution are

$$u_t = -au_x - bu_y + \epsilon(u_{xx} + u_{yy}) \quad (19)$$

$$\begin{aligned}u_{tt} &= a^2 u_{xx} + b^2 u_{yy} + 2abu_{xy} - 2a\epsilon(u_{xxx} + u_{xyy}) - 2b\epsilon(u_{yyy} + u_{xxy}) \\ &\quad + \epsilon^2(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) .\end{aligned} \quad (20)$$

For the time derivatives of the discretised solution we get

$$\begin{aligned} (\mathbf{R}_2)_t &= -\nabla \cdot (\mathbf{R}_2 \mathbf{a}^T) = -au_x - bu_y \\ \frac{1}{k} \{(\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i)_t\} &= -\nabla \cdot ((\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i)(\mathbf{a} - \epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T) = \epsilon(u_{xx} + u_{yy}). \end{aligned}$$

Summing this  $(k + 1)$  equations up we get

$$u_t = -au_x - bu_y + \epsilon(u_{xx} + u_{yy}).$$

The same procedure for the second derivatives yields

$$u_{tt} = a^2 u_{xx} + b^2 u_{yy} + 2abu_{xy} - 2a\epsilon(u_{xxx} + u_{xyy}) - 2b\epsilon(u_{yyy} + u_{xxy}).$$

The difference between the Taylor expansions of the exact and discretised solutions is

$$\epsilon^2 (u_{xxxx} + 2u_{xyyy} + u_{yyyy}).$$

This term is contributed symmetrically to the  $x$  and  $y$  component of the correction term so that

$$\mathbf{K} = \begin{pmatrix} k^x \\ k^y \end{pmatrix} = \begin{pmatrix} \frac{\Delta t \Delta x}{2l^x} \epsilon(u_{xxx} + u_{xyy}) \\ \frac{\Delta t \Delta y}{2l^y} \epsilon(u_{yyy} + u_{xxy}) \end{pmatrix}$$

leads to a second order approximation of (11).

### 4.3 Stability analysis

Up to now the choice of  $l$  is not clear. As mentioned in section 4.1  $l$  will depend on the spatial discretisation since we have to treat a parabolic equation with infinite propagation speeds. Furthermore, it is necessary to investigate the time-step restriction for the discretisation of equations (12) and (13).

We consider the Method of Transport as described in section 3 in one space dimension. The control volume  $\Omega_i$  is the interval  $[x_{i-1/2}, x_{i+1/2}]$ , then the contributions in one space dimension for the advection equation read

$$F_{\Omega_i \Omega_{i+1}} = \int_{x_{i+1/2}}^{x_{i+1/2} + \Delta t a} u(x + \Delta t a) dx = \int_{x_{i+1/2} - \Delta t a}^{x_{i+1/2}} u(x) dx$$

for the fluxes from cell  $\Omega_i$  into  $\Omega_{i+1}$  and

$$F_{\Omega_{i+1} \Omega_i} = \int_{x_{i+1/2} - \Delta t a}^{x_{i+1/2}} u(x - \Delta t a) dx = \int_{x_{i+1/2}}^{x_{i+1/2} + \Delta t a} u(x) dx$$

from cell  $\Omega_{i+1}$  into  $\Omega_i$ .

For the advection-diffusion-equation we have to sum up the fluxes of the advection and the diffusion parts. For a first order approximation we choose  $u$ ,  $a$  and  $\tilde{\epsilon}$  constant in  $\Omega_i$  then the fluxes become

$$F_{\Omega_i, \Omega_{i+1}} = \begin{cases} \Delta t a u_i + \Delta t \frac{a - \tilde{\epsilon}}{2\Delta x} (l D_0 u)_i + \Delta t \frac{a + \tilde{\epsilon}}{2\Delta x} (-l D_0 u)_i & \text{if } a > 0, \tilde{\epsilon} > 0, a - \tilde{\epsilon} > 0 \\ \Delta t a u_i & + \Delta t \frac{a + \tilde{\epsilon}}{2\Delta x} (-l D_0 u)_i & \text{if } a > 0, \tilde{\epsilon} > 0, a - \tilde{\epsilon} < 0 \\ \Delta t \frac{a + \tilde{\epsilon}}{2\Delta x} (-l D_0 u)_i & \text{if } a < 0, \tilde{\epsilon} > 0, a + \tilde{\epsilon} > 0 \\ \Delta t \frac{a - \tilde{\epsilon}}{2\Delta x} (l D_0 u)_i & \text{if } a < 0, \tilde{\epsilon} < 0, a - \tilde{\epsilon} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{\Omega_i, \Omega_{i-1}} = \begin{cases} \Delta t a u_i + \Delta t \frac{a - \tilde{\epsilon}}{2\Delta x} (l D_0 u)_i + \Delta t \frac{a + \tilde{\epsilon}}{2\Delta x} (-l D_0 u)_i & \text{if } a < 0, \tilde{\epsilon} < 0, a - \tilde{\epsilon} < 0 \\ \Delta t a u_i & + \Delta t \frac{a + \tilde{\epsilon}}{2\Delta x} (-l D_0 u)_i & \text{if } a < 0, \tilde{\epsilon} < 0, a - \tilde{\epsilon} > 0 \\ \Delta t \frac{a + \tilde{\epsilon}}{2\Delta x} (-l D_0 u)_i & \text{if } a > 0, \tilde{\epsilon} < 0, a + \tilde{\epsilon} < 0 \\ \Delta t \frac{a - \tilde{\epsilon}}{2\Delta x} (l D_0 u)_i & \text{if } a > 0, \tilde{\epsilon} > 0, a - \tilde{\epsilon} < 0 \\ 0 & \text{otherwise} \end{cases}$$

with  $(D_0 u)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$  as the central difference operator.

For second order accuracy we reconstruct linearly. For simplicity we only give the fluxes for the advection equation

$$\begin{aligned} F_{\Omega_i, \Omega_{i+1}} &= \begin{cases} \Delta t (a u_i + \frac{\Delta x}{2} (1 - \lambda) a (D u)_i) & \text{if } a > 0 \\ 0 & \text{otherwise} \end{cases} \\ F_{\Omega_{i+1}, \Omega_i} &= \begin{cases} \Delta t (a u_{i+1} - \frac{\Delta x}{2} (1 + \lambda) a (D u)_{i+1}) & \text{if } a < 0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

with  $\lambda = \frac{a \Delta t}{\Delta x}$ .

