Floer homology and surface diffeomorphisms

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Abstract

In this thesis we treat symplectic Floer theory in two dimensions. According to Seidel, this theory assigns a module to each mapping class of a closed connected oriented two-manifold of genus bigger than one. The notion of monotone symplectomorphisms is thereby central.

We compute this module for algebraically finite mapping classes, i.e. classes which do not have pseudo-Anosov components in the sense of Thurston’s theory of surface diffeomorphisms. The Nielsen-Thurston representative of such a class is shown to be monotone. The formula for the Floer homology is obtained from a topological separation of fixed points and a separation mechanism for Floer connecting orbits.

As an example, we consider the geometric monodromy map of an isolated plane curve singularity. We prove a refined version of the classical theorem of A’Campo and Lê which states that the Lefschetz number of such a map vanishes. Our proof uses, besides the A’Campo-Lê Theorem, the theory of splice diagrams which was developed by Eisenbud and Neumann.

This result has two applications in the realm of Floer theory. First, we think of the Milnor fiber of an isolated plane curve singularity as being contained in a closed oriented two-manifold and consider the mapping class which is obtained by extending the geometric monodromy trivially to the ambient space. In this case, our formula for the Floer homology takes a particularly simple form. Second, we show that the Floer homology of the geometric monodromy in itself, as a mapping class of a two-manifold with boundary, vanishes. This confirms a conjecture of Seidel.

In the second part of this thesis, which is independent of the first one, we prove a series of assertions on subgroups of surface groups, i.e. fundamental groups of closed connected two-manifolds.

Our main theorem states that if the rank of a subgroup of a surface group is smaller than the rank of the surface group, then the subgroup is free. This result is related to the Freiheitssatz, which a classical theorem in combinatorial group theory. Our proof uses the classification of compact two-manifolds and a topological lemma of Epstein.

As an application of this theorem, we prove an assertion about homotopy classes of maps from the torus to closed oriented two-manifolds, which is used in the first part of the thesis.
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1 Introduction

Around 1988, Floer introduced in a series of articles an infinite dimensional version of Morse homology into symplectic geometry. Inspired by Witten’s approach to Morse theory, Floer combined the seminal works of Gromov and Conley-Zehnder and was able to prove the non-degenerate Arnold conjecture for monotone symplectic manifolds. During the past fourteen years, Floer’s work has been carried on and extended into various directions. To name a few:

- Generalization of the original construction in order to eliminate certain assumptions.
- Connection with other invariants in symplectic geometry and low-dimensional topology.
- Revealing of algebraic structures, i.e. products, $A_\infty$ structures, Frobenius structures.
- Computation of Floer homology; exact sequences; combinatorial approaches.

Each of these aspects is an active research area. The present thesis is placed in the last one.

Today, one can think of symplectic Floer homology theory in terms of a set of axioms. Here, we focus on what is also called Floer homology of symplectic fixed points. We would like to point out that our objective is not to be as general as possible. Instead, we adopt the point of view which best fits the subsequent work.

Consider a compact connected symplectic manifold $(X, \omega)$ with vanishing second homotopy group, i.e. $\pi_2(X) = 0$. Denote by $\text{Symp}$ the group of symplectomorphisms of $(X, \omega)$, that is $\omega$-preserving diffeomorphisms of $X$, and by $\text{Vect}$ the category of $\mathbb{Z}_2$-graded finite dimensional vector spaces over $\mathbb{Z}_2$. There exists a set

$$\text{Symp}^m \subseteq \text{Symp},$$

called set of monotone symplectomorphisms, and a functor

$$HF_* : \text{Symp}^m \rightarrow \text{Vect},$$

with the following properties. By $\phi$ we denote an element of $\text{Symp}^m$.

(Identity) $\text{id}_X \in \text{Symp}^m$ and $HF_*(\text{id}_X)$ is canonically isomorphic to $H_*(X; \mathbb{Z}_2)$.

(Isotopy) If $\phi'$ is Hamiltonian isotopic to $\phi$, then $\phi' \in \text{Symp}^m$ and any Hamiltonian isotopy induces an isomorphism $HF_*(\phi') \cong HF_*(\phi)$. $^\dagger$

(Naturality) If $\psi \in \text{Symp}$, then $\psi^{-1}\phi' \psi \in \text{Symp}^m$ and $\psi$ induces an isomorphism $HF_*(\psi^{-1}\phi' \psi) \cong HF_*(\phi)$.

(Lefschetz number) $\chi(HF_*(\phi)) = \Lambda(\phi)$, where $\chi$ denotes the Euler characteristic.

$^\dagger$Consider $\text{Symp}^m$ as a category with Hamiltonian isotopies as morphisms and $HF_*$ is a functor in the strict sense of the word.
and $\Lambda$ the Lefschetz number.

(Fixed points) If $\phi$ only has non-degenerate fixed points, i.e. $\det(id - d\phi_x) \neq 0$ for all $x \in \text{Fix}(\phi)$, then $\# \text{Fix}(\phi) \geq \dim HF_* (\phi)$.

(Product) There exists a bilinear map, called quantum cap product,

$$\cap : H^*(X; \mathbb{Z}_2) \otimes HF_*(\phi) \rightarrow HF_*(\phi),$$

which equips $HF_*(\phi)$ with a $H^*(X; \mathbb{Z}_2)$-module structure and agrees with the ordinary cap product for $\phi = id_X$.

(Duality) $\phi^{-1} \in \text{Symp}^m$ and there exists a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : HF_*(\phi) \otimes HF_*(\phi^{-1)} \rightarrow \mathbb{Z}_2,$$

such that $\langle \alpha \cap \beta, \gamma \rangle = \langle \beta, \alpha \cap \gamma \rangle$ for all $\alpha \in H^*(X; \mathbb{Z}_2), \beta \in HF_*(\phi)$ and $\gamma \in HF_*(\phi^{-1})$.

As the name suggests, $HF_*(\phi)$ is the homology of a chain complex, the Floer complex $(CF_*(\phi), \partial_F)$. If $\phi$ has only non-degenerate fixed points, which is generically the case, then

$$CF_*(\phi) = \mathbb{Z}_2^{\text{Fix}(\phi)}$$

and the $\mathbb{Z}_2$-grading is given by the sign of $\det(id - d\phi)$. The Floer boundary operator is defined by counting solutions $u : \mathbb{R}^2 \rightarrow X$ of the Floer equations

$$u(s, t) = \phi(u(s, t + 1)),$$
$$\partial_s u + J_t(u)\partial_t u = 0,$$
$$\lim_{s \rightarrow \pm \infty} u(s, t) \in \text{Fix}(\phi).$$

Here, $(J_t)_{t \in \mathbb{R}}$ is a smooth path of $\omega$-compatible almost complex structures on $X$ satisfying $J_{t+1} = \phi^* J_t$. Monotonicity and the vanishing of $\pi_2$ guarantee compactness of the relevant moduli spaces.

Typical extensions of this definition of $HF_*$ include a functor from $\text{Symp}$ to the category of modules over the universal Novikov ring, monotone symplectic manifolds and symplectic Floer homology theory for Lagrangian intersections.

Our goal, however, is not to generalize but to specialize to the case where $X$ is two-dimensional. It was observed by Seidel, that in this case Floer homology exhibits additional structure:

(Inclusion) The inclusion of $\text{Symp}$ into the group of orientation preserving diffeomorphisms of $X$, equipped with the $C^\infty$ topology, is a homotopy equivalence.

(Monotone invariance) If $\phi, \phi' \in \text{Symp}^m$ are isotopic, then $HF_*(\phi')$ is naturally isomorphic to $HF_*(\phi)$.

Using these two properties, Seidel was able to define the functor

$$HF_* : \mathcal{M} \rightarrow \text{Mod},$$
where \( \mathcal{M} \) denotes the mapping class group of \( X \) and \( \text{Mod} \) the category of \( \mathbb{Z}_2 \)-graded \( H^\ast(X; \mathbb{Z}_2) \)-modules. This functor is the central object of the present work. The main result is a formula for the restriction of \( HF^\ast \) to a certain subset of \( \mathcal{M} \). Note that due to Moser’s isotopy theorem and naturality, Floer homology in two dimensions is rather a topological than symplectic invariant.

As already mentioned, the first application of Floer’s theory was his proof of the Arnold conjecture in the monotone case: If a Hamiltonian symplectomorphism has only non-degenerate fixed points, then the number of fixed points is bounded from below by the sum of the Betti numbers of the manifold. This can be viewed as an existence result for periodic orbits of Hamiltonian systems. By now, the non-degenerate Arnold conjecture has been proven in full generality.

In his thesis, Seidel applied Floer homology theory to study the symplectic isotopy problem which asks whether symplectic isotopy is a finer relation between symplectomorphisms than smooth isotopy. His answer was positive: The square of the generalized Dehn twist along a Lagrangian two-sphere in a compact symplectic four-manifold is smoothly isotopic, but, under certain circumstances, not symplectically isotopic to the identity. Moreover, Seidel discovered an exact sequence in Floer homology which allowed him to compute \( HF^\ast \) of the generalized Dehn twist explicitly. Using the duality axiom of Floer homology, he found conditions on the four-manifold under which \( HF^\ast \) of the generalized Dehn twist and its inverse are not isomorphic.

In view of these results and also from a conceptual perspective, Seidel’s exact sequence seems to be a promising tool for a systematic approach to the computation of Floer homology groups. Over the last years, Seidel has extended the exact sequence to include more general symplectomorphisms and Lagrangian intersections.

Besides that, he has made several other contributions to the computation of \( HF^\ast \). Of particular interest for this thesis is the computation of the Floer homology of a Dehn twist in dimension two. Let \( \Sigma \) denote a compact connected and oriented two-manifold of genus bigger than one. Let \( C \subset \Sigma \) be an embedded non-contractible circle and denote by \( \tau \) the positive Dehn twist along \( C \). Seidel proved that

\[
HF^\ast(\tau) \cong H_\ast(\Sigma, C; \mathbb{Z}_2),
\]

where the \( H^\ast(\Sigma; \mathbb{Z}_2) \)-module structure on the right hand side is given by the ordinary cap product. The result generalizes to products of disjoint positive and negative Dehn twists. This time, Seidel derived the formula for the Floer homology by explicitly computing the Floer complex. For this, the following two results were crucial:

(i) Topological separation of fixed points: if \( x \) and \( y \) are fixed points of \( \tau \) which lie in different connected components of \( \Sigma \setminus C \), then \( x \) and \( y \) are in different Nielsen fixed point classes.

\[\text{\textsuperscript{1}}\text{The Lagrangian two-sphere itself represents a non-trivial class in } \pi_2. \text{ Floer homology is therefore only defined with coefficients in the Novikov ring.}\]
(ii) Separation mechanism for Floer connecting orbits: after suitably pertur-

b- 

ging \( \tau \), every Floer connecting orbit starting and ending in the same connected component of \( \Sigma \setminus C \) does not touch the curve \( C \).

This approach to the computation of Floer homology groups was the starting point of the present thesis. Later on it turned out, that (i) and (ii) are principles with a much wider range of application than products of disjoint Dehn twists.

The central result of this work is a formula for the Floer homology of algebraically finite mapping classes. The term algebraically finite goes back to Nielsen. In the language of Thurston’s theory of surface diffeomorphisms, a mapping class is algebraically finite, if it does not have any pseudo-Anosov components. Such a mapping class is represented by a diffeomorphism \( \phi : \Sigma \to \Sigma \) which is of the following special form and which we call a diffeomorphism of finite type.

Think of \( \Sigma \) as obtained by piecing together two-manifolds \( \Sigma' \) with boundary; \( \Sigma' \) need not be connected, but their connected components are all diffeomorphic. Moreover, \( \Sigma' \) is not a union of disks. Think of \( \phi \) as obtained by piecing together diffeomorphisms \( \phi' : \Sigma' \to \Sigma' \). Each of the \( \phi' \) cyclically permutes the connected components of the corresponding \( \Sigma' \) and is either a periodic, a flip-twist or a twist map without fixed points. The precise definition is given in the next chapter.

A finite type diffeomorphism \( \phi \) is a symplectomorphism with respect to some area form \( \omega \) on \( \Sigma \). We show that \( \omega \) can be chosen such that \( \phi \) is a monotone symplectomorphism. Denote by \( \Sigma_0 \) the union of those \( \Sigma' \) where \( \phi \) restricts to the identity, and by \( \partial_+ \Sigma_0 \) the union of boundary components of \( \Sigma_0 \) where \( \phi \) twists in the positive sense.

**Theorem A.** Let \( \phi \) be a diffeomorphism of finite type and \( \Sigma_0 \) as above. Then

\[
HF_*(\phi) \cong H_*(\Sigma_0, \partial_+ \Sigma_0; \mathbb{Z}_2) \oplus \mathbb{Z}_2^\Lambda(\phi|\Sigma_0),
\]

where \( H^*(\Sigma; \mathbb{Z}_2) \) acts by ordinary cap product on the first summand and trivially on the second. Here, \( \Lambda \) denotes the Lefschetz number.

By the trivial action, we mean that \( 1 \in H^0(\Sigma; \mathbb{Z}_2) \) acts by the identity and any element of \( H^1(\Sigma; \mathbb{Z}_2) \oplus H^2(\Sigma; \mathbb{Z}_2) \) by the zero map.

As suggested by the formula above, the Floer complex of \( \phi \) splits into a complex associated to \( \phi|\Sigma_0 \) and one associated to \( \phi|\Sigma \setminus \Sigma_0 \). This phenomenon is due to the topological separation of fixed points and the separation mechanism for Floer connecting orbits which carry over, if spelled out suitably, to diffeomorphisms of finite type.

