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Abstract

We consider the electric field integral equation on the surface of polyhedral domains and its Galerkin-discretization by means of divergence-conforming boundary elements. With respect to a Hodge-decomposition the continuous variational problem is shown to be coercive. However, this does not immediately carry over to the discrete setting, as discrete Hodge decompositions fail to possess essential regularity properties. Introducing an intermediate semidiscrete Hodge decomposition we can bridge the gap and come up with asymptotically optimal a-priori error estimates. Hitherto, those had been elusive, in particular for non-smooth boundaries.

Keywords: Electric field integral equation, Rumsey's principle, Raviart-Thomas elements, Hodge decomposition, discrete coercivity

Subject Classification: 65N12, 65N38, 78M15

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1. Introduction. One of the main tasks in computational electromagnetism is the computation of the scattering of electromagnetic waves at a perfectly conducting body $\Omega \subset \mathbb{R}^3$. It boils down to solving the time-harmonic Maxwell's equations in the exterior $\Omega' = \mathbb{R}^3 \setminus \overline{\Omega}$ of Ω for a fixed frequency, subject to vanishing tangential trace of the total electric field on the surface of the scatterer and the Silver-Müller radiation conditions at ∞ . It is known that the exterior scattering problem for Maxwell's equations has a unique solution (see, e.g., [26, Ch. 6] and [22]). In most technical applications the boundary Γ of Ω will only be piecewise smooth.

Starting from the Stratton-Chu representation formulas in Ω' (see, e.g., [19, Sect. 3]), an indirect method yields the well-known electrical field boundary integral equation (EFIE) for the unknown jump \mathbf{j} of the magnetic field [19, Sect. 4]. Cast in variational form, this integral equation is sometimes referred to as Rumsey's principle [9] and reads: find a complex amplitude $\mathbf{j} \in \mathbf{H}^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma)$ such that

$$\langle V_\varsigma \text{div}_\Gamma \mathbf{j}, \text{div}_\Gamma \mathbf{v} \rangle_{\frac{1}{2}, \Gamma} - \varsigma^2 \langle \mathbf{A}_\varsigma \mathbf{j}, \mathbf{v} \rangle_{\parallel, \Gamma} = f(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^{-\frac{1}{2}}(\text{div}_\Gamma, \Gamma). \quad (1.1)$$

Here, $\varsigma \in \mathbb{R}_+$ is the nondimensional wave number, the continuous linear functional $f : \mathbf{H}^{-\frac{1}{2}}(\text{curl}_\Gamma, \Gamma) \mapsto \mathbb{C}$ represents the excitation due to an incident wave, and $V_\varsigma, \mathbf{A}_\varsigma$ stand for scalar and vectorial single layer potential integral operators, respectively. In (1.1), the (sesqui-linear) duality pairings $\langle \cdot, \cdot \rangle_{\frac{1}{2}, \Gamma}$ and $\langle \cdot, \cdot \rangle_{\parallel, \Gamma}$ coincide with the usual $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ duality when Γ is smooth. On polyhedra, however, there are several, nonequivalent definitions of these dualities. Details will be explained below. For the well-posedness of the integral equation formulation (1.1), we adopt the following assumption throughout this paper.

ASSUMPTION 1.1. *The wavenumber ς is bounded away from the spectrum of the interior Maxwell problem.*

This implies the injectivity of the boundary integral operator in (1.1).

Recalling the derivation of (1.1), the unknown \mathbf{j} emerges as the jump of tangential traces $\mathbf{H} \times \mathbf{n}$ of magnetic field solutions for the full Maxwell equations in the interior and exterior of Ω . When stating Maxwell's equations concisely in the language of differential forms [6, 14] the magnetic field is modeled by a twisted 1-form. The same will hold for its trace on Γ . This suggests that two-dimensional discrete twisted 1-forms built upon a triangulation of Γ should be used to approximate \mathbf{j} . Those are provided by the boundary element counterparts of the 2D Raviart-Thomas $\mathbf{H}(\text{div}; \Omega)$ -conforming finite elements. We could also reason in an entirely discrete setting: It is no longer a moot point that a suitable discretization of magnetic fields is provided by $\mathbf{H}(\text{curl}; \Omega)$ -conforming edge elements [45], which are discrete 1-forms in 3D. Taking a look at their tangential trace, again, we discover Raviart-Thomas elements mapped onto the surface [36]. Thus, we argue that the latter offer a “natural” boundary element discretization of (1.1): find $\mathbf{j}_h \in \mathcal{RT}_0(\Gamma_h)$ such that

$$\langle V_\varsigma \text{div}_\Gamma \mathbf{j}_h, \text{div}_\Gamma \mathbf{v}_h \rangle_{\frac{1}{2}, \Gamma} - \varsigma^2 \langle \mathbf{A}_\varsigma \mathbf{j}_h, \mathbf{v}_h \rangle_{\parallel, \Gamma} = f(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{RT}_0(\Gamma_h) \quad (1.2)$$

where $\mathcal{RT}_0(\Gamma_h)$ denotes the lowest order Raviart-Thomas boundary element space on a triangulation Γ_h on Γ .

The Galerkin discretization (1.2) is commonplace in engineering codes. The first convergence analysis of this scheme was given by Bendali in [7, 8] based on a saddle point formulation and elliptic regularization, which is inherently confined to smooth surfaces. Using parametric variants of the Raviart-Thomas boundary elements, he could establish asymptotic a-priori convergence estimates. Attempts to adapt his approach to non-smooth surfaces have not been successful. Recently, Buffa, Costabel and Schwab succeeded in showing convergence of a mixed discretization of (1.1) which, however, is different from the “natural” scheme (1.2) used in engineering practice.

Obstructions to a convergence analysis on non-smooth surfaces are threefold: First, the correct function spaces and relevant surface differential operators have to be

properly characterized. For smooth domains, using smooth charts and trace theorems for the entire scale of Sobolev spaces, this is not hard to do [2, 22]. It becomes a challenge in a non-smooth setting, as is vividly conveyed in the introduction of [20]. The situation on polyhedral boundaries Γ was clarified only recently by A. Buffa and P. Ciarlet, Jr., in [16–18] and general Lipschitz-boundaries were admitted in [20]. We emphasize that only these results made possible the progress reported in the current paper.

Secondly, with (1.1) we recognize the typical difficulty faced when dealing with variational problems arising from Maxwell’s equations: Owing to the large kernel of the surface divergence operator $\operatorname{div}_\Gamma$, it becomes impossible to assign one term the role of a principal part and, thus, the sesqui-linear form of (1.1) fails to be coercive. A remedy was first found in the case of the Maxwell differential equations [40, 43] and it is marked by the use of *Hodge decompositions*. Also for boundary integral equations the idea is fruitful and was exploited many times in order to recover coercive problems [3, 19, 31].

Unfortunately, Hodge decompositions and the divergence conforming boundary elements do not match easily. This is the third obstacle and it is also faced in the analysis of $\mathbf{H}(\operatorname{curl}; \Omega)$ -conforming finite element schemes. In that context a solution has been devised, relying on judiciously combining discrete and continuous Hodge decompositions. The idea was successful in the analysis of multigrid methods for edge elements [5, 37, 39] and in the numerical analysis of schemes for Maxwell eigenproblems that are free of spurious modes [10–12, 21, 44].

It is this idea that permits us to launch a successful attack on Galerkin discretizations of the EFIE (1.1) on polyhedral domains. Yet, numerous adjustments of the technique are necessary to cope with the low regularity of the function spaces of interest on Γ . Whereas for problems on domains $\subset \mathbb{R}^3$ all fields are at least square integrable, here we find that surface vectorfields in $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ do not necessarily have this property. In this paper we aim to elucidate how to handle this difficulty. S. Christiansen in [23] pursues a policy partly similar to ours, but with a different objective and confined to smooth domains.

The paper is organized as follows: In the next section we summarize important results about spaces of tangential vectorfields on polyhedra. The third and fourth section establish the coercivity of the continuous variational problem with respect to a Hodge decomposition of $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$. Then we introduce divergence conforming boundary elements and review their main properties. In the sixth section we define and scrutinize mappings that create a link between discrete and continuous Hodge decompositions. The seventh section is dedicated to proving a discrete inf-sup condition. The final section covers asymptotic a-priori estimates of the discretization error.

It was our objective to keep the treatment as focused and self-contained as possible. To that end we forgo any generalizations and investigate only the lowest order RT boundary elements and Lipschitz polyhedra with plane faces. By and large, generalizations are straightforward. Numerical experiments are skipped, since the Galerkin discretization of the EFIE by RT-boundary elements is widely and successfully used in electrical engineering practice.

2. Spaces. The domain $\Omega \subset \mathbb{R}^3$ is supposed to be a Lipschitz-polyhedron (cf. the introduction of [29]). In particular, we assume that the Lipschitz boundary Γ can be written as union of a finite number of plane faces Γ_j , $j = 1, \dots, N_\Gamma$, i.e. $\bar{\Gamma} = \bigcup_j \bar{\Gamma}_j$. For each face Γ_j we find a constant unit normal vector \mathbf{n}_i pointing into the exterior of Ω . These vectors can be blended into an exterior unit normal vector field $\mathbf{n} \in L^\infty(\Gamma)$, defined almost everywhere on Γ . In addition, we can fix two orthogonal unit vectors $\mathbf{e}_j^1, \mathbf{e}_j^2$ that span the tangential plane for Γ_i . It goes without saying that each Γ_j can be identified with a bounded subset of \mathbb{R}^2 .

Next, we introduce two different tangential surface trace operators [20, Sect. 2]: The tangential components trace $\pi_{\mathbf{t}}$ is defined for $\mathbf{u} \in \mathbf{C}^\infty(\bar{\Omega})$ by $\pi_{\mathbf{t}} \mathbf{u}(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \times$

$(\mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}))$ for almost all $\mathbf{x} \in \Gamma$. Accordingly, the tangential surface trace $\gamma_{\mathbf{t}}$ can be computed through $\gamma_{\mathbf{t}} \mathbf{u}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})$. The same traces from Ω' are $\pi_{\mathbf{t}}'$ and $\gamma_{\mathbf{t}}'$. To begin with, the trace operators supply functions in

$$\mathbf{L}_{\mathbf{t}}^2(\Gamma) := \{\mathbf{u} \in (L^2(\Gamma))^3, \mathbf{u} \cdot \mathbf{n} = 0\}.$$

The usual Sobolev spaces of scalar functions and related functionals, $H^s(\Gamma)$ and $H^{-s}(\Gamma)$, can be defined invariantly for $0 \leq s \leq 1$ [35, Sect. 1.3.3]. For indices $s > 1$ we resort to the piecewise definition

$$H^s(\Gamma) := \{u \in H^1(\Gamma), u|_{\Gamma_j} \in H^s(\Gamma_j), j = 1, \dots, N_{\Gamma}\}.$$

We equip this space with the graph norm

$$\|u\|_{H^s(\Gamma)}^2 := \|u\|_{H^1(\Gamma)}^2 + \sum_{j=1}^{N_{\Gamma}} \|u\|_{H^s(\Gamma_j)}^2.$$

Using the local coordinate systems introduced above, spaces of tangential vectorfields that feature certain Sobolev regularity in a piecewise sense, are readily available

$$\mathbf{H}_{\mathbf{t}}^s(\Gamma) := \{\mathbf{u} \in \mathbf{L}_{\mathbf{t}}^2(\Gamma), \mathbf{u}|_{\Gamma_j} \cdot \mathbf{e}_j^i \in H^s(\Gamma_j), j = 1, \dots, N_{\Gamma}, i = 1, 2\}.$$

By localization to the Γ_j we can define the tangential surface gradient \mathbf{grad}_{Γ} [20, Def. 3.1]. Its continuity as a mapping $H^{s+1}(\Gamma) \mapsto \mathbf{H}_{\mathbf{t}}^s(\Gamma)$, $s \geq 0$, is straightforward. The surface divergence is obtained as formal $\mathbf{L}_{\mathbf{t}}^2(\Gamma)$ -adjoint $\text{div}_{\Gamma} : \mathbf{L}_{\mathbf{t}}^2(\Gamma) \mapsto H_*^{-1}(\Gamma)$. Its range space is

$$H_*^{-s}(\Gamma) := \{\phi \in H^{-s}(\Gamma), \langle \chi, \phi \rangle_{s,\Gamma} = 0, \forall \chi \in Z\} \quad (2.1)$$

where Z is the space of piecewise constants on connected components of Γ and $\langle \cdot, \cdot \rangle_{s,\Gamma}$ denotes the $H^s(\Gamma) \times H^{-s}(\Gamma)$ duality pairing.

The two operators can be used to define the surface Laplace-Beltrami operator $\Delta_{\Gamma} : H^1(\Gamma) \mapsto H_*^{-1}(\Gamma)$ by $\Delta_{\Gamma} := \text{div}_{\Gamma} \mathbf{grad}_{\Gamma}$. It will be a key tool as theorem 5.3 of [19] reveals the following lifting property.

