A scalar boundary integrodifferential equation for eddy current problems using an impedance bound condition

Author(s):
Sterz, O.; Schwab, Christoph

Publication Date:
2000

Permanent Link:
https://doi.org/10.3929/ethz-a-004289451 →

1 Introduction

The numerical solution of electromagnetic problems has numerous applications in electrical engineering, see e.g. [12,16,20,30], and the design and analysis of numerical methods has attracted attention in recent years. An important class of problems are the quasi-static problems and, particularly in power engineering, the eddy-current approximation which is obtained by neglecting the displacement current in the Maxwell-equations. The problem is to find time harmonic electric and magnetic fields, i.e.

$$\mathbf{E}(\mathbf{x},t) = \mathrm{Re}(e^{i\omega t}\mathbf{E}(\mathbf{x})), \quad \mathbf{H}(\mathbf{x},t) = \mathrm{Re}(e^{i\omega t}\,\mathbf{H}(\mathbf{x}))\,,$$

which are the solution of a transmission problem for the eddy current approximation of the Maxwell-equations in $\Omega^+ \cup \Omega^-$, where by $\Omega^- \subset \mathbb{R}^3$, we denote a bounded, connected domain with smooth boundary Γ occupied by a lossy, highly conductive medium, and by $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega^-}$ its exterior which is assumed to be non-conductive. The eddy current model reads

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\,\mathbf{H}, \tag{1.1}$$

$$\mathbf{curl}\,\mathbf{H} = \kappa\,\mathbf{E} + \mathbf{J_0}. \tag{1.2}$$

Here $\mathbf{J_0}$ is an impressed current density, where $\mathrm{supp}\,\mathbf{J_0}$ is a bounded subset of Ω^+. As usual $\omega \in \mathbb{R}^+$ is the angular frequency. κ and $\mu = \mu_0\mu_r$ are positive and real valued L^∞-integrable scalar fields on $\mathbb{R}^3$, where κ is the conductivity and μ is the permeability, $\mu_0 = 4\pi\,10^{-7}\,\frac{Vs}{Am}$ is a constant (the permeability of free space), μ_r is dimensionless.

If, in addition, one is dealing with a highly conductive body and the so-called *penetration depth*

$$\delta = \sqrt{\frac{2}{\omega\mu\kappa}}$$

is small enough, one can approximately describe the behaviour of the fields at the conductor's boundary by impedance boundary conditions (IBCs),

$$\mathbf{n} \times \mathbf{E} = \eta\;\mathbf{n} \times (\mathbf{n} \times \mathbf{H}) \;\text{ on }\; \Gamma\,, \tag{1.3}$$

which allows one to deal with an exterior problem instead of a transmission problem ([3,7,16,20,27,30]). Here

$$\eta = (1+i)\,\sqrt{\frac{\omega\mu}{2\kappa}} \tag{1.4}$$

is the so-called *surface impedance* and $\mathbf{n}$ is the unit normal vector field on Γ pointing into Ω^+. Note that μ and κ in (1.4), which can be discontinuous at Γ, are the limits from Ω^-. For simplicity we assume them to be constant on Γ from this point on, but our proofs can be extended to the more general case easily.

The exterior impedance boundary value problem for the full Maxwell-equations and the eddy current model has been proposed by integral equation methods for example in [4,8,12]. These integral equations are vector valued.

It is a well known peculiarity of the eddy current model that it can be solved in the magnetic field $\mathbf{H}$ only (e.g. [9]), and the electric field can be obtained by (1.2) inside the conductor and in its exterior by solving a second boundary value problem in the variable $\mathbf{E}$. It is worth mentioning that this second BVP has no unique solution unless the electrostatic part of the field is fixed, e.g. by fixing its divergence and the total charge in the system.

The advantage of modelling in $\mathbf{H}$ only is that in the non-conductive region the eddy current problem can be reduced to a scalar one (see, e.g., [10]) — and thus by using impedance boundary conditions the whole problem becomes scalar. Recently, this idea has been used by Mayergoyz [22] in a finite element context. Here, we analyse this reformulation and show how it can be reduced to a scalar, strongly elliptic pseudo-differential equation of order 1 on the surface of the conductor. We present a Galerkin boundary element method for its numerical solution and show that the Ohmic losses converge with order $h^{\frac{5}{2}}$ where h is the mesh width on the boundary. A numerical example is given in the last section.

Remark 1.1. In the derivation of the IBC (1.3) it is essential that the curvature of Γ is bounded ([27,30]). We shall therefore assume in the following that Γ is smooth.

Remark 1.2. The IBC is a first order approximation to the eddy current model if $\delta k \ll 1$, i.e. the error is $\mathcal{O}\big(\delta\,k\big)$, where $k = \max(|k_1|, |k_2|)$ with k_1, k_2 being the principal curvatures of Γ. Furthermore, for the validity of the IBC the wavelength in free space and the distance to a source must be large compared to δ. With respect to these quantities the IBC is a second order approximation [23].

2 Elimination of the electric field

The electric field can be eliminated from the eddy current model. Here we present the derivation in [22] for completeness: Let $\boldsymbol{\chi} : \mathbb{R}^2 \to \Gamma \subset \mathbb{R}^3$ be a local parametric representation (chart) of the C^∞-manifold Γ; this chart induces coordinates $(\tau_1, \tau_2) \in \mathbb{R}^2$ in Γ. The exterior unit normal vector field $\mathbf{n}$ induces a third coordinate n. The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$ tangent to the τ_1, τ_2, n coordinate lines respectively, are assumed to be orthonormal. By h_1, h_2, we denote the associate tangential metric coefficients. Then (1.3) takes the form

$$E_{\tau_1} = -\eta\,H_{\tau_2}, \quad E_{\tau_2} = \eta\,H_{\tau_1} \tag{2.1}$$

where $E_{\tau_\alpha} = \mathbf{E}\cdot\mathbf{a}_\alpha$ is the projection of $\mathbf{E}$ onto $\mathbf{a}_\alpha$. The normal component of (1.1), restricted to Γ, reads

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial\tau_1}\,(h_2\,E_{\tau_2}) - \frac{\partial}{\partial\tau_2}\,(h_1\,E_{\tau_1})\right) = -i\omega\mu_0\,H_n\,. \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$0 = \eta\frac{1}{h_1 h_2}\Big(\frac{\partial}{\partial\tau_1}\,(h_2\,H_{\tau_1}) + \frac{\partial}{\partial\tau_2}\,(h_1\,H_{\tau_2})\Big) + i\omega\mu_0\,H_n\,,$$

a boundary condition equivalent to (1.3), which in vector notation is

$$\mathrm{div}_\Gamma\,\mathbf{H}_t + \alpha\,H_n = 0 \;\text{ on }\; \Gamma\,, \tag{2.3}$$

where $\mathbf{H}_t = -\mathbf{n}\times(\mathbf{n}\times\mathbf{H})$ is the projection of $\mathbf{H}$ into the tangent plane at Γ, $H_n = \mathbf{H}\cdot\mathbf{n}$ is its normal component, and $\operatorname{div}_\Gamma$ is the surface divergence (see, e.g., [11]), and

$\alpha := i\omega\mu_0/\eta\,.$

Note that we have by $\omega,\mu,\kappa > 0$

$$\begin{aligned}\alpha &= \frac{i\omega\mu_0}{\eta} = \frac{i\omega\mu_0\sqrt{2}}{(1+i)\sqrt{\frac{\omega\mu}{\kappa}}}\\ &= (1+i)\sqrt{\frac{\omega^2\mu_0^2\,\kappa}{2\,\omega\mu}} = (1+i)\sqrt{\frac{\omega\mu\kappa}{2\,\mu_r^2}}\\ &= (1+i)\,\frac{1}{\mu_r\delta} =: (1+i)\,\beta\,, \quad \text{where } 0<\beta\in\mathbb{R}\,.\end{aligned}$$

