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Abstract

We use difference methods to prove the existence of global $W^{1,\infty}$-solutions to the isentropic gas equations with some special initial data, in which the density is non-negative but not necessarily positive. The difference scheme employed is also shown to be convergent.

Keywords: isentropic gas equation, $W^{1,\infty}$-solutions, difference scheme, weak*-convergence

Subject Classification: 35L65, 65M06, 76M20

In this note we use difference methods to prove the existence of a global $W^{1,\infty}$-solution to the isentropic gas equations in Eulerian coordinates

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= 0
\end{aligned}
\tag{1.1}
$$

with some kind of special initial data

$$
(\rho(x,0), u(x,0)) = (\rho_0(x), u_0(x)). \tag{1.2}
$$

Here $\rho_0(x)$ is non-negative but not necessarily positive. We also show the convergence of the difference scheme under consideration.

The existence result is first established by Lu in [1] with a variant of the viscosity method under some assumptions about the pressure function $p(\rho)$. Unlike the earlier existence results on smooth solutions (see [2] and [3]), this one allows that the density ρ takes the value zero. We note that at $\rho = 0$, the system is not hyperbolic any more.

About $p(\rho)$, we make a slightly more general assumption than that in [1]. The assumption reads as follows:

(a). The sound speed $c(\rho) = (\frac{\partial p}{\partial \rho}(\rho))^{\frac{1}{2}}$ and $\frac{\rho c'(\rho)}{c(\rho)}$ are continuous in $\rho \in [0, \infty)$, and $\rho \mid c'(\rho) \mid \leq c(\rho)$.

(b). $\int_0^\rho \frac{c(s)}{s} ds$, as a function of $\rho \in [0, +\infty)$, is well-defined and has a continuously differentiable inverse.

The reader can easily verify that for polytropic gas, in which $p(\rho) = \text{const.}\rho^\gamma$, the assumption is satisfied for $1 < \gamma \leq 3$.

The result is mainly based on the following features of the system (1.1) with $p(\rho)$ satisfying the above assumption:

(1). The system can be rewritten as the following diagonal form

$$
\begin{aligned}
w_t + \lambda_2(w,z) w_x &= 0 \\
z_t + \lambda_1(w,z) z_x &= 0.
\end{aligned}
\tag{1.3}
$$

Here $w = u + \int_0^\rho \frac{c(s)}{s} ds$, and $z = u - \int_0^\rho \frac{c(s)}{s} ds$.

(2). Both $\lambda_1(w,z) (= u + c(\rho))$ and $\lambda_2(w,z) (= u - c(\rho))$ are continuous differentiable and non-decreasing with respect to their arguments.

non-decreasing smooth initial data have classical solutions. The present existence result can be viewed as an extension of the scalar conclusion for the system. Indeed, we assume that the initial data $(\rho_0(x), u_0(x))$ are such that the corresponding $w_0(x)$ and $z_0(x)$ are non-decreasing. That is the specialness of initial data we mentioned before.

We will prove the existence result by constructing difference solutions to the diagonal system (1.3). It will be seen that our approach is much more elementary and simpler. Moreover, the time-derivatives are uniformly bounded with respect to the time variable (This fact is not derived in [1]).

One advantage of using the diagonal system (1.3), which is not used in [1], is that (1.3) with some initial data has at most one $W^{1,\infty}$-solution under the above assumptions on $p(\rho)$. This fact simply follows from the classical local energy inequality for symmetrizable hyperbolic systems with $W^{1,\infty}$-coefficients ([4]). It is this uniqueness result that enables us to conclude the convergence of the difference solutions (not only a subsequence).

Unlike the viscosity approach, the difference one enables us to make detailed pointwise estimates without having *a priori* smoothness and positiveness information (see [1]). Another advantage of our approach is that initial-boundary value problems can be handled in a similar fashion including complex moving boundary problems (see [5], [3] and references cited therein). On the other hand, the viscosity approach seems to have difficulties for such problems because auxiliary boundary conditions need to be introduced and cause boundary-layer problems. This is left for the future.

