Boundary element methods for Maxwell's equations on non-smooth domains

Report

Author(s):
Buffa, Annalisa; Costabel, Martin; Schwab, Christoph

Publication date:
2001-01

Permanent link:
https://doi.org/10.3929/ethz-a-004289387

Originally published in:
SAM Research Report 2001-01

Boundary element methods for Maxwell's equations on non-smooth domains *

A. Buffa[†], M. Costabel[‡] and C. Schwab

Research Report No. 2001-01
January 2001

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

*This work has been partially supported by I.A.N. - C.N.R. Pavia, Italy
[†]Dipartimento di Matematica, Università degli studi di Pavia, Italy
[‡]IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France

Boundary element methods for Maxwell's equations on non-smooth domains *

A. Buffa†, M. Costabel‡ and C. Schwab

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 2001-01 January 2001

Abstract

Variational boundary integral equations for Maxwell's equations on Lipschitz surfaces in $\mathbb{R}^3$ are derived and their well-posedness in the appropriate trace spaces is established. An equivalent, stable mixed reformulation of the system of integral equations is obtained which admits discretization by Galerkin boundary elements based on standard spaces. On polyhedral surfaces, quasioptimal asymptotic convergence of these Galerkin boundary element methods is proved. A sharp regularity result for the surface multipliers on polyhedral boundaries with plane faces is established.

*This work has been partially supported by I.A.N. - C.N.R. Pavia, Italy
†Dipartimento di Matematica, Università degli studi di Pavia, Italy
‡IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France

The numerical solution of boundary value problems arising in electromagnetics has received increasing attention in recent years. The equations to be solved are the time-harmonic Maxwell equations and derived simplified formulations such as the time-harmonic eddy current model. For the calculation of waves radiated e.g. from antenna or conductors, exterior boundary value problems in a homogeneous ambient medium must be solved. It has long been recognized (e.g. [8] and the references there) that in this case the boundary reduction of the problem to a system of Fredholm integral equations on the surface of the conductor is advantageous. In [8], several possible boundary reductions have been described and the mapping properties of the resulting boundary integral operators in Hölder spaces were established with emphasis on the classical integral equations of the second kind. Variational integral equation formulations of the first kind have also been considered in recent years. As an example, we consider a perfect conductor occupying the bounded domain $\Omega \subset \mathbb{R}^3$ which is externally irradiated by a plane wave $\mathbf{E}^{in} = \mathbf{d}e^{ik\mathbf{c}\cdot\mathbf{x}}$, with $|\mathbf{d}| = |\mathbf{c}| = 1$ and $\mathbf{c}\cdot\mathbf{d} = 0$. The scattered electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$, respectively, then solve the following equations in the exterior domain $\Omega_e := \mathbb{R}^3\backslash\overline{\Omega}$

$$\begin{cases} \mathbf{curl}\,\mathbf{E} - i\omega\mu\mathbf{H} = 0 & \text{in} \quad \Omega_e \\ \mathbf{curl}\,\mathbf{H} + i\varepsilon\omega\mathbf{E} = 0 & \text{in} \quad \Omega_e \\ \text{Silver-Müller radiation condition at } \infty & \\ \gamma_\tau(\mathbf{E}) = -\gamma_\tau(\mathbf{E}^{in}) & \text{at} \quad \Gamma. \end{cases}$$

Here, γ_τ denotes the tangential trace on the boundary $\Gamma = \partial\Omega$ which we assume to be smooth for now.

With the Stratton-Chu representation formula, we may represent $\mathbf{E}(\mathbf{x})$ in Ω_e in the form

$$\mathbf{E}(\mathbf{x}) = i\omega\mu \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathbf{j}(\mathbf{y})\,ds(\mathbf{y}) + \frac{i}{\varepsilon\omega}\nabla \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma(\mathbf{j})(\mathbf{y})\,ds(\mathbf{y})$$

where

$$\Phi(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi\,|\mathbf{x}-\mathbf{y}|},$$

denotes the Helmholtz fundamental solution with wave number $k = \omega\sqrt{\varepsilon\mu}$ associated to the frequency ω and where $\mathbf{j}$ is the jump of the total magnetic field across Γ. Inserting the representation into the boundary condition, we arrive at the following variational boundary integral equation of the first kind for the unknown tangential component $\mathbf{j}_\gamma$ (the jump in the normal component being zero) which is called the Rumsey Variational Principle: find $\mathbf{j}_\gamma \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ such that for all $\mathbf{j}^t_\gamma \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ there holds

$$i\omega\mu\,\langle\mathbf{j}^t_\gamma,\mathcal{V}\mathbf{j}_\gamma\rangle - \frac{i}{\varepsilon\omega}\langle\mathrm{div}_\Gamma(\mathbf{j}^t_\gamma),\mathcal{V}\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\rangle = -\langle\mathbf{j}^t_\gamma,\mathbf{f}\rangle.$$

Here, div_Γ denotes the surface divergence and $\mathcal{V}$ the single layer potential corresponding to the Helmholtz fundamental solution and $\langle\cdot,\cdot\rangle$ the $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ duality pairing and $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ denotes the set of tangential fields with weak surface divergence in $H^{-1/2}(\Gamma)$ (we refer to [19] for a detailed discussion of these spaces on smooth surfaces Γ).

A Galerkin discretization of this boundary integral equation by means of the simplest Raviart-Thomas finite elements on Γ is used in commercial codes. Its convergence analysis properties. A convergence proof for smooth surfaces was given by Bendali in [2]. The situation on polyhedra or general Lipschitz surfaces is considerably more complicated. In particular, the meaning of the dualities in the Rumsey principle are not clear on such surfaces and the original convergence proof in [2] does not apply immediately on nonsmooth surfaces.

Using different techniques from harmonic analysis, Mitrea et al. [17, 16] developed a theory of boundary integral equations for the time-harmonic Maxwell equations on Lipschitz domains. So far, these techniques do not give enough information for the analysis of variational formulations of the integral equations and hence for their numerical analysis. In particular, no Hodge decomposition on the boundary is obtained.

An approach equivalent to the one presented here is being investigated in the thesis [20]. For the case of smooth closed and open surfaces, wavelet bases in standard nodal spline spaces are considered that allow the analysis of fast algorithms for the numerical solution of our integral equations.

The purpose of the present paper is to justify the Rumsey variational principle on Lipschitz polyhedra and to derive a convergent boundary element discretization. Its outline is as follows: in Section 2, we present the functional framework for our analysis. We use in particular recent results from [4, 5, 6] on the trace spaces of Maxwell's equations to clarify the meaning of the dualities in the Rumsey principle and to prepare the principal tool for its analysis, namely the boundary Hodge decomposition. Section 3 justifies the Stratton-Chu representation formula on a Lipschitz surface Γ. Section 4 is then devoted to the derivation of the Rumsey principle, and to a mixed reformulation by means of a Hodge decomposition of $\mathbf{j}_\gamma$ which we prove to be strongly elliptic in the sense that it satisfies a Gårding-inequality on Γ. Finally, Section 5 is devoted to the analysis of Galerkin Boundary Element discretizations of our mixed reformulation of Rumsey's principle. We establish quasioptimal asymptotic convergence rates of the Galerkin discretization and give explicit and sharp bounds on the convergence rates in terms of the Maxwell singularities described in [10]. An additional complication arises since the Lagrange multiplier used for the weak formulation of the Laplace-Beltrami operator on Γ exhibits vertex singularities. We examine these singularities which may be of interest in their own right and show that in 'typical' situations they do not downgrade the asymptotic convergence rate.

2 Preliminaries

We shall make use of some recent results on the characterization of traces associated to Sobolev spaces of interest for Maxwell's equations. We present here a synopsis of these results and refer to [4], [5] and to [6] for details and proofs.

We begin by introducing some definitions and notations. We denote by $\mathcal{D}(\mathbb{R}^3)^3$ the space of the $3D$ vector fields with each component belonging to $C^\infty_{comp}(\mathbb{R}^3)$ and by $\mathcal{D}'(\mathbb{R}^3)^3$ the corresponding dual space. The duality is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{D}}$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$. We suppose that Ω is connected and simply connected, i.e., all its Betti numbers are zero. We denote by Γ its boundary and, thanks to the assumption on Ω, Γ is connected and simply connected also. Ω_e denotes the complementary $\mathbb{R}^3 \setminus \bar{\Omega}$, and $\mathbf{n}$ the outer unit normal vector to Ω. Moreover we denote by $H^s(\Omega)$, $H^s_{loc}(\Omega_e)$ $\forall s \in \mathbb{R}$ and $H^t(\Gamma)$, $\forall t \in [-1, 1]$ the standard (local in the case of the exterior domain) complex valued, Hilbertian Sobolev space defined on Ω, Ω_e and Γ respectively (with the convention $H^0 = L^2$.)

$$\mathbf{H}^s(\Omega) := \left(H^s(\Omega)\right)^3, V = \left(H^{\frac{1}{2}}(\Gamma)\right)^3, V' = \left(H^{-\frac{1}{2}}(\Gamma)\right)^3. \tag{1}$$
$$\mathbf{H}(\mathbf{curl}\,,\Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\}; \tag{2}$$
$$\mathbf{H}_{loc}(\mathbf{curl}\,,\Omega_e) = \{\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\Omega_e}) \mid \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2_{loc}(\overline{\Omega_e})\}; \tag{3}$$
$$\mathbf{H}(\mathbf{curl}\,,\mathbb{R}^3 \setminus \Gamma) = \mathbf{H}(\mathbf{curl}\,,\Omega) \cup \mathbf{H}_{loc}(\mathbf{curl}\,,\Omega_e) \tag{4}$$
$$\mathbf{L}^2_t(\Gamma) = \{\mathbf{v} \in \mathbf{L}^2(\Gamma) \mid \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma\}; \tag{5}$$
$$H^{-s}_\star(\Gamma) := \{u \in H^{-s}(\Gamma) \mid \langle u, 1\rangle_{s,\Gamma} = 0\} \quad (s \in [0,1]) \tag{6}$$
$$H^{\frac{3}{2}}(\Gamma) := \{u_{|\Gamma} \mid u \in H^2(\Omega)\}. \tag{7}$$

The space $\mathbf{L}^2_t(\Gamma)$ is identified with the space of fields belonging to the tangent bundle $T\Gamma$ of Γ for almost every $\mathbf{x} \in \Gamma$ and which are square integrable.

The space $H^{\frac{3}{2}}(\Gamma)$ has no intrinsic definition on the surface Γ. Nevertheless it is a Hilbert space endowed with the norm: $||\lambda||_{\frac{3}{2},\Gamma} := \inf_{u\in H^2(\Omega)}\{||u||_{2,\Omega} \text{ such that } u_{|\Gamma} = \lambda\}$. We denote by $H^{-\frac{3}{2}}(\Gamma)$ its dual space with $L^2(\Gamma)$ as pivot space. Finally, when Ω is a polyhedron this space can be characterized face by face. We refer to [4] for this characterization and to Section 2.2 for a brief presentation.

Definition 2.1 *The "tangential components trace" mapping $\pi_\tau : \mathcal{D}(\bar\Omega)^3 \to \boldsymbol{L}^2_t(\Gamma)$ and the "tangential trace" mapping $\gamma_\tau : \mathcal{D}(\bar\Omega)^3 \to \boldsymbol{L}^2_t(\Gamma)$ are defined as $\boldsymbol{u} \mapsto \boldsymbol{n} \wedge (\boldsymbol{u} \wedge \boldsymbol{n})_{|\Gamma}$ and $\boldsymbol{u} \mapsto \boldsymbol{u}_{|\Gamma} \wedge \boldsymbol{n}$, respectively.*

We denote by γ the standard trace operator acting on vectors: $\gamma : \mathbf{H}^1(\Omega) \to V$, $\gamma(\mathbf{u}) = \mathbf{u}_{|\Gamma}$. Let γ^{-1} be one of its right inverses. We will also use the notation π_τ (resp. γ_τ) for the composite operator $\pi_\tau \circ \gamma^{-1}$ (resp. $\gamma_\tau \circ \gamma^{-1}$) which acts only on traces. By density of $\mathcal{D}(\bar\Omega)^3_{|\Gamma}$ into $\mathbf{L}^2(\Gamma)$, the operators π_τ and γ_τ can be extended to linear continuous operators in $\mathbf{L}^2(\Gamma)$.

We define:

Definition 2.2 *Let $V_\gamma := \gamma_\tau(V)$ and $V_\pi := \pi_\tau(V)$.*

V_γ and V_π are Hilbert spaces endowed with norms that assure the continuity of the operators γ_τ and π_τ, respectively. We set:

$$||\boldsymbol{\lambda}||_{V_\gamma} = \inf_{\mathbf{u}\in V}\{||\mathbf{u}||_V \mid \quad \gamma_\tau(\mathbf{u}) = \boldsymbol{\lambda}\} \tag{8}$$
$$||\boldsymbol{\lambda}||_{V_\pi} = \inf_{\mathbf{u}\in V}\{||\mathbf{u}||_V \mid \quad \pi_\tau(\mathbf{u}) = \boldsymbol{\lambda}\} \tag{9}$$

Note that $\pi_\tau : V \to V_\pi$ and $\gamma_\tau : V \to V_\gamma$ are isomorphisms by construction. The spaces V_γ and V_π will be the bases of our construction. We denote by V'_γ and V'_π their dual spaces respectively with $\mathbf{L}^2_t(\Gamma)$ as pivot. V'_γ and V'_π are Hilbert spaces endowed with their natural norms.

Let $i_\pi : \mathbf{L}^2_t(\Gamma) \to \mathbf{L}^2(\Gamma)$ be the adjoint operator of π_τ. This operator is nothing but the identification of two-dimensional tangential vectors fields, sections of the tangent bundle $T\Gamma$ of Γ, with three-dimensional vector fields on Γ (with zero normal component). Thanks to the Lipschitz assumption, a local system of orthonormal coordinates $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \mathbf{n})$ can be defined at almost every $\mathbf{x} \in \Gamma$. Here $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are two orthonormal vectors belonging to the tangent plane for almost every $\mathbf{x} \in \Gamma$, while $\mathbf{n}$ is the outer normal to Ω. Of course, the vectors $\boldsymbol{\tau}_1$ and of clarity, we denote by $\tilde{\boldsymbol{\tau}}_1$ and $\tilde{\boldsymbol{\tau}}_2$ this basis of tangent fields. We have:

$$\mathbf{u} \in \mathbf{L}^2_t(\Gamma) \quad \mathbf{u} = u_1\tilde{\boldsymbol{\tau}}_1 + u_2\tilde{\boldsymbol{\tau}}_2 \qquad i_\pi(\mathbf{u}) = u_1\boldsymbol{\tau}_1 + u_2\boldsymbol{\tau}_2. \tag{10}$$

This operator can be extended in the following way:

$$i_\pi : V'_\pi \to \big(ker\{\pi_\tau\} \cap V\big)^\circ \subset V' \tag{11}$$

where $\cdot^\circ$ denotes the polar set. The following proposition obviously holds:

Proposition 2.3 *The operator $i_\pi : V'_\pi \to \big(ker\{\pi_\tau\} \cap V\big)^\circ$ is an isomorphism.*

A suitable characterization of the space $\big(ker\{\pi_\tau\} \cap V\big)^\circ$ can be found in [6].