Assuming that in $\overline{\Omega^+}$ the field $\mathbf{H}$ can be split into a (known) exciting field $\mathbf{H}_0$ and an unknown secondary field $\mathbf{H}_s$, we find for (1.1)–(1.3) the formulation

$$\operatorname{div}\mathbf{H}_s = 0 \quad \text{in } \Omega^+\,, \tag{2.4a}$$

$$\mathbf{curl}\,\mathbf{H}_s = 0 \quad \text{in } \Omega^+\,, \tag{2.4b}$$

$$\operatorname{div}_\Gamma \mathbf{H}_{st} + \alpha H_{sn} = g \quad \text{on } \Gamma \tag{2.4c}$$

and the condition at infinity

$$\mathbf{H}_s(\mathbf{x}) = \mathcal{O}\big(|\mathbf{x}|^{-2}\big) \quad \text{as } |\mathbf{x}|\to\infty\,, \tag{2.4d}$$

where

$g := -\operatorname{div}_\Gamma \mathbf{H}_{0t} - \alpha\,\mathbf{H}_{0n}\,.$

Condition (2.4d) is a consequence of $\mathbf{H}_s$ being a harmonic vector field with zero value at infinity (see [21]).

Remark 2.1. The exciting field can be computed by the Biot-Savart law:

$$\mathbf{H_0}(\mathbf{x}) = \mathbf{curl}\,\frac{1}{4\pi}\int_{\Omega_0}\mathbf{J_0}(\mathbf{y})\,\frac{1}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}\,,$$

where Ω_0 is the support of the impressed currents $\mathbf{J_0}$.

3 Scalar potential formulation

As mentioned before an advantage of (2.3) is that (2.4) can be reduced to a scalar problem, as will be shown. By (2.4b) and the assumption of Ω^- being simply connected, we have in Ω^+ that

$$\mathbf{H}_s = -\mathbf{grad}\,\phi,\ \ \mathbf{H} = \mathbf{H}_0 - \mathbf{grad}\,\phi\,. \tag{3.1}$$

Inserting into (2.4), we get the BVP:

$$\Delta\,\phi = 0 \ \text{ in } \ \Omega^+\,, \tag{3.2a}$$

$$B\phi := -\,\Delta_\Gamma\,\phi - \alpha\,\frac{\partial\phi}{\partial n} = g \ \text{ on } \ \Gamma\,, \tag{3.2b}$$

$$\phi = \mathcal{O}(|\mathbf{x}|^{-1}) \ \text{ as } \ |\mathbf{x}|\to\infty \tag{3.2c}$$

where $\Delta_\Gamma = \operatorname{div}_\Gamma \mathbf{grad}_\Gamma$ denotes the surface Laplacian, i.e., the Laplace-Beltrami operator on Γ.

Problem (3.2) is a nonstandard problem, since the principal part of the PDE lives on the boundary Γ. In particular, the boundary operator is not subordinate to the Laplacian in Ω^+.

Proposition 3.1. *Any solution ϕ of (3.2) gives, via (3.1), a solution of (2.4).*

For a variational formulation of problem (3.2), we introduce the space $\mathcal{V}^+$ as closure of

$$\mathcal{D}(\Omega^+) = \{\phi\in C^\infty(\overline{\Omega^+}) :\ \phi \text{ has bounded support in } \Omega^+\}$$

with respect to the norm $|||\phi|||$ defined by

$$|||\phi|||^2 := \|\,\mathbf{grad}\,\phi\|^2_{L^2(\Omega^+)} + \left\|\frac{\phi}{\sqrt{1+|\mathbf{x}|^2}}\right\|^2_{L^2(\Omega^+)} + \|\phi\|^2_{H^1(\Gamma)}\,. \tag{3.3}$$

Evidently, $\mathcal{V}^+$ is a Hilbert-space with respect to the inner product corresponding to (3.3):

$$(\phi,\psi) := \int_{\Omega^+}\mathbf{grad}\,\phi\cdot\overline{\mathbf{grad}\,\psi}\,d\mathbf{x} + \int_{\Omega^+}\frac{\phi\,\overline{\psi}}{1+|\mathbf{x}|^2}\,d\mathbf{x} + \int_\Gamma \mathbf{grad}_\Gamma\,\phi\cdot\overline{\mathbf{grad}_\Gamma\,\psi}\,ds + \int_\Gamma \phi\,\overline{\psi}\,ds\,.$$

In the following we will write $\langle\cdot\,,\cdot\rangle$ indexed by Ω or Γ for the usual real, symmetric L^2 scalar product (*not* the Hermitian one) on Ω and Γ respectively for space-saving reasons. All restrictions to the boundary have to be understood in the sense of traces.

The weak form of (3.2) reads: find $\phi\in\mathcal{V}^+$, that

$$\Big\langle \mathbf{grad}_\Gamma\ \phi,\ \overline{\mathbf{grad}_\Gamma\ \widetilde{\phi}}\Big\rangle_\Gamma + \alpha\Big\langle \mathbf{grad}\ \phi,\ \overline{\mathbf{grad}\ \widetilde{\phi}}\Big\rangle_{\Omega^+} = \Big\langle g,\overline{\widetilde{\phi}}\Big\rangle_\Gamma \tag{3.4}$$

for all $\widetilde{\phi}\in\mathcal{V}^+$, using that (3.2a) and $\phi\in\mathcal{V}^+ \Longrightarrow$

$$0 = \int_{\Omega^+}\overline{\psi}\,\Delta\,\phi\,d\mathbf{x} = -\int_{\Omega^+}\mathbf{grad}\,\phi\cdot\overline{\mathbf{grad}\,\psi}\,d\mathbf{x} - \int_\Gamma\overline{\psi}\,\frac{\partial\phi}{\partial n}\,ds$$

(recall that $\mathbf{n}$ points into Ω^+).

Proposition 3.2. *Denote by $a(\phi,\widetilde{\phi})$ the sesquilinear form on the LHS of (3.4). Then, for every $\phi\in\mathcal{V}^+$ the Gårding inequality*

$$\operatorname{Re}\,a(\phi,\phi) \ge C(\Omega^+)\left(|||\phi|||^2 - \|\phi\|^2_{L^2(\Gamma)}\right)$$

holds.

Proof. For every $\phi\in\mathcal{V}^+$, we have

$$\|\,\mathbf{grad}\,\phi\|^2_{L^2(\Omega^+)} \ge C(\Omega^+)\,\left\|\frac{\phi}{\sqrt{1+|\mathbf{x}|^2}}\right\|^2_{L^2(\Omega^+)}$$

(see e.g. [24]). Hence, since $\alpha = (1+i)\beta$, $\beta>0$, we get

$$\begin{aligned}\operatorname{Re}\,a(\phi,\phi) &= \beta\,\|\,\mathbf{grad}\ \phi\|^2_{L^2(\Omega^+)} + \|\,\mathbf{grad}_\Gamma\ \phi\|^2_{L^2(\Gamma)}\\ &\ge C(\Omega^+)\big(|||\phi|||^2 - \|\phi\|^2_{L^2(\Gamma)}\big)\,.\end{aligned}$$

□

Proposition 3.3. *For $g \equiv 0$ on Γ, (3.4) admits only the trivial solution.*

Proof. We have in (3.4) that Re $a(\phi,\phi) = 0$, hence we get $\phi \equiv$ const. in $\overline{\Omega^+}$. If $\phi \in \mathcal{V}^+$, $|||\phi||| < \infty$, hence $\phi \equiv 0$. □

In the following we denote by $H^s(\Omega)$, $H^s_{\mathrm{loc}}(\mathbb{R}^3)$ and $H^s(\Gamma)$, $s \geq 0$, the usual Sobolev spaces on Ω, $\mathbb{R}^3$ and Γ respectively. For $s < 0$, $H^s(\Gamma)$ denotes the space of continuous, linear functionals on $H^{-s}(\Gamma)$, i.e. $H^{-s}(\Gamma) = (H^s(\Gamma))^*$ (see, e.g., [19]).

