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Abstract

We analyze the hp Streamline Diffusion Finite Element Method (SDFEM) and the standard Galerkin FEM for one dimensional stationary convection-diffusion problems. Under the assumption of analyticity of the input data, a mesh is exhibited on which approximation with continuous piecewise polynomials of degree p allows for resolution of the boundary layer. On such meshes, both the SDFEM and the Galerkin FEM lead to robust exponential convergence in the "energy norm" and in the L^∞ norm.

Next, we show that even in the case that the boundary layers are not resolved, robust exponential convergence on compact subsets "upstream" of the layer can be achieved with the hp-SDFEM. This is possible on sequences of meshes that would typically be generated by an hp-adaptive scheme.

Detailed numerical experiments confirm our convergence estimates.

1 Introduction

The Streamline Diffusion Finite Element Method (SDFEM) was introduced by T. Hughes, C. Johnson and their coworkers to improve the stability of Galerkin Finite Element discretizations of advection dominated diffusion problems since standard Galerkin FEM were known to produce oscillatory solutions for these non-selfadjoint problems. In the pioneering papers ([2, 3, 5, 6]), the SDFEM was introduced and its convergence rate as the meshwidth h of the FE triangulation $\mathcal{T}$ tends to zero was analyzed. In the meantime, numerous papers have appeared showing how the SDFEM can be combined with the related SUPG techniques [4] to obtain stable discretizations of convection dominated, incompressible flow problems and of problems with analogous mathematical structure in solid mechanics [16, 17].

All these works considered the h-version FEM where convergence is achieved by refining the mesh $\mathcal{T}$ at fixed, low polynomial degree p. The convergence rates were consequently at best *algebraic*. In the 1980ies, the p- and hp-FEM were introduced by I. Babuška and B.A. Szabó and their coworkers, and it was shown that the hp-FEM achieves *exponential convergence* for elliptic problems with *piecewise analytic* solutions (cf. the survey paper [1] and the references therein).

For singularly perturbed reaction-diffusion problems, it was shown recently in [14, 15, 8, 9] that the hp-FEM can achieve *robust exponential convergence*. Analogous to the standard h-version, the main problem in the convection dominated case is *stability*.

A rigorous proof of robust exponential convergence for the hp-streamline diffusion and Galerkin FEM for *convection dominated* problems in one dimension is the purpose of the present paper. We confine our analysis to the one-dimensional case since there the asymptotic structure of the exact solution is known in detail [9]. We prove *robust exponential convergence* of the hp-SDFEM and hp-Galerkin FEM in global norms provided that the boundary layers and fronts of the solution are resolved. Whereas in the h-version FEM this amounts to the use of so-called Shishkin meshes (cf. [11, 12]), in the hp context this can be achieved very efficiently by inserting just one element of the proper size into the layer (see, e.g., [14, 15, 8, 9]). Furthermore, we investigate the behavior of the hp-SDFEM under the assumption that layers are not resolved which may happen, for example, when the precise location of the layer is unknown. In this case, we show that the hp-SDFEM leads to robust exponential convergence on compact subsets upstream of the layer/shock for certain types of mesh sequences. These mesh sequences are those that would typically be generated by an adaptive scheme that locates the layers and tries to resolve them. Finally, for a model problem we study numerically the optimal p-dependence of the SDFEM parameter and find the choice $O(h_i/\sqrt{p})$ to give the best performance.

Our theoretical results are in agreement with our numerical experiments which also show that hp-SDFEM and standard hp-FEM are comparable if all small scale features are resolved. In itself, this is already known to CFD practitioners. The main impetus for the development of SDFEM and related subgrid scale models has in fact come from the need for stabilization because of the inability of standard FEM to resolve all small scales of the flow. The main conclusions of the present work are twofold:

1. hp-FEMs *are* able to resolve localized small scale features of the solution such as viscous boundary layers and shock profiles highly accurately at very reasonable cost [14, 8, 9, 15].

2. In the preasymptotic range when small scale features are not resolved, the hp-SDFEM can lead to robust exponential convergence on compact subsets if the increase of the polynomial degree is coupled with an appropriate mesh refinement in low order elements towards the layers.

The present work has natural extensions to two and three dimensions as well as to systems—these issues shall be dealt with elsewhere.

The outline of the paper is as follows. In Sections 1.1, 1.2 we present our model problem and introduce two types of Streamline Diffusion Methods, the "L^2-stabilized" and "bubble-stablized" method. In Section 2 we show that piecewise polynomials on appropriate meshes can resolve small scale features such as layers and fronts at a robust exponential rate. Section 3 is devoted to a detailed analysis of both the "L^2-" and the "bubble-stabilized" SDFEM; their performance is measured in the global "energy norm". Our theoretical results are corroborated by numerical examples in Section 4. Section 4 also contains the statement that the hp-SDFEM can lead to robust exponential convergence of compact subsets if the increase of the polynomial degree is coupled with an appropriate mesh refinement towards the layer.

Section 4 concludes with a numerical investigation of the optimal choice of the SDFEM parameter.

1.1 Model Problem

We consider the one-dimensional convection–diffusion equation

$$L_\varepsilon u \equiv -\varepsilon u'' + a(x)u' + b(x)u = f \quad \text{on } \Omega := (-1,1), \qquad u(\pm 1) = 0. \tag{1.1}$$

Here the parameter $\varepsilon \in (0,1]$ may approach zero, $f \in L^2(\Omega)$, and the coefficients $a \in W^{1,\infty}(\Omega)$ and $b \in L^\infty(\Omega)$ are assumed to satisfy

$$a(x) \geq \underline{a} > 0 \qquad \text{on } \Omega, \tag{1.2}$$
$$b(x) \geq \underline{b} \in \mathbb{R} \quad \text{on } \Omega, \qquad \underline{a}^2 + 4\varepsilon\underline{b} > 0 \qquad \forall \varepsilon \in (0,1]. \tag{1.3}$$

Conditions (1.2), (1.3) guarantee that (1.1) does have a unique solution for all $\varepsilon \in (0,1]$. The prototypical analysis of this paper is performed in Section 3 under the following additional assumption:

$$b(x) - \frac{1}{2}a'(x) \geq \gamma > 0 \qquad \text{on } \Omega. \tag{1.4}$$

Remark 1.1 Condition (1.4) can always be achieved for problems of the form (1.1) by the substitution $u(x) = e^{\omega x}\tilde{u}(x)$ for some bounded, appropriately chosen ω and ε sufficiently small.

A weak formulation of (1.1) reads: Find $u_\varepsilon \in H_0^1(\Omega)$ such that

$$B_\varepsilon(u_\varepsilon, v) := \varepsilon \int_\Omega u_\varepsilon' v'\, dx + \int_\Omega (a u_\varepsilon' + b u_\varepsilon) v\, dx = F(v) := \int_\Omega f v\, dx \qquad \forall v \in H_0^1(\Omega). \tag{1.5}$$

It is natural to introduce the following "energy norm" on the space $H_0^1(\Omega)$

$$|||u|||^2 := \varepsilon \|u'\|^2_{L^2(\Omega)} + \gamma \|u\|^2_{L^2(\Omega)}. \tag{1.6}$$

Proposition 1.2 *Under the assumptions* (1.2), (1.4) *there holds*

$$\begin{aligned} |||u|||^2 &\leq B_\varepsilon(u,u) \qquad \forall u \in H_0^1(\Omega), \\ |B_\varepsilon(u,v)| &\leq \left[2\max\left(1, \|a\|_{L^\infty(\Omega)}/\sqrt{\gamma\varepsilon}, \|b\|_{L^\infty(\Omega)}/\gamma\right)\right] |||u|||\,|||v||| \qquad \forall u,v \in H_0^1(\Omega). \end{aligned}$$

In particular, therefore, for every $f \in L^2(\Omega)$, there exists a unique solution u_ε of (1.5).

Proof: For the first estimate, we note that an integration by parts yields

$$\int_\Omega a u' u\, dx + \int_\Omega b u^2\, dx = \int_\Omega \left(b - \frac{1}{2}a'\right) u^2\, dx \geq \gamma \|u\|^2_{L^2(\Omega)}.$$

The second estimate follows from the Cauchy-Schwarz inequality.

□

1.2 *hp* FEM

In the classical Galerkin FEM, the infinite dimensional space $H_0^1(\Omega)$ is replaced by finite dimensional spaces of piecewise polynomials of degree $p \geq 1$. In order to make this more precise, we introduce for a collection of nodes $-1 = x_0 < x_1 < \cdots < x_N = 1$ the notation $I_i := (x_{i-1}, x_i)$ and $h_i = |I_i| = x_i - x_{i-1}$ for $i = 1, \ldots, N$. The elements I_i form a mesh $\mathcal{T} = \{I_i \,|\, i = 1, \ldots, N\}$ on Ω. The classical spaces of piecewise polynomials of degree p are then given by

$$S^{p,1}(\mathcal{T}) = \{u \in H^1(\Omega) \,|\, u|_{I_i} \in \Pi_p(I_i), \quad i = 1, \ldots, N\}, \tag{1.7}$$
$$S_0^{p,1}(\mathcal{T}) = H_0^1(\Omega) \cap S^{p,1}(\mathcal{T}), \tag{1.8}$$

where $\Pi_p(J)$ denotes the space of all polynomials of degree p on the interval J. We restrict ourselves here to a uniform polynomial degree distribution for simplicity of exposition—*mutatis mutandis* the polynomial degree may vary from element to element.

The standard Galerkin FEM for (1.5) reads:

$$\text{Find } u_G \in S_0^{p,1}(\mathcal{T}) \text{ such that } B_\varepsilon(u_G, v) = F(v) \qquad \forall v \in S_0^{p,1}(\mathcal{T}). \tag{1.9}$$

With a given mesh $\mathcal{T}$, let us associate a collection of non-negative numbers $(\rho_i)_{i=1}^N$ to be selected and weight functions d_i which can be either $d_i \equiv 1$ or $d_i(x) = b_i(x)$ where b_i is the "quadratic bubble"

$$b_i(x) := \frac{4}{h_i^2}(x - x_i)(x_{i+1} - x). \tag{1.10}$$

For these weights $(\rho_i, d_i)_{i=1}^N$, the SDFEM reads:

$$\text{Find } u_{SD} \in S_0^{p,1}(\mathcal{T}) \text{ such that } B_{SD}(u_{SD}, v) = F_{SD}(v) \qquad \forall v \in S_0^{p,1}(\mathcal{T}), \tag{1.11}$$

where

$$B_{SD}(u,v) \;:=\; B_\varepsilon(u,v) + \sum_{i=1}^N \rho_i \int_{I_i} d_i \left(-\varepsilon u'' + a u' + b u\right) a v' \, dx, \tag{1.12}$$

$$F_{SD}(v) \;:=\; F(v) + \sum_{i=1}^N \rho_i \int_{I_i} d_i f a v' \, dx. \tag{1.13}$$

If $d_i = 1$ for all i, (1.11) will be referred to as the "L^2-stabilized" SDFEM whereas (1.11) with the d_i given by (1.10) will be called the "bubble-stabilized" SDFEM (cf. Definition 3.1 ahead for the precise definition). Note that the choice $\rho_i = 0$ for all i reduces the SDFEM to the usual Galerkin FEM.

Proposition 1.2 implies that the Galerkin FEM satisfies the inf-sup condition and hence (1.9) is uniquely solvable. Existence and uniqueness of the SDFEM solution of problem (1.11) is guaranteed, if the bilinear form B_{SD} is coercive on $S_0^{p,1}(\mathcal{T})$; this will be proved in Theorem 3.5.

2 Polynomial Approximability of the Solution

We prove now that the solution u_ε of (1.1) can be approximated from the spaces $S_0^{p,1}(\mathcal{T})$ at an exponential rate, uniformly in ε. To this end, we first recapitulate some regularity results from [9].

2.1 Regularity of the solution u_ε

Let us consider (1.1) on $\Omega = (-1, 1)$ with *analytic* input data $a(x)$, $b(x)$, $f(x)$ satisfying

$$\|a^{(n)}\|_{L^\infty(\Omega)} \leq C_a \gamma_a^n \qquad \forall n \in \mathbb{N}_0, \tag{2.1}$$

$$\|b^{(n)}\|_{L^\infty(\Omega)} \leq C_b \gamma_b^n \qquad \forall n \in \mathbb{N}_0, \tag{2.2}$$

$$\|f^{(n)}\|_{L^\infty(\Omega)} \leq C_f \gamma_f^n \qquad \forall n \in \mathbb{N}_0, \tag{2.3}$$

for some constants C_a, C_b, C_f, γ_a, γ_b, $\gamma_f > 0$. The purpose of this subsection is to illuminate the regularity properties of u_ε in dependence on the parameter ε. These regularity results are necessary for the proof of *robust exponential convergence* of the *hp*-FEM obtained in the present paper. The proof of the assertions of this section can be found in [9].

The solution u_ε of (1.1) is analytic on Ω; however, for small values of ε, it exhibits a boundary layer at the outflow boundary. This boundary behavior can be characterized with the aid of asymptotic expansions: For any expansion order $M \in \mathbb{N}_0$, we decompose in the standard way into a smooth part w_M, a boundary layer part u_M^{BL}, and a (small) remainder r_M:

$$u_\varepsilon = w_M + u_M^{BL} + r_M. \tag{2.4}$$

Concerning these three parts, it was shown in [9] that the following holds true:

Theorem 2.1 *Let u_ε be the solution of* (1.1) *and assume that* (1.2), (1.3) *hold. Then there are constants C, K depending only on the constants in (2.1)–(2.3) and on the constants $\underline{a}$, $\underline{b}$, such that*

$$\|u_\varepsilon^{(n)}\|_{L^\infty(\Omega)} \leq CK^n \max{(n, \varepsilon^{-1})^n} \qquad \forall n \in \mathbb{N}_0. \tag{2.5}$$

Furthermore, under the assumption $0 < \varepsilon MK \leq 1$, the terms in the decomposition (2.4) satisfy

$$\begin{aligned}
\|w_M^{(n)}\|_{L^\infty(\Omega)} &\leq CK^n n! \qquad \forall n \in \mathbb{N}_0, & (2.6)\\
|\left(u_M^{BL}\right)^{(n)}(x)| &\leq CK^n \max{(n, \varepsilon^{-1})^n} e^{-\underline{a}(1-x)/(2\varepsilon)} \qquad \forall n \in \mathbb{N}_0, \quad x \in I, & (2.7)\\
\|r_M^{(n)}\|_{L^\infty(\Omega)} &\leq C\varepsilon^{1-n}(\varepsilon MK)^M \qquad n = 0, 1, 2, & (2.8)\\
r_M(\pm 1) &= 0. & (2.9)
\end{aligned}$$

2.2 hp-Approximation of boundary layers

As the regularity of the solution u_ε is now available, we are in position to formulate results for the approxmation of u_ε by piecewise polynomials of degree p. We will be interested in *robust exponential* approximation of u_ε by elements of $S_0^{p,1}(\mathcal{T})$. Clearly, as u_ε exhibits in general a boundary layer at the outflow boundary $x = 1$, the mesh $\mathcal{T}$ has to be chosen in dependence on ε. The simplest scheme that leads to robust exponential approximability is the "two-element" approach introduced in [14]. There, piecewise polynomials of degree p on a mesh with two elements are used where one elements of size $O(p\varepsilon)$ is located in the layer therefore able to resolve the boundary layer.

Definition 2.2 *For $\varepsilon > 0$, $\kappa > 0$, and $p \in \mathbb{N}$ define the "two-element mesh" $\mathcal{T} = \mathcal{T}_{\varepsilon,\kappa,p}$ as*

$$\begin{aligned}
\mathcal{T} = \{(-1, 1 - \kappa p\varepsilon), (1 - \kappa p\varepsilon, 1)\} &\quad \text{if } \kappa p\varepsilon < 1,\\
\mathcal{T} = \{\Omega\} &\quad \text{if } \kappa p\varepsilon \geq 1.
\end{aligned}$$

In [9] (cf. also [14, 8]) the following theorem was proved.

Theorem 2.3 *Let u_ε be the solution of* (1.1) *and assume that* (1.2), (1.3), (2.1)–(2.3) *hold. Then there are C, σ, $\kappa_0 > 0$ independent of ε such that for $0 < \kappa < \kappa_0$*

$$\inf_{\pi_p} \left\{\|u_\varepsilon - \pi_p\|_{L^\infty(\Omega)} + \kappa p\varepsilon \|\left(u_\varepsilon - \pi_p\right)'\|_{L^\infty(\Omega)}\right\} \leq C(1 + \ln p)p^2 e^{-\sigma\kappa p}$$

where the infimum is taken over all $\pi_p \in S_0^{p,1}(\mathcal{T}_{\kappa,p,\varepsilon})$.

Theorem 2.3 follows from the ensuing Lemma 2.4. The key ingredient is the ability to decompose u_ε into a "regular" part u_{reg} and a "singular" part u_{BL} which can be approximated separately by piecewise polynomials $u_{reg,p}$, $u_{BL,p} \in S^{p,1}(\mathcal{T})$ at a robust exponential rate.

Lemma 2.4 *There are C, σ, $\kappa_0 > 0$ depending only on a, b, and f (in particular* independent *of ε) such that the following holds. Assume that for every $p \in \mathbb{N}$ a mesh $\mathcal{T}$ is given such that for some $\kappa \in (0, \kappa_0)$ we have $S_0^{p,1}(\mathcal{T}_{\kappa,p,\varepsilon}) \subset S_0^{p,1}(\mathcal{T})$. Then, for each $p \in \mathbb{N}$ the solution u_ε of* (1.1) *admits a splitting*

$$u_\varepsilon = u_{reg} + u_{BL}$$

and there is $u_p \in S_0^{p,1}(\mathcal{T})$ with a corresponding splitting

$$u_p = u_{reg,p} + u_{BL,p} \qquad \text{with } u_{reg,p},\ u_{BL,p} \in S^{p,1}(\mathcal{T})$$

such that the errors

$$\eta_{reg} := u_{reg} - u_{reg,p}, \qquad \eta_{BL} := u_{BL} - u_{BL,p}, \qquad \eta := \eta_{reg} + \eta_{BL}$$

satisfy $\eta_{reg}(\pm 1) = \eta_{BL}(\pm 1) = 0$ *and*

$$\begin{aligned}
\|\eta'_{reg}\|_{L^\infty(I_i)} + \|\eta_{reg}\|_{L^\infty(I_i)} &\leq C h_i e^{-\sigma p}, \qquad i = 1, \ldots, N, &(2.10)\\
\|\eta_{BL}^{(l)}\|_{L^\infty(\Omega)} &\leq C(\kappa p \varepsilon)^{-l} e^{-\sigma\kappa p}, \qquad l = 0, 1, &(2.11)\\
(\kappa p)^{1/2}\varepsilon\|\eta'_{BL}\|_{L^2(\Omega)} + \|\eta_{BL}\|_{L^2(\Omega)} &\leq C\varepsilon^{1/2} e^{-\sigma\kappa p}, \qquad l = 0, 1, &(2.12)\\
\sum_{i=1}^{N} \min\left\{1, \frac{h_i}{\varepsilon}\right\}\left[(\kappa p \varepsilon)^2 \|\eta'\|^2_{L^\infty(I_i)} + \|\eta\|^2_{L^\infty(I_i)}\right] &\leq C e^{-\sigma\kappa p}. &(2.13)
\end{aligned}$$

Proof: We will only sketch the proof as it is very similar to that in [9, 8]. In particular, $\kappa_0 < 1$ is chosen sufficiently small (but independent of p, ε) as in [9, 8]. We will restrict ourselves to the case $\kappa p \varepsilon < 1$ as in the complementary case, one can take the specific splitting $u_{reg} = 0$, $u_{BL} = u_\varepsilon$ and conclude as in [8] that we obtain the desired results.