For the stability analysis we use the von Neumann method. Therefore we consider the time evolution of a single harmonic  $E_k^n e^{Ii\phi}$  where  $E_k^n$  is the amplitude of the  $k$ th harmonic and  $\phi$  its phase angle. Inserting this form into the numerical scheme (12)

and (13) and dividing by  $E_k^n e^{Ii\phi}$  the absolute value of the amplification factor

$$|G| = \left| \frac{E_k^{n+1}}{E_k^n} \right|$$

has to be lower or equal one for all wave-numbers  $k$ . For the advection equation we obtain the restriction  $\lambda \leq 1$ . This restriction continues to hold for the advection-diffusion equation and we look for a restriction for  $\xi = \frac{\Delta t \tilde{\epsilon}}{\Delta x}$ .

We consider the first order scheme and assume  $a > 0, \tilde{\epsilon} > 0$ . For the case  $a - \tilde{\epsilon} > 0$  the scheme is

$$\begin{aligned} u_i^{n+1} &= u_i^n - \lambda(u_i^n - u_{i-1}^n) \\ &\quad - \frac{\sigma}{2} \frac{l}{2\Delta x} (u_{i+1}^n - u_{i-1}^n - u_i^n + u_{i-2}^n) \\ &\quad - \frac{\theta}{2} \frac{l}{2\Delta x} (-u_{i+1}^n + u_{i-1}^n + u_i^n - u_{i-2}^n) \\ &= u_i^n - \lambda(u_i^n - u_{i-1}^n) + \xi \frac{l}{2\Delta x} (u_{i+1}^n - u_{i-1}^n - u_i^n + u_{i-2}^n) \end{aligned}$$

with  $\sigma = \lambda - \xi$  and  $\theta = \lambda + \xi$ . Then we obtain for the amplification factor

$$G = 1 - \lambda(1 - e^{-I\phi}) + \xi \frac{l}{2\Delta x} (e^{I\phi} - e^{-I\phi} - 1 + e^{-2I\phi}),$$

where  $\xi \frac{l}{2\Delta x}$  represents the parabolic part.

Here we obtain the term  $\frac{l}{2\Delta x}$  which represents an additional grid dependence of  $G$ . Since we want all these dependencies to be included in  $\lambda$  and  $\xi$  only, we choose  $l = \Delta x$  so that  $\tilde{\epsilon} = \frac{\epsilon}{\Delta x}$ .

Then we get

$$\begin{aligned} G^2 = G\bar{G} &= \cos^3\phi(2\xi^2) \\ &\quad + \cos^2\phi(2\xi - 2\xi^2) \\ &\quad + \cos\phi(2\lambda - 2\lambda^2 - 2\xi^2) \\ &\quad + 1 - 2\lambda + 2\lambda^2 - 2\xi + 2\xi^2 \\ &= (\cos^3\phi - 1)2\xi^2 \\ &\quad + (\cos\phi - 1)^2(-\lambda + \lambda^2 + \xi^2) \\ &\quad + (\cos^2\phi - 1)(\lambda - \lambda^2 + 2\xi - 3\xi^2) \\ &\quad + 1. \end{aligned}$$

We have rearranged the terms so that a positive and a negative term have been combined. Hence to achieve  $|G| \leq 1$  we obtain a CFL restriction  $\lambda + \xi \leq 1$  and since  $\xi < \lambda$  it is clear that  $\xi \leq \frac{1}{2}$ .

For the second case  $(a - \tilde{\epsilon}) < 0$  the scheme has the form

$$\begin{aligned}
u_i^{n+1} &= u_i^n - \lambda(u_i^n - u_{i-1}^n) \\
&\quad - \frac{\sigma}{2} \frac{l}{2\Delta x} (u_{i+2}^n - u_i^n - u_{i+1}^n + u_{i-1}^n) \\
&\quad - \frac{\theta}{2} \frac{l}{2\Delta x} (-u_{i+1}^n + u_{i-1}^n + u_i^n - u_{i-2}^n) \\
&= u_i^n - \lambda(u_i^n - u_{i-1}^n) \\
&\quad - \frac{l}{4\Delta x} \left( 2\lambda(-u_{i+1}^n + u_{i-1}^n) + 2\xi u_i^n + (\lambda - \xi)u_{i+2}^n - (\lambda + \xi)u_{i-2}^n \right).
\end{aligned}$$

Here the amplification factor is

$$G = 1 - \lambda(1 - e^{-I\phi}) - \frac{l}{4\Delta x} \left( 2\lambda(-e^{I\phi} + e^{-I\phi}) + 2\xi + (\lambda - \xi)e^{2I\phi} - (\lambda + \xi)e^{-2I\phi} \right).$$

Multiplying by its conjugate we get

$$\begin{aligned}
G^2 = G\bar{G} &= \cos^4\phi(-\lambda^2 + \xi^2) \\
&\quad + \cos^3\phi(2\lambda\xi) \\
&\quad + \cos^2\phi(2\lambda^2 + 2\xi - 2\xi^2 - 2\lambda\xi) \\
&\quad + \cos\phi(2\lambda - 2\lambda^2 - 2\lambda\xi) \\
&\quad + 1 - 2\lambda + \lambda^2 - 2\xi + \xi^2 + 2\lambda\xi \\
&= \cos^2(\cos^2\phi - 1)(-\lambda^2 + \xi^2) \\
&\quad + (\cos^3\phi - 1)(2\lambda\xi) \\
&\quad + (\cos\phi - 1)^2(-\lambda + \lambda^2 + \lambda\xi) \\
&\quad + (\cos^2\phi - 1)^2(\lambda + 2\xi - \xi^2 - 3\lambda\xi) \\
&\quad + 1.
\end{aligned}$$

To achieve  $G^2 \leq 1$  we get the restrictions  $\lambda + \xi \leq 1$  and  $\xi \leq \frac{1}{2}$ . Note that although  $\xi > \lambda$  we obtain directly from the analysis the restriction  $\xi \leq \frac{1}{2}$ .