Theorem 3.1. *For every $g \in (H^1(\Gamma))^*$, problem (3.4) admits a unique solution $\phi \in \mathcal{V}^+ \cap C^\infty(\Omega^+)$. If $g \in C^\infty(\Gamma)$, we even have $\phi \in C^\infty(\overline{\Omega^+})$.*

Proof. We have for every $\phi \in \mathcal{V}^+$

Re $a(\phi,\phi) \geq C(\Omega^+)((\phi,\phi) - k(\phi,\phi))$

where $k(\phi,\widetilde{\phi}) := \int_\Gamma \phi\,\overline{\widetilde{\phi}}\,ds$ is a compact form on $H^1(\Gamma)$. By the boundedness of Γ and Rellich's Lemma, it is also compact on $\mathcal{V}^+$. Thus uniqueness implies existence. By Proposition 3.3, there exists for every $g \in (H^1(\Gamma))^*$ a unique solution $\phi \in \mathcal{V}^+$ of (3.4) and we have the a-priori estimate

$$|||\phi||| \leq C(\Omega^+,\beta)\,\|g\|_{H^{-1}(\Gamma)}\,.$$

□

Remark 3.1. The above arguments could be extended to the case where (3.2a) is replaced by

$$P(D)\phi = f \ \text{ in } \ \Omega^+\,,$$

where $P(D)$ is any second order, strongly elliptic differential operator with constant coefficients and $f \in L^2_{\mathrm{comp}}(\mathbb{R}^3)$, say. Since it is not of interest in the considered problem, we shall not elaborate on it here.

Remark 3.2. The existence also holds for Lipschitz domains, but the impedance boundary condition only describes physical phenomena for smooth domains [6,27].

4 Boundary integral equation

We reduce the problem (3.2) to an integral equation on Γ. We satisfy (3.2a) and (3.2c) by a single layer potential

$$\phi(\mathbf{x}) = (\mathcal{S}\,\sigma)(\mathbf{x}) := \frac{1}{4\pi}\int\limits_{\mathbf{y}\in\Gamma} \frac{1}{|\mathbf{x}-\mathbf{y}|}\,\sigma(\mathbf{y})\,ds_{\mathbf{y}}\,, \quad \mathbf{x}\notin\Gamma \tag{4.1}$$

for some unknown density σ on Γ. Inserting (4.1) in (3.2b), we get

$$-\Delta_\Gamma\,\mathrm{V}\,\sigma - \alpha\,\frac{\partial}{\partial n}\,(\mathrm{V}\,\sigma) = g\,, \tag{4.2}$$

where $\mathrm{V}\,\sigma := (\mathcal{S}\,\sigma)|_\Gamma$ is the single layer operator on the boundary Γ. This is an integrodifferential equation on Γ for the density σ. The jump relations for $(\mathcal{S}\,\sigma)(\mathbf{x})$ imply

$$A\sigma := -\Delta_\Gamma\,V\sigma + \alpha\left(\frac{1}{2}\,I - K'\right)\sigma = g \ \text{ on } \ \Gamma \tag{4.3}$$

where

$$(K'\sigma)(\mathbf{x}) := \int\limits_{\mathbf{y}\in\Gamma} \frac{\partial}{\partial n_{\mathbf{x}}}\left(\frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|}\right)\sigma(\mathbf{y})\,ds_{\mathbf{y}} \tag{4.4}$$

is the adjoint operator to the classical double layer potential (apart from a factor $\frac{1}{2}$). Since Γ is smooth, the integrand in (4.4) is weakly singular if $\sigma \in C^0(\Gamma)$.

Let us give a variational formulation of (4.2). Integrating parts on Γ, we get:
Find $\sigma \in H^{\frac{1}{2}}(\Gamma)$ such that

$$\begin{aligned} a(\sigma,\widetilde{\sigma}) := {}& \left\langle \mathbf{grad}_\Gamma\ \mathrm{V}\,\sigma,\ \overline{\mathbf{grad}_\Gamma\,\widetilde{\sigma}}\right\rangle_\Gamma \\ &+ \alpha\left\langle\left(\frac{1}{2}\,I - K'\right)\sigma, \overline{\widetilde{\sigma}}\right\rangle_\Gamma = \left\langle g, \overline{\widetilde{\sigma}}\right\rangle_\Gamma \end{aligned} \tag{4.5}$$

for all $\widetilde{\sigma} \in H^{\frac{1}{2}}(\Gamma)$.

Proposition 4.1. *The sesquilinear form $a(\cdot,\cdot)$ in (4.5) is continuous on $H^{\frac{1}{2}}(\Gamma)$ and satisfies a Gårding inequality, i.e. there is $c > 0$ and a compact form $k(\cdot,\cdot)$ on $H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ such that*

$$\forall\sigma \in H^{\frac{1}{2}}(\Gamma) : \mathrm{Re}\ a(\sigma,\sigma) \geq c\,\|\sigma\|^2_{H^{\frac{1}{2}}(\Gamma)} - k(\sigma,\sigma)\,. \tag{4.6}$$

Proof.

i) **Continuity**
It is known (e.g. [13]) that for $\Gamma \in C^\infty$, $\mathrm{V} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ for any $s \in \mathbb{R}$, and $K' : H^s(\Gamma) \to H^s(\Gamma)$ for all $s \in \mathbb{R}$, since K' has a weakly singular kernel if Γ is smooth. Therefore, for $\sigma,\widetilde{\sigma} \in H^{\frac{1}{2}}(\Gamma)$ by duality

$$\begin{aligned} &\left|\left\langle \mathbf{grad}_\Gamma\ \mathrm{V}\,\sigma,\ \overline{\mathbf{grad}_\Gamma\,\widetilde{\sigma}}\right\rangle_\Gamma\right| \\ &\quad\leq \|\mathbf{grad}_\Gamma\ \mathrm{V}\,\sigma\|_{H^{\frac{1}{2}}(\Gamma)} \|\mathbf{grad}_\Gamma\,\widetilde{\sigma}\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\quad\leq C\,\|\,\mathrm{V}\,\sigma\|_{H^{\frac{3}{2}}(\Gamma)} \|\widetilde{\sigma}\|_{H^{\frac{1}{2}}(\Gamma)} \\ &\quad\leq C\,\|\sigma\|_{H^{\frac{1}{2}}(\Gamma)}\,\|\widetilde{\sigma}\|_{H^{\frac{1}{2}}(\Gamma)} \end{aligned}$$

since $\mathbf{grad}_\Gamma : H^s(\Gamma) \to H^{s-1}(\Gamma)$ and $\mathrm{V} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ continuously for all $s \in \mathbb{R}$.

ii) **Gårding's inequality**
The single layer operator V is a strongly elliptic pseudo-differential operator of order -1 on $H^{-\frac{1}{2}}(\Gamma)$, and

$$\forall q \in H^{-\frac{1}{2}}(\Gamma) :\ \left\langle \mathrm{V}\,q, \overline{q}\right\rangle_\Gamma \geq C\,(\Gamma)\,\|q\|^2_{H^{-\frac{1}{2}}(\Gamma)}\,.$$