A natural source of algebraically finite mapping classes is provided by the theory of singularities. It assigns to an isolated plane curve singularity a compact connected oriented two-manifold with boundary, the Milnor fiber, and an isotopy class of orientation preserving diffeomorphisms of the Milnor fiber which are the identity near the boundary, called geometric monodromy.
Theorem B. Let $M \subset \Sigma$ be the Milnor fiber of an isolated plane curve singularity and $g$ be the mapping class which is obtained by extending the geometric monodromy trivially to $\Sigma$. Then

$$HF_\ast (g) \cong H_\ast (\Sigma; M; \mathbb{Z}_2),$$

where $H^\ast (\Sigma; \mathbb{Z}_2)$ acts by cap product.

A special case of this result is the following generalization of Seidel’s formula for the Floer homology of a Dehn twist.

Corollary C. Let $(C_1, \ldots, C_k)$ be an $A_k$-configuration of circles in $\Sigma$. Let $g$ be the mapping class of the product $\tau_1 \circ \cdots \circ \tau_k$, where $\tau_i$ denotes the positive Dehn twist along $C_i$. Then

$$HF_\ast (g) \cong H_\ast (\Sigma; C_1 \cup \cdots \cup C_k; \mathbb{Z}_2),$$

where $H^\ast (\Sigma; \mathbb{Z}_2)$ acts by cap product. The same formula holds, if $g$ is the mapping class of $\tau_{\sigma 1} \circ \cdots \circ \tau_{\sigma k}$, where $\sigma$ is a cyclic permutation of $k$ elements.

The proof of Theorem B relies on Theorem A and an additional result about the geometric monodromy.

Theorem D. Let $M$ be the Milnor fiber and $g$ be the geometric monodromy of an isolated plane curve singularity. There exists a diffeomorphism $\phi : M \to M$ of finite type which is isotopic relative to the boundary to an element of $g$ and satisfies $\text{Fix}(\phi) = \partial M$. Moreover, $\phi$ has only positive twists.

This result combines two classical facts in the theory of plane curve singularities, namely that the geometric monodromy is algebraically finite and that its Lefschetz number vanishes. The latter is a well-known theorem of A’Campo and Lê, which holds for holomorphic hypersurface singularities, isolated or non-isolated, in any dimension.

To prove Theorem D we use, besides the A’Campo-Lê result, the work of Eisenbud and Neumann on invariants of plane curve singularities. The combinatorial approach of Eisenbud-Neumann, based on splice diagrams, gives a rather explicit description of the geometric monodromy in terms of periodic and twist maps; it therefore suits our purpose.

Theorem D has a further application in the realm of Floer homology. There is a version of $HF_\ast$ for diffeomorphisms of compact oriented two-manifolds with boundary. It assigns to every isotopy class $g$ of orientation preserving diffeomorphisms which are the identity near the boundary, a pair $HF_\ast (g, +), HF_\ast (g, -)$ of $\mathbb{Z}_2$-vector spaces. The sign corresponds to two different ways of perturbing $g$ near the boundary.

Theorem E. If $g$ is the geometric monodromy of an isolated plane curve singularity, then

$$HF_\ast (g, +) = 0.$$
This was conjectured by Seidel for isolated holomorphic hypersurface singularities in any dimension. Following Seidel, Theorem E determines another invariant of the singularity. This invariant and the conjectural relation to the Floer homology of the monodromy involves the theory of $A_{\infty}$ structures, Fukaya categories, and Hochschild cohomology.

Let $\Gamma$ be an ordered collection of Lagrangian submanifolds in general position of a compact symplectic manifold. Generalizing Donaldson’s definition of the product structure in Floer theory, Fukaya introduced a sequence of product type operations

$$\mu : CF(L_1, L_2) \otimes \cdots \otimes CF(L_{k-1}, L_k) \longrightarrow CF(L_1, L_k),$$

where $(L_1, \ldots, L_k)$ is an ordered subcollection of $\Gamma$ and

$$CF(L, L') = Z_{2}^{L \cap L'},$$

for all $L \neq L' \in \Gamma$. The operation $\mu$ is defined by counting pseudoholomorphic polygons and satisfies a family of relations reminiscent of $A_{\infty}$ structures. This was carried on by Seidel who associated a directed Fukaya $A_{\infty}$ category $A$ to $\Gamma$ and introduced its Hochschild cohomology $HH^{*}(A, A)$.

Combining this with Picard-Lefschetz theory, he obtained an invariant of exact Morse fibrations \footnote{This is a generalization of the notion of exact symplectic fibrations, which allows critical points of Morse-like type. The base space is the unit disk in $\mathbb{C}$.}, in particular of isolated hypersurface singularities. In this case, $\Gamma$ is a distinguished basis of vanishing cycles in the fiber and the Hochschild cohomology does not depend on the choice of such a basis.

Motivated by certain ideas of Donaldson, Seidel conjectured the existence of a long exact sequence \footnote{We think of all vector spaces as being $\mathbb{Z}$-graded. Seidel lifted the gradings to $\mathbb{Z}$}:

$$\begin{array}{c}
HF^{*}(\phi, +) \longrightarrow H^{*}(E; Z_{2}) \longrightarrow HH^{*}(A, A) \longrightarrow HH^{*}(B, B)
\end{array}$$

Here, $E$ denotes the total space of the exact Morse fibration, $A$ the directed Fukaya $A_{\infty}$ category associated to a distinguished basis of vanishing cycles and $\phi$ the global monodromy. In the case of an isolated plane curve singularity, $E$ is homeomorphic to a closed ball in $\mathbb{C}^{2}$ and $HF^{*}(\phi, +)$ vanishes, by Theorem E. The conjecture, if true, would therefore imply that

$$HH^{*}(A, A) \cong Z_{2}.$$

Apart from this implication, which can be viewed as a rigidity result for $A$, the conjectural sequence is expected to serve the computation of $HF^{*}(\phi, +)$. This seems to be most promising in the case where the fiber is two-dimensional, i.e
when \( \phi \) is a diffeomorphism of a two-manifold with boundary. The reason for this is that in two dimensions, there is an alternative approach to the definition of \( \mathcal{A} \) which is purely combinatorial in nature and does not involve the usual analysis of pseudoholomorphic curves. Instead, the operation \( \mu \) is defined by counting immersed polygons where every corner point is mapped to \( L \cap L' \) for some \( L, L' \in \Gamma \) and every boundary arc to some \( L \in \Gamma \). This observation has its origin in de Silva’s thesis and is also referred to as combinatorial Floer homology.

**Outlook.** The developments and speculations outlined in the last paragraphs all appeared in Seidel’s articles “Vanishing cycles and mutation” [49] and “More on vanishing cycles and mutation” [48]. Clearly, they form a circle of ideas which will be central for the computation of Floer homology groups in the future.

What is the significance of the present thesis in the light of these developments? We think of two different aspects. The proof of Theorem A shows that the Floer homology of many surface diffeomorphisms can be computed following the same pattern which was used by Seidel in the case of a Dehn twist. We expect this to carry on and include further examples, e.g. products of Dehn twists along trees of embedded circles. We have taken particular care stating our result about Floer connecting orbits in such a way that its degree of generality is apparent.

The second point is that the two-dimensional theory is a model case for the study of Floer homology in higher dimensions, where computations are far more difficult. No doubt, our techniques are restricted to two dimensions. Our results, however, Theorems A, B and E, might serve as “experimental data” with which a combinatorial approach can be tested.

**References.** The following list does not claim completeness. Floer’s original work appeared in [16]–[20]. The non-degenerate Arnold conjecture for monotone symplectic manifolds was proven in [19]. The extension to general symplectic manifolds was completed by Fukaya-Ono [22] and Liu-Tian [32]. Further references for Floer homology of Hamiltonian symplectomorphisms and the Arnold conjecture are Hofer-Salamon [28], Salamon [42] and Salamon-Zehnder [43]. The Frobenius structure in Floer homology was studied by Piunikhin-Salamon-Schwarz [40] and Schwarz [45]. Floer homology for general symplectomorphisms was applied by Dostoglou-Salamon [12] in connection with the Atiyah-Floer conjecture and by Seidel [46] in connection with the symplectic isotopy problem. Seidel’s exact sequence is proven in [50].

Of outmost importance for this thesis is Seidel’s work on symplectic Floer homology and the mapping class group [51] as well as Seidel’s computation of the Floer homology of the Dehn twist [47]. Our result on the geometric monodromy is a refinement of a classical result of A’Campo [1] and Lê [31]. The proof heavily relies on the theory of splice diagrams, which was developed by Eisenbud-Neumann [14], [38].

\( A_{\infty} \) categories in Floer theory were discovered by Fukaya, e.g. see [23]. We have already mentioned Seidel’s articles [49], [48], which contain a variety of ideas on
this subject. The conjectural exact sequence outlined above is contained in the latter article. We would like to add that Seidel’s work on exact Morse fibrations is related to Donaldson’s work on vanishing cycles [11]. Combinatorial Floer homology was first studied by de Silva [8]. In a current project, this is carried on by de Silva, Salamon, Robbin and the author [9], [10].

Remarks on Chapter 3. Chapter 3 is independent, though not disconnected, from Chapter 2. It contains a series of results on subgroups of fundamental groups of two-manifolds which are related to the Freiheitssatz, a classical theorem in combinatorial group theory. For a detailed account we refer to Section 3.1. Corollary 3.4 is used in the proof of Proposition 2.20.

A few words to the origin of Chapter 3. The starting point was the study of Nielsen fixed point classes of certain surface diffeomorphisms. Not surprisingly, this problem showed up in connection with Floer homology. We only discovered at a later stage of the research, that our results fit into the context of the Freiheitssatz.

Outline of the thesis. The results stated in the introduction are all proven in Chapter 2. In Section 2.1 we define the notion of monotonicity and recall the basic facts about symplectic Floer homology in two dimensions. Section 2.2 is devoted to diffeomorphisms of finite type and their properties relevant for Floer homology. We compute the fixed point classes, establish monotonicity and show that the symplectic action is exact. At several points, we use a topological result on products of disjoint Dehn twists. This result is proven in Appendix A, i.e. Section 2.6.

In 2.3 we discuss the separation mechanism for Floer connecting orbits and in 2.4 the results from the previous sections are put together to prove Theorem A. At the beginning of Section 2.5, we recall the definition of the Milnor fiber and geometric monodromy of an isolated plane curve singularity. Then it is shown how Theorem B follows from Theorems A and D and Theorem E from Theorem D. Theorem D is proven in Appendix B, i.e. Section 2.7. This section starts with an introduction to the theory of splice diagrams. Appendix C, that is Section 2.8, addresses Floer homology on two-manifolds with boundary.

Chapter 3 begins with an introduction to the Freiheitssatz and the statement of the results. Section 3.2 summarizes some background material on surface groups and Section 3.3 contains the proofs of the results. In Appendix D, Section 3.4, we give a characterization of the cylinder. This result is used in the previous sections.
2 Floer homology of algebraically finite mapping classes

This chapter is the core of the thesis, where we prove the results stated in the introduction.

We begin with an outline of the main concepts of Floer homology in dimension two. The two proximate sections are aimed towards the proof of Theorem A, which is given in the fourth section. Finally, we draw our attention to the geometric monodromy and prove Theorems B, E and Corollary C.

The last three sections collect results and proofs that are needed in the foregoing sections, in particular the proof of Theorem D.

2.1 Monotonicity and Floer homology

In this section we discuss the notion of monotonicity as defined in [51] and its significance for Floer homology. For a detailed account we refer to Seidel’s article. At the end of this section we give two criteria for monotonicity which we use in the next section. Throughout this chapter, $\Sigma$ denotes a closed connected and oriented 2-manifold of genus $\geq 2$. In this section, we also fix an area form $\omega$ on $\Sigma$.

Let $\phi \in \text{Symp}(\Sigma, \omega)$, the group of $\omega$-preserving diffeomorphisms of $\Sigma$. The mapping torus of $\phi$,

$$T_\phi = \mathbb{R} \times \Sigma / (t + 1, x) \sim (t, \phi(x)),$$

is a 3-manifold fibered over $S^1 = \mathbb{R}/\mathbb{Z}$. There are two natural second cohomology classes on $T_\phi$, denoted by $[\omega_\phi]$ and $c_\phi$. The first one is represented by the closed two-form $\omega_\phi$ which is induced from the pullback of $\omega$ to $\mathbb{R} \times \Sigma$. The second is the Euler class of the vector bundle

$$V_\phi = \mathbb{R} \times T\Sigma / (t + 1, \xi_x) \sim (t, d\phi_x \xi_x),$$

which is of rank 2 and inherits an orientation from $T\Sigma$.

**Definition 2.1.** $\phi \in \text{Symp}(\Sigma, \omega)$ is called **monotone**, if

$$[\omega_\phi] = (\text{area}_\omega(\Sigma)/\chi(\Sigma)) \cdot c_\phi$$

in $H^2(T_\phi; \mathbb{R})$; $\text{Symp}^m(\Sigma, \omega)$ denotes the set of monotone symplectomorphisms.
Now $H^2(T\phi; \mathbb{R})$ fits into the following short exact sequence \(\text{(1)}\)

\[
0 \longrightarrow \frac{H^1(\Sigma; \mathbb{R})}{\text{im}(\text{id} - \phi^*)} \xrightarrow{\delta} H^2(T\phi; \mathbb{R}) \xrightarrow{\iota^*} H^2(\Sigma; \mathbb{R}) \longrightarrow 0.
\]

The map $\delta$ is defined as follows. Let $\rho : [0, 1] \to \mathbb{R}$ be a smooth function which vanishes near 0 and 1 and satisfies $\int_0^1 \rho \, dt = 1$. If $\theta$ is a closed 1-form on $\Sigma$, then $\rho \cdot \theta \wedge dt$ defines a closed 2-form on $T\phi$; indeed

\[
\delta[\theta] = [\rho \cdot \theta \wedge dt].
\]

The map $\iota : \Sigma \hookrightarrow T\phi$ assigns to each $x \in \Sigma$ the equivalence class of $(1/2, x)$. Note, that $\iota^* \omega_\phi = \omega$ and $\iota^* c_\phi$ is the Euler class of $T\Sigma$. Hence, by (1), there exists a unique class $m(\phi) \in H^1(\Sigma; \mathbb{R})/\text{im}(\text{id} - \phi^*)$ satisfying

\[
d\iota^* m(\phi) = [\omega_\phi] - (\text{area}_\omega(\Sigma)/\chi(\Sigma)) \cdot c_\phi,
\]

where $\chi$ denotes the Euler characteristic. Therefore, $\phi$ is monotone if and only if $m(\phi) = 0$.