2.1 Definition of tangential differential operators

In the following we need various differential operators defined on the surface Γ, a closed Lipschitz surface without boundary. The tangential functional spaces defined here above are suitable for their definition. The operators:

$$\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma), \qquad \mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$$

are defined on Γ in the usual way by a localization argument (see [18] or [6]). The adjoint operators of $-\nabla_\Gamma$ and $\mathbf{curl}_\Gamma$ are:

$$\mathrm{div}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}_\star(\Gamma), \qquad \mathrm{curl}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}_\star(\Gamma)$$

respectively, and they are linear and continuous for these choices of spaces. The operators ∇_Γ and curl_Γ can be restricted to more regular spaces. In [6] (see [4]-[5] for the case of polyhedra) the following operators are proved to be continuous:

$$\begin{array}{ll} \nabla_\Gamma : H^{\frac{3}{2}}(\Gamma) \to V_\pi & \nabla_\Gamma : H^{\frac{1}{2}}(\Gamma) \to V'_\gamma \\ \mathbf{curl}_\Gamma : H^{\frac{3}{2}}(\Gamma) \to V_\gamma & \mathbf{curl}_\Gamma : H^{\frac{1}{2}}(\Gamma) \to V'_\pi. \end{array} \tag{12}$$

Moreover they verify:

$$||p||_{H^{\frac{1}{2}}(\Gamma)/\mathbb{R}} \le C||\nabla_\Gamma p||_{V'_\gamma} \qquad ||p||_{H^{\frac{1}{2}}(\Gamma)/\mathbb{R}} \le C||\mathbf{curl}_\Gamma p||_{V'_\pi} \tag{13}$$

As a consequence, their adjoint operators $\mathrm{div}_\Gamma : V_\gamma \to H_\star^{-\frac{1}{2}}(\Gamma)$ and $\mathrm{curl}_\Gamma : V_\pi \to H_\star^{-\frac{1}{2}}(\Gamma)$ are linear continuous and surjective operators.

Finally, we define the Laplace-Beltrami operator on the Lipschitz manifold Γ as $\Delta_\Gamma u = \mathrm{div}_\Gamma(\nabla_\Gamma u)$ for any $u \in H^1(\Gamma)$. It is easy to see that $\Delta_\Gamma : H^1(\Gamma) \to H^{-1}_\star(\Gamma)$ is linear, continuous and admits a right inverse.

When Ω is a polyhedron, the spaces V_γ, V_π and $H^{\frac{3}{2}}(\Gamma)$ can be fully characterized. To this end, we introduce some notation. We denote by Γ_j, $j = 1,..,N_\Gamma$ the boundary faces of the polyhedron Ω and by $e_{ij} = \bar{\Gamma}_j \cap \bar{\Gamma}_i$ (for some i, j) the set of edges. Let $\boldsymbol{\tau}_{ij}$ be a unit vector parallel to e_{ij} and $\mathbf{n}_j = \mathbf{n}_{|\Gamma_j}$; $\boldsymbol{\tau}_i := \boldsymbol{\tau}_{ij} \wedge \mathbf{n}_j$. The couple $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij})$ is an orthonormal basis of the plane generated by Γ_i (resp. Γ_j); $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij}, \mathbf{n}_i)$ is an orthonormal basis of $\mathbb{R}^3$. Finally, we denote by $\mathcal{I}_j$ the set of indices i such that Γ_i shares an edge (namely e_{ij}) with Γ_j.

For any $\varphi \in L^2(\Gamma)$ we adopt the notation $\varphi_j = \varphi_{|\Gamma_j}$. This notation is used whenever the restriction to a face is considered, that is as regards to any functional space in which the restriction to a face is meaningful.

We set $\mathbf{H}_-^{\frac{1}{2}}(\Gamma) := \{\boldsymbol{\varphi} \in \mathbf{L}_t^2(\Gamma) \text{ such that } \boldsymbol{\varphi}_j \in H^{\frac{1}{2}}(\Gamma)^2\}$. For any $\boldsymbol{\varphi} \in \mathbf{H}_-^{\frac{1}{2}}(\Gamma)$, we define:

$$\mathcal{N}_{ij}^{||}(\boldsymbol{\varphi}) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\boldsymbol{\varphi}_i(\mathbf{x})\cdot\boldsymbol{\tau}_{ij} - \boldsymbol{\varphi}_j(\mathbf{y})\cdot\boldsymbol{\tau}_{ij}|^2}{||\mathbf{x}-\mathbf{y}||^3}\, d\sigma(\mathbf{x})d\sigma(\mathbf{y}) \quad \forall i \in \mathcal{I}_j\ \forall j$$
$$\mathcal{N}_{ij}^{\perp}(\boldsymbol{\varphi}) := \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\boldsymbol{\varphi}_i(\mathbf{x})\cdot\boldsymbol{\tau}_{i} - \boldsymbol{\varphi}_j(\mathbf{y})\cdot\boldsymbol{\tau}_{j}|^2}{||\mathbf{x}-\mathbf{y}||^3}\, d\sigma(\mathbf{x})d\sigma(\mathbf{y}) \quad \forall i \in \mathcal{I}_j\ \forall j$$

and we adopt the notation $\boldsymbol{\varphi}_i \cdot \boldsymbol{\tau}_{ij} \stackrel{\frac{1}{2}}{=} \boldsymbol{\varphi}_j \cdot \boldsymbol{\tau}_{ij}$ at e_{ij}, $i \in \mathcal{I}_j$ (resp. $\boldsymbol{\varphi}_i \cdot \boldsymbol{\tau}_i \stackrel{\frac{1}{2}}{=} \boldsymbol{\varphi}_j \cdot \boldsymbol{\tau}_j$ at e_{ij}) if and only if $\mathcal{N}_{ij}^{||}(\boldsymbol{\varphi})$ (resp. $\mathcal{N}_{ij}^{\perp}(\boldsymbol{\varphi})$) is finite.

The proof of the following lemma can be found in [5].

Lemma 2.4 *Let Ω be a polyhedron. The spaces V_π and V_γ can be characterized in the following way:*

$$\begin{aligned} V_\pi \equiv \boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma) &:= \left\{\boldsymbol{\psi} \in \boldsymbol{H}_-^{\frac{1}{2}}(\Gamma) \mid \boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_{ij} \stackrel{\frac{1}{2}}{=} \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_{ij} \text{ at } e_{ij}\ \forall i \in \mathcal{I}_j,\ \forall j\right\}. \\ V_\gamma \equiv \boldsymbol{H}_{\perp}^{\frac{1}{2}}(\Gamma) &:= \left\{\boldsymbol{\psi} \in \boldsymbol{H}_-^{\frac{1}{2}}(\Gamma) \mid \boldsymbol{\psi}_i \cdot \boldsymbol{\tau}_{i} \stackrel{\frac{1}{2}}{=} \boldsymbol{\psi}_j \cdot \boldsymbol{\tau}_{j} \text{ at } e_{ij}\ \forall i \in \mathcal{I}_j,\ \forall j\right\} \end{aligned} \tag{14}$$

The norms

$$||\boldsymbol{\psi}||^2_{||,\frac{1}{2},\Gamma} := \sum_{j=1}^{N} ||\boldsymbol{\psi}_j||^2_{\frac{1}{2},\Gamma_j} + \sum_{j=1}^{N}\sum_{i\in\mathcal{I}_j} \mathcal{N}_{ij}^{||}(\boldsymbol{\psi}). \tag{15}$$

$$||\boldsymbol{\psi}||^2_{\perp,\frac{1}{2},\Gamma} := \sum_{j=1}^{N} ||\boldsymbol{\psi}_j||^2_{\frac{1}{2},\Gamma_j} + \sum_{j=1}^{N}\sum_{i\in\mathcal{I}_j} \mathcal{N}_{ij}^{\perp}(\boldsymbol{\psi}). \tag{16}$$

are equivalent to (8) and (9) respectively.

In Section 5.2, we shall make use also of more "regular" spaces that we define here for convenience. For any $t > 1$, we define the space:

$$H^t(\Gamma) = \{u \in H^1(\Gamma) \mid u_j \in H^t(\Gamma_j)\} \tag{17}$$

endowed with its natural norm $\|u\|_{t,\Gamma} := \left(\|u\|^2_{1,\Gamma} + \sum_{j=1}^{N_\Gamma} \|u_j\|^2_{t,\Gamma_j}\right)^{\frac{1}{2}}$. We define:

$$\begin{aligned}
\mathbf{H}^s_-(\Gamma) &= \{\varphi \in \mathbf{L}^2_t(\Gamma) \mid \varphi_j \in H^s(\Gamma_j)^2\} \quad (s \geq 0) \ ; \\
\mathbf{H}^s_{\|}(\Gamma) &= \{\varphi \in \mathbf{H}^s_-(\Gamma) \mid \varphi_i \cdot \boldsymbol{\tau}_{ij} = \varphi_j \cdot \boldsymbol{\tau}_{ij} \text{ at } e_{ij}\} \quad (s > \tfrac{1}{2}) \ ; \\
\mathbf{H}^s_{\perp}(\Gamma) &= \{\varphi \in \mathbf{H}^s_-(\Gamma) \mid \varphi_i \cdot \boldsymbol{\tau}_{i} = \varphi_j \cdot \boldsymbol{\tau}_{i} \text{ at } e_{ij}\} \quad (s > \tfrac{1}{2}) \ .
\end{aligned} \tag{18}$$

The space $\mathbf{H}^s_-(\Gamma)$ is endowed with its natural norm $\|\varphi\|_{s,-,\Gamma} := \left(\sum_{j=1}^{N_\Gamma} \|\varphi_j\|^2_{s,\Gamma_j}\right)^{\frac{1}{2}}$. The spaces $\mathbf{H}^s_{\|}(\Gamma)$ and $\mathbf{H}^s_{\perp}(\Gamma)$ are closed subspaces of $\mathbf{H}^s_-(\Gamma)$ for any $s > \frac{1}{2}$. Finally it is easy to see that, for any $s \geq \frac{1}{2}$, the operators

$$\nabla_\Gamma : H^{s+1}(\Gamma) \to \mathbf{H}^s_{\|}(\Gamma) \ ; \qquad \mathbf{curl}_\Gamma : H^{s+1}(\Gamma) \to \mathbf{H}^s_{\perp}(\Gamma)$$

are linear and continuous. Moreover, it holds:

$$\|p\|_{s+1,\Gamma} \leq \|\nabla_\Gamma p\|_{s,-,\Gamma} \qquad \|p\|_{s+1,\Gamma} \leq \|\mathbf{curl}_\Gamma p\|_{s,-,\Gamma} \quad \forall p \in H^{s+1}(\Gamma)/\mathbb{R}. \tag{19}$$

Remark 2.5 *The inequalities (19) correspond to (13), but they hold true for a wider range of indices. Moreover, the definition (17) seems natural for polyhedra, but cannot be extended to the general case of Lipschitz surfaces. In particular, in the case $s = 3/2$, in [4] it is shown that the two definitions (7) and (17) give the same space both algebraically and topologically.*

2.3 Traces of $\mathbf{H}(\mathbf{curl}, \Omega)$

We are now in the position to introduce the spaces of interest in the characterization of the space of tangential traces and tangential components for vector fields in $\mathbf{H}(\mathbf{curl}, \Omega)$, or analogously in $\mathbf{H}_{loc}(\mathbf{curl}, \Omega_e)$. Let

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \ := \ \{\boldsymbol{\lambda} \in {V_\pi}' \mid \quad \mathrm{div}_\Gamma(\boldsymbol{\lambda}) \in H^{-\frac{1}{2}}(\Gamma)\} \tag{20}$$

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \ := \ \{\boldsymbol{\lambda} \in {V_\gamma}' \mid \quad \mathrm{curl}_\Gamma(\boldsymbol{\lambda}) \in H^{-\frac{1}{2}}(\Gamma)\}. \tag{21}$$

They are Hilbert spaces endowed with the induced graph norms, e.g.,

$$||\mathbf{u}||_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} := ||\mathbf{u}||_{V'_\pi} + ||\mathrm{div}_\Gamma(\mathbf{u})||_{-\frac{1}{2},\Gamma}. \tag{22}$$

The following theorem holds true. The proof can be found in [6] (see also [5] for the case of polyhedra):

Theorem 2.6 *The operators π_τ and γ_τ can be extended to linear continuous operators acting on $\boldsymbol{H}(\mathbf{curl}, \Omega)$. Namely, $\pi_\tau : \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ and $\gamma_\tau : \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ are linear continuous and surjective. Defining*

$$T := \{\boldsymbol{\xi} \in V' \mid \exists \eta \in H^{-\frac{1}{2}}(\Gamma) \ : \ \forall \phi \in H^2(\Omega) \ : \qquad {}_{V'}\langle \boldsymbol{\xi}, \gamma(\nabla\phi)\rangle_V = \langle \eta, \gamma\phi\rangle_{\frac{1}{2},\Gamma}\}, \tag{23}$$

the isomorphism i_π verifies:

$$i_\pi\left(\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)\right) \equiv T$$

Finally, the spaces $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ verify the following:

$$\mathcal{H}(\Gamma) := \{\alpha \in H^1(\Gamma) \text{ such that } \Delta_\Gamma \alpha \in H^{-\frac{1}{2}}(\Gamma)\}. \tag{24}$$

The following Hodge decompositions hold:

$$\begin{aligned} \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) &= \nabla_\Gamma \mathcal{H}(\Gamma) \oplus \mathbf{curl}_\Gamma H^{\frac{1}{2}}(\Gamma) \\ \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) &= \mathbf{curl}_\Gamma \mathcal{H}(\Gamma) \oplus \nabla_\Gamma H^{\frac{1}{2}}(\Gamma). \end{aligned} \tag{25}$$

Moreover these spaces can be put in duality. Let $\boldsymbol{u} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\boldsymbol{v} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ *such that* $\boldsymbol{u} = \nabla_\Gamma \alpha_{\boldsymbol{u}} + \mathbf{curl}_\Gamma \beta_{\boldsymbol{u}}$ *and* $\boldsymbol{v} = \nabla_\Gamma \alpha_{\boldsymbol{v}} + \mathbf{curl}_\Gamma \beta_{\boldsymbol{v}}$*; we define*

$$_\gamma\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\pi := -\langle \Delta_\Gamma \alpha_{\boldsymbol{u}}, \alpha_{\boldsymbol{v}}\rangle_{\frac{1}{2},\Gamma} + \langle \Delta_\Gamma \beta_{\boldsymbol{v}}, \beta_{\boldsymbol{u}}\rangle_{\frac{1}{2},\Gamma}. \tag{26}$$

The following integration by parts formula holds true:

$$\int_\Omega \{\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}\} = {}_\gamma\langle \gamma_\tau(\boldsymbol{v}), \pi_\tau(\boldsymbol{u})\rangle_\pi. \tag{27}$$

The proof of this theorem can be found in [5] in the case of Lipschitz polyhedra and in [6] in the case of general Lipschitz domains.