The analysis of the second order scheme yields the same restrictions. In order to adjust  $\xi$  and  $\lambda$  and thus adjust  $a$  and  $\tilde{\epsilon}$  we choose  $l = \frac{\Delta x}{2}$  so that the CFL restriction for  $\xi$  reads

$$\xi = \frac{2\epsilon\Delta t}{\Delta x^2} \leq 1.$$

Since we are using an explicit time stepping we get an inherent stiffness problem for large  $\epsilon$  with respect to  $\Delta x$ .

#### 4.4 Two-step time integration

The introduction of third derivatives into the scheme to obtain second order accuracy in time is not very satisfying because the computational stencil gets large and they



could lead to dispersive effects. They had to be used because the one stage time integration does not yield the second order term of the diffusion part. Hence another possibility is to replace it by a two-step integration method. However we have to be careful because in the case of constant coefficient the numerical scheme gives the exact solution for a CFL number equal to one even in first order. Hence the modification of the time integration can result in a worse approximation.

Let us consider again the Taylor expansion (14) and the time derivatives (15) and (16). The second derivative is given by

$$u_{tt} = -a u_{xt} + \epsilon u_{xxt} .$$

The sequence of time and space derivatives can be exchanged and with (15) we obtain

$$\begin{aligned} u_{tt} &= -a (-a u_x + \epsilon u_{xx})_x + \epsilon (-a u_x + \epsilon u_{xx})_{xx} \\ &= \underbrace{a^2 u_{xx} - a \epsilon u_{xxx}}_{\text{Advection part}} + \underbrace{-a \epsilon u_{xxx} + \epsilon^2 u_{xxxx}}_{\text{Diffusion part}} . \end{aligned}$$

The first two terms on the right hand side are dominated by the advection and the others by the diffusion. For that reason the advection part shall be captured by the advection equation (12) and the diffusion part by the equations (13). Since the term  $\epsilon u_{xxxx}$  which caused the correction term is contained in (13) the improvement for the time integration has to be done for this part. This will be achieved by a two-step time integration. In the first step the solution  $u^n$  will be decomposed into  $\mathbf{R}_1$ ,  $\mathbf{R}_2$  and  $\mathbf{L}$ . For solving the advection part we introduce the term

$$\mathbf{S} = \frac{\Delta t}{2} \nabla \cdot (\epsilon (\nabla u)^T)$$

to take the term  $\epsilon u_{xx}$  into account and solve the equation

$$(\mathbf{R}_2)_t + \nabla \cdot ((\mathbf{R}_2 + \mathbf{S})\mathbf{a}^T) = 0 . \quad (21)$$

Still in the first step with the same decomposition we solve the diffusion part

$$\frac{1}{k} \left\{ (\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i)_t + \nabla \cdot ((\mathbf{R}_1 + \alpha \mathbf{L} \mathbf{n}_i) (-\epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T) \right\} = 0 \quad i = 1, \dots, k . \quad (22)$$

From the entire solution of this first step a new decomposition is constructed and equation (22) is solved with this new solution for a time step  $\frac{\Delta t}{2}$ . Note that we have replaced the propagation velocity  $(\mathbf{a} - \epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T$  of (13) by  $(-\epsilon \mathbf{H}^{-1} \mathbf{n}_i)^T$  because the superposition of  $\mathbf{a}$  and  $\epsilon$  is considered by  $\mathbf{S}$  for the advection part and by the two-step integration for the diffusion part.

The scheme looks as follows: Let  $F_{\Omega_i \Omega_j}^{\text{adv}}(u^n, \Delta t)$  be the numerical flux of equation (21) and  $F_{\Omega_i \Omega_j}^{\text{diff}}(u^n, \Delta t)$  of equation (22). Then the numerical time integration can be written as

- First step:

$$\hat{u}_{\Omega_i}^{n+1} = u_{\Omega_i}^n - \frac{1}{|\Omega_i|} \sum_{j \in V} \left( F_{\Omega_i \Omega_j}^{\text{adv}}(u^n, \Delta t) - F_{\Omega_j \Omega_i}^{\text{adv}}(u^n, \Delta t) \right)$$

$$\tilde{u}_{\Omega_i}^{n+1} = \hat{u}_{\Omega_i}^{n+1} - \frac{1}{|\Omega_i|} \sum_{j \in V} \left( F_{\Omega_i \Omega_j}^{\text{diff}}(u^n, \Delta t) - F_{\Omega_j \Omega_i}^{\text{diff}}(u^n, \Delta t) \right)$$

- Second step:

$$u_{\Omega_i}^{n+1} = \frac{\hat{u}_{\Omega_i}^{n+1} + \tilde{u}_{\Omega_i}^{n+1}}{2} - \frac{1}{|\Omega_i|} \sum_{j \in V} \left( F_{\Omega_i \Omega_j}^{\text{diff}}(\tilde{u}^{n+1}, \frac{\Delta t}{2}) - F_{\Omega_j \Omega_i}^{\text{diff}}(\tilde{u}^{n+1}, \frac{\Delta t}{2}) \right).$$

The discretisation of this scheme yields a solution which is a second order approximation of the exact solution of (11) yet the viscous fluxes have to be approximated of first order in time and space only. Since the error and stability analysis for the equations (21) and (22) is the same as for the equations (12) and (13), they will not be carried out here.

With regard to the Navier-Stokes equations we note that the more general equation

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(a(\rho u)) + \frac{\partial}{\partial y}(b(\rho u)) = \frac{\partial}{\partial x} \left( \epsilon \rho \frac{\partial}{\partial x}(u + v) \right) + \frac{\partial}{\partial y} \left( \epsilon \rho \frac{\partial}{\partial y}(u + v) \right), \quad (23)$$

where  $\rho = \rho(x, t)$  and  $v = v(x, t)$  are known functions, will be solved with second order accuracy, too.