The paper is organized as follows. In section 2 we consider the explicit upwind difference scheme for the diagonal system and estimate the difference solution and the first-order difference quotients. Having established the *a priori* estimates, the convergence of the difference solutions and the existence of $W^{1,\infty}$-solutions are concluded in section 3. Finally in section 4, a few remarks are made.

2. *A Priori* Estimates on Difference Solutions

In this section we consider a difference approximation for the system (1.3), and we estimate the difference solutions and the first-order difference quotients in the maximum norm.

First of all, we assume that

(A). $\lambda_1(w, z)$ and $\lambda_2(w, z)$ in (1.3) belong to $C^1([C_1, C_2] \times [C_3, C_4])$ and are non-decreasing with respect to the arguments.

system (1.3) does not necessarily come from the isentropic gas equations.

Let h, τ be the increments respectively in x and t. The difference solutions (w_j^n, z_j^n) are then computed from the explicit upwind scheme

$$
\begin{aligned}
\frac{w_j^{n+1} - w_j^n}{\tau} + \lambda_{2,j}^{+,n} \frac{w_j^n - w_{j-1}^n}{h} + \lambda_{2,j}^{-,n} \frac{w_{j+1}^n - w_j^n}{h} &= 0 \\
\frac{z_j^{n+1} - z_j^n}{\tau} + \lambda_{1,j}^{+,n} \frac{z_j^n - z_{j-1}^n}{h} + \lambda_{1,j}^{-,n} \frac{z_{j+1}^n - z_j^n}{h} &= 0
\end{aligned} \tag{2.1}
$$

together with

$$
(w_j^0, z_j^0) = (w_0(jh), z_0(jh)) \tag{2.2}
$$

for $j = 0, \pm 1, \pm 2, \cdots$ and $n = 0, 1, 2, \cdots$. Here $\lambda^{\pm} = \frac{1}{2}(\lambda \pm |\lambda\,|)$ and $\lambda_j^n = \lambda(w_j^n, z_j^n)$.

At the outset, it is not obvious that the scheme is well defined since the values (w_j^n, z_j^n) might lie beyond the domain of λ_1(or λ_2). The following Lemma 2.1 will establish this.

Lemma 2.1. Let (w_j^0, z_j^0) satisfy

$$
C_1 \leq w_j^0 \leq C_2, \quad C_3 \leq z_j^0 \leq C_4
$$

for all j. Then under the CFL condition

$$
\frac{\tau}{h} \max_{(w,z)} \{|\lambda_1(w,z)|, |\lambda_2(w,z)|\} \leq 1,
$$

the difference system (2.1) has a unique solution

$$
\{(w_j^n, z_j^n) : n = 0, 1, 2, \cdots; j = 0, \pm 1, \pm 2, \cdots\}.
$$

And the solution satisfies the following estimates

$$
C_1 \leq w_j^n \leq C_2, \quad C_3 \leq z_j^n \leq C_4 \tag{2.3}
$$

for all j and all n.

Proof. We use induction on n. Obviously, $\{(w_j^0, z_j^0)\}_j$ is well-defined and (w_j^0, z_j^0) satisfies (2.3) for each j. Assume that $\{(w_j^n, z_j^n)\}_j$ has been uniquely determined by the scheme (2.1) and (w_j^n, z_j^n) satisfies (2.3) for each j. Then $\lambda_{2,j}^n$ is defined and

$$
\frac{\tau}{h} |\lambda_{2,j}^n| \leq 1
$$

under the CFL condition.