Let therefore $\kappa p \varepsilon < 1$. The key idea is to use the asymptotic expansion (2.4) and to choose the expansion order M proportional to the polynomial degree p. Let $0 < q < 1$ and $\mu := q/K$ with K of Theorem 2.1 and chooose $1 \leq M = \mu\kappa p$ (strictly speaking, M should be chosen as the integer part of $\mu\kappa p$—for notational convenience, however, we will not pursue this point further). We note that with this choice of μ, and the assumption $\kappa p \varepsilon < 1$, (2.6) gives bounds on the derivatives of w_M which are independent of ε, M. We therefore set $u_{reg} := w_M$, $u_{BL} := u_M^{BL} + r_M$ and approximate each term separately. First, for the approximation of u_{reg} we note that standard piecewise polynomial approximation theory (e.g., the piecewise Gauss–Lobatto interpolant) gives the existence of the piecewise polynomials $u_{reg,p}$ such that $\eta_{reg} = u_{reg} - u_{reg,p}$ has the properties (2.10).

Let us now turn to the construction of $u_{BL,p}$, an approximation of $u_{BL} := u_\varepsilon^{BL} + r_M$. Equation (2.8) gives that

$$\begin{aligned}
\|r_M^{(l)}\|_{L^\infty(\Omega)} &\leq C\varepsilon^{1-l}(\varepsilon M K)^M \leq C\varepsilon^{1-l}(\varepsilon M K)(\varepsilon M K)^{M-1} \leq C\varepsilon^{2-l}\kappa p\, q^{\kappa p - 1}\\
&\leq C\varepsilon^{2-l} e^{-\sigma\kappa p}, \qquad l = 0, 1, 2 \qquad (2.14)
\end{aligned}$$

for some appropriate $\sigma > 0$ independent of ε, p. Next, let x_n, $1 < n < N$ be the mesh point $x_n = 1 - \kappa p \varepsilon$. As κ_0 can be chosen sufficiently small, we may assume that $0 \leq x_n$. Checking the proof of Theorem 16 of [8] shows that for $\sigma > 0$ sufficiently small, the Gauss-Lobatto interpolant π_p of u_M^{BL} on $[x_n, 1]$ satisfies

$$\|(u_M^{BL} - \pi_p)^{(l)}\|_{L^\infty([x_n,1])} \leq C(\kappa p \varepsilon)^{-l} e^{-\sigma p}, \qquad l = 0, 1. \qquad (2.15)$$

Following an idea of [14], we now define

$$u_{BL,p} := \begin{cases} u_M^{BL}(-1)\left(1 - \frac{1+x}{1+x_n}\right) & \text{on } [-1, x_n] \\ \pi_p - u_M^{BL}(x_n)\frac{1-x}{1-x_n} & \text{on } [x_n, 1]. \end{cases} \qquad (2.16)$$

Note that $u_{BL,p} \in S^{p,1}(\mathcal{T})$ and that, by (2.9) $u_{BL,p}(\pm 1) = u_{BL}(\pm 1)$. Furthermore, we observe that by (2.7)

$$|u_M^{BL}(x_n)| \leq C e^{-\underline{a}\kappa p/2}, \qquad |u_M^{BL}(-1)| \leq C e^{-\underline{a}/\varepsilon} \leq C'\varepsilon e^{-\underline{a}/(2\varepsilon)} \qquad (2.17)$$

for some C, $C' > 0$ independent of ε. Introducing the shorthand

$$z := u_M^{BL} - u_{BL,p}$$

and using (2.7), (2.15), (2.17) we have for some $\sigma > 0$ independent of ε, p:

$$\begin{aligned}
\kappa p \varepsilon\|z'\|_{L^\infty(I_i)} + \|z\|_{L^\infty(I_i)} &\leq C(1 + \kappa p) e^{-(1-x_i)\underline{a}/(2\varepsilon)}, \qquad i = 1, \ldots, n &(2.18)\\
\kappa p \varepsilon\|z'\|_{L^\infty(I_i)} + \|z\|_{L^\infty(I_i)} &\leq C e^{-\sigma\kappa p}, \qquad i = n+1, \ldots, N. &(2.19)
\end{aligned}$$

As $\|(\eta_B)^{(l)}\|_{L^\infty(\Omega)} \leq \|(u_M^{BL} - u_{BL,p})^{(l)}\|_{L^\infty(\Omega)} + \|r_M^{(l)}\|_{L^\infty(\Omega)}$, it is now easy to see that (2.11) is satisfied by combining (2.18), (2.19), and (2.14). One proceeds similarly to obtain (2.12): Combining (2.14) and

(2.19) yields bounds on $(x_n, 1)$ and (2.14), (2.16), (2.7) together give bounds on $(-1, x_n)$, so that we obtain, for appropriate C, $\sigma > 0$:

$$\begin{aligned}(\varepsilon\kappa p)^2\|\eta'_{BL}\|^2_{L^2(x_n,1)} + \|\eta_{BL}\|^2_{L^2(x_n,1)} &\leq C\kappa p\varepsilon\, e^{-\sigma\kappa p},\\ \varepsilon^2\|\eta'_{BL}\|^2_{L^2(-1,x_n)} + \|\eta_{BL}\|^2_{L^2(-1,x_n)} &\leq C\varepsilon e^{-\sigma\kappa p}.\end{aligned}$$

The desired estimate (2.12) can be obtained easily from these two bounds.

We finally turn to the proof of (2.13). We observe that the expression

$$S(\eta) := \left\{\sum_{i=1}^N \min\{1, h_i/\varepsilon\}\left[(\kappa p\varepsilon)^2\|\eta'\|^2_{L^\infty(I_i)} + \|\eta\|^2_{L^\infty(I_i)}\right]\right\}^{1/2}$$

defines a norm on $W^{1,\infty}(\Omega)$. The triangle inequality therefore gives $S(\eta) \leq S(\eta_{reg}) + S(z) + S(r_M)$. Together with the fact that $\sum_{i=1}^N h_i = 2$, it follows easily from (2.10), (2.14) that $S(\eta_{reg}) + S(r_M)$ satisfies the desired estimate. For the remaining term, $S(z)$, we proceed as follows. First, we recall that n is chosen such that $x_n = 1 - \kappa p\varepsilon$. This implies that

$$\text{card}\,\{i\,|\,i \geq n+1, \quad h_i \leq \varepsilon\} \leq \kappa p\varepsilon/\varepsilon = \kappa p$$

and, together with $\sum_{i=n+1}^N h_i = \kappa p\varepsilon$, this implies that $\sum_{i=n+1}^N \min\{1, h_i/\varepsilon\} \leq 2\kappa p$. Hence, upon using the shorthand

$$z_i := (\kappa p\varepsilon)^2\|z'\|^2_{L^\infty(I_i)} + \|z\|^2_{L^\infty(I_i)}$$

we get using (2.19)

$$\sum_{i=n+1}^N \min\{1, h_i/\varepsilon\} z_i \leq Ce^{-\sigma\kappa p}\sum_{i=n+1}^N \min\{1, h_i/\varepsilon\} \leq C\kappa p e^{-\sigma\kappa p}.$$

Finally, from (2.18) we get

$$\sum_{i=1}^n \min\{1, h_i/\varepsilon\} z_i \leq C(1+\kappa p)^2 \sum_{i=1}^n \min\{1, h_i/\varepsilon\} e^{-\underline{a}(1-x_i)/\varepsilon}.$$

A simple calculation shows that there is $C > 0$ (independent of ε, i) with

$$e^{-\underline{a}(1-x_i)/\varepsilon}\min\{1, h_i/\varepsilon\}\left(\int_{x_{i-1}}^{x_i} e^{-\underline{a}(1-t)/\varepsilon}\,dt\right)^{-1} \leq C\varepsilon^{-1}.$$

Whence

$$\sum_{i=1}^n \min\{1, h_i/\varepsilon\} z_i \leq C\varepsilon^{-1}\sum_{i=1}^n \int_{x_{i-1}}^{x_i} e^{-\underline{a}(1-t)/\varepsilon}\,dt \leq C\varepsilon^{-1}\int_{-1}^{x_n} e^{-\underline{a}(1-t)/\varepsilon}\,dt \leq Ce^{-\underline{a}\kappa p}.$$

Combining these estimates allows us to conclude the proof.

□

3 Analysis of the SDFEM

3.1 Preliminaries and Notation

For the analysis of the SDFEM, it is convenient to introduce the following mesh-dependent semi norm and full norm:

$$|||u|||^2_\rho := \sum_{i=1}^N \rho_i\|\sqrt{d_i} a u'\|^2_{L^2(I_i)}, \tag{3.1}$$

$$|||u|||^2_{SD} := |||u|||^2 + |||u|||^2_\rho. \tag{3.2}$$

Here, the functions d_i are either $d_i \equiv 1$ or given by (1.10). For the purpose of our analysis, we will assume that the weights $(\rho_i, d_i)_{i=1}^N$ are of the following form:

$$\rho_i = \delta_i h_i \begin{cases} \dfrac{1}{2p} & \text{if } d_i = b_i \text{ (given by (1.10))} \\ \dfrac{1}{2\sqrt{3}p^2} & \text{if } d_i = 1 \end{cases} \tag{3.3}$$

where the numbers δ_i satisfy

$$0 \le \delta_i \le \delta_0 < \delta_0' := \min\left\{\frac{1}{\|a\|_{L^\infty}}, \frac{\gamma}{\sqrt{2}\|a\|_{L^\infty}\|b\|_{L^\infty}}\right\}. \tag{3.4}$$

As we will restrict ourselves in the remainder of the paper mostly to the special cases $d_i = 1$ or $d_i = b_i$ for all i, we introduce the following terminology:

Definition 3.1 *Let the pairs $(\rho_i, d_i)_{i=1}^N$ satisfy* (3.3).

1. *If $d_i = b_i$ for $i = 1, \ldots, N$, then the SDFEM* (1.11) *is called "bubble-stabilized" SDFEM;*
2. *if $d_i = 1$ for $i = 1, \ldots, N$, then the SDFEM* (1.11) *is called "L^2-stabilized" SDFEM.*

The SDFEM (1.11) *is said to be "non-degenerate", if the following "non-degeneracy" condition is satisfied:*

$$\rho_i \text{ is of the form (3.3) and } \exists\, c, \underline{\delta} > 0 \quad \text{such that} \quad h_i \ge c\varepsilon p \Longrightarrow \delta_i \ge \underline{\delta}. \tag{3.5}$$

The Galerkin and the SDFEM errors share the fundamental *orthogonality* property:

$$B_\varepsilon(u_\varepsilon - u_G, v) = 0 \qquad \forall v \in S_0^{p,1}(\mathcal{T}), \tag{3.6}$$

$$B_{SD}(u_\varepsilon - u_{SD}, v) = 0 \qquad \forall v \in S_0^{p,1}(\mathcal{T}). \tag{3.7}$$

Remark 3.2 It should be noted that the orthogonality relation (3.7) for the SDFEM holds because of the assumptions on the data ($f \in L^2(\Omega)$).

For the analysis of the Galerkin FEM and the SDFEM, it will be more convenient to analyze the error $u_\varepsilon - u_{SD}$ indirectly by analyzing the difference between the SDFEM solution and a nearby interpolant. In order to formalize this idea, we introduce

Definition 3.3 *A decomposition $u_p = u_{reg,p} + u_{BL,p} \in S_0^{p,1}(\mathcal{T})$ is said to be an* admissible splitting, *if $u_{reg,p}$, $u_{BL,p} \in S^{p,1}(\mathcal{T})$ and there exists a corresponding decomposition $u_\varepsilon = u_{reg} + u_{BL}$ such that $u_{reg}(\pm 1) = u_{reg,p}(\pm 1)$, $u_{BL}(\pm 1) = u_{BL,p}(\pm 1)$. Denoting u_{SD} the solution of the SDFEM, we introduce the functions e, η, η_{reg}, η_{BL} as:*

$$e := u_{SD} - (u_{reg,p} + u_{BL,p}) \in S_0^{p,1}(\mathcal{T}), \tag{3.8}$$

$$\eta := \eta_{reg} + \eta_{BL} := (u_{reg} - u_{reg,p}) + (u_{BL} - u_{BL,p}) \tag{3.9}$$

$$= u_\varepsilon - (u_{reg,p} + u_{BL,p}). \tag{3.10}$$

For weights (ρ_i, d_i) of the form (3.3) and admissible splitting we define for the error $\eta = \eta_{reg} + \eta_{BL}$ the following mesh-dependent *norms:*

$$E_{SD}(\eta) := \left\{\sum_{i=1}^N \delta_i^2 \frac{1}{\rho_i + \varepsilon}\left[\varepsilon^2 \|\eta'\|^2_{L^2(I_i)} + \|\eta\|^2_{L^2(I_i)}\right]\right\}^{1/2}, \tag{3.11}$$

$$E_{SD,\infty}(\eta) := \left\{\sum_{i=1}^N \delta_i \frac{\rho_i}{\rho_i + \varepsilon}\left[\varepsilon^2 \|\eta'\|^2_{L^\infty(I_i)} + \|\eta\|^2_{L^\infty(I_i)}\right]\right\}^{1/2}, \tag{3.12}$$

$$E_{G,d}(\eta_{reg}, \eta_{BL}) := \min\left\{\left\{\sum_{i=1}^N \frac{1}{\rho_i + \varepsilon}\|d_i^{-1/2}\eta\|^2_{L^2(I_i)}\right\}^{1/2}; \right.$$
$$\left. \|\eta'_{reg}\|_{L^2(\Omega)} + \varepsilon^{-1/2}\|\eta_{BL}\|_{L^2(\Omega)}\right\}. \tag{3.13}$$

Let us finally note the following standard *inverse estimates* (cf., e.g., [13]).

Lemma 3.4 *Let $I_i \in \mathcal{T}$ be an interval of length h_i and b_i be given by (1.10). Then for all polynomials π_p of degree $p \in \mathbb{N}$ there holds for some $C_d > 0$ independent of p, h_i*

$$\|\sqrt{b_i}\pi_p'\|_{L^2(I_i)} \leq 2\sqrt{2}\frac{p}{h_i}\|\pi_p\|_{L^2(I_i)}, \qquad \|\pi_p\|_{L^2(I_i)} \leq C_d p\|\sqrt{b_i}\pi_p\|_{L^2(I_i)},$$
$$\|b_i\pi_p''\|_{L^2(I_i)} \leq \frac{2p}{h_i}\|\sqrt{b_i}\pi_p'\|_{L^2(I_i)},$$
$$\|\pi_p'\|_{L^2(I_i)} \leq 2\sqrt{3}\frac{p^2}{h_i}\|\pi_p\|_{L^2(I_I)}, \qquad \|\pi_p\|_{L^\infty(I_i)} \leq \frac{p+1}{\sqrt{h_i}}\|\pi_p\|_{L^2(I_I)}.$$

For brevity of notation in some of the proofs, it will be convenient to use the following shorthand: For functions u and elements $I_i \subset \Omega$ we write

$$\|u\|_{I_i} := \|u\|_{L^2(I_i)}, \qquad \|u\|_{L^\infty} := \|u\|_{L^\infty(\Omega)}, \qquad \|u\| := \|u\|_{L^2(\Omega)}. \tag{3.14}$$

3.2 Error estimates in the Energy norm and in the mesh-dependent norm

3.2.1 Stability of the SDFEM

We start by showing that the bilinear form B_{SD} of (1.12) is coercive if the weights $(\rho_i, d_i)_{i=1}^N$ are of the form (3.3):

Theorem 3.5 *Assume that the weights $(\rho_i, d_i)_{i=1}^N$ are of the form (3.3). Then there holds*

$$(1 - \delta_0/\delta_0')|||u|||_{SD}^2 \leq B_{SD}(u,u) \qquad \forall u \in S_0^{p,1}(\mathcal{T}).$$

Proof: We will only prove the "bubble-stabilized" case (i.e., $d_i = b_i$)—the "L^2-stabilized" or the general case begin analogous. For $u \in S_0^{p,1}(\mathcal{T})$ we obtain with the Cauchy-Schwarz inequality, the inverse estimates of Lemma 3.4 and the fact that $0 \leq d_i \leq 1$:

$$\begin{aligned}
\sum_{i=1}^N \rho_i \int_{I_i} d_i (L_\varepsilon u)(au')\,dx &= \sum_{i=1}^N \rho_i \int_{I_i} -\varepsilon d_i u'' a u'\,dx + \rho_i \int_{I_i} d_i (au')^2\,dx + \rho_i \int_{I_i} d_i abu'u\,dx \\
&\geq \sum_{i=1}^N -\varepsilon\rho_i \frac{2p}{h_i}\|\sqrt{d_i}u'\|_{I_i}\,\|au'\|_{I_i} + \rho_i \left\|\sqrt{d_i}au'\right\|_{I_i}^2 - \rho_i \frac{2\sqrt{2}p}{h_i}\|a\|_{L^\infty}\|b\|_{L^\infty}\|u\|_{I_i}^2 \\
&\geq \sum_{i=1}^N -\varepsilon\delta_i\|a\|_{L^\infty}\|u'\|_{I_i}^2 + \rho_i \left\|\sqrt{d_i}au'\right\|_{I_i}^2 - \delta_i \frac{\sqrt{2}\|a\|_{L^\infty}\|b\|_{L^\infty}}{\gamma}\gamma\,\|u\|_{I_i}^2
\end{aligned}$$

Recalling from the proof of Proposition 1.2 that

$$B_\varepsilon(u,u) \geq \sum_{i=1}^N \varepsilon\|u'\|_{I_i}^2 + \gamma\|u\|_{I_i}^2$$

we obtain a lower bound for $B_{SD}(u,u)$)

$$B_{SD}(u,u) \geq \min\left\{1 - \delta_0\|a\|_{L^\infty}, 1 - \delta_0\sqrt{2}\|a\|_{L^\infty}\|b\|_{L^\infty}/\gamma\right\}\,|||u|||_{SD}^2,$$

and the claim of the theorem follows by the definition of δ_0' in (3.4).