In order to give a precise meaning to the objects that will be introduced in the jump relations and integral representations of the next section, we need to use a different notation for elements in T (three-dimensional vectors on the surface Γ) and the elements in $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. If $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$, we denote by $\gamma_\tau(\mathbf{u}) \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ the tangential trace interpreted as two-dimensional vector fields. We adopt the notation $\mathbf{u} \wedge \mathbf{n} = i_\pi(\gamma_\tau(\mathbf{u}))$. By construction, $\mathbf{u} \wedge \mathbf{n} \in T$, and it is a three dimensional vector field on the surface Γ. Note that for general Lipschitz surfaces, the space T can indeed consist of general three-dimensional fields, see [6] Section 5. In the case of polyhedral manifolds, loosely speaking, the vectors in T have, in general, a third non-zero component at edges and vertices.

3 Representation Formula

Let $\mathbf{E},\ \mathbf{H} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$ be such that $(\mathbf{E}_i, \mathbf{H}_i) = (\mathbf{E}_{|\Omega}, \mathbf{H}_{|\Omega})$ and $(\mathbf{E}_e, \mathbf{H}_e) = (\mathbf{E}_{|\Omega_e}, \mathbf{H}_{|\Omega_e})$ are solutions of the interior and exterior Maxwell problem, respectively:

$$\begin{cases} \mathbf{curl}\, \mathbf{E} - i\omega\mu \mathbf{H} = 0 \\ \mathbf{curl}\, \mathbf{H} + i\varepsilon\omega \mathbf{E} = 0 \end{cases} \text{in } \Omega; \quad \begin{cases} \mathbf{curl}\, \mathbf{E} - i\omega\mu \mathbf{H} = 0 \\ \mathbf{curl}\, \mathbf{H} + i\varepsilon\omega \mathbf{E} = 0 \\ \text{Silver-Müller radiation condition at } \infty \end{cases} \text{in } \Omega_e \tag{28}$$

We set $\mathbf{j} = \mathbf{n} \wedge \mathbf{H}_i - \mathbf{n} \wedge \mathbf{H}_e$ and $\mathbf{m} = \mathbf{n} \wedge \mathbf{E}_i - \mathbf{n} \wedge \mathbf{E}_e$ and we set $\mathbf{j}_\gamma := i_\pi^{-1}(\mathbf{j})$ and $\mathbf{m}_\gamma := i_\pi^{-1}(\mathbf{m})$, $\mathbf{j}_\gamma,\ \mathbf{m}_\gamma \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. We would like to express then the whole fields $\mathbf{E}$ and $\mathbf{H}$ in terms of their "jumps" $\mathbf{j}_\gamma$ and $\mathbf{m}_\gamma$, or $\mathbf{j}$ and $\mathbf{m}$, across Γ. In order to do that some preliminary results and definitions are needed.

Lemma 3.1 ***(Jump relation)*** *Let* $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)$ *and set* $[\boldsymbol{n} \wedge \boldsymbol{u}] = \boldsymbol{n} \wedge \boldsymbol{u}_i - \boldsymbol{n} \wedge \boldsymbol{u}_e$*. We denote by* $\mathbf{curl}\, \boldsymbol{u}$ *the* $\mathbf{curl}$ *of* $\boldsymbol{u}$ *in the sense of distributions in* $\mathbb{R}^3$*, and we set*

$$(\mathbf{curl}\, \boldsymbol{u}) = \begin{cases} \mathbf{curl}\,(\boldsymbol{u}_i) & \text{in } \Omega \\ \mathbf{curl}\,(\boldsymbol{u}_e) & \text{in } \Omega_e \end{cases}$$

$$\mathbf{curl}\,\boldsymbol{u} = (\mathbf{curl}\,\boldsymbol{u}) - \big[\boldsymbol{n}\wedge\boldsymbol{u}\big]\,\delta_\Gamma \tag{29}$$

where $\langle\big[\boldsymbol{n}\wedge\boldsymbol{u}\big]\,\delta_\Gamma, \boldsymbol{v}\rangle_{\mathcal{D}} = {}_{V'}\langle\big[\boldsymbol{n}\wedge\boldsymbol{u}\big], \boldsymbol{v}_{|\Gamma}\rangle_V$.

Proof: This proof is standard (see, e.g., [7]) on regular surfaces. We report here the (short) proof, only with the aim of showing that it holds even on Lipschitz surfaces. The integration by parts formulas

$$\begin{aligned}
&\int_\Omega \mathbf{u}_i\cdot\mathbf{curl}\,\mathbf{v} - \int_\Gamma \mathbf{v}\cdot\mathbf{curl}\,\mathbf{u}_i = {}_{V'}\langle\mathbf{u}_i\wedge\mathbf{n}, \mathbf{v}_{|\Gamma}\rangle_V,\\
&\int_\Omega \mathbf{u}_e\cdot\mathbf{curl}\,\mathbf{v} - \int_\Gamma \mathbf{v}\cdot\mathbf{curl}\,\mathbf{u}_e = -\,{}_{V'}\langle\mathbf{u}_e\wedge\mathbf{n}, \mathbf{v}_{|\Gamma}\rangle_V
\end{aligned} \tag{30}$$

hold true for any Lipschitz domain $\Omega\subseteq\mathbb{R}^3$, $\mathbf{u}_i\in\mathbf{H}(\mathbf{curl}\,,\Omega)$, $\mathbf{u}_e\in\mathbf{H}_{loc}(\mathbf{curl}\,,\Omega_e)$, $\mathbf{v}\in\mathcal{D}(\mathbb{R}^3)^3$ (see e.g., [12]).

For any $\mathbf{v}\in\mathcal{D}(\mathbb{R}^3)^3$, using the integrations by parts (30) and the definition of the jumps $[\cdot]$, we obtain:

$$\langle\mathbf{curl}\,\mathbf{u}, \mathbf{v}\rangle_{\mathcal{D}} = \int_{\mathbb{R}^3}(\mathbf{curl}\,\mathbf{u})\mathbf{v} - {}_{V'}\langle\big[\mathbf{n}\wedge\mathbf{u}\big], \mathbf{v}_{|\Gamma}\rangle_V.$$

Finally, using the definition of $[\mathbf{n}\wedge\mathbf{u}]\delta_\Gamma$,we see that this is just (29), and the proof is achieved. □

Let now $k=\omega\sqrt{\varepsilon\mu}$ be the wave number associated to the frequency ω and

$$\Phi(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi\,|\mathbf{x}-\mathbf{y}|}, \quad \Phi_0(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|} \tag{31}$$

be the fundamental solutions associated to the scalar Helmholtz equation in $\mathbb{R}^3$ with wave number k (i.e. to $(\Delta+k)\Phi=\delta$) and to the Laplace operator, respectively.

We are now in the position to prove the validity of the so-called *Stratton-Chu* representation formula for non–smooth domains:

Theorem 3.2 *Let* $\boldsymbol{E},\boldsymbol{H}\in\boldsymbol{H}_{loc}(\mathbf{curl}\,,\mathbb{R}^3\setminus\Gamma)$ *be the unique solution of the system (28) with assigned jumps:* $\boldsymbol{j}=\boldsymbol{n}\wedge\boldsymbol{H}_i-\boldsymbol{n}\wedge\boldsymbol{H}_e$ *and* $\boldsymbol{m}=\boldsymbol{n}\wedge\boldsymbol{E}_i-\boldsymbol{n}\wedge\boldsymbol{E}_e$*. As before,* $\boldsymbol{j}=i_\pi(\boldsymbol{j}_\gamma)$, $\boldsymbol{m}=i_\pi(\boldsymbol{m}_\gamma)$, *with* $\boldsymbol{j}_\gamma$, $\boldsymbol{m}_\gamma\in\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$*.* $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *can be formally represented in the following way for almost every* $\boldsymbol{x}\in\Omega_e\cup\Omega$*:*

$$\begin{aligned}
\boldsymbol{E}(\boldsymbol{x}) = {} & i\omega\mu\textstyle\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{j}(\boldsymbol{y})\,ds(\boldsymbol{y}) + \dfrac{i}{\varepsilon\omega}\nabla\textstyle\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\mathrm{div}_\Gamma(\boldsymbol{j}_\gamma)(\boldsymbol{y})\,ds(\boldsymbol{y})+\\
& \mathbf{curl}\textstyle\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{m}(\boldsymbol{y})\,ds(\boldsymbol{y})\\
\boldsymbol{H}(\boldsymbol{x}) = {} & -i\omega\varepsilon\textstyle\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{m}(\boldsymbol{y})\,ds(\boldsymbol{y}) - \dfrac{i}{\mu\omega}\nabla\textstyle\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\mathrm{div}_\Gamma(\boldsymbol{m}_\gamma)(\boldsymbol{y})\,ds(\boldsymbol{y})+\\
& \mathbf{curl}\textstyle\int_\Gamma\Phi(\boldsymbol{x},\boldsymbol{y})\boldsymbol{j}(\boldsymbol{y})\,ds(\boldsymbol{y})
\end{aligned} \tag{32}$$

is a direct consequence of the surjectivity of the trace operators stated in Theorem 2.6 and standard results in functional analysis.

Based on Theorems 2.6, 4.2 and Lemma 3.1, we can prove (32) in the usual way, see e.g., [19], or [8].

Due to the the jump relation (29), $(\mathbf{E},\mathbf{H})$ verifies the following system of equations in the sense of distributions:

$$\mathbf{curl}\,\mathbf{E} - i\omega\mu\mathbf{H} = \mathbf{m}\delta_\Gamma \qquad \mathbf{curl}\,\mathbf{H} + i\omega\varepsilon\mathbf{E} = \mathbf{j}\delta_\Gamma$$

where $\langle \mathbf{m}\delta_\Gamma, \mathbf{v}\rangle_{\mathcal{D}} = {}_{V'}\langle \mathbf{m}, \mathbf{v}_{|\Gamma}\rangle_V$. We first prove the following equality

$$\mathrm{div}\,(\mathbf{j}\,\delta_\Gamma) = \mathrm{div}_\Gamma(\mathbf{j}_\gamma)\delta_\Gamma \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^3). \tag{33}$$

Actually, recalling that $\mathbf{j} = i_\pi(\mathbf{j}_\gamma)$ with $\mathbf{j}_\gamma \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, for any $\boldsymbol{\xi} \in \mathcal{D}(\mathbb{R}^3)^3$

$$\langle \mathrm{div}\,(\mathbf{j}\,\delta_\Gamma), \xi\rangle_{\mathcal{D}} = -\,{}_{V'}\langle i_\pi(\mathbf{j}_\gamma), \nabla\xi_{|\Gamma}\rangle_V = -{}_{V'_\pi}\langle \mathbf{j}_\gamma, \nabla_\Gamma\xi\rangle_{V_\pi} = \langle \mathrm{div}_\Gamma(\mathbf{j}_\gamma), \xi\rangle_{\frac{1}{2},\Gamma}.$$

We concentrate now on the integral representation of the electric field since the one for the magnetic field is analogous. We set first $\mathbf{m} = 0$ and we use the following Hodge decomposition in $\mathbb{R}^3$

$$\mathbf{E} = \mathbf{A} + \nabla W \quad \text{with} \quad \mathrm{div}\,(\mathbf{A}) - k^2 W = 0.$$

By standard manipulation and using (33), we obtain that the scalar potential W verifies $\Delta W + k^2\,W = -\dfrac{i}{\omega\varepsilon}\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\delta_\Gamma$ which implies the following:

$$W(\mathbf{x}) = \frac{i}{\omega\varepsilon}\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\,ds(\mathbf{y}). \tag{34}$$

On the other side, the vector potential $\mathbf{A}$ verifies $\Delta\mathbf{A} + k^2\mathbf{A} = -i\omega\mu\mathbf{j}\delta_\Gamma$ and it admits then the following representation:

$$\mathbf{A}(\mathbf{x}) = i\omega\mu\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathbf{j}(\mathbf{y})\,ds(\mathbf{y}).$$

Now, using the equation (28), by symmetry with the magnetic field, we deduce that the part of the electric field depending on $\mathbf{m}$ reads

$$\mathbf{curl}\int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathbf{m}(\mathbf{y})\,ds(\mathbf{y}). \tag{35}$$

□

4 The perfect conductor

We consider the scattering problem associated to Maxwell's equations when the scatterer Ω is a perfectly conducting body with Lipschitz boundary.