Consider the operator

$$A_0 = -\,\Delta_\Gamma\,\mathrm{V}\ .$$

Clearly, $A_0 : H^s(\Gamma) \to H^{s-1}(\Gamma)$ continuously and A_0 is, as composition of the pseudo-differential operator V and the Laplace-Beltrami operator $-\,\Delta_\Gamma$, itself a pseudo-differential operator of order $+1$ on Γ. By the calculus of pseudo-differential operators (e.g. [31]), its principal symbol is given by

$$\sigma_0(A_0) = \sigma_0(-\Delta_\Gamma)\,\sigma_0(\mathrm{V}) = |\boldsymbol{\xi}|^2\,|\boldsymbol{\xi}|^{-1} = |\boldsymbol{\xi}|$$

which is positive for $\boldsymbol{\xi} \neq 0$, hence A_0 is strongly elliptic on $H^{\frac{1}{2}}(\Gamma)$: there is $c > 0$ and a compact operator T_0 on $H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ such that

$$\mathrm{Re}\left\langle (A_0 + T_0)\,\sigma, \overline{\sigma}\right\rangle_\Gamma \geq c\|\sigma\|^2_{H^{\frac{1}{2}}(\Gamma)} \forall\sigma \in H^{\frac{1}{2}}(\Gamma)\,.$$

Setting $T = T_0 + \alpha(\frac{1}{2}\, I + K')$, we get

$$\mathrm{Re}\,\langle (A+T)\,\sigma, \overline{\sigma}\rangle_\Gamma = \mathrm{Re}\,\langle (A_0+T_0)\,\sigma, \overline{\sigma}\rangle_\Gamma \geq c\|\sigma\|^2_{H^{\frac{1}{2}}(\Gamma)} \forall \sigma \in H^{\frac{1}{2}}(\Gamma)\,.$$

The assertion follows with

$$k(\sigma, \widetilde{\sigma}) := \langle T\sigma, \widetilde{\sigma}\rangle_\Gamma\,.$$

Hence A in (4.3) is a strongly elliptic pseudo-differential operator of order $+1$ on Γ and (4.6) implies that it is Fredholm. □

Proposition 4.2. *The homogeneous problem $A\sigma = 0$ admits only $\sigma = 0$.*

Proof. Assume not. Then $\phi(\mathbf{x}) = (\mathcal{S}\,\sigma)(\mathbf{x})$, $\mathbf{x} \in \Omega^- \cup \Omega^+$ satisfies (3.2a) in $\Omega^+ \cup \Omega^-$ and also (3.2c). Further, it also satisfies (3.2b) with $g = 0$ and the jump relation $[\phi] = 0$, where we will denote by $[u]$ the (downward counted) jump in the function u as it crosses Γ in positive normal direction.

Since the PDE problem with $g = 0$ admits only the zero solution in Ω^+, we have for $\phi^+ = \phi|_{\Omega^+}$ that $\phi^+|_\Gamma = 0$. By $[\phi] = 0$ the function $\phi^- = \phi|_{\Omega^-}$ satisfies $\phi^-|_\Gamma = 0$. Thus, in Ω^- we have

$$\Delta\,\phi^- = 0 \;\text{ in }\; \Omega^-, \;\; \phi^-|_\Gamma = 0\,,$$

which implies $\phi^- = 0$ in Ω^- by the uniqueness of the Dirichlet problem, hence $\phi \equiv 0$ in $\Omega^+ \cup \Omega^-$. Since $\phi(\mathbf{x}) = (\mathcal{S}\,\sigma)(\mathbf{x})$, this implies that $\sigma \equiv 0$, since by the representation formula

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \int\limits_{\mathbf{y}\in\Gamma} \left[\frac{\partial\phi}{\partial n}\right] \frac{1}{|\mathbf{x}-\mathbf{y}|}\, ds_{\mathbf{y}}\,,$$

$\sigma = [\frac{\partial\phi}{\partial n}] = 0$ on Γ, a contradiction. □

Theorem 4.1. *For every $g \in H^{-\frac{1}{2}}(\Gamma)$, the variational problem* (4.5) *admits a unique solution $\sigma \in H^{\frac{1}{2}}(\Gamma)$ and its potential $\phi(\mathbf{x}) = (\mathcal{S}\,\sigma)(\mathbf{x})$ is the unique solution of the problem* (3.2)*.*

Proof. The existence and uniqueness follow from (4.6) and Proposition 4.2 in the usual way. The potential $(\mathcal{S}\,\sigma)$ is, by construction, a solution of (3.2a,b,c). By the continuity of $\mathcal{S}$, $\sigma \in H^{\frac{1}{2}}(\Gamma) \Longrightarrow \phi \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ (see, e.g., [13], Theorem 1 (i)), whence $\phi|_\Gamma \in H^{\frac{3}{2}}(\Gamma)$ and $\phi \in \mathcal{V}^+$. Since $H^{-\frac{1}{2}}(\Gamma) \subset (H^1(\Gamma))^*$, $g \in (H^1(\Gamma))^*$ and by Theorem 3.1 the PDE problem (3.2) has a solution $\widetilde{\phi} \in \mathcal{V}^+$. By the uniqueness of this solution, $\widetilde{\phi} = \phi$. □

5 Regularity

The BVP is not elliptic in the classical sense, since the boundary operator $B\phi = -\Delta_\Gamma\,\phi - \alpha\,\frac{\partial\phi}{\partial n}$ is not subordinate to the differential operator Δ in $\Omega^+ \cup \Omega^-$. Thus, standard elliptic regularity theory does not apply. Nevertheless, the reformulation of (3.2) as a pseudo-differential equation (4.3) on Γ and the strong ellipticity of A allow us to prove

Theorem 5.1. *Assume $\Gamma \in C^\infty$ and $g \in H^{-\frac{1}{2}+s}(\Gamma)$, $s \geq 0$ in* (3.2b)*. Then the solution ϕ of* (3.4) *satisfies $\phi \in H^{2+s}_{\mathrm{loc}}(\mathbb{R}^3)$, $\phi|_\Gamma \in H^{\frac{3}{2}+s}(\Gamma)$.*

Proof. If $g \in H^{-\frac{1}{2}+s}(\Gamma)$, strong ellipticity and continuity of A in (4.3) imply that $\sigma \in H^{\frac{1}{2}+s}(\Gamma)$, hence we get

$$\phi = (\mathcal{S}\,\sigma) \in H^{2+s}_{\mathrm{loc}}(\mathbb{R}^3)\,,$$

by the mapping properties of the single layer potential [13]. □

Remark 5.1. Since $\Delta\,\phi = 0$ in Ω^+, we have of course $\phi \in C^\infty(\Omega^+)$. The essential point of Theorem 5.1 is the regularity of ϕ up to Γ.

6 Galerkin discretization

Let $\{\mathcal{T}_h\}_h$ be a quasi-uniform, shape-regular family of triangulations on Γ consisting of triangular and/or quadrilateral curvilinear boundary elements.

Remark 6.1. Here and in what follows, we shall assume exact element mappings — i.e. there is no surface discretization, unless stated explicitly otherwise.