$\square$

3.2.2 Consistency of the SDFEM

Theorem 3.5 guarantees that the SDFEM formulation (1.11) does have a unique solution u_{SD}. For the error $u_\varepsilon - u_{SD}$ the following estimates hold true (Note that the case $\rho_i = 0$, i.e., the case of a pure Galerkin FEM is not excluded):

Lemma 3.6 *let $\mathcal{T}$ be any mesh. Then there is a constant $C > 0$ such that the following holds: For the "bubble-stabilized" SDFEM (cf. Definition 3.1) and any admissible splitting in the sense of Definition 3.3 there holds:*

$$(1-\delta_0/\delta_0')|||e|||_{SD} \le CK\Big[|||\eta||| + E_{G,d}(\eta_{reg},\eta_{BL}) + E_{SD}(\eta)\Big], \tag{3.15}$$

where the constant $K > 0$ depends only a and b and is given in (3.17). For the "L^2-stabilized" SDFEM (cf. Definition 3.1) and any admissible splitting there holds for some $C' > 0$ depending only on a and b:

$$(1-\delta_0/\delta_0')|||e|||_{SD} \le C'\Big[|||\eta||| + E_{G,d}(\eta_{reg},\eta_{BL}) + E_{SD}(\eta) + E_{SD,\infty}(\eta)\Big]. \tag{3.16}$$

Proof: Let us begin with the first estimate, the "bubble-stabilized" SDFEM. Let $u_p = u_{reg,p} + u_{BL,p}$ be any admissible splitting in the sense of Definition 3.3. The orthogonality (3.7) gives

$$B_{SD}(e,e) = B_{SD}(\eta,e).$$

By Theorem 3.5, we obtain therefore

$$\begin{aligned}
&(1-\delta_0/\delta_0')|||e|||_{SD}^2 \le B_{SD}(e,e) = B_{SD}(\eta,e) = \\
&\quad \left\{\varepsilon\int_\Omega \eta' e'\,dx + \int_\Omega b\eta e\,dx\right\} + \left\{\int_\Omega a\eta' e\,dx\right\} + \left\{\sum_{i=1}^N \rho_i \int_{I_i} d_i\,(L_\varepsilon \eta)\,ae'\,dx\right\} \\
&\quad =: S_1 + S_2 + S_3.
\end{aligned}$$

Let us estimate each of these three terms. We immediately have

$$|S_1| \le \max\{1, \|b\|_{L^\infty}/\gamma\}\,|||\eta|||\,|||e||| \le CK_1\,|||\eta|||\,|||e|||_{SD},$$

where we set $K_1 := \max\{1, \|b\|_{L^\infty}/\gamma\}$. As $e(\pm 1) = 0$, an integration by parts yields for S_2

$$S_2 = \int_\Omega a\eta' e\,dx = -\int_\Omega a'\eta e\,dx - \int_\Omega \eta a e'\,dx.$$

Hence, we obtain for S_2

$$\begin{aligned}
|S_2| &\le \|a'\|_\infty \|\eta\|\,\|e\| + \sum_{i=1}^N \left(\|d_i^{-1/2}\eta\|_{I_i}\frac{1}{(\rho_i+\varepsilon)^{1/2}}\right)\left((\rho_i+\varepsilon)^{1/2}\|d_i^{1/2}ae'\|_{I_i}\right) \\
&\le \|a'\|_{L^\infty}\gamma^{-1}\,|||\eta|||\,|||e|||_{SD} + (1+\|a\|_{L^\infty})\left\{\sum_{i=1}^N \frac{1}{\rho_i+\varepsilon}\|d_i^{-1/2}\eta\|_{I_i}^2\right\}^{1/2} |||e|||_{SD} \\
&\le CK_2\left[|||\eta||| + \left\{\sum_{i=1}^N \frac{1}{\rho_i+\varepsilon}\|d_i^{-1/2}\eta\|_{I_i}^2\right\}^{1/2}\right]|||e|||_{SD},
\end{aligned}$$

where we set $K_2 := \max\{1, \|a'\|_{L^\infty}/\gamma, \|a\|_{L^\infty}\}$. We note that the term in curly braces gives the first entry of the minimum in the definition of $E_{G,d}(\eta_{reg},\eta_{BL})$. In order to get the second entry of that minimum, we estimate S_2 slightly differently: We write $\eta = \eta_{reg} + \eta_{BL}$ and integrate by parts to arrive at

$$\begin{aligned}
S_2 &= \int_\Omega a\eta_{reg}' e\,dx + \int_\Omega a\eta_{BL}' e\,dx = \int_\Omega a\eta_{reg}' e\,dx - \int_\Omega a'\eta_{BL} e\,dx - \int_\Omega a\eta_{BL} e'\,dx \\
|S_2| &\le \left(\|a\|_\infty\gamma^{-1/2}\|\eta_{reg}'\| + \|a'\|_\infty\gamma^{-1/2}\|\eta_{BL}\|\right)\gamma^{1/2}\|e\| + \left(\varepsilon^{-1/2}\|a\|_{L^\infty}\|\eta_{BL}\|\right)\varepsilon^{1/2}\|e'\| \\
&\le CK_3\left(\|\eta_{reg}'\| + \|\eta_{BL}\| + \varepsilon^{-1/2}\|\eta_{BL}\|\right)|||e|||_{SD},
\end{aligned}$$

where $K_3 := (\|a\|_{L^\infty} + \|a'\|_{L^\infty})/\sqrt{\gamma} + \|a\|_{L^\infty}$. This last estimate on S_2 gives the second entry in the minimum defining $E_{G,d}(\eta_{reg}, \eta_{BL})$. We also note that up to now all the estimates are valid for the "L^2-stabilized" SDFEM as well.

Finally, in order to estimate S_3, elementwise integration by parts, Lemma 3.4 and the observation that $|d_i'| \leq 4/h_i$ allow us to estimate

$$
\begin{aligned}
\left|\int_{I_i} d_i a\eta' ae'\,dx\right| &= \left|-\int_{I_i} \eta\left(d_i a^2 e'\right)'\,dx\right| = \left|-\int_{I_i} \eta d_i' a^2 e'\,dx - \int_{I_i} \eta d_i 2aa'e'\,dx - \int_{I_i} \eta d_i a^2 e''\,dx\right| \\
&\leq \|\eta\|_{I_i}\left[\frac{4p}{h_i}C_d\frac{\|a\|_{L^\infty}^2}{\underline{a}} + 2\|a'\|_{L^\infty} + \frac{2p}{h_i}\frac{\|a\|_{L^\infty}^2}{\underline{a}}\right]\|\sqrt{d_i}ae'\|_{I_i}, \\
\left|\varepsilon\int_{I_i} d_i\eta'' ae'\,dx\right| &= \left|-\varepsilon\int_{I_i} \eta'\left(d_i ae'\right)'\,dx\right| = \left|-\varepsilon\int_{I_i} \eta' d_i' ae'\,dx - \varepsilon\int_{I_i} \eta' d_i a'e'\,dx - \varepsilon\int_{I_i} \eta' d_i ae''\,dx\right| \\
&\leq \varepsilon\|\eta'\|_{I_i}\left[\frac{4p}{h_i}C_d\frac{\|a\|_{L^\infty}}{\underline{a}} + \|a'\|_{L^\infty} + \frac{2p}{h_i}\frac{\|a\|_{L^\infty}}{\underline{a}}\right]\|\sqrt{d_i}ae'\|_{I_i}, \\
\left|\int_{I_i} d_i b\eta ae'\,dx\right| &\leq \|b\|_{L^\infty}\|a\|_{L^\infty}\|\eta\|_{I_i}\|\sqrt{d_i}e'\|_{I_i} \leq 2\sqrt{2}\frac{p}{h_i}\frac{\|b\|_{L^\infty}\|a\|_{L^\infty}}{\gamma}\gamma\|\eta\|_{I_i}\cdot\|e\|_{I_i}
\end{aligned}
$$

Exploiting that $\delta_i\|b\|_{L^\infty}\|a\|_{L^\infty}/\gamma \leq 1$ by (3.4) and setting $K_4 := \max\{\|a'\|_{L^\infty}, \|a\|_{L^\infty}^2/\underline{a}, \|a\|_{L^\infty}/\underline{a}\}$, we conclude with the Cauchy-Schwarz inequality for sums

$$
\begin{aligned}
|S_3| &\leq CK_4\sum_{i=1}^N\Big(\rho_i\frac{p}{h_i}(\rho_i+\varepsilon)^{-1/2}\Big[\varepsilon\|\eta'\|_{I_i} + \|\eta\|_{I_i}\Big](\rho_i+\varepsilon)^{1/2}\|\sqrt{d_i}ae'\|_{I_i} + \gamma\|\eta\|_{I_i}\|e\|_{I_i}\Big) \\
&\leq CK_4(1+\|a\|_{L^\infty})\Big[E_{SD}(\eta) + \gamma^{1/2}\|\eta\|_{L^2(\Omega)}\Big]|||e|||_{SD} \\
&\leq CK_4(1+\|a\|_{L^\infty})\left[E_{SD}(\eta) + |||\eta|||\right]|||e|||_{SD}.
\end{aligned}
$$

Upon setting

$$
\begin{aligned}
K &:= \max\{K_1, K_2, K_3, K_4(1+\|a\|_{L^\infty})\} \\
&\leq 1 + \frac{\|b\|_{L^\infty} + \|a'\|_{L^\infty}}{\gamma} + \frac{\|a\|_{L^\infty} + \|a'\|_{L^\infty}}{\sqrt{\gamma}} + \left(\|a'\|_{L^\infty} + \frac{\|a\|_{L^\infty} + \|a\|_{L^\infty}^2}{\underline{a}}\right)(1+\|a\|_{L^\infty}),
\end{aligned}
\tag{3.17}
$$

we can conclude the proof of the first estimate.

Let us now turn to the second estimate, the case of the "L^2-stabilized" SDFEM. The terms S_1, S_2 are estimated as above leading immediately to the first two terms in (3.16). The SDFEM term, S_3, however, is estimated differently now as elementwise integration by parts yield additional terms that can be controlled by the term $E_{SD,\infty}(\eta_{reg}, \eta_{BL})$. More precisely, we obtain the additional terms

$$
\begin{aligned}
&2\sum_{i=1}^N \rho_i\Big[\|\eta a^2 e'\|_{L^\infty(I_i)} + \varepsilon\|\eta' ae'\|_{L^\infty(I_i)}\Big] \\
&\quad\leq 2\sum_{i=1}^N \rho_i\frac{p+1}{\sqrt{h_i}}\Big[\|a\|_{L^\infty}^2\|\eta\|_{L^\infty(I_i)}\|e'\|_{I_i} + \varepsilon\|a\|_{L^\infty}\|\eta'\|_{L^\infty(I_i)}\|e'\|_{I_i}\Big] \\
&\quad\leq 2\max\left\{\frac{\|a\|_{L^\infty}^2}{\underline{a}}, \frac{\|a\|_{L^\infty}}{\underline{a}}\right\}\sum_{i=1}^N \rho_i\frac{p+1}{\sqrt{h_i}}\Big[\|\eta\|_{L^\infty(I_i)} + \varepsilon\|\eta'\|_{L^\infty(I_i)}\Big]\|ae'\|_{I_i} \\
&\quad\leq C\left\{\sum_{i=1}^N \rho_i^2\frac{(p+1)^2}{h_i}\frac{1}{\rho_i+\varepsilon}\Big[\varepsilon^2\|\eta'\|_{L^\infty(I_i)}^2 + \|\eta\|_{L^\infty(I_i)}^2\Big]\right\}^{1/2}|||e|||_{SD}
\end{aligned}
$$

where we appealed to Lemma 3.4 in the first estimate. As $\rho_i = \delta_i h_i/p^2$ for the "L^2-stabilized" SDFEM, the expression in curly braces can easily be bounded by $CE_{SD,\infty}(\eta)|||e|||_{SD}$.

□

Lemma 3.6 is formulated such that a variety of results may be obtained from it. Let us first show that in the case of smooth solutions, the SDFEM (on a quasi uniform mesh) is half a power of h/p away from being quasi-optimal. For the h version SDFEM, this is a well-known fact; the following corollary therefore extends this fact to the p version with quasi-uniform meshes:

Theorem 3.7 *Let $\mathcal{T}$ be a quasi-uniform mesh with mesh width h and let $(\rho_i, d_i)_{i=1}^N$ be the weights of a non-degenerate "bubble-stabilized" SDFEM (cf. Definition 3.1). Let $u_\varepsilon \in H^k(\Omega)$, $k > 1$, be the solution of* (1.1)*. Then there is $C > 0$ independent of ε, h, p such that for the solution u_{SD} of* (1.11) *there holds*

$$|||u_\varepsilon - u_{SD}|||_{SD}^2 \leq C\left[\varepsilon + \frac{h}{p}\right]\left(\frac{h}{p}\right)^{2(k-1)} \|u_\varepsilon\|_{H^k(\Omega)}^2.$$

Proof: Let us choose a particular admissible splitting as follows: Set $u_{reg} = u_\varepsilon$, $u_{BL} = 0$, $u_{BL,p} = 0$ and let $u_{reg,p}$ be an interpolant of u_ε satisfying

$$\|(u_\varepsilon - u_{reg,p})'\|_{L^2(I_i)} \leq C\left(\frac{h_i}{p}\right)^{k-1}\|u_\varepsilon\|_{H^k(I_i)}, \qquad \|d_i^{-1/2}(u_\varepsilon - u_{reg,p})\|_{L^2(I_i)} \leq C\left(\frac{h_i}{p}\right)^{k}\|u_\varepsilon\|_{H^k(I_i)} \tag{3.18}$$

for some $C > 0$ depending only on k. Such an interpolant exists, cf. [13]. We obtain therefore from Theorem 3.6

$$|||u_\varepsilon - u_{SD}|||_{SD} \leq |||\eta|||_{SD} + |||e|||_{SD} \leq |||\eta|||_{SD} + C\left[|||\eta||| + E_{G,d}(\eta_{reg}, 0) + E_{SD}(\eta)\right]. \tag{3.19}$$

Using $E_{G,d}^2(\eta_{reg}, 0) \leq \sum_{i=1}^N (\rho_i + \varepsilon)^{-1}\|\sqrt{d_i}\eta_{reg}\|_{I_i}^2$ and observing that (3.5) implies the existence of $C > 0$ depending only on c, $\underline{\delta}$ of (3.5) such that

$$\frac{1}{\rho_i + \varepsilon} \leq C\frac{p}{h_i}, \qquad \frac{1}{\rho_i + \varepsilon} \leq \frac{1}{\varepsilon}, \tag{3.20}$$

we obtain the result by inserting the estimates (3.18) on $\eta = \eta_{reg} = u_\varepsilon - u_{reg,p}$ in (3.19).

□

In the presence of boundary layers, Lemma 3.6 allows us to take advantage of the freedom to choose among all admissible splittings. In particular, for meshes that "contain" the two-element mesh of Definition 2.2, we may use the splitting of Lemma 2.4 to obtain robust exponential convergence of the *hp*-SDFEM as well as the *hp* Galerkin FEM:

Theorem 3.8 *There is $\kappa_0 > 0$ depending on a, b, f such that following holds. Assume that there is $c' > 0$ such that for each $p \in \mathbb{N}$ there is $\kappa \in (0, \kappa_0)$ with $\kappa p \geq c'$ such that the mesh $\mathcal{T}$ satisfies $S_0^{p,1}(\mathcal{T}_{\varepsilon,\kappa,p}) \subset S_0^{p,1}(\mathcal{T})$. Assume furthermore that the weights $(\rho_i, d_i)_{i=1}^N$ fall into one of the following three categories:*

(i) The pure Galerkin FEM, i.e., $\rho_i = 0$ for $i = 1, \ldots, N$;

(ii) the non-degenerate "bubble-stabilized" SDFEM (cf. Definition 3.1);

(iii) the non-degenerate "L^2-stabilized" SDFEM (cf. Definition 3.1).

Then the solution u_{SD} of (1.11) *exists and there are C, $\sigma > 0$ depending only on a, b, f, c', and δ_0, δ_0', (and on c, $\underline{\delta}$ of* (3.5) *in the cases (ii), (iii)) such that*

$$|||u_\varepsilon - u_{SD}||| \leq Cp^2 e^{-\sigma\kappa p}.$$

Proof: For any splitting in the sense of Definition 3.3 we can estimate

$$|||u_\varepsilon - u_{SD}||| \leq |||\eta||| + |||e|||_{SD}$$

and then apply Lemma 3.6. For case (i), the case of the pure Galerkin FEM, we observe that $E_{SD}(\eta) = 0$ for any admissible splitting. Using the splitting $u_\varepsilon = u_{reg,p} + u_{BL,p}$ of Lemma 2.4, we can bound

$$|||\eta||| + E_{G,d}(\eta_{reg}, \eta_{BL}) \leq |||\eta||| + \|\eta_{reg}'\|_{L^2(\Omega)} + \varepsilon^{-1/2}\|\eta_{BL}\|_{L^2(\Omega)}$$

and therefore the claim of the theorem follows from Lemma 2.4 and Lemma 3.6 and the fact $\kappa p \geq c'$. For cases (ii) and (iii), we first note that the non-degeneracy condition (3.5) implies the existence of $C > 0$ such that

$$E_{SD}(\eta) \leq C p^{\alpha} E_{SD,\infty}(\eta),$$

where $\alpha = 1$ for case (ii) and $\alpha = 2$ for case (iii). Next, we have

$$\frac{\rho_i}{\rho_i + \varepsilon} \leq \min\{1, \rho_i/\varepsilon\} \leq C \min\{1, h_i/\varepsilon\}$$

so that we can estimate

$$E_{SD}(\eta) + E_{SD,\infty}(\eta) \leq (1 + Cp^{\alpha}) \left(\sum_{i=1}^{N} \min\{1, h_i/\varepsilon\} \left[\varepsilon^2 \|\eta'\|^2_{L^\infty(I_i)} + \|\eta\|^2_{L^\infty(I_i)} \right] \right\}^{1/2}.$$

Using now the fact that $\kappa p \geq c' > 0$, we obtain the desired result by appealing to Lemma 2.4.

□

3.3 L^∞ bounds

Let us finally show that in this one-dimensional setting, robust estimate in the L^∞ norm easily be obtained. We begin with the following

Lemma 3.9 *Let $\mathcal{T}$ be any mesh. Assume that the weights $(\rho_i, d_i)_{i=1}^N$ corresponend to either a non-degenerate "bubble-stabilized" or a non-degenerate "L^2-stabilized" SDFEM. Then there is $C > 0$ depending only on c, $\underline{\delta}$, $\underline{a}$ such that for all $n = 1, \ldots, N$*

$$\|u\|^2_{L^\infty(-1,x_n)} \leq Cnp \left(\sum_{i=1}^{n} \rho_i p^{\alpha} \|\sqrt{d_i} a u'\|^2_{L^2(I_i)} + \varepsilon \|u'\|^2_{L^2(I_i)} \right) \qquad \forall u \in S_0^{p,1}(\mathcal{T}),$$

where $\alpha = 1$ in the case of the "L^2-stabilized" SDFEM and and $\alpha = 2$ in the case of the "bubble-stabilized" SDFEM. In particular, we have

$$\|u\|^2_{L^\infty(\Omega)} \leq CNp^{1+\alpha} |||u|||^2_{SD} \qquad \forall u \in S_0^{p,1}(\mathcal{T}).$$

Proof: For any $x \in (-1, x_n)$, $u \in S_0^{p,1}(\mathcal{T})$ we write

$$|u(x)| = \left| \int_{-1}^{x} u'(t)\,dt \right| \leq \sum_{i=1}^{n} h_i^{1/2} \|u'\|_{L^2(I_i)} \leq \sqrt{n} \left\{ \sum_{i=1}^{n} h_i \|u'\|^2_{L^2(I_i)} \right\}^{1/2}.$$

In the case of "bubble-stabilization", the assumptions and Lemma 3.4 allow us to conclude that

$$\begin{aligned} h_i \|u'\|^2_{L^2(I_i)} &\leq c\varepsilon p \|u'\|^2_{L^2(I_i)} && \text{if } h_i < c\varepsilon p \\ h_i \|u'\|^2_{L^2(I_i)} &\leq \rho_i \frac{C_d^2 p^3}{\underline{a}^2 \underline{\delta}} \|\sqrt{d_i} a u'\|^2_{L^2(I_i)} && \text{if } h_i \geq c\varepsilon p \end{aligned}$$

and the result follows from these estimates. We may proceed similarly in the case of "L^2-stabilization".

□

Remark 3.10 It should be noted that no term involving $\|u\|_{L^2(I_i)}$ appears on the right hand side of the estimates in Lemma 3.9. One can make use of this observation to get L^∞ bounds in the special case $a = 1$, $b = 0$.

Lemma 3.9 allows us immediately to obtain robust convergence in the L^∞ norm for the SDFEM if the mesh "contains" a small element of size $O(\varepsilon p)$ in the boundary layer:

Proposition 3.11 *Let the assumptions of Theorem 3.8, (ii) or (iii) be satisfied. Then there are constants* C, $\sigma > 0$ *such that*

$$\|u_\varepsilon - u_{SD}\|_{L^\infty(\Omega)} \le C\sqrt{N}p^{7/2}e^{-\sigma\kappa p}$$

Proof: We use the splitting of Theorem 3.8, (ii) (resp. (iii)) and write $\|u_\varepsilon - u_{SD}\|_{L^\infty(\Omega)} \le \|\eta\|_{L^\infty(\Omega)} + \|e\|_{L^\infty(\Omega)}$. The term $\|\eta\|_{L^\infty(\Omega)}$ can be estimated using Lemma 2.4 and $\|e\|_{L^\infty(\Omega)}$ can be estimated by $Cp^{3/2}\sqrt{N}|||e|||_{SD}$ using Lemma 3.9. Upon checking the proof of Theorem 3.8, we notice that in fact $|||e|||_{SD} \le Cp^2e^{-\sigma\kappa p}$ and thus the desired result follows.