We recall the fundamental properties of $\text{Symp}^m(\Sigma, \omega)$ from [51]. By $\text{Diff}^+(\Sigma)$, we denote the group of orientation preserving diffeomorphisms of $\Sigma$.

(Naturality) If $\phi \in \text{Symp}^m(\Sigma, \omega)$, $\psi \in \text{Diff}^+(\Sigma)$, then $\psi^{-1} \phi \psi \in \text{Symp}^m(\Sigma, \psi^* \omega)$.

(Isotopy) Let $(\psi_t)_{t \in [0, 1]}$ be an isotopy in $\text{Symp}(\Sigma, \omega)$, i.e. a smooth path with $\psi_0 = \text{id}$. Then

\[
m(\phi \circ \psi_1) = m(\phi) + [\text{Flux}(\psi_t)_{t \in [0, 1]}]
\]

in $H^1(\Sigma; \mathbb{R})/\text{im}(\text{id} - \phi^*)$; see [51, Lemma 6]. The flux is defined by

\[
\text{Flux}(\psi_t)_{t \in [0, 1]} := \int_0^1 \omega(X_t, \cdot) \, dt \in H^1(\Sigma; \mathbb{R}),
\]

where $X_t = (\partial_t \psi_t) \circ \psi_t^{-1}$. The flux of an isotopy vanishes if and only if the isotopy is homotopic to a Hamiltonian isotopy; see [35].

(Inclusion) The inclusion $\text{Symp}^m(\Sigma, \omega) \hookrightarrow \text{Diff}^+(\Sigma)$ is a homotopy equivalence. This follows from the isotopy property, surjectivity of the flux homomorphism and Moser’s isotopy theorem [37]. Furthermore, the Earle-Eells Theorem [13] implies that every connected component of $\text{Symp}^m(\Sigma, \omega)$ is contractible.

(Floer homology) To every $\phi \in \text{Symp}^m(\Sigma, \omega)$ symplectic Floer homology theory assigns a $\mathbb{Z}_2$-graded vector space $HF_* (\phi)$ over $\mathbb{Z}_2$, with an additional multiplicative structure, called the quantum cap product,

\[
H^* (\Sigma; \mathbb{Z}_2) \otimes HF_* (\phi) \longrightarrow HF_* (\phi).
\]

---

\[\text{(1)}\]This sequence is obtained from the Mayer-Vietoris sequence of the pair $(p(U), p(V))$, where $p$ denotes the projection $\mathbb{R} \times \Sigma \to T\phi$ and

\[
U = (-\varepsilon, 1/2 + \varepsilon) \times \Sigma, \quad V = (1/2 - \varepsilon, 1 + \varepsilon) \times \Sigma
\]

with $\varepsilon > 0$ small.
Each $\psi \in \text{Diff}^+(\Sigma)$ induces an isomorphism of $H^*(\Sigma; \mathbb{Z}_2)$-modules $HF_*(\psi) \cong HF_*(\psi^{-1}\phi\psi)$.

(Invariance) If $\phi, \phi' \in \text{Symp}^m(\Sigma, \omega)$ are isotopic, then $HF_*(\phi)$ and $HF_*(\phi')$ are naturally isomorphic as $H^*(\Sigma; \mathbb{Z}_2)$-modules. This is proven in [51, Page 7].

Now let $g$ be a mapping class of $\Sigma$, i.e. an isotopy class of $\text{Diff}^+(\Sigma)$. Pick an area form $\omega$ and a representative $\phi \in \text{Symp}^m(\Sigma; \omega)$ of $g$. Then $HF_*(\phi)$ is an invariant of $g$, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form $\omega$ by Moser’s isotopy theorem [37] and naturality of Floer homology.

Let $\phi \in \text{Symp}^m(\Sigma, \omega)$. We give a brief outline of the definition of $HF_*(\phi)$ in the special case where $\phi$ has only non-degenerate fixed points. If we denote the set of $y \in \Sigma$ with $\phi(y) = y$ by $\text{Fix}(\phi)$, this means that

$$\det(\text{id} - d\phi_y) \neq 0, \quad \forall y \in \text{Fix}(\phi).$$

In particular, it follows that $\text{Fix}(\phi)$ is a finite set and the $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\text{Fix}(\phi)}$ admits a $\mathbb{Z}_2$-grading with

$$(-1)^{\text{deg}y} = \text{sign}(\det(\text{id} - d\phi_y)), \quad \forall y \in \text{Fix}(\phi).$$

The Floer boundary operator is defined as follows. Let $J = (J_t)_{t \in \mathbb{R}}$ be a smooth path of $\omega$-compatible complex structures on $\Sigma$ such that $J_{t+1} = \phi^*J_t$. For $y^\pm \in \text{Fix}(\phi)$, denote by $M(y^-, y^+; J, \phi)$ the set of smooth maps $u : \mathbb{R}^2 \to \Sigma$ which have finite energy, see below, and satisfy the Floer equations

$$\begin{cases}
  u(s, t) = \phi(u(s, t + 1)), \\
  \partial_s u + J_t(u)\partial_t u = 0, \\
  \lim_{s \to \pm \infty} u(s, t) = y^\pm. 
\end{cases}
$$

The energy of the map $u$ is defined by

$$E(u) = \int_{\mathbb{R}} \int_0^1 \omega(\partial_s u(s, t), J_t \partial_t u(s, t)) \, dt \, ds.
$$

One way to think of the Floer equations is in terms of the symplectic action. Let

$$\Omega_\phi = \{ y \in C^\infty(\mathbb{R}, \Sigma) \mid y(t) = \phi(y(t + 1)) \}
$$

denote the twisted loop space. The action form is the one-form $\alpha_\omega$ on $\Omega_\phi$ defined by

$$\alpha_\omega(y)\xi = \int_0^1 \omega\left(\frac{dy}{dt}(t), \xi(t)\right) \, dt,
$$

where $y \in \Omega_\phi$ and $\xi \in T_y\Omega_\phi$, i.e. $\xi(t) \in T_{y(t)}\Sigma$ and $\xi(t) = d\phi_{y(t+1)}\xi(t + 1)$ for all $t \in \mathbb{R}$. If $\xi, \xi' \in T_y\Omega_\phi$, then

$$\int_0^1 \omega(\xi'(t), J_t\xi(t)) \, dt$$

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defines a metric on $\Omega_{\omega}$. The negative gradient lines of $\alpha_{\omega}$ with respect to this metric are solutions of (2).

Now to every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is associated a Fredholm operator $D_u$ which linearizes (2) in suitable Sobolev spaces. The index of $D_u$ is given by the so-called Maslov index $\mu(u)$, which satisfies

$$\mu(u) = \deg(y^+) - \deg(y^-) \mod 2.$$  

For a generic $J$, every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is regular, meaning that $D_u$ is onto. Hence, by the implicit function theorem, $M_k(y^-, y^-; J, \phi)$ is a smooth $k$-dimensional manifold, where $M_k(y^-, y^+; J, \phi)$ denotes the subset of those $u \in \mathcal{M}(y^-, y^+; J, \phi)$ with $\mu(u) = k \in \mathbb{Z}$.

Translation of the $s$-variable defines a free $\mathbb{R}$-action on $M_1(y^-, y^+; J, \phi)$ and hence the quotient is a discrete set of points. Assume for the moment that for all $y \in \text{Fix}(\phi)$ this quotient is a finite set and let $n(y^-, y^+)$ denote its cardinality mod 2. Define the linear map

$$\partial_J : CF_1(\phi) \longrightarrow CF_{1+1}(\phi) \quad \text{by} \quad \text{Fix}(\phi) \ni y \longmapsto \sum_z n(y, z)z.$$  

That $\partial_J$ is of degree 1 follows from the equation relating the index and the degree. That $\partial_J$ is a boundary operator, i.e. that

$$\partial_J \circ \partial_J = 0,$$

is due to the so-called gluing theorem. For this theorem to hold, as well as for $\mathcal{M}_1(y^-, y^+; J, \phi)/\mathbb{R}$ to be a finite set, one needs certain bounds on the energy. Note that bubbling is not an issue here, since $\pi_2(\Sigma) = 0$. It is proven in [51, Lemma 9] that if $\phi$ is monotone, then the energy is constant on each $\mathcal{M}_k(y^-, y^+; J, \phi)$. It follows that $(CF_*(\phi), \partial_J)$ is a chain complex and that its homology is an invariant of $\phi$, denoted by $HF_*(\phi)$, i.e. it is independent of $J$.

Next we introduce the quantum cap product on $HF_*(\phi)$. For this, choose a Morse function $f : \Sigma \rightarrow \mathbb{R}$ and set

$$CM^*(f) := \mathbb{Z}^\text{Crit}(f),$$

with a $\mathbb{Z}$-grading given by the Morse index $\text{ind}_f$. Choose a Riemannian metric on $\Sigma$ such that $\nabla f$ is a Morse-Smale vector field. If $x^\pm \in \text{Crit}(f)$ and $\text{ind}_f(x^+) = \text{ind}_f(x^-) + 1$, denote by $l(x^-, x^+) \in \mathbb{Z}_2$ the number mod 2 of positive gradient lines going from $x^-$ to $x^+$. Define the Morse coboundary operator

$$\delta_{\partial_J} : CM^*(f) \longrightarrow CM^{*+1}(f) \quad \text{by} \quad \text{Crit}(f) \ni x \longmapsto \sum_y l(x, y)y.$$  

The cohomology of $(CM^*(f), \delta_{\partial_J})$ is naturally isomorphic to $H^*(\Sigma; \mathbb{Z}_2)$, see [44]. Now by a suitable choice of the function $f$ or the metric, we may assume that for all $y^\pm \in \text{Fix}(\phi), x \in \text{Crit}(f)$ and $k \in \mathbb{Z}$, the evaluation map

$$\eta_k : \mathcal{M}_k(y^-, y^+; J, \phi) \longrightarrow \Sigma, \quad u \longmapsto u(0, 0),$$

defines a metric on $\Omega_{\omega}$. The negative gradient lines of $\alpha_{\omega}$ with respect to this metric are solutions of (2).
is transverse to the unstable manifold $W^u(\nabla f, x) \subset \Sigma$. Note that the dimension of $W^u(\nabla f, x)$ is $2 - \text{ind}_f(x)$. Hence, if $k = \text{ind}_f(x)$, then $\eta_k^{-1}(W^u(f, x))$ is a discrete set of points. It is in fact a finite set, which is proven in [51, page 8]. The proof uses the Gromov-Floer compactification of the moduli spaces and the fact that $\pi_2(\Sigma) = 0$. Denote by $q(x; y^-, y^+) \in \mathbb{Z}_2$ the cardinality mod 2 of $\eta_k^{-1}(W^u(f, x)) \subset \mathcal{M}_k(y^-, y^+; J, \phi)$, where $k = \text{ind}_f(x)$. Define the linear map

$$CM^*(f) \otimes CF_*(\phi) \longrightarrow CF_*(\phi), \quad x \otimes y \longmapsto \sum z q(x; y, z) z. \quad (4)$$

It can be shown that this is a chain map and that the induced map on homology is independent of $\nabla f$ and $J$. It is called the quantum cap product. For details we refer to [51] and the references given therein.

If $\phi$ has degenerate fixed points one needs to perturb equations (2) in order to define the Floer homology. Equivalently, one could say that the action form needs to be perturbed. At this point Seidel’s approach differs from the usual one. He uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed. As a consequence, the usual invariance of Floer homology under Hamiltonian isotopies is extended to the stronger property stated above. This ends the general discussion of Floer homology.

To compute the Floer homology of a mapping class, one needs to pick a monotone representative. We now give two criteria for monotonicity which we use later on. Let $\omega$ be an area form on $\Sigma$ and $\phi \in \text{Symp}(\Sigma, \omega)$.

**Lemma 2.2.** Assume that every class $\alpha \in \ker(\text{id} - \phi_\ast) \subset H_1(\Sigma; \mathbb{Z})$ is represented by a map $\gamma : S \to \text{Fix}(\phi)$, where $S$ is a compact oriented 1-manifold. Then $\phi$ is monotone.

**Proof.** By dualizing the exact sequence (1), we get the following exact sequence for homology with real coefficients

$$0 \longrightarrow H_2(\Sigma; \mathbb{R}) \xrightarrow{\gamma_\ast} H_2(T_\phi; \mathbb{R}) \xrightarrow{\delta} \ker(\text{id} - \phi_\ast) \subset H_1(\Sigma; \mathbb{R}), \quad (5)$$

where $\delta$ is dual to $\delta$. Hence, $\phi$ is monotone if and only if

$$\langle m(\phi), \alpha \rangle = 0, \quad \forall \alpha \in \ker(\text{id} - \phi_\ast) \subset H_1(\Sigma; \mathbb{R}).$$

If we think of $H_1(\Sigma; \mathbb{Z})$ as a lattice in $H_1(\Sigma; \mathbb{R})$, it is furthermore enough to consider $\alpha \in H_1(\Sigma; \mathbb{Z}) \cap \ker(\text{id} - \phi_\ast)$.

