# Isoperimetric inequalities in a boundary value problem in an unbounded domain

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## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $\Omega^*$ the exterior of $\Omega$. We consider the boundary value problem

$$\left.\begin{array}{rcll} \Delta u &=& \gamma^2 f(u) & \text{in} \quad \Omega^* \\ u &=& 1 & \text{on} \quad \partial\Omega \\ u &\to& 0 \ \text{ as } |x| \to \infty & \end{array}\right\} . \tag{1.1}$$

Here $x$ is a generic point of $\mathbb{R}^N$, $\gamma$ is a given positive parameter and the nonlinearity is assumed to satisfy

$$f(0) = 0, \quad f'(0) > 0, \quad f''(u) \geq 0 \ \text{ for } \ u \geq 0 \, . \tag{1.2}$$

The assumptions on $f(u)$ guarantee that the solution decays to zero when $|x| \to \infty$ at least as fast as the solution of the linearised problem. Two possible backgrounds for problem (1.1) are described in the following:

(a) *Poisson-Boltzmann problem:*
With this application in mind $u$ represents the potential of a charge distribution. Important choices for $f(u)$ are then

$$f(u) = \sin h\, u \ \text{ or } \ f(u) = u \ (\text{Debye-Hückel approximation}) \, .$$

The parameter $\gamma$ then involves a number of physical constants (see e.g. Garrett & Poladian [2]).

(b) *Reaction-Diffusion:*
In this interpretation $u$ is the concentration of a reactant. The reactant is fed into $\Omega$ and diffuses through $\partial\Omega$ into the reaction region $\Omega^*$ where a simultaneous diffusion-reaction process takes place. The concentration inside $\Omega$ is held constant by a continuous supply of reactant. An important case is again $f(u) = u$ (e.g. for a first order degradation process). Another frequent choice is (Michaelis-Menten kinetics)

$$f(u) = \frac{u}{a+u}, \quad \text{where } a \text{ is a positive constant} \, .$$

Especially with the second interpretation of problem (1.1) in mind one is mostly not interested in the values $u(x)$, but rather in some functional of the solution. If (1.1) describes a diffusion reaction process then one has the relation

$$\gamma^2 = \frac{k}{D} \, ,$$

where $k$ is the reaction rate and $D$ the diffusion coefficient. Very often then (see Aris [1] one is interested in a pure number characterizing the influence of diffusion in the chemical reaction. One such possible number may be defined as

$$\eta := \frac{\oint_{\partial\Omega} |\nabla u| \, d\sigma}{\gamma^2 |\Omega|} \, . \tag{1.3}$$

Here $d\sigma$ is the surface element of $\partial\Omega$ and $|\Omega|$ the volume of $\Omega$. We will concentrate mainly on optimal estimates for $\eta$ in terms of geometrical data of $\Omega$.

Note that for $\gamma = 0$ and the condition $u(x) = O(|x|^{2-N})$, for $|x| \to \infty$, $N \geq 3$, (1.1) is the classical electrostatic problem and $\oint_{\partial\Omega} |\nabla u| ds$ is up to normalization the electrostatic capacity of $\Omega$.

## 2. Bounds derived from optimal sub- or supersolutions

Suppose we can find a function $\underline{u}(x)$ with the properties (subsolution)

$$\left.\begin{array}{rcl} \Delta \underline{u} & \geq & \gamma^2 f(\underline{u}) \text{ in } \Omega^* \\ \underline{u} & \leq & 1 \text{ on } \partial\Omega,\ \underline{u}(x) \to 0 \text{ for } |x| \to \infty \end{array}\right\} \tag{2.1}$$

for the limiting case $\gamma = 0$ we have to replace the boundary condition at infinity by

$$\underline{u}(x) = O(|x|^{2-N}) \cdot |x|, \quad N \geq 3\ (N = \text{ Number of dimensions }.$$

Then one has $\underline{u}(x) \leq u(x)$ for any $x \in \Omega^*$. For a supersolution $\overline{u}(x)$ one just has to reverse all inequality signs.