□

Let us conclude this section with a proof that also the Galerkin FEM leads to robust exponential convergence in the L^∞ norm, if a true "two-element mesh" is used:

Lemma 3.12 *Let* κ_0 *be given as in the statement of Lemma 2.4. Then for each* $\kappa \in (0, \kappa_0)$ *there are* C, $\sigma > 0$ *such that for the meshes* $\mathcal{T}_{\kappa,\varepsilon,p}$ *of Definition 2.2 there holds for the Galerkin solution* u_G

$$\|u_\varepsilon - u_G\|_{L^\infty(\Omega)} \le Cpe^{-\sigma\kappa p}.$$

Proof: For the splitting of Lemma 2.4 we have

$$\|u_\varepsilon - u_{SD}\|_{L^\infty(\Omega)} \le \|e\|_{L^\infty(\Omega)} + \|\eta\|_{L^\infty(\Omega)}.$$

We note that $\|\eta\|_{L^\infty(\Omega)}$ can be estimated in the desired fashion by Lemma 2.4. For $\|e\|_{L^\infty(\Omega)}$ we consider the two elements $I_1 = [-1, 1-\kappa p\varepsilon]$, $I_2 = [1-\kappa p\varepsilon, 1]$ separately. On the large element I_1 with $h_1 \ge 1$, we have by Lemma 3.4 and Theorem 3.8

$$\|e\|_{L^\infty(I_1)} \le \frac{p+1}{\sqrt{h_1}}\|e\|_{L^2(I_1)} \le Cp\,|||e||| \le Cpe^{-\sigma\kappa p}.$$

On the small element I_2, we exploit the fact that $e(\pm 1) = 0$ and obtain

$$\|e\|_{L^\infty(I_2)} \le \sqrt{h_2}\|e'\|_{L^2(I_2)} = (\kappa p\varepsilon)^{1/2}\|e'\|_{L^2(I_2)} \le C(\kappa p)^{1/2}e^{-\sigma\kappa p}.$$

□

3.4 Remarks on the choice of the weights

In our analysis of the "bubble-stabilized" and the "L^2-stabilized" SDFEM we assumed that the factors ρ_i were of size $O(h_i/p)$ or $O(h_i/p^2)$ as suggested by the inverse estimates of Lemma 3.4. For the "bubble-stabilized" SDFEM, this choice maximized the power of h/p for smooth solutions (cf. Theorem 3.7). However, if small scale features like boundary layers are not resolved, then stabilization of the scheme can be more important than maximizing the exponent of h/p for smooth solutions: In the case that the layer cannot be solved, Section 4 provides numerical evidence that the proper amount of "extra stability" can dramatically improve the convergence on compact subsets upstream of the layer. In this subsection we therefore want to show briefly that other choices of the weights $(\rho_i, d_i)_{i=1}^N$ than those of (3.3) are possible. These other choices lead to more stable methods in the sense that the bilinear form B_{SD} is coercive in a stronger mesh dependent norm than the one used so far. We illustrate this for the "bubble-stabilized" SDFEM—similar results can easily be obtained for the "L^2-stabilized" SDFEM.

For some fixed $q \in (0,1)$ and $\delta_0 < 1/\|a\|_{L^\infty(\Omega)}$, choose ρ_0 such that

$$1 - q - \frac{\|b\|^2_{L^\infty(\Omega)}}{2\gamma}\rho_0 > 0. \tag{3.21}$$

Let $(\rho_i)_{i=1}^N \subset [0, \rho_0]$ satisfy with C_d of Lemma 3.4

$$\begin{cases} \rho_i \frac{2\sqrt{2}p}{h_i} \le \delta_0 & \text{if } 2\sqrt{2}C_d\underline{a}^{-1}\left(\varepsilon\dfrac{p^2}{h_i}\right) < q \\ \rho_i \le \rho_0 & \text{else} \end{cases} \tag{3.22}$$

Then we have the following stability result:

Theorem 3.13 *For the factors $(\rho_i)_{i=1}^N$ satisfying (3.22) the bilinear form B_{SD} based on the weights $(\rho_i, b_i)_{i=1}^N$ satisfies the following: There is $C > 0$ depending only on a, b, and q, δ_0, ρ_0 such that*

$$C|||u|||_{SD}^2 \leq B_{SD}(u,u) \qquad \forall u \in S_0^{p,1}(\mathcal{T}).$$

Proof: For $u \in S_0^{p,1}(\mathcal{T})$ we obtain with the Cauchy-Schwarz inequality

$$\begin{aligned}
\sum_{i=1}^N \rho_i \int_{I_i} d_i(L_\varepsilon u)(au')\,dx &= \sum_{i=1}^N \rho_i \int_{I_i} -\varepsilon d_i u'' a u'\,dx + \rho_i \int_{I_i} d_i (au')^2\,dx + \rho_i \int_{I_i} d_i a b u' u\,dx \\
&\geq \sum_{i=1}^N -\varepsilon \rho_i 2\sqrt{2}\frac{p}{h_i}\|u'\|_{I_i} \left\|\sqrt{d_i} a u'\right\|_{I_i} + \rho_i\left(1 - \rho_i \frac{\|b\|^2_{L^\infty(\Omega)}}{2\gamma}\right)\left\|\sqrt{d_i} a u'\right\|^2_{I_i} - \frac{\gamma}{2\|b\|^2_{L^\infty(\Omega)}}\|bu\|^2_{I_i} \\
&\geq \sum_{i=1}^N -\varepsilon \rho_i 2\sqrt{2}\frac{p}{h_i}\|u'\|_{I_i} \left\|\sqrt{d_i} a u'\right\|_{I_i} + \rho_i\left(1 - \rho_0 \|b\|^2_{L^\infty(\Omega)}(2\gamma)^{-1}\right)\left\|\sqrt{d_i} a u'\right\|^2_{I_i} - \frac{\gamma}{2}\|u\|^2_{I_i}
\end{aligned}$$

We estimate

$$\begin{aligned}
\varepsilon\rho_i 2\sqrt{2}\tfrac{p}{h_i}\|u'\|_{I_i}\|\sqrt{d_i} a u'\|_{I_i} &\leq \rho_i q \|\sqrt{d_i} a u'\|^2_{I_i} \qquad &&\text{if } \left(\varepsilon p^2/h_i\right) 2\sqrt{2} C_d \underline{a}^{-1} < q \\
\varepsilon\rho_i 2\sqrt{2}\tfrac{p}{h_i}\|u'\|_{I_i}\|\sqrt{d_i} a u'\|_{I_i} &\leq \varepsilon \delta_i \|a\|_\infty \|u'\|^2_{I_i} &&\text{else.}
\end{aligned}$$

Recalling from the proof of Proposition 1.2 that $B_\varepsilon(u,u) \geq \sum_{i=1}^N \varepsilon\|u'\|^2_{I_i} + \gamma\|u\|^2_{I_i}$. we obtain a lower bound for $B_{SD}(u,u))$

$$B_{SD}(u,u) \geq \min\left\{1 - \delta_0\|a\|_\infty, 1 - q - \rho_0\|b\|^2_{L^\infty(\Omega)}/(2\gamma), \gamma/2\right\} |||u|||^2_{SD} \tag{3.23}$$

□

Theorem 3.13 shows that, in order to obtain a coercive bilinear form B_{SD}, a restriction on the factors ρ_i to be $O(h_i/p)$ has to be placed only for those elements where $h_i/p^2 = O(\varepsilon)$; if $h_i/p^2 \gg \varepsilon$, then the factors ρ_i merely have to be bounded. Essentially this means that for very small values of ε (relative to h_i/p^2), any choice of the factors ρ_i leads to a coercive bilinear form B_{SD}. The choice $\rho_i = O(h_i p^{-\alpha})$ for various $\alpha \in [0,2]$ will be studied numerically in Section 4 ahead.

Remark 3.14 Let us finally point out that the bilinear form of a "bubble-stabilized" SDFEM in the sense of Definition 3.1 has a coercivity constant that is robust with respect to the size of the function b. For example, if an implicit Euler scheme for the problem

$$u_t + L_\varepsilon u = f, \qquad u(\pm 1) = 0 \tag{3.24}$$

is considered, one has to solve in each time step an elliptic boundary value problem of the form

$$-\varepsilon u''_{n+1} + a u'_{n+1} + \left(b + \frac{1}{\Delta t}\right) u_{n+1} = f + \frac{u_n}{\Delta t}. \tag{3.25}$$

We notice that the differential operator on the left hand side is of the form considered in this paper with a modified coefficient for the reaction term: $b_{\Delta t} := b + \frac{1}{\Delta t}$. We observe that for $\Delta t \to 0$

$$\|b_{\Delta t}\|_{L^\infty} = O(\Delta t^{-1}), \qquad \gamma_{\Delta t} = O(\Delta t^{-1}), \qquad \frac{\|b_{\Delta t}\|_{L^\infty}}{\gamma_{\Delta t}} = O(1),$$

and hence that δ_0' of (3.4) and K of (3.17) can be controlled uniformly in Δt. For the choice of weights (3.22), estimate (3.23) shows that one needs to control $\rho_0\|b_{\Delta t}\|^2_{L^\infty}/\gamma_{\Delta t}$ which entails a strong coupling of the step width Δt and the size of the weights ρ_i.

3.5 The case $a = 1,\ b = 0$

So far, we assumed we assumed $\gamma > 0$. As our numerical examples include the case $a = 1$, $b = 0$, we state here the approximation results corresponding to Theorem 3.8. Theorem 3.15 can be proved in the same way as Theorem 3.8; for the proof of Theorem 3.16 we refer to [10]. We assume that the factors δ_i now satisfy

$$0 \leq \delta_i \leq \delta_0 < 1 \tag{3.26}$$

Theorem 3.15 *Let $a = 1$, $b = 0$ and let the assumptions of Theorem 3.8, (ii) or (iii) be given but that the factors δ_i satisfy* (3.26) *instead of* (3.4). *Then there are constants C, $\sigma > 0$ such that*

$$|||u_\varepsilon - u_{SD}||| \leq Cp^2 e^{-\sigma\kappa p}, \qquad \|u_\varepsilon - u_{SD}\|_{L^\infty(\Omega)} \leq C\sqrt{N}p^{7/2}e^{-\sigma\kappa p}.$$

For the pure Galerkin method, we have

Theorem 3.16 *Let $a = 1$, $b = 0$ and let the assumptions of Theorem 3.8, (i) be given. Assume additionally the mesh restricted to $(-1, 1 - \kappa p\varepsilon)$ is quasi-uniform and that $p \geq 2$. Then there are C, $\sigma > 0$ such that the pure Galerkin solution u_G satisfies*

$$|||u_\varepsilon - u_G||| \leq Ce^{-\sigma\kappa p}.$$

4 Computational Experiments

In this section, we illustrate our theoretical findings with numerical examples. Our aims are

1. to illustrate the theoretical results obtained above, in particular the ability of the *hp*-FEM to resolve very narrow fronts and layers, leading to the asymptotic exponential convergence with few degrees of freedom;
2. to compare *hp*-SDFEM and *hp*-Galerkin FEM in the preasymptotic phase, i.e., if the small scales of the solution are not resoled. In particular, we will see that the appropriate choice of mesh sequences lead to robust exponential convergence on compact subsets for the *hp*-SDFEM. Furthermore, we will study numerically the optimal choice of the weights ρ_i in the pre-asymptotic regime.

We consider two types of problems, the boundary layer case ($\underline{a} > 0$) and the case of a turning point problem which satisfies the crucial assumption (1.4).

4.1 The boundary layer case

We consider for a, $b \in \mathbb{R}$ the problem

$$-\varepsilon u'' + au' + bu = e^{\omega x}, \qquad u(\pm 1) = 0. \tag{4.1}$$

The exact solution has a boundary layer at the outflow boundary $x = 1$ and is given by

$$u(x) = u_0(x) + \alpha e^{\lambda_1(1+x)} + \beta e^{-\lambda_2(1-x)}, \tag{4.2}$$

where

$$\begin{aligned}
u_0(x) &= \frac{1}{-\varepsilon\omega^2 + a\omega + b}e^{\omega x}, \\
\lambda_1 &= \frac{-2b}{a + \sqrt{a^2 + 4b\varepsilon}} = O(1), \qquad \lambda_2 = \frac{a + \sqrt{a^2 + 4b\varepsilon}}{2\varepsilon} = O(\varepsilon^{-1}), \\
c &= \left(1 - e^{2(\lambda_1 - \lambda_2)}\right)^{-1} = O(1), \\
\alpha &= c\left(u_0(1)e^{-2\lambda_2} - u_0(-1)\right) = O(1), \qquad \beta = c\left(u_0(-1)e^{2\lambda_1} - u_0(1)\right) = O(1).
\end{aligned}$$

Note that both $\|u_\varepsilon\|_{L^2(\Omega)}$ and $\sqrt{\varepsilon}\|u'_\varepsilon\|_{L^2(\Omega)}$ are $O(1)$ independently of ε. In our numerical experiments, we will always choose $\omega = 1$, $a = 1$. We use $b = 0$ in Sections 4.1.1, 4.2.1 and $b = 1$ in Section 4.2.2.

4.1.1 Global SDFEM performance

We consider the model problem (4.1) for $a = 1$, $b = 0$, $\omega = 1$. We present numerical results for a "L^2-stabilized" SDFEM, where the weights are of the form $(\rho_i, 1)_{i=1}^N$ with

$$\rho_i = \begin{cases} \frac{1}{2}h_i & \text{if } \varepsilon p^2/h_i \leq \frac{1}{4} \\ 0 & \text{otherwise.} \end{cases}$$

We remark at this point that this choice of the factors ρ_i was made to illustrate our claim on robustness in Section 3.4. We report, however, that the numerical results are very similar for the choice $(\rho_i, b_i)_{i=1}^N$ as well as for the "L^2-stabilized" and the "bubble-stabilized" SDFEM in the sense of Definition 3.1.

In our first series of numerical experiments, we resolve the boundary layer with the two-element mesh of Definition 2.2 with $\kappa = 1$, i.e., $\mathcal{T}_{1,\varepsilon,p}$. Fig. 2 compares the behavior of the Galerkin and the SDFEM in the L^2 norm and the energy norm $|||\cdot|||$ (which is $\sqrt{\varepsilon}|\cdot|_{H^1(\Omega)}$) for $\varepsilon = 10^{-8}$ where p ranges from 1 to 27. The theory of Section 3.5 yields robust exponential convergence in the energy norm for the SDFEM as well as the Galerkin FEM on this two element mesh. This exponential convergence is visible in the right figure of Fig. 2. Furthermore, for the SDFEM, we have robust exponential convergence in L^∞ (Theorem 3.15) and thus in L^2 (cf. the left figure of Fig. 2); we also observe robust exponential convergence in L^2 for the standard Galerkin FEM, Fig. 2. We note that the qualitative behavior of the schemes is comparable although the error of the hp-SDFEM is slightly smaller than that of the Galerkin FEM for this problem.

We conclude that the two-element mesh scheme is able to resolve the boundary layer at the outflow boundary and that no stabilization is required in this case.

Our next experiment is geared towards getting insight in the behavior of the Galerkin method and the "L^2-stabilized" SDFEM, if the boundary layer has not been resolved. To that end, we consider the performance of the p version on a *uniform* mesh with $h = 0.5$ (i.e., 4 elements). Here, p ranges from 1 to 27 and $\varepsilon = 10^{-4}$. The weights are given by $(\rho_i, 1)_{i=1}^N$ with

$$\rho_i = \frac{1}{4}\frac{h_i}{p^2}.$$

Fig. 3 shows the behavior in the L^2 and the energy norm $|||\cdot|||$. The error in the hp-SDFEM is considerably smaller than that of the Galerkin method, but the rate of convergence of the SDFEM is very poor also—in the energy norm, no convergence can be observed!

Finally, Fig. 4 shows the performance of a uniform mesh ($h = 0.5$) augmented by one small element of size ε in the outflow boundary layer (i.e., the mesh given by the nodes $\{-1, -0.5, 0, 0.5, 1-\varepsilon, 1\}$). As to be expected, inserting one small element of size ε greatly alleviates the problems of the standard Galerkin method (cf. Corollary A.6 of [10] for a detailed analysis). Comparing Fig. 3 with Fig. 4, the error of the Galerkin FEM is reduced by two orders of magnitude. Nevertheless, both the Galerkin method and the SDFEM yield poor rates of convergence as the p version on a mesh with one small element of size ε in the layer cannot resolve the boundary layer properly. Hence, comparing the results with those in Fig. 2, we see that the proper element length εp in the boundary layer is essential for the boundary layer resolutions as well as for exponential convergence.

4.2 Local p-SDFEM performance — pollution control

The conclusion of the preceding section is that both Galerkin FEM and SDFEM perform similarly if the small scale boundary layer is resolved; if the layer is not resolved, then one cannot expect convergence of either method in the global L^2 norm and energy norm. In the context of the h version, it is known that the SDFEM performs much better on compact subsets upstream of the layers (cf. [7]). One can therefore view the stabilization of the SDFEM as a means to controll *pollution*. Similar results can be observed in a p-version context as well as we will show now.

4.2.1 Local SDFEM performance on geometric mesh sequences

We want to show here that the hp-SDFEM leads to robust exponential convergence on compact supsets if an increase of the polynomial degree is combined with a mesh refinement towards the layer. We therefore

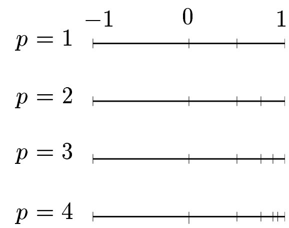

Figure 1: sequence of meshes generated by successively halving rightmost element

consider the following scheme: For $q \in (0,1)$ let

$$p_0 \in \mathbb{N} \quad \text{be the smallest integer s.t.} \quad q^{p_0} < p_0 \varepsilon$$

and let for each polynomial degree p a geometrically refined mesh with p layers be given by the points

$$\{-1, 1, 1 - q^i \,|\, i = 0, \ldots, \min{(p, p_0)}\}. \tag{4.3}$$

On such meshes, we will consider as trial spaces the space $S_0^{p,1}(\mathcal{T})$ (cf. Fig. 1). We note that such mesh sequences would typically be generated by adaptive schemes that locate and try to resolve the layers. It can be shown using ideas of [7, 18, 19] (cf. [10] for the details) that the "bubble-stabilized" *hp*-SDFEM converges robustly and exponentially on compact subset of Ω for such mesh sequences:

Theorem 4.1 *Let $a = 1$, $b = 0$, $q \in (0,1)$, $\xi \in (-1,1)$ be fixed. For $p \in \mathbb{N}$ consider the meshes $\mathcal{T}$ defined by the nodes* (4.3)*. For a non-degenerate "bubble-stabilized" SDFEM in the sense of Definition 3.1 there are constants C, $\sigma > 0$ independent of ε, p such that*

$$\|u_\varepsilon - u_{SD}\|_{H^1(-1,\xi)} \leq Ce^{-\sigma p}, \quad p = 1, 2, \ldots$$

The following numerical experiments are performed for both the "L^2-stabilized" and the "bubble-stabilized" SDFEM. The refinement factor q is chosen as $q = 1/2$ and the weights $(\rho_i)_{i=1}^N$ are given in both cases by

$$\rho_i = \begin{cases} \frac{1}{4}\frac{h_i}{p} & \text{if } \varepsilon < \frac{1}{4}\frac{h_i}{p} \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Again, we point out that choosing the factors $(\rho)_{i=1}^N$ as $O(h_i)$ or $O(h_i/p^2)$ leads to qualitatively similar numerical results. For $\varepsilon = 10^{-8}$ and p going from 1 to 22 Figs. 5-8 show the performance of the "L^2-stablized" and the "bubble-stablized" SDFEM in comparison with the Galerkin FEM. Figs. 5, 7 depict their behavior in global norms (L^2 and energy norm) whereas Figs. 6, 8 show the relative error (measured in the L^2 and H^1 norm) in the first element $I_1 = (-1,0)$. Figs. 5, 7 illustrate once more that both Galerkin FEM and *hp*-SDFEM do not lead to convergence in the energy norm until the layer is resolved, that is, $q^p \approx \varepsilon p$ (for $q = 0.5$ and $\varepsilon = 10^{-8}$ this happens for $p \approx 22$). The behavior of the Galerkin FEM is, however, completely different from that of the *hp*-SDFEM if the error on the first element $I_1 = (-1,0)$ is of interest (cf. Figs. 6, 8). The Galerkin FEM is highly prone to *pollution*: The local error in I_1 cannot be controlled until p is so large that the smallest element in the layer has width $q^p \approx p\varepsilon$. In contrast to this, the SDFEM is *pollution-free* as robust exponential convergence on the compact subset $(-1,0)$ can be achieved according to Theorem 4.1 and in fact is visible in Figs. 6, 8. It is noteworthy, however, that the local behavior of the "L^2-stabilized" SDFEM is strikingly superior to that of the "bubble-stablized" SDFEM (cf. also Section 4.2.2 ahead for a comparison of the local performance of the two stabilization schemes).