THEOREM 2.1. *If $f \in H_*^s(\Gamma)$ for $s \geq -1$, the (unique) solution $u \in H^1(\Gamma)/Z$ of $-\Delta_{\Gamma} u = f$ belongs to $H^{1+r}(\Gamma)$ for $0 \leq r \leq \min\{s+1, s^*\}$, where $s^* > 0$ depends on the geometry of Γ in neighborhoods of vertices only.*

In other words, with $\tilde{C} = \tilde{C}(t, \Gamma)$ and $0 \leq r \leq \min\{s+1, s^*\}$

$$f \in H_*^s(\Gamma) \quad , \quad -\Delta_{\Gamma} u = f \quad \Rightarrow \quad \|u\|_{H^{r+1}(\Gamma)} \leq \tilde{C} \|f\|_{H^s(\Gamma)}. \quad (2.2)$$

We adopt the convention that C and c stand for generic positive constants, whose values might be different between different occurrences, but must not depend on any concrete function. When tagged with a tilde on top, they may only depend on ς , continuous function spaces, and the geometry of Γ .

Note that there exist polyhedral vertices for which $s^* > 0$ is arbitrarily small (see [19]). Nevertheless, reasonable geometries will allow for s^* well bounded above zero. For instance, if only three edges meet at a vertex O , we find $s^* = 2\pi/(\varphi_1 + \varphi_2 + \varphi_3) - \varepsilon$, for any $\varepsilon > 0$, where $\varphi_1, \varphi_2, \varphi_3$ are the opening angles at vertex O of the three plane faces Γ_j meeting at O .

Owing to Thm. 2.1 the space

$$H^{-\frac{1}{2}}(\Delta_{\Gamma}, \Gamma) := \{u \in H^1(\Gamma), \Delta_{\Gamma} u \in H^{-\frac{1}{2}}(\Gamma)\}$$

will actually be embedded in $H^{1+r}(\Gamma)$ for all $0 \leq r \leq \min\{\frac{3}{2}, s^*\}$. Based on div_{Γ} , we get the Hilbert spaces ($s \geq 0$)

$$\mathbf{H}^s(\text{div}_{\Gamma}; \Gamma) := \{\mathbf{u} \in \mathbf{H}_{\mathbf{t}}^s(\Gamma), \text{div}_{\Gamma} \mathbf{u} \in H^s(\Gamma)\}.$$

Tangential traces of vectorfields in $\mathbf{H}_{\text{loc}}^1(\Omega)$ form the spaces $\mathbf{H}_{\parallel}^{\frac{1}{2}}(\Gamma)$ and $\mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma)$ which were characterized in [17, Prop. 1.6]. Loosely speaking, $\mathbf{H}_{\parallel}^{\frac{1}{2}}(\Gamma)$ contains the tangential surface vectorfields that are in $\mathbf{H}^{\frac{1}{2}}(\Gamma_i)$ for each smooth component Γ_i of Γ and feature a suitable “weak tangential continuity” across the edges of the Γ_i . A corresponding “weak normal continuity” is satisfied by surface vectorfields in $\mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma)$. For smooth Γ these spaces coincide with the spaces of tangential surface vectorfields in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. The associated dual spaces will be denoted by $\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma)$ and $\mathbf{H}_{\perp}^{-\frac{1}{2}}(\Gamma)$, respectively, where the duality pairings are taken with $\mathbf{L}_{\mathbf{t}}^2(\Gamma)$ as pivot space. Further, we denote by $\langle \cdot, \cdot \rangle_{\parallel, \Gamma}$ and $\langle \cdot, \cdot \rangle_{\perp, \Gamma}$ the respective duality pairings. A fundamental result of [17] asserts that the tangential trace mapping $\pi_{\mathbf{t}} : \mathbf{H}_{\text{loc}}^1(\Omega) \mapsto \mathbf{H}_{\parallel}^{\frac{1}{2}}(\Gamma)$ is continuous, surjective and possesses a continuous right inverse (see proposition 1.7 in [17]).

One of the crucial insights gained in [17] and [20] was that the tangential surface gradient $\mathbf{grad}_{\Gamma} : H^1(\Gamma) \mapsto \mathbf{L}_{\mathbf{t}}^2(\Gamma)$ can be both extended and restricted to continuous and *injective* linear operators

$$\mathbf{grad}_{\Gamma} : \tilde{H}^{\frac{3}{2}}(\Gamma)/Z \mapsto \mathbf{H}_{\parallel}^{\frac{1}{2}}(\Gamma) \quad , \quad \mathbf{grad}_{\Gamma} : H^{\frac{1}{2}}(\Gamma)/Z \mapsto \mathbf{H}_{\perp}^{-\frac{1}{2}}(\Gamma)$$

(cf. propositions 3.4 and 3.6 in [20]), where $\tilde{H}^{\frac{3}{2}}(\Gamma)$ is the space of traces of functions in $H^2(\Omega)$. Consequently, div_{Γ} can also be read as continuous and *surjective* operator

$$\text{div}_{\Gamma} : \mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma) \mapsto \tilde{H}_{*}^{-\frac{3}{2}}(\Gamma) \quad , \quad \text{div}_{\Gamma} : \mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma) \mapsto H_{*}^{-\frac{1}{2}}(\Gamma) .$$

This is important for the definition of the space $\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$ introduced in [17] as

$$\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma) = \{ \mathbf{v} \in \mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma), \text{div}_{\Gamma} \mathbf{v} \in H^{-\frac{1}{2}}(\Gamma) \} .$$

It is endowed with the natural graph norm $\|\cdot\|_{\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)}$.

The key role of *Hodge decompositions* was emphasized in the introduction. The following theorem reveals the nature of the Hodge decomposition that we will need. More details are given in [20, Sect. 5], [18], and [16].

THEOREM 2.2. *The space $\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$ has the direct and stable decomposition*

$$\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma) := \mathbf{grad}_{\Gamma} H^{-\frac{1}{2}}(\Delta_{\Gamma}, \Gamma) \oplus (\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma) \cap \text{Ker}(\text{div}_{\Gamma})) .$$

Moreover, when restricted to $\mathbf{L}_{\mathbf{t}}^2(\Gamma) \cap \mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$ the decomposition is $\mathbf{L}_{\mathbf{t}}^2(\Gamma)$ -orthogonal.

Proof. Any function in $\mathbf{grad}_{\Gamma} H^{-\frac{1}{2}}(\Delta_{\Gamma}, \Gamma) \cap \text{Ker}(\text{div}_{\Gamma})$ must be the gradient of a function in the kernel of Δ_{Γ} on Γ . The latter only contains piecewise constants with respect to the connected components of Γ and, therefore, the decomposition is direct.

Next, pick some $\mathbf{v} \in \mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$. Since $\text{div}_{\Gamma} : \mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma) \mapsto H_{*}^{-\frac{1}{2}}(\Gamma)$ is surjective, we can find $\boldsymbol{\psi} \in \mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma)$ such that $\text{div}_{\Gamma} \boldsymbol{\psi} = \text{div}_{\Gamma} \mathbf{v} \in H_{*}^{-\frac{1}{2}}(\Gamma)$. Define φ by

$$\varphi \in H^1(\Gamma)/Z : \quad (\mathbf{grad}_{\Gamma} \varphi, \mathbf{grad}_{\Gamma} \eta)_{0, \Gamma} = (\boldsymbol{\psi}, \mathbf{grad}_{\Gamma} \eta)_{0, \Gamma} \quad \forall \eta \in H^1(\Gamma)/Z ,$$

that is, as the unique weak solution of $\Delta_{\Gamma} \varphi = \text{div}_{\Gamma} \boldsymbol{\psi}$. This yields the decomposition

$$\mathbf{v} = \mathbf{grad}_{\Gamma} \varphi + (\boldsymbol{\psi} - \mathbf{grad}_{\Gamma} \varphi + \mathbf{v} - \boldsymbol{\psi}) ,$$

whose second part is readily seen to be divergence-free. Since div_{Γ} is surjective, the open mapping theorem ensures that $\boldsymbol{\psi}$ can be chosen such that

$$\|\boldsymbol{\psi}\|_{\mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma)} \leq \tilde{C} \|\text{div}_{\Gamma} \boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\Gamma)} .$$

This implies

$$\|\mathbf{grad}_\Gamma \varphi\|_{L^2(\Gamma)} \leq \|\psi\|_{L^2(\Gamma)} \leq \|\psi\|_{\mathbf{H}_\perp^{\frac{1}{2}}(\Gamma)} \leq \tilde{C} \|\operatorname{div}_\Gamma \psi\|_{H^{-\frac{1}{2}}(\Gamma)} ,$$

which confirms the stability of the decomposition. For $\mathbf{v} \in \mathbf{L}_\mathbf{t}^2(\Gamma) \cap \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ the $\mathbf{L}_\mathbf{t}^2(\Gamma)$ -orthogonality is immediate from the definition of $\operatorname{div}_\Gamma$. \square

In the sequel we write

$$\mathbf{X} := \mathbf{grad}_\Gamma H^{-\frac{1}{2}}(\Delta_\Gamma, \Gamma) \quad \text{and} \quad \mathbf{N} := \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) \cap \operatorname{Ker}(\operatorname{div}_\Gamma).$$

From the stability of the Hodge decomposition we conclude that both \mathbf{X} and \mathbf{N} are closed subspaces of $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$.

LEMMA 2.3. *If $\mathbf{v} \in \mathbf{X}$ satisfies $\operatorname{div}_\Gamma \mathbf{v} \in H^s(\Gamma)$ for some $s \geq -\frac{1}{2}$ then for all $0 \leq r \leq \min\{s+1, s^*\}$*

$$\mathbf{v} \in \mathbf{H}_\mathbf{t}^r(\Gamma) \quad \text{and} \quad \|\mathbf{v}\|_{\mathbf{H}^r(\Gamma)} \leq \tilde{C} \|\operatorname{div}_\Gamma \mathbf{v}\|_{H^s(\Gamma)} ,$$

with a constant $\tilde{C} = \tilde{C}(r, s)$ and $s^* > 0$ as in Theorem 2.1.

Proof. $\mathbf{v} \in \mathbf{X}$ means $\mathbf{v} = \mathbf{grad}_\Gamma \varphi$ for some $\varphi \in H^1(\Gamma)$. By definition of \mathbf{X} we see $\Delta_\Gamma \varphi = \operatorname{div}_\Gamma \mathbf{v}$, and the assertion follows from theorem 2.1. \square

In particular, we conclude

$$\|\mathbf{v}\|_{\mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma)} \leq \|\mathbf{v}\|_{L^2(\Gamma)} \leq \tilde{C} \|\operatorname{div}_\Gamma \mathbf{v}\|_{H^{-\frac{1}{2}}(\Gamma)} , \quad \forall \mathbf{v} \in \mathbf{X} . \quad (2.3)$$

3. Continuous variational problem. We recall the scalar single layer potential $\Psi_\zeta^V : H^{-\frac{1}{2}}(\Gamma) \mapsto H_{\text{loc}}^1(\mathbb{R}^3)$ for the Helmholtz operator $\Delta + \zeta^2$. Its relative, the vectorial Helmholtz single layer potential $\Psi_\zeta^A(\mathbf{v})$ for $\mathbf{v} \in \mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma)$ is given by

$$\Psi_\zeta^A(\mathbf{v})(\mathbf{x}) := \int_\Gamma \Phi_\zeta(\mathbf{x}, \mathbf{y}) \mathbf{v}(\mathbf{y}) dS(\mathbf{y}) \quad , \quad \Phi_\zeta(\mathbf{x}, \mathbf{y}) := \frac{\exp(i\zeta|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} .$$

For every $\mathbf{v} \in \mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma)$ it defines a function in $\mathbf{H}_{\text{loc}}^1(\mathbb{R}^3)$ and, as a consequence of the trace theorem for $\pi_\mathbf{t}$, we can introduce the vectorial single layer boundary operator

$$\mathbf{A}_\zeta : \mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma) \mapsto \mathbf{H}_\parallel^{\frac{1}{2}}(\Gamma) \quad , \quad \mathbf{A}_\zeta := \pi_\mathbf{t} \circ \Psi_\zeta^A ,$$

in analogy to the scalar single layer integral operator

$$V_\zeta : H^{-1/2}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma) \quad , \quad V_\zeta := \gamma \circ \Psi_\zeta^V ,$$

where $\gamma : H_{\text{loc}}^1(\mathbb{R}^3) \mapsto H^{\frac{1}{2}}(\Gamma)$ is the standard trace operator. In the static case, i.e. at wavenumber $\zeta = 0$, these operators are coercive.

LEMMA 3.1. *The operators V_0 and \mathbf{A}_0 are continuous, selfadjoint and elliptic, i.e. there are constants $\tilde{c}_1, \tilde{c}_2 > 0$ only depending on Γ such that for all $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and all $\boldsymbol{\mu} \in \mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma)$, $\operatorname{div}_\Gamma \boldsymbol{\mu} = 0$,*

$$\langle V_0 \mu, \mu \rangle_{\frac{1}{2}, \Gamma} \geq \tilde{c}_1 \|\mu\|_{H^{-\frac{1}{2}}(\Gamma)}^2 \quad , \quad \langle \mathbf{A}_0 \boldsymbol{\mu}, \boldsymbol{\mu} \rangle_{\parallel, \Gamma} \geq \tilde{c}_2 \|\boldsymbol{\mu}\|_{\mathbf{H}_\parallel^{-\frac{1}{2}}(\Gamma)}^2 .$$

Proof. See Cor. 8.13 in [41] or Thm. 3 in [30, Vol. IV, Ch. XI, § 2], and Thm. 6.2 in [38] or Prop. 4.1 in [19]. \square

Along with the following result, this yields the coercivity of V_ζ and \mathbf{A}_ζ (compare the proof of Thm. 4.4 in [19]).