In the following we construct an optimal subsolution in two steps. First we define $\varphi(\rho)$ as the solution of problem (1.1) for the exterior of an $N$-ball of radius $R$. It is easy to see that we can take $\varphi(\rho)$ as the solution of (prime denotes $\frac{d}{d\rho}$)

$$\left.\begin{array}{rcl} \frac{1}{\rho^{N-1}} \left(\rho^{N-1} \varphi'\right)' & = & (\gamma R)^2 f(\varphi) \text{ for } \rho \in (1, \infty) \\ \varphi(1) & = & 1,\ \varphi(\rho) \to 0 \text{ for } \rho \to \infty \end{array}\right\} . \tag{2.2}$$

The second important ingredient is the (exterior) distance function $d(x) = \min_{y \in \partial\Omega} |x - y|$, $x \in \Omega^*$. We then set

$$s(x) = \left(1 + H_0\, d(x)\right)^{2-N},\ N \geq 3 . \tag{2.3}$$

Here

$$H_0 = \max_{y \in \partial\Omega} \left\{ \frac{1}{N-1} \sum_{j=1}^{N-1} k_j(y) \right\} ,$$

where $k_j(y)$ denote the principal curvatures at a point $y$ of $\partial\Omega$. Thus $H_0$ is the maximum of the mean curvature, which is assumed to be finite. Note that for the $N$-ball $H_0 = \frac{1}{R}$. The next result will enable us to derive optimal bounds for $\eta$ as defined in Eq. (1.3).

**Lemma 1** *Let $\Omega$ be a convex domain, i.e. $k_j \geq 0$ for $j = 1, ..., N-1$. Then the function $\underline{u}(x) = \tilde{\varphi}(s(x))$, $s(x)$ defined by Eq. (2.3) and where $\tilde{\varphi}(s) = \varphi(\rho)$, $\varphi(\rho)$ being the solution of (2.2) and $s = \frac{1}{\rho^{N-2}}$, is a subsolution to problem (1.1).*

In order to prove Lemma 1 it is convenient to prove first an auxiliary result stated as:

**Lemma 2** *Let $\Omega$ be a convex domain. Then the function $s(x)$ defined in Eq. (2.3) satisfies $\Delta s \geq 0$ in $\Omega^*$.*

**Proof:** We have

$$\Delta s = \frac{(2-N)\,H_0}{(1+H_0\,d)^{N-1}}\;\Delta d + \frac{(N-2)(N-1)}{(1+H_0\,d)^N}\;H_0^2\cdot|\nabla d|^2\;.$$

But one has (see e.g. Gibarg-Trudinger [3], p. 355)

$$\Delta d = \sum_{i=1}^{N-1}\frac{k_i}{1+k_i\,d}\,,\;|\nabla d| = 1\;,$$

so that

$$\Delta s = \frac{(N-2)\,H_0}{(1+H_0\,d)^N}\;\{(N-1)\,H_0 - (1+H_0\,d)\,\Delta d\}\;.$$

We now rewrite the term$\{(N-1)\,H_0 - (1-H_0\,d)\,\Delta d\}$ putting the expression for $\Delta d$ over a common denominator. Then we may write

$$\Delta d = \frac{A(d)}{B(d)}\;,$$

with

$$A(d) = \sum_{j=0}^{N-1}\;j\cdot P_j\,d^{j-1},\;\;B(d) = \sum_{j=0}^{N-1}\;P_j\cdot d^j\;.$$

Here we have used the abbreviation $P_j$, where $P_0 \equiv 1$ and $P_j$ = sum of all products with $j$ different factors $k_\ell$ and $1 \leq j,\,\ell \leq N-1$.