4.2.2 Impact of weights ρ_i on local performance of SDFEM

We have seen in Section 3.4 that especially for very small values of ε (relative to h_i/p^2) other choices for the weights ρ_i are possible than those given by inverse estimates. In the present section, we will explore these possibilities numerically by studying the performance of the p-version of the SDFEM on compact subsets in dependence of the choice of the factors $(\rho_i)_{i=1}^N$. To that end, we consider the model problem (4.1) for $a = 1$, $b = 1$, $\omega = 1$.

For exponents α given by

$$\alpha = 0, \quad \alpha = 0.25, \quad \alpha = 0.5, \quad \alpha = 1, \quad \alpha = 1.5, \quad \alpha = 2,$$

we define the factors $(\rho_i)_{i=1}^N$ as

$$\rho_i = \frac{1}{4} h_i p^{-\alpha}. \tag{4.5}$$

We consider both the "L^2-stabilized" SDFEM with weights $(\rho_i, 1)_{i=1}^N$ and the "bubble-stablized" SDFEM with weights $(\rho_i, b_i)_{i=1}^N$. Our experiments were performed on a fixed uniform mesh with 4 elements (i.e., $h_i = 0.5$) for $\varepsilon = 2 \cdot 10^{-3}$ and $\varepsilon = 10^{-8}$. In Figs. 9–12 we report the relative error in the H^1 semi norm on the first element $(-1, -1/2)$ as a function of the polynomial degree p for these various cases. For comparison purposes, the corresponding performance of the pure Galerkin method is also included. We mention here that the results using the L^2 norm instead of the H^1 semi norm on the first element are qualitatively similar.

Let us consider the case of "L^2-stabilization" first (cf. Figs. 9, 10). We notice a great variation in the performance as the exponent α varies. The choice of $\alpha = 0$ (i.e., ρ_i is independent of p), although yielding fairly accurate solutions, does not seem to lead to a convergent method. If $0 < \alpha \leq 1.5$, then the SDFEM seems to converge exponentially on $(-1, -1/2)$ with $\alpha = 0.5$ being the optimal choice for both "large" values of ε ($\varepsilon = 2 \cdot 10^{-3}$) and small values of ε ($\varepsilon = 10^{-8}$)).

In the case of "bubble-stabilization" (cf. Figs. 11, 12), the difference in performance between the different choices of α is less pronounced as in the "L^2-stabilized" scheme, i.e., the method seems to be more robust with respect to the choice of the weight. However, the "bubble-stablized" SDFEM is far less accurate than the "L^2-stabilized" SDFEM.

4.3 Turning point problems

Let us now consider a problem with a turning point at $x = 0$. We consider

$$-\varepsilon u_\varepsilon'' + a x u_\varepsilon' + u_\varepsilon = 1 \quad \text{on } (-1,1), \qquad a = \pm 1, \tag{4.6}$$

$$u_\varepsilon(\pm 1) = 0. \tag{4.7}$$

In the case $a = 1$, the exact solution has boundary layers at both endpoints ± 1; for $a = -1$, the exact solution exhibits an internal layer at the turning points $x = 0$. The exact solutions are given by

$$u_\varepsilon(x) = 1 - \exp\{(x^2-1)/(2\varepsilon)\} \qquad \text{for } a = 1, \tag{4.8}$$

$$u_\varepsilon(x) = 1 - c\,x\,\mathrm{erf}\left(x/\sqrt{2\varepsilon}\right) - \sqrt{2/\pi}\,c\sqrt{\varepsilon}\exp\{-x^2/(2\varepsilon)\} \qquad \text{for } a = -1, \tag{4.9}$$

$$c := \left(\mathrm{erf}\,(1/\sqrt{2\varepsilon}) + \sqrt{2\varepsilon/\pi}\exp(-1/(2\varepsilon))\right)^{-1} \approx 1 \qquad \text{for small } \varepsilon,$$

$$\mathrm{erf}\,(x) := \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\,dt, \qquad \mathrm{erf}\,(x) \to 1 \quad \text{for x} \to \infty.$$

Equation (4.6) satisfies the crucial assumption (1.4) and upon checking the proof of Theorem 3.6, we see that the fact that the coefficient a is a polynomial allows us to modify the arguments as to accomodate the case of (4.6) as well. We consider the "L^2-stabilized" SDFEM with weights $(\rho_i, 1)_{i=1}^N$ where

$$\rho_i = \frac{1}{4}\frac{h_i}{p^2}.$$

The solution given by (4.8) (i.e., the case $a = 1$) has two boundary layers at both endpoints with length scale $O(\varepsilon)$. The structure of the boundary layers is essentially of the form analyzed in Section 2 so that the approximation results with the "two-element" meshes introduced there apply. In fact, a "three-element" mesh consisting of two small elements of size $p\varepsilon$ at the boundary points and one large element in the middle (that is, the mesh is given by the points $\{-1, -1+p\varepsilon, 1-p\varepsilon, 1\}$) is well-suited to resolve the layers in both the Galerkin as well as the SDFEM (cf. Figs. 13 where $\varepsilon = 10^{-8}$).

In the case $a = -1$, the solution is given by (4.9) and has an internal layer of width $O(\sqrt{\varepsilon})$. Again, the "two-element" ideas of Section 2 can be applied successfully for the approximation of the internal layer if at least one element of size $O(p\sqrt{\varepsilon})$ is introduced at the turning point $x = 0$. Figs. 14 show the performance of the Galerkin FEM and the SDFEM for a "four-element" mesh based on the points $\{-1, -p\sqrt{\varepsilon}, 0, p\sqrt{\varepsilon}, 1\}$ and $\varepsilon = 10^{-8}$. Although the error graphs do not behave monotonically, the overall convergence of the "four-element" *hp*-SDFEM shows exponential convergence rates.

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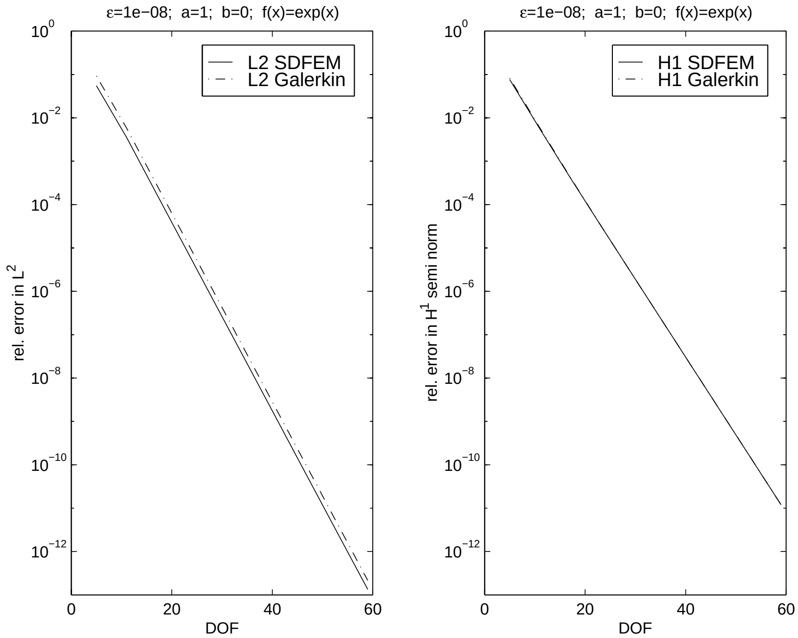

Figure 2: L^2 and energy performance of "two-element mesh" for Galerkin FEM and "L^2-stabilized" SDFEM, $\varepsilon = 10^{-8}$;

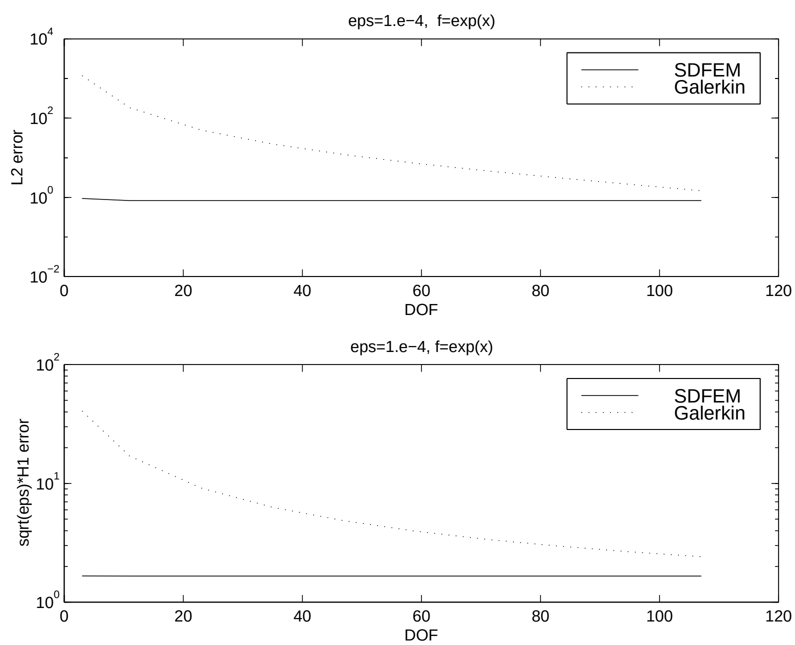

Figure 3: L^2 and energy performance of p version on uniform mesh with $h = 0.5$ for Galerkin FEM and "L^2-stabilized" SDFEM, $\varepsilon = 10^{-4}$

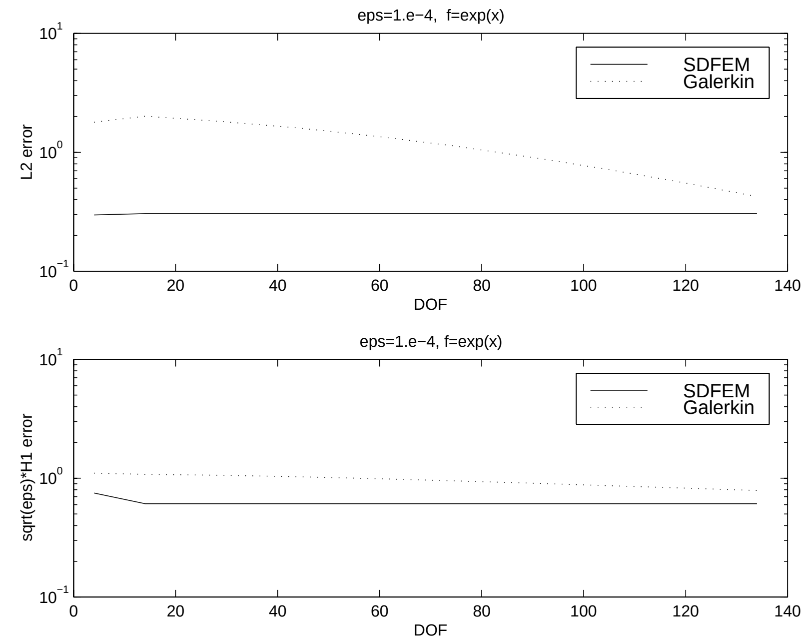

Figure 4: L^2 and energy performance of p version on uniform mesh with $h = 0.5$ + small element size ε, $\varepsilon = 10^{-4}$

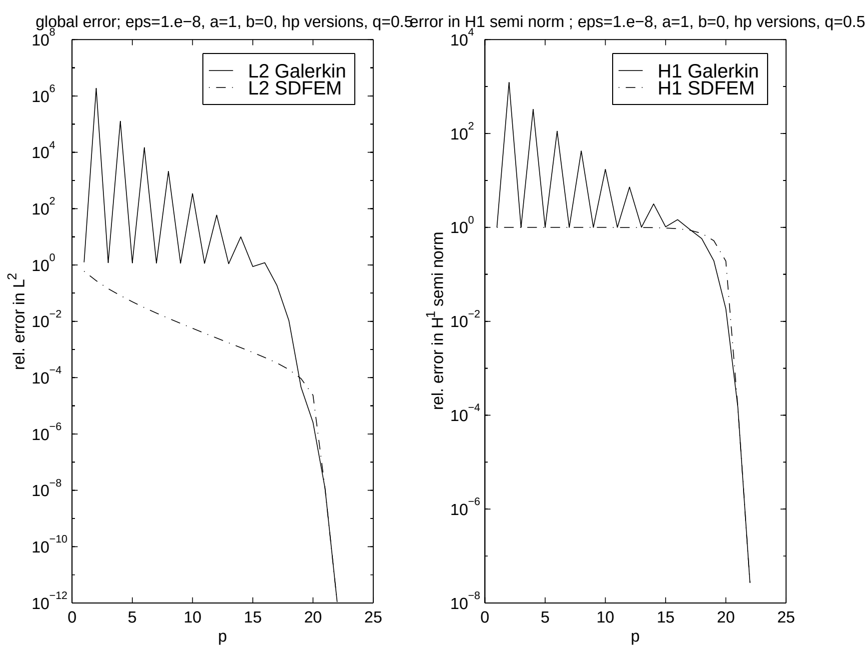

Figure 5: L^2 and energy performance for “hp”-mesh; “L^2-stabilized” SDFEM; $q = 0.5$, $\varepsilon = 10^{-8}$

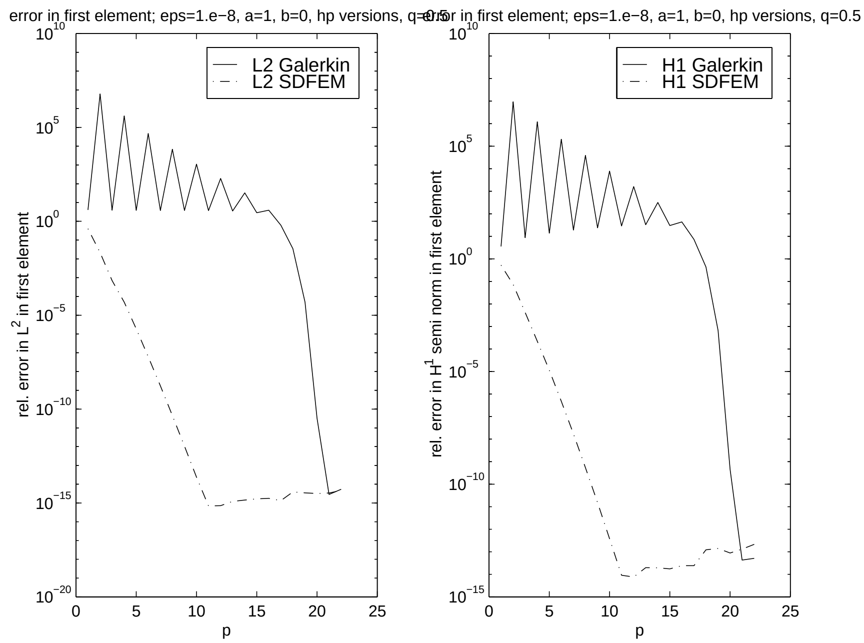

Figure 6: L^2 and H^1 performance on first element $(-1,0)$ for "hp"-mesh; "L^2-stabilized" SDFEM; $q = 0.5$, $\varepsilon = 10^{-8}$

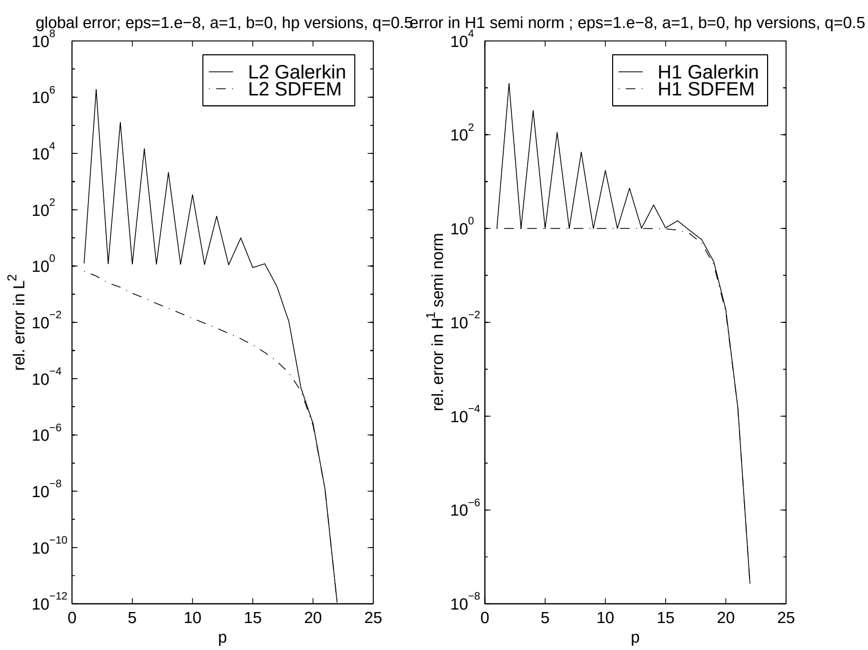

Figure 7: L^2 and H^1 performance on L^2 and energy performance for "hp"-mesh; "bubble-stabilized" SDFEM; $q = 0.5$, $\varepsilon = 10^{-8}$

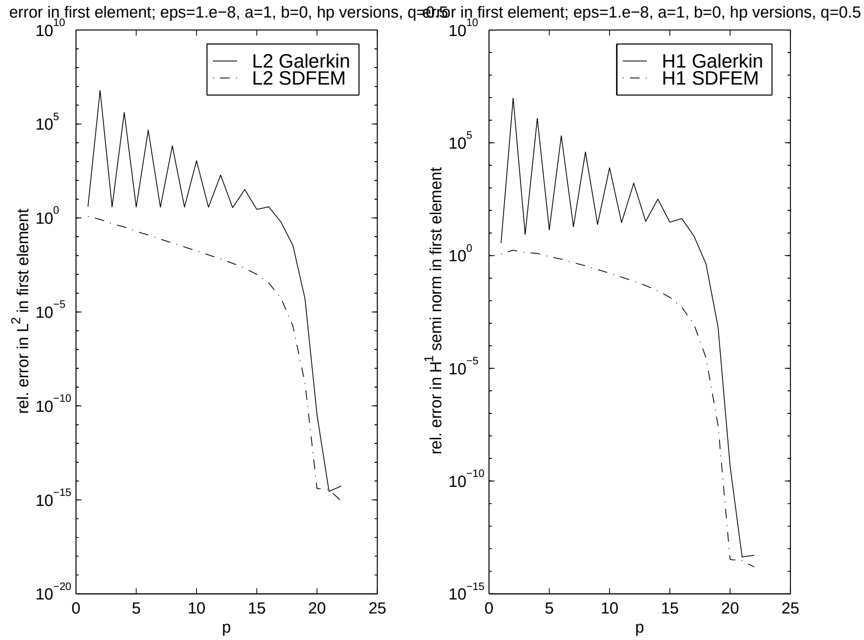

Figure 8: L^2 and H^1 performance on first element $(-1, 0)$ for "hp"-mesh; "bubble-stabilized" SDFEM; $q = 0.5$, $\varepsilon = 10^{-8}$

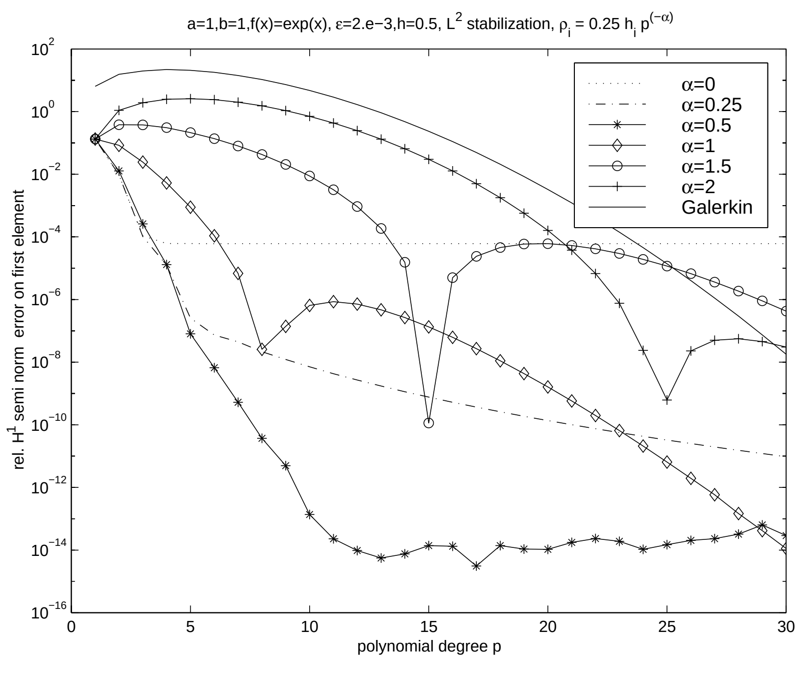

Figure 9: rel. error in H^1 semi norm performance on first element $(-1, -1/2)$ for "L^2-stabilized" SDFEM for various choices of ρ_i; $\varepsilon = 2 \cdot 10^{-3}$