Let $\gamma : S \to \text{Fix}(\phi)$ and define $u : S \times S^1 \to T_\phi$ by

$$u(s, t) = (t, \gamma(s)).$$

From the definition of $\delta$ on page 12, it is straightforward to check that

$$\langle \delta \alpha, [u] \rangle = \langle \alpha, [\gamma] \rangle,$$

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for all $\alpha \in H_1(\Sigma; \mathbb{R})$, i.e. that $\hat{\partial}[u] = [\gamma]$. Here, the brackets denote homology classes. Now on one hand, since $\partial u(s, t) = (1, 0)$, we have that $u^*\omega_\phi = 0$ and hence $\langle [\omega_\phi], [u] \rangle = 0$. On the other hand, $\langle c_\phi, [u] \rangle = 0$. This is because the bundle $u^*V_\phi$ is isomorphic to the bundle $\gamma^*T\Sigma \times S^1$, which is trivial. Hence, it follows that

$$\langle m(\phi), [\gamma] \rangle = \langle [\omega_\phi], [u] \rangle - \langle \text{area}_\omega(\Sigma)/\chi(\Sigma) \rangle \langle c_\phi, [u] \rangle = 0.$$  

This proves the lemma.

**Lemma 2.3.** If $\phi^k$ is monotone for some $k > 0$, then $\phi$ is monotone. If $\phi$ is monotone, then $\phi^k$ is monotone for all $k > 0$.

**Proof.** Recall that $T_\phi$ is the orbit space of the $\mathbb{Z}$-action

$$n \cdot (t, x) = (t + n, \phi^{-n}(x)),$$

where $n \in \mathbb{Z}$ and $\langle t, x \rangle \in \mathbb{R} \times \Sigma$. If we only divide out by the subgroup $k\mathbb{Z}$, for $k \in \mathbb{N}_{>0}$, we naturally get the mapping torus of $\phi^k$. Further dividing by $\mathbb{Z}/k\mathbb{Z}$ defines the $k$-fold covering map $p_k : T_{\phi^k} \to T_\phi$. It is straight forward to check that

$$p_k^*[\omega_\phi] = [\omega_{\phi^k}] \quad \text{and} \quad p_k^*[c_\phi] = c_{\phi^k}.$$  

The first equality follows immediately from the definitions. To prove the second, note that

$$p_k^*([\mathbb{R} \times T\Sigma]/\mathbb{Z}) \cong ([\mathbb{R} \times T\Sigma]/k\mathbb{Z}) \cong V_{\phi^k},$$

where the $\mathbb{Z}$-action on $\mathbb{R} \times T\Sigma$ is given by $n \cdot (t, x) = (t + n, d\phi^{-n}(x))$, for $n \in \mathbb{Z}$ and $x \in T_x\Sigma$. The lemma follows from (6) and the fact that $p_k^*$ is injective. To prove injectivity, define the map

$$a_k^* : H^2(T_{\phi^k}; \mathbb{R}) \to H^2(T_\phi; \mathbb{R})$$

by averaging differential forms. Since $a_k^*$ is a left inverse of $p_k^*$, that is $a_k^* \circ p_k^* = \text{id}$, this ends the proof of the lemma.

### 2.2 Diffeomorphisms of finite type

We begin with the basic definition. Note that $S^1$ is always identified with $\mathbb{R}/\mathbb{Z}$.

**Definition 2.4.** We call $\phi \in \text{Diff}_+(\Sigma)$ of **finite type** if the following holds. There is a $\phi$-invariant finite union $N \subset \Sigma$ of disjoint non-contractible annuli such that:

1. $\phi|\Sigma \setminus N$ is periodic, i.e. there exists $\ell > 0$ such that $\phi^\ell|\Sigma \setminus N = \text{id}$.
2. Let $N'$ be a connected component of $N$ and $\ell'$ be the smallest positive integer such that $\phi^{\ell'}$ maps $N'$ to itself. Then $\phi^{\ell'}|N'$ is given by one of the following two models with respect to some coordinates $(q, p) \in [0, 1] \times S^1$:
   
   **(twist map)** $(q, p) \mapsto (q, p - f(q))$
(flip-twist map) \((q, p) \mapsto (1 - q, -p - f(q))\),

where \(f : [0, 1] \to \mathbb{R}\) is smooth and strictly monotone. A twist map is called **positive/negative**, if \(f\) is increasing/decreasing.

(3) Let \(N'\) and \(\ell'\) be as in (2). If \(\ell' = 1\) and \(\phi|N'\) is a twist map, then \(\text{im}(f) \subset [0, 1]\), i.e. \(\phi|\text{int}(N')\) has no fixed points.

(4) If two connected components of \(N\) are homotopic, then the corresponding local models of \(\phi\) are either both positive or both negative twists.

**Remarks.**

(i) Let \(\phi\) be a diffeomorphism of finite type and \(\ell\) be as in (1). Then \(\phi^\ell\) is the product of (multiple) **Dehn twists** “along \(N\)”. Moreover, two parallel Dehn twists have the same sign, by (4). We say that \(\phi\) has **uniform twists**, if \(\phi^\ell\) is the product of only positive, or only negative Dehn twists.

(ii) A mapping class of \(\Sigma\) is called **algebraically finite** if it does not have any pseudo-Anosov components in the sense of Thurston’s theory of surface diffeomorphism. Every such class is represented by a diffeomorphism of finite type. To see this, recall Thurston’s classification theorem [53, Theorem 4]: for every mapping class of \(\Sigma\), there exists a diffeomorphism \(\phi\) representing the class and a \(\phi\)-invariant finite union \(C \subset \Sigma\) of non-contractible disjoint circles such that:

1. The components of \(C\) are pairwise non-homotopic,
2. If \(\Sigma'\) is a \(\phi\)-invariant union of connected components of \(\Sigma \setminus C\), then \(\phi|\Sigma'\) is isotopic to either a periodic or pseudo-Anosov map.

The set \(C\) is called a reducing set. Starting with a mapping class without pseudo-Anosov components, one first chooses a minimal reducing set \(C\), meaning that it has the minimal number of components of all reducing sets. Minimality guarantees that after isotopying the Nielsen-Thurston representative \(\phi\) on a complement of a tubular neighborhood \(N\) of \(C\) to a periodic map, \(\phi|N\) does not have periodic components. One can thus achieve condition (2) above, by isotopying \(\phi|N\) relative to \(\partial N\). If (3) is not satisfied, this is achieved in a last step by introducing further components of \(C\), violating (1'), but such that (4) still holds.

(iii) The term algebraically finite goes back to Nielsen [39]. Fried [21] defined the notion of algebraically finite diffeomorphism in any dimension. In two dimensions, these are special representatives of algebraically finite mapping classes. Fried’s definition, however, is adopted to the theory of dynamical systems. For our purpose, a representative which is of the special type defined above is most convenient.

(iv) The term flip-twist map is taken from [30].

The rest of this section is devoted to the study of diffeomorphisms of finite type. The points of interest for Floer homology are: fixed point classes, monotonicity and action. The results we obtain are used in Section 2.4 to compute the Floer homology.

**Convention.** From now on, \(\phi\) denotes a diffeomorphism of finite type and \(N\) the associated \(\phi\)-invariant union of annuli. By \(\Sigma_0\) we denote the union of the
components of $\Sigma \setminus \text{int}(N)$, where $\phi$ restricts to the identity. Furthermore, we denote by $\ell$ the smallest positive integer such that $\phi^\ell$ restricts to the identity on $\Sigma \setminus N$.

The first proposition describes the set of fixed point classes of $\phi$. It is a special case of a theorem by B. Jiang and J. Guo [30] which gives a representative for any mapping class, that realizes its Nielsen number.

**Proposition 2.5 (Fixed point classes).** Each fixed point class of $\phi$ is either a connected component of $\Sigma_0$ or consists of a single fixed point. A fixed point $x$ of the second type satisfies $\det(\text{id} - d\phi_x) > 0$.

The crucial step in our proof of this proposition is to prove it in the special case of products of disjoint Dehn twists. For this, we refer to Appendix A, i.e. Section 2.6.

**Proof of Proposition 2.5.** First note, that if $x \in \text{Fix}(\phi) \cap \text{int}(N)$, then $\phi$ restricted to the component of $N$ containing $x$, is a flip-twist map and $x = (\frac{1}{2}, \frac{1}{2} f(\frac{1}{2}))$ or $(\frac{1}{2}, \frac{1}{2} - f(\frac{1}{2}))$. Now let $x \neq y$ be arbitrary fixed points in the same fixed point class. We prove in three steps, that $x$ and $y$ are in the same connected component of $\Sigma_0$.

**Claim 1.** $x$ and $y$ are in the same component of either int$(N)$ or $\Sigma \setminus \text{int}(N)$.

Note that every connected component of $\Sigma \setminus \text{int}(N)$ is a connected component of $\text{Fix}(\phi^\ell)$. Similarly if $x \in \text{int}(N)$, then $x$ is contained in a fixed circle of $\phi^\ell$. Such a circle is also a connected component of $\text{Fix}(\phi^\ell)$. By Corollary 2.21 in Appendix B, however, every connected component of $\text{Fix}(\phi^\ell)$ is a fixed point class of $\phi^\ell$. Since $x$ and $y$ are in the same fixed point class of $\phi^\ell$, this proves claim 1. Denote by $M$ the connected component of $N$ or $\Sigma \setminus \text{int}(N)$, containing $x$ and $y$.

**Claim 2.** $x$ and $y$ are in the same fixed point class of $\phi|M$.

By assumption, there exists a map $u : [0,1]^2 \rightarrow \Sigma$ with

$$u(0,t) = x, \quad u(1,t) = y \quad \text{and} \quad u(s,1) = \phi(u(s,0)),$$

for all $s, t \in [0,1]$. By Corollary 2.21, we can assume that $u(s,0) \in M$, for all $s \in [0,1]$. Let $M'$ be the union of $M$ and a tubular neighborhood of $\partial M$. We prove that $u$ can be deformed in the interior of $[0,1]^2$ such that its image is contained in a $M'$. First note, that by a small perturbation, we may assume that $u$ is transverse to $\partial M'$. Hence,

$$u^{-1}(\partial M') \subset [0,1]^2$$

is a 1-dimensional submanifold with boundary

$$\partial u^{-1}(\partial M') = u^{-1}(\partial M') \cap \partial [0,1]^2 = \emptyset.$$
Every component of $u^{-1}(\partial M')$ is therefore a circle and bounds a disk in $[0,1]^2$. The restriction of $u$ to such a disk represents an element of $\pi_2(\Sigma, \partial M')$. Since $\pi_2(\Sigma, \partial M') = 0$, $u$ can be deformed in the interior of $[0,1]^2$ to a map $v$ such that the number of components of $v^{-1}(\partial M')$ is less than that of $u^{-1}(\partial M')$. It follows inductively, that $u$ can be deformed in the interior of $[0,1]^2$ to a map $w$ with $w^{-1}(\partial M') = \emptyset$. This proves claim 2 and we are left with

**Claim 3.** Let $\varphi$ be either a flip-twist map or a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface $M$ of Euler characteristic $\leq 0$. Then each fixed point class of $\varphi$ consists of a single point.

For a flip-twist map, this is checked explicitly by using the model. The other case was first proven in [29]. We repeat the argument here. First assume that $M$ is closed. The uniformization theorem states that in every conformal class of metrics on $M$, there is a unique metric of constant curvature $-1$ if $\chi(M) < 0$ or $0$ if $\chi(M) = 0$. This implies that the unique representative of a $\varphi$-invariant conformal class, such a class exists since $\varphi$ is finite order, is itself $\varphi$-invariant. Hence we can pick a $\varphi$-invariant metric of constant curvature $-1$ or $0$ on $M$ and lift $\varphi$ to an isometry $\tilde{\varphi}$ of the universal cover $\tilde{M}$ of $M$. $\tilde{M}$ is either isometric to the hyperbolic plane $H^2$ or the Euclidean plane $\mathbb{R}^2$.

Let $x \in \text{Fix}(\varphi)$ and let $\tilde{x}, \tilde{y}$ be lifts of $\varphi, x$ to $\tilde{M}$, such that $\tilde{\varphi}(\tilde{x}) = \tilde{x}$. Note, that a fixed point of $\varphi$ is in the same class as $x$ if and only if it can be lifted to a fixed point of $\tilde{\varphi}$. Assume by contradiction that $\tilde{y} \neq \tilde{x}$ is a fixed point of $\tilde{\varphi}$. It follows that the unique geodesic going through $\tilde{x}$ and $\tilde{y}$ is pointwise fixed by $\tilde{\varphi}$. In particular, since $\tilde{\varphi}$ preserves orientation, $d\tilde{\varphi}_{\tilde{x}} = \text{id}$. This implies that $\tilde{\varphi} = \text{id}$, because an isometry of $H^2$ or $\mathbb{R}^2$ is determined by its value and differential at one point. This proves claim 3 in the case that $M$ is closed.

The case $\partial M \neq \emptyset$ is reduced to the above case by gluing together two copies of $M$ along a $\varphi$-invariant tubular neighborhood of $\partial M$. The glued manifold is closed and of Euler characteristic $\leq 0$; $\varphi$ extends to a non-trivial diffeomorphism $\varphi'$, which is orientation preserving and of finite order. Hence, every fixed point class of $\varphi'$ is a single point. The same holds therefore for $\varphi$. This ends the proof of claim 3. Finally we have

**Claim 4.** If $x \in \text{Fix}(\varphi) \setminus \Sigma_0$, then $\det(\text{id} - d\phi_x) > 0$.

The point $x$ is a fixed point of either a flip-twist map or an orientation preserving non-trivial isometry. In the first case, the assertion is checked by using the local model. Similarly, one checks in the second case, that $\det(\text{id} - d\phi_x) \leq 0$ if and only if $d\phi_x = \text{id}$. As shown in the proof of claim 3, however, $d\phi_x = \text{id}$ implies that $x \in \Sigma_0$, which is a contradiction. \qed

The next issue is monotonicity. First note that if $\omega$ is an area form on $\Sigma$, which is the standard form $dq \wedge dp$ with respect to the $(q,p)$-coordinates on $N$, then

$$\omega' := \frac{1}{L} \sum_{i=1}^L (\phi^i)\ast \omega$$
is standard on $N$ and $\phi$-invariant, i.e. $\phi \in \text{Symp}(\Sigma, \omega)$. To prove that $\omega$ can be chosen such that $\phi \in \text{Symp}^m(\Sigma, \omega)$, we distinguish two cases: uniform and non-uniform twists. In the first case we have the following stronger statement.