With this notation we can write

$$\Delta s = \frac{(N-2)\,H_0}{(1+H_0\,d)^N\;B(d)}\;\Big\{\underbrace{(N-1)\,H_0\,B(d) - (1+H_0\,d)\,A(d)}_{f(d)}\Big\}\;.$$

The expression $f(d)$ can be put into the form

$$f(d) = \sum_{\ell=0}^{N-1}\;C_\ell\cdot d^\ell\;,$$

where

$$C_\ell = H_0\,P_\ell\Big(1-\frac{\ell}{N-1}\Big) - (\ell+1)\,P_{\ell+1}\;.$$

We finally show that $C_\ell \geq 0$ for all $\ell$. It is clear for $\ell = 0$ and $\ell = N-1$, noting that by definition

$$P_1 = \sum_{j=1}^{N-1}\;k_j \leq (N-1)\,H_0\;.$$

We consider $C_\ell$ as a function of the $N-1$ variables $k_j$. Since $C_\ell$ is symmetric in all variables the minimum is attained for values $k_j$ such that

$$k_1 = k_2 = \ldots = k_{N-1} = k^0 \ .$$

But

$$P_\ell(k^0, \ldots, k^0) = \binom{N-1}{\ell} (k^0)^\ell \ ,$$

and hence

$$C_\ell(k, \ldots, k_{N-1}) \geq C_\ell(k^0, \ldots, k^0) = 0 \ ,$$

which implies that $\Delta s \geq 0$.

**Remarks on Lemma 2:**

(a) For the special case $\gamma = 0$ Lemma 2 shows that

$$s(x) = (1 + H_0\, d(x))^{2-N}$$

is a subsolution in the "capacity problem", i.e.

$$u(x) \geq (1 + H_0\, d(x))^{2-N}, \quad N \geq 3 \ , \tag{2.4}$$

and the equality sign holds in (2.4) if $\Omega$ is the $N$-ball of radius $R = \frac{1}{H_0}$. Inequality (2.4) was derived by Payne & Philippin [4] by a different method.

(b) The convexity assumption is needed in order to ensure that $\Delta d$ is finite for all $x \in \Omega^*$. Inequality (2.4) however holds without the convexity assumption, as shown by Payne & Philippin.

**Proof of Lemma 1:** We have

$$\Delta \overline{u} = \frac{d\tilde{\varphi}}{ds} \cdot \Delta s + \frac{d^2\tilde{\varphi}}{ds^2} \cdot |\nabla s|^2 \ ,$$

and

$$|\nabla s|^2 = \frac{((N-2)\, H_0)^2}{(1 + H_0\, d)^{2N-2}}\, |\nabla s|^2 = \frac{((N-2)\, H_0)^2}{\rho^{2N-2}} \ . \tag{2.5}$$

Furthermore

$$\frac{d\tilde{\varphi}}{ds} = \frac{1}{N-2}\, \rho^{N-1}\, \frac{d\varphi}{d\rho} \ , \tag{2.6}$$

and hence

$$\begin{aligned} \Delta \underline{u} - \gamma^2 f(\underline{u}) \ &\geq \ \frac{((N-2)\, H_0)^2}{\rho^{2N-2}}\ \frac{d^2\tilde{\varphi}}{ds^2} - \gamma^2 f(\varphi) \\ &= \ \frac{H_0^2}{\rho^{N-1}}\ \frac{d}{d\rho}\ \Big(\rho^{N-1}\, \frac{d\varphi}{d\rho}\Big) - \gamma^2 f(\varphi) = 0 \ . \end{aligned}$$

On $\partial\Omega$ we have $\rho = 1$ and thus
$\underline{u} = \varphi(1) = 1$ on $\partial\Omega$
as well as
$u(x) \to 0$ as $|x| \to \infty$.