LEMMA 3.2. *The following operators are compact:*

$$\delta V_\zeta := V_\zeta - V_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma), \quad \delta \mathbf{A}_\zeta := \mathbf{A}_\zeta - \mathbf{A}_0 : \mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma) \mapsto \mathbf{H}_{\parallel}^{\frac{1}{2}}(\Gamma).$$

Proof. We write G_ζ for the Green's operator in \mathbb{R}^3 for the Helmholtz equation, defined by $(G_\zeta \varphi)(\mathbf{x}) = \int_{\mathbf{y} \in \mathbb{R}^3} \Phi_\zeta(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d\mathbf{y}$ for $\varphi \in C_0^\infty(\mathbb{R}^3)$. With γ^* denoting a right inverse of the trace operator and appealing to the continuity of the trace map $\gamma : H_{loc}^s(\mathbb{R}^3) \rightarrow H^{s-1/2}(\Gamma)$, $s \in (1/2, 3/2)$ (Lemma 3.6 in [27]), we find $\delta V_\zeta = \gamma \circ (G_\zeta - G_0) \circ \gamma^* : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ compactly, since the kernel $\Phi_\zeta(\mathbf{x}, \mathbf{y}) - \Phi_0(\mathbf{x}, \mathbf{y})$ of the operator $G_\zeta - G_0$ has essentially bounded derivatives in \mathbf{x} and \mathbf{y} . The vectorial case follows in the same way. \square

The main tool in the analysis of the variational problem (1.1) are Hodge decompositions according to theorem 2.2 (cf. [19, Sect. 4.3]). Based on Theorem 2.2 we Hodge-decompose $\mathbf{j} := \mathbf{j}^\perp + \mathbf{j}^0$, $\mathbf{j}^\perp \in \mathbf{X}$, $\mathbf{j}^0 \in \mathbf{N}$, and $\mathbf{v} := \mathbf{v}^\perp + \mathbf{v}^0$, $\mathbf{v}^\perp \in \mathbf{X}$, $\mathbf{v}^0 \in \mathbf{N}$ in (1.1). In this way, we end up with the equivalent variational problem: Find $\mathbf{j}^\perp \in \mathbf{X}$, $\mathbf{j}^0 \in \mathbf{N}$ such that for all $\mathbf{v}^\perp \in \mathbf{X}$, $\mathbf{v}^0 \in \mathbf{N}$

$$\begin{aligned} \langle V_\zeta \operatorname{div}_\Gamma \mathbf{j}^\perp, \operatorname{div}_\Gamma \mathbf{v}^\perp \rangle_{\frac{1}{2}, \Gamma} - \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^\perp, \mathbf{v}^\perp \rangle_{\parallel, \Gamma} - \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^0, \mathbf{v}^\perp \rangle_{\parallel, \Gamma} &= f(\mathbf{v}^\perp), \\ \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^\perp, \mathbf{v}^0 \rangle_{\parallel, \Gamma} + \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^0, \mathbf{v}^0 \rangle_{\parallel, \Gamma} &= -f(\mathbf{v}^0). \end{aligned} \quad (3.1)$$

Remember that $\langle \cdot, \cdot \rangle_{\parallel, \Gamma}$ stands for the $\mathbf{H}_{\parallel}^{\frac{1}{2}}(\Gamma) \times \mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma)$ duality pairing.

The natural setting for formulation (3.1) is the Hilbert space $\mathcal{G} := \mathbf{X} \oplus \mathbf{N}$ endowed with the graph norm

$$\|(\mathbf{v}^\perp, \mathbf{v}^0)\|_{\mathcal{G}}^2 := \|\mathbf{v}^\perp\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)}^2 + \|\mathbf{v}^0\|_{\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma)}^2, \quad (\mathbf{v}^\perp, \mathbf{v}^0) \in \mathcal{G}.$$

Thanks to Thm. 2.2 the space \mathcal{G} is isomorphic to $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ algebraically and topologically. The sesqui-linear form $a : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$ that belongs to (3.1) reads

$$\begin{aligned} a((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) &:= \langle V_\zeta \operatorname{div}_\Gamma \mathbf{j}^\perp, \operatorname{div}_\Gamma \mathbf{v}^\perp \rangle_{\frac{1}{2}, \Gamma} - \\ &\quad - \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^\perp, \mathbf{v}^\perp \rangle_{\parallel, \Gamma} - \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^0, \mathbf{v}^\perp \rangle_{\parallel, \Gamma} + \\ &\quad + \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^\perp, \mathbf{v}^0 \rangle_{\parallel, \Gamma} + \zeta^2 \langle \mathbf{A}_\zeta \mathbf{j}^0, \mathbf{v}^0 \rangle_{\parallel, \Gamma}, \end{aligned} \quad (3.2)$$

and is continuous, i.e.

$$|a(\varphi, \eta)| \leq \tilde{C}_a \|\varphi\|_{\mathcal{G}} \|\eta\|_{\mathcal{G}} \quad \forall \varphi, \eta \in \mathcal{G}. \quad (3.3)$$

Using the form $a(\cdot, \cdot)$, we can express the variational problem (3.1) succinctly as: Find $\iota \in \mathcal{G}$ such that

$$a(\iota, \eta) = f(\eta) \quad \forall \eta \in \mathcal{G}, \quad (3.4)$$

where $f(\eta) := f(\mathbf{v}^\perp) - f(\mathbf{v}^0)$, $\eta := (\mathbf{v}^\perp, \mathbf{v}^0)$. We point out that (3.4) is entirely equivalent to (1.1) in the sense that, if $\mathbf{j} \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ is a solution of (1.1), then $\iota := (\mathbf{j}^\perp, \mathbf{j}^0) \in \mathcal{G}$ will solve (3.4). In particular, assertions on existence and uniqueness of solutions of (1.1) instantly carry over to (3.4) and vice versa.

4. Strong ellipticity. To establish strong ellipticity of the form $a(\cdot, \cdot)$ in (3.4), we proceed as in [19] and write $a = a_0 - k_0$, where $k_0 : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$ reads

$$\begin{aligned} k_0((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) &:= - \langle \delta V_\zeta \operatorname{div}_\Gamma \mathbf{j}^\perp, \operatorname{div}_\Gamma \mathbf{v}^\perp \rangle_{\frac{1}{2}, \Gamma} + \zeta^2 \langle \delta \mathbf{A}_\zeta \mathbf{j}^\perp, \mathbf{v}^\perp \rangle_{\parallel, \Gamma} + \\ &\quad + \zeta^2 \langle \delta \mathbf{A}_\zeta \mathbf{j}^0, \mathbf{v}^\perp \rangle_{\parallel, \Gamma} - \zeta^2 \langle \delta \mathbf{A}_\zeta \mathbf{j}^\perp, \mathbf{v}^0 \rangle_{\parallel, \Gamma} - \zeta^2 \langle \delta \mathbf{A}_\zeta \mathbf{j}^0, \mathbf{v}^0 \rangle_{\parallel, \Gamma}, \end{aligned}$$

and where $a_0 : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$ emerges from a by replacing $V_\zeta \rightarrow V_0$ and $\mathbf{A}_\zeta \rightarrow \mathbf{A}_0$. The next lemma is crucial for establishing the strong ellipticity of the variational problem (3.4).

LEMMA 4.1. *The operator $L : \mathbf{X} \mapsto \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)'$, defined by $L\mathbf{u}^\perp(\mathbf{z}) := \langle \mathbf{A}_0 \mathbf{u}^\perp, \mathbf{z} \rangle_{\|\cdot\|_\Gamma}$, for all $\mathbf{u}^\perp \in \mathbf{X}$, $\mathbf{z} \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$, is compact.*

Proof. Consider a bounded sequence $(\mathbf{u}_n^\perp)_{n \in \mathbb{N}}$ in \mathbf{X} . By Lemma 2.3 it is also bounded in $\mathbf{H}_t^1(\Gamma)$ for some $0 < t \leq \min\{s+1, s^*\}$. By Rellich's theorem we can find a subsequence, also designated by $(\mathbf{u}_n^\perp)_n$ that converges in $\mathbf{L}_t^2(\Gamma)$. Observe that due to the continuity of the vectorial single layer boundary integral operator

$$\begin{aligned} \|L\mathbf{z}^\perp\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)'} &= \sup_{\mathbf{v} \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)} \frac{|(L\mathbf{z}^\perp)(\mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)}} \leq \sup_{\mathbf{v} \in \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)} \frac{|\langle \mathbf{A}_0 \mathbf{z}^\perp, \mathbf{v} \rangle_{\|\cdot\|_\Gamma}|}{\|\mathbf{v}\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)}} \leq \\ &\leq \|\mathbf{A}_0 \mathbf{z}^\perp\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \leq \tilde{C} \|\mathbf{z}^\perp\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \leq \tilde{C} \|\mathbf{z}^\perp\|_{L^2(\Gamma)}. \end{aligned}$$

Thus $(L\mathbf{u}_n^\perp)_n$ will converge in \mathbf{X}' . \square

At once, from $\mathbf{X} \subset \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ and $\mathbf{N} \subset \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$, we deduce that $L : \mathbf{X} \mapsto \mathbf{X}'$ and $L : \mathbf{X} \mapsto \mathbf{N}'$ are compact, as well.

To establish the strong ellipticity of the form $a(\cdot, \cdot)$, we accordingly split $a_0(\cdot, \cdot)$ as $a_0 = d - k_1$ where the sesqui-linear form $k_1 : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$ is defined by

$$k_1((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) := \varsigma^2 \langle \mathbf{A}_0 \mathbf{j}^\perp, \mathbf{v}^\perp \rangle_{\|\cdot\|_\Gamma} + \varsigma^2 \langle \mathbf{A}_0 \mathbf{j}^0, \mathbf{v}^\perp \rangle_{\|\cdot\|_\Gamma} - \varsigma^2 \langle \mathbf{A}_0 \mathbf{j}^\perp, \mathbf{v}^0 \rangle_{\|\cdot\|_\Gamma},$$

and the definite part $d : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$ reads

$$d((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) := \langle V_0 \operatorname{div}_\Gamma \mathbf{j}^\perp, \operatorname{div}_\Gamma \mathbf{v}^\perp \rangle_{\frac{1}{2}, \Gamma} + \varsigma^2 \langle \mathbf{A}_0 \mathbf{j}^0, \mathbf{v}^0 \rangle_{\|\cdot\|_\Gamma}.$$

THEOREM 4.2. *The sesqui-linear form $a : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$ is coercive, that is, it can be written as the difference of a \mathcal{G} -elliptic sesqui-linear form d and a compact sesqui-linear form $k : \mathcal{G} \times \mathcal{G} \mapsto \mathbb{C}$.*

Proof. Recall $a = a_0 - k_0$. Lemma 3.2 reveals that k_0 is a compact perturbation of a_0 . Further, $a_0 = d - k_1$ and Lemma 4.1 implies that k_1 is a compact perturbation of d . From the ellipticity of the single layer boundary integral operators in lemma 3.1 we immediately get

$$\begin{aligned} |d((\mathbf{v}^\perp, \mathbf{v}^0), (\mathbf{v}^\perp, \mathbf{v}^0))| &= |\langle V_0 \operatorname{div}_\Gamma \mathbf{v}^\perp, \operatorname{div}_\Gamma \mathbf{v}^\perp \rangle_{\frac{1}{2}, \Gamma} + \varsigma^2 \langle \mathbf{A}_0 \mathbf{v}^0, \mathbf{v}^0 \rangle_{\|\cdot\|_\Gamma}| \geq \\ &\geq \tilde{c}_1 \|\operatorname{div}_\Gamma \mathbf{v}^\perp\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^2 + \tilde{c}_2 \varsigma^2 \|\mathbf{v}^0\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)}^2 \end{aligned}$$

for all $(\mathbf{v}^\perp, \mathbf{v}^0) \in \mathcal{G}$. Now, we can appeal to (2.3) and obtain

$$|d(\varphi, \varphi)| \geq \tilde{c}_a \|\varphi\|_{\mathcal{G}}^2 \quad \forall \varphi \in \mathcal{G}.$$

Setting $k = k_0 + k_1$ yields $a = d - k$ with a principal part d which is positive on \mathcal{G} and a compact perturbation k , as claimed. \square

The strong ellipticity of the form $a(\cdot, \cdot)$ together with its injectivity ensured by Ass. 1.1 implies, as usual, the unique solvability of the EFIE (3.4) (and hence of (1.1)) for any admissible right hand side. Moreover, there holds the continuous inf-sup condition for $a(\cdot, \cdot)$

$$\sup_{\boldsymbol{\nu} \in \mathcal{G}} \frac{|a(\boldsymbol{\nu}, \boldsymbol{\nu})|}{\|\boldsymbol{\nu}\|_{\mathcal{G}}} \geq \tilde{c}_a \|\boldsymbol{\nu}\|_{\mathcal{G}} \quad \forall \boldsymbol{\nu} \in \mathcal{G}. \quad (4.1)$$

5. Boundary element spaces. We equip Γ with a family of shape-regular, quasi-uniform triangulations $(\Gamma_h)_{h>0}$ [24] comprising only flat triangles. The parameter h designates the meshwidth, that is, the length of the longest edge. Let \mathbb{H} stand for the collection of meshwidths occurring in $(\Gamma_h)_{h \in \mathbb{H}}$ and assume that $\mathbb{H} \subset \mathbb{R}^+$ forms a decreasing sequence converging to zero. The set \mathcal{T}_h will include all triangles of Γ_h , and \mathcal{E}_h stands for the set of edges of Γ_h .