$$
\begin{aligned}
w_j^{n+1} &= (1-\frac{\tau}{h}\,\lambda_{2,j}^{+,n} + \frac{\tau}{h}\,\lambda_{2,j}^{-,n})w_j^n + \frac{\tau}{h}\,\lambda_{2,j}^{+,n}w_{j-1}^n - \frac{\tau}{h}\,\lambda_{2,j}^{-,n}w_{j+1}^n \\
&= (1-\frac{\tau}{h}\mid \lambda_{2,j}^n \mid)w_j^n + \frac{\tau}{h}\,\lambda_{2,j}^{+,n}w_{j-1}^n - \frac{\tau}{h}\,\lambda_{2,j}^{-,n}w_{j+1}^n.
\end{aligned}
$$

Thus, w_j^{n+1} is defined as a convex combination of w_{j-1}^n, w_j^n and w_{j+1}^n, and therefore satisfies the estimate.

The same argument applies to z_j^n. Hence the lemma is proved. □

Next, we turn to estimate the difference quotients. To do this, we set

$$
W_j^n = \frac{w_j^n - w_{j-1}^n}{h} \quad \text{and} \quad Z_j^n = \frac{z_j^n - z_{j-1}^n}{h},
$$

and compute from the scheme that W_j^n and Z_j^n satisfy

$$
\begin{aligned}
&\frac{W_j^{n+1}-W_j^n}{\tau} + \lambda_{2,j-1}^{+,n}\,\frac{W_j^n - W_{j-1}^n}{h} + \\
&\qquad + \lambda_{2,j}^{-,n}\,\frac{W_{j+1}^n - W_j^n}{h} + \frac{\lambda_{2,j}^n - \lambda_{2,j-1}^n}{h}W_j^n = 0 \\
&\frac{Z_j^{n+1}-Z_j^n}{\tau} + \lambda_{1,j-1}^{+,n}\,\frac{Z_j^n - Z_{j-1}^n}{h} + \\
&\qquad + \lambda_{1,j}^{-,n}\,\frac{Z_{j+1}^n - Z_j^n}{h} + \frac{\lambda_{1,j}^n - \lambda_{1,j-1}^n}{h}\,Z_j^n = 0.
\end{aligned}
\tag{2.4}
$$

In addition, we note that

$$
\begin{aligned}
\lambda_j^{\pm} - \lambda_{j-1}^{\pm} &= \frac{1}{2}\,(\lambda_j \pm \mid \lambda_j \mid -(\lambda_{j-1} \pm \mid \lambda_{j-1} \mid)) \\
&= \frac{1}{2}\,((\lambda_j - \lambda_{j-1}) \pm (\mid \lambda_j \mid - \mid \lambda_{j-1} \mid)) \\
&= \frac{1}{2}\,((\lambda_j - \lambda_{j-1}) \pm \frac{\lambda_j^2 - \lambda_{j-1}^2}{\mid \lambda_j \mid + \mid \lambda_{j-1} \mid}) \\
&= \frac{1}{2}\,(\lambda_j - \lambda_{j-1})(1 \pm \frac{\lambda_j + \lambda_{j-1}}{\mid \lambda_j \mid + \mid \lambda_{j-1} \mid})
\end{aligned}
\tag{2.5}
$$

and

$$
\alpha_j^{\pm} \equiv 1 \pm \frac{\lambda_j + \lambda_{j-1}}{\mid \lambda_j \mid + \mid \lambda_{j-1} \mid} \geq 0.
\tag{2.6}
$$

Lemma 2.2. Suppose (W_j^0, Z_j^0) satisfy

$$
0 \leq W_j^0 \leq C_5, \quad 0 \leq Z_j^0 \leq C_6
$$

$$\frac{\tau}{h}\max_{(w,z)}\{|\lambda_1(w,z)|,|\lambda_2(w,z)|\}\leq 1-\delta$$

holds. Then for τ sufficiently small, the following estimates

$$0\leq W_j^n\leq C_5,\quad 0\leq Z_j^n\leq C_6 \tag{2.7}$$

hold for all j and all n.

Proof. Obviously, (2.7) holds for $n=0$ and all j. Assume (2.7) holds for n and all j. We prove (2.7) for $n+1$ and all j.