For any $\mathbf{u} \in C^0(\Gamma)^3$, we denote the (vector) single layer potential by:

$$\mathcal{S}\mathbf{u}(\mathbf{x}) = \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{x})ds(\mathbf{y})$$

and we set $\mathcal{V}\mathbf{u} = \gamma(\mathcal{S}\mathbf{u})$. The same operators corresponding to $k = 0$ are denoted by $\mathcal{S}_0$ and $\mathcal{V}_0$, respectively. We report some properties of these operators which will be useful in the sequel. The following result was stated and proved in [9], for example:

Proposition 4.1 *The operators*

$$\mathcal{S} : \left(H^{-\frac{1}{2}+\sigma}(\Gamma)\right)^3 \to \boldsymbol{H}^{1+\sigma}_{loc}(\Omega)\ , \qquad \mathcal{V} : \left(H^{-\frac{1}{2}+\sigma}(\Gamma)\right)^3 \to \left(H^{\frac{1}{2}+\sigma}(\Gamma)\right)^3 \tag{36}$$

are linear and continuous for any $\sigma \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Moreover, it holds:

$$\exists \alpha > 0 \ : \ {}_{V'}\langle \boldsymbol{u}, \mathcal{V}_0 \boldsymbol{u}\rangle_V \geq \alpha ||\boldsymbol{u}||^2_{V'}\ , \qquad \forall \boldsymbol{u} \in V'.$$

We need to study the coercivity property of the single layer potential when acting on tangential traces. We prove the following proposition:

Theorem 4.2 *The operators $\mathcal{S}$ and $\mathcal{V}$ act on vectors $\boldsymbol{\lambda} \in V'_\pi$ according to:*

$$\mathcal{S}(\boldsymbol{\lambda}) = \mathcal{S}(i_\pi(\boldsymbol{\lambda}))\ , \qquad \mathcal{V}(\boldsymbol{\lambda}) = \gamma(\mathcal{S}(\boldsymbol{\lambda})). \tag{37}$$

Correspondingly, the operator $\mathcal{V}_0 : V'_\pi \to V$ is linear and continuous and it verifies:

$$\forall \boldsymbol{\lambda} \in V'_\pi : \quad {}_{V'_\pi}\langle \boldsymbol{\lambda}, \pi_\tau(\mathcal{V}_0\boldsymbol{\lambda})\rangle_{V_\pi} \geq C||\boldsymbol{\lambda}||^2_{V'_\pi}. \tag{38}$$

Proof: For any $\boldsymbol{\lambda} \in \mathbf{L}^2_t(\Gamma)$, the operator i_π is defined by (10) and it is then obvious that $\mathcal{S}(\boldsymbol{\lambda}) = \mathcal{S}(i_\pi(\boldsymbol{\lambda}))$. Since V_π is dense in $\mathbf{L}^2_t(\Gamma)$, the equality (37) holds true. Using (36) and Proposition 2.3, we immediately deduce that also

$${}_{V'_\pi}\langle \boldsymbol{\lambda}, \pi_\tau\mathcal{V}_0(\boldsymbol{\lambda})\rangle_{V_\pi} = {}_{V'}\ \langle i_\pi(\boldsymbol{\lambda}), \mathcal{V}_0(\boldsymbol{\lambda})\rangle_V \geq C||i_\pi(\boldsymbol{\lambda})||^2_{V'} \geq C'||\boldsymbol{\lambda}||^2_{V'_\pi}. \tag{39}$$

□

Remark 4.3 *Theorem 4.2 allows to replace the vectors $\boldsymbol{j}$ and $\boldsymbol{m}$ by the corresponding $\boldsymbol{j}_\gamma = i_\pi^{-1}(\boldsymbol{j})$, $\boldsymbol{m}_\gamma = i_\pi^{-1}(\boldsymbol{m})$ in the integrals appearing in (32).*

4.2 Boundary reduction

The conductor Ω is irradiated by an external source which is as usual assumed to be a plane wave $\mathbf{E}^{in} = \mathbf{d}e^{ik\mathbf{c}\cdot\mathbf{x}}$, with $|\mathbf{d}| = |\mathbf{c}| = 1$ and $\mathbf{c}\cdot\mathbf{d} = 0$. The scattered fields $\mathbf{E}$ and $\mathbf{H}$ solve the following equations in the exterior domain:

$$\begin{cases} \mathbf{curl}\,\mathbf{E} - i\omega\mu\mathbf{H} = 0 & \text{in} \quad \Omega_e \\ \mathbf{curl}\,\mathbf{H} + i\varepsilon\omega\mathbf{E} = 0 & \text{in} \quad \Omega_e \\ \text{Silver-Müller radiation condition at } \infty & \\ \gamma_\tau(\mathbf{E}) = -\gamma_\tau(\mathbf{E}^{in}) & \text{at} \quad \Gamma. \end{cases} \tag{40}$$

conductor and the total electromagnetic field inside must be equal to zero.

As a consequence, the jump at the interface Γ of the electric field is equal to zero, while the jump $\mathbf{j}$ of the magnetic fields turns out to be equal to the tangential component of the total magnetic field:

$$\mathbf{j} = -\mathbf{n} \wedge \mathbf{H} - \mathbf{n} \wedge \mathbf{H}^{in} = -\mathbf{n} \wedge \mathbf{H}^{tot}$$

By using the Stratton-Chu representation formula (32) and (37), we obtain (recall that $i_\pi(\mathbf{j}_\gamma) = \mathbf{j}$):

$$\mathbf{E}(\mathbf{x}) = i\omega\mu \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathbf{j}_\gamma(\mathbf{y})\, ds(\mathbf{y}) + \frac{i}{\varepsilon\omega}\nabla \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\mathrm{div}_\Gamma(\mathbf{j}_\gamma)(\mathbf{y})\, ds(\mathbf{y}). \tag{41}$$

In the remainder of this section, we deduce a boundary integral equation from (41) and we prove that it is uniquely solvable under suitable conditions on the frequency. We set $\mathbf{f} := \pi_\tau(\mathbf{E}^{in})$, $\mathbf{f} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$.

Using standard continuity properties across the interface Γ of the single layer potential, and multiplying by a test function $\mathbf{j}_\gamma^t \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, we obtain:

$${}_\gamma\langle \mathbf{j}_\gamma^t, \pi_\tau(\mathbf{E})\rangle_\pi = i\omega\mu\ {}_\gamma\langle \mathbf{j}_\gamma^t, \pi_\tau \mathcal{V}\mathbf{j}_\gamma\rangle_\pi + \frac{i}{\varepsilon\omega}\, {}_\gamma\langle \mathbf{j}_\gamma^t, \nabla_\Gamma\big(\mathcal{V}\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\big)\rangle_\pi \tag{42}$$

where the duality ${}_\gamma\langle\cdot,\cdot\rangle_\pi$ is the one defined in Theorem 2.7. Using now that $\mathrm{div}_\Gamma(\mathbf{j}_\gamma^t) \in H^{-\frac{1}{2}}(\Gamma)$, the definition of the divergence operator and the fact that $\pi_\tau(\mathbf{E}) = -\pi_\tau(\mathbf{E}^{in}) = -\mathbf{f}$, we easily obtain the following variational BIE, sometimes also referred to as Rumsey Variational Principle: find $\mathbf{j}_\gamma \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ such that for all $\mathbf{j}_\gamma^t \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ holds

$$i\omega\mu\ {}_\gamma\langle \mathbf{j}_\gamma^t, \pi_\tau\mathcal{V}\mathbf{j}_\gamma\rangle_\pi - \frac{i}{\varepsilon\omega}\langle \mathrm{div}_\Gamma(\mathbf{j}_\gamma^t), \mathcal{V}\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\rangle_{\frac{1}{2},\Gamma} = -{}_\gamma\langle \mathbf{j}_\gamma^t, \mathbf{f}\rangle_\pi. \tag{43}$$

Finally, we introduce some nomenclature that we shall use in the next sections. We set:

$$B(\mathbf{j}_\gamma, \mathbf{j}_\gamma^t) := i\omega\mu\ {}_\gamma\langle \mathbf{j}_\gamma^t, \pi_\tau\mathcal{V}\mathbf{j}_\gamma\rangle_\pi, \quad C(\mathrm{div}_\Gamma(\mathbf{j}_\gamma), \mathrm{div}_\Gamma(\mathbf{j}_\gamma^t)) := \frac{i}{\varepsilon\omega}\langle \mathrm{div}_\Gamma(\mathbf{j}_\gamma^t), \mathcal{V}\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\rangle_{\frac{1}{2},\Gamma}. \tag{44}$$

We write:

$$B(\cdot,\cdot) = B_0(\cdot,\cdot) + (B(\cdot,\cdot) - B_0(\cdot,\cdot)), \quad C(\cdot,\cdot) = C_0(\cdot,\cdot) + (C(\cdot,\cdot) - C_0(\cdot,\cdot)),$$

where B_0 and C_0 are the principal parts of the bilinear forms $B(\cdot,\cdot)$ and $C(\cdot,\cdot)$ which are given, respectively, by

$$B_0(\mathbf{j}_\gamma, \mathbf{j}_\gamma^t) := i\omega\mu_\gamma\langle \mathbf{j}_\gamma^t, \pi_\tau\mathcal{V}_0\mathbf{j}_\gamma\rangle_\pi\ ,\ C_0(\mathrm{div}_\Gamma(\mathbf{j}_\gamma), \mathrm{div}_\Gamma(\mathbf{j}_\gamma^t)) := \frac{i}{\varepsilon\omega}\langle \mathrm{div}_\Gamma(\mathbf{j}_\gamma^t), \mathcal{V}_0\mathrm{div}_\Gamma(\mathbf{j}_\gamma)\rangle_{\frac{1}{2},\Gamma}. \tag{45}$$

4.3 Strong ellipticity

Theorem 4.4 *Let ω be bounded away from the spectrum of the interior Maxwell problem. Then BIE (43) admits a unique solution $\boldsymbol{j}_\gamma \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and we have continuous dependence on the data:*

$$||\boldsymbol{j}_\gamma||_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \leq C||\boldsymbol{f}||_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)}.$$

Proposition 4.5 *Let H be a separable Hilbert space, H' its dual space and $a : H \times H \to \mathbb{C}$ a continuous sesquilinear form on H. If there exist a positive constant $\alpha > 0$, an isomorphism $\Theta : H \to H$ and a compact sesquilinear form $c : H \times H \to \mathbb{C}$ such that, for any $u \in H$*

$$|a(u, \Theta(u))| \geq \alpha ||u||_H^2 - |c(u,u)| \tag{46}$$

and if

$$\sup_{v \in H} |a(u,v)| > 0 \quad \forall u \in H\,,\ u \neq 0_H \tag{47}$$

then, for any $f \in H'$, the variational problem $a(u,v) = {}_{H'}\langle f, v\rangle_H$ admits a unique solution $u \in H$ verifying:

$$||u||_H \leq C||f||_{H'}.$$

The proof of this result can be deduced from [14], for example. In the following, we shall refer to (46) as Generalized Gårding inequality.

Proof of Theorem 4.4: In the right hand side of (43), none of the terms represents the principal part of the boundary integral operator and, moreover, since they have different sign, choosing $\mathbf{j}_\gamma = \mathbf{j}_\gamma^t$ will not establish coercivity of the boundary integral operator. Therefore, we will establish the more general condition (46) with Θ different from the identity is required.

In the case of regular surfaces a proof of existence and uniqueness of solutions to the BIE (43) can be found in e.g., [19] (see also [2] or [8]). Here, we use the Hodge decomposition (25) for the space $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

By Theorem 2.7, we may decompose both test and trial functions as

$$\mathbf{j}_\gamma = \nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi \quad \text{ and } \quad \mathbf{j}_\gamma^t = \nabla_\Gamma q + \mathbf{curl}_\Gamma \psi$$

for unique $p,\ q \in \mathcal{H}(\Gamma)/\mathbb{R}$ and $\varphi,\ \psi \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$. Using these decompositions, we obtain the following equivalent reformulation of (43): find $(p,\varphi) \in H := \mathcal{H}(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ such that

$$\begin{aligned} B(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi, \nabla_\Gamma q) - C(\Delta_\Gamma p, \Delta_\Gamma q) &= -{}_\gamma\langle \nabla_\Gamma q, \mathbf{f}\rangle_\pi \quad \forall q \in \mathcal{H}(\Gamma) \\ B(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi, \mathbf{curl}_\Gamma \psi) &= -{}_\gamma\langle \mathbf{curl}_\Gamma \psi, \mathbf{f}\rangle_\pi \quad \forall \psi \in H^{\frac{1}{2}}(\Gamma). \end{aligned} \tag{48}$$

where the bilinear forms $B(\cdot,\cdot)$ and $C(\cdot,\cdot)$ have been defined in (44) (see also (45)). Selecting the form $a(\cdot,\cdot)$ in (46) as

$$a(p,\varphi; q,\psi) = B(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi, \nabla_\Gamma q + \mathbf{curl}_\Gamma \psi) - C(\Delta_\Gamma p, \Delta_\Gamma q)$$

and since the terms $B(\cdot,\cdot) - B_0(\cdot,\cdot)$ and $C(\cdot,\cdot) - C_0(\cdot,\cdot)$ are compact perturbations of the principal parts, it is sufficient to prove the generalized inf-sup condition (46) for the principal part $B_0(\cdot,\cdot) - C_0(\cdot,\cdot)$ of $a(\cdot,\cdot)$. To prove (46), for given $u = (p,\varphi) \in H$, we choose $(q,\psi) = \Theta(u) = (-\bar{p}, \bar{\varphi})$. Then there exist positive c_1, c_2, c_3 such that

$$\begin{aligned} \Im(B_0(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi, -\nabla_\Gamma \bar{p})) &+ \Im(C_0(\Delta_\Gamma p, \Delta_\Gamma \bar{p})) \\ &\geq c_1||\Delta_\Gamma p||^2_{-\frac{1}{2}} - c_2||\nabla_\Gamma p||^2_{V_\pi'} - |B_0(\mathbf{curl}_\Gamma \varphi, \nabla_\Gamma \bar{p},)|, \\ \Im(B_0(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi, \mathbf{curl}_\Gamma \bar{\varphi})) &\geq c_3||\mathbf{curl}_\Gamma \varphi||^2_{V_\pi'} - |B_0(\mathbf{curl}_\Gamma \varphi, \nabla_\Gamma \bar{p})| \end{aligned}$$

Now, the term $||\nabla_\Gamma p||^2_{V'_\pi}$ is compact with respect to $||\Delta_\Gamma p||^2_{-\frac{1}{2},\Gamma}$, since

$$||\nabla_\Gamma p||^2_{V'_\pi} \leq ||\nabla_\Gamma p||^2_{\mathbf{L}^2_t(\Gamma)} \leq c||\Delta_\Gamma p||^2_{-1,\Gamma}.$$

This implies immediately that the norm $||\Delta_\Gamma p||_{-\frac{1}{2},\Gamma} + ||\mathbf{curl}_\Gamma \varphi||_{V'_\pi}$ is equivalent to the norm defined in (22). By the continuity of the bilinear form $B_0 : V'_\pi \times V'_\pi \to \mathbb{C}$, the term $|B_0(\mathbf{curl}_\Gamma \varphi, \nabla_\Gamma \bar{p})|$ is also compact. This proves (46).