Using the local coordinate systems on the faces Γ_j , $j = 1, \dots, N_\Gamma$, each $T \in \mathcal{T}_h$ can be embedded in \mathbb{R}^2 . Then we can define the local spaces (cf. [46])

$$\mathcal{RT}_0(T) := \{\mathbf{x} \mapsto \mathbf{a} + \beta \mathbf{x}, \mathbf{a} \in \mathbb{C}^2, \beta \in \mathbb{C}\}, \quad T \in \mathcal{T}_h.$$

They give rise to the global boundary element space

$$\mathcal{RT}_0(\Gamma_h) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div}_\Gamma; \Gamma), \mathbf{v}|_T \in \mathcal{RT}_0(T) \forall T \in \mathcal{T}_h\}.$$

Keep in mind that this definition is based on a weak notion of $\operatorname{div}_\Gamma$. So Green's formula applied to the surface triangles can be used to confirm that the "edge-normal" components of the tangential vectorfields in $\mathcal{RT}_0(\Gamma_h)$ must be continuous across inter-element edges. This renders the following degrees of freedom well defined

$$\phi_e : \mathcal{RT}_0(\Gamma_h) \mapsto \mathbb{C}, \quad \phi_e(\mathbf{v}_h) := \int_e (\mathbf{v}_h \times \mathbf{n}_j) \cdot d\vec{s}, \quad e \in \mathcal{E}_h,$$

where \mathbf{n}_j is the normal of a face Γ_j in whose closure e is contained. Given the degrees of freedom we have nodal interpolation operators Π_h onto $\mathcal{RT}_0(\Gamma_h)$ at our disposal. To begin with, those can be declared for $\{\Gamma_j\}$ -piecewise continuous tangential surface vectorfields, whose edge-normal components are continuous, too. It turns out that this is not enough and we badly need to apply Π_h to less regular surface vectorfields. A first step towards this goal is the following lemma (cf. formula (3.40) in [15]).

LEMMA 5.1. *For any $s > 0$ the local interpolation operator $\Pi_T : \mathbf{H}^s(T) \cap \mathbf{H}(\operatorname{div}; T) \mapsto \mathcal{RT}_0(T)$, $T \in \mathcal{T}_h$, is continuous.*

Proof. Only the case $s \leq \frac{1}{2}$ is of interest. We consider a single degree of freedom on T : Pick an edge $e \subset \partial T$ and regard its characteristic function χ_e as an element in $W_q^{1-\frac{1}{q}}(e)$ for $q := 1 + s$. As $1 < q < 2$, Thm. 1.4.5.2 of [35] reveals that extension by zero of χ_e onto all of ∂T will provide a function $\tilde{\psi}$ in $W_q^{1-\frac{1}{q}}(\partial T)$. Then we can use the trace theorem [35, Thm. 1.5.1.3] to extend $\tilde{\psi}$ to a function $\psi \in W_q^1(T)$ in a continuous fashion. Using Green's formula, extended by continuity, we estimate for any smooth vectorfield \mathbf{v}

$$\begin{aligned} \int_e \mathbf{v} \cdot \mathbf{n}_e ds &= \int_{\partial T} \tilde{\psi} \mathbf{v} \cdot \mathbf{n} ds = \int_T \mathbf{grad} \psi \cdot \mathbf{v} + \psi \operatorname{div} \mathbf{v} d\mathbf{x} \leq \\ &\leq \|\mathbf{grad} \psi\|_{L^q(T)} \|\mathbf{v}\|_{L^p(T)} + \|\psi\|_{L^2(T)} \|\operatorname{div} \mathbf{v}\|_{L^2(T)}, \end{aligned}$$

where p is the exponent conjugate to q , i.e. $p^{-1} + q^{-1} = 1$. The Sobolev embedding theorem [1, Thm. 4.5] gives the continuous inclusions $W_q^1(T) \hookrightarrow L^2(T)$ and $\mathbf{H}^s(T) \hookrightarrow L^p(T)$. This implies, with $\tilde{C} = \tilde{C}(s, T)$,

$$\int_e \mathbf{v} \cdot \mathbf{n}_e ds \leq \tilde{C} \left(\|\mathbf{grad} \psi\|_{L^q(T)}^2 + \|\psi\|_{W_q^1(T)}^2 \right)^{\frac{1}{2}} \left(\|\mathbf{v}\|_{\mathbf{H}^s(T)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(T)}^2 \right)^{\frac{1}{2}}$$

for all $\mathbf{v} \in \mathbf{H}^s(T) \cap \mathbf{H}(\operatorname{div}; T)$ and the assertion of the lemma, since ψ is fixed. \square

The importance of the interpolation operators Π_h can be traced back to the *commuting diagram property* [15, Prop. 3.7]:

$$\operatorname{div}_\Gamma \Pi_h \mathbf{v} = Q_h \operatorname{div}_\Gamma \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{H}(\operatorname{div}; \Gamma) \cap \operatorname{Dom}(\Pi_h), \quad (5.1)$$

where Q_h is the $L^2(\Gamma)$ -orthogonal projection onto the space

$$\mathcal{Q}_0(\Gamma_h) := \{\mu \in L^2(\Gamma), \mu|_T = \text{const.}, \forall T \in \mathcal{T}_h\}.$$

Identity (5.1) is a simple consequence of the definition of the degrees of freedom and Gauß' theorem applied to elements. An important consequence is that

$$\text{div}_\Gamma \mathbf{v} = 0 \quad \wedge \quad \mathbf{v} \in \text{Dom}(\Pi_h) \quad \Rightarrow \quad \text{div}_\Gamma(\Pi_h \mathbf{v}) = 0.$$

The relationship (5.1) also reveals that $\text{div}_\Gamma \mathcal{RT}_0(\Gamma_h) = \mathcal{Q}_0(\Gamma_h)$.

Remark. The reader should be aware that we have restricted ourselves to lowest order Raviart-Thomas elements only for the sake of simplicity. All other $\mathbf{H}(\text{div}; \Omega)$ -conforming finite elements in 2D that provide valid discrete 1-forms could be used as well. A rich collection is offered in [15, Sect. III.3]. All arguments in the sequel will carry over to these elements with only slight alterations.

The Raviart-Thomas elements form an affine family of finite elements in the sense of [24] with respect to Piola's transformation [15, §III.1.3]

$$\mathfrak{P}_T : \mathbf{L}^2(\hat{T}) \mapsto \mathbf{L}_t^2(T), \quad \mathfrak{P}_T(\hat{\mathbf{v}}_h)(\mathbf{x}) := |\det D\Phi_T|^{-1} D\Phi_T \hat{\mathbf{v}}_h(\Phi_T^{-1}(\mathbf{x})), \quad \mathbf{x} \in T,$$

where \hat{T} is the reference triangle $\hat{T} := \{\mathbf{x} \in \mathbb{R}^2, x_1, x_2 > 0, x_1 + x_2 < 1\}$, $T \in \mathcal{T}_h$, and Φ_T the unique affine mapping that takes \hat{T} to T . The Piola transform preserves the values of degrees of freedom. Shape-regularity and quasi-uniformity guarantee that $|\det D\Phi_T| \asymp h^2$ and $\|D\Phi_T\| \asymp h$ uniformly in $T \in \mathcal{T}_h$ and $h \in \mathbb{H}$. Here and in the sequel, we are using the symbol \asymp to indicate equivalence up to constants that may depend on Γ and the shape regularity of $\{\Gamma_h\}_h$, but are independent of h . The same is expected of all generic constants unless they bear a tilde.

Now, using standard affine equivalence techniques, the effect of Piola's transform on fractional Sobolev norms can be controlled:

LEMMA 5.2. *The Piola transform \mathfrak{P}_T , $T \in \mathcal{T}_h$, satisfies for $0 \leq s \leq 1$*

$$|\hat{\mathbf{u}}|_{\mathbf{H}^s(\hat{T})} \asymp h^s |\mathfrak{P}_T \hat{\mathbf{u}}|_{\mathbf{H}^s(T)} \quad \forall \hat{\mathbf{u}} \in \mathbf{H}^s(\hat{T}),$$

with constants only depending on the shape-regularity of T .

Proof. See lemma 3 in [46] for the cases $s = 0$ and $s = 1$. The rest follows by interpolation. \square

Remark. Using Piola's transform one easily constructs parametric divergence-conforming surface elements [8, 32] for piecewise smooth Γ . Thus, our approach can be instantly extended to curved Lipschitz-polyhedra.

6. Hodge mapping. Coercivity of the sesqui-linear form related to (1.1) could only be established in the split space \mathcal{G} arising from the Hodge decomposition. This means that, though the boundary element spaces $\mathcal{RT}_0(\Gamma_h)$ are conforming and natural for the EFIE (1.1), Thm. 4.2, i.e. the validity of a Gårding inequality on the continuous level, gives no immediate information about the convergence of the Galerkin discretization. The reason is that we would need conforming finite element subspaces of both \mathbf{X} and \mathbf{N} in order to apply the usual results (cf. e.g. [48, Sect. 2.3]) about the convergence of Galerkin schemes for strongly elliptic variational problems.

A discrete $\mathbf{L}_t^2(\Gamma)$ -orthogonal Hodge decomposition

$$\mathcal{RT}_0(\Gamma_h) = \mathbf{X}_h \oplus \mathbf{N}_h, \quad \mathbf{N}_h := \text{Ker}(\text{div}_\Gamma) \cap \mathcal{RT}_0(\Gamma_h), \quad (6.1)$$

yields $\mathbf{N}_h \subset \mathbf{N}$, but generally we cannot expect $\mathbf{X}_h \subset \mathbf{X}$. In short, \mathbf{X}_h provides only a *non-conforming* discretization of \mathbf{X} and a discrete inf-sup condition does not follow from the strong ellipticity of the continuous problem. On the other hand, no modification of the sesqui-linear form $a(\cdot, \cdot)$ is necessary if we consider the variational problem (3.4) over $\mathcal{G}_h := \mathbf{X}_h \times \mathbf{N}_h$. This is simply due to the fact that everything

remains perfectly conforming in $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$. In particular, \mathcal{G}_h can be equipped with the norm $\|\cdot\|_{\mathcal{G}}$. However, embedding and regularity properties of \mathbf{X} (cf. lemmas 2.3 and 4.1) are crucial and the space \mathbf{X}_h lacks them. We deal with this by introducing semi-discrete spaces arising from the *continuous* Hodge decomposition of the discrete boundary element space: We split $\mathbf{v}_h \in \mathcal{RT}_0(\Gamma_h)$ in two ways

$$\mathbf{v}_h = \mathbf{v}_h^\perp + \mathbf{v}_h^0, \quad \mathbf{v}_h^\perp \in \mathbf{X}_h, \mathbf{v}_h^0 \in \mathbf{N}_h, \quad \text{and} \quad \mathbf{v}_h = \mathbf{v}^\perp + \mathbf{v}^0, \quad \mathbf{v}^\perp \in \mathbf{X}, \mathbf{v}^0 \in \mathbf{N}.$$

The discrete field \mathbf{v}_h^\perp is a genuine boundary element function, but only the *semidiscrete* field \mathbf{v}^\perp has the desired properties. We have labeled it semi-discrete because $\operatorname{div}_\Gamma \mathbf{v}^\perp = \operatorname{div}_\Gamma \mathbf{v}_h^\perp$ is still piecewise constant and, hence, \mathbf{v}^\perp still depends on the triangulation. To bridge the gap between \mathbf{v}_h^\perp and \mathbf{v}^\perp we need the following device (cf. Def. 4.1 in [39]):

DEFINITION 6.1. *We define the Hodge mapping $H_h : \mathcal{RT}_0(\Gamma_h) \mapsto \mathbf{X}$ by*

$$H_h \mathbf{v}_h \in \mathbf{X} : \quad \operatorname{div}_\Gamma H_h \mathbf{v}_h := \operatorname{div}_\Gamma \mathbf{v}_h \quad , \quad \mathbf{v}_h \in \mathcal{RT}_0(\Gamma_h).$$

Owing to (2.3), H_h is well-defined. The Hodge mappings are uniformly continuous with respect to $h \in \mathbb{H}$. They create the desired link between \mathbf{X}_h and \mathbf{X} (cf. Lemma 4.2 in [39]):

LEMMA 6.2. *For any $s \geq -\frac{1}{2}$ the Hodge mappings satisfy the estimate*

$$\|\mathbf{v}_h - H_h \mathbf{v}_h\|_{L^2(\Gamma)} \leq Ch^r \|\operatorname{div}_\Gamma \mathbf{v}_h\|_{H^s(\Gamma)} \quad \forall \mathbf{v}_h \in \mathbf{X}_h \quad \forall h \in \mathbb{H}.$$

with $0 \leq r \leq \min\{s+1, 1, s^*\}$ and constants only depending on s, r, Γ , and the shape-regularity of the surface triangulations.