**Proposition 2.6 (Monotone 1).** If $\phi$ has uniform twists and $\omega$ is a $\phi$-invariant area form, then $\phi \in \text{Symp}^m(\Sigma, \omega)$.

**Proof.** By Lemma 2.3, it is enough to prove that $\phi^f$ is monotone with respect to any $\phi$-invariant area form. Replace $\phi$ by $\phi^f$. By the uniform twist condition, $\phi$ is the product of disjoint Dehn twists which are all, say positive. We prove that $\phi$ satisfies the hypothesis of Lemma 2.2 and is therefore monotone. We use the Picard-Lefschetz formula for the action of a Dehn twist on $H_1(\Sigma; \mathbb{Z})$: if $\tau$ denotes a positive Dehn twist along $C$, then

$$\tau_* \alpha = \alpha - (\alpha \cdot [C]) [C],$$

for all $\alpha \in H_1(\Sigma; \mathbb{Z})$. Here, $[C]$ is the homology class of $C$ with respect to some orientation, and $\alpha \cdot [C]$ is the intersection number.

Let $C_1, \ldots, C_n \subset \Sigma$ be the disjoint non-contractible circles along which $\phi$ twists. Choose orientations of the $C_i$. Let $\alpha \in \ker(id - \phi) \subset H_1(\Sigma; \mathbb{Z})$. We claim that for all $i = 1, \ldots, n$, $\alpha \cdot [C_i] = 0$. This is equivalent to the condition that $\alpha$ can be represented by a map $S \to \text{Fix}(\phi)$, where $S$ is a compact oriented 1-manifold, and therefore ends the proof of the proposition. Since the $C_i$ are pairwise disjoint, it follows from the Picard-Lefschetz formula that

$$\alpha = \phi_* \alpha = \alpha - \sum_{i=1}^n (\alpha \cdot [C_i]) [C_i],$$

and hence that $\sum_{i=1}^n (\alpha \cdot [C_i]) [C_i] = 0$. Pairing with $\alpha$, we get

$$\sum_{i=1}^n (\alpha \cdot [C_i])^2 = 0,$$

which implies that $\alpha \cdot [C_i] = 0$, for all $i = 1, \ldots, n$. \square

In the non-uniform case, monotonicity is a more subtle point and does not hold for arbitrary $\phi$-invariant area forms.

**Proposition 2.7 (Monotone 2).** If $\phi$ does not have uniform twists, there exists a $\phi$-invariant area form $\omega$ such that $\phi \in \text{Symp}^m(\Sigma, \omega)$. Moreover, $\omega$ can be chosen such that it equals the form $dq \wedge dp$ on $N$.

**Proof.** The strategy of the proof is the following. Assume first that $\phi|\Sigma \setminus N = id$. We begin by defining a $\phi$-invariant area form $\omega$ with $\text{area}_\omega(\Sigma) = -\chi(\Sigma)$. Then we construct for every class $[\gamma] \in \ker(id - \phi)$, a class $[\Gamma] \in H_2(T_\phi; \mathbb{Z})$ such that

$$\partial [\Gamma] = [\gamma] \quad \text{and} \quad \langle [\omega], [\Gamma] \rangle = -\langle c_\phi, [\Gamma] \rangle,$$

where $c_\phi$ is the $\phi$-invariant curve class.
with \( \partial \) as defined in the sequence (5). Finally, we show how the general case is reduced to the case above.

We start with the following set-up. Fix a union \( \tilde{N} \subset \Sigma \) of non-contractible and pairwise non-homotopic disjoint annuli such that \( \phi|\Sigma \setminus \tilde{N} = \text{id} \). Moreover, let for every connected component \( N' \) of \( \tilde{N} \), \( \ell' \) be a positive integer and \( f : [0, 1] \to \mathbb{R} \) be a smooth monotone function with \( f(0) = 0, f(1) = \ell' \) and such that \( \phi|N' \) is (an \( \ell' \)-fold Dehn twist) given by the model \((q, p) \mapsto (q, p - f(q)) \) for \((q, p) \in [0, 1] \times S^1 \). We emphasize here, that not only the function \( f \) but also the local chart of \( N' \) is fixed for the rest of the proof.

In a first step, we choose for every component \( N' \) of \( \tilde{N} \), an embedded circle \( C_0 \subset N_0 \) as follows. Look at the set \( \text{graph}(f) \) on \( [0, 1] \times [-\ell', 0] \) if \( \ell' > 0 \), respectively \( [0, 1] \times [0, -\ell'] \) if \( \ell' < 0 \). See the figures below.

**Figure 1:** graph(\(-f\)) for a positive twist

**Figure 2:** graph(\(-f\)) for a negative twist

For any \( a \in (0, 1) \), the complement of the union of graph(\(-f\)) and the set \( \{q = a\} \) has four components. If \( \phi|N' \) is a positive twist, we choose \( a \) such that the left upper component (indicated with a \(-\) sign in Figure 1) and the right lower component (indicated with a \(+\) sign) have the same area with respect to the standard area form on \( [0, 1] \times [-\ell', 0] \). If \( \phi|N' \) is a negative twist, left upper is replaced by left lower and right lower by right upper and the sings are interchanged. In both cases, we set \( C_0 := f(a) \times S^1 \subset N', \) with orientation induced from \( S^1 \). The purpose of this construction will become clear below. Let \( C \) denote the union of the loops \( C' \). Let \( \Sigma_1, \ldots, \Sigma_m \subset \Sigma \) denote the closures of the connected components of \( \Sigma \setminus C \). Since the \( C' \) are disjoint non-contractible and pairwise non-homotopic, it follows that \( \chi(\Sigma_j) < 0 \) for all \( j = 1, \ldots, m \). Now choose an area form \( \omega \) on \( \Sigma \) such that

\[
\text{area}_\omega(\Sigma_j) = -\chi(\Sigma_j) \quad \text{for all } j = 1, \ldots, m, \\
\omega|\tilde{N} = \epsilon \cdot dq \wedge dp, \tag{8}
\]

where \( \epsilon > 0 \) is sufficiently small. By the first condition, we have that \( \text{area}_\omega(\Sigma) = -\chi(\Sigma) \) and from the second it follows that \( \phi^* \omega = \omega \). We now prove in several steps that \([\omega_\phi] = -c_\phi \) in \( H^2(S^2; \mathbb{R}) \).

Let \( S \) be a compact oriented 1-manifold and \( \gamma : S \to \Sigma \) be an immersion which is transverse to \( C \). Moreover, assume that \([\gamma] = [\phi \circ \gamma] \) in \( H_1(\Sigma; \mathbb{Z}) \). The goal is
to lift the 1-cycle $\gamma$ to a 2-cycle $\Gamma$ in $T_\phi$. For this, we first define a 2-chain $A$ in $\Sigma$ which satisfies, compare (7),

$$\partial A = \gamma - \phi \circ \gamma - \sum_{i=1}^n \ell_i([\gamma] \cdot [C_i])C_i,$$

where we think of the right hand side as a 1-chain. Here, we have introduced a numbering of the components of $C$. The chain $A$ can be described as follows, compare Figure 1, 2: At every intersection point of $\gamma$ and $C$ where $\gamma$ runs in the positive $q$-direction, there is a local contribution to $A$ given by the regions in Figure 1, 2 which are labelled by $\pm$. The sign of the contribution is as indicated in the figure. If $\gamma$ runs in the negative $q$-direction, the signs are interchanged. Note that by our choice of $\omega$, we have that $\int_A \omega = 0$.

Next, we use that $\gamma$ is homologous to $\gamma$, i.e. that

$$\sum_{i=1}^n \ell_i([\gamma] \cdot [C_i])C_i = 0.$$

This means that there exist integers $k_1, \ldots, k_m$ such that

$$\partial\left( \sum_{j=1}^m k_j \Sigma_j \right) = \sum_{i=1}^n \ell_i([\gamma] \cdot [C_i])C_i.$$

We can now define the 2-chain $\Gamma$ in $T_\phi = [0,1] \times \Sigma/(0,\phi(x)) \sim (1,x)$, by

$$\Gamma := -[0,1/2] \times (\phi \circ \gamma) - \{1/2\} \times (A + \sum_{j=1}^m k_j \Sigma_j) - [1/2,1] \times \gamma.$$

By construction, $\Gamma$ is a cycle; indeed

$$\partial \Gamma = \{0\} \times (\phi \circ \gamma) - \{1/2\} \times (\phi \circ \gamma) - \{1/2\} \times \partial A$$

$$- \{1/2\} \times \partial(\sum_{j=1}^m k_j \Sigma_j) + \{1/2\} \times \gamma - \{1\} \times \gamma$$

$$= 0.$$

By a similar calculation as in the proof of Lemma 2.2, it furthermore follows that $\partial[\Gamma] = [\gamma]$.

Claim 1. $\langle [\omega_\phi], [\Gamma] \rangle = - \sum_{j=1}^m k_j \text{area}_\omega(\Sigma_j)$.

Only the middle summand of $\Gamma$ contributes to $\langle [\omega_\phi], [\Gamma] \rangle$. Since $A$ has vanishing $\omega$-area, this already proves claim 1.

Claim 2. $\langle e_\phi, [\Gamma] \rangle = - \sum_{j=1}^m k_j \chi(\Sigma_j)$.

To prove this, we use the following property of the Euler class. If a smooth section $s : T_\phi \rightarrow V_\phi$ is transverse to the zero-section, then $s^{-1}(0) \subset T_\phi$ is a submanifold of codimension 2 and its homology class, with respect to a suitable orientation, is Poincaré-dual to $e_\phi$. In particular, $\langle e_\phi, [u] \rangle$ equals the intersection number $[s^{-1}(0)] \cdot [u]$, for any $[u] \in H_2(T_\phi; \mathbb{Z})$. The orientation of $s^{-1}(0)$ at a point $x$ is defined as follows. Let $\{e_1, e_2, e_3\}$ be an oriented basis of $T_x T_\phi$ such that $e_1$
is tangent to $s^{-1}(0)$. Then $e_1$ is said to be oriented, if \{ $e_1, e_2, e_3, ds_2 e_2, ds_2 e_3$ \} is an oriented basis of 

$$T_{(x,0)} V_{\phi} \cong \mathbb{R} \oplus T_x \Sigma \oplus T_x \Sigma.$$ 

We now define a smooth section of $V_{\phi}$. To begin with, we choose a vector field $\xi$ on $\Sigma$ with only non-degenerate zeros and such that $\xi|\hat{\mathcal{N}} = \partial/\partial t$. Furthermore, we require that $\xi^{-1}(0)$ is disjoint from $\text{im}(\gamma)$. That such a vector field exists is a standard result in differential topology. By the Poincaré-Hopf Theorem, the sum of indices of $\xi$ over all zeros in $\Sigma_j$ equals $\chi(\Sigma_j)$, for all $j = 1, \ldots, m$.

Note, that the vector field $\phi^* \xi - \xi$ is supported in $\hat{\mathcal{N}}$, where it is given by $(0, -f'(q))$ with respect to the local model. Hence, there exists a smooth path $(\xi_t)_{t \in \mathbb{R}}$ of vector fields such that $\xi_{t+1} = \phi^* \xi_t$, $\xi_t = \xi$ on $\Sigma \setminus \mathcal{N}$ and $\xi_t^{-1}(0) = \xi^{-1}(0)$. Let the section $s : T_{\phi} \to V_{\phi}$ be defined by $s([t, x]) := [t, \xi_t(x)]$; recall that

$$V_{\phi} = \mathbb{R} \times T\Sigma/(t + 1, \xi_x) \sim (t, d\phi_x \xi_x).$$

By our choice of the vector field $\xi$, $s$ in transverse to the zero-section and thus, $[s^{-1}(0)]$ is Poincaré-dual to $c_\phi$. Moreover,

$$[s^{-1}(0)] \cdot [\Gamma] = -\sum_{j=1}^m k_j (s^{-1}(0) \cdot \Sigma_j)$$

$$= -\sum_{j=1}^m k_j \cdot \chi(\Sigma_j).$$

The numbers $s^{-1}(0) \cdot \Sigma_j$ are well defined because $s^{-1}(0)$ intersects $\Sigma_j$ transversally and in the interior of $\Sigma_j$. Note, that the sign of an intersection point equals the index of $\xi$ at that point. This proves claim 2.

From claim 1, 2 and the first equation in (8), we conclude that

$$\langle [\omega_{\phi}], [\Gamma] \rangle = -\langle c_\phi, [\Gamma] \rangle,$$

and hence that $\phi \in \text{Symp}^m(\Sigma, \omega)$. We end the proof of the proposition with the following observation. Let $\phi$ be a diffeomorphism of finite type; apply the above construction to $\phi^f$ and let $\omega$ be an area form which satisfies (8). It follows that

$$\omega' := \frac{1}{\ell} \sum_{i=1}^\ell (\phi^i)^* \omega$$

also satisfies (8) and hence that $\phi^f \in \text{Symp}^m(\Sigma, \omega')$. On the other hand, $\phi^* \omega' = \omega'$, and therefore $\phi \in \text{Symp}^m(\Sigma, \omega')$, by Lemma 2.3.

Next we consider the symplectic action $\alpha_{\omega}$ on the twisted loop space $\Omega_{\phi}$ and prove that it is exact. See (3) for the definition of $\alpha_{\omega}$. This result is crucial for the computation of the Floer homology, in particular for the use of the connecting
orbits proposition proved in the next section.

We need the following lemma, which holds for any \( \phi \in \text{Symp}(\Sigma, \omega) \). First note that a loop in \( \Omega_\phi \) is represented by a map \( u : S^1 \times [0, 1] \to \Sigma \) with

\[
u(s, 0) = \phi(u(s, 1)) \quad \text{for all } (s, t) \in S^1 \times [0, 1].\]

By \([u]\) we denote the homology class of the loop \( u \) in \( \Omega_\phi \).