By S^p_h, we shall denote the space of continuous, piecewise polynomial functions in local coordinates on Γ of total (separate) degree p, $p \geq 1$. Then, for any $\varepsilon > 0$,

$$S^p_h \subset H^{\frac{3}{2}-\varepsilon}(\Gamma),\; \dim S^p_h = \mathcal{O}\big(p^2|\mathcal{T}_h|\big)$$

where $|\mathcal{T}_h| = \#$ of elements in $\mathcal{T}_h$. The space S^p_h has the **approximation property**

$$\min_{v\in S^p_h} \|\sigma - v\|_{H^s(\Gamma)} \leq C\, h^{\min(p+1,k)-s}\|\sigma\|_{H^k(\Gamma)} \qquad (6.1)$$

for $\sigma \in H^k(\Gamma)$, where C is independent of h and σ, and $0 \leq s \leq 1$, $k \geq s$.

The Galerkin-BEM for the problem (4.5) reads:
Find $\sigma_N \in S^p_h$ such that

$$a(\sigma_N, \widetilde{\sigma}) = \langle g, \overline{\widetilde{\sigma}}\rangle_\Gamma \quad \forall \widetilde{\sigma} \in S^p_h\,. \qquad (6.2)$$

By the Gårding inequality and the injectivity of A we have from Theorem 5.1 and (6.1):

Proposition 6.1. *For h sufficiently small,* (6.2) *admits a unique solution $\sigma_N \in S^p_h$ and*

$$\|\sigma - \sigma_N\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \inf_{v\in S^p_h} \|\sigma - v\|_{H^{\frac{1}{2}}(\Gamma)}\,.$$

If, in particular, $g \in H^{-\frac{1}{2}+s}(\Gamma)$, $s \geq 0$, as $h \to 0$:

$$\|\sigma - \sigma_N\|_{H^{\frac{1}{2}}(\Gamma)} \leq C\, h^{\min(p+\frac{1}{2},s)}\|g\|_{H^{-\frac{1}{2}+s}(\Gamma)}\,. \qquad (6.3)$$

For the calculation of losses, we shall be also interested in $H^{-\frac{1}{2}}(\Gamma)$-error estimates. Here we have

Proposition 6.2. *If $g \in H^{-\frac{1}{2}+s}(\Gamma)$, $s \geq 1$, then*

$$\|\sigma - \sigma_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\, h^{\min(p+\frac{3}{2},s+1)}\|g\|_{H^{-\frac{1}{2}+s}(\Gamma)}\,. \qquad (6.4)$$

Proof. We use the usual duality argument: Let $e_N = \sigma - \sigma_N$. Then

$$\|e_N\|_{H^{-\frac{1}{2}}(\Gamma)} = \sup_{\xi \in H^{\frac{1}{2}}(\Gamma)} \frac{\langle e_N, \xi \rangle}{\|\xi\|_{H^{\frac{1}{2}}(\Gamma)}} .$$

For $\xi \in H^{\frac{1}{2}}(\Gamma)$ let $\phi_\xi \in H^{\frac{1}{2}}(\Gamma)$ solve

$$A^* \phi_\xi = \xi .$$

Since $\xi \in H^{\frac{1}{2}}(\Gamma)$ and the adjoint

$$A^* : H^s(\Gamma) \longrightarrow H^{s-1}(\Gamma)$$

is an isomorphism, we have

$$\|\phi_\xi\|_{H^{\frac{3}{2}}(\Gamma)} \leq C \, \|\xi\|_{H^{\frac{1}{2}}(\Gamma)} .$$

Using the Galerkin orthogonality, we write

$$\begin{aligned} \langle e_N, \xi \rangle_\Gamma &= \langle A^{-1} A \, e_N, \xi \rangle_\Gamma = \langle A \, e_N, A^{*-1} \, \xi \rangle_\Gamma \\ &= \langle A \, e_N, \phi_\xi \rangle_\Gamma = a(e_N, \phi_\xi - \eta) \end{aligned}$$

for any $\eta \in S_h^p$. By the continuity of $a(\cdot, \cdot)$ and (6.1)

$$\begin{aligned} |\langle e_N, \xi \rangle_\Gamma| &\leq C \, \|e_N\|_{H^{\frac{1}{2}}} \, \|\phi_\xi - \eta\|_{H^{\frac{1}{2}}} \\ &\leq C \, h \, \|e_N\|_{H^{\frac{1}{2}}} \, \|\phi_\xi\|_{H^{\frac{3}{2}}} \\ &\leq C \, h \, \|e_N\|_{H^{\frac{1}{2}}} \, \|\xi\|_{H^{\frac{1}{2}}} . \end{aligned}$$

Hence it follows that

$$\|e_N\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \, h \, \|e_N\|_{H^{\frac{1}{2}}(\Gamma)}$$

and using (6.3) we obtain (6.4). □

Proposition 6.2 gives immediately a result on the convergence of the Ohmic losses, i.e. the quantities

$$\begin{aligned} P &= \frac{\operatorname{Re} \eta}{2} \int_\Gamma |\mathbf{H}_t|^2 \, ds \\ &= \frac{\operatorname{Re} \eta}{2} \int_\Gamma |\mathbf{H}_{0t} - \mathbf{grad}_\Gamma \, \mathrm{V} \, \sigma|^2 \, ds . \end{aligned} \tag{6.5}$$

Let P_N be as in (6.5) with σ replaced by σ_N. Then

$$\begin{aligned} \frac{2}{\operatorname{Re} \eta} |P - P_N| &= \Big| \int_\Gamma |\mathbf{H}_t|^2 - |\mathbf{H}_t^N|^2 \, ds \Big| \\ &= \Big| \int_\Gamma \operatorname{Re} \left(|\mathbf{H}_t|^2 - |\mathbf{H}_t^N|^2 - 2 \, i \, \operatorname{Im}(\overline{\mathbf{H}_t} \cdot \mathbf{H}_t^N) \right) ds \Big| \\ &= \Big| \int_\Gamma \operatorname{Re} \Big(\big(\mathbf{H}_{st} - \mathbf{H}_{st}^N\big) \cdot \big(\overline{\mathbf{H}_t + \mathbf{H}_t^N}\big) \Big) \, ds \Big| \\ &\leq \int_\Gamma \Big| \big(\mathbf{grad}_\Gamma \, \mathrm{V}(\sigma - \sigma_N)\big) \cdot \big(\overline{\mathbf{H}_t + \mathbf{H}_t^N}\big) \Big| \, ds \\ &\leq \| \, \mathbf{grad}_\Gamma \, \mathrm{V}(\sigma - \sigma_N) \|_{H^{-\frac{1}{2}}(\Gamma)} \|\mathbf{H}_t + \mathbf{H}_t^N\|_{H^{\frac{1}{2}}(\Gamma)} \\ &\leq C \Big(\|\mathbf{H}_t\|_{H^{\frac{1}{2}}(\Gamma)} + \|\mathbf{H}_t^N\|_{H^{\frac{1}{2}}(\Gamma)} \Big) \|\sigma - \sigma_N\|_{H^{-\frac{1}{2}}(\Gamma)} . \end{aligned}$$

By the boundedness of σ_N in $H^{\frac{1}{2}}(\Gamma)$ and the stability of the Galerkin scheme (6.2),

$$\|\mathbf{H}_t\|_{H^{\frac{1}{2}}(\Gamma)} + \|\mathbf{H}_t^N\|_{H^{\frac{1}{2}}(\Gamma)} \leq C$$

with C independent of N and (6.4) gives

Proposition 6.3. *If $g \in H^{-\frac{1}{2}+s}(\Gamma)$, the Ohmic losses converge, as $h \to 0$, like*

$$|P - P_N| \leq C \, h^{\min(p+\frac{3}{2}, s+1)} \|g\|_{H^{-\frac{1}{2}+s}(\Gamma)} .$$

In particular, if $p = 1$ and g is smooth, the Ohmic losses converge as $\mathcal{O}(h^{\frac{5}{2}})$.