Proof. We follow the proof of lemma 4.2 from [39], pick $\mathbf{u}_h \in \mathcal{RT}_0(\Gamma_h)$, and focus on an single triangle $T \in \Gamma_h$. Take $H_h \mathbf{u}_h|_T$ to the reference element and set $\hat{\mathbf{w}} := \mathfrak{P}_T^{-1} H_h \mathbf{u}_h$. By (2.3) we see $\hat{\mathbf{w}} \in \mathbf{H}^r(\hat{T})$, so that the assumptions of lemma 5.1 are satisfied. We have for any $r > 0$

$$\|\hat{\Pi} \hat{\mathbf{w}}\|_{L^2(\hat{T})} \leq \tilde{C}(r) (\|\hat{\mathbf{w}}\|_{\mathbf{H}^r(\hat{T})} + \|\operatorname{div} \hat{\mathbf{w}}\|_{L^2(\hat{T})}),$$

where $\hat{\Pi}$ is the local interpolation operator on \hat{T} . Remember that $\operatorname{div}_\Gamma H_h \mathbf{u}_h$ is piecewise constant, which renders $\operatorname{div} \hat{\mathbf{w}}$ constant. Exploiting the equivalence of all norms on finite dimensional spaces, we can easily bound $\|\operatorname{div} \hat{\mathbf{w}}\|_{L^2(\hat{T})}$ and arrive at

$$\|\hat{\Pi} \hat{\mathbf{w}}\|_{L^2(\hat{T})} \leq \tilde{C}(r) \|\hat{\mathbf{w}}\|_{\mathbf{H}^r(\hat{T})}.$$

Constant vectorfields on \hat{T} are preserved by the interpolation $\hat{\Pi}$. Thus, for any $\mathbf{p} \in \mathbb{C}^2$

$$\|\hat{\mathbf{w}} - \hat{\Pi} \hat{\mathbf{w}}\|_{L^2(\hat{T})} = \|\hat{\mathbf{w}} - \mathbf{p} - \hat{\Pi}(\hat{\mathbf{w}} - \mathbf{p})\|_{L^2(\hat{T})} \leq \|\hat{\mathbf{w}} - \mathbf{p}\|_{L^2(\hat{T})} + \tilde{C}(r) \|\hat{\mathbf{w}} - \mathbf{p}\|_{\mathbf{H}^r(\hat{T})}.$$

From the definition of the fractional Sobolev norm [35, Def. 1.3.2.1] and $0 \leq r \leq 1$, it is immediate that

$$\|\hat{\mathbf{w}} - \mathbf{p}\|_{\mathbf{H}^r(\hat{T})}^2 = \|\hat{\mathbf{w}} - \mathbf{p}\|_{L^2(\hat{T})}^2 + |\hat{\mathbf{w}}|_{\mathbf{H}^r(\hat{T})}^2.$$

As, according to Prop. 6.1 in [33], a Bramble-Hilbert-type estimate of the form

$$\inf_{c \in \mathbb{R}} \|f - c\|_{L^2(\hat{T})} \leq \tilde{C}(r) |f|_{\mathbf{H}^r(\hat{T})} \quad \forall f \in \mathbf{H}^r(\hat{T}).$$

also holds in fractional Sobolev spaces, we end up with the estimate

$$\|\hat{\mathbf{w}} - \hat{\Pi} \hat{\mathbf{w}}\|_{L^2(\hat{T})} \leq \tilde{C}(r) |\hat{\mathbf{w}}|_{\mathbf{H}^r(\hat{T})}.$$

Since interpolation and the Piola transform commute, we may use lemma 5.2 to pull the estimate back to the element T

$$\|H_h \mathbf{u}_h - \Pi_h H_h \mathbf{u}_h\|_{L^2(T)} \leq Ch^r \|H_h \mathbf{u}_h\|_{\mathbf{H}^r(T)} .$$

At this stage shape-regularity starts affecting the constants. Squaring and summing over all elements yields

$$\|H_h \mathbf{u}_h - \Pi_h H_h \mathbf{u}_h\|_{L^2(\Gamma)} \leq Ch^r \|H_h \mathbf{u}_h\|_{\mathbf{H}^r(\Gamma)} ,$$

which, in light of lemma 2.3, involves

$$\|H_h \mathbf{u}_h - \Pi_h H_h \mathbf{u}_h\|_{L^2(\Gamma)} \leq Ch^r \|\operatorname{div}_\Gamma \mathbf{u}_h\|_{H^s(\Gamma)} . \quad (6.2)$$

By the commuting diagram property of Π_h we conclude from $\operatorname{div}_\Gamma(\mathbf{v}_h - H_h \mathbf{v}_h) = 0$ that also $\operatorname{div}_\Gamma(\mathbf{v}_h - \Pi_h H_h \mathbf{v}_h) = 0$. This means $\mathbf{v}_h - \Pi_h H_h \mathbf{v}_h \in \mathbf{N}_h$ and makes it possible for us to apply Nedelec's trick [45, Sect. 3.3]

$$\begin{aligned} \|\mathbf{u}_h - H_h \mathbf{u}_h\|_{L^2(\Gamma)}^2 &= (\mathbf{u}_h - H_h \mathbf{u}_h, \mathbf{u}_h - \Pi_h H_h \mathbf{u}_h + \Pi_h H_h \mathbf{u}_h - H_h \mathbf{u}_h)_{0;\Gamma} = \\ &= (\mathbf{u}_h - H_h \mathbf{u}_h, \Pi_h H_h \mathbf{u}_h - H_h \mathbf{u}_h)_{0;\Gamma} . \end{aligned}$$

Together with (6.2) this shows the assertion of the lemma. \square

Now, we fix $t \leq \min\{\frac{1}{2}, s^*\}$ and keep it constant for the remainder of this paper. A legal choice for r in the previous lemma is $r = t$ for $s = -\frac{1}{2}$, and we denote the associated constant by C_3 .

LEMMA 6.3. *The decomposition $\mathcal{RT}_0(\Gamma_h) = \mathbf{X}_h \oplus \mathbf{N}_h$ is uniformly $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ -stable.*

Proof. For $\mathbf{u}_h \in \mathbf{X}_h$ we can use the Hodge mapping and the previous lemma to estimate

$$\|\mathbf{u}_h\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq \|\mathbf{u}_h - H_h \mathbf{u}_h\|_{L^2(\Gamma)} + \|H_h \mathbf{u}_h\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq C(h^t + 1) \|\operatorname{div}_\Gamma \mathbf{u}_h\|_{H^{-\frac{1}{2}}(\Gamma)} ,$$

as $H_h \mathbf{u}_h \in \mathbf{X}$. Since $\operatorname{div}_\Gamma H_h \mathbf{u}_h = \operatorname{div}_\Gamma \mathbf{u}_h$ and \mathbb{H} is bounded, the proof is finished. \square

We shall also require the following right inverse of the Hodge mapping.

DEFINITION 6.4. *We define the linear continuous mappings $T_h : \mathbf{X} \mapsto \mathbf{X}_h$, $h \in \mathbb{H}$, by*

$$T_h \mathbf{w} \in \mathbf{X}_h : \quad \operatorname{div}_\Gamma T_h \mathbf{w} = Q_h^{-\frac{1}{2}} \operatorname{div}_\Gamma \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{X} ,$$

where $Q_h^{-\frac{1}{2}} : H^{-\frac{1}{2}}(\Gamma) \mapsto Q_0(\Gamma_h)$ are the $H^{-\frac{1}{2}}(\Gamma)$ -orthogonal projections.

Note that only due to the preceding stability result this definition makes real sense. Besides, lemma 6.3 guarantees that the family of operators $(T_h)_{h \in \mathbb{H}}$ is uniformly continuous, as

$$\|T_h \mathbf{w}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq C \|\operatorname{div}_\Gamma T_h \mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \|\operatorname{div}_\Gamma \mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} .$$

LEMMA 6.5. *For any fixed $\mathbf{w} \in \mathbf{X}$ we have*

$$\lim_{h \rightarrow 0} \|\mathbf{w} - T_h \mathbf{w}\|_{\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)} = 0 .$$

Proof. We resort to the same trick as in the proof of lemma 6.3 and use $H_h T_h \mathbf{w} - \mathbf{w} \in \mathbf{X}$

$$\begin{aligned} \|T_h \mathbf{w} - \mathbf{w}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} &\leq \|H_h T_h \mathbf{w} - T_h \mathbf{w}\|_{L^2(\Gamma)} + \|H_h T_h \mathbf{w} - \mathbf{w}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \\ &\leq C_3 h^t \|\operatorname{div}_\Gamma T_h \mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} + \tilde{C} \|\operatorname{div}_\Gamma (H_h T_h \mathbf{w} - \mathbf{w})\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq Ch^t \|\operatorname{div}_\Gamma \mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} + \tilde{C} \inf_{\mu_h \in Q_0(\Gamma_h)} \|\operatorname{div}_\Gamma \mathbf{w} - \mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} . \end{aligned}$$

As $\bigcup_{h \in \mathbb{H}} Q_0(\Gamma_h)$ is dense in $L^2(\Gamma)$ which in turn is dense in $H^{-\frac{1}{2}}(\Gamma)$, the lemma holds true. \square

7. Stability of the Galerkin scheme. Galerkin discretization of (3.4) leads to the discrete variational problem: Seek $\boldsymbol{u}_h \in \mathcal{G}_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{\eta}_h) = f(\boldsymbol{\eta}_h) \quad \forall \boldsymbol{\eta}_h \in \mathcal{G}_h. \quad (7.1)$$

The discrete Hodge decomposition (6.1) shows (7.1) to be equivalent to the Galerkin discretization (1.2) of the EFIE (1.1). From theorem 4.2 we saw that problem (3.4) is strongly elliptic, i.e. $a = d - k$, with a \mathcal{G} -elliptic sesqui-linear form d and a \mathcal{G} -compact form k . Discretization of (3.4) by a dense and conforming family of finite dimensional subspaces would therefore imply quasioptimal asymptotic convergence of the approximate solutions. The problem here is that \mathcal{G}_h is generally non-conforming, i.e. $\mathcal{G}_h \not\subset \mathcal{G}$. Therefore, coercivity in the discrete setting must be established by a separate argument. For the proof, we draw on an idea of A. Schatz [47].

To get compact formulas, we replace bilinear forms by the associated Riesz operators. First, $A : \mathcal{G} \mapsto \mathcal{G}'$ is associated to the sesqui-linear form a . Next, the operator $K : \mathcal{G} \mapsto \mathcal{G}'$ is associated with the sesqui-linear form k defined in the proof of theorem 4.2. Both operators are continuous from $\mathcal{G} \mapsto \mathcal{G}'$. However, since \mathcal{G}_h is non-conforming, these operators are not defined on \mathcal{G}_h a priori. To extend them, we use Hodge mappings on \mathcal{G}_h which are defined through

$$\mathbf{H}_h : \mathcal{G}_h \mapsto \mathcal{G} \quad , \quad \mathbf{H}_h(\mathbf{v}_h^\perp, \mathbf{v}_h^0) := (H_h \mathbf{v}_h^\perp, \mathbf{v}_h^0) \in \mathcal{G} \quad , \quad (\mathbf{v}_h^\perp, \mathbf{v}_h^0) \in \mathcal{G}_h.$$

Lemma 6.2 ensures the uniform boundedness in h of this family of operators. We also define the extension $\mathbf{T}_h : \mathcal{G} \mapsto \mathcal{G}_h$ to \mathcal{G}_h of the right inverses T_h , $h \in \mathbb{H}$, of the Hodge mappings from Definition 6.4:

$$\mathbf{T}_h(\mathbf{v}^\perp, \mathbf{v}^0) := (T_h \mathbf{v}^\perp, \mathbf{Q}_h^{-\frac{1}{2}} \mathbf{v}^0) \in \mathcal{G}_h \quad (\mathbf{v}^\perp, \mathbf{v}^0) \in \mathcal{G} \quad ,$$

where $\mathbf{Q}_h^{-\frac{1}{2}}$ is the $\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma)$ -orthogonal projection $\mathbf{N} \mapsto \mathbf{N}_h$. The operator \mathbf{T}_h is well defined, since $\mathbf{N}_h \subset \mathbf{N}$ and \mathbf{N} is a closed subspace of $\mathbf{H}_{\parallel}^{-\frac{1}{2}}(\Gamma)$. Density of $\bigcup_{h \in \mathbb{H}} \mathbf{N}_h$ in \mathbf{N} and lemma 6.5 confirm pointwise convergence

$$\lim_{h \rightarrow 0} \|\varphi - \mathbf{T}_h \varphi\|_{\mathcal{G}} = 0 \quad \forall \varphi \in \mathcal{G}. \quad (7.2)$$

Next, we consider the operator $S : \mathcal{G}' \mapsto \mathcal{G}$ defined as the solution operator of the \mathcal{G} -elliptic variational problem

$$\boldsymbol{\eta}' \in \mathcal{G}' : \quad d(S\boldsymbol{\eta}', \varphi) = \boldsymbol{\eta}'(\varphi) \quad \forall \varphi \in \mathcal{G}.$$

Continuity and ellipticity of the sesqui-linear form d ensure that S is well-defined and give

$$\tilde{C}_d^{-1} \|\boldsymbol{\eta}'\|_{\mathcal{G}'} \leq \|S\boldsymbol{\eta}'\|_{\mathcal{G}} \leq \tilde{c}_d^{-1} \|\boldsymbol{\eta}'\|_{\mathcal{G}'} \quad \forall \boldsymbol{\eta}' \in \mathcal{G}', \quad (7.3)$$

where $\tilde{C}_d := \|d\|$. Note also that the operator S is confined to the continuous setting.