## 5 The Navier-Stokes equations

The Euler equations are used to simulate non-viscous fluids. They are obtained from the Navier-Stokes equations by neglecting the second order terms like shear stresses and heat conduction terms. But as Prandtl showed in his boundary layer approximation this is not the approximation of physical flows in the limit of vanishing viscosity. However, they are a valid approximation for flows at high Reynolds numbers outside this viscous boundary region. Since the Navier-Stokes equations are the most general description of a newtonian fluid they have to be used if the boundary layer cannot be neglected or clearly if the viscosity and the heat conduction are large. The complete Navier-Stokes equations are

$$\mathbf{U}_t + \nabla \cdot \mathcal{F} = \nabla \cdot \mathcal{G},$$

with

$$\mathcal{G}(\mathbf{U}, \nabla \mathbf{U}) = \begin{pmatrix} \mathbf{0}^T \\ \tau \\ \mathbf{u}^T \tau \end{pmatrix} + \begin{pmatrix} \mathbf{0}^T \\ \mathbf{0} \\ \kappa (\nabla T)^T \end{pmatrix}, \quad (24)$$

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \lambda \mathbf{I} \nabla \cdot \mathbf{u} \quad \text{and} \quad \nabla \mathbf{u} := \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}.$$

$\mu$ ,  $\lambda$  and  $\kappa$  are the dynamic viscosity, second viscosity and thermal conductivity coefficients respectively. They are functions of the fluid state essentially of the temperature and only weakly coupled to the pressure. To determine the coefficients we assume only an influence by the temperature and use Sutherland's formula to determine the dynamic viscosity

$$\mu = \mu_0 T^{3/2} \frac{1 + S_l}{T + S_l} \quad \text{with} \quad S_l = \frac{110K}{T_0},$$

where  $\mu_0$  and  $T_0$  are to be given dependent on the physical problem.

For the relation between  $\mu$  and  $\lambda$  we can assume that the Stokes relation

$$\lambda = -\frac{2}{3}\mu$$

holds. Furthermore we assume a constant Prandtl number so that we achieve a simple relation between the viscosity and the thermal conductivity coefficient

$$\kappa = \frac{\mu c_p}{Pr}.$$

The equation of state combines the temperature and the pressure

$$p = \rho RT,$$

with the gas constant  $R$ .

## 5.1 Decomposition into advection equations

In this section we give a decomposition of  $\mathcal{G}$  into a finite number of advection equations which corresponds to the ansatz in section 4.1. Therefore we first need a propagation velocity which is related to the diffusion. As mentioned in section 4 the thermal diffusivity and the kinematic viscosity represent the  $\epsilon$  of the diffusion equation. Since the relation between the kinematic viscosity and the dynamic viscosity is

$$\nu = \frac{\mu}{\rho},$$

and between the thermal diffusivity and conductivity

$$\chi = \frac{\kappa}{\rho c_p}$$

we can rewrite  $\mathcal{G}$  as

$$\mathcal{G}(\mathbf{U}, \nabla \mathbf{U}) = \nu \begin{pmatrix} \mathbf{0}^T \\ \tilde{\tau} \\ \mathbf{u}^T \tilde{\tau} \end{pmatrix} + \chi \begin{pmatrix} \mathbf{0}^T \\ \mathbf{0} \\ c_p \rho (\nabla T)^T \end{pmatrix}$$

with  $\tau = \nu \tilde{\tau}$ .

The coefficients  $\nu$  and  $\chi$  have a fixed proportion because we are assuming a fixed Prandtl number. Therefore it is sufficient to choose one "diffusion-velocity" only. If we take the quantity  $\nu$  we can introduce as transport value of the diffusive fluxes

$$\mathbf{L}^v = \left[ \begin{pmatrix} \mathbf{0}^T \\ \tilde{\tau} \\ \mathbf{u}^T \tilde{\tau} \end{pmatrix} + \begin{pmatrix} \mathbf{0}^T \\ \mathbf{0} \\ \frac{\rho}{\text{Pr}} (\nabla T)^T \end{pmatrix} \right] \mathbf{H}.$$

The state vector  $\mathbf{U}$  is

$$\mathbf{U} = \mathbf{R}_1 + \mathbf{R}_2 + \frac{1}{k} \sum_{i=1}^k (\mathbf{L} + \mathbf{L}^v) \mathbf{n}_i.$$

From this term we get additionally to the equations (7) and (8)  $k$  advection equations of the form

$$\Phi_v(\mathbf{n}_i) = \frac{1}{k} \left\{ (\alpha \mathbf{L}^v \mathbf{n}_i)_t + \nabla \cdot ((\alpha \mathbf{L}^v \mathbf{n}_i)(\mathbf{u} + \mathbf{H}^{-1} \mathbf{n}_i \nu)^T) \right\} = 0, \quad (25)$$

so that the Navier-Stokes equations can be rewritten as

$$\Phi_2 + \sum_{i=1}^k \Phi_n(\mathbf{n}_i) = \sum_{i=1}^k \Phi_v(\mathbf{n}_i).$$

Therefore the  $(2k + 1)$  advection equations (7), (8) and (25) are a linearisation of the Navier-Stokes equations.

Note that the discretisation in time of this equations yields only a first order approximation of the exact solution for two reasons: first due to the linearisation process of the system like for the Euler equations and second due to the fact that the discretisation of equations (25) leads to a first order approximation of the solution only corresponding to the problem considered in section 2.2.

To achieve second order accuracy a correction term  $\mathbf{K}^v$  is introduced and  $\mathbf{L}^v$  is replaced by  $(\mathbf{L}^v + \mathbf{K}^v)$ . But this correction term is too complicated to be printed here completely. The correction term for the mass equation is a correction for the linearisation error only since this equation is hyperbolic. It has the form

$$\mathbf{K}^v = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \\ \frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} \end{pmatrix}.$$

Adding all correction terms the local approximation error is  $O(\Delta t^3)$ . Note that the correction terms for the momentum and energy include third derivatives of the conservative variables.

The CFL condition for the Navier-Stokes equation is

$$\Delta t \leq \frac{\Delta x}{v_{\max}}$$

with

$$v_{\max} = \max(|\mathbf{u}|_{\infty} + c, |\mathbf{u} + \epsilon \mathbf{H}^{-1} \mathbf{1}|_{\infty}),$$

where  $\mathbf{H}$  is different from the advection-diffusion case. Here we have to choose

$$l^x = \frac{\Delta x}{4 + \frac{c\Delta x}{\epsilon}} \quad \text{and} \quad l^y = \frac{\Delta y}{4 + \frac{c\Delta x}{\epsilon}}$$

to achieve a stable scheme.