The injectivity (47) required in Proposition 4.5 is proved by going back to the original differential problem. If

$$B(\mathbf{j}_\gamma, \mathbf{j}^t_\gamma) - C(\mathrm{div}_\Gamma(\mathbf{j}_\gamma), \mathrm{div}_\Gamma(\mathbf{j}^t_\gamma)) = 0 \quad \forall \mathbf{j}^t_\gamma \in H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

then the electric field $\mathbf{E}$ in (41) solves the equation in Ω_e:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - k^2\mathbf{E} = \mathbf{0} & \text{in } \Omega_e \\ \gamma_\tau(\mathbf{E}) = \mathbf{0} & \text{on } \Gamma \\ \text{Silver-Müller Radiation condition} & \text{at } \infty\,. \end{cases}$$

By our hypotheses, this problem admits as unique solution $\mathbf{E} = \mathbf{0}$. Using again Maxwell's equation, we obtain that also $\mathbf{H} = \mathbf{0}$ which means $\mathbf{j}_\gamma = \mathbf{0}$. By means of the representation Theorem, Proposition 4.5, the assertion follows. □

Remark 4.6 *From Theorem 4.4 and the continuity with respect to the norm (22) of the bilinear form*

$$a(\boldsymbol{j}_\gamma, \boldsymbol{j}^t_\gamma) := B(\boldsymbol{j}_\gamma, \boldsymbol{j}^t_\gamma) - C(\mathrm{div}_\Gamma(\boldsymbol{j}_\gamma), \mathrm{div}_\Gamma(\boldsymbol{j}^t_\gamma))$$

in the Rumsey principle (42) it follows in particular that the form $a(\cdot,\cdot)$ satisfies an inf-sup condition on $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ equipped with the norm (22). This is an immediate consequence of the fact that an inf-sup condition is necessary and sufficient for the unique solvability of the variational problem (42) shown in Theorem 4.4.

4.4 Mixed formulation

In this section, we propose a mixed formulation of the Rumsey principle (43), or equivalently of (48). Our aim is to write the variational integral equation (43) in such a way that the discretization by means of standard Galerkin boundary elements can be easily analyzed and stability can be shown. Unfortunately this is neither possible for (43) nor for (48).

In [2] a mixed discretization of (43) by $\mathbf{H}(\mathrm{div}\,, \Gamma)$ conforming finite elements of Raviart-Thomas type was proposed for C^∞ surfaces. In Remark 4.6, we observed that the form $b(\cdot,\cdot)$ in (42) is continuous and satisfies an inf-sup condition on $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. For the stability of Bendali's $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$-conforming Galerkin discretization of the Rumsey principle (43), a discrete inf-sup condition must be proved for the Raviart-Thomas boundary elements. This cannot be done with arguments in the proof of Theorem 4.4.

On the other hand, the discretization of (48) with two unknowns, namely $p \in \mathcal{H}(\Gamma)$ and $\varphi \in H^{\frac{1}{2}}(\Gamma)$ would involve the construction of C^1 continuous finite elements on a boundary with edges and corners. Although this is in principle possible, from an implementational point difficult to realize.

In the following we propose therefore a mixed formulation which does not contain the Laplace Beltrami operator explicitly. It will be shown that stable Galerkin approximations of this formulation can be obtained by means of standard, low order boundary elements on the surface. The drawbacks of our approach are:

- two extra unknowns are added.
- some regularity on the datum $\mathbf{f}$ is necessary to ensure the equivalence of the mixed and primal formulations of the boundary integral equations.

Concerning the regularity of the data, we need to assume $\mathbf{f} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \cap \mathbf{L}^2_t(\Gamma)$. In terms of the Hodge decomposition, let $\mathbf{f} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta$, $\alpha \in H^{\frac{1}{2}}(\Gamma)$, $\beta \in \mathcal{H}(\Gamma)$, our assumption corresponds to $\alpha \in H^1(\Gamma)$. If the conductor is irradiated by a plane wave, then $\mathbf{f} = \pi_\tau(\mathbf{E}^{in})$ and this hypothesis is satisfied. We remark also that the two extra unknowns introduced by the mixed formulation of the Beltrami operator lead only to sparse blocks of the global stiffness matrix.

We now introduce the multipliers. We assume that $\mathbf{j}_\gamma$ and $\mathbf{j}^t_\gamma$ are Helmholtz-decomposed, i.e.

$$\mathbf{j}_\gamma = \nabla_\Gamma p' + \mathbf{curl}_\Gamma \varphi' \quad \text{and} \quad \mathbf{j}^t_\gamma = \nabla_\Gamma q + \mathbf{curl}_\Gamma \psi,$$

with p', $q \in \mathcal{H}(\Gamma)$ and $\varphi', \psi \in H^{\frac{1}{2}}(\Gamma)$. We set

$$m = -\Delta_\Gamma p' \qquad m^t = -\Delta_\Gamma q \tag{49}$$

and we substitute them in equations (48). We then consider m, $m^t \in H^{-\frac{1}{2}}(\Gamma)$ as new unknown and new test function, respectively. We impose conditions (49) weakly by means of Lagrange multipliers. This leads to a saddle point problem that we will prove to be equivalent to (48). To formulate it, we introduce the space

$$X = H^1(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}}(\Gamma)/\mathbb{R} \times H_\star^{-\frac{1}{2}}(\Gamma) \times H^1(\Gamma)/\mathbb{R}$$

and we denote by $|||\cdot|||_X$ the norm associated to X. With $\mathbf{f} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta$, we set

$$RHS = -\int_\Gamma \nabla_\Gamma \alpha \cdot \nabla_\Gamma q - \langle \Delta_\Gamma \beta, \psi \rangle_{\frac{1}{2},\Gamma}.$$

The saddle point problem reads:

Find $\left(p', \varphi', m, \lambda\right) \in X$ such that $\forall \left(q, \psi, m^t, \lambda^t\right) \in X$

$$\begin{array}{lllll} B(\nabla_\Gamma p' + \mathbf{curl}_\Gamma \varphi', \nabla_\Gamma q + \mathbf{curl}_\Gamma \psi) & & -\int_\Gamma \nabla_\Gamma q \cdot \nabla_\Gamma \lambda & = RHS \\ & -C(m, m^t) & +\langle m^t, \lambda \rangle_{\frac{1}{2},\Gamma} & = 0 \\ -\int_\Gamma \nabla_\Gamma p' \cdot \nabla_\Gamma \lambda^t & +\langle m, \lambda^t \rangle_{\frac{1}{2},\Gamma} & & = 0 \end{array} \tag{50}$$

with $B(.,.)$ and $C(.,.)$ as in (44). Note that we do not assume that $q \in \mathcal{H}(\Gamma)$ and therefore, in general, $\mathbf{j}^t_\gamma = \nabla_\Gamma q + \mathbf{curl}_\Gamma \psi$ does not belong to $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ anymore. This is the reason

$$\begin{pmatrix} i\omega\mu\,\mathrm{div}_\Gamma\pi_\tau(\mathcal{V}\nabla_\Gamma) & i\omega\mu\,\mathrm{div}_\Gamma\pi_\tau(\mathcal{V}\mathbf{curl}_\Gamma) & 0 & -\Delta_\Gamma \\ i\omega\mu\,\mathrm{curl}_\Gamma\pi_\tau(\mathcal{V}\nabla_\Gamma) & i\omega\mu\,\mathrm{curl}_\Gamma\pi_\tau(\mathcal{V}\mathbf{curl}_\Gamma) & 0 & 0 \\ 0 & 0 & -\frac{i}{\omega\varepsilon}\mathcal{V} & 1 \\ -\Delta_\Gamma & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ \varphi \\ m \\ \lambda \end{pmatrix} = \begin{pmatrix} \Delta_\Gamma\alpha \\ -\Delta_\Gamma\beta \\ 0 \\ 0 \end{pmatrix}. \tag{51}$$

We next prove the strong ellipticity of the system (50). To this end, we introduce the bilinear form

$$\begin{aligned} \mathcal{B}((p',\varphi',m,\lambda),(q,\psi,m^t,\lambda^t)) \quad = \quad & -C(m,m^t) + B(\nabla_\Gamma p' + \mathbf{curl}_\Gamma\varphi', \nabla_\Gamma q + \mathbf{curl}_\Gamma\psi) + \qquad (52) \\ & -\int_\Gamma \nabla_\Gamma q\cdot\nabla_\Gamma\lambda + \langle m^t,\lambda\rangle_{\frac{1}{2},\Gamma} - \int_\Gamma \nabla_\Gamma p'\cdot\nabla_\Gamma\lambda^t + \langle m,\lambda^t\rangle_{\frac{1}{2},\Gamma}. \end{aligned}$$

Theorem 4.7 *The bilinear form $\mathcal{B} : X\times X \to \mathbb{C}$ is continuous and strongly elliptic, i.e. there exists $\alpha > 0$, an isomorphism $\Theta : X \to X$ and a compact form $c : X\times X \to \mathbb{C}$ such that for every $(p,\varphi,m,\lambda)\in X$ there holds*

$$\begin{aligned} |\mathcal{B}((p,\varphi,m,\lambda),\Theta(p,\varphi,m,\lambda))| \quad &\geq \quad \alpha\,|||p,\varphi,m,\lambda|||_X^2 - |c(p,\varphi,m,\lambda)| \qquad (53) \\ \sup_{(q,\psi,m^t,\lambda^t)\in X} |\mathcal{B}(p,\varphi,m,\lambda,q,\psi,m^t,\lambda^t)| \quad &> \quad 0 \quad \forall\,(p,\varphi,m,\lambda)\neq \mathbf{O}_X. \qquad (54) \end{aligned}$$

In particular, for any $\boldsymbol{f}\in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)\cap \boldsymbol{L}_t^2(\Gamma)$ the problem (50) admits a unique solution $(p',\varphi',m,\lambda)\in X$.

Finally, let $(p,\varphi)\in\mathcal{H}(\Gamma)/\mathbb{R}\times H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ be the solution of (48); then the following holds:

$$p'\equiv p \quad \varphi'\equiv\varphi \quad m = -\Delta_\Gamma p. \tag{55}$$

Proof: Choosing $\Theta : X\to X$ as

$$\Theta(p',\varphi',m,\lambda) = (-\bar\lambda,\bar\varphi',-\bar m,-\bar p')$$

we obtain:

$$\begin{aligned} \mathcal{B}(p',\varphi',m,\lambda,-\bar\lambda,\bar\varphi',-\bar m,\bar p') = C(m,\bar m) + B(\nabla_\Gamma p' + \mathbf{curl}_\Gamma\varphi', -\nabla_\Gamma\bar\lambda + \mathbf{curl}_\Gamma\bar\varphi') + \\ + \int_\Gamma \nabla_\Gamma\lambda\cdot\nabla_\Gamma\bar\lambda - \langle\bar m,\lambda\rangle_{\frac{1}{2},\Gamma} + \int_\Gamma \nabla_\Gamma p'\cdot\nabla_\Gamma\bar p' + \langle m,\bar p'\rangle_{\frac{1}{2},\Gamma}. \end{aligned}$$

The terms

$$B(\nabla_\Gamma p' + \mathbf{curl}_\Gamma\varphi', -\nabla_\Gamma\bar\lambda)\,, \quad B(\nabla_\Gamma p', \mathbf{curl}_\Gamma\bar\varphi')\,, \quad -\langle\bar m,\lambda\rangle_{\frac{1}{2},\Gamma}\,, \quad \langle m,\bar p'\rangle_{\frac{1}{2},\Gamma}$$

are compact in X. Arguing as in the proof of Theorem 4.4, we obtain (53).

We prove now injectivity, i.e. (54), and at the same time (55). The third equation in (50), imposes exactly that $-\Delta_\Gamma p' = m$. By using as test functions $(q,0,-\Delta_\Gamma q,\cdot)$ and $(0,\psi,0,\cdot)$ in the system (50), we recover immediately equations (48) and this implies (55). Now, since any solution $(p',\varphi',m,\lambda)\in X$ of (50) verifies (55) and th solution of (48) is unique, in order to prove injectivity, we simply have to show that, given p' , φ' and m, there is only one possible multiplier $\lambda\in H^1(\Gamma)$ solving (50).

Choosing now $\psi = 0$ in the first equation of (50), we find

$$-\int_\Gamma \nabla_\Gamma\lambda\cdot\nabla_\Gamma q = B(\nabla_\Gamma p + \mathbf{curl}_\Gamma\varphi, \nabla_\Gamma q) - \int_\Gamma \nabla_\Gamma\alpha\cdot\nabla_\Gamma q \qquad \forall\, q\in H^1(\Gamma). \tag{56}$$

By means of Proposition 4.5 the proof is complete. □

We now present a discretization of the saddle point form (50) of the Rumsey principle by boundary elements and analyze its convergence. Throughout, we assume that $\Omega \subset \mathbb{R}^3$ is a simply connected polyhedron with Lipschitz boundary Γ which is moreover a finite union of planar sides Γ_j, straight edges e_k and vertices v_ℓ.

5.1 Galerkin Discretization

The Galerkin discretization of the problem (50) is based on a family $\{X_h\}_h$ of finite dimensional spaces satisfying the following properties:

- Density: $\overline{\bigcup_{h\downarrow 0} X_h} = X$ where the closure is taken w.r.t. $|||\cdot|||_X$;
- both variables p and λ are discretized with the same subspace of $H^1(\Gamma)$.