**Lemma 2.8.** Let \( u \) and \( v \) be two loops in \( \Omega_\phi \). If \( u(\cdot, 0) \) and \( v(\cdot, 0) \) are freely homotopic loops in \( \Sigma \), then \( \langle [\alpha_\omega], [u] \rangle = \langle [\alpha_\omega], [v] \rangle \).

**Proof.** Let \( w : S^1 \times [0, 1] \to \Sigma \) be such that \( w(\cdot, 0) = u(\cdot, 0) \) and \( w(\cdot, 1) = v(\cdot, 0) \). Define the map \( u' := w^{-1} \# u \# (\phi^{-1} \circ w) : S^1 \times [0, 1] \to \Sigma \) by

\[
u' \begin{cases} u(s, 3t - 1) & \text{if } t \in [1/3, 2/3] \\ \phi^{-1}(w(s, 3t - 2)) & \text{if } t \in [2/3, 1]. \end{cases}
\]

Since for all \( t \in [0, 1/3] \) and all \( s \in S^1 \),

\[
u'(s, 1 - t) = \phi^{-1}(w(s, 1 - 3t)) = \phi^{-1}(u'(s, t)),
\]

\( u' \) and \( u \) are homotopic loops in \( \Omega_\phi \); in particular \( [u] = [u'] \). Note, that since \( u'(s, 0) = v(s, 0) \) and \( u'(s, 1) = v(s, 1) \) for all \( s \in S^1 \), the map \( u' \# v^{-1} \) descends to a map \( S^1 \times S^1 \to \Sigma \). Now if \( w \) is an arbitrary loop in \( \Omega_\phi \), then

\[
\langle [\alpha_\omega], [w] \rangle = - \int_{S^1 \times [0, 1]} w^* \omega.
\]

Therefore

\[
\langle [\alpha_\omega], [u'] \rangle - \langle [\alpha_\omega], [v] \rangle = - \int_{S^1 \times S^1} (u' \# v^{-1})^* \omega = 0.
\]

In the last equality we use the fact ** that the mapping degree of \( u' \# v^{-1} \) vanishes, since the genus of \( \Sigma \) is \( \geq 2 \). This proves the proposition. \( \square \)

We return to the situation where \( \phi \) is a diffeomorphism of finite type. The proof of the following proposition relies on the results discussed in Appendix A.

**Proposition 2.9 (Action).** If \( \omega \) is a \( \phi \)-invariant area form, then \( \alpha_\omega \) has vanishing periods.

**Proof.** For any loop \( u \) in \( \Omega_\phi \), define \( v : S^1 \times [0, \ell] \to \Sigma \) by

\[
v(s, t) = \phi^{-j}(u(s, t - j)) \quad \text{for } (s, t) \in S^1 \times [j, j + 1], j < \ell.
\]

**This is due to Poincaré-duality.**
Since \( v(s, 0) = \phi^\ell(v(s, \ell)) \) for all \( s \in S^1 \), \( v \) can be considered a loop in \( \Omega_{\phi^\ell} \). Note that
\[
\int_{S^1 \times [0, \ell]} v^* \omega = \ell \cdot \langle [\alpha_\omega], [u] \rangle.
\]
Now observe, that \( v(\cdot, 0) \) is freely homotopic to \( \phi^{-\ell}(v(\cdot, 0)) = v(\cdot, \ell) \). By Corollary 2.21 in Appendix A, we therefore know that \( v(\cdot, 0) \) is freely homotopic to a loop \( \gamma : S^1 \to \text{Fix}(\phi^\ell) \). Set \( v_0(s, t) = \gamma(s) \) for all \( (s, t) \in S^1 \times [0, \ell] \). It follows from the previous lemma with \( \phi \) replaced by \( \phi^\ell \), that
\[
\int_{S^1 \times [0, \ell]} v^* \omega = \int_{S^1 \times [0, \ell]} v_0^* \omega = 0.
\]
Therefore \( \langle [\alpha_\omega], [u] \rangle = 0 \), which ends the proof of the proposition.

2.3 Connecting orbits

The main result of this section is a separation mechanism for Floer connecting orbits. Together with the topological separation of fixed points discussed in Proposition 2.5, it allows us to compute the Floer homology of diffeomorphisms of finite type. We expect, however, that the results of this section are applicable to a larger class of surface diffeomorphisms.

We start by introducing the setup. The notation is reminiscent of that in Section 2.2. However, it is only in the next section where we return our attention to diffeomorphisms of finite type.

Let \( \Sigma_0 \subset \Sigma \) be a compact submanifold, not necessarily connected. Let \( N_0 \subset \Sigma_0 \) be a collar neighborhood of \( \partial \Sigma_0 \). On every connected component of \( N_0 \), we choose coordinates \( (q, p) \in [0, 1] \times S^1 \) such that \( \partial \Sigma_0 = \bigcup \{1\} \times S^1 \). Let \( \omega \) be an area form on \( \Sigma \) which is given by \( dq \wedge dp \) on \( N_0 \). Let \( \Phi \in \text{Symp}(\Sigma, \omega), H : \Sigma \to \mathbb{R} \) a smooth function and \( J = (J_t)_{t \in \mathbb{R}} \) such that the following holds:

(H1) \( \Sigma_0 \) is \( \Phi \)-invariant. Moreover, \( \Phi(x) = \psi_1(x) \) for all \( x \in \Sigma_0 \), where \( (\psi_t)_{t \in \mathbb{R}} \) denotes the Hamiltonian flow generated by \( H \). This means, that \( \partial_t \psi_t = X \circ \psi_t \), where the vector field \( X \) is defined by \( dH = \omega(X, \cdot) \).

(H2) There exists a constant \( 0 < \delta < 1/4 \) such that on each connected component of \( N_0 \), we have \( \Phi(q, p) = (q, p + \delta) \). The sign may depend on the component.

(H3) \( \text{Fix}(\Phi) \cap \Sigma_0 = \text{Crit}(H) \cap \Sigma_0 \).

(H4) If \( \omega' \) is a \( \Phi \)-invariant area form such that \( \omega' = \omega \) on \( \Sigma \setminus N_0 \), then \( \alpha_{\omega'} \) has vanishing periods on \( \Omega_\Phi \).

(H5) For all \( t \in \mathbb{R} \), \( J_t \) is an \( \omega \)-compatible complex structure which restricts to the standard complex structure on \( N_0 \) with respect to the \( (q, p) \)-coordinates. Moreover, \( J_{t+1} = \Phi^* J_t \).

Assuming (H1–5) we prove:
Proposition 2.10 (Connecting orbits). Let $x^-, x^+ \in \text{Fix}(\Phi) \cap \Sigma_0$ be in the same connected component of $\Sigma_0$. If $u \in \mathcal{M}(x^-, x^+; J, \Phi)$, then $\text{im} u \subset \Sigma_\delta$, where $\Sigma_\delta$ denotes the $\delta$-neighborhood of $\Sigma_0 \setminus N_0$ with respect to any of the metrics $\omega(\cdot, J_t \cdot)$.

To prove this proposition we vary the symplectic form. Fix $\varepsilon > 0$ sufficiently small and set $N_\varepsilon := \bigcup [\varepsilon, 1 - \varepsilon] \times S^1 \subset N_0$. For every $R > 0$, let $\lambda_R : \Sigma \to \mathbb{R}_{>0}$ be a $\Phi$-invariant smooth function such that

$$
\lambda_R \equiv \begin{cases} R & \text{on } N_\varepsilon, \\
1 & \text{on } \Sigma \setminus N_0.
\end{cases}
$$

Set

$$
\omega_R := \lambda_R^2 \cdot \omega \quad \text{and} \quad g_{R,t} := \omega_R(\cdot, J_t \cdot).
$$

Note that $\omega_R$ is $\Phi$-invariant and $\omega_R = \omega$ on $\Sigma \setminus N_0$. By (H4), we can define an action functional $A_R : \Omega \Phi \to \mathbb{R}$ such that

$$
A_R(y') - A_R(y) = \int_0^1 \alpha_{\lambda_R}(u(s, \cdot)) \partial_s u(s, \cdot) \, ds
$$

$$
= \int_0^1 \int_0^1 \omega_R(\partial_t u(s, t), \partial_s u(s, t)) \, dt \, ds
$$

for all $y, y' \in \Omega \Phi$ and $u : [0, 1] \times \mathbb{R} \to \Sigma$ with $u(s, t) = \Phi(u(s, 1 + t))$ and $u(0, \cdot) = y, u(1, \cdot) = y'$. The following observation is crucial for the proof of the proposition.

Lemma 2.11. Let $x^-, x^+ \in \text{Fix}(\Phi) \cap \Sigma_0$ be in the same connected component of $\Sigma_0$. Then

$$
A_R(x^+) - A_R(x^-) = H(x^-) - H(x^+),
$$

for every $R > 0$.

Proof. Choose a path $\gamma : [0, 1] \to \Sigma_0 \setminus N_0$ from $x^-$ to $x^+$ and define $h : [0, 1] \times \mathbb{R} \to \Sigma$ by

$$
h(s, t) := \psi_{-t}(\gamma(s)).
$$

By (H1), we have $h(s, t) = \Phi(h(s, t + 1))$ and furthermore, by (H3), $h(0, \cdot) = x^-, h(1, \cdot) = x^+$. Hence,

$$
A_R(x^+) - A_R(x^-) = \int_0^1 \int_0^1 \omega_R(\partial_t h(s, t), \partial_s h(s, t)) \, ds \, dt
$$

$$
= -\int_0^1 \left( \int_0^1 dH(h(s, t)) \partial_s h(s, t) \, ds \right) dt
$$

$$
= \int_0^1 (H(h(0, t)) - H(h(1, t))) \, dt
$$

$$
= H(x^-) - H(x^+).
$$
In the second line we use that
\[ \omega_R(\cdot, \partial_t h(s, t)) = \omega(\cdot, \partial_t h(s, t)) = dH(h(s, t)), \]
since \( \text{im}(h) \subset \Sigma_0 \setminus N_0. \)

Proof of Proposition 2.10. Let \( u \in \mathcal{M}(x^-, x^+; J, \Phi) \), i.e. \( u : \mathbb{R}^2 \to \Sigma \) is a smooth function satisfying
\[
\begin{align*}
\left\{
\begin{array}{l}
u(s, t) = \phi(u(s, t + 1)), \\
\partial_s u + J_t(u)\partial_t u = 0,
\end{array}
\right.
\lim_{s \to \pm \infty} u(s, t) = \pm
\end{align*}
\]
and
\[
\int_{\mathbb{R}} \int_0^1 \omega(\partial_t u(s, t), J_t \partial u(s, t)) \, dt \, ds < \infty.
\]
We prove that
\[ \text{im}(u) \cap N'_s = \emptyset, \]
where \( N'_s = \bigcup \{ s + \delta, 1/2 \} \times S^1 \subset N_0. \) Assume by contradiction that there exist \( s_1 < s_2 \) and \( t' \) such that \( u(s, t') \in N'_s \) for all \( s \in [s_1, s_2]. \)

Denote by \( g \) the standard Euclidean metric on \( N_0. \) Note, that if \( x \in N'_s, \) then \( B_g(x, \delta) \subset N_s. \) Here \( B_g(x, \delta) \) denotes the \( g \)-disk of radius \( \delta \) around \( x. \) Moreover, if \( x = (q, p), \) then \( \Phi^{-1}(x) = (q, p \pm \delta) \in \partial B_g(x, \delta). \)

Hence, it follows from the first equation in (9), that \( u(s, t' + 1) \in \partial B_g(u(s, t')) \) for all \( s \in [s_1, s_2]. \) Fix \( s \in [s_1, s_2] \) and let \( r \in (0, 1] \) be such that
\[ u([s] \times [t', t' + r]) \subset B_g(u(s, t'), \delta), \quad u(s, t' + r) \in \partial B_g(u(s, t'), \delta). \]
This implies that
\[ \delta \leq \int_{t'}^{t' + r} |\partial_t u(s, t)|_g \, dt. \]
By our choice of \( J_t, \) we know that \( g_{R,t} = R^2 g \) on \( N_s, \) and hence that \( R \cdot |\partial_t u(s, t)|_{g_{R,t}} = |\partial_t u(s, t)|_{g_{R,t}} \) for all \( t \in [t', t' + r]. \) Therefore,
\[
R \cdot \delta \leq \int_{t'}^{t' + r} |\partial_t u(s, t)|_{g_{R,t}} \, dt
\]
\[
\leq \int_{t'}^{t' + 1} |\partial_t u(s, t)|_{g_{R,t}} \, dt = \int_0^1 |\partial_t u(s, t)|_{g_{R,t}} \, dt.
\]
In the last step we use that \( |\partial_t u(s, t)|_{g_{R,t}} \) is a 1-periodic function in \( t. \) By Hölder’s inequality we get that
\[ R^2 \cdot \delta^2 \leq \int_0^1 |\partial_t u(s, t)|_{g_{R,t}}^2 \, dt, \]
for all $R > 0$ and $s \in [s_1, s_2]$. Now integrate over $[s_1, s_2]$:

\[
R^2 \cdot \delta^2 \cdot |s_1 - s_2| \leq \int_{s_1}^{s_2} \int_0^1 |\partial_t u(s, t)|^2_{g_{R, t}} \, ds \, dt \\
\leq \int_0^1 \int R |\partial_t u(s, t)|^2_{g_{R, t}} \, ds \, dt.
\]

(10)

Finally, we use the energy identity

\[
|\partial_t u(s, t)|^2_{g_{R, t}} = \omega_R(\partial_s u(s, t), \partial_t u(s, t)),
\]

which follows from the second equation in (9). Note, that since $u$ has finite energy with respect to $\omega_R$, the energy identity implies that

\[
\int_0^1 \int \partial_t u(s, t)|^2_{g_{R, t}} \, ds \, dt = \mathcal{A}_R(x^-) - \mathcal{A}_R(x^+).
\]

From (10) and Lemma 2.11 we therefore get

\[
R^2 \cdot \delta^2 \cdot |s_1 - s_2| \leq H(x^+) - H(x^-),
\]

for all $R > 0$. For large $R$ this is a contradiction and proves that $\text{im}(u)$ is disjoint from $N'_\varepsilon$. Since $\varepsilon$ can be chosen arbitrarily small by an appropriate choice of functions $\lambda_R$, $\text{im}(u)$ is disjoint from $\bigcup (\delta, 1/2] \times S^1$. This proves the proposition. \hfill \Box

**Remark.** The advantage of our approach to the connecting orbits proposition compared to Seidel’s original approach in [47] is, that we do not have to make the function $H$ depend on $R$. Moreover, the bubbling argument completely disappears. This is relevant in the case where $\Phi$ twists with different signs at different “ends” of $\Sigma_0$. In that case, when the ends are stretched as in [47, Lemma 4], the energy difference of certain fixed points may go to infinity and the bubbling argument fails.