LEMMA 7.1. *There is a function $b : \mathbb{H} \mapsto \mathbb{R}^+$ with $b(h) \rightarrow 0$ as $h \rightarrow 0$ such that*

$$\|(\mathbf{T}_h - Id)SK\boldsymbol{\eta}\|_{\mathcal{G}} \leq b(h) \|\boldsymbol{\eta}\|_{\mathcal{G}} \quad \forall \boldsymbol{\eta} \in \mathcal{G}.$$

Proof. Set $B_1(\mathcal{G}) := \{\varphi \in \mathcal{G} : \|\varphi\|_{\mathcal{G}} \leq 1\}$. As $K : \mathcal{G} \mapsto \mathcal{G}'$ is compact, the set $KB_1(\mathcal{G})$ is precompact in \mathcal{G}' . Thanks to the continuity of S the closure w.r.t. the $\|\cdot\|_{\mathcal{G}}$ -norm $M := \overline{SKB_1(\mathcal{G})}$ is compact in \mathcal{G} . Pick some $\epsilon > 0$ and write $B_\epsilon(\boldsymbol{\nu})$ for the ϵ -neighborhood of $\boldsymbol{\nu}$ in \mathcal{G} . We can find finitely many $\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_L$, $L = L(\epsilon) \in \mathbb{N}$, in M such that $M \subset \bigcup_l B_\epsilon(\boldsymbol{\nu}_l)$. From (7.2) we learn that there is $h_0 = h_0(\epsilon) \in \mathbb{H}$ such that

$$\|\mathbf{T}_h \boldsymbol{\nu}_l - \boldsymbol{\nu}_l\|_{\mathcal{G}} \leq \epsilon \quad \forall h < h_0, \quad l = 1, \dots, L.$$

For any $\boldsymbol{\eta} \in M$ there exists a $\boldsymbol{\nu}_l$ such that $\boldsymbol{\eta} \in B_\epsilon(\boldsymbol{\nu}_l)$. Hence

$$\|\mathbf{T}_h \boldsymbol{\eta} - \boldsymbol{\eta}\|_{\mathcal{G}} \leq \|\mathbf{T}_h \boldsymbol{\eta} - \mathbf{T}_h \boldsymbol{\nu}_l\|_{\mathcal{G}} + \|\mathbf{T}_h \boldsymbol{\nu}_l - \boldsymbol{\nu}_l\|_{\mathcal{G}} + \|\boldsymbol{\nu}_l - \boldsymbol{\eta}\|_{\mathcal{G}} \leq (\|\mathbf{T}_h\|_{\mathcal{G} \rightarrow \mathcal{G}} + 2)\epsilon,$$

if $h < h_0$. Undoing the substitutions, we get

$$\|(\mathbf{T}_h - Id)SK\boldsymbol{\eta}\|_{\mathcal{G}} \leq (\|\mathbf{T}_h\|_{\mathcal{G} \rightarrow \mathcal{G}} + 2)\epsilon \quad \forall \boldsymbol{\eta} \in B_1(\mathcal{G}), \quad h < h_0.$$

A homogeneity argument finishes the proof. \square

Next, we prove the discrete inf-sup condition for the form $a(\cdot, \cdot)$: Given $\boldsymbol{\eta}_h \in \mathcal{G}_h$, we set

$$\boldsymbol{\varphi}_h := (Id - \mathbf{T}_h SK \mathbf{H}_h) \boldsymbol{\eta}_h \in \mathcal{G}_h.$$

The uniform boundedness with respect to h of the operators involved ensures that there is $C_4 > 0$ independent of $h \in \mathbb{H}$ and $\boldsymbol{\eta}_h$ such that

$$\|\boldsymbol{\varphi}_h\|_{\mathcal{G}} \leq C_4 \|\boldsymbol{\eta}_h\|_{\mathcal{G}}. \quad (7.4)$$

We therefore estimate

$$\begin{aligned} |a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)| &= |a(\boldsymbol{\eta}_h, (Id - \mathbf{T}_h SK \mathbf{H}_h) \boldsymbol{\eta}_h)| \\ &= |a(\boldsymbol{\eta}_h, ((Id - \mathbf{T}_h)(SK \mathbf{H}_h) + (Id - SK \mathbf{H}_h)) \boldsymbol{\eta}_h)| \\ &\geq |a(\boldsymbol{\eta}_h, (Id - SK \mathbf{H}_h) \boldsymbol{\eta}_h)| - |a(\boldsymbol{\eta}_h, (Id - \mathbf{T}_h) SK \mathbf{H}_h \boldsymbol{\eta}_h)| \\ &\geq |a(\boldsymbol{\eta}_h, (Id - SK \mathbf{H}_h) \boldsymbol{\eta}_h)| - \tilde{C}_a \|\boldsymbol{\eta}_h\|_{\mathcal{G}} \|(Id - \mathbf{T}_h) SK \mathbf{H}_h \boldsymbol{\eta}_h\|_{\mathcal{G}} \\ &\geq |a(\boldsymbol{\eta}_h, (Id - SK \mathbf{H}_h) \boldsymbol{\eta}_h)| - b(h) \tilde{C}_a \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2, \end{aligned}$$

the final inequality being a consequence of lemma 7.1. Further, we estimate the first term

$$\begin{aligned} |a(\boldsymbol{\eta}_h, (Id - SK \mathbf{H}_h) \boldsymbol{\eta}_h)| &= |a(\boldsymbol{\eta}_h, ((Id - \mathbf{H}_h) + (Id - SK) \mathbf{H}_h) \boldsymbol{\eta}_h)| \\ &\geq |a(\boldsymbol{\eta}_h, (Id - SK) \mathbf{H}_h \boldsymbol{\eta}_h)| - |a(\boldsymbol{\eta}_h, (Id - \mathbf{H}_h) \boldsymbol{\eta}_h)| \\ &\geq |a(\boldsymbol{\eta}_h, S(S^{-1} - K) \mathbf{H}_h \boldsymbol{\eta}_h)| - \tilde{C}_a \|\boldsymbol{\eta}_h\|_{\mathcal{G}} \|(Id - \mathbf{H}_h) \boldsymbol{\eta}_h\|_{\mathcal{G}} \\ &\geq |a(\boldsymbol{\eta}_h, S(S^{-1} - K) \mathbf{H}_h \boldsymbol{\eta}_h)| - \tilde{C}_a C_3 h^t \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2, \end{aligned}$$

by lemma 6.2. Now, we note that $\boldsymbol{\psi} := S(S^{-1} - K) \boldsymbol{\lambda} \in \mathcal{G}$, $\boldsymbol{\lambda} \in \mathcal{G}$, satisfies

$$d(\boldsymbol{\psi}, \boldsymbol{\nu}) = \langle (S^{-1} - K) \boldsymbol{\lambda}, \boldsymbol{\nu} \rangle = d(\boldsymbol{\lambda}, \boldsymbol{\nu}) - k(\boldsymbol{\lambda}, \boldsymbol{\nu}) = a(\boldsymbol{\lambda}, \boldsymbol{\nu})$$

for all $\boldsymbol{\nu} \in \mathcal{G}$. In short, $S(S^{-1} - K) = SA$. This enables us to continue the estimates

$$\begin{aligned} |a(\boldsymbol{\eta}_h, S(S^{-1} - K) \mathbf{H}_h \boldsymbol{\eta}_h)| &= |a(\boldsymbol{\eta}_h - \mathbf{H}_h \boldsymbol{\eta}_h + \mathbf{H}_h \boldsymbol{\eta}_h, SA \mathbf{H}_h \boldsymbol{\eta}_h)| \\ &\geq |a(\mathbf{H}_h \boldsymbol{\eta}_h, SA \mathbf{H}_h \boldsymbol{\eta}_h)| - \tilde{C}_a \|(Id - \mathbf{H}_h) \boldsymbol{\eta}_h\|_{\mathcal{G}} \|SA \mathbf{H}_h \boldsymbol{\eta}_h\|_{\mathcal{G}} \\ &\geq |d(SA \mathbf{H}_h \boldsymbol{\eta}_h, SA \mathbf{H}_h \boldsymbol{\eta}_h)| - \tilde{C}_a^2 \tilde{c}_d^{-1} C_3 h^t \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2. \end{aligned}$$

For the last time we target the first term

$$\begin{aligned} |d(SA \mathbf{H}_h \boldsymbol{\eta}_h, SA \mathbf{H}_h \boldsymbol{\eta}_h)| &\geq \tilde{c}_d \|SA \mathbf{H}_h \boldsymbol{\eta}_h\|_{\mathcal{G}}^2 \geq \tilde{c}_d \tilde{C}_d^{-1} \|A \mathbf{H}_h \boldsymbol{\eta}_h\|_{\mathcal{G}}^2 \\ &\geq \tilde{c}_d \tilde{C}_d^{-1} \tilde{c}_a \|\boldsymbol{\eta}_h - (Id - \mathbf{H}_h) \boldsymbol{\eta}_h\|_{\mathcal{G}}^2 \\ &\geq \tilde{c}_4 (\|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2 - \|(Id - \mathbf{H}_h) \boldsymbol{\eta}_h\|_{\mathcal{G}}^2) \\ &\geq \tilde{c}_4 \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2 - \tilde{c}_4 C_3 h^t \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2, \end{aligned}$$

with $\tilde{c}_4 := \tilde{c}_d \tilde{C}_d^{-1} \tilde{c}_a$. Summing up, we have obtained

$$|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)| \geq \left(\tilde{c}_4 - (\tilde{c}_4 + \tilde{C}_a^2 \tilde{c}_d^{-1} + \tilde{C}_a) C_3 h^t - \tilde{C}_a b(h) \right) \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2.$$

If $h < h_*$ with $(\tilde{c}_4 + \tilde{C}_a^2 \tilde{c}_d^{-1} + \tilde{C}_a) C_3 h_*^t + \tilde{C}_a b(h_*) < \frac{1}{2} \tilde{c}_4$, we obtain the lower bound

$$|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)| \geq \frac{1}{2} \tilde{c}_4 \|\boldsymbol{\eta}_h\|_{\mathcal{G}}^2 \quad \forall h < h_* .$$

This is valid for any $\boldsymbol{\eta}_h$. Recalling (7.4), an immediate consequence is the discrete inf-sup condition

$$\sup_{\boldsymbol{\varphi}_h \in \mathcal{G}_h} \frac{|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)|}{\|\boldsymbol{\varphi}_h\|_{\mathcal{G}}} \geq \frac{\tilde{c}_4}{2C_4} \|\boldsymbol{\eta}_h\|_{\mathcal{G}} \quad \forall \boldsymbol{\eta}_h \in \mathcal{G}_h, h < h_* \quad (7.5)$$

Based on this discrete stability condition, the stability (4.1) of the continuous problem and the continuity (3.3) of the bilinear forms involved we obtain quasioptimal asymptotic convergence for the sequence of Galerkin solutions.

THEOREM 7.2. *There exists a constant $C > 0$ and a meshwidth $h_* \in \mathbb{H}$ only depending on Γ , ς , and on the shape-regularity of the triangulations Γ_h such that for any $h < h_*$ the discrete problem (7.1) has a unique solution $\boldsymbol{\iota}_h$ and the family $\{\boldsymbol{\iota}_h\}_h$ converges quasioptimally:*

$$\|\boldsymbol{\iota} - \boldsymbol{\iota}_h\|_{\mathcal{G}} \leq C \inf_{\boldsymbol{\varphi}_h \in \mathcal{G}_h} \|\boldsymbol{\iota} - \boldsymbol{\varphi}_h\|_{\mathcal{G}} .$$

Proof. We denote by $\hat{a}(\mathbf{j}, \mathbf{v}) := \langle V_\varsigma \operatorname{div}_\Gamma \mathbf{j}, \operatorname{div}_\Gamma \mathbf{v} \rangle_{\frac{1}{2}, \Gamma} - \varsigma^2 \langle \mathbf{A}_\varsigma \mathbf{j}, \mathbf{v} \rangle_{\|\cdot\|_\Gamma}$ the bilinear form in (1.1) and (1.2). Since $\mathcal{RT}_0(\Gamma_h)$ is $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ -conforming, the $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ -stable Hodge decompositions $\mathcal{RT}_0(\Gamma_h) = \mathbf{X}_h \oplus \mathbf{N}_h$, $\mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) = \mathbf{X} \oplus \mathbf{N}$ and the equivalence of (1.1) with (3.4) and of (1.2) with (7.1) imply that for every $\boldsymbol{\iota}_h = (\mathbf{v}_h^\perp, \mathbf{v}_h^0) \in \mathcal{G}_h$ holds: $\mathbf{v}_h := \mathbf{v}_h^\perp - \mathbf{v}_h^0 \in \mathcal{RT}_0(\Gamma_h) \subset \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ and

$$a((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}_h^\perp, \mathbf{v}_h^0)) = \hat{a}(\mathbf{j}, \mathbf{v}_h) = f(\mathbf{v}_h) = f(\mathbf{v}_h^\perp - \mathbf{v}_h^0) = a((\mathbf{j}_h^\perp, \mathbf{j}_h^0), (\mathbf{v}_h^\perp, \mathbf{v}_h^0)) .$$