As a consequence of Lemma 1 we have

**Theorem 3** *Let $\Omega$ be a finite convex domain in $\mathbb{R}^N$, $N \geq 3$. Denote by $H_0$ the maximum of the mean curvature of $\partial\Omega$ and by $\eta_0$ the effectiveness of a ball of radius $H_0^{-1}$ (as defined by Eq. (1.3)). Then the effectiveness $\eta$ of $\Omega$ satisfies*

$$\eta \leq \eta_0 \, \frac{|\partial\Omega|}{N H_0 |\Omega|} \tag{2.7}$$

*and the equality sign holds if $\Omega$ is a ball.*

**Proof:** Since the subsolution $\underline{u}$ defined in Lemma 1 satisfies the boundary conditions required in problem (1.1) we have

$$\oint_{\partial\Omega} |\nabla u| d\sigma \leq \oint_{\partial\Omega} |\nabla \underline{u}| d\sigma \; . \tag{2.8}$$

But from Eq. (2.5) and (2.6) we find

$$\oint_{\partial\Omega} |\nabla \underline{u}| d\sigma = -H_0 \cdot \frac{d\varphi}{d\rho} \Big|_{\rho=1} \cdot |\partial\Omega| \; . \tag{2.9}$$

From (2.2) it follows that ($R = H_0^{-1}$)

$$\eta_0 = \frac{N H_0^2}{\gamma^2} \, |\varphi'(1)| \; . \tag{2.10}$$

Eqs. (2.9) and (2.10) then imply inequality (2.7).

**Remarks on Theorem 1**

(a) An important case for applications is $N = 3$ and $f(u) = u$. Then the subsolution can be written down explicitly and one has at distance $d$ from $\partial\Omega$

$$u(d) \geq \frac{e^{-\gamma d}}{1 + H_0 \, d} \; . \tag{2.11}$$

Inequality (2.7) then takes the form

$$\eta \leq \frac{(H_0 + \gamma)|\partial\Omega|}{\gamma^2 |\Omega|} \; . \tag{2.12}$$

(b) We can also derive an explicit lower bound for $\eta$ from Lemma 1 in the case $N = 3, f(u) = u$. From the differential equation and the boundary conditions it follows that

$$\eta = \frac{1}{|\Omega|} \int_{\Omega^*} u\,dx \geq \frac{1}{|\Omega|} \int_{\Omega^*} \underline{u}\,dx\ .$$

We integrate over $\Omega^*$ using parallel surfaces to $\partial\Omega$. One has for the surface area $S(\delta)$ of a parallel surface at distance $\delta$ from $\partial\Omega$ (see e.g. Polya-Szegö [5], p.66)

$$S(\delta) = S(0) + 2M \cdot \delta + 4\pi\,\delta^2\ . \tag{2.13}$$

Here $M$ is the Minkowski constant of $\partial\Omega$ defined by

$$M = \int_{\partial\Omega} H\,d\sigma, \quad H = \text{mean curvature}\ , \tag{2.14}$$

and of course $S(0) = |\partial\Omega|$.

Hence we have

$$\eta \geq \frac{1}{|\Omega|} \int_0^\infty \frac{e^{-\gamma\delta}}{1 + H_0\,\delta}(|\partial\Omega| + 2M\delta + 4\pi\,\delta^2)\,d\sigma\ . \tag{2.15}$$

The evaluation of the integral gives after some routine calculation

$$\eta \geq \frac{1}{\nu}\left\{\frac{4\pi}{\mu} + \lambda + \mu(H_0^2\,|\partial\Omega| - \lambda)\,e^\mu \cdot E_1(\mu)\right\}\ , \tag{2.16}$$

where the following abbreviations have been used:

$$\nu = H_0^2\,\gamma|\Omega|, \quad \mu = \frac{\gamma}{H_0}, \quad \lambda = 2M\,H_0 - 4\pi\ . \tag{2.17}$$

$E_1(\mu)$ denotes the exponential integral defined by

$$E_1(\mu) = \int_\mu^\infty \frac{e^{-t}}{t}\,dt\ .$$

Inequality (2.16) is again isoperimetric since the equality sign holds if $\Omega$ is a ball.

(c) One could also derive a version of Theorem 1 valid for plane domains $\Omega$. However then the limiting case $\gamma = 0$ does not make any sense. In addition problem (1.1) seems less interesting in the two dimensional case.