### 2.4 Floer homology of finite type diffeomorphisms

We return to the notation of Section 2.2, i.e. $\phi$ is a diffeomorphism of finite type and $\Sigma_0$ denotes the union of connected components of $\Sigma \setminus \text{int}(N)$, where $\phi$ restricts to the identity.

For every connected component of $\partial \Sigma_0$, there is an outside collar neighborhood where $\phi$ is given by $(q, p) \mapsto (q, p - f(q))$ for some strictly monotone function with $f(0) = 0$. By $\partial_+ \Sigma_0, \partial_- \Sigma_0$, we denote the union of components of $\partial \Sigma_0$, where $f$ in increasing, respectively decreasing.

In this section we prove
Theorem 2.12. If \( \phi \) is a diffeomorphism of finite type and \( \Sigma_0 \) as above, then
\[
H^*_\Sigma(\phi) \cong H_*(\Sigma_0, \partial_+ \Sigma_0; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{N(\Sigma; \Sigma_0)}.
\]
The \( H^*(\Sigma; \mathbb{Z}_2) \)-action on the first summand is given by the ordinary cap product.
On the second summand, \( 1 \in H^0(\Sigma; \mathbb{Z}_2) \) acts by the identity and any element of \( H^1(\Sigma; \mathbb{Z}_2) \oplus H^2(\Sigma; \mathbb{Z}_2) \) by the zero map.

We begin by introducing the set-up.

(Monotonicity) By Propositions 2.6 and 2.7 we can choose an area form \( \omega \) such that \( \phi \in \text{Symp}^m(\Sigma, \omega) \). We impose an additional condition on \( \omega \) in the next paragraph. Note, that by the isotopy axiom, if \( (\psi_t)_{t \in [0,1]} \) is a Hamiltonian isotopy, then \( \phi \circ \psi_t \in \text{Symp}^m(\Sigma, \omega) \).

(Hamiltonian perturbation) As a preparation, let \( f_1, f_2 : [0,3] \to \mathbb{R} \) be two functions which are constant on \([0,1]\) and such that \( f_1(q) = f_2(q) \) for all \( q \in [2,3] \). Define the function
\[
h(q) = \int_0^3 (f_1(r) - f_2(r)) \, dr,
\]
for \( q \in [0,3] \). It follows that
\[
h(q) = \begin{cases} 
\delta \cdot q + c & \text{if } q \in [0,1], \\
0 & \text{if } q \in [2,3],
\end{cases}
\]
where \( \delta = f_2(0) - f_1(0) \) and \( c = \int_0^3 (f_1(r) - f_2(r)) \, dr \). Now consider \( h(q) \) as a function on \([0,3] \times S^1\) with coordinates \((q,p)\). The Hamiltonian vector field of \( h \) with respect to \( dq \wedge dp \) is simply \( (f_1(q) - f_2(q)) \cdot \partial / \partial p \) at the point \((q,p)\). The time-1-maps of the flow is thus the twist map \((q,p) \mapsto (q,p + f_1(q) - f_2(q))\). In particular, \((q,p) \mapsto (q,p - \delta)\) if \( q \in [0,1] \).

This has the following application. Let \( \tilde{N} \subset \Sigma \) be a \( \phi \)-invariant closed tubular neighborhood of \( \partial \Sigma_0 \). On every connected component of \( \tilde{N} \), we choose coordinates \((q,p) \in [0,3] \times S^1\) such that \( \phi \) is given by \((q,p) \mapsto (q,p + f(q))\), for some monotone increasing \( f : [0,3] \to [0,1] \). Moreover, we assume that \( N_0 := \Sigma_0 \cap \tilde{N} \cong \bigcup [0,1] \times S^1 \). Note that \( f|\{0,1\} \equiv 0 \). Furthermore note, that we can assume that \( \omega = dq \wedge dp \) on \( \tilde{N} \). It now follows from the preliminary remarks, that for every \( 0 < \delta < 1/4 \) there exists \( h : \tilde{N} \to \mathbb{R} \) such that

(i) On a connected component of \( N_0 \), \( h(q,p) = \pm \delta \cdot q + c \). Here, the sign and the constant \( c \) may depend on the component.

(ii) \( h \equiv 0 \) on \( \bigcup [2,3] \times S^1 \).

(iii) Let \( \psi \) denote the time-1 map of the Hamiltonian flow generated by \( h \) with respect to \( dq \wedge dp \). For every connected component of \( N_0 \), there exists a monotone increasing function \( g : [0,3] \to [\delta, 1) \), such that \( \phi \circ \psi(q,p) = (q,p + g(q)) \).

As a consequence of (i) and (ii), there exists a function \( H : \Sigma \to \mathbb{R} \) with
\[
H(x) = \begin{cases} 
h(x) & \text{if } x \in \tilde{N}, \\
0 & \text{if } x \in \Sigma \setminus (\Sigma_0 \cup \tilde{N}),
\end{cases}
\]
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and such that $H|\text{int}(\Sigma_0)$ is a Morse function, meaning that all the critical points are non-degenerate. We refer to [44, Lemma 4.15] for the extension of Morse functions.

Let $(\psi_t)_{t \in \mathbb{R}}$ denote the Hamiltonian flow generated by $H$ with respect to the fixed area form $\omega$ and set

$$\Phi := \phi \circ \psi_1.$$ 

By construction, $\omega, \Phi, H$ and $N_0$ satisfy the hypothesis (H1.2) of the last section. Moreover, by choosing $H$ and $\delta$ such that the $C^2$-norm of $H$ is sufficiently small, we also guarantee (H3).

(Fixed points) By (iii), $\Phi|\tilde{N}$ has no fixed points. Since $\Phi = \phi$ on $\Sigma \setminus (\Sigma_0 \cup \tilde{N})$, we therefore have

$$\text{Fix}(\Phi) = (\text{Crit}(H) \cap \Sigma_0) \cup (\text{Fix}(\phi) \setminus \Sigma_0).$$

In particular, $\Phi$ only has non-degenerate fixed points and the $\mathbb{Z}_2$-degree of a fixed point is given by

$$\deg(y) = \text{ind}_H(y) \mod 2 \quad \forall y \in \text{Crit}(H) \cap \Sigma_0,$$

and

$$\deg(y) = 0 \mod 2 \quad \forall y \in \text{Fix}(\phi) \setminus \Sigma_0.$$ 

The first equality follows from [43, Lemma 7.2], the second from Proposition 2.5. Moreover, Proposition 2.5 implies that every $y \in \text{Fix}(\phi) \setminus \Sigma_0$ forms a different fixed point class of $\Phi$. This has an immediate consequence for the Floer complex $(\text{CF}_*(\Phi), \partial_J)$ with respect to a generic $J = (J_t)_{t \in \mathbb{R}}$.

**Lemma 2.13.** $(\text{CF}_*(\Phi), \partial_J)$ splits into two subcomplexes $(\mathcal{C}_1, \partial_1)$ and $(\mathcal{C}_2, \partial_2)$, where $\mathcal{C}_1$ is generated by $\text{Crit}(H) \cap \Sigma_0$ and $\mathcal{C}_2$ by $\text{Fix}(\phi) \setminus \Sigma_0$. Moreover, $\mathcal{C}_2$ is graded by 0 and $\partial_2 = 0$. The splitting is respected by the quantum cap action (4).

**Proof.** If $y^\pm \in \text{Fix}(\Phi)$ are in different fixed point classes, then

$$\mathcal{M}(y^-, y^+; J, \phi) = \emptyset.$$ 

This follows from the first equation in (2) and proves the lemma. \hfill \square

Next we show that hypothesis (H4) holds.

**Lemma 2.14 (Action 2).** If $\omega'$ is a $\Phi$-invariant area form such that $\omega' = \omega$ on $\Sigma \setminus N_0$, then $\alpha_{\omega'}$ has vanishing periods.

**Proof.** This proof is an extension of the proof of Proposition 2.9. Let $u$ be a loop in $\Omega_{\Phi}$. We claim that $\langle [\alpha_{\omega'}], [u] \rangle = 0$. Since $\Phi'$ is isotopic to $\phi'$, we can assume that $u(., 0)$ is contained either in $\Sigma \setminus (\Sigma_0 \cup \tilde{N})$ or in $\Sigma_0 \setminus N_0$. The argument is similar as in the above mentioned proof and relies on Lemma 2.8 and
Corollary 2.21. The first case reduces to the case considered in Proposition 2.9.
In the second case, define $h : S^1 \times \mathbb{R} \to \Sigma_0 \setminus N_0$ by $h(s, t) := \psi_\tau(u(s, 0))$; $h$ is a loop in $\Omega \Phi$. Since $\omega' = \omega$ on $\text{im}(h)$, it follows that $\int h^* \omega' = \int h^* \omega = 0$ and hence, from Lemma 2.8, that $\int u^* \omega' = 0$.

(Path of complex structures) Let $J_0$ be a $\omega$-compatible complex structure on $\Sigma$ which restricts to the standard complex structure on $N_0$. Let $J = (J_t)_{t \in \mathbb{R}}$ be a smooth path of $\omega$-compatible complex structures such that $J_{t+1} = \Phi^* J_t$ and $J_t(x) = (\psi^*_t J_0)(x)$ for all $t \in \mathbb{R}$ and $x \in \Sigma_0$. The existence of such a $J$ relies on the contractibility of the space of $\omega$-compatible complex structures on $\Sigma$. Note that $J$ satisfies (H5). Below, we impose an additional regularity condition on $J_0$.

We are now in position to apply Proposition 2.10 and compute the homology of $(C_1, \partial_1)$.

**Lemma 2.15.** The homology of $(C_1, \partial_1)$ is isomorphic to
\[ H_*(\Sigma_0, \partial_1; \mathbb{Z}_2). \]

The quantum cap product is given by the ordinary cap product
\[ H^*(\Sigma; \mathbb{Z}_2) \otimes H_*(\Sigma_0, \partial_1; \mathbb{Z}_2) \to H_*(\Sigma_0, \partial_1; \mathbb{Z}_2). \]

**Proof.** The proof uses the same technique as in [47]. By modifying $J_0$ in a neighborhood of $\text{Crit}(H) \cap \Sigma_0$, we can assume that $\nabla H$ is a Morse-Smale vector field on $\Sigma_0$, where the gradient is taken with respect to the metric $\omega(\cdot, J_0 \cdot)$. This means that the stable and unstable manifolds are all transverse to each other and that the stable, unstable manifolds are all transverse to $\partial \Sigma_0$.

Note that by Proposition 2.5, $(C_1, \partial_1)$ splits into subcomplexes generated by fixed points of $\Phi$ which are in the same connected component of $\Sigma_0$. Let $x^\pm$ be a pair of such fixed points and set $M := M(x^- \cup x^+; J, \Phi)$. For every $u \in M$, it follows from Proposition 2.10 that $\text{im}(u) \subset \Sigma_0$. Define the map
\[ \tilde{u} : \mathbb{R}^2 \to \Sigma_0, \quad \tilde{u}(s, t) := \psi_t(u(s, t)). \]

A straightforward calculation, using that $u(s, t) = \psi_1(u(s, t+1))$, $\psi_t \circ \psi_1 = \psi_{t+1}$ and $J_1 = \psi^*_1 J_0$ on $\text{im}(u)$, shows that
\[
\begin{cases}
\tilde{u}(s, t) = \tilde{u}(s, t + 1), \\
\partial_s \tilde{u} + J_0(\tilde{u}) (\partial_t \tilde{u} - X_H(\tilde{u})) = 0, \\
\lim_{s \to \pm \infty} \tilde{u}(s, t) = x^\pm,
\end{cases}
\tag{11}
\]

where $X_H$ denotes the Hamiltonian vector field of $H$. The system (11) was studied in [43, Theorem 7.3]. There it is shown that if $H|\Sigma_0$ is replaced by $\varepsilon \cdot H|\Sigma_0$ with $\varepsilon > 0$ sufficiently small, then every solution of (11) is independent of the $t$-variable and is therefore a solution of
\[ d\tilde{u}/ds = \varepsilon \cdot \nabla H(\tilde{u}). \]
Moreover, \(\tilde{u}\) is regular in the sense that the operator which is defined by linearizing the equations (11) is surjective; see page 13. If we go back to the definition of \(H\), we can assume from now on that \(\varepsilon = 1\). That the ambient space \(\Sigma_0\) has non-empty boundary does not affect the argument in [43, Theorem 7.3]. It is essential however, that \(\pi_2(\Sigma_0) = 0\).