Because of the assumed truth for n and the assumption (A), there is a constant K, depending only on $C_i (i=1,2,\cdots,6)$, such that

$$\frac{\lambda_{2,j}^n-\lambda_{2,j-1}^n}{h}\leq K.$$

Moreover, since $\lambda_2(w,z)$ is non-decreasing with respect to its arguments, $w_j^n\geq w_{j-1}^n$ and $z_j^n\geq z_{j-1}^n$, we have

$$0\leq\frac{\lambda_{2,j}^n-\lambda_{2,j-1}^n}{h}.$$

Furthermore,

$$\lambda_{2,j}^{-,n}\geq\lambda_{2,j-1}^{-,n},$$

where (2.5)-(2.6) have been used.

Next, from (2.4) and the assumed truth for n we deduce that

$$\begin{aligned}W_j^{n+1} &\geq (1-\frac{\tau}{h}\lambda_{2,j-1}^{+,n}+\frac{\tau}{h}\lambda_{2,j}^{-,n})W_j^n+\frac{\tau}{h}\lambda_{2,j-1}^{+,n}W_{j-1}^n-\frac{\tau}{h}\lambda_{2,j}^{-,n}W_{j+1}^n-\tau KW_j^n\\ &\geq (1-\frac{\tau}{h}\lambda_{2,j-1}^{+,n}+\frac{\tau}{h}\lambda_{2,j-1}^{-,n}-\tau K)W_j^n\\ &\geq (1-\frac{\tau}{h}\mid\lambda_{2,j-1}^n|-\tau K)W_j^n\\ &\geq 0,\end{aligned}$$

whenever $\tau K\leq\delta$.

Again from (2.4) we have

$$\begin{aligned}W_j^{n+1}\leq\ &(1-\frac{\tau}{h}\lambda_{2,j-1}^{+,n}+\frac{\tau}{h}\lambda_{2,j}^{-,n})W_j^n+\\ &+\frac{\tau}{h}\lambda_{2,j-1}^{+,n}W_{j-1}^n-\frac{\tau}{h}\lambda_{2,j}^{-,n}W_{j+1}^n.\end{aligned}\tag{2.8}$$

$$1 - \frac{\tau}{h}\lambda_{2,j-1} + \frac{\tau}{h}\lambda_{2,j} \geq 1 - \frac{\tau}{h}|\lambda_{2,j-1}| \geq 0,$$

the right-hand side in (2.8) is a convex combination of W_{j-1}^n, W_j^n and W_{j+1}^n, and therefore $W_j^{n+1} \leq C_5$.

Similarly, $0 \leq Z_j^{n+1} \leq C_6$. This completes the proof. $\square$

Corollary. Under the conditions of Lemma 2.2, there is a positive constant M, depending only on $C_i (i = 1, 2, \cdots, 6)$, such that for τ sufficiently small, the following estimates

$$\left|\frac{w_j^{n+1} - w_j^n}{\tau}\right|, \quad \left|\frac{z_j^{n+1} - z_j^n}{\tau}\right| \leq M$$

hold for all j and all n.

Proof. The estimates simply follow from the scheme (2.1) and Lemma 2.2. $\square$

3. Convergence and Existence

The objective here is to prove our convergence and existence theorems by investigating the limits of the difference approximations as the mesh sizes, which satisfy the CFL condition in Lemma 2.2, tend to zero.