Next we derive an estimate for $\eta$ which is based on the construction of an optimal supersolution $\overline{u}$. For this purpose we use as an auxiliary problem the "electrostatic problem" $(N \geq 3)$

$$\left.\begin{array}{rcll} \Delta h &=& 0 & \text{in} \quad \Omega^* \\ ]h &=& 1 & \text{on} \quad \partial\Omega \\ h(x) &=& O(|x|^{2-N}) & \text{as} \quad |x| \to \infty \end{array}\right\} \tag{2.18}$$

In order to get an explicit inequality we restrict our attention to the special case $f(u) = u$.

**Theorem 4** *Let $f(u) = u$ in problem (1.1) and $\Omega$ be a finite domain with boundary of class $C^{2+\epsilon}$, but not necessarily convex. Let $\tau = \max_{\partial\Omega} |\nabla h|$, $h$ being the solution of (2.18), and $C = w_N^{-1} \oint_{\partial\Omega} |\nabla h| d\sigma$ the "capacity" of $\partial\Omega (w_N = (N-1)$ dimensional surface area of the unit $N$-sphere). Then we have*

$$\eta \geq \left[1 + \frac{\kappa}{N-2} \; \frac{K_{\frac{N-4}{2}}(\kappa)}{K_{\frac{N-2}{2}}(\kappa)}\right] \; \frac{C \cdot w_N}{\gamma^2 |\Omega|} \tag{2.19}$$

*where $\kappa = (N-2)\,\frac{\gamma}{\tau}$ and $K_p(\kappa)$ denote Bessel functions. The equality sign holds if $\Omega$ is the $N$-ball.*

**Proof:** We use the solution of (1.1) with $f(u) = u$ and $\Omega$ the unit ball as an auxiliary function:

$$\left.\begin{array}{rcll} \frac{1}{\rho^{N-1}} \left(\rho^{N-1} \varphi'(\rho)\right)' & = & c^2 \varphi & \text{in} \quad (1,\infty) \\ \varphi(1) & = & 1, \; \varphi(\rho) \to 0 & \text{for} \quad \rho \to \infty \end{array}\right\} . \tag{2.20}$$

The value of $c$ will be chosen later. The solution of (2.20) is

$$\varphi(\rho) = \rho^{\frac{N-2}{2}} \cdot \frac{K_{\frac{N-2}{2}}(c\rho)}{K_{\frac{N-2}{2}}(c)} . \tag{2.21}$$

We then set $\rho = h^{-\frac{1}{N-2}}$, $h$ being the solution of problem (2.18). It is convenient to write

$$\begin{array}{rcl} \hat{\varphi}(h) & = & \varphi(\rho) \text{ and to use the relation} \\ \frac{d}{dh} & = & -\frac{1}{N-2}\, h^{-\frac{N-1}{N-2}} \; \frac{d}{d\rho} = -\frac{1}{N-2}\, \rho^{N-1} \; \frac{d}{d\rho} . \end{array} \tag{2.22}$$

We now choose $\overline{u}(x) = \varphi(\rho(x)) = \hat{\varphi}(h(x))$ and calculate

$$\Delta \overline{u} = \frac{d\hat{\varphi}}{dh} \cdot \Delta h + \frac{d^2\hat{\varphi}}{dh^2} \cdot |\nabla h|^2 = \frac{d^2 \varphi}{dh^2} \cdot |\nabla h|^2 . \tag{2.23}$$

At this point we use a result of Payne & Philippin [4] stating that in $N$ dimensions the solution $h$ of (2.18) satisfies (the smoothness of $\partial\Omega$ is used here!)