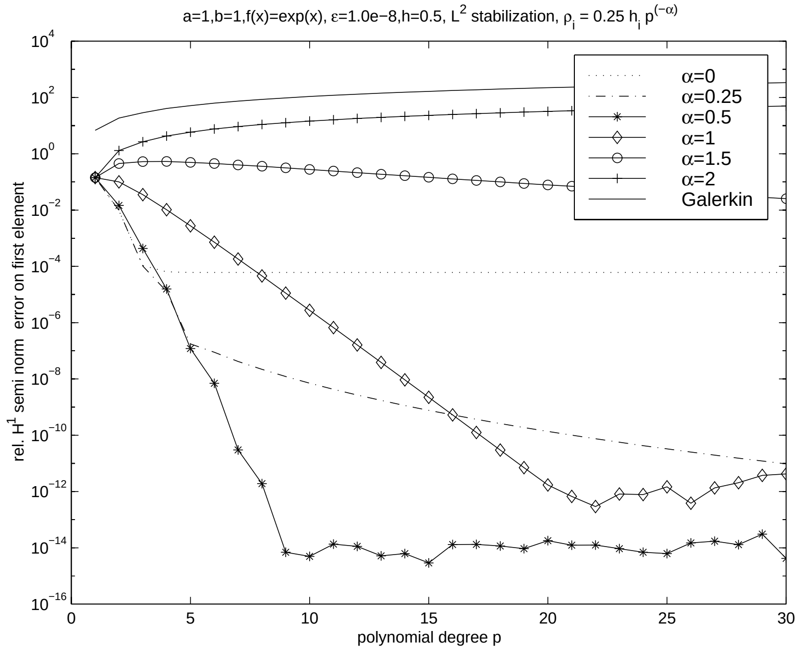

Figure 10: rel. error in H^1 semi norm performance on first element $(-1,-1/2)$ for "L^2-stabilized" SDFEM for various choices of ρ_i; $\varepsilon = 10^{-8}$

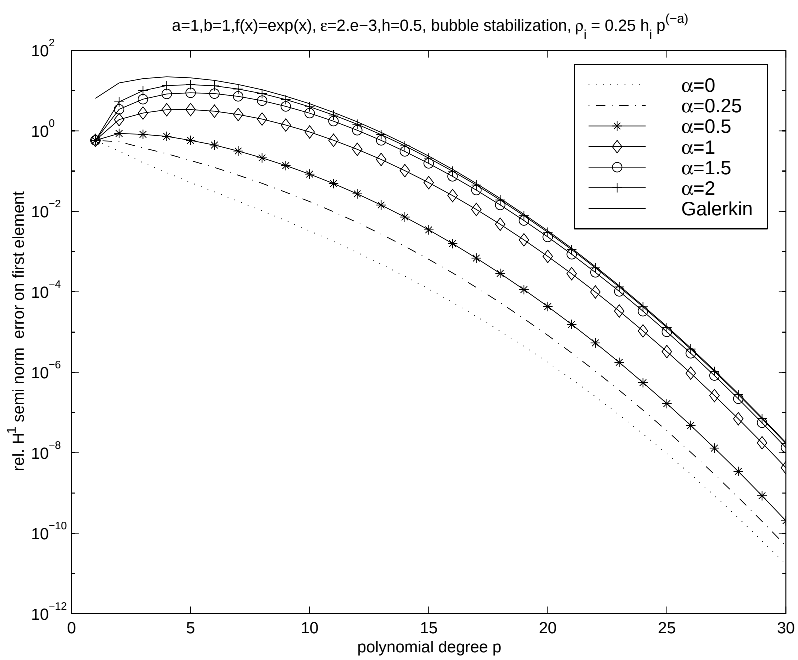

Figure 11: rel. error in H^1 semi norm performance on first element $(-1,-1/2)$ for "bubble-stabilized" SDFEM for various choices of ρ_i; $\varepsilon = 2 \cdot 10^{-3}$

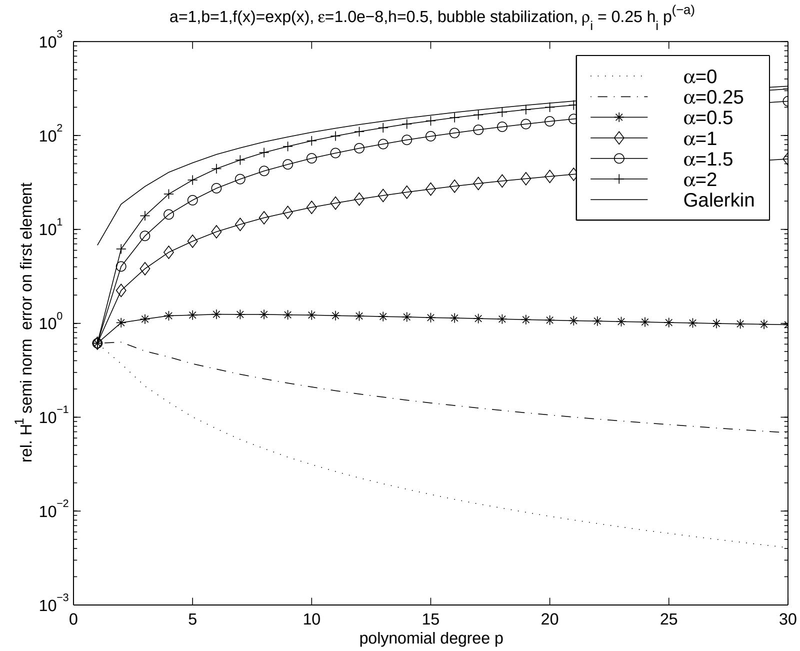

Figure 12: rel. error in H^1 semi norm performance on first element $(-1,-1/2)$ for "bubble-stabilized" SDFEM for various choices of ρ_i; $\varepsilon = 10^{-8}$

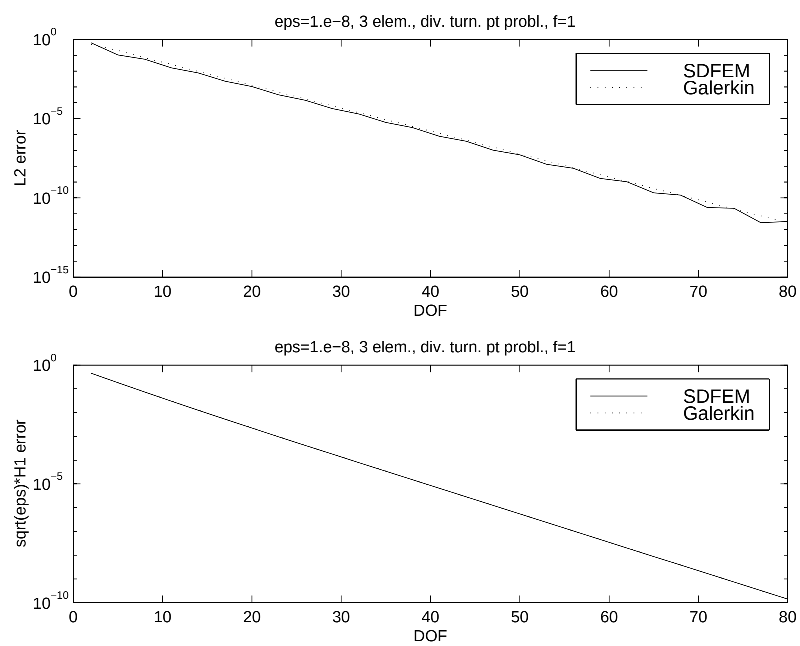

Figure 13: L^2 and energy error for turning pt. problem, $a = 1$, $\varepsilon = 10^{-8}$, 3 elem.

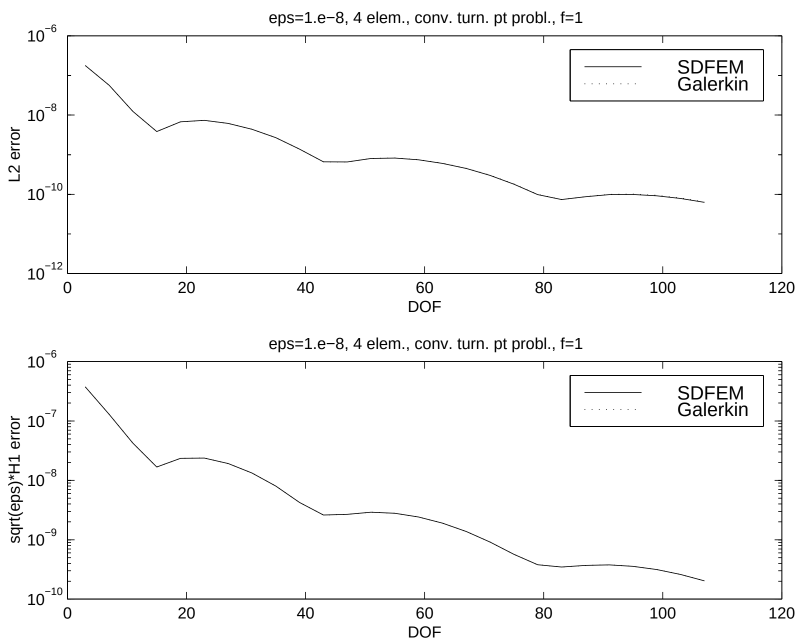

Figure 14: L^2 and energy error for turning pt. problem, $a = -1$, $\varepsilon = 10^{-8}$, 4 elem.

A The case $b \equiv 0$, $a \equiv 1$: Global Error Analysis

Appendix A is devoted to the analysis of the special case $b \equiv 0$, $a(x) \equiv 1$ for $\varepsilon > 0$. The norm $|||\cdot|||$ and the semi norm $|||\cdot|||_\rho$ are defined as

$$|||u|||^2 := \varepsilon \|u'\|^2_{L^2(\Omega)}, \qquad |||u|||^2_\rho := \sum_{i=1}^{N} \rho_i \|\sqrt{d_i} u'\|^2_{L^2(I_i)}, \qquad |||u|||^2_{SD} := |||u|||^2 + |||u|||^2_\rho. \tag{A.1}$$

Proceeding as in the proof of Theorem 3.5, we obtain the following stability result:

Theorem A.1 *Assume that the weights* $(\rho_i, d_i)_{i=1}^N$ *are of the form* (3.3) *with the condition* (3.26) *replacing* (3.4)*. Then there holds*

$$(1 - \delta_0)|||u|||^2_{SD} \leq B_{SD}(u, u) \qquad \forall u \in S_0^{p,1}(\mathcal{T}).$$

Due to the fact that energy norm $|||\cdot|||$ completely degenerates as $\varepsilon \to 0$, the analysis of the Galerkin FEM is more delicate than that of the SDFEM, and different analyses for these two methods are necessary. The different behavior of the two methods can be seen already in the following a priori bounds.

Lemma A.2 *Let* $b \equiv 0$, $a = 1$, $\mathcal{T}$ *be any mesh. Assume the hypotheses of Lemma 3.9. Then there is* $C > 0$ *such that the SDFEM solution* u_{SD} *satisfies*

$$|||u_{SD}|||_{SD} \leq C\sqrt{N} p^{3/2} \|f\|_{L^2(\Omega)}, \qquad \|u_{SD}\|_{L^\infty(\Omega)} \leq CNp^3 \|f\|_{L^2(\Omega)}.$$

In the case of the pure Galerkin method, we have

$$|||u_G||| \leq C\varepsilon^{-1/2} \|f\|_{L^2(\Omega)}.$$

Proof: The estimate for the Galerkin method is standard. For the SDFEM, we have by Theorem A.1 and the orthogonality relation (3.7)

$$(1 - \delta_0)|||u_{SD}|||^2_{SD} \leq B_{SD}(u_G, u_G) = F_{SD}(u_{SD}) \leq \|f\|_{L^2(\Omega)} \sqrt{N} p^{3/2} |||u_{SD}|||_{SD}$$

by Lemma 3.9. Applying again Lemma 3.9 yields the estimate for u_{SD}.

□

Remark A.3 The factors $\sqrt{N}$, $p^{3/2}$, p^3 in the estimates may not be optimal; however, the main point of the estimate on u_{SD} is that it is independent of ε in contrast to the estimate for u_G. The factor $\varepsilon^{-1/2}$ in the Galerkin estimate seems to be optimal from computational experiments (cf. Figs. 3, 5).

A.1 The SDFEM

Using the same techniques as in the proof of Theorem 3.6, we obtain

Theorem A.4 *Let* $a = 1$, $b = 0$. *Assume the weights* $(\rho_i, d_i)_{i=1}^N$ *are of the form* (3.3) *with* (3.26) *replacing* (3.4)*. Then there is* $C > 0$ *such that for any mesh* $\mathcal{T}$ *and any admissible splitting in the sense of Definition 3.3 there holds:*

1. *In the case of the "bubble-stabilized" SDFEM (cf. Definition 3.1) there holds*

$$(1 - \delta_0)|||e|||_{SD} \leq C\left\{|||\eta||| + \widetilde{E}_{G,d}(\eta_{reg}) + \varepsilon^{-1/2}\|\eta_{BL}\|_{L^2} + E_{SD}(\eta)\right\}.$$

2. *For the "L^2-stabilized" SDFEM (cf. Definition 3.1) there holds*

$$(1 - \delta_0)|||e|||_{SD} \leq C\left\{|||\eta||| + \widetilde{E}_{G,d}(\eta_{reg}) + \varepsilon^{-1/2}\|\eta_{BL}\|_{L^2} + E_{SD}(\eta) + E_{SD,\infty}(\eta).\right\}$$

Here

$$\widetilde{E}_{G,d}(\eta) = \left\{ \sum_{i=1}^{N} (\rho_i + \varepsilon)^{-1} \| \sqrt{d_i} \eta \|^2_{L^2(I_i)} \right\}^{1/2}.$$

The first estimate of Theorem A.4 allows us to conclude that Theorem 3.7 holds for the case $a \equiv 1$, $b \equiv 0$ as well. We can also obtain from Theorem A.4 robust exponential convergence if we consider meshes that "contain" the "two-element" mesh of Definition 2.2 under the assumption that the weights ρ_i satisfy the non-degeneracy condition (3.5):

Corollary A.5 *Let the hypotheses of Theorem A.4 be satisfied and assume additionally that the "non-degeneracy" condition* (3.5) *holds. Let κ_0 be given by Lemma 2.4. Assume that for some $\kappa \in (0, \kappa_0)$ the mesh $\mathcal{T}$ satisfies $S_0^{p,1}(\mathcal{T}_{\varepsilon,\kappa,p}) \subset S_0^{p,1}(\mathcal{T})$. Denote by u_{SD} the SDFEM solution of* (1.11). *Then there are C, $\sigma > 0$ depending on κ, $\underline{\delta}$, δ_0, and f, such that*

$$|||u_\varepsilon - u_{SD}||| \leq C e^{-\sigma p}, \qquad \|u_\varepsilon - u_{SD}\|_{L^\infty(\Omega)} \leq C\sqrt{N} e^{-\sigma p}.$$

Proof: Use the splitting of Lemma 2.4 to obtain the energy norm bound and invoke Lemma 3.9 for the L^∞ bound.

□

A.2 The Galerkin FEM

We consider now the case of the Galerkin FEM, i.e., all ρ_i vanish. Corollary A.5 does not cover this case as the non-degeneracy assumption (3.5) was instrumental in controlling the term $E_{G,d}(\eta_{reg})$ robustly in ε: For the pure Galerkin FEM, this term will introduce an additional factor of $\varepsilon^{-1/2}$ in the final estimate. The aim of the present section is to show that a more refined analysis allows us to obtain robust exponential convergence in the 'energy norm' $|||\cdot|||$ for the Galerkin FEM as well.

The analogue of Corollary A.5 reads

Corollary A.6 *Assume the same hypotheses as in Corollary A.5. Let u_G be the Galerkin solution of* (1.11) *with $b \equiv 0$, $a = 1$. Then for each $\kappa \in (0, \kappa_0)$ there are C, $\sigma > 0$*

$$|||u_\varepsilon - u_G||| \leq C e^{-\sigma p} \left(1 + \left\{ \sum_{i=1}^{N} \varepsilon h_i^{-1} \right\}^{1/2} \right).$$

Remark A.7 Corollary A.6 gives robust exponential convergence in the energy norm $|||\cdot|||$ for the Galerkin method for the "two-element" mesh. ($h_1 = O(1)$, $h_2 = O(p\varepsilon)$). This robust exponential convergence is indeed observed in the numerical experiments of Section 4.

In order to prove this theorem, we need two lemmata, the first of which is standard.

Lemma A.8 *Let u be analytic on $\overline{\Omega} = [-1, 1]$. Denote P_{p,I_i} the L^2 projection of u onto $\Pi_p(I_i)$, the space of polynomials of degree $p \in \mathbb{N}_0$ on I_i. Then there are constants C, $\sigma > 0$ such that for each element I_i*

$$\begin{aligned} \| (u - P_{p,I_i} u)^{(l)} \|_{L^2(I_i)} &\leq C e^{-\sigma p} h_i^{3/2 - l}, \qquad l = 0, 1, \\ \|u - P_{p,I_i}\|_{L^\infty(I_i)} &\leq C e^{-\sigma p} h_i. \end{aligned} \tag{A.2}$$

Lemma A.9 *Let u be analytic on $\overline{\Omega} = [-1, 1]$. Then there are C, $\sigma > 0$ depending only on u such that the following holds. There is $u_p \in S^{p,1}(\mathcal{T})$ with $u(\pm 1) = u_p(\pm 1)$ and*

$$\begin{aligned} \left| \int_\Omega (u - u_p) v' \, dx \right| &\leq C e^{-\sigma p} h_N^{1/2} \|v'\|_{L^2(I_N)} \qquad \forall v \in S^{p,1}(\mathcal{T}), \\ \|(u - u_p)^{(l)}\|_{L^2(I_i)} &\leq C e^{-\sigma p} h_i^{1/2 - l}, \qquad l = 0, 1. \end{aligned}$$

Here I_N denotes the last element abutting on $x = 1$.