## 5.2 Decomposition into advection-diffusion equations

The decomposition in the previous section yields third derivatives into the numerical scheme like for the decomposition of the advection-diffusion equation in section 4.1. To avoid them the Navier-Stokes equations are decomposed into advection-diffusion equations of the form of (23). These equations will be solved with the two-step time integration of section 4.4 which is of second order accuracy. There remains an error due to the linearisation of the system which will be captured again by correction terms. The advantage compared with the linearisation of section 5.1 is that this correction terms get much simpler and do not include third derivatives.

Two types of diffusion mechanisms are contained in the Navier-Stokes equations: the kinematic and the thermal diffusion. Since these diffusion coefficients and therefore the diffusion velocities are in general different, the corresponding advection-diffusion equations have to be found. Therefore the term  $\mathbf{R}_1$  of the Euler equations will be split into a kinetic and an internal energy part

$$\mathbf{R}_1 = \mathbf{R}_1^k + \mathbf{R}_1^i.$$

The kinetic energy term  $\mathbf{R}_1^k$  is similar to  $\mathbf{R}_2$ . It differs only by a constant factor

$$\mathbf{R}_1^k(\mathbf{U}) = \frac{1}{\gamma} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho \mathbf{u}^2 / 2 \end{pmatrix}.$$

The term  $\mathbf{R}_1^i$  represents the internal energy

$$\mathbf{R}_1^i(\mathbf{U}) = \frac{1}{\gamma} \begin{pmatrix} 0 \\ \mathbf{0} \\ \rho c_p T \end{pmatrix},$$

where  $T$  is the temperature.

With this decomposition we get the following linear advection-diffusion equations

$$(\mathbf{R}_2)_t + \nabla \cdot (\mathbf{R}_2 \mathbf{u}^T) = \nabla \cdot (\mathbf{V}_2 \nu) \quad (26)$$

$$(\mathbf{R}_1^k + \mathbf{L} \mathbf{n}_i)_t + \nabla \cdot \left( (\mathbf{R}_1^k + \mathbf{L} \mathbf{n}_i)(\mathbf{u} + c \mathbf{n}_i)^T \right) = \nabla \cdot (\mathbf{V}_1^k \nu) \quad (27)$$

$$(\mathbf{R}_1^i)_t + \nabla \cdot \left( (\mathbf{R}_1^i)(\mathbf{u} + c \mathbf{n}_i)^T \right) = \nabla \cdot (\mathbf{V}_1^i \chi) \quad (28)$$

with

$$\mathbf{V}_1^k = \frac{\rho}{\gamma} \left[ \begin{pmatrix} \mathbf{0}^T \\ (\nabla \mathbf{u})^T \\ \mathbf{u}^T (\nabla \mathbf{u})^T \end{pmatrix} + (1 + \lambda') \begin{pmatrix} \mathbf{0}^T \\ (\nabla \mathbf{u}) \\ \mathbf{u}^T (\nabla \mathbf{u}) \end{pmatrix} \right],$$

$$\mathbf{V}_1^i = \rho c_p \begin{pmatrix} \mathbf{0}^T \\ \mathbf{0} \\ (\nabla T)^T \end{pmatrix},$$

$$\mathbf{V}_2 = \rho \frac{\gamma - 1}{\gamma} \left[ \begin{pmatrix} \mathbf{0}^T \\ (\nabla \mathbf{u})^T \\ \mathbf{u}^T (\nabla \mathbf{u})^T \end{pmatrix} + (1 + \lambda') \begin{pmatrix} \mathbf{0}^T \\ (\nabla \mathbf{u}) \\ \mathbf{u}^T (\nabla \mathbf{u}) \end{pmatrix} \right]$$

and  $\lambda' = \frac{\lambda}{\mu}$ .

The Navier-Stokes equations can be rewritten as sum of these equations. Hence with (26), (27) and (28) we obtain a linearisation of the Navier-Stokes equations with  $(2k + 1)$  advection-diffusion equations, instead of  $(k + 1)$  advection equations for the Euler equations.

Note that the advection part for  $\mathbf{R}_1^k$  and  $\mathbf{R}_1^i$  can be treated together as for the Euler equations, only the diffusion part has to be solved separately.

Although the truncation error of the semi-discretised advection-diffusion equations is of order  $\Delta t^3$  the linearisation yields only a first order approximation of the numerical solution. This lack can be solved in the same manner as for the Euler equations by introducing correction terms. Therefore we replace  $\mathbf{L}$  by  $(\mathbf{L} + \mathbf{K})$  and get with the right choice of  $K$  a second order approximation in time. As mentioned before the correction terms become much simpler compared to the case in section 5.1:

$$k_1^\rho = - \frac{\Delta t (\rho c c_x + \rho u_y v \gamma + u u_x \rho \gamma - \tau_{11;x} \gamma - \tau_{12;y} \gamma)}{2 c \gamma}$$

$$k_2^\rho = - \frac{\Delta t (v v_y \rho \gamma + c c_y \rho + v_x u \rho \gamma - \tau_{12;x} \gamma - \tau_{22;y} \gamma)}{2 c \gamma}$$