$$|\nabla h|^2 \leq \tau^2 \cdot h^{\frac{2(N-1)}{N-2}} . \tag{2.24}$$

Hence

$$\Delta \overline{u} - \gamma^2 \overline{u} \leq \tau^2 \; \frac{d^2 \hat{\varphi}}{dh^2} \cdot h^{\frac{N-2}{2(N-1)}} - \gamma^2 \, \hat{\varphi} ,$$

but because of Eq. (2.22) this can be put into the form

$$\Delta \overline{u} - \gamma^2 \overline{u} \leq \frac{\tau^2}{(N-2)^2} \; \frac{1}{\rho^{2(N-1)}} \, \rho^{N-1} \; \frac{d}{d\rho} \left(\rho^{N-1} \; \frac{d\varphi}{d\rho}\right) - \gamma^2 \, \varphi(\rho) = 0 ,$$

if we choose $c = \frac{(N-2)\gamma}{\tau}$ in problem (2.20). It is easy to see that $\overline{u}$ satisfies the boundary conditions. We therefore have

$$\oint_{\partial\Omega} |\nabla u| d\sigma \geq \oint_{\partial\Omega} |\nabla \overline{u}| d\sigma \ . \tag{2.25}$$

The relations

$$\overline{u} = \sqrt{h}\ \frac{K_{\frac{N-2}{2}}(\kappa\, h^p)}{K_{\frac{N-2}{2}}(\kappa)}, \quad \kappa = \frac{(N-2)\gamma}{\tau}, \quad p = -\frac{1}{N-2}$$

together with well known identities for Bessel functions then show after some routine calculations that inequality (2.25) leads to the statement of Theorem 2.

**Remarks on Theorem 2**

(a) Inequality (2.19) is still not quite explicit. It is not hard to check that the function

$$s\, \frac{K_{\frac{N-4}{2}}(s)}{K_{\frac{N-2}{2}}(s)}$$

is increasing for $s > 0$. Hence we need an upper bound for $\tau$ and a lower bound for the capacity $C$. It was shown by Payne & Philippin [4] that

$$\tau \leq (N-2)\, H_0 \ . \tag{2.26}$$

For the capacity $C$ one has the classical result of Poincaré-Szegö (see [6]) that

$$C \geq (N-2)\Big(\frac{N|\Omega|}{w_N}\Big)^{\frac{N-2}{N}} \ . \tag{2.27}$$

Inequality (2.19) therefore implies the weaker but still isoperimetric inequality

$$\eta \geq \left(1 + \frac{\gamma}{H_0} \cdot \frac{K_{\frac{N-4}{2}}\ (\frac{\gamma}{H_0})}{K_{\frac{N-2}{2}}\ (\frac{\gamma}{H_0})}\right)\ \frac{(N-2)}{\gamma^2 (\frac{N|\Omega|}{w_N})^{2/N}} \tag{2.28}$$

which for $N = 3$ reduces to

$$\eta \geq 3\Big(\frac{1}{\gamma^2} + \frac{1}{\gamma\, H_0}\Big)\ \Big(\frac{4\pi}{3|\Omega|}\Big)^{2/3} \ . \tag{2.29}$$

(b) For $N = 3$ the supersolution $\overline{u}$ can be written as

$$\overline{u}(x) = h(x)\ \exp\Big\{-\frac{\gamma}{\tau}\ \Big(\frac{1}{h(x)} - 1\Big)\Big\} \ . \tag{2.30}$$

An explicit upper solution $\overline{h}(x)$ for $h(x)$ would then lead to an explicit upper bound for $\overline{u}(x)$. However for general domains it seems difficult to give an optimal choice of $\overline{h}(x)$.

### 3. Concluding Remarks

(a) The method of integration over level surfaces (see Sperb [7]) could also be used in order to derive an optimal inequality for the "effectiveness" $\eta$. One can show then that for given $N$-volume of $\Omega$ $\eta$ is a minimum for the $N$-ball. In order to keep this note at a reasonable length we mention this result without proof.

(b) A number of additional bounds could be proven by exploiting the fact that the function

$$P := |\nabla u|^2 - 2\gamma^2 F(u) + \beta u ,$$

where $F(u) = \int_0^u f(v)dv$ and $\beta$ is chosen appropriately, is non positive in $\Omega^*$. Here techniques described in Sperb [7] can be used.

(c) The case $N = 2$ has been excluded here since it is of less interest in applications. It is however not hard to extend Theorem 1 (but not Theorem 2) to this case. Also for the remarks (a),(b) above the two-dimensional case is no exception.

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