To begin with, we construct several families of piecewise constant functions defined in the upper-half plane $R \times R_+ (\equiv (-\infty, +\infty) \times [0, +\infty) \ni (x, t))$ as follows:

$$\begin{aligned}
(w_{h,\tau}, z_{h,\tau})(x, t) &= (w, z)_j^n, \\
(\hat{W}_{h,\tau}, \hat{Z}_{h,\tau})(x, t) &= \frac{(w, z)_j^{n+1} - (w, z)_j^n}{\tau}, \\
(W_{h,\tau}, Z_{h,\tau})(x, t) &= \frac{(w, z)_j^n - (w, z)_{j-1}^n}{h}, \\
(\check{W}_{h,\tau}, \check{Z}_{h,\tau})(x, t) &= \frac{(w, z)_{j+1}^n - (w, z)_j^n}{h}
\end{aligned}$$

for $(x, t) \in [jh, (j+1)h) \times [n\tau, (n+1)\tau)$. From the lemmas and the corollary in the previous section, these functions are uniformly bounded with respect to h and τ. Thus, we can extract a mesh sequence $\{(h_k, \tau_k)\}_{k=1}^{\infty}$, in which each (h_k, τ_k) satisfies the CFL condition in Lemma 2.2, such that as $k \longrightarrow \infty$, h_k goes to zero and the corresponding function sequences constructed above converge respectively to $(w, z), (\hat{W}, \hat{Z}), (W, Z)$ and $(\check{W}, \check{Z})$

$$(w_t, z_t) = (\hat{W}, \hat{Z}) \quad \text{and} \quad (w_x, z_x) = (W, Z) = (\check{W}, \check{Z}),$$

where the subscripts denote the generalized derivatives. Consequently,

$$(w, z) \in W^{1,\infty}(R \times R_+)$$

and as $k \longrightarrow \infty$,

$$\begin{aligned}
(w_k, z_k) &\overset{*}{\rightharpoonup} (w, z), \\
(\hat{W}_k, \hat{Z}_k) &\overset{*}{\rightharpoonup} (w_t, z_t), \\
(W_k, Z_k) &\overset{*}{\rightharpoonup} (w_x, z_x), \\
(\check{W}_k, \check{Z}_k) &\overset{*}{\rightharpoonup} (w_x, z_x)
\end{aligned}$$

in $L^\infty(R \times R_+)$. Here for notational convenience we have used w_k for w_{h_k,τ_k}, and so on.

Next, we show that $\{(w_k, z_k)\}$ has a subsequence converging uniformly to (w, z) in each bounded subset of $R \times R_+$. To this end, we construct another family of functions $(w^\triangle, z^\triangle)(x, t)$ defined in $R \times R_+$ as follows:

$$\begin{aligned}
(w^\triangle, z^\triangle)(x, t) = {}& (1 - \frac{t - n\tau}{\tau})\,(1 - \frac{x - jh}{h})\,(w, z)_j^n \quad + \quad (1 - \tfrac{t-n\tau}{\tau})\tfrac{x-jh}{h}(w, z)_{j+1}^n \\
& + \frac{t - n\tau}{\tau}\,(1 - \frac{x - jh}{h})\,(w, z)_j^{n+1} \quad + \quad \frac{t - n\tau}{\tau}\,\frac{x - jh}{h}\,(w, z)_{j+1}^{n+1}
\end{aligned}$$

for $(x, t) \in [jh, (j+1)h) \times [n\tau, (n+1)\tau)$. First, we easily deduce from Lemma 2.2 that

$$|\,(w^\triangle, z^\triangle) - (w_{h,\tau}, z_{h,\tau})\,| \leq C(h + \tau), \tag{3.1}$$

where C depends only on $C_i (i = 1, 2, \cdots, 6)$. On the other hand, because each $(w^\triangle, z^\triangle)$ is continuous and because of the *a priori* estimates in the previous section, $\{(w^\triangle, z^\triangle)\}$ lie in a bounded subset of $W^{1,\infty}(R \times R_+)$. Therefore, it follows from the well-known embedding theorem that $\{(w^{\triangle_k}, z^{\triangle_k})\}$, with $\triangle_k = (h_k, \tau_k)\}$, contains a subsequence, still denoted by $\{(w^{\triangle_k}, z^{\triangle_k})$, uniformly convergent in each bounded domain as $k \longrightarrow \infty$. Thus, we conclude from (3.1) that $\{(w_k, z_k)\}$ (a subsequence) converges to (w, z) uniformly in each bounded domain. In particular,