$$\begin{aligned}
k_1^{\rho u} &= -\frac{\Delta t}{2c\gamma}(\rho vcc_y + \rho v_y c^2 \gamma - \rho c^2 v_y + \rho u_x c^2 \gamma - 2\rho c^2 u_x + u\rho cc_x - \\
&\quad \gamma^2 u_x \tau_{11} + \gamma u_x \tau_{11} - \tau_{12} v_x \gamma^2 + \gamma v_x \tau_{12} - \tau_{12} \gamma^2 u_y + \gamma u_y \tau_{12} - \gamma^2 v_y \tau_{22} + \\
&\quad \gamma v_y \tau_{22} - \gamma^2(\kappa_x T_x + \kappa T_{xx}) + \gamma(\kappa_x T_x + \kappa T_{xx}) - (\kappa_y T_y + \kappa T_{yy})\gamma^2 + \\
&\quad \gamma(\kappa_y T_y + \kappa T_{yy})) + uk_1^\rho \\
k_2^{\rho u} &= \frac{\Delta t c \rho u_y}{2\gamma} + uk_2^\rho \\
k_1^{\rho v} &= \frac{\Delta t c \rho v_x}{2\gamma} + vk_1^\rho \\
k_2^{\rho v} &= -\frac{\Delta t}{2c\gamma}(\rho vcc_y + \rho v_y c^2 \gamma - 2\rho c^2 v_y + \rho u_x c^2 \gamma - \rho c^2 u_x + u\rho cc_x - \\
&\quad \gamma^2 u_x \tau_{11} + \gamma u_x \tau_{11} - \tau_{12} v_x \gamma^2 + \gamma v_x \tau_{12} - \tau_{12} \gamma^2 u_y + \gamma u_y \tau_{12} - \gamma^2 v_y \tau_{22} + \\
&\quad \gamma v_y \tau_{22} - \gamma^2(\kappa_x T_x + \kappa T_{xx}) + \gamma(\kappa_x T_x + \kappa T_{xx}) - (\kappa_y T_y + \kappa T_{yy})\gamma^2 + \\
&\quad \gamma(\kappa_y T_y + \kappa T_{yy})) + vk_2^\rho \\
k_1^{\rho E} &= \frac{\Delta t c(\rho cc_x - u_y \rho v - u\rho u_x + \tau_{11;x}\gamma + \tau_{12;y}\gamma)}{2\gamma(\gamma - 1)} + uk_1^m + vk_1^n + \frac{(u^2 + v^2)k_1^\rho}{2} \\
k_2^{\rho E} &= \frac{\Delta t c(-v v_y \rho + c c_y \rho - v_x \rho u + \tau_{12;x}\gamma + \tau_{22;y}\gamma)}{2\gamma(\gamma - 1)} + uk_2^m + vk_2^n + \frac{(u^2 + v^2)k_2^\rho}{2}
\end{aligned}$$

with  $\mathbf{K} = (\mathbf{k}_1, \mathbf{k}_2)$ .

## 6 Numerical results

The first set of calculations verifies that the approximation for the advection-diffusion equation is of second order. In the second part the validation of the Navier-Stokes equations is performed. An analytic solution of the incompressible Navier-Stokes equations will be investigated and the approximation order will be checked. Furthermore, the flow formation in a Couette motion is calculated. For the velocity distributions in this flow an analytic solution can be obtained which will be checked against the numerical solution.

### 6.1 Advection-diffusion equation

For the advection-diffusion equation the solutions of the two approaches, with correction terms or two-step time integration, will be compared with an exact solution. Therefore we consider the  $C_\infty$ -function

$$u = \exp(x - at + y - bt - 2 + 2\epsilon t)$$

which we choose for the initial and boundary conditions. The computational domain is  $[-2, 2] \times [-2, 2]$ . We set  $a = 0.2$ ,  $b = -0.2$  and  $\epsilon = 10^{-4}$ . The solution is computed

on different meshes beginning with  $20 \times 20$  cells and two steps with  $\Delta t = 0.75$ . For the first approach two calculations will be performed. In the first case no correction terms are used, but the advection equations are solved with second order accuracy. In the second case the correction terms are switched on. We compare the numerical solution with the exact solution and calculate the error in the  $L_1$ - and  $L_\infty$ -norm:

Without correction terms				
$n \times m$	Error		Order	
	$L_1$	$L_\infty$	$L_1$	$L_\infty$
$20 \times 20$	1.12e-7	1.45e-6		
$40 \times 40$	7.80e-8	1.06e-6	0.53	0.46
$80 \times 80$	2.32e-8	3.32e-7	1.75	1.67
$160 \times 160$	6.34e-9	9.31e-8	1.87	1.83
$320 \times 320$	1.82e-9	2.70e-8	1.79	1.78
$640 \times 640$	5.65e-10	8.35e-9	1.69	1.69

With correction terms				
$n \times m$	Error		Order	
	$L_1$	$L_\infty$	$L_1$	$L_\infty$
$20 \times 20$	1.22e-7	1.58e-6		
$40 \times 40$	7.34e-8	9.98e-7	0.73	0.66
$80 \times 80$	2.10e-8	3.00e-7	1.81	1.73
$160 \times 160$	5.25e-9	7.71e-8	2.00	1.96
$320 \times 320$	1.28e-9	1.90e-8	2.03	2.02
$640 \times 640$	2.95e-10	4.37e-9	2.12	2.12

We observe that without the use of the correction terms the order of the method is clearly distinct from two.

For the second approach there are no correction terms so that only one calculation will be carried out to show that the approximation is of second order:

$n \times m$	Error		Order	
	$L_1$	$L_\infty$	$L_1$	$L_\infty$
$20 \times 20$	1.25e-7	1.62e-6		
$40 \times 40$	7.18e-8	9.71e-7	0.80	0.73
$80 \times 80$	2.02e-8	2.87e-7	1.83	1.76
$160 \times 160$	4.85e-9	7.05e-8	2.06	2.03
$320 \times 320$	1.09e-9	1.58e-8	2.15	2.15
$640 \times 640$	2.06e-10	2.93e-9	2.40	2.43



## 6.2 Navier-Stokes equations

For validating the Navier-Stokes equations the first test is to set the viscosities to zero and compare with the solution of the Euler equations. It turns out that we get two identical solutions.

For the validation of the viscous terms an analytical test problem is chosen. Assuming density and viscosity constant an exact solution of the incompressible Navier-Stokes equations is defined by

$$\begin{aligned} u &= -A \cos(x) \sin(y) \exp(-2\mu/\rho t) \\ v &= A \sin(x) \cos(y) \exp(-2\mu/\rho t) \\ p &= p_0 - A^2/4\rho (\cos(2x) + \cos(2y)) \exp(-4\mu/\rho t), \end{aligned}$$

where  $p_0$  is a reference pressure and  $A$  is the amplitude. Since we are solving the compressible Navier-Stokes equations we will obtain a different solution. This errors can be computed by comparing the time derivatives of the Navier-Stokes equations when the exact solution is applied with the time derivative of the analytical solution itself. The initial solution is divergence free so that after one time step the solutions for the velocities are still correct. However, the density will not stay constant as in the analytic solution which leads to a first order error. The energy of the compressible solution has to be determined out of the pressure for the incompressible solution. This leads after one time-step to a deviation

$$\begin{aligned} &A^2 \frac{2}{(\gamma - 1)} \mu \exp(-4\mu/\rho t) (\cos(y)^2 - 1 \\ &\quad + 2\gamma (1 - \cos(y)^2 - \cos(x)^2 + \cos(x)^2 \cos(y)^2) + \cos(x)^2 - 2 \cos(x)^2 \cos(y)^2) \\ &\quad - A^3 \frac{\rho}{(\gamma - 1)} \exp(-6\mu/\rho t) \sin(y) \sin(x) (\cos(y)^2 - \cos(x)^2). \end{aligned}$$

This term is independent from the grid spacing and its size is mainly determined by the amplitude  $A$ .