As $\mathbf{X}_h - \mathbf{N}_h$ spans all of $\mathcal{RT}_0(\Gamma_h)$, we have the Galerkin orthogonality

$$a(\boldsymbol{\iota} - \boldsymbol{\iota}_h, \boldsymbol{\varphi}_h) = 0 \quad \forall \boldsymbol{\varphi}_h \in \mathcal{G}_h .$$

Now the stability (7.5) of the Galerkin scheme allows us to proceed in the classical fashion: For all $\boldsymbol{\eta}_h \in \mathcal{G}_h$, with $C_5 := \tilde{c}_4/2C_4$,

$$\begin{aligned} \|\boldsymbol{\iota} - \boldsymbol{\iota}_h\|_{\mathcal{G}} &\leq \|\boldsymbol{\iota} - \boldsymbol{\eta}_h\|_{\mathcal{G}} + \|\boldsymbol{\eta}_h - \boldsymbol{\iota}_h\|_{\mathcal{G}} \leq \|\boldsymbol{\iota} - \boldsymbol{\eta}_h\|_{\mathcal{G}} + C_5 \sup_{\boldsymbol{\varphi}_h \in \mathcal{G}_h} \frac{|a(\boldsymbol{\eta}_h - \boldsymbol{\iota}_h, \boldsymbol{\varphi}_h)|}{\|\boldsymbol{\varphi}_h\|_{\mathcal{G}}} \\ &\leq \|\boldsymbol{\iota} - \boldsymbol{\eta}_h\|_{\mathcal{G}} + C_5 \sup_{\boldsymbol{\varphi}_h \in \mathcal{G}_h} \frac{|a(\boldsymbol{\eta}_h - \boldsymbol{\iota}, \boldsymbol{\varphi}_h)|}{\|\boldsymbol{\varphi}_h\|_{\mathcal{G}}} \leq (1 + C_5 \tilde{C}_a) \|\boldsymbol{\iota} - \boldsymbol{\eta}_h\|_{\mathcal{G}} . \end{aligned}$$

□

8. Convergence Rates. Eventually, we are interested in getting a convergence estimate depending on the smoothness of the continuous solution \mathbf{j} of (1.1) only. The next lemma is a first step towards this goal.

LEMMA 8.1. *If $\mathbf{j} \in \mathbf{H}^\sigma(\operatorname{div}_\Gamma; \Gamma)$, $0 < \sigma$, and $h < h_*$, then*

$$\|\boldsymbol{\iota} - \boldsymbol{\iota}_h\|_{\mathcal{G}} \leq \|\mathbf{j} - \Pi_h \mathbf{j}\|_{\mathbf{H}_{\|\cdot\|}^{-\frac{1}{2}}(\Gamma)} + Ch^{\min\{1, \frac{1}{2} + \sigma, s^*\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)} .$$

Proof. We owe the powerful estimate of the lemma to duality techniques introduced by V. Girault in [34] and by P. Monk in [42]: We start with the Hodge decomposition

$$\mathbf{j} = \mathbf{j}^\perp + \mathbf{j}^0, \quad \mathbf{j}^\perp \in \mathbf{X}, \quad \mathbf{j}^0 \in \mathbf{N},$$

and fix $\mathbf{v}_h^\perp \in \mathbf{X}_h$ by $\mathbf{v}_h^\perp := \mathbf{T}_h \mathbf{j}^\perp$. This will give us the \mathbf{X}_h -component of a suitable φ_h for the estimate of Thm. 7.2. By definition 6.4 of \mathbf{T}_h , \mathbf{v}_h^\perp fulfills

$$(\operatorname{div}_\Gamma(\mathbf{j}^\perp - \mathbf{v}_h^\perp), \operatorname{div}_\Gamma \mathbf{w}_h)_{H^{-\frac{1}{2}}(\Gamma)} = 0 \quad \forall \mathbf{w}_h \in \mathcal{RT}_0(\Gamma_h), \quad (8.1)$$

with $(\cdot, \cdot)_{H^{-\frac{1}{2}}(\Gamma)}$ designating the inner product in the Hilbert space $H^{-\frac{1}{2}}(\Gamma)$. This orthogonality and the commuting diagram property (5.1) yield

$$\begin{aligned} \|\operatorname{div}_\Gamma(\mathbf{j}^\perp - \mathbf{v}_h^\perp)\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \|\operatorname{div}_\Gamma \mathbf{j} - \operatorname{div}_\Gamma \Pi_h \mathbf{j}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq \|(Id - Q_h) \operatorname{div}_\Gamma \mathbf{j}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq Ch^{\min\{\frac{3}{2}, \frac{1}{2} + \sigma\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)}. \end{aligned} \quad (8.2)$$

This is a consequence of approximation estimates for the $L^2(\Gamma)$ -orthogonal projections Q_h in negative norms, which can be verified by simple duality techniques.

Completeness arguments show that $\tilde{\mathbf{N}} := \mathbf{L}_t^2(\Gamma) \cap \mathbf{N}$ is a closed subspace of $\mathbf{L}_t^2(\Gamma)$, since it is isometrically isomorphic to $\operatorname{Ker}(\operatorname{div}_\Gamma)$ in $\mathbf{H}(\operatorname{div}_\Gamma; \Gamma)$. Looking at the proof of Thm. 2.2, we also find the stable $\mathbf{L}_t^2(\Gamma)$ -orthogonal decomposition

$$\mathbf{L}_t^2(\Gamma) \cap \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma) = \mathbf{X} \oplus \tilde{\mathbf{N}}.$$

Thus, in conjunction with lemma 2.3, it is easy to verify existence and uniqueness of solutions of the following auxiliary variational problems: For $\mathbf{g} \in \mathbf{L}_t^2(\Gamma)$ seek $\mathbf{q}^\perp = \mathbf{q}^\perp(\mathbf{g}) \in \mathbf{X}$ and $\mathbf{z}^0 = \mathbf{z}^0(\mathbf{g}) \in \tilde{\mathbf{N}}$ such that for all $\mathbf{w} \in \mathbf{L}_t^2(\Gamma) \cap \mathbf{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$

$$(\operatorname{div}_\Gamma \mathbf{q}^\perp, \operatorname{div}_\Gamma \mathbf{w})_{H^{-\frac{1}{2}}(\Gamma)} + (\mathbf{z}^0, \mathbf{w})_{0;\Gamma} = (\mathbf{g}, \mathbf{w})_{0;\Gamma}. \quad (8.3)$$

Obviously, we have $\|\mathbf{z}^0\|_{L^2(\Gamma)} \leq \|\mathbf{g}\|_{L^2(\Gamma)}$. Moreover, the strong form of (8.3) is

$$-\mathbf{grad}_\Gamma V_0 \operatorname{div}_\Gamma \mathbf{q}^\perp + \mathbf{z}^0 = \mathbf{g} \quad \text{on } \Gamma.$$

The mapping properties of \mathbf{grad}_Γ (cf. Sect. 2) give evidence of

$$-\mathbf{grad}_\Gamma V_0 \operatorname{div}_\Gamma \mathbf{q}^\perp \in \mathbf{L}_t^2(\Gamma) \quad \Leftrightarrow \quad V_0 \operatorname{div}_\Gamma \mathbf{q}^\perp \in H^1(\Gamma).$$

By the lifting properties of the scalar single layer potential operator on polyhedra (cf. Ch. 6 of [41]) we get

$$\operatorname{div}_\Gamma \mathbf{q}^\perp \in \mathbf{L}_t^2(\Gamma) \quad , \quad \|\operatorname{div}_\Gamma \mathbf{q}^\perp\|_{L^2(\Gamma)} \leq \tilde{C} \|\mathbf{g} - \mathbf{z}^0\|_{L^2(\Gamma)} \leq \tilde{C} \|\mathbf{g}\|_{L^2(\Gamma)}.$$

To prepare the crucial duality estimates we introduce the $\mathbf{L}_t^2(\Gamma)$ -orthogonal projections $\mathbf{Q}_h^0 : \tilde{\mathbf{N}} \rightarrow \mathbf{N}_h$, $h \in \mathbb{H}$. For an arbitrary $\mathbf{g} \in \mathbf{L}_t^2(\Gamma)$ we can then exploit the orthogonalities (8.1) and $\mathbf{N}_h \perp (\mathbf{X} + \mathbf{X}_h)$, and obtain

$$\begin{aligned} (\mathbf{g}, \mathbf{j}^\perp - \mathbf{v}_h^\perp)_{0;\Gamma} &= (\operatorname{div}_\Gamma \mathbf{q}^\perp, \operatorname{div}_\Gamma(\mathbf{j}^\perp - \mathbf{v}_h^\perp))_{H^{-\frac{1}{2}}(\Gamma)} + (\mathbf{z}^0, \mathbf{j}^\perp - \mathbf{v}_h^\perp)_{0;\Gamma} \\ &= (\operatorname{div}_\Gamma(\mathbf{q}^\perp - \Pi_h \mathbf{q}^\perp), \operatorname{div}_\Gamma(\mathbf{j}^\perp - \mathbf{v}_h^\perp))_{H^{-\frac{1}{2}}(\Gamma)} + (\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \mathbf{j}^\perp - \mathbf{v}_h^\perp)_{0;\Gamma} \\ &= (\operatorname{div}_\Gamma(\mathbf{q}^\perp - \Pi_h \mathbf{q}^\perp), \operatorname{div}_\Gamma(\mathbf{j}^\perp - \mathbf{v}_h^\perp))_{H^{-\frac{1}{2}}(\Gamma)} + \\ &\quad + (\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \mathbf{j}^\perp - \Pi_h \mathbf{j}^\perp)_{0;\Gamma} + (\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \Pi_h \mathbf{j}^\perp - \mathbf{v}_h^\perp)_{0;\Gamma}. \end{aligned}$$

We point out that lemma 5.1 tells us that functions in \mathbf{X} are sufficiently regular to render all the above inner products and nodal interpolation by Π_h well defined. We carry on with another Hodge decomposition

$$\Pi_h \mathbf{j}^\perp - \mathbf{v}_h^\perp = \mathbf{u}^\perp + \mathbf{u}^0, \quad \mathbf{u}^\perp \in \mathbf{X}, \quad \mathbf{u}^0 \in \mathbf{N},$$

and note that also, since Π_h is a projector,

$$\Pi_h \mathbf{j}^\perp - \mathbf{v}_h^\perp = \Pi_h \mathbf{u}^\perp + \Pi_h \mathbf{u}^0, \quad \text{where, by (5.1), } \Pi_h \mathbf{u}^0 \in \mathbf{N}_h.$$

As $(\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \Pi_h \mathbf{u}^0)_{0;\Gamma} = 0$ and $(\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \mathbf{u}^\perp)_{0;\Gamma} = 0$, the last term in the above expression for $(\mathbf{g}, \mathbf{j}^\perp - \mathbf{v}_h^\perp)_{0;\Gamma}$ can be converted into

$$(\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \Pi_h \mathbf{j} - \mathbf{v}_h^\perp)_{0;\Gamma} = (\mathbf{z}^0 - \mathbf{Q}_h^0 \mathbf{z}^0, \Pi_h \mathbf{u}^\perp - \mathbf{u}^\perp)_{0;\Gamma}.$$

Since \mathbf{u}^\perp is a semi-discrete function we see, as in the proof of lemma 6.2, that

$$\|\mathbf{u}^\perp - \Pi_h \mathbf{u}^\perp\|_{\mathbf{L}^2(\Omega)} \leq Ch^{\min\{\frac{1}{2}, s^*\}} \|\operatorname{div}_\Gamma \mathbf{u}^\perp\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

We use the triangle inequality, estimate (8.2), and continue

$$\begin{aligned} \|\mathbf{u}^\perp - \Pi_h \mathbf{u}^\perp\|_{\mathbf{L}^2(\Omega)} &\leq Ch^{\min\{\frac{1}{2}, s^*\}} \|\operatorname{div}_\Gamma (\Pi_h \mathbf{j}^\perp - \mathbf{v}_h^\perp)\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq Ch^{\min\{\frac{1}{2}, s^*\}} \left(\|\operatorname{div}_\Gamma (\Pi_h \mathbf{j}^\perp - \mathbf{j}^\perp)\|_{H^{-\frac{1}{2}}(\Gamma)} + \|\operatorname{div}_\Gamma (\mathbf{j}^\perp - \mathbf{v}_h^\perp)\|_{H^{-\frac{1}{2}}(\Gamma)} \right) \\ &\leq Ch^{\min\{\frac{1}{2}, s^*\}} \cdot h^{\min\{\frac{3}{2}, \frac{1}{2} + \sigma\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)} \\ &\leq Ch^{\min\{2, 1 + \sigma, \frac{3}{2} + s^*, \frac{1}{2} + \sigma + s^*\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)}. \end{aligned}$$