$$(w, z)(x, 0) = (w_0(x), z_0(x)).$$

R_+). Then we have from the scheme (2.1)

$$\sum_{j,n} h\tau\varphi(jh,n\tau)\,(\frac{w_j^{n+1}-w_j^n}{\tau}\;+\;\lambda_{2,j}^{+,n}\,\frac{w_j^n-w_{j-1}^n}{h}\;+\;\lambda_{2,j}^{-,n}\,\frac{w_{j+1}^n-w_j^n}{h})=0,$$

or

$$\int\varphi_{h,\tau}(\hat{W}_{h,\tau}+\lambda_{2,h,\tau}^{+}W_{h,\tau}+\lambda_{2,h,\tau}^{-}\check{W}_{h,\tau})dxdt=0,$$

where $\varphi_{h,\tau}(x,t)=\varphi(jh,n\tau)$ for $(x,t)\in[jh,(j+1)h)\times[n\tau,(n+1)\tau)$ and $\lambda_{2,h,\tau}^{\pm}=\lambda_2^{\pm}(w_{h,\tau},z_{h,\tau})$. Since $\hat{W}_k$ is uniformly bounded and converges to w_t in the weak*-topology of L^∞, and φ_k converges uniformly to φ as $k\longrightarrow\infty$, we have

$$\int\varphi_k\hat{W}_kdxdt=\int\varphi\hat{W}_kdxdt+\int(\varphi_k-\varphi)\hat{W}_kdxdt$$

$$\longrightarrow\int\varphi w_tdxdt.$$

Furthermore, because $\lambda_2^+(w,z)$ is continuous with respect to its arguments and (w_k,z_k) converges uniformly to (w,z) in each bounded domain, $\lambda_{2,k}^+$ converges uniformly to λ_2^+ in each bounded domain. Therefore, we have

$$\int\varphi_k\lambda_{2,k}^+W_kdxdt\longrightarrow\int\varphi\lambda_2^+w_xdxdt.$$

Similarly,

$$\int\varphi_k\lambda_{2,k}^-\check{W}_kdxdt\longrightarrow\int\varphi\lambda_2^-w_xdxdt.$$

Consequently, we arrive at

$$\int\varphi(w_t+\lambda_2w_x)dxdt=0.$$

Similarly,

$$\int\varphi(z_t+\lambda_1z_x)dxdt=0.$$

Because φ is arbitrary, the last two equations give

$$w_t+\lambda_2w_x=0\quad\text{and}\quad z_t+\lambda_1z_x=0$$

for almost every $(x,t)\in R\times R_+$. Thus, we have proved

Theorem 1. Let λ_1 and λ_2 satisfy the assumption (A), $(w_0,z_0)\in W^{1,\infty}(R)$ are non-decreasing and take values in $[C_1,C_2]\times[C_3,C_4]$. Then the diagonal system (1.3) with initial data $(w_0(x),z_0(x))$ has a solution in $W^{1,\infty}(R\times R_+)$.

we easily know that the diagonal system has at most one $W^{1,\infty}$-solution. Thus, when the mesh sizes tend to zero, the difference solutions converge to the unique solutions. Therefore we have

Theorem 2. Under the assumptions of Theorem 1, the difference solutions converge uniformly to the unique solution in each bounded domain, as the mesh sizes, satisfying the CFL condition in Lemma 2.2, go to zero.

The above discussions are all on the diagonal system (1.3). We now return to the isentropic gas equations (1.1). The next theorem can be viewed as a corollary of Theorem 1. We prove that under the assumption about $p(\rho)$, any $W^{1,\infty}$-solution to the diagonal system induces a $W^{1,\infty}$-solution to the isentropic gas equations, whenever

$$w \geq C_1 \geq C_4 \geq z.$$

Recall that the last inequalities are fulfilled as long as the density $\rho \geq 0$.