Proof: Denote L_j the jth Legendre polynomial normalized to $L_j(1)=1$. Writing $m_i=(x_{i-1}+x_i)/2$, we set $\widetilde{L}_{ij}(x):=(-1)^j L_j(2(x-m_i)/h_i)$, $i=1,\ldots,N$. First, choose a discontinuous approximation of u by polynomials of degree $p-1$ on all elements

$$\widetilde{u}_{p-1,i}:=P_{p-1,I_i}u, \qquad i=1,\ldots,N.$$

Note that $\int_{I_i}(u-\widetilde{u}_{p-1,i})\widetilde{L}_{ij}\,dx=0$ for $j=0,\ldots,p-1$. Correct the inter-element discontinuities inductively by setting

$$\begin{array}{rclrcl}
\alpha_1 & := & u(-1)-\widetilde{u}_{p-1,1}(-1), & u_p|_{I_1} & := & \widetilde{u}_{p-1,1}+\alpha_1\widetilde{L}_{1p},\\
\alpha_i & := & u_p|_{I_{i-1}}(x_{i-1})-\widetilde{u}_{p-1,i}(x_{i-1}), & u_p|_{I_i} & := & \widetilde{u}_{p-1,i}+\alpha_i\widetilde{L}_{ip}, \qquad i=2,\ldots,N-1,\\
\alpha_N & := & u_p|_{I_{N-1}}(x_{N-1})-\widetilde{u}_{p-1,N}(x_{N-1}), & u_p|_{I_N} & := & \widetilde{u}_{p-1,N}+\alpha_N\widetilde{L}_{Np}+l(x),\\
 & & & l(x) & := & \Big(u(1)-\widetilde{u}_{p-1,N}(1)-\alpha_N\widetilde{L}_{Np}(1)\Big)\cdot\frac{x-x_{N-1}}{h_N}
\end{array}$$

We observe that $u_p\in S^{p,1}(\mathcal{T})$ and that $u_p(\pm1)=u_p(\pm1)$. Furthermore, by the orthogonality of the Legendre polynomials, we still have for all elements but the last one

$$\int_{I_i}(u-u_p)\widetilde{L}_{ij}\,dx=\int_{I_i}(u-\widetilde{u}_{p-1,i})\widetilde{L}_{ij}\,dx-\alpha_i\int_{I_i}\widetilde{L}_{ip}\widetilde{L}_{ij}\,dx=0, \qquad j=0,\ldots,p-1, \quad i=1,\ldots,N-1.$$

Hence, we conclude

$$\int_{\Omega\setminus I_N}(u-u_p)v'\,dx=0 \qquad \forall v\in S^{p,1}(\mathcal{T}).$$

This yields

$$\left|\int_{\Omega}(u-u_p)v'\,dx\right|=\left|\int_{I_N}(u-u_p)v'\,dx\right|\le\|u-u_p\|_{L^2(I_N)}\|v'\|_{L^2(I_N)} \qquad \forall v\in S^{p,1}(\mathcal{T}).$$

Let us now analyze the difference $u-u_p$. As $|\widetilde{L}_{ip}|\le 1$ and

$$\alpha_i=\Big(u_p|_{I_{i-1}}(x_{i-1})-u(x_{i-1})\Big)+\Big(u(x_{i-1})-\widetilde{u}_{p-1,i}(x_{i-1})\Big), \qquad i=2,\ldots,N,$$

we obtain with Lemma A.8

$$\begin{array}{rcl}
|\alpha_1| & \le & Ce^{-\sigma p}h_1\\
|\alpha_i| & \le & \|u-u_p\|_{L^\infty(I_{i-1})}+Ce^{-\sigma p}h_i\\
 & \le & Ce^{-\sigma p}h_i+Ce^{-\sigma p}h_{i-1}+|\alpha_{i-1}|, \qquad i=2,\ldots,N.
\end{array}$$

Inductively, we conclude therefore

$$|\alpha_i|\le 2Ce^{-\sigma p}\sum_{j=1}^{i}h_j\le 4Ce^{-\sigma p}, \qquad i=1,\ldots,N. \tag{A.3}$$

This allows us to obtain for $(u-u_p)|_{I_i}=(u-\widetilde{u}_{p-1,i})-\alpha_i\widetilde{L}_{ip}$ on all elements but the last one:

$$\begin{array}{rcl}
\|(u-u_p)^{(l)}\|_{L^2(I_i)} & \le & Ce^{-\sigma p}h_i^{3/2-l}+|\alpha_i|\,Cp^{2l}h_i^{1/2-l}\\
 & \le & Ce^{-\sigma p}h_i^{1/2-l}, \qquad l=0,1, \quad i=1,\ldots,N-1
\end{array} \tag{A.4}$$

For the last element I_N, we estimate the linear function $l(x)$ by

$$\|l^{(l)}\|_{L^2(I_N)}\le\left(Ce^{-\sigma p}h_N+|\alpha_N|\right)h_N^{1/2-l}, \qquad l=0,1,$$

and therefore we arrive at

$$\|(u-u_p)^{(l)}\|_{L^2(I_N)}\le Ce^{-\sigma p}h_N^{1/2-l}, \qquad l=0,1.$$

This concludes the proof of the lemma.

□

Remark A.10 The element I_N in the statement of Lemma A.9 can be replaced by any non-trivial contiguous submesh abutting on $x = 1$.

Proof of Corollary A.6: We restrict ourselves here to the case of interest $\kappa p\varepsilon < 1$. For any admissible splitting in the sense of Definition 3.3 we may use the coercivity (Theorem A.1) and the orthogonality (3.6) to write

$$C\,|||e|||^2 \quad\leq\quad B_\varepsilon(e,e) = B_\varepsilon(\eta, e) = \varepsilon\int_\Omega \eta' e'\,dx + \int_\Omega \eta' e\,dx = \varepsilon\int_\Omega \eta' e'\,dx - \int_\Omega \eta e'\,dx$$

by an integration by parts. We split $u_\varepsilon = u_{reg} + u_{BL}$ as in Lemma 2.4. u_{BL} is approximated by $u_{BL,p}$ as in Lemma 2.4; $u_{reg,p}$ will be chosen below. On writing $\eta = \eta_{reg} + \eta_{BL}$ we conclude by exploiting $|||\eta_{BL}||| \leq Ce^{-\sigma p}$, $\|\eta_{BL}\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}e^{-\sigma p}$

$$|B_\varepsilon(\eta_{BL}, e)| \leq Ce^{-\sigma p}\,|||e|||\,.$$

Let us now turn to $B_\varepsilon(\eta_{reg}, e)$. To that end, let us construct $u_{reg,p}$ as an approximation of u_{reg} (which is analytic) as given by Lemma A.9, and we obtain

$$|B_\varepsilon(\eta_{reg}, e)| \leq |||\eta_{reg}|||\;|||e||| + Ch_N^{1/2}e^{-\sigma p}e\|e'\|_{L^2(I_N)}.$$

As $h_N \leq \kappa p\varepsilon$ we therefore arrive at

$$|B_\varepsilon(\eta_{reg}, e)| \leq |||\eta_{reg}|||\;|||e||| + C(\kappa p)^{1/2}e^{-\sigma p}\,|||e|||\,.$$

Finally, also by Lemma A.9

$$|||\eta_{reg}|||^2 = \sum_{i=1}^N \varepsilon\|\eta'_{reg}\|^2_{L^2(I_i)} \leq Ce^{-2\sigma p}\sum_{i=1}^N \frac{\varepsilon}{h_i}$$

which allows us to conclude the proof of Corollary A.6 by means of the triangle inequality.

$\square$

Under extra assumptions on the regularity of the mesh in the region outside the boundary layer, the term $\sum_{i=1}^N \varepsilon/h_i$ in Corollary A.6 can be removed:

Corollary A.11 *Let κ_0 be given by Theorem 2.4. Assume that for each $p \in \mathbb{N}$, $p \geq 2$, there is a mesh $\mathcal{T}$ satisfying the following two conditions: (i) For each p, there is $\kappa \in (0, \kappa_0)$ such that $1 - \kappa p\varepsilon$ is a mesh point, and (ii) $\mathcal{T}$ restricted to $(-1, 1-\kappa p\varepsilon)$ is a quasiuniform mesh. ($\mathcal{T}$ restricted to $(1-\kappa p\varepsilon, 1)$ may be any mesh.) Then there are C, $\sigma > 0$ depending only on the domain of analyticity of f and the constant a such that*

$$|||u_\varepsilon - u_G||| \leq C\left(\varepsilon + (\kappa p)^{-1/2}\right)e^{-\sigma\kappa p}, \qquad p \geq 2.$$

Proof: Let h be the mesh width of the quasiuniform mesh on $[-1, 1-\kappa p\varepsilon]$. First, we observe that for $p \geq 2$, we may replace the factor h_i in (A.2) by h_i^2. From Remark A.10, we may assume without loss of generality that the last element $I_N = (1-\kappa p\varepsilon, 1)$. Then, upon inspection of the proof of Lemma A.9, we see that we may replace the $\sum_{j=1}^i h_j$ by $\sum_{j=1}^i h_j^2$ in (A.3). By the quasi-uniformity of the mesh restricted to $\Omega\setminus I_N$, we may estimate $\sum_{j=1}^i h_j^2 \leq Ch$ and hence all the estimates for the coefficients α_i, $i = 1, \ldots, N-1$ are improved by a factor h. Thus, the estimate (A.4) is improved by a factor $h \sim h_i$. In the proof of Corollary A.6, the term $\sum_{i=1}^N \varepsilon/h_i$ is due to $|||\eta|||_{reg}$. Inserting these improved bounds, we obtain

$$|||\eta|||^2_{reg} = \sum_{i=1}^{N-1}\varepsilon\|\eta'_{reg}\|^2_{L^2(I_i)} + \varepsilon\|\eta'_{reg}\|^2_{L^2(I_N)} \leq Ce^{-2\sigma p}\sum_{i=1}^{N-1}\varepsilon h_i + Ce^{-2\sigma p}\varepsilon h_N^{-1}.$$

As $h_N = \kappa p\varepsilon$, the claim of the corollary follows.

$\square$

B The case $b \equiv 0$, $a \equiv 1$: Local Error Analysis

The crucial feature of the SDFEM is, of course, that even if layers/fronts are not resolved the SDFEM still yields good results in certain parts of the computational domain *away* from the layers/fronts. In other words, the SDFEM can be viewed as a technique to control pollution.

Our main focus will be on the analysis of the following meshes which results from an hp adaptive scheme that locates and tries to resolve layers. As the solutions of our model equation (1.1) have a layer at the outflow boundary $x = 1$, an adaptive scheme would create a sequence of meshes that are obtained by successively halving the rightmost element until the element abutting on the boundary has size $O(p\varepsilon)$, i.e., the mesh can resolve the boundary layer. Thereafter the mesh is fixed and merely the polynomial degree p is increased. We formalize this procedure as follows: Fix $q \in (0, 1)$. Let

$$p_0 \in \mathbb{N} \quad \text{be the smallest integer such that } q^{p_0} < p_0\varepsilon. \tag{B.5}$$

We consider meshes $\mathcal{T}$ with geometric refinement towards the layer determined by the nodes

$$\{-1, 1, 1 - q^i \,|\, i = 0, \ldots, \min(p, p_0)\}, \tag{B.6}$$

and we will consider as trial spaces the space $S_0^{p,1}(\mathcal{T})$ (cf. Fig. 1) On such meshes, the hp-SDFEM converges robustly and exponentially on compact subsets of Ω:

Theorem B.1 *Let $a = 1$, $b = 0$, $\xi \in (-1, 1)$ fixed. Consider the meshes $\mathcal{T}$ defined by* (B.6). *Assume that the weights are of the form $(\rho_i, b_i)_{i=1}^N$ where the factors $(\rho_i)_{i=1}^N$ are of the form* (3.3) *with (3.4) replaced by* (3.26). *Furthermore, assume that the non-degeneracy condition* (3.5) *holds. Then there are constants C, $\sigma > 0$ independent of ε, p such that*

$$\|u_\varepsilon - u_{SD}\|_{H^1(-1,\xi)} \leq Ce^{-\sigma p}, \quad p = 1, 2, \ldots$$

The proof of Theorem B.1 is the object of the remainder of this section.

Remark B.2 We consider here a "bubble-stabilized" SDFEM. This is not essential here. A similar result can be obtain for the "L^2-stabilized" SDFEM and numerically observed (cf. the numerical results in Section 4).

B.1 Preliminaries

Our analysis proceeds along the lines of [7]. The idea is to obtain estimates in *weighted* spaces. To that end, one proves stability in these weighted spaces (Proposition B.5) and then estimates various error terms (Lemmata B.8–B.10).

Let $\mathcal{T}$ be a mesh whose elements have lengths h_i. Let ξ be a fixed mesh point and let $\lambda > 0$ be a fixed number to be chosen sufficiently small below. The weights $(\rho_i, b_i)_{i=1}^N$ are assumed to of the form (3.3) with (3.26) in place of (3.4). In order to introduce the piecewise linear weight function ω^2, we need to introduce the functions ψ, $\widetilde{\omega}$ by

$$\psi|_{I_i} := \psi_i := \frac{\delta_i}{h_i}, \tag{B.7}$$

$$\widetilde{\omega}(x) := \begin{cases} 1 & \text{if } x \leq \xi \\ \exp\left[-\lambda \int_\xi^x \psi(t)\,dt\right] & \text{if } x > \xi. \end{cases} \tag{B.8}$$

Remark B.3 A few remarks concerning the properties of ψ are in order. We have $\psi_i h_i = \delta_i \leq \delta_0$. Furthermore, $(\psi_i h_i)h_i/p = \rho_i$, a relation of which we will make use later on. If $\delta_i \geq \underline{\delta} > 0$ for all i, then $\int_\xi^x \psi(t)\,dt$ measures (up to a constant) the number of elements between ξ and x.

The weight function ω^2 is then finally given as the piecewise linear interpolant of $\widetilde{\omega}^2$.

We note that for all meshes and $\lambda > 0$ the function $\widetilde{\omega}$ is Lipschitz continuous on I and satisfies $\widetilde{\omega}' \leq 0$. Furthermore, on each element $I_i = (x_{i-1}, x_i)$ we have in fact that

$$\widetilde{\omega}(x_{i-1})/\widetilde{\omega}(x_i) \leq e^{\lambda\delta_i}, \qquad |\widetilde{\omega}'\widetilde{\omega}| \leq \lambda\psi\,\widetilde{\omega}^2.$$

These properties are shared by the piecewise linear interpolant ω^2:

$$\omega' \leq 0 \quad \text{on } I, \qquad \omega(x_{i-1})/\omega(x_i) \leq e^{\lambda\delta_0}, \qquad |\omega'\omega| \leq \lambda\psi e^{\lambda\delta_0}\,\omega^2. \tag{B.9}$$

We introduce the following mesh dependent weighted norm

$$|||u|||^2_{SD,\omega} := \varepsilon\|\omega u'\|^2_{L^2(\Omega)} + \|\,|\omega\omega'|^{1/2}u\|^2_{L^2(\Omega)} + \sum_{i=1}^{N}\rho_i\|\sqrt{d_i}\omega u'\|^2_{L^2(\Omega)}. \tag{B.10}$$

We need the following "superapproximation" result:

Lemma B.4 *Let $(\rho_i, b_i)_{i=1}^N$ be of the form (3.3) with (3.26) in place of (3.4). There is $C > 0$ depending only on δ_0 such that for all $\lambda \in [0,1]$, $e \in S_0^{p,1}(\mathcal{T})$ there is $W \in S_0^{p,1}(\mathcal{T})$ satisfying*

$$\|b_i^{-1/2}\omega^{-1}(\omega^2 e - W)\|_{L^2(I_i)} \leq C\lambda\rho_i\|\sqrt{b_i}\omega e'\|_{L^2(I_i)}, \qquad \|\omega^{-1}(\omega^2 e - W)'\|_{L^2(I_i)} \leq C\lambda\delta_i\|\sqrt{b_i}\omega e'\|_{L^2(I_i)}.$$

Furthermore, W can be chosen such that $\omega^2 e = W$ at the nodes of $\mathcal{T}$.

Proof: On the reference element $(-1,1)$, expansion in Legendre series and the orthogonality of the Jacobi polynomials allow us to estimate for all polynomials $\hat{e}$ of degree p:

$$\int_{-1}^{1}(1-x^2)^p|\hat{e}^{(p)}(x)|^2\,dx \leq \frac{(2p)!}{p(p+1)}\int_{-1}^{1}(1-x^2)|\hat{e}'(x)|^2\,dx \tag{B.11}$$

(see, e.g., [13]). Consider now a fixed element I_i. Denote $\hat{\omega}^2$, $\hat{e}$ the functions corresponding to ω^2, e on the reference element $(-1,1)$ via the linear mapping. The standard p-version argument (see, e.g., [13]) allows us to construct a polynomial $\hat{W}$ of degree p such that $\hat{W}(\pm 1) = (\hat{\omega}^2\hat{e})(\pm 1)$ and

$$\begin{aligned}
\|(1-x^2)^{-1/2}(\hat{\omega}^2\hat{e} - \hat{W})\|^2_{L^2(-1,1)} &\leq \frac{1}{(2p)!p(p+1)}\int_{-1}^{1}(1-x^2)^p\left|(\hat{\omega}^2\hat{e})^{(p+1)}\right|^2\,dx,\\
\|(\hat{\omega}^2\hat{e} - \hat{W})'\|^2_{L^2(-1,1)} &\leq \frac{1}{(2p)!}\int_{-1}^{1}(1-x^2)^p\left|(\hat{\omega}^2\hat{e})^{(p+1)}\right|^2\,dx.
\end{aligned}$$

As $\hat{\omega}^2$ is a linear function and $\hat{e}$ a polynomial of degree p we observe that $(\hat{\omega}^2\hat{e})^{(p+1)} = (p+1)(\hat{\omega}^2)'\hat{e}^{(p)}$ and thus by invoking (B.11)

$$\begin{aligned}
\|(1-x^2)^{-1/2}(\hat{\omega}^2\hat{e} - \hat{W})\|^2_{L^2(-1,1)} &\leq \frac{1}{p^2}|(\hat{\omega}^2)'|^2\int_{-1}^{1}(1-x^2)|\hat{e}'(x)|^2\,dx,\\
\|(\hat{\omega}^2\hat{e} - \hat{W})'\|^2_{L^2(-1,1)} &\leq \frac{p+1}{p}|(\hat{\omega}^2)'|^2\int_{-1}^{1}(1-x^2)|\hat{e}'(x)|^2\,dx.
\end{aligned}$$

(B.9) implies that $|(\hat{\omega}^2)'| \leq \lambda e^{\lambda\delta_0}\psi_i\frac{h_i}{2}\hat{\omega}^2$. Exploiting the fact that ω is of bounded variation on I_i (cf. (B.9)) we obtain the desired result by a scaling argument.

□

B.2 Error Analysis

Proposition B.5 *Let $c > 0$ be fixed and $\mathcal{T}$ be any mesh with the property that $h_i \geq c\varepsilon$ for $i = 1,\ldots,N$.Assume the weights $(\rho_i, b_i)_{i=1}^N$ are of the form (3.3) with satisfy* (3.26) *replacing (3.4). Then there are λ_0, $\gamma > 0$ depending only on c and δ_0 such that for all $0 \leq \lambda \leq \lambda_0$*

$$B_{SD}(u,\omega^2 u) \geq \gamma|||u|||^2_{SD,\omega} \qquad \forall u \in S_0^{p,1}(\mathcal{T}).$$

Proof: We have

$$B_{SD}(u,\omega^2 u) = \varepsilon \int_\Omega u'(\omega^2 u)'\,dx + \int_\Omega u'(\omega^2 u)\,dx + \sum_{i=1}^N \rho_i \int_{I_i} d_i(-\varepsilon u'' + u')(\omega^2 u)'\,dx.$$

We estimate each of these terms separately. An integration by parts yields for the second term

$$\int_\Omega u'\omega^2 u\,dx = \frac{1}{2}\int_\Omega \left(u^2\right)'\omega^2\,dx = -\int_\Omega u^2\omega\omega'\,dx = \|\,|\omega\omega'|^{1/2}u\|^2_{L^2(\Omega)}.$$

Cauchy's inequality together with the assumption that $h_i \geq c\varepsilon$ for all i (implying $\psi\varepsilon \leq \delta_0/c$) allows us to estimate the first term by

$$\begin{aligned}
\varepsilon \int_\Omega u'(\omega^2 u)'\,dx &= \varepsilon \int_\Omega |u'|^2\omega^2\,dx + \varepsilon\int_\Omega (\omega^2)'uu'\,dx \\
&\geq \varepsilon\|\omega u'\|^2_{L^2(\Omega)} - \left[\frac{1}{\tau}\|\,|\omega\omega'|^{1/2}u\|^2_{L^2(\Omega)} + \tau\varepsilon^2\lambda e^{\lambda\delta_0}\|\psi^{1/2}\omega u'\|^2_{L^2(\Omega)}\right] \\
&\geq \varepsilon\|\omega u'\|^2_{L^2(\Omega)} - \left[\frac{1}{\tau}\|\,|\omega\omega'|^{1/2}u\|^2_{L^2(\Omega)} + \tau\varepsilon\lambda e^{\lambda\delta_0}\frac{\delta_0}{c}\|\omega u'\|^2_{L^2(\Omega)}\right]
\end{aligned}$$

for any $\tau > 0$ which we will choose sufficiently large below. For the SDFEM terms, we estimate, using the bounds (B.9)

$$\sum_{i=1}^N \rho_i \int_{I_i} d_i u'(\omega^2 u)'\,dx \geq \sum_{i=1}^M \rho_i \|\sqrt{d_i}\omega u'\|^2_{L^2(I_i)} - \sum_{i=1}^N \left[\frac{1}{\tau}\|\,|\omega\omega'|^{1/2}u\|^2_{L^2(I_i)} + \rho_i^2\tau\lambda e^{\lambda\delta_0}\psi_i\|\sqrt{d_i}\omega u'\|^2_{L^2(I_i)}\right]$$

for any $\tau > 0$ to be chosen sufficiently large below. We note that $\rho_i\psi_i \leq \delta_0^2$. Using Lemma 3.4 and again (B.9) we bound furthermore

$$\begin{aligned}
&\left|\sum_{i=1}^N \rho_i \int_{I_i} -d_i\varepsilon u''(\omega^2 u)'\,dx\right| \\
&\quad \leq \sum_{i=1}^N \varepsilon\rho_i\frac{2p}{h_i}e^{\lambda\delta_0}\|\omega u'\|^2_{L^2(I_i)} + \frac{1}{\tau}\|\,|\omega\omega'|^{1/2}u\|^2_{L^2(I_i)} + \tau\left(\frac{2p\rho_i}{h_i}\right)^2\varepsilon^2\psi_i\lambda e^{\lambda\delta_0}\|\sqrt{d_i}\omega u'\|^2_{L^2(I_i)} \\
&\quad \leq \sum_{i=1}^N \varepsilon(\delta_i e^{\lambda\delta_0})\|\omega u'\|^2_{L^2(I_i)} + \frac{1}{\tau}\|\,|\omega\omega'|^{1/2}u\|^2_{L^2(I_i)} + \tau\delta_i^2\varepsilon\frac{\delta_i}{c}\lambda e^{\lambda\delta_0}\|\omega u'\|^2_{L^2(I_i)}
\end{aligned}$$

for any $\tau > 0$. We obtain the desired result by first choosing $\tau > 3$ and then choosing λ_0 suffiently small.