7 Numerical Results

By defining the integral operator

$$(\mathbf{K} \, \sigma)(\mathbf{x}) := -p.v. \int_\Gamma \sigma(\mathbf{y}) \, \frac{\mathbf{x} - \mathbf{y}}{4\pi \, |\mathbf{x} - \mathbf{y}|^3} \, ds_{\mathbf{y}}$$

we rewrite the sesquilinear form (4.5) for computational purposes:

$$\begin{aligned} a(\sigma, \widetilde{\sigma}) &= \Big\langle \mathbf{grad}_\Gamma \, \mathrm{V} \, \sigma, \, \overline{\mathbf{grad}_\Gamma \, \widetilde{\sigma}} \Big\rangle_\Gamma \\ &\quad + \alpha \Big\langle \Big(\frac{1}{2} \, I - K' \Big) \, \sigma, \overline{\widetilde{\sigma}} \Big\rangle_\Gamma \\ &= \Big\langle \mathbf{K} \, \sigma, \, \overline{\mathbf{grad}_\Gamma \, \widetilde{\sigma}} \Big\rangle_\Gamma \\ &\quad + \alpha \Big\langle \Big(\frac{1}{2} \, I - \mathbf{n} \cdot \mathbf{K} \Big) \, \sigma, \overline{\widetilde{\sigma}} \Big\rangle_\Gamma \end{aligned}$$

Here we have pulled the surface gradient under the single layer operator and utilized the fact that the normal component of $\mathbf{K} \, \sigma$ does not contribute to the scalar product integral. On the RHS we do an integration by parts, which yields

$$\begin{aligned} \Big\langle g, \overline{\widetilde{\sigma}} \Big\rangle_\Gamma &= -\Big\langle \operatorname{div}_\Gamma \, \mathbf{H}_{0t} + \alpha \, \mathbf{H}_{0n}, \overline{\widetilde{\sigma}} \Big\rangle_\Gamma \\ &= \Big\langle \mathbf{H}_0, \, \mathbf{grad}_\Gamma \, \overline{\widetilde{\sigma}} \Big\rangle_\Gamma - \alpha \Big\langle \mathbf{H}_{0n}, \overline{\widetilde{\sigma}} \Big\rangle_\Gamma . \end{aligned}$$

For a computational realisation of the presented method we apply triangular boundary elements with piecewise quadratic geometry representation and a piecewise linear ansatz for the unknown density and the test-functions. That is for every element $K_j \in \mathcal{T}_h$ we have a map which allows one to pull functions defined on Γ back to a reference element, in our case to the unit simplex $\mathbb{R}^2 \supset U = \{\mathbf{u} : 0 < u_1 < 1, 0 < u_2 < u_1\}$ (see figure 6.1)[1]:

$$\boldsymbol{\chi}_j : U \to \Gamma \subset \mathbb{R}^3 ,$$

$$\boldsymbol{\chi}_j(\mathbf{u}) = \sum_{i=1}^{6} N_i(\mathbf{u}) \, \mathbf{x}_{j_i} ,$$

with $\mathbf{u} = (u_1, u_2)^T \in U$ and $\mathbf{x}_{j_1}, \mathbf{x}_{j_2}, \mathbf{x}_{j_3}$ being the vertices of element K_j and $\mathbf{x}_{j_4}, \mathbf{x}_{j_5}, \mathbf{x}_{j_6}$ the mid points of the element edges. The N_i are either 2nd order polynomial or bilinear functions in $\mathbf{u}$. With $u_3 := 1 - u_1 - u_2$ they read

$$\begin{aligned} N_1(\mathbf{u}) &= u_1 \, (2 \, u_1 - 1) & \qquad N_4(\mathbf{u}) &= 4 \, u_1 \, u_2 \\ N_2(\mathbf{u}) &= u_2 \, (2 \, u_2 - 1) & \qquad N_5(\mathbf{u}) &= 4 \, u_2 \, u_3 \\ N_3(\mathbf{u}) &= u_3 \, (2 \, u_3 - 1) & \qquad N_6(\mathbf{u}) &= 4 \, u_3 \, u_1 . \end{aligned}$$

[1] All graphical output displaying the approximated unit sphere is generated by CADfix, FEGS Ltd., www.fegs.co.uk

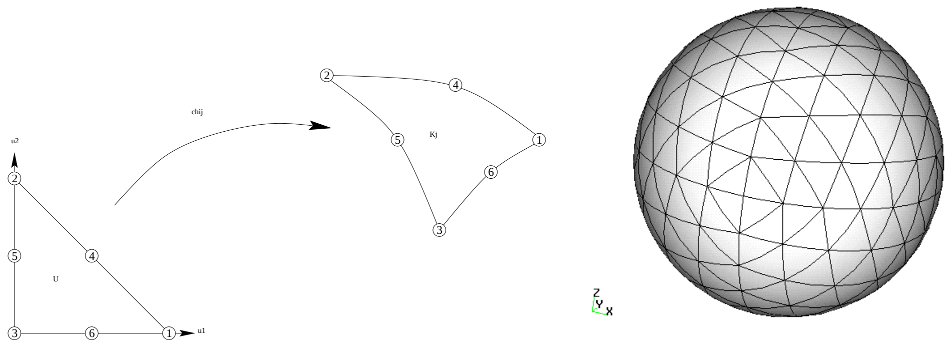

Fig. 6.1. Left: $\boldsymbol{\chi}_j$ maps the unit simplex onto element K_j. Right: Mesh No. 4 with 288 elements

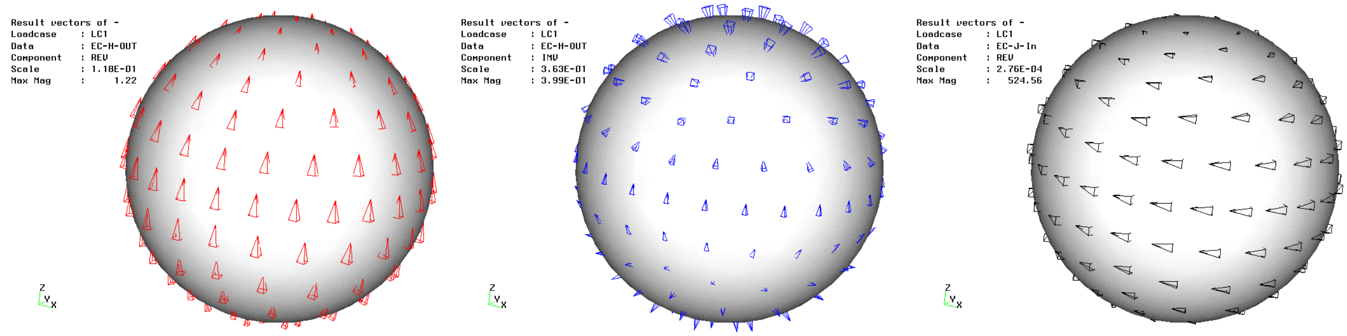

Fig. 7.1. From left to the right: real and imaginary part of the magnetic field at the surface and the real part of the current density at the surface. The imaginary part of the current density qualitatively looks like the real part, but has a maximum magnitude 730.0

Furthermore, we use $\sigma_N, \overline{\sigma} \in S_h^1$,

$$\sigma_N(\mathbf{x}) = \sum_{j=1}^{N} \varphi_j(\mathbf{x})\, \sigma_{Nj}\,,$$

with φ_j being the so called hat functions, which form a basis of S_h^1, with the property $\varphi_j(x_i) = \delta_{ji}$ for all nodes $\mathbf{x}_i$ of a given triangulation $\mathcal{T}_h$ and $\varphi_i|_{K_j} \circ \boldsymbol{\chi}_\mathbf{j}$ being linear for all $K_j \in \mathcal{T}_h$. $N = |\mathcal{N}(\mathcal{T})| = \#$ of vertices of $\mathcal{T}_h$.