Theorem 3. Suppose $p(\rho)$ satisfies the assumption, the initial data $(\rho_0(x), u_0(x))$ are such that the corresponding $(w_0(x), z_0(x))$ satisfy the condition in Theorem 1 and $C_1 \geq C_4$. Then the isentropic gas equations (1.1) has a $W^{1,\infty}$-solution with the prescribed initial data.

Proof. The proof is an elementary calculation, which is carried out very carefully in order to avoid the troubles caused by $\rho = 0$ or $w = z$.

Let Φ be the inverse function of $\int_0^\rho \frac{c(s)}{s} ds$, that is,

$$\Phi(\int_0^\rho \frac{c(s)}{s} ds) \equiv \rho$$

for $\rho \in [0, +\infty)$. Differentiating the last identity with respect to ρ yields

$$c(\rho)\Phi' = \rho$$

for $\rho > 0$. This equation still holds at $\rho = 0$, since the existence of the integral implies $c(0) = 0$.

Let $(w, z) \in W^{1,\infty}$ solve the diagonal system induced from the isentropic gas equations. Note that $w \geq C_1 \geq C_4 \geq z$. We define

$$\rho = \Phi(\frac{w - z}{2}) \quad \text{and} \quad u = \frac{w + z}{2}.$$

$$
\begin{aligned}
\rho_t + (\rho u)_x &= \Phi' \frac{w_t - z_t}{2} + \rho u_x + u \rho_x \\
&= \Phi' \frac{w_t - z_t}{2} + \rho \frac{w_x + z_x}{2} + u \Phi' \frac{w_x - z_x}{} \\
&= \Phi' \left(\frac{w_t - z_t}{2} + u \frac{w_x - z_x}{2}\right) + \rho \frac{w_x + z_x}{2} \\
&= \Phi'(-c(\rho) \frac{w_x + z_x}{2}) + \rho \frac{w_x + z_x}{2} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(u_t + u u_x) + p(\rho)_x &= \rho(-c(\rho) \frac{w_x - z_x}{2}) + c^2(\rho) \rho_x \\
&= -\rho c(\rho) \frac{w_x - z_x}{2} + c^2(\rho) \Phi' \frac{w_x - z_x}{2} \\
&= 0. \qquad \square
\end{aligned}
$$

Theorem 4. Under the assumptions of Theorem 3 and as the mesh sizes tend to zero, $\frac{w_{h,\tau}+z_{h,\tau}}{2}$ and $\Phi(\frac{w_{h,\tau}-z_{h,\tau}}{2})$ converge uniformly to the velocity u and the density ρ in each bounded domain, respectively. The same is true of $\frac{w^{\triangle}+z^{\triangle}}{2}$ and $\Phi(\frac{w^{\triangle}-z^{\triangle}}{2})$.

4. Remarks

A few remarks are made in this section.

(1). Because $W^{1,\infty}$ (compactly) embeds into $C^{0,\alpha}$ for any $\alpha \in (0,1)$ on compacta, (w,z) is in fact locally Hölder continuous.

(2). From the proof of Theorem 3, any $W^{1,\infty}$-solution to the diagonal system induces a $W^{1,\infty}$-solution to the original system under the assumption about the pressure function. It is, however, not clear if the inverse is true. Therefore, although the diagonal system has a unique solution in $W^{1,\infty}$, we don't claim any uniqueness result for the isentropic gas equations. The latter is not hyperbolic at $\rho = 0$.

(3). For the polytropic gas with $\gamma \geq 3$, one can easily verify that a $W^{1,\infty}$-solution to the isentropic gas system induces a $W^{1,\infty}$-solution to the diagonal one. Because the latter has at most one $W^{1,\infty}$-solution, $W^{1,\infty}$-solutions to the isentropic gas equations are unique, where the density is non-negative but not necessarily positive.

(4). The above arguments and results can be easily extended to the diagonal systems of n equations. The only requirement is that the corresponding eigenvalues are smooth and in [6] about the system of three equations can be established with the above arguments.

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