If $\mathbf{f} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta$, set $RHS_h = -\int_\Gamma (\nabla_\Gamma \alpha \cdot \nabla_\Gamma q_h + \mathbf{curl}_\Gamma \beta \cdot \mathbf{curl}_\Gamma \psi_h)$. The Galerkin discretization of problem (50) reads

Find $\left(p_h, \varphi_h, m_h, \lambda_h\right) \in X_h$ such that $\forall \left(q_h, \psi_h, m_h^t, \lambda_h^t\right) \in X_h$

$$\begin{array}{llll} B(\nabla_\Gamma p_h + \mathbf{curl}_\Gamma \varphi_h, \nabla_\Gamma q_h + \mathbf{curl}_\Gamma \psi_h) & & -\int_\Gamma \nabla_\Gamma q_h \cdot \nabla_\Gamma \lambda_h & = RHS_h \\ & -C(m_h, m_h^t) & +\int_\Gamma m_h^t \lambda_h & = 0 \\ -\int_\Gamma \nabla_\Gamma p_h \cdot \nabla_\Gamma \lambda_h^t & +\int_\Gamma m_h \lambda_h^t & & = 0 \end{array} \tag{57}$$

Theorem 5.1 *There exists a value $h_0 > 0$ such that for any $h \le h_0$ the discretized problem (57) admits a unique solution $\boldsymbol{u}_h = \left(p_h, \varphi_h, m_h, \lambda_h\right) \in X_h$. This Galerkin solution is quasioptimal, i.e. there is a constant C independent of h and of $\boldsymbol{f}$ such that if $\boldsymbol{u} = (p, \varphi, m, \lambda) \in X$ is the solution of the continuous problem (50),*

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_X \le C \inf_{\boldsymbol{\xi}_h \in X_h} ||\boldsymbol{u} - \boldsymbol{\xi}_h|||_X . \tag{58}$$

Proof: Since $X_h \subset X$, the proof of Theorem 4.7 can be applied also to the well-posedeness of the discrete problem (57). We sketch the argument for completeness: By Theorem 4.7, there exists a continuous operator $\tilde{\Theta} : X \to X$ realizing the inf-sup condition

$$|B(u, \tilde{\Theta}(u))| \ge \tilde{\alpha} |||u|||_X^2 \quad \forall u \in X ,$$

and such that $\Theta - \tilde{\Theta} : X \to X$ is compact. Given $\mathbf{u}_h \in X_h$, let $\mathbf{v}_h \in X_h$ be the best approximation of $\mathbf{v} = \tilde{\Theta}(\mathbf{u}_h) \in X$. According to our assumptions, we have $\Theta(\mathbf{u}_h) \in X_h$, and with the density of $\{X_h\}_h$ in X and the compactness of $\Theta - \tilde{\Theta}$ it follows then easily that $|||\mathbf{v}_h - \mathbf{v}|||_X \le \delta_h |||\mathbf{u}_h|||_X$ with δ_h independent of $\mathbf{u}_h$ and $\delta_h \to 0$ as $h \to 0$. For $h < h_0$ sufficiently small, one finds from this the discrete inf-sup condition

$$|B(\mathbf{u}_h, \mathbf{v}_h)| \ge \frac{\tilde{\alpha}}{2} |||\mathbf{u}_h|||_X |||\mathbf{v}_h|||_X .$$

The quasioptimal error estimate (58) is then straightforward. □

convergence of the Galerkin approximations as $h \to 0$.

We emphasize that an advantage of the saddle-point formulation (50) is that suitable spaces X_h can be built from standard finite element spaces on the surface Γ. We illustrate this by the easiest choice. Let $\mathcal{T}^p_{h_p}$, $\mathcal{T}^\varphi_{h_\varphi}$ and $\mathcal{T}^m_{h_m}$ be three possibly different, regular meshes consisting of shape-regular (triangular or quadrilateral) elements with meshwidths h_p, h_φ, h_m, respectively, on Γ. We set

$$\mathcal{S}^{k_\ell,i}(\mathcal{T}^\ell_{h_\ell},\Gamma) = \{u \in H^i(\Gamma) \text{ such that } u_{|K} \in \mathbb{P}^{k_\ell}(K) \quad \forall K \in \mathcal{T}^\ell_{h_\ell}\}$$

for the fields $\ell = p, \varphi, m$ and $i = 0, 1$. Here $\mathbb{P}^{k_\ell}(K)$ denotes the space of polynomials of degree k_ℓ on K if K is a triangle and, respectively, the space of polynomials of degree p_ℓ in each variable if K is a quadrangle. We remark that for $i = 1$ we have continuous finite elements, while for $i = 0$ we have discontinuous ones. For the Galerkin discretization (57), the lowest order choice of Boundary Element space is

$$X_h = \left(\mathcal{S}^{1,1}(\mathcal{T}^p_{h_p},\Gamma) \times \mathcal{S}^{1,1}(\mathcal{T}^\varphi_{h_\varphi},\Gamma) \times \mathcal{S}^{0,0}(\mathcal{T}^m_{h_m},\Gamma) \times \mathcal{S}^{1,1}(\mathcal{T}^p_{h_p},\Gamma)\right)/\mathbb{R}^4 \tag{59}$$

Note again that the finite element space for the variable λ **must** be equal to the one for the variable p. Without this condition the discrete inf-sup condition and hence the validity of Theorem 5.1 is not assured.

Remark 5.2 *Looking at the system in its matrix form (51), it is not hard to see that only two kinds of operators must be discretized: the Laplace-Beltrami operator and the single layer potential integrals. The blocks can be built in a fast fashion and the Laplace-Beltrami part turns out to be sparse. Moreover, if $\mathcal{T}^p_{h_p} = \mathcal{T}^\varphi_{h_\varphi} = \mathcal{T}^m_{h_m} = \mathcal{T}_h$ which consists only of triangles T, the matrix setup for (57) with the subspace (59) requires only the evaluation of the integrals*

$$\int_T\int_{T'} \Phi(\boldsymbol{x},\boldsymbol{y})\, ds(\boldsymbol{y}) ds(\boldsymbol{x}), \quad T, T' \in \mathcal{T}_h.$$

5.2 Regularity

In this section, we discuss the regularity of the solution of the system (50). Before tackling directly this problem, we need the classification of singularities of the Laplace-Beltrami operators on Γ.

To describe the singularities, we require some geometric notions. For any vertex v, we denote by $\omega_v \subset S^2$ the domain on the unit sphere in $\mathbb{R}^3$ cut out by the tangent cone K_v to Γ with vertex at v. Then ω_v is a curvilinear polygon on S^2, the boundary of which is a union of arcs of great circles.

5.2.1 Regularity of Δ_Γ

Here, we consider regularity of the Laplace-Beltrami operator on Γ, i.e. of the boundary value problem: *Find*

$$u \in H^1(\Gamma)/\mathbb{R} \qquad \langle \nabla_\Gamma u, \nabla_\Gamma v\rangle_{0,\Gamma} = \langle f, v\rangle_{0,\Gamma} \qquad \forall v \in H^1(\Gamma)/\mathbb{R}. \tag{60}$$

This problem admits, for every $f \in H^{-1}_\star(\Gamma)$, a unique solution $u \in H^1(\Gamma)/\mathbb{R}$. Assume now that $f \in H^s(\Gamma)$ for some $s > -1$. Then we are interested in whether u belongs to $H^{2+s}(\Gamma)$.

The following result addresses this.

(60) belongs to $H^{1+t}(\Gamma)$ *for* $0 \le t < s^*(s)$ *where*

$$s^*(s) = \min\left\{\frac{2\pi}{L}, s+1\right\} \tag{61}$$

with $L = \max_{v\in\Gamma}\{|\partial\omega_v|\}$ *denoting the maximal boundary length (in radians) of the spherical domains* $\omega_v \subset S^2$ *corresponding to the vertices* v.

Proof: We are going to use the nomenclature introduced in Subsection 2.2. The strong form of (60) reads:

$$-\Delta u_j = f_j \quad \text{in} \quad \Gamma_j, \tag{62}$$

and on any edge $e_{ij} = \bar\Gamma_i \cap \bar\Gamma_j$ there holds

$$u_i = u_j, \quad \boldsymbol{\tau}_i \cdot \nabla_\Gamma u = \boldsymbol{\tau}_j \cdot \nabla_\Gamma u \quad \text{on} \quad e_{ij}\,. \tag{63}$$

Let $\bar\Gamma_{ij} = \bar\Gamma_i \cup \bar\Gamma_j$ and $u_{ij} = u_{|\Gamma_{ij}}$ and χ_{ij} be any regular function in the plane parametrized by $(\boldsymbol{\tau}_{ij}, \boldsymbol{\tau}_i)$ on Γ_i and $(\boldsymbol{\tau}_{ij}, \boldsymbol{\tau}_i)$ on Γ_j. We assume that χ_{ij} has compact support on Γ_{ij}.

Using (62)-(63), we deduce that $\Delta(u_{ij}\chi_{ij}) \in H^{s-1}(\Gamma_{ij})$ for $s < \frac{3}{2}$ (on the parametric plane). By the standard shift theorem, we have that $u_{ij}\chi_{ij} \in H^{s+1}(\Gamma_{ij})$.

Therefore, singularities of u can only arise in the vertices v_ℓ. By localization, it is sufficient to consider a generic vertex v which we assume w.l.o.g. to coincide with the origin O in $\mathbb{R}^3$ and denote Γ_j and e_k only the faces and edges meeting at O. To determine the regularity, we compute the dominant singular form. It is well known (see, e.g., [15],[11]), that the corner singularities of Δ_Γ in Γ_j are of power-logarithmic type. We look for nontrivial solutions $U = U(|x|, x/|x|)$ of the homogeneous problem

$$\Delta_\Gamma U = 0 \quad \text{on} \quad K_O \tag{64}$$

subject to the transmission conditions (63) on all edges $e_k \subset K_O$ meeting at O. By homogeneity of Δ_Γ, we separate variables and express the restriction U_j of U to the face Γ_j in polar coordinates $r = |x|$ and θ centered at O in the face Γ_j:

$$\Delta_\Gamma U_j = (r^{-1}\partial_r(r\partial_r) + r^{-2}\partial_\theta^2)u_\lambda^j(r,\theta)$$

As $U_j = u_\lambda^j(r,\theta) = r^\lambda U^j(\theta)$ with $U^j(\theta) = u_\lambda^j(1,\theta)$, this gives on each face Γ_j that

$$\partial_\theta^2 U^j + \lambda^2 U^j = 0 \quad \text{on} \quad (0,\theta_j) \tag{65}$$

where θ_j denotes the opening angle of Γ_j at O. This gives in Γ_j:

$$U^j(\theta) = C_{1j}e^{-i\lambda\theta} + C_{2j}e^{i\lambda\theta} \quad \theta \in (0,\theta_j)$$

We denote the sum of opening angles of Γ_j at O by $L =: \sum_j \theta_j$ and by $U(\theta)$ the function composed of the U^j: $U|_{(0,\theta_j)} = U^j$. Note that $U(\theta)$ is a function of $\theta \in (0, L)$. The transmission conditions (63) imply that $U \in C^0_{per}([0,L])$. Further, since U^j is analytic, U is piecewise analytic in $[0, L]$. The transmission conditions (63) also imply that

$$\boldsymbol{\tau}_i \cdot \nabla_\Gamma u_i(1,\theta) = \partial_\theta u(1,\theta) = U'(\theta)$$

Evidently, $\lambda = 0$ is a simple eigenvalue of (65) with eigenfunction $U = const.$. Consider now an eigenvalue $\lambda \neq 0$. The continuity and the piecewise analyticity of U and (65) imply that $U'' \in C^0_{per}([0,L])$. Iterating this argument, we obtain that $U(\theta) \in C^\infty_{per}([0,L])$ and that U is piecewise a trigonometric function. It follows that $U(\theta)$ is globally on $(0,L)$ a trigonometric function, i.e. $U(\theta) = C\exp(\pm i\lambda\theta)$. The L-periodicity of U implies the value of λ:

$$\lambda = k\frac{2\pi}{L}, \qquad k = 1,2,..., \quad u_\lambda(r,\theta) = r^{\frac{2k\pi}{L}} e^{\pm i2k\pi\theta/L} \tag{66}$$

The dominant singularity in the solution occurs for $k = 1$ which proves the assertion. □

We remark that for polyhedra with a finite number of edges meeting at any vertex v, L remains finite. This means that for $f \in H^s_\star(\Gamma)$ $s > -1$, the solution of problem (60), always belongs to $H^{1+t}(\Gamma)$ for some strictly positive t. However, there are examples where t can be arbitrarily small, i.e. L arbitrarily large. For example, if the number of edges meeting in a vertex gets large, it is possible that $L \to \infty$. These cases are rather pathologic, however.

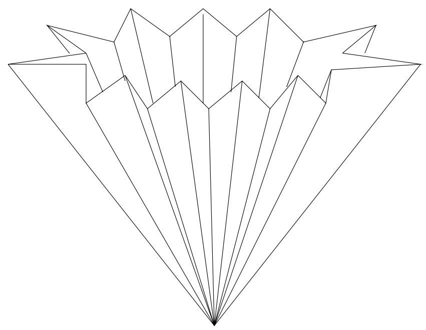

Figure 1: Example of a vertex where L is large.

5.2.2 Maxwell Singularities

In this section, we study the actual regularity of the solution (p, φ, m, λ) of the problem (50). Once the regularity result is settled, we shall immediately know the convergence rate of the boundary element discretization proposed in Section 5.1.

First of all, thanks to Theorem 4.4, we know that $\mathbf{j}_\gamma = -\gamma_\tau(\mathbf{H}^{tot})$. Now, from Theorem 4.7, we also know that:

$$-\gamma_\tau(\mathbf{H}^{tot}) = \nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi \qquad \mathrm{div}_\Gamma(\gamma_\tau(\mathbf{H}^{tot})) = m. \tag{67}$$

Since all the considerations concerning regularity are independent of the radiation condition at infinity, we denote by $B_R \subset \mathbb{R}^3$ the ball centered at the origin and with radius R large enough to ensure that $\bar{\Omega} \subset B_R$.

Finally, we need the following functional spaces for any $\mu \geq 0$:

$$\begin{aligned} \mathbf{H}^\mu(\mathrm{curl}_\Gamma, \Gamma) &:= \{\boldsymbol{\lambda} \in \mathbf{H}^\mu_-(\Gamma) \ : \ \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in H^\mu(\Gamma)\} \\ \mathbf{H}^\mu(\mathrm{div}_\Gamma, \Gamma) &:= \{\boldsymbol{\lambda} \in \mathbf{H}^\mu_-(\Gamma) \ : \ \mathrm{div}_\Gamma \boldsymbol{\lambda} \in H^\mu(\Gamma)\}. \end{aligned}$$

The main result of this section is the following:

$$\boldsymbol{f} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta \qquad \alpha\,,\ \beta \in H^k(\Gamma)$$

for $k \in \mathbb{R}$, $k \geq 2$.

Then there exists a real $\sigma^\star$, $0 < \sigma^\star \leq \frac{1}{2}$, *such that the solution* $\boldsymbol{j}_\gamma$ *of the problem (43) belongs to* $\boldsymbol{H}^\sigma(\mathrm{div}_\Gamma, \Gamma)$ *for any* $\sigma < \sigma^\star$.

Moreover, let $(p, \varphi, m, \lambda) \in X$ *be the solution of the problem (50). Then the following holds:*

$$p \in H^{1+t}(\Gamma) \qquad m \in H^\sigma(\Gamma) \qquad \varphi \in H^s(\Gamma) \qquad \lambda \in H^{t^\star}(\Gamma); \tag{68}$$

where $0 \leq \sigma < \sigma^\star$, $t < s^\star(\sigma)$, $s = \min\{1+\sigma, 1+t\}$ *and* $t^\star = \max\{1+\sigma, 1+t\}$.