□

Remark B.6 Checking the proof of Proposition B.5 shows that the result also holds true for $\lambda = 0$ and that in fact in that case the additional assumption $h_i \geq c\varepsilon$ can be removed. Hence the result of Theorem A.4 is contained as a special case.

The stability estimate allows us to formulate the following error estimate for the solution in terms of the weighted norms $|||\cdot|||_{SD,\omega}$:

Proposition B.7 *Under the assumptions of Proposition B.5 there are $\lambda_0 > 0$, $C > 0$ such that for all $\lambda \in [0,\lambda_0]$ and for any admissible splitting in the sense of Definition 3.3 there holds*

$$|||e|||^2_{SD,\omega} \leq C\left\{\left|B_{SD}(e,\omega^2 e - W)\right| + \left|B_{SD}(\eta,\omega^2 e)\right| + \left|B_{SD}(\eta,\omega^2 e - W)\right|\right\} \qquad \forall W \in S_0^{p,1}(\mathcal{T}).$$

Proof: This is just an application of the Galerkin orthogonality (3.7)

$$\begin{aligned}
\gamma|||e|||^2_{SD,\omega} &\leq B_{SD}(e,\omega^2 e) = B_{SD}(e,\omega^2 e - W) + B_{SD}(\eta, W) \\
&= B_{SD}(e,\omega^2 e - W) + B_{SD}(\eta, W - \omega^2 e) + B_{SD}(\eta,\omega^2 e).
\end{aligned}$$

□

Lemma B.8 *Let $\mathcal{T}$ be any mesh and assume that the weights $(\rho_i, b_i)_{i=1}^N$ are of the form (3.3) with* (3.26) *replacing (3.4). Then there is $C > 0$ depending only on δ_0 such that for all $\lambda \in [0,1]$*

$$\left|B_{SD}(e, \omega^2 e - W)\right| \leq C\lambda \, |||e|||_{SD,\omega}^2 \qquad \forall e \in S_0^{p,1}(\mathcal{T}),$$

where, for each $e \in S_0^{p,1}(\mathcal{T})$ the function $W \in S_0^{p,1}(\mathcal{T})$ is given by Lemma B.4.

Proof: We write

$$B_{SD}(e, \omega^2 e - W) = \varepsilon \int_\Omega e'(\omega^2 e - W)' \, dx + \int_\Omega e'(\omega^2 e - W) + \sum_{i=1}^N \rho_i \int_{I_i} (-\varepsilon e'' + e')(\omega^2 e - W)' \, dx.$$

The first term can be estimated in the desired fashion immediately by invoking Lemma B.4. For the second term, we also use Lemma B.4 to arrive at

$$\left|\int_\Omega e'(\omega^2 e - W) \, dx\right| \leq \sum_{i=1}^N \|\sqrt{d_i}\omega e'\|_{I_i} C\lambda\rho_i \|\sqrt{d_i}\omega e'\|_{I_i} \leq C\lambda |||e|||_{SD,\omega}^2.$$

Finally, for the SDFEM terms we use Lemmata 3.4, B.4, and the bounds (B.9) to get

$$\begin{aligned} \rho_i \left|\int_{I_i} d_i(-\varepsilon e'' + e')(\omega^2 e - W)' \, dx\right| &\leq \rho_i \left[\varepsilon \|\sqrt{d_i}\omega e''\|_{I_i} + \|\sqrt{d_i}\omega e'\|_{I_i}\right] C\lambda\delta_i \|\sqrt{d_i}\omega e'\|_{I_i} \\ &\leq C\lambda\delta_0 \left[e^{\lambda\delta_0}\varepsilon(\rho_i \frac{2p}{h_i})\|\omega e'\|_{I_i}^2 + \rho_i \|\omega e'\|_{I_i}^2\right] \end{aligned}$$

As $\rho_i(2p/h_i) \leq \delta_0$ we obtain the desired result.

□

In order to formulate the next two lemmata, we need to introduce some weighted mesh-dependent norms on $H^1(\Omega)$ which are analogous to $|||\cdot|||$, (3.11),(3.13):

$$\begin{aligned} E_\omega(\eta) &:= \left\{\varepsilon \|\omega\eta'\|_{L^2(\Omega)}^2 + \|\,|\omega'\omega|^{1/2}\eta\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\,|\omega'\omega|^{1/2}\eta'\|_{L^2(\Omega)}^2\right\}^{1/2}, \\ E_{SD,\omega}(\eta) &:= \left\{\sum_{i=1}^N \rho_i \frac{p^2}{h_i^2}\left[\varepsilon^2 \|\omega\eta'\|_{L^2(I_i)}^2 + \|\omega\eta\|_{L^2(I_i)}^2\right]\right\}^{1/2}, \\ E_{G,\omega}(\eta) &:= \left\{\sum_{i=1}^N \frac{p^2}{\rho_i + \varepsilon}\|\omega\eta\|_{L^2(I_i)}^2\right\}^{1/2}. \end{aligned}$$

Lemma B.9 *Under the same hypotheses as in Lemma B.8 there exists $C > 0$ depending only on δ_0 such that for all $\lambda \in [0,1]$*

$$\left|B_{SD}(\eta, \omega^2 e)\right| \leq C|||e|||_{SD,\omega}\left[E_\omega(\eta) + E_{G,\omega}(\eta) + E_{SD,\omega}(\eta)\right] \qquad \forall e \in S_0^{p,1}(\mathcal{T}), \quad \eta \in H_0^1(\Omega).$$

Proof: We start again by recalling that

$$B_{SD}(\eta, \omega^2 e) = \varepsilon \int_\Omega \eta'(\omega^2 e)' \, dx + \int_\Omega \eta' \omega^2 e \, dx + \sum_{i=1}^N \rho_i \int_{I_i} d_i(-\varepsilon\eta'' + \eta')(\omega^2 e)' \, dx.$$

The Cauchy-Schwarz inequality allows us to estimate the first term by

$$\varepsilon \|\omega\eta'\|_{L^2(\Omega)} \|\omega e'\|_{L^2(\Omega)} + 2\varepsilon \|\,|\omega\omega'|^{1/2}\eta'\|_{L^2(\Omega)} \|\,|\omega\omega'|^{1/2} e\|_{L^2(\Omega)}$$

which yields the term $E_\omega(\eta)|||e|||_{SD,\omega}$. For the second term, we integrate by parts and obtain by splitting the integral into sums over elements and using the Cauchy-Schwarz inequality for sums:

$$2\|\,|\omega\omega'|^{1/2}\eta\|_{L^2(\Omega)} \|\,|\omega\omega|^{1/2} e\|_{L^2(\Omega)} + \left\{\sum_{i=1}^N \|\omega\eta\|_{I_i}^2 \frac{p^2}{\rho_i + \varepsilon}\right\}^{1/2} \left\{\sum_{i=1}^N \|\omega e'\|_{I_i}^2 \frac{\rho_i + \varepsilon}{p^2}\right\}^{1/2}$$

which yields the term $(E_\omega(\eta)+E_{G,\omega}(\eta))|||e|||_{SD,\omega}$ after an application of Lemma 3.4 and an invocation of (B.9). Let us now turn to the third term. An integration by parts on the element level gives

$$\left|\int_{I_i} d_i(-\varepsilon\eta''+\eta')(\omega^2 e)'\,dx\right| \leq \|-\varepsilon\eta'+\eta\|_{L^2(I_i)}\left[\frac{4}{h_i}\|\omega^2 e'\|_{L^2(I_i)}+\|d_i(\omega^2 e)''\|_{L^2(I_i)}\right].$$

Observing that $\omega^2 e'$ is a polynomial of degree p, we may invoke Lemma 3.4 to estimate

$$\|\omega^2 e'\|_{L^2(I_i)} \leq C_d p\|\sqrt{d_i}\omega^2 e'\|_{L^2(I_i)}, \qquad \|d_i(\omega^2 e')'\|_{L^2(I_i)} \leq 2p/h_i\|\sqrt{d_i}\omega^2 e'\|_{L^2(I_i)}$$

and furthermore make use of (B.9) to arrive at

$$\left|\int_{I_i} d_i(-\varepsilon\eta''+\eta')(\omega^2 e)'\,dx\right| \leq C\|\omega(-\varepsilon\eta'+\eta)\|_{L^2(I_i)}\frac{p}{h_i}\|\sqrt{d_i}\omega e'\|_{L^2(I_i)}.$$

Hence, we obtain the term $E_{SD,\omega}(\eta)|||e|||_{SD,\omega}$ by using the Cauchy inequality for sums.

□

Finally, we have the following lemma:

Lemma B.10 *Under the assumptions of Lemma B.8 there is $C>0$ depending only on δ_0 such that for all $\lambda\in[0,1]$*

$$\left|B_{SD}(\eta,\omega^2 e-W)\right| \leq C\lambda|||e|||_{SD,\omega}\left[E_\omega(\eta)+E_{G,\omega}(\eta)+E_{SD,\omega}(\eta)\right] \qquad \forall e\in S_0^{p,1}(\mathcal{T}), \quad \eta\in H_0^1(\Omega).$$

Here, for each $e\in S_0^{p,1}(\mathcal{T})$ the function $W\in S_0^{p,1}(\mathcal{T})$ is given by Lemma B.4.

Proof: The proof is essentially a repetition of the preceding proof and an application of Lemma B.4.

□

Lemma 2.4 made crucial use of the assumption that a "two-element" mesh is a submesh in order to resolve the boundary layer. In the present context, we cannot make this assumption. For our approximation result on compact subsets upstream of the layer we therefore need the following result:

Proposition B.11 *Let $c,\tilde{c}>0$ be fixed. Let $N\geq 3$ and let $\mathcal{T}=\{I_i\}_{i=1}^N$ be a mesh such that the length of the rightmost element I_N satisfies $h_N\geq cp\varepsilon$ and such that there is a meshpoint x_M with $-1<x_M\leq 1-\tilde{c}$. Then there are constants C, $\sigma>0$ depending only on the coefficients of* (1.1) *and $c,\tilde{c}>0$ such that the splitting $u_\varepsilon=u_{reg}+u_{BL}$ of Lemma 2.4 satisfies the following:*

1. *u_{reg} can be approximated by piecewise polynomials of degree p such that the assertions of Lemma 2.4 about η_{reg} hold true verbatim.*
2. *Denoting by $u_{BL,p}$ the piecewise linear function $u_{BL,p}$ given by the conditions $u_{BL}(\pm1)=u_{BL,p}(\pm1)$, $u_{BL,p}(x_{N-1})=0$, we have for $\eta_{BL}:=u_{BL}-u_{BL,p}$*

$$\begin{aligned}
\|\eta_{BL}^{(l)}\|_{L^2(-1,x_{N-1})} &\leq C\varepsilon^{1/2-l}e^{-\sigma p}, \qquad l=0,1,\\
\|\eta_{BL}^{(l)}\|_{L^\infty(-1,x_{N-1})} &\leq C\varepsilon^{-l}e^{-\sigma p}, \qquad l=0,1,\\
\|\eta_{BL}^{(l)}\|_{L^\infty(x_{N-1},1)} &\leq C\left\{\varepsilon^{-l}+h_N^{-l}\right\}, \qquad l=0,1,\\
\|\eta_{BL}^{(l)}\|_{L^2(x_{N-1},1)} &\leq C\left\{\varepsilon^{1/2-l}+h_N^{1/2-l}\right\}, \qquad l=0,1,\\
\|\eta_{BL}^{(l)}\|_{L^\infty(-1,x_M)} &\leq Ce^{-\sigma p}, \qquad l=0,1.
\end{aligned}$$

Proof: The assertions about η_{BL} follow by estimating u_{BL} and $u_{BL,p}$ separately. Note that both u_{BL} and $u_{BL,p}$ are exponentially small (in $1/\varepsilon$) on $(-1,x_M)\subset(-1,1-\tilde{c})$.

□

Proof of Theorem B.1: By possibly making ξ slightly larger, we may assume that ξ is a mesh point $\xi = 1 - \sigma^{p_1}$ for some fixed p_1. We may restrict ourselves to the case $p \geq p_1$.

Let us first consider the pre-asymptotic case, i.e., $p \leq p_0$. Choosing λ sufficiently small, we obtain from Propositions B.5, B.7, and Lemmata B.8– B.10 that there is $C > 0$ such that for any admissible splitting in the sense of Definition 3.3

$$|||e|||_{SD,\omega} \leq C \left[E_\omega(\eta) + E_{G,\omega}(\eta) + E_{SD,\omega}(\eta) \right]. \tag{B.12}$$

Let us now see that we may choose the admissible splitting such that

$$E_\omega(\eta) + E_{G,\omega}(\eta) + E_{SD,\omega}(\eta) \leq Ce^{-\sigma p} \tag{B.13}$$

some C, $\sigma > 0$ independent of ε, p. To that end, let the admissible splitting $u_p = u_{reg,p} + u_{BL,p}$ be given by Proposition B.11. Estimating $\omega \leq 1$, $\rho_i + \varepsilon \geq Ch_i/p$ for some $C > 0$ (which follows from (3.5)), and $\varepsilon/h_i \leq C$ (which follows from the assumption $p \leq p_0$), we get by Proposition B.11 $E_\omega(\eta_{reg}) + E_{SD,\omega}(\eta_{reg}) + E_{G,\omega}(\eta_{reg}) \leq Ce^{-\sigma p}$ for some $\sigma > 0$. Let us now obtain corresponding estimates for η_{BL}. As noted in Remark B.3 we have

$$\omega|_{I_N} \leq e^{-\lambda\tilde{c}(p-p_1)} \tag{B.14}$$

for some $\tilde{c} > 0$ ($\tilde{c}$ depends only on c of (3.5) and $\underline{\delta}$). Using this bound, estimating $\omega \leq 1$ on $\Omega \setminus I_N$, and recalling that $\rho_i \leq \delta_0 h_i$ we get $E_\omega(\eta_{BL}) + E_{SD,\omega}(\eta_{BL}) + E_{G,\omega}(\eta_{BL}) \leq Ce^{-\sigma p}$ for some suitable $\sigma > 0$ by Proposition B.11. Hence $|||e|||_{SD,\omega} \leq Ce^{-\sigma p}$. As $\omega \equiv 1$ on the $p_1 + 1$ elements that comprise $(-1, \xi)$, this implies with Lemma 3.9

$$\|e\|_{L^\infty(-1,\xi)} \leq Cp^{3/2}(p_1 + 1)|||e|||_{SD,\omega} \leq Ce^{-\sigma p}.$$

after suitably adjusting σ. Together with the triangle inequality, we finally obtain

$$\|u_\varepsilon - u_{SD}\|_{L^\infty(-1,\xi)} \leq Ce^{-\sigma p}$$

In the asymptotic case, i.e., $p \geq p_0$, we obtain directly from Corollary A.5 (noting that the number of elements can be bounded by $p + 2$)

$$\|u_\varepsilon - u_{SD}\|_{L^\infty(-1,\xi)} \leq Ce^{-\sigma p}$$

for some $\sigma > 0$. This gives the desired estimate for $k = 0$. For $k = 1$, we proceed in the standard fashion: The splitting $u_p = u_{reg,p} + u_{BL,p}$ of Proposition B.11 satisfies on $(-1, \xi)$

$$\|u_\varepsilon - u_p\|_{H^1(-1,\xi)} \leq Ce^{-\sigma p}$$

for some $C, \sigma > 0$. Hence, using the fact that the mesh restricted to $(-1, \xi)$ is fixed the inverse estimates of Lemma 3.4 yield

$$\|u_\varepsilon - u_{SD}\|_{H^1(-1,\xi)} \leq \|u_\varepsilon - u_p\|_{H^1(-1,\xi)} + Cp^2\|u_\varepsilon - u_{SD}\|_{L^\infty(-1,\xi)}$$

and the result follows.

□

Remark B.12 The proof of Theorem B.1 relies on Lemma 3.9 which exploits heavily the fact that we restrict our attention to 1-d. This was done for notational convenience only. Choosing the weight function ω so that $\omega'\omega \neq 0$ on $(-1, \xi)$ would allow us to estimate $\|u_\varepsilon - u_{SD}\|_{L^2(-1,\xi)}$ directly from $|||e|||_{SD,\omega}$ and hence there is no need to appeal to Lemma 3.9. For example, one could choose the weight function $\widetilde{\omega}$ as $\widetilde{\omega}(x) = \int_\xi^x \psi(t)\,dt$ where the function ψ is given by (B.7) for $x > \xi$ and by $-\delta_i$ for $x \in I_i$ with $x < \xi$.

Remark B.13 We note that in the proof of Theorem B.1 we only made use of the fact that $\omega \leq 1$ on the whole element and that ω is exponentially small (in p) on the last element. Exploiting the decay properties of ω on the elements between ξ and 1 would allow us to let the polynomial degree go down linearly from p to 1 on the $O(p)$ elements between ξ and I_N and the assertions (B.13) still hold true.

Theorem B.1 concentrates on a compact subset $(-1,\xi)$. However, the techniques developed for the proof of Theorem B.1 also yield estimates on variable sets $(-1,\xi_p)$ where $\xi_p \to 1$ as $p \to \infty$. Prototypical may be the following

Corollary B.14 *Let the mesh $\mathcal{T}$ satisfy the assumptions as in Theorem B.1. Fix $\alpha \in (0,1)$. For each $p \in \mathbb{N}$ choose $j \in \mathbb{N}$ as the integer part of αp and set $\xi_p := 1 - q^j$. Then for ε sufficiently small, there are C, $\sigma > 0$ independent of ε, p such that*

$$\|u_\varepsilon - u_{SD}\|_{L^\infty(-1,\xi_p)} \leq C \exp(-\sigma p)$$

Proof: For $p \geq p_0$ the statement follows from the proof of Theorem B.1. Let us consider the pre-asymptotic range $p \leq p_0$. We choose the weight function ω as in (B.8) with ξ replaced with ξ_p and check that the estimates of (B.14) hold true with the constants C, $\tilde{c}$ depending additionally on α. Hence we obtain with the same arguments as in the proof (B.13) that $E_\omega(\eta)$, $E_{G,\omega}(\eta)$, $E_{SD,\omega}(\eta)$ can now be bounded by $C\exp(-\sigma p)$. We may then conclude the proof by repeating the arguments of the proof of Theorem B.1.

$\square$