The discrete Galerkin form then reads

$$\sum_{j=1}^{N} a_{Nij}\, \sigma_{Nj} = g_{Ni}\,, \qquad i = 1..N\,, \tag{7.1}$$

where

$$a_{Nij} := \int\limits_{\operatorname{supp}\varphi_i} \Big(\big(\mathbf{grad}_\Gamma\, \varphi_i - \alpha\, \varphi_i\, \mathbf{n}\big) \cdot \mathbf{K}\, \varphi_j + \frac{\alpha}{2}\, \varphi_i\, \varphi_j \Big)\, ds$$

$$g_{Ni} := \int\limits_{\operatorname{supp}\varphi_i} \big(\mathbf{grad}_\Gamma\, \varphi_i - \alpha\, \varphi_i\, \mathbf{n}\big) \cdot \mathbf{H}_0\, ds\,.$$

Note that the numerical integration of the singular elements must be carried out with care because the tangential part of $\mathbf{K}\,\sigma_N$ shows logarithmic singularities at the element boundaries (see, e.g., [29]). At first glance for smooth surfaces Γ this problem might appear to be only apparent due to the fact that the singularities will cancel by integrating over the whole surface. However, it is quite obvious that every numerical quadrature scheme necessitates a subdivision of Γ, and thus a numerical integration over a single element must deal with this singularity. This is the reason why standard quadrature is not applicable here. An adequate quadrature scheme can be found in [14] and [25]. There the double surface integrals of the Galerkin form are considered as integrals over four dimensional domains and the singularities are cancelled by domain transformations introducing relative coordinates. Using this integration method our implementation shows the validity of the presented method.

The linear equation system (7.1) has been solved by a GMRES solver. The calculation of the magnetic field strength on the boundary Γ

$$\mathbf{H}_N(\mathbf{x}) = \big(\tfrac{1}{2}\,\mathbf{n} - \mathbf{K}\big)\, \sigma_N + \mathbf{H_0}\,,$$

is a post processing step. Here we adopt a method presented in [15] for the integration of the singular elements that changes the strongly singular integrals into a sum of regular integrals.

From the magnetic field $\mathbf{H}$ the current density $\mathbf{J}$ can easily be obtained via (1.3) and Ohm's law

$$\mathbf{J} = \kappa\, \mathbf{E}\,.$$

Remark 7.1. It is obvious that the surface approximation by piecewise quadratic interpolation is neither C^∞ nor C^1. The theory in the previous sections ignores the surface approximation as well as the effects of numerical quadrature. Nevertheless, the following numerical example shows the predicted convergence rate.

Numerical example. As a test problem we consider an approximated sphere of radius 1 m (see figure 6.1) in a constant exciting field $\mathbf{H}_0 = 1\,\mathrm{A/m}\,\mathbf{e}_z$ and $\omega = 2\,\pi\, 50\,\mathrm{s}^{-1}$ with material parameters typical for steel, i.e.

Table 7.1. Hierarchy of grids

mesh No.	unknowns ($\lvert\mathcal{N}_h\rvert$)	elements ($\lvert\mathcal{T}_h\rvert$)
1	6	8
2	18	32
3	66	128
4	146	288
5	258	512
6	402	800
7	578	1152
8	1026	2048
9	2118	4232

$\kappa = 0.666 \cdot 10^7\,\mathrm{S/m}$ and $\mu_r = 200$. Thus we have a penetration depth $\delta = 1.95\,\mathrm{mm}$ and the eddy current model with an IBC is a reasonable assumption (in the sense that there is a physical meaning). For this case the constant in the sesquilinear form yields $\beta = 2.56\,\mathrm{m}^{-1}$ and both terms of the form do contribute numerically. Numerical values of all physical quantities are represented in SI-units.

We generated a family of meshes to verify the predicted convergence of the total power loss (see table 7.1).

Our numerical experiments show good agreement with the analytical solution of this problem which can be found in many textbooks, e.g. [32]. For a plot of the surface density see figure 7.2. The corresponding magnetic field and current density at the surface are shown in 7.1. Moreover figure 7.3 confirms the predicted convergence rate of the total power loss, which is $\mathcal{O}(h^{\frac{5}{2}})$.

For the sake of completeness we present the two limiting cases for the constant $\alpha = (1+i)\,\beta$:

i) $\beta \to \infty$
This is the **perfect conductor limit** as $\kappa \to \infty$. Here only the compact part of the sesquilinear form is left. The PDE problem changes to a standard exterior Neumann problem, which is well posed. See figure 7.4 for a plot of the magnetic field and the surface density σ for this case. The IBC (1.3) degenerates to the perfect conductor boundary condition

$$\mathbf{n} \times \mathbf{E} = 0$$

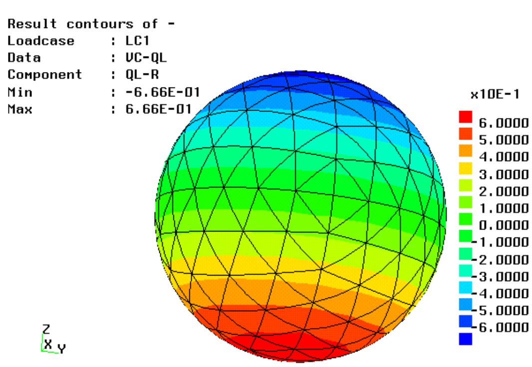

Fig. 7.2. The real part of the surface density σ. The imaginary part looks the same but $\max(\sigma_N) = 0.594$.

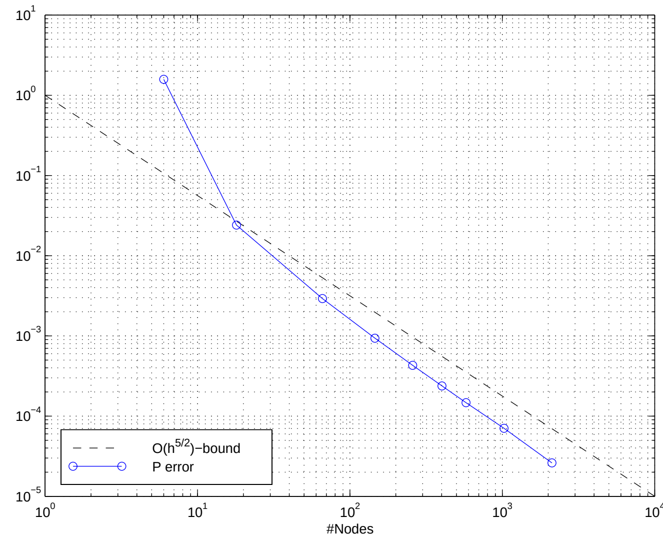

Fig. 7.3. Relative error of the total power loss depending on the number of unknowns

which is often used in scattering problems.

ii) $\beta \to 0$
This can be designated as the **magnetostatic limit** if $\mu \to \infty$, while ω, κ remain bounded. Since here the compact part is missing the problem becomes ill posed. There exist nontrivial solutions of the homogeneous problem with a constant potential on Γ and one observes a very high condition number of the stiffness matrix.
The solution of the problem becomes unique by imposing

$$\int_{\Gamma} \mathbf{H}\, ds = 0$$

for example. Due to the symmetry of our numerical example this can simply be done by enforcing antisymmetry of the unknown surface density σ. The solution is shown in figure 7.5.