It follows that every \(u \in \mathcal{M}\) is regular and that the map \(u \mapsto \tilde{u}\) induces a diffeomorphism of \(\mathcal{M}\) and the space of (parameterized) slow lines of \(\nabla H\) which are contained in \(\Sigma_0\) and connect the critical points \(x^\pm\). Furthermore, these diffeomorphisms, one for each pair \(x^\pm\), induce an isomorphism

\[
(C_1, \partial_1) \cong (CM_*(H|\Sigma_0), \partial_{\nabla H})
\]

of chain complexes. Here, \(CM_*(H|\Sigma_0)\) is freely generated by \(\text{Crit}(H) \cap \Sigma_0\) and \(\partial_{\nabla H}\) is defined by counting index-1 slow lines of \(\nabla H\) which are contained in \(\Sigma_0\). Note that \(\nabla H\) points outwards/inwards at a component of \(\partial_+ \Sigma_0\). The homology of \((CM_*(H|\Sigma_0), \partial_{\nabla H})\) is therefore isomorphic to \(H_*(\Sigma_0, \partial_+ \Sigma_0; \mathbb{Z}_2)\). See [44] for details on relative Morse homology.

Similarly, we can identify the quantum cap product defined in (4). Note that the image of the evaluation map \(\mathcal{M} \to \Sigma, u \mapsto u(0,0)\), is \(W^u(\nabla H, x^-) \cap W^s(\nabla H, x^+)\). Choose a Morse function \(f : \Sigma \to \mathbb{R}\) such that the evaluation map is transverse to \(W^u(\nabla f, x)\) for all \(x \in \text{Crit}(f)\). For \(x \in \text{Crit}(f)\) and \(x^\pm \in \text{Crit}(H) \cap \Sigma_0\) satisfying

\[
\text{ind}_H(x^+) = \text{ind}_H(x^-) + \text{ind}_f(x),
\]

let \(q(x; x^-, x^+) \in \mathbb{Z}_2\) be the cardinality mod 2 of

\[
W^u(\nabla f, x) \cap W^u(\nabla H, x^-) \cap W^s(\nabla H, x^+).
\]

The map

\[
CM^*(f) \otimes CM_*(H|\Sigma_0) \longrightarrow CM_*(H|\Sigma_0), \quad x \otimes y \longmapsto \sum_z q(x; y, z)z,
\]

which induces the quantum cap product on homology, is therefore given in purely Morse theoretical terms. On the level of homology, it is the ordinary cap product. This finishes the proof of the lemma.

**Proof of Theorem 2.12.** From Lemma 2.13 and 2.15, it follows that

\[
HF_*(\phi) \cong H_*(\Sigma_0, \partial_+ \Sigma_0; \mathbb{Z}_2) \oplus \mathbb{Z}^\#(\text{Fix}(\phi|\Sigma) \setminus \Sigma_0).
\]

Moreover, \(H^*(\Sigma; \mathbb{Z}_2)\) acts on the first summand by ordinary cap product. Since every fixed point of \(\phi|\Sigma \setminus \Sigma_0\) has fixed point index 1, the Lefschetz fixed point formula implies that

\[
\#(\text{Fix}(\phi) \setminus \Sigma_0) = \Lambda(\phi|\Sigma \setminus \Sigma_0).
\]

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It remains to show that $1 \in H^0(\Sigma; \mathbb{Z}_2)$ acts on $\mathbb{Z}_2^\Delta(\phi; \Sigma_0)$ by the identity and any element of $H^1(\Sigma; \mathbb{Z}_2) \oplus H^2(\Sigma; \mathbb{Z}_2)$ by the zero map.

From the proof of Lemma 2.13, we know that if $y^-, y^+ \in \text{Fix}(\phi) \setminus \Sigma_0$ and $\mathcal{M}(y^-, y^+; J, \Phi) \neq \emptyset$, then $y^- = y^+$. Since the action $\alpha_x$ has vanishing periods on $\Omega_\Phi$, by Lemma 2.14, it follows that every $u \in \mathcal{M}(y, y; J, \Phi)$ has energy zero. Hence, $\mathcal{M}(y, y; J, \Phi) = \mathcal{M}_0(y, y; J, \Phi)$ consists of the constant map. Now choose a Morse function $f : \Sigma \to \mathbb{R}$ with only one critical point $x_0$ of index 0. Generically, $\text{Fix}(\phi|\Sigma \setminus \Sigma_0) \subset W_u(\nabla f, x_0)$. It follows that $q(x; y^-, y^+) \neq 0$ if and only if $x = x_0$ and $y^- = y^+$. This ends the proof.

Remark. Using Theorem 2.12, we can verify [51, Theorem 1] in the special case of an algebraically finite mapping class: if $g$ is a non-trivial mapping class, the quantum cap product is trivial on $H^2(\Sigma; \mathbb{Z}_2) \otimes HF_\ast(g)$.

To see this, assume that $g$ is algebraically finite. By Theorem 2.12, the only possibly non-trivial part of the quantum cap product is given by the cap product on $H^*(\Sigma; \mathbb{Z}_2) \otimes H_c(\Sigma_0, \partial, \Sigma_0; \mathbb{Z}_2)$. The submanifold $\Sigma_0 \subset \Sigma$ has non-trivial boundary since $g$ is non-trivial. Now the cap product on $H^*(\Sigma; \mathbb{Z}_2) \otimes H_c(\Sigma_0, \partial, \Sigma_0; \mathbb{Z}_2)$ factors through the homomorphism $\iota : H^*(\Sigma; \mathbb{Z}_2) \to H^*(\Sigma_0; \mathbb{Z}_2)$ which is induced by the inclusion $\iota : \Sigma_0 \hookrightarrow \Sigma$. Since $H^2(\Sigma_0; \mathbb{Z}_2) = 0$, this proves the claim.

Similarly, it follows from Theorem 2.12, that if $\alpha \in H^1(\Sigma; \mathbb{Z}_2)$ acts non-trivially on $HF_\ast(g)$, then there exists a map $\gamma : S^1 \to \Sigma_0$ such that $\langle \alpha, [\gamma] \rangle = 1$. This is a special case of [51, Theorem 2].

### 2.5 Isolated plane curve singularities

We begin with a brief summary of the basic facts on isolated plane curve singularities. The standard reference is Milnor’s book [36].

An isolated plane curve singularity is a germ $[f]$ of holomorphic functions $f : (U, 0) \to (\mathbb{C}, 0)$, where $U \subset \mathbb{C}^2$ is a neighborhood of 0, with $(d f)^{-1}(0) = \{0\}$. Let $f : U \to \mathbb{C}$ be such a function. For $\varepsilon > 0$ sufficiently small, the singular fiber $f^{-1}(0)$ intersects the 3-sphere $S_\varepsilon := \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = \varepsilon\}$ transversally. The intersection

$$L := f^{-1}(0) \cap S_\varepsilon \subset S_\varepsilon$$

is a compact oriented 1-manifold, i.e. a link. A link obtained in this way is called an algebraic link. An algebraic link is a fibred link: The map

$$\pi : S_\varepsilon \setminus L \to \{z \in \mathbb{C} : |z| = 1\}, \quad z \mapsto f(z)/|f(z)|,$$

is a fibration, the Milnor fibration, and the Milnor fiber

$$M := \pi^{-1}(1) \cup L$$

is a Seifert surface of $L$. This means that $M \subset S_\varepsilon$ is a compact connected oriented embedded 2-manifold with $\partial M = L$. 

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The geometric monodromy is an isotopy class of the group $\text{Diff}^+ c(M)$ of orientation preserving diffeomorphisms which are the identity near $\partial M$ and is defined as follows. Given a connection on $S_\varepsilon \setminus L$, i.e. a rank-1 subbundle of the tangent bundle of $S_\varepsilon \setminus L$ which is transverse to $\ker(d\pi)$, parallel transport induces an orientation preserving diffeomorphism of $\pi^{-1}(1)$; a so-called characteristic diffeomorphism. To extend this diffeomorphism to $M$, we specify the connection in a neighborhood of $L$. For this observe the following. Let $L_0$ be a connected component of $L$ and $T \subset S_\varepsilon$ be a tubular neighborhood of $L_0$. A standard meridian of $(T;L_0)$ is an embedded circle in $T_\varepsilon \setminus L_0$ which is homologically trivial in $T$ and has linking number 1 with $L_0$ in $S_\varepsilon$. There is a fibration of $T_\varepsilon \setminus L_0$ such that every fiber is a standard meridian of $(T,L_0)$. This fibration is unique up to isotopy and if $T$ is sufficiently small, induces a connection on $T_\varepsilon \setminus L_0$. In this way, we get a standard connection in neighborhood of $L$. A connection on $S_\varepsilon \setminus L$ which restricts to this standard connection, induces a characteristic diffeomorphism which is compactly supported and hence extends trivially to $M$. Moreover, any two such diffeomorphisms are isotopic in $\text{Diff}^+ c(M)$. The isotopy class obtained in this way is therefore an invariant of $\pi$.

The main result of this section is

**Theorem 2.16.** Let $M \subset \Sigma$ be the Milnor fiber of an isolated plane curve singularity and $g$ be the mapping class which is obtained by extending the geometric monodromy trivially to $\Sigma$. Then

$$HF^*_g(\Sigma) \cong H^*_\varepsilon(\Sigma;M;\mathbb{Z}_2),$$

where $H^*(\Sigma;\mathbb{Z}_2)$ acts by cap product.

The proof of Theorem 2.16 relies on the following result which is a refinement of the classical result of A’Campo [1] and Lê [31] that the Lefschetz number of the geometric monodromy of an isolated plane curve singularity vanishes. We use the following notation: We denote by $\iota: \text{Diff}^+ c(M) \rightarrow \text{Diff}^+(M,\partial M)$ the inclusion, where $\text{Diff}^+(M,\partial M)$ denotes the group of orientation preserving diffeomorphisms which are the identity on $\partial M$.

**Theorem 2.17.** Let $M$ be the Milnor fiber and $g$ be the geometric monodromy of an isolated plane curve singularity. There is a representative $\phi \in \iota_* g$ which is a diffeomorphism of finite type (same definition as for closed surfaces) and such that $\text{Fix}(\phi) = \partial M$. Moreover, $\phi$ only has positive twists.

The proof of Theorem 2.17 is given in Appendix B, i.e. Section 2.7. It heavily relies on the work of Eisenbud and Neumann [14] on the monodromy of plane curve singularities. The relevant results are summarized in Appendix B. We remark, that it is a classical result in the theory of singularities that the geometric monodromy is an algebraically finite mapping class and that it only has positive twists. The vanishing of the Lefschetz number, however, is not enough to conclude the statement of the proposition. In particular, it does not exclude the existence of pointwise fixed annuli.
Proof of Theorem 2.16. Assume for the moment that no component of $\Sigma \setminus \text{int}(M)$ is a disk. From Theorem 2.17, it follows that there exists a representative $\phi \in g$ which is of finite type and such that $\Sigma_0 = \Sigma \setminus \text{int}(M)$, $\partial_\ast \Sigma_0 = \partial M$ and $\Lambda(\phi|\text{int}(M)) = 0$. Theorem 2.12 therefore implies that

$$HF_*(\phi) \cong H_*(\Sigma \setminus \text{int}(M), \partial M; \mathbb{Z}_2).$$

By excision, it follows that $HF_*(\phi) \cong H_*(\Sigma, M; \mathbb{Z}_2)$.

Now assume that $D_1, \ldots, D_n$ are disk components of $\Sigma \setminus \text{int}(M)$. Set $M_0 := M \setminus D_1 \cup \cdots \cup D_n$. We claim that there exists a representative $\phi \in g$ which is of finite type and such that $\Sigma_0 = \Sigma \setminus \text{int}(M_0)$, $\partial_\ast \Sigma_0 = \partial M_0$ and $\Lambda(\phi|\text{int}(M_0)) = n$. Theorem 2.12 then implies that

$$HF_*(\phi) \cong H_*(\Sigma \setminus \text{int}(M_0), \partial M_0; \mathbb{Z}_2) \oplus \mathbb{Z}_2^n.$$

Since $H_*(\Sigma, M_0; \mathbb{Z}_2) \oplus \mathbb{Z}_2^n \cong H_*(\Sigma, M; \mathbb{Z}_2)$, this proves Theorem 2.16 up to the claim above. The claim follows from Theorem 2.17 by “collapsing” each disk component of $\Sigma \setminus \text{int}(M)$ to a point. 

As an example of Theorem 2.16 we consider the $A_k$-singularity. First, we recall some terminology. Let $k > 0$; an $A_k$-configuration in $\Sigma$ is a $k$-tuple $(C_1, \ldots, C_k)$ of embedded circles in $\Sigma$ such that

$$\#(C_i \cap C_{i+1}) = 1 \quad \text{for } i = 1, \ldots, k - 1,$$

$$\#(C_i \cap C_j) = 0 \quad \text{if } |i - j| > 1.$$  

The $A_k$-singularity is the germ of the function $f(x, y) = x^2 + y^{k+1}$. It is a classical result in the theory of singularities, that the Milnor fiber $M$ of this singularity contains an $A_k$-configuration, which is (i) a spine of $M$ and (ii) a distinguished basis of vanishing cycles. See [4, Section 2.9], [2].

Corollary 2.18. Let $(C_1, \ldots, C_k)$ be an $A_k$-configuration of circles in $\Sigma$. Let $g$ be the mapping class of the product $\tau_1 \circ \cdots \circ \tau_k$, where $\tau_i$ denotes the positive Dehn twist along $C_i$. Then

$$HF_*(g) \cong H_*(\Sigma, C_1 \cup \cdots \cup C_k; \mathbb{Z}_2),$$

where $H^*(\Sigma; \mathbb{Z}_2)$ acts by cap product. The same formula holds, if $g$ is the class of $\tau_{\sigma 1} \circ \cdots \circ \tau_{\sigma k}$, where $\sigma$ is a cyclic permutation of $k$ elements.

Proof of Corollary 2.18. Let $(C_1, \ldots, C_k)$ be an $A_k$-configuration in $\Sigma$. From the remark (i) above, it follows that a tubular neighborhood $N$ of $C_1 \cup \cdots \cup C_k$ can be identified with the Milnor fiber of the $A_k$-singularity such that, by (ii), $C_1 \cup \cdots \cup C_k