The proof of this theorem requires the following lemma:

Lemma 5.5 *Let* $s > \frac{1}{2}$, $\boldsymbol{\varphi} \in \boldsymbol{H}^s_\parallel(\Gamma)$, $\boldsymbol{\psi} \in \boldsymbol{H}^s_\perp(\Gamma)$ *and* $\mathcal{H}^{s-1}(\Gamma) = \{u \in H^1(\Gamma)$ *such that* $\Delta_\Gamma u \in H^{s-1}(\Gamma)\}$. *Then the following Hodge decompositions hold:*

$$\exists!\, \alpha \in H^{1+t}(\Gamma)/\mathbb{R}\,,\ \beta \in \mathcal{H}^{s-1}(\Gamma)/\mathbb{R} \quad \boldsymbol{\varphi} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta \qquad \forall\, t < s^\star(s-1)\,; \tag{69}$$

$$\exists!\, \alpha \in \mathcal{H}^{s-1}(\Gamma)/\mathbb{R}\,,\ \beta \in H^{1+t}(\Gamma)/\mathbb{R} \quad \boldsymbol{\psi} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta \qquad \forall\, t < s^\star(s-1). \tag{70}$$

Proof of Lemma 5.5: We focus our attention only on the proof of (69). In [5], it is proved that

$$\exists!\, \alpha\,,\ \beta \in H^1(\Gamma)/\mathbb{R} \text{ such that } \boldsymbol{\varphi} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta.$$

Moreover, β is the solution of the problem

$$\mathrm{curl}_\Gamma \boldsymbol{\varphi} = \mathrm{curl}_\Gamma \mathbf{curl}_\Gamma \beta = \Delta_\Gamma \beta.$$

Now, since $\mathrm{curl}_\Gamma \boldsymbol{\varphi} \in H^{s-1}(\Gamma)$, we deduce $\beta \in \mathcal{H}^{s-1}(\Gamma)$. Using Theorem 5.3, we also know that $\beta \in H^{1+t}(\Gamma)$, $0 \leq t < s^\star(s-1)$. By difference, we deduce $\nabla_\Gamma \alpha \in \mathbf{H}^t_-(\Gamma)$. Using (19), we deduce $\alpha \in H^{1+t}(\Gamma)/\mathbb{R}$. □

Proof of Theorem 5.4: To shorten the notation, we rename $\boldsymbol{\Psi} := \mathbf{H}^{tot}_{|B_R\setminus\bar{\Omega}}$. Of course $\mathbf{j}_\gamma = \gamma_\tau(\boldsymbol{\Psi})$ holds. Using equations (40), the known regularity results for Maxwell's equations [10], and the assumption on the datum, we have $\boldsymbol{\Psi} \in \mathbf{H}^{\frac{1}{2}+\sigma}(B_R \setminus \bar{\Omega})$ and $\mathbf{curl}\,\boldsymbol{\Psi} \in \mathbf{H}^{\frac{1}{2}+\sigma}(B_R \setminus \bar{\Omega})$, for any $\sigma < \sigma^\star$ and where $\sigma^\star$ is the singularity exponent associated to the magnetic problem [10]. By standard decomposition in regular and singular part [1], [12], we have that for any $\sigma < \sigma^\star$:

$$\exists\, \boldsymbol{\xi} \in \mathbf{H}^{\frac{3}{2}+\sigma}(B_R \setminus \bar{\Omega})\,,\ q \in H^{\frac{3}{2}+\sigma}(B_R \setminus \bar{\Omega}) \quad \text{such that } \boldsymbol{\Psi} = \boldsymbol{\xi} + \nabla q \tag{71}$$

Taking now the tangential trace, and using (67), we have:

$$\gamma_\tau(\boldsymbol{\Psi}) = \gamma_\tau(\boldsymbol{\xi}) + \mathbf{curl}_\Gamma q \qquad \mathrm{div}_\Gamma(\gamma_\tau(\boldsymbol{\xi})) = m,$$

which immediately implies $m \in H^\sigma(\Gamma)$.

We focus now our attention on $\gamma_\tau(\boldsymbol{\xi})$. Using Lemma 5.5, we have that

$$\gamma_\tau(\boldsymbol{\xi}) = \nabla_\Gamma p' + \mathbf{curl}_\Gamma q' \qquad p', q' \in H^{1+t}(\Gamma) \qquad 0 \leq t < s^\star(\sigma).$$

Using this decomposition in (71) and comparing with (67), we obtain:

$$p = p'\,, \quad \varphi = q + q'\,, \quad m = \Delta_\Gamma p',$$

Concerning now λ, using the second equation of (50), we obtain:

$$\lambda = \frac{i}{\omega\varepsilon}\mathcal{V}(m). \tag{72}$$

Since we are working on a polyhedron, $\sigma < \sigma^\star$ and $\sigma^\star$ is smaller than the singularity exponent for the Laplace operator with Neumann boundary conditions, we deduce that $\lambda \in H^{1+\sigma}(\Gamma)$.

In order to prove that $\lambda \in H^{1+t}(\Gamma)$, for any $0 \le t < s^\star(\sigma)$, we consider equation (56), where without loss of generality we set $\alpha = 0$. Rearranging terms, we have:

$$\int_\Gamma \nabla_\Gamma \lambda \cdot \nabla_\Gamma q = i\omega\mu \int_\Gamma \nabla_\Gamma q \cdot \pi_\tau(\mathcal{V}(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi)). \tag{73}$$

Using the previous result on the regularity of p and φ, we have $\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi \in \mathbf{H}_{-}^{s-1}(\Gamma)$. Since $s - 1 < \frac{1}{2}$, it is easy to see that $i_\pi(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi) \in \mathbf{H}_{-}^{s-1}(\Gamma) = H^{s-1}(\Gamma)^3$. Using standard properties of the single layer potential on the polyhedral domains and the fact that $s - 1 < \sigma$, we deduce $\mathcal{V}(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi) \in \mathbf{H}^s(\Gamma)$ and then $\boldsymbol{\xi} := \pi_\tau(\mathcal{V}(\nabla_\Gamma p + \mathbf{curl}_\Gamma \varphi)) \in \mathbf{H}_{||}^s(\Gamma)$. Using now Lemma 5.5, we decompose $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = \nabla_\Gamma u + \mathbf{curl}_\Gamma v \qquad u\,,\ v \in H^{1+t}(\Gamma) \qquad 0 \le t < s^\star(\sigma),$$

since $s^\star(s-1) \ge s^\star(\sigma)$. Plugging this decomposition into equation (73), we see that $\lambda = i\omega\mu\, u$, thus $\lambda \in H^{1+t}(\Gamma)$. The proof is complete.

□

The assumption on the datum $\mathbf{f}$ in Theorem 5.4 is unrealistic. In the next lemma, precise and realistic assumptions are furnished on $\mathbf{f}$. The influence of the regularity of the datum is analyzed.

Lemma 5.6 *The same regularity stated in Theorem 5.4, holds for any right hand side $\boldsymbol{f}$ verifying:*

$$\boldsymbol{f} \in \boldsymbol{H}^\sigma(\mathrm{curl}_\Gamma, \Gamma) \cap \boldsymbol{H}_{||}^t(\Gamma) \qquad \forall\, \sigma < \sigma^\star\,,\ t < s^\star(\sigma). \tag{74}$$

Moreover, if only $\boldsymbol{f} \in \boldsymbol{H}^\mu(\mathrm{curl}_\Gamma, \Gamma)$ for any fixed $\mu \ge \sigma^\star$, then the first three components p, φ, m of the solution $\boldsymbol{u} = (p, \varphi, m, \lambda)$ of (50) verify the same regularity as in Theorem 5.4, while $\lambda \in H^{t_\star}(\Gamma)$ with $t_\star = \min\{1 + \mu, 1 + t\}$, for any $t < s^\star(\sigma)$.

Proof: The assumption, $\mathbf{f} \in \mathbf{H}^\sigma(\mathrm{curl}_\Gamma, \Gamma)$ for any $\sigma < \sigma^\star$, ensures that the solution of the problem (40) belongs to $\mathbf{H}_{loc}^{\frac{1}{2}+\sigma}(\mathbf{curl}, \mathbb{R}^3 \setminus \bar{\Omega})$. The proof of the regularity of the first three components p, φ, m of the solution $\mathbf{u}$ of (50) works with no change. Concerning λ, the regularity $\lambda \in H^{1+\sigma}(\Gamma)$ comes from (72). Let now

$$\mathbf{f} = \nabla_\Gamma \alpha + \mathbf{curl}_\Gamma \beta \qquad \alpha\,,\ \beta \in H^1(\Gamma).$$

Using equation (56), the proof of Theorem 5.4, proves that $\lambda - \alpha \in H^{1+t}(\Gamma)$, for any $t < s^\star(\sigma)$. Now, the assumption (74) ensures that $\alpha \in H^{1+t}(\Gamma)$. If we only have that $\mathbf{f} \in \mathbf{H}^\mu(\mathrm{curl}_\Gamma, \Gamma)$ for any $\mu \ge \sigma^\star$, then, $\alpha \in H^s(\Gamma)$, for $s = \min\{1 + \mu, 1 + \ell\}$, $\forall\, 0 < \ell < s^\star(\mu)$. The assertion of the Lemma is a consequence. □

Remark 5.7 *The two different regularity assumptions on the datum in Lemma 5.6 are somehow technical, but natural. The first assumption (74), corresponds to the situation when $\boldsymbol{f}$ is the tangential component of a "regular field", as $\boldsymbol{E}^{in}$. The assumption $\boldsymbol{f} \in \boldsymbol{H}^\mu(\mathrm{curl}_\Gamma, \Gamma)$, corresponds to a general charge density distributed on the boundary of the conductor Ω.*

In order to deduce from the a-priori error estimate (58) asymptotic convergence rates, we shall use the regularity results proved in the previous Subsection.

We set, for any $s \geq 0$,

$$X^s = H^{1+s}(\Gamma)/\mathbb{R} \times H^{\frac{1}{2}+s}(\Gamma)/\mathbb{R} \times H^{-\frac{1}{2}+s}(\Gamma)/\mathbb{R} \times H^{1+s}(\Gamma)/\mathbb{R}$$

with the convention $X \equiv X^0$.

Using the approximation properties of X_h in X, Theorem 5.1, and Theorem 5.4, we have:

Proposition 5.8 *Let t, σ, s be defined in Theorem 5.4. We denote by $\boldsymbol{u} := (p, \varphi, m, \lambda) \in X$ and $\boldsymbol{u}_h := \big(p_h, \varphi_h, m_h, \lambda_h\big) \in X_h$ be the (unique) solutions of (50) and (57) respectively. The following holds:*

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_X \leq C\left\{h_p^{t_-}\left(||p||_{1+t_-,\Gamma} + ||\lambda||_{1+t_-,\Gamma}\right) + h_m^{\sigma+\frac{1}{2}}||m||_{\sigma,\Gamma} + h_\varphi^{s-\frac{1}{2}}||\varphi||_{s,\Gamma}\right\} \tag{75}$$

where $t_- = \min\{1, t\}$. Moreover, let $\eta = \min\{\frac{1}{2} + \sigma, t\}$ for any $t < s^\star(\sigma)$ and $\sigma < \sigma^\star$, and $h = \max\{h_p, h_\varphi, h_m\}$, we have:

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_X \leq Ch^\eta. \tag{76}$$

C stand as a uniform constant both in (75) and (76).

When looking at the estimates (75) (76), it is clear that the convergence rate can be arbitrary small since t_- (and η consequently) can be arbitrary close to zero. Moreover, these error estimates are quasioptimal with respect to the considered norm. The error we are interested in computing is actually $\nabla_\Gamma(p - p_h) + \mathbf{curl}_\Gamma(\varphi - \varphi_h)$ in V_π' and $m - m_h$ in $H^{-\frac{1}{2}}(\Gamma)$. The next Proposition has the purpose to study the asymptotic rate for the quantity:

$$\|\nabla_\Gamma(p - p_h)\|_{V_\pi'} + \|\varphi - \varphi_h\|_{\frac{1}{2},\Gamma} + \|m - m_h\|_{-\frac{1}{2},\Gamma}.$$

By means of an Aubin-Nitsche duality argument we will now prove:

Proposition 5.9 *As in Theorem 5.1, we denote by $\boldsymbol{u} := (p, \varphi, m, \lambda) \in X$ and $\boldsymbol{u}_h := \big(p_h, \varphi_h, m_h, \lambda_h\big) \in X_h$ be the (unique) solutions of (50) and (57) respectively. Let σ, s as in Theorem 5.4 and $h = \max\{h_p, h_\varphi, h_m\}$. The following holds for any $0 < \mu < s^\star(-\frac{1}{2})$:*

$$||p - p_h||_{\frac{1}{2},\Gamma} + \|\nabla_\Gamma(p - p_h)\|_{V_\pi'} \leq Ch^\mu|||\boldsymbol{u} - \boldsymbol{u}_h|||_X \tag{77}$$

$$||\varphi - \varphi_h||_{\frac{1}{2},\Gamma} \leq Ch^\mu|||\boldsymbol{u} - \boldsymbol{u}_h|||_X + C'h_\varphi^{s-\frac{1}{2}}\|\varphi\|_{s,\Gamma} \tag{78}$$

$$||m - m_h||_{-\frac{1}{2},\Gamma} \leq Ch^\mu|||\boldsymbol{u} - \boldsymbol{u}_h|||_X + C'h_m^{\frac{1}{2}+\sigma}\|m\|_{\sigma,\Gamma}. \tag{79}$$

where C and C' are uniform constants with respect to the mesh sizes.

Proof: First of all, we observe that the bilinear form $\mathcal{B}$ defined in (52) is symmetric. This means that the corresponding differential operator is self-adjoint.