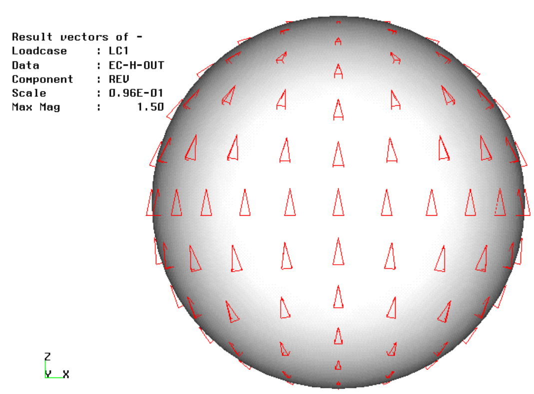

Fig. 7.4. The magnetic field if $\kappa \to \infty$

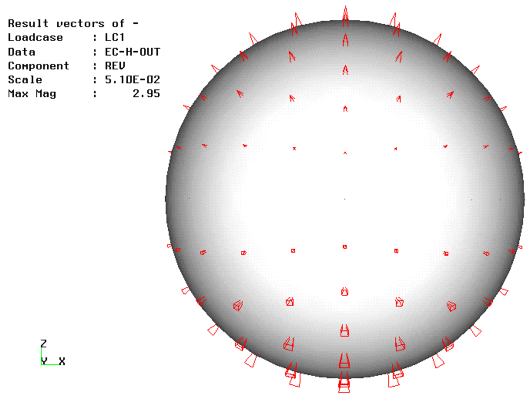

Fig. 7.5. The magnetic field if $\mu \to \infty$

In both cases the Poynting vector becomes tangent to Γ and there is no power loss in the body.

Acknowledgements. The authors wish to acknowledge Prof. Dr. K.V. Boos and Dr. B. Schaub, ABB Corporate Research Center Heidelberg, for making this work possible and Prof. Dr. Z. Andjelic, ABB Corporate Research Center Heidelberg, for many helpful discussions concerning the numerical implementation of the presented method.

References

1. K.F. Ali, M. Taher Ahmet and P.E. Burke: Surface impedance-BEM technique for nonlinear TM-eddy current problems. J. Appl. Phys. **61** (1987), 3925-3927.
2. H. Ammari, A. Buffa and J.-C. Nédélec: A justification of eddy currents model for the Maxwell equations. Tech. rep. (IAN, University of Pavia, Pavia, Italy, 1998).
3. H. Ammari and J.C. Nédélec: Sur les conditions d'impédance généralisées pour les couches minces. Comptes Rendus de l'Académie des Sciences de Paris. Série I, **322** (1996), 995-1000.
4. H. Ammari, M. Laouadi and J.C. Nédélec: Low frequency behaviour of solutions to electromagnetic scattering problems in chiral media. SIAM Journal on Applied Mathematics, to appear.
5. H. Ammari and J.-C. Nédélec: Generalized impedance boundary conditions for the Maxwell equations as singular perturbation problems. Comm. Part. Diff. Equat. **24** (1999), no. 5&6, 821-850.
6. H. Ammari, C. Latiri-Grouz and J.C. Nédélec: The Leontovich boundary value problem for the time-harmonic Maxwell equations in an inhomogeneous medium: a singular perturbation problem. SIAM Journal on Applied Mathematics **59** (1999), no. 4, 1322-1334.
7. H. Ammari, C. Latiri-Grouz and J.C. Nédélec: The Leontovich boundary value problem for the time-harmonic Maxwell equations. Asymp. Anal. **18** (1998), no. 1&2, 33-48.
8. A. Bendali: Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element, Part I: the continuous problem, Part II: the discrete problem. Mathematics of Computation, **43** (1984), 29-46 and 47-68.
9. A. Bossavit: Two dual formulations of the 3D eddy-currents problem. COMPEL, **4** (1985).
10. A. Bossavit: *Computational electromagnetism.* Academic Press, San Diego (1998).
11. M. Cessenat: *Mathematical methods in electromagnetism.* World Scientific, Singapore (1996).
12. D. Colton and R. Kress: The impedance boundary value problem for the time-harmonic Maxwell equations. Mathematical Methods in the Applied Sciences, **3** (1981), 475-487.
13. M. Costabel: Boundary integral operators on Lipschitz domains. SIAM J. Math. Anal., **19** (1988), 613-626.
14. S. Erichsen and S. Sauter: Efficient automatic quadrature in 3-d Galerkin BEM. Computer methods in applied mechanics and engineering, **157** (1998), 215-224.
15. M. Guiggiani and A. Gigante: A general algoithm for multidimensional Cauchy principal value integrals in the Boundary Element Method. Journal of Applied Mechanics **57** (1990), 906-915.
16. S.I. Hariharan and R.C. MacCamy: Integral equation procedures for eddy current problems. Journal of Computational Physics, **45** (1982), 80-99.
17. St. Hildebrandt and N. Wienholtz: Constructive Proofs of representation theorems in separable Hilbert Space. Comm. Pure Appl. Math. **XVII** (1964) 369-373.
18. L.M. Leontovich: On approximate boundary conditions for electromagnetic fields on the surface of highly conducting bodies (in russian), *Research in the propagation of radio waves* (Academy of Sciences of the USSR, Moscow 1948).
19. J.L. Lions and E. Magenes: *Non-homogeneous boundary value problems and applications I.* Springer, New York (1972).
20. R.C. MacCamy and E. Stephan: Solution procedures for three-dimensional eddy-current problems. Journal of Mathematical Analysis and Applications, **101** (1984), 348-379.
21. E. Martensen: *Potentialtheorie.* Teubner, Stuttgart (1968).
22. Mayergoyz and G. Bredosian: On finite element implementation of impedance boundary conditions. J. Appl. Phys. **75**, 10 (1994), 6027-6029.
23. K.M. Mitzner: An integral equation approach to scattering from a body of finite conductivity. Radio Science **2** (1967), 1459-1470.
24. J.C. Nédélec: Integral Equations, Chapter IX, in Vol. 4 of *Mathematical analysis and numerical methods for science and technology* (Dautray and Lions, Berlin 1990), 114 ff.
25. T. v. Petersdorff and C. Schwab: Fully discrete multiscale Galerkin BEM, in Vol. 6 of *Wavelet analysis and applications* (Dahmen, Kurdila and Oswald, Academic Press 1997), 287 ff.
26. M. Reissel: On a transmission boundary-value problem for the time-harmonic Maxwell equations without displacement currents. SIAM J. Math. Anal. **24**, no. 6, (1993).
27. S.M. Rytov: Calcul du skin-effet par la méthode des perturbations. Journal de Physique USSR, **2** (1940), 233-242.
28. G. Sande: Computation of induced currents in three dimensions. Thesis (University of Trondheim, Norway, 1993).
29. C. Schwab and W.L. Wendland: On numerical cubatures of singular surface integrals in boundary element methods. Numerische Mathematik **62** (1992), 343-369.

30. T.B.A. Senior and J.L. Volakis: Approximate boundary conditions in electromagnetics. IEE Electromagnetic Waves Series 41, IEE Publications, London (1995).
31. M. Taylor: *Pseudodifferential operators.* Princeton University Press, Princeton (1981).
32. J. Van Bladel: *Electromagnetic fields (rev. ed.).* Hemisphere Pub, (1988).
33. N. Weck and K.J. Witsch: Low-frequency asymptotics for dissipative Maxwell's equations in bounded domains. Mathematical Methods in the Applied Sciences **13** (1990), 81-93.