For the calculation we introduce nondimensional variables, where the transformation into nondimensional variables should lead to values of about 1 for the quantities  $\rho$ ,  $\mathbf{u}$  and  $T$ . This can be achieved by setting  $R = \frac{1}{\gamma \text{Ma}^2}$  with  $\gamma = 1.4$  for air, so that the flow is characterized by a reference Mach number and Reynolds number. We choose

$$\begin{aligned} \text{Ma} &= 0.2 \\ \text{Re} &= 1 \\ \kappa &= 0 \text{ (Pr} = \infty) \\ A &= 0.04. \end{aligned}$$

Due to the restriction of the explicit time integration the time step size is for that configuration about one to two orders smaller than the grid spacing. Hence we can only check the spatial error by computing one time step with the same  $\Delta t$  on grids

with a different number of cells. For a second order approximation the error has to be a quarter of the error on the coarser grid when the number of cells will be doubled. We choose the domain  $[0, 2\pi] \times [0, 2\pi]$ :

$\Delta t = 2 \cdot 10^{-4}$								
$n \times m$	Error ( $L_1$ )				Order ( $L_1$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$20 \times 20$	3.92e-8	4.17e-8	4.17e-8	2.78e-6				
$40 \times 40$	1.96e-8	1.02e-8	1.02e-8	1.77e-6	1.00	2.03	2.03	/
$80 \times 80$	9.72e-9	1.85e-9	1.81e-9	1.35e-6	1.01	2.47	2.49	/
$n \times m$	Error ( $L_\infty$ )				Order ( $L_\infty$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$20 \times 20$	9.05e-8	1.10e-7	1.09e-7	4.89e-6				
$40 \times 40$	4.77e-8	2.74e-8	2.73e-8	3.45e-6	0.93	2.00	1.99	/
$80 \times 80$	2.35e-8	5.13e-9	5.13e-9	3.00e-6	1.02	2.42	2.41	/

$\Delta t = 2 \cdot 10^{-5}$								
$n \times m$	Error ( $L_1$ )				Order ( $L_1$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$80 \times 80$	9.90e-10	2.65e-10	2.63e-10	1.36e-7				
$160 \times 160$	4.94e-10	6.41e-11	6.29e-11	1.21e-7	1.00	2.05	2.06	/
$320 \times 320$	2.46e-10	7.77e-12	6.46e-12	1.16e-7	1.01	3.04	3.29	/
$n \times m$	Error ( $L_\infty$ )				Order ( $L_\infty$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$80 \times 80$	2.38e-9	7.35e-10	7.37e-10	3.01e-7				
$160 \times 160$	1.17e-9	1.92e-10	1.92e-10	2.90e-7	1.02	1.93	1.94	/
$320 \times 320$	5.96e-10	2.58e-11	2.57e-11	2.88e-7	0.98	2.90	2.90	/

To illustrate the influence of the correction terms for the Navier-Stokes equations we consider a further test case. An initial solution with small perturbations in density, velocity and pressure is taken so that the solution keeps smooth in the considered time interval. The computational domain is  $[-2, 2] \times [-2, 2]$  and the time-step on the coarsest mesh is  $\Delta t = 0.3$ . We use the radial symmetric function

$$z = \begin{cases} z_i + \left(\frac{r}{0.4}\right)^3 (z_o - z_i) \left(10 - 15 \left(\frac{r}{0.4}\right) + 6 \left(\frac{r}{0.4}\right)^2\right) & ; r \leq 0.4 \\ z_o & ; r > 0.4 \end{cases}$$

where  $r$  is the radius around the center of the perturbations which are located at

$$\left(\frac{1}{2}, \frac{1}{2}\right)^T, \left(\frac{1}{2}, -\frac{1}{2}\right)^T, \left(-\frac{1}{2}, -\frac{1}{2}\right)^T, \left(-\frac{1}{2}, \frac{1}{2}\right)^T.$$

For each of this point we take for  $z$  one of the following quantity

$$\begin{aligned}\rho_i &= 1.1 \quad , \quad \rho_o = 1.0 \\ u_i &= 0.1 \quad , \quad u_o = 0.0 \\ v_i &= -0.1 \quad , \quad v_o = 0.0 \\ p_i &= 1.0 \quad , \quad p_o = 0.9 .\end{aligned}$$

We interpret them again as non-dimensional variables. To obtain a temperature of about 1 and fulfill the state equation we get  $R = 1$  because the pressure is already fixed. This leads to a reference Mach number  $\text{Ma} \approx 0.85$ . For the reference values  $T_0$  and  $\mu_0$  we set

$$T_0 = 273K \quad \text{and} \quad \mu_0 = 10^{-3}$$

which corresponds to a Reynolds number of about 100. The viscous velocities therefore varies in a range from about  $5 \cdot 10^{-3}$  for the  $20 \times 20$  grid to  $1.6 \cdot 10^{-1}$  for the  $640 \times 640$  grid. For comparison the speed of sound has values of about 1.1, so that the time-step will be mainly restricted by the inviscid part of the equations. We consider only the approach of the decomposition into advection-diffusion equations. The results are shown below:

With correction terms for Euler equations only								
$n \times m$	Error ( $L_1$ )				Order ( $L_1$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$40 \times 40$	2.28e-4	1.70e-4	1.42e-4	7.23e-4	2.18	2.19	2.41	1.96
$80 \times 80$	8.52e-5	5.27e-5	5.38e-5	1.84e-4	1.42	1.69	1.41	1.98
$160 \times 160$	2.80e-5	1.80e-5	1.79e-5	5.99e-5	1.60	1.55	1.59	1.62
$320 \times 320$	8.38e-6	5.38e-6	5.37e-6	1.71e-5	1.74	1.74	1.74	1.81
$n \times m$	Error ( $L_\infty$ )				Order ( $L_\infty$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$40 \times 40$	3.85e-3	3.41e-3	3.26e-3	1.38e-2	2.20	1.79	1.82	0.83
$80 \times 80$	1.86e-3	1.34e-3	1.34e-3	3.33e-3	1.05	1.35	1.28	2.05
$160 \times 160$	4.59e-4	4.95e-4	4.95e-4	1.40e-3	2.02	1.44	1.44	1.25
$320 \times 320$	1.62e-4	1.77e-4	1.77e-4	5.79e-4	1.50	1.48	1.48	1.27