The regularity of \mathbf{q}^\perp and the commuting diagram property make possible the estimate

$$\|\operatorname{div}_\Gamma (\mathbf{q}^\perp - \Pi_h \mathbf{q}^\perp)\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Ch^{\frac{1}{2}} \|\operatorname{div}_\Gamma \mathbf{q}^\perp\|_{\mathbf{L}^2(\Gamma)} \leq Ch^{\frac{1}{2}} \|\mathbf{g}\|_{\mathbf{L}^2(\Gamma)},$$

which, along with the previous one and (8.2), translates into

$$\begin{aligned} (\mathbf{g}, \mathbf{j}^\perp - \mathbf{v}_h^\perp)_{0;\Gamma} &\leq Ch^{\min\{2, 1 + \sigma\}} \|\mathbf{g}\|_{\mathbf{L}^2(\Gamma)} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)} + \\ &+ \|\mathbf{g}\|_{\mathbf{L}^2(\Gamma)} \left(\|\mathbf{j}^\perp - \Pi_h \mathbf{j}^\perp\|_{\mathbf{L}^2(\Gamma)} + Ch^{\min\{2, 1 + \sigma, \frac{3}{2} + s^*, \frac{1}{2} + \sigma + s^*\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)} \right). \end{aligned}$$

Appealing to lemma 2.3 and, in turns, to interpolation error estimates for $\mathcal{RT}_0(\Gamma_h)$, we end up with

$$\|\mathbf{j}^\perp - \Pi_h \mathbf{j}^\perp\|_{\mathbf{L}^2(\Gamma)} \leq Ch^r \left(\|\mathbf{j}^\perp\|_{\mathbf{H}^r(\Gamma)} + \|\operatorname{div}_\Gamma \mathbf{j}^\perp\|_{H^\sigma(\Gamma)} \right) \leq Ch^r \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)} \quad (8.4)$$

for $r = \min\{1, \sigma + 1, s^*\}$. Finally, we set $\mathbf{g} := \mathbf{j}^\perp - \mathbf{v}_h^\perp$ and use all of the previous estimates:

$$\|\mathbf{j}^\perp - \mathbf{v}_h^\perp\|_{\mathbf{L}^2(\Gamma)} \leq Ch^{\min\{1, 1 + \sigma, s^*\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)}. \quad (8.5)$$

As \mathbf{N}_h -component of φ_h from Thm. 7.2 we pick $\mathbf{v}_h^0 := \Pi_h \mathbf{j}^0 \in \mathbf{N}_h$. By (8.4) and the triangle inequality

$$\begin{aligned} \|\mathbf{j}^0 - \mathbf{v}_h^0\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} &\leq \|\mathbf{j} - \Pi_h \mathbf{j}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} + \|\mathbf{j}^\perp - \Pi_h \mathbf{j}^\perp\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \\ &\leq \|\mathbf{j} - \Pi_h \mathbf{j}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} + Ch^{\min\{1, \sigma + 1, s^*\}} \|\operatorname{div}_\Gamma \mathbf{j}\|_{H^\sigma(\Gamma)}. \end{aligned} \quad (8.6)$$

At long last, we collect all estimates. Taking into account (8.2), (8.4), (8.6), Thm. 7.2 and the definition of the graph norm $\|\cdot\|_{\mathcal{G}}$ the assertion follows. \square

Remark. If Γ is simply connected, \mathbf{j}^0 can be represented as vectorial surface $\operatorname{curl}_\Gamma$ of a scalar *stream function* $\varphi \in H^{\frac{1}{2}}(\Gamma)$. If $\mathbf{j} \in \mathbf{H}^\sigma(\operatorname{div}_\Gamma; \Gamma)$, we deduce $\mathbf{j}^0 \in \mathbf{H}^r(\Gamma)$, $r := \min\{\sigma, s^*\}$, and, thus, $\varphi \in H^{r+1}(\Gamma)$. Then we can simply take the $\operatorname{curl}_\Gamma$ of the Γ_h -piecewise linear interpolant $I_h \varphi$ of φ to approximate \mathbf{j}^0 in \mathbf{N}_h . This is possible thanks

to a commuting diagram property connecting piecewise linear continuous functions on Γ_h and $\mathcal{RT}_0(\Gamma_h)$. Eventually we have

$$\|\mathbf{j}^0 - \mathbf{curl}_\Gamma(I_h \varphi)\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} = \|\varphi - I_h \varphi\|_{H^{\frac{1}{2}}(\Omega)} \leq Ch^{\min\{\frac{3}{2}, r + \frac{1}{2}\}} \|\mathbf{j}^0\|_{\mathbf{H}^r(\Gamma)} .$$

For general topology a stream function representation is no longer feasible and harmonic surface vectorfields have to be taken into account [16]. They disrupt the preceding simple argument. Hence, we have decided to pursue a different strategy, targeting \mathbf{j} directly.

We point out that plain interpolation error estimates based on affine equivalence techniques and a Bramble-Hilbert type result (cf. proof of lemma 6.2) leave us with the suboptimal estimate

$$\|\mathbf{j} - \Pi_h \mathbf{j}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq Ch^{\min\{1, \sigma\}} \|\mathbf{j}\|_{\mathbf{H}^\sigma(\text{div}_\Gamma; \Gamma)} .$$

An improvement is only possible, if we can lift interpolation off the boundary. This is by no means wishful thinking, because \mathbf{j} is the jump $[\gamma_t \mathbf{H}]_\Gamma$ of the tangential traces of magnetic field solutions of Maxwell boundary value problems in Ω and its complement $\Omega' := \mathbb{R}^3 \setminus \Omega$. If the source term (incident wave) has minimal smoothness, regularity theory for the solutions of Maxwell's equations in polyhedra [28] shows that

$$\mathbf{H} \in \mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; \Omega \cup \Omega') \quad , \quad \mathbf{curl} \mathbf{H} \in \mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; \Omega \cup \Omega') \quad , \quad (8.7)$$

for some $\sigma > 0$, where we defined

$$\mathbf{H}^\sigma(\mathbf{curl}; \Omega) := \{\mathbf{V} \in \mathbf{H}_{\text{loc}}^\sigma(\Omega), \mathbf{curl} \mathbf{V} \in \mathbf{H}_{\text{loc}}^\sigma(\Omega)\} .$$

Prerequisite to exploiting the information about \mathbf{H} is an ‘‘extension’’ of the surface triangulations.

DEFINITION 8.2. *We call the family of surface meshes $\{\Gamma_h\}_{h \in \mathbb{H}}$ extensible, if there is $R > 0$ such that Ω is contained in a cube B_R with diameter R , and a family of tetrahedral meshes $\{\Omega_h\}_{h \in \mathbb{H}}$ covering B_R that satisfy*

- $\{\Omega_h\}_h$ is shape-regular and quasi-uniform, and h retains its meaning as the meshwidth of Ω_h .
- Γ_h is composed of those simplices of Ω_h that are located on Γ .

Extensibility is not far-fetched, considering practical ways to obtain a family $\{\Gamma_h\}_h$: Whenever the meshes Γ_h are created by regular refinement of some coarse initial mesh, an extensible family will naturally emerge. This property makes it possible to switch to three-dimensional interpolation by means of Nédélec's edge elements [45] temporarily.

LEMMA 8.3. *If $\{\Gamma_h\}_{h \in \mathbb{H}}$ is extensible and $\mathbf{j} := \gamma_t' \mathbf{H}' - \gamma_t \mathbf{H}$ for magnetic fields $\mathbf{H} \in \mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; \Omega)$, $\mathbf{H}' \in \mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; B_R \setminus \Omega)$, $0 < \sigma$, then*

$$\|\mathbf{j} - \Pi_h \mathbf{j}\|_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq Ch^{\min\{1, \frac{1}{2} + \sigma\}} \left(\|\mathbf{H}\|_{\mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; \Omega)} + \|\mathbf{H}'\|_{\mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; B_R \setminus \Omega)} \right)$$

Proof. We are going to write Ω_h for the meshes obtained by extending Γ_h into Ω , and Ω_h' for the exterior extensions into $\Omega_R' := \Omega' \cap B_R$. Recall Nédélec's lowest order \mathbf{curl} -conforming elements on tetrahedral, the so-called edge elements. We refer to [13, 25, 45] for the definition of the local spaces. The resulting global spaces feature tangential continuity across inter-element faces and will be denoted by $\mathcal{ND}_1(\Omega_h)$ and $\mathcal{ND}_1(\Omega_h')$. Suitable degrees of freedom are given by path integrals along the edges of the triangulations. This defines nodal interpolation operators $\Theta_h : \mathbf{C}^\infty(\Omega) \mapsto \mathcal{ND}_1(\Omega_h)$ and $\Theta_h' : \mathbf{C}^\infty(\Omega_R') \mapsto \mathcal{ND}_1(\Omega_h')$. As explained in [36], there holds

$$\gamma_t \Theta_h \mathbf{H} = \Pi_h \gamma_t \mathbf{H} \quad , \quad \forall \mathbf{H} \in \mathbf{H}^{\frac{1}{2} + \sigma}(\mathbf{curl}; \Omega) . \quad (8.8)$$

Next, Lemma 4.7 from [4] teaches that edge element interpolation is well defined for vectorfields in $\mathbf{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega)$ and $\mathbf{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega'_R)$. Besides, a fundamental trace theorem for $\mathbf{H}(\mathbf{curl}; \Omega)$ (Theorem 4.1 of [20]) states that $\gamma_{\mathbf{t}} : \mathbf{H}(\mathbf{curl}; \Omega \cup \Omega'_R) \mapsto \mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)$, is continuous with a continuous right inverse. Combined with $\mathbf{j} = \gamma'_{\mathbf{t}} \mathbf{H}' - \gamma_{\mathbf{t}} \mathbf{H}$ and (8.8), this means

$$\begin{aligned} \|\mathbf{j} - \Pi_h \mathbf{j}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} &= \|\gamma'_{\mathbf{t}} \mathbf{H}' - \gamma_{\mathbf{t}} \mathbf{H} - (\gamma'_{\mathbf{t}} \Theta'_h \mathbf{H}' - \gamma_{\mathbf{t}} \Theta_h \mathbf{H})\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)} \\ &\leq \tilde{C} \left(\|\mathbf{H} - \Theta_h \mathbf{H}\|_{\mathbf{H}(\mathbf{curl}; \Omega)} + \|\mathbf{H} - \Theta'_h \mathbf{H}\|_{\mathbf{H}(\mathbf{curl}; \Omega'_R)} \right). \end{aligned}$$

The last step is based on an estimate for the interpolation error in edge element space [25, Lemma 3.2]

$$\|\mathbf{H} - \Theta_h \mathbf{H}\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \leq Ch^{\min\{1, \frac{1}{2}+\sigma\}} \|\mathbf{H}\|_{\mathbf{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega)}.$$

It is applied to both Ω and Ω'_R . \square

Recall that according to [20, Formula (29)]

$$\text{div}_{\Gamma}(\mathbf{j}) = \mathbf{curl} \mathbf{H} \cdot \mathbf{n} - \mathbf{curl} \mathbf{H}' \cdot \mathbf{n}.$$

Now, we appeal to (8.7) and standard trace theorems for Sobolev spaces, and see that $\text{div}_{\Gamma} \mathbf{j} \in H^{\sigma}(\Gamma)$. Thus, we can merge lemmas 8.1 and 8.3 into the final convergence result.

THEOREM 8.4. *Assume σ -regularity, $\sigma > 0$, according to (8.7) for the interior and exterior magnetic field solutions of Maxwell's equations subject to some excitation. The family of triangular surface meshes $\{\Gamma_h\}_{h \in \mathbb{H}}$ with meshwidths h is to be shape-regular, quasi-uniform, and extensible. Then there is $h_* > 0$ such that all $h < h_*$*

$$\begin{aligned} \|\mathbf{j} - \mathbf{j}_h\|_{\mathbf{H}^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)} &\leq C \left(h^{\min\{1, \frac{1}{2}+\sigma\}} \left(\|\mathbf{H}\|_{\mathbf{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega)} + \|\mathbf{H}'\|_{\mathbf{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega'_R)} \right) + \right. \\ &\quad \left. h^{\min\{1, \frac{1}{2}+\sigma, s^*\}} \left(\|\mathbf{curl} \mathbf{H}\|_{\mathbf{H}^{\frac{1}{2}+\sigma}(\Omega)} + \|\mathbf{curl} \mathbf{H}'\|_{\mathbf{H}^{\frac{1}{2}+\sigma}(\Omega'_R)} \right) \right), \end{aligned}$$

with $C > 0$ depending on Γ , ς , and the shape-regularity of the surface and volume meshes.

Remark. Barring awkward boundaries with tiny s^* , for $0 < \sigma \leq 1$ this estimate is optimal in the sense that the spread of Sobolev scales occurs as the exponent of h . In a sense, the result of Thm. 8.4 is far superior to the rates proven in [8], though the latter work assumes smooth boundaries.

Remark. Keep in mind that lowest order divergence-conforming boundary elements cannot give us more than first order convergence. Yet, lemmas 8.1 and 8.3 can be extended to k th order Raviart-Thomas boundary elements, $k \geq 0$, with little effort, yielding convergence of order $\min\{k + 1, \frac{1}{2} + \sigma, s^*\}$.

Remark. In [19] an equivalent mixed formulation of (1.1) is proposed that takes the variational problem to classical Sobolev spaces. In that setting duality techniques are available that also provide even better rates of asymptotic convergence, provided that s^* is sufficiently large.

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