Let $\xi \in H^1(\Gamma)$ and $\mathbf{v} = (q, \psi, m^t, \lambda^t)$ be any function in X. We consider the problem: Find $\boldsymbol{\chi}(\xi) \in X$ such that

$$\mathcal{B}(\mathbf{v}, \boldsymbol{\chi}(\xi)) = \int_\Gamma \nabla_\Gamma q \cdot \nabla_\Gamma \xi. \tag{80}$$

write, by duality:

$$||p-p_h||_{\frac{1}{2},\Gamma} = \sup_{\boldsymbol{\lambda}\in V_\gamma} \frac{\langle \mathrm{div}_\Gamma \boldsymbol{\lambda}, p-p_h\rangle_{\frac{1}{2},\Gamma}}{||\mathrm{div}_\Gamma \boldsymbol{\lambda}||_{-\frac{1}{2},\Gamma}}.$$

Integrating by parts and using the properties of the Laplace-Beltrami operator, we deduce:

$$||p-p_h||_{\frac{1}{2},\Gamma} \leq \sup_{\xi\in\mathcal{H}(\Gamma)} \frac{\int_\Gamma \nabla_\Gamma \xi \cdot \nabla_\Gamma (p-p_h)}{||\xi||_{\mathcal{H}(\Gamma)}}.$$

Now, we use the adjoint problem (80) with $\mathbf{v} = \mathbf{u}-\mathbf{u}_h$ and we obtain by Galerkin orthogonality:

$$\|p-p_h\|_{\frac{1}{2},\Gamma} \leq \sup_{\xi\in\mathcal{H}(\Gamma)} \frac{\mathcal{B}(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\chi}(\xi)-\boldsymbol{\chi}_h)}{\|\xi\|_{\mathcal{H}(\Gamma)}}$$

for any $\boldsymbol{\chi}_h \in X_h$. Using the continuity of the bilinear form $\mathcal{B}$ in X, we have:

$$\|p-p_h\|_{\frac{1}{2},\Gamma} \leq C|||\mathbf{u}-\mathbf{u}_h|||_X \inf_{\boldsymbol{\chi}_h\in X_h} \frac{|||\boldsymbol{\chi}(\xi)-\boldsymbol{\chi}_h|||_X}{\|\xi\|_{\mathcal{H}(\Gamma)}}.$$

By means of Theorem 5.3, $\mathcal{H}(\Gamma) \subseteq H^{1+\mu}(\Gamma)$, for any $0 \leq \mu < s^\star(-\frac{1}{2})$. This implies that $\xi \in H^{1+\mu}(\Gamma)$, for any $0 \leq \mu < s^\star(-\frac{1}{2})$ and $\|\xi\|_{1+\mu,\Gamma} \leq \|\xi\|_{\mathcal{H}(\Gamma)}$. We are in the situation of Lemma 5.6 with

$$\mathbf{f} = \nabla_\Gamma \xi\,.$$

The proof of that lemma shows that $\boldsymbol{\chi}(\xi) \in X^\mu$.

Using standard approximation properties of the space X_h, we obtain:

$$\|p-p_h\|_{\frac{1}{2},\Gamma} \leq Ch^\mu|||\mathbf{u}-\mathbf{u}_h|||_X$$

where C is a uniform constant. We estimate now the quantity $\|\nabla_\Gamma(p-p_h)\|_{V_\pi'}$, by duality:

$$||\nabla_\Gamma(p-p_h)||_{V_\pi'} = \sup_{\boldsymbol{\lambda}\in V_\pi} \frac{{}_{V_\pi'}\langle \nabla_\Gamma(p-p_h), \boldsymbol{\lambda}\rangle_{V_\pi}}{||\boldsymbol{\lambda}||_{V_\pi}}.$$

By Lemma 5.5 and using the same argument as before, we have:

$$\|\nabla_\Gamma(p-p_h)\|_{V_\pi'} \leq \sup_{\alpha\in H^{\mu+1}(\Gamma)} \frac{\int_\Gamma \nabla_\Gamma(p-p_h)\cdot\nabla_\Gamma\alpha}{\|\alpha\|_{\mu+1,\Gamma}}.$$

Applying again the Aubin-Nitsche trick, we deduce (77).

For the estimate of $||\varphi-\varphi_h||_{\frac{1}{2},\Gamma}$, it is enough to choose in the first equation of (50) and of (57) the test function $(0,\psi_h)$. By subtraction we obtain the Galerkin orthogonality:

$$B(\nabla_\Gamma(p-p_h)+\mathbf{curl}_\Gamma(\varphi-\varphi_h), \mathbf{curl}_\Gamma\psi_h) = 0 \quad \forall\psi_h \in \mathcal{S}^{1,1}(\mathcal{T}^\varphi_{h_\varphi},\Gamma).$$

By standard argument, we obtain that:

$$\|\varphi-\varphi_h\|_{\frac{1}{2},\Gamma} \leq C_1\|\nabla_\Gamma(p-p_h)\|_{V_\pi'} + C_2 \inf_{\psi_h\in\mathcal{S}^{1,1}(\mathcal{T}^\varphi_{h_\varphi},\Gamma)} \|\varphi-\psi_h\|_{\frac{1}{2},\Gamma}.$$

In order to prove now the estimate (79), we have to pass through the estimate of $||\lambda-\lambda_h||_{\frac{1}{2},\Gamma}$. The discrete solution verifies:

$$B(\nabla_\Gamma p_h + \mathbf{curl}_\Gamma \varphi_h, \nabla_\Gamma q_h) - \int_\Gamma \nabla_\Gamma q_h \nabla_\Gamma \lambda_h = {}_\gamma\langle \nabla_\Gamma q_h, \mathbf{f}\rangle_\pi. \quad \forall q_h \in \mathcal{S}^{1,1}(\mathcal{T}^p_{h_p}, \Gamma)$$

Consider the solution $\tilde{\lambda} \in H^1(\Gamma)$ to the problem

$$B(\nabla_\Gamma p_h + \mathbf{curl}_\Gamma \varphi_h, \nabla_\Gamma q) - \int_\Gamma \nabla_\Gamma q \nabla_\Gamma \tilde{\lambda} = {}_\gamma\langle \nabla_\Gamma q, \mathbf{f}\rangle_\pi. \quad \forall q \in H^1(\Gamma).$$

By a duality argument applied to the operator $\int_\Gamma \nabla_\Gamma p \cdot \nabla_\Gamma q = \langle f, q\rangle_{\frac{1}{2},\Gamma}$ and using Theorem 5.3, we obtain:

$$\begin{aligned} ||\tilde{\lambda} - \lambda_h||_{\frac{1}{2},\Gamma} &\leq Ch^\mu ||\tilde{\lambda} - \lambda_h||_{1,\Gamma} \quad \forall \mu < s^\star(-\frac{1}{2}) \\ ||\tilde{\lambda} - \lambda||_{1,\Gamma} &\leq C\{||\nabla_\Gamma(p - p_h)||_{V'_\pi} + ||\mathbf{curl}_\Gamma(\varphi - \varphi_h)||_{V'_\pi}\}. \end{aligned}$$

As a consequence, $||\lambda - \lambda_h||_{\frac{1}{2},\Gamma} \leq Ch^\mu |||\mathbf{u} - \mathbf{u}_h|||_X$.

Now, using the second equation of (50) and of (57) and taking their difference, we obtain:

$$-C(m - m_h, m_h^t) + \langle m_h^t, \lambda - \lambda_h\rangle_{\frac{1}{2},\Gamma} = 0.$$

Using the same reasoning as in the estimate of $||\varphi - \varphi_h||_{\frac{1}{2},\Gamma}$ and the previous estimate on the quantity $||\lambda - \lambda_h||_{\frac{1}{2},\Gamma}$, the inequality (79) is finally proved. □

Corollary 5.10 *Let $\eta = \min\{\frac{1}{2} + \sigma, t\}$ for any $t < s^\star(\sigma)$, and μ be any value satisfying $\mu < s^\star(-\frac{1}{2})$. The following holds:*

$$\|\nabla_\Gamma(p - p_h)\|_{V'_\pi} + \|\varphi - \varphi_h\|_{\frac{1}{2},\Gamma} + \|m - m_h\|_{-\frac{1}{2},\Gamma} \leq C(h^\mu h^\eta + h^{\frac{1}{2}+\sigma}).$$

Remark 5.11 *The leading singularities are the one of Laplace Beltrami operator only in "pathological vertices" as the one in Figure 5.1. When such a situation occurs, the use of the Aubin-Nitsche trick doubles the convergence rate. More in detail:*

$$\begin{array}{ll} s^\star(\sigma) > \frac{1}{2} + \sigma & \begin{cases} |||\boldsymbol{u} - \boldsymbol{u}_h|||_X \leq Ch^{\frac{1}{2}+\sigma} \\ \|\nabla_\Gamma(p - p_h)\|_{V'_\pi} + \|\varphi - \varphi_h\|_{\frac{1}{2},\Gamma} + \|m - m_h\|_{-\frac{1}{2},\Gamma} \leq Ch^{\frac{1}{2}+\sigma}. \end{cases} \\ \frac{1}{2}\left(\frac{1}{2} + \sigma\right) < s^\star(\sigma) < \frac{1}{2} + \sigma & \begin{cases} |||\boldsymbol{u} - \boldsymbol{u}_h|||_X \leq C_\varepsilon h^{s^\star(\sigma)-\varepsilon} \\ \|\nabla_\Gamma(p - p_h)\|_{V'_\pi} + \|\varphi - \varphi_h\|_{\frac{1}{2},\Gamma} + \|m - m_h\|_{-\frac{1}{2},\Gamma} \leq Ch^{\frac{1}{2}+\sigma} \end{cases} \\ s^\star(\sigma) < \frac{1}{2}\left(\frac{1}{2} + \sigma\right) & \begin{cases} |||\boldsymbol{u} - \boldsymbol{u}_h|||_X \leq C_\varepsilon h^{s^\star(\sigma)-\varepsilon} \\ \|\nabla_\Gamma(p - p_h)\|_{V'_\pi} + \|\varphi - \varphi_h\|_{\frac{1}{2},\Gamma} + \|m - m_h\|_{-\frac{1}{2},\Gamma} \leq C_\varepsilon h^{2(s^\star(\sigma)-\varepsilon)} \end{cases} \end{array} \tag{81}$$

where $\varepsilon > 0$, C stands for a uniform constant and C_ε stands for a constant exposing with ε. The leading singularity is the one of Laplace Beltrami operator only when $2s^\star(\sigma) \ll \frac{1}{2} + \sigma$.

tion of singularities at neighborhood of vertices and edges. The worse singularity exponent is considered in any case.

Of course, far from the geometric singularities the local convergence will be much faster than what announced in the present section. On the other hand, the exact singularity exponents for edges and corners could be deduced for each variable using the results in [10], [11] and the ones in this section. The convergence rate could be then improved considering local refined meshes.

Acknowledgements: Thanks are due to Monique Dauge for the argument leading to the explicit formula (61) for the singularity exponents of the Beltrami-Operator on Γ. A. Buffa acknowledges the support of the ERASMUS fellowship by which her stay at ETH Zürich was sponsored, M. Costabel acknowledges the support of the Mathematics Research Institute (FIM) at ETH Zürich for supporting a visit during which this work was initiated and finally C. Schwab acknowledges partial support by the TMR-network "Wavelets and multiscale methods in numerical analysis" for travel to Paris and Rennes.

References

[1] **C. Amrouche, C. Bernardi, M. Dauge and V. Girault (1998)**. Vector potentials in three-dimensional non-smooth domains. *Math. Meth. Appl. Sci.* **21**, pp. 823-864.

[2] **A. Bendali (1984)**. Numerical Analysis of the exterior Boundary Value problem for Time-Harmonic Maxwell Equations by a Boundary Finite Element Method. Part 2: The discrete problem. *Math. of Comp.*, **43**, n. 167, pp. 47-68.

[3] **A. de la Bourdonnaye (1993)**. Décomposition de $H^{-\frac{1}{2}}_{div}(\Gamma)$ et nature de l'opérateur de Steklov-Poincaré du problème extérieur de l'électromagnétisme. *C. R. Acad. Sci. Paris, t. 316, Série I*, pp. 369–372.

[4] **A. Buffa and P. Ciarlet Jr.**. On traces for functional spaces related to Maxwell's Equations. Part I: An integration by parts formula in Lipschitz polyhedra. *Math. Meth. Appl. Sciences,* **24***, pp. 9-30.*

[5] **A. Buffa and P. Ciarlet, Jr.**. On traces for functional spaces related to Maxwell's equations. Part II: Hodge decompositions on the boundary of Lipschitz Polyhedra and Applications. *Math. Meth. Appl. Sciences,* **24***, pp. 31-48.*

[6] **A. Buffa, M. Costabel and D. Sheen (2000)**. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *Preprint IAN-CNR* **1185**, University of Pavia.

[7] **M. Cessenat (1996)**, *Mathematical Methods in Electromagnetism. Linear Theory and Applications.* World Scientific publishing, **41**.

[8] **D. Colton and R. Kress (1992)**, *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer Verlag, Berlin.

[9] **M. Costabel (1988)** . Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal.* **19**, No. 3, pp. 613–626.

[10] **M. Costabel and M. Dauge (2000)**. Singularities of the Maxwell Equations in non-smooth domains. *Arch. Rational Mech. Anal.* **151**, pp. 221–276.

[11] **M. Dauge (1988)**, *Elliptic boundary value problems on corner domains*, Lecture Notes in Mathematics. Springer Verlag, Berlin.

[12] **V. Girault and P.-A. Raviart (1986)**, *Finite Element Methods for Navier-Stokes Equations.* Springer Series in Computational Mathematics, Springer Verlag, Berlin.

A, pp. 1581-1583.

[14] **S. Hildebrandt and E. Wienholtz (1964)** Constructive Proofs of Representation Theorems in Separable Hilbert space. *Comm. Pure Appl. Maths*, **XVII**, pp. 369–373.

[15] **V. A. Kondrat'ev and O. A. Oleinik (1983)** Boundary value problems for partial differential equations in non–smooth domains. *Uspekhi Mat. Nauk.* **38**, n.2, pp. 3-76 (and (1983) *Russian Mathematical surveys* **38**, pp. 1-86.)

[16] **D. Mitrea and M. Mitrea (1999)**. Finite energy solutions of Maxwell's equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds, to appear.

[17] **D. Mitrea, M. Mitrea and J. Pipher (1997)**. Vector potential theory on nonsmooth domains in $\mathbb{R}^3$ and applications to electromagnetic scattering. *J. Fourier Anal. Appl.* **3**, No. 2, pp. 131–192.

[18] **J. Necăs (1967)** *Les Méthodes directes en Théorie des Equations Elliptiques.* Masson, Paris.

[19] **J.C. Nédélec (2001)** Acoustic and electromagnetic equations: Integral representations for harmonic problems. Springer Verlag, Berlin (to appear).

[20] **C. Safa** *Résolution d'équations intégrales pour un problème d'antenne par des méthodes d'ondelettes.* Ph.D. thesis, University of Rennes, in preparation.

[21] **L. Tartar (1997)**, On the characterization of traces of a Sobolev space used for Maxwell's equation. *Proceedings of a meeting held in Bordeaux, in honour of Michel Artola*, November 1997.