With correction terms for Navier-Stokes equations								
$n \times m$	Error ( $L_1$ )				Order ( $L_1$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$40 \times 40$	1.30e-3	7.89e-4	7.47e-4	2.29e-3	1.01	1.02	1.03	1.12
$80 \times 80$	2.31e-4	1.65e-4	1.70e-4	4.56e-4	2.49	2.26	2.14	2.33
$160 \times 160$	4.27e-5	2.91e-5	2.84e-5	9.27e-5	2.44	2.50	2.58	2.30
$320 \times 320$	1.20e-5	5.80e-6	5.49e-6	2.18e-5	1.83	2.33	2.37	2.09
$n \times m$	Error ( $L_\infty$ )				Order ( $L_\infty$ )			
	$\rho$	$\rho u$	$\rho v$	$\rho E$	$\rho$	$\rho u$	$\rho v$	$\rho E$
$40 \times 40$	2.58e-2	8.50e-3	8.34e-3	3.17e-2	/	/	/	/
$80 \times 80$	2.70e-3	2.02e-3	2.10e-3	7.79e-3	3.26	2.07	1.99	2.02
$160 \times 160$	1.35e-3	8.12e-4	8.12e-4	1.48e-3	1.00	1.32	1.37	2.40
$320 \times 320$	3.04e-4	2.25e-4	2.25e-4	3.37e-4	2.15	1.85	1.85	2.13

As the last test problem the flow formation in Couette motion is calculated. The configuration consists of two parallel walls with a distance  $h$ . The gas is at rest and at the time  $t = 0$  one wall will be impulsively accelerated to the velocity  $u_0$ . If we assume the flow to be incompressible an exact solution of the time dependent velocity profiles is known (see [9]). In the limit  $t \rightarrow \infty$  the velocity profile will be the linear distribution of the steady state. With the boundary conditions

$$\begin{aligned} u &= u_0 & \text{for } y &= h \\ u &= 0 & \text{for } y &= 0, \end{aligned}$$

the velocity profiles are given by

$$\frac{u}{u_0} = \sum_{n=0}^{\infty} \operatorname{erfc}(2n\eta_1 + \eta) - \sum_{n=0}^{\infty} \operatorname{erfc}(2(n+1)\eta_1 - \eta),$$

where  $\eta = \frac{y}{2\sqrt{\nu t}}$ ,  $\eta_1 = \frac{h}{2\sqrt{\nu t}}$  and  $\operatorname{erfc}$  is the complementary error function.

We set the reference values

$$\begin{aligned} \text{Ma} &= 0.2 \\ \text{Re} &= 1000 \\ T_0 &= 273K. \end{aligned}$$

The profiles are approximately similar and will remain so as long as the boundary layer has not spread to the stationary wall but the simulation has not been carried out so far. The comparison of the numerical and exact solution for different times is presented in Fig. 3, where the velocities are scaled with respect to  $u_0$  and  $y$  with respect to the distance  $h$ . The numerical solution coincides in all cases very well with the exact one. This indicates that the implementation is also time accurate and describes the correct viscosity.

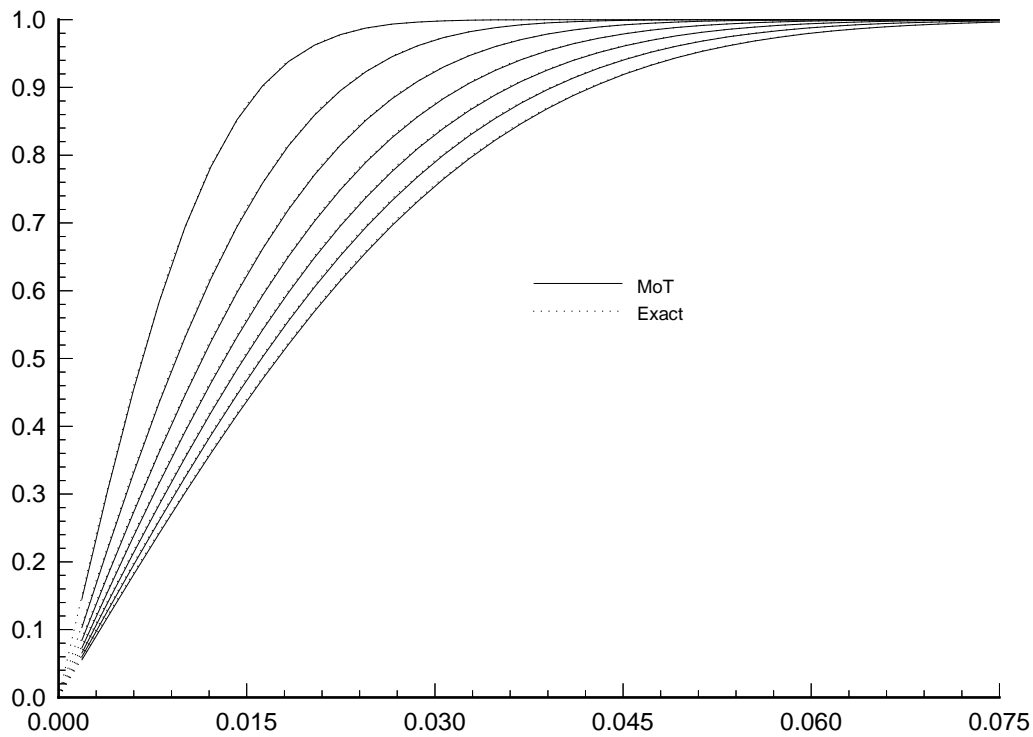


Figure 3: Flow formation in Couette motion at various time levels: Comparison of numerical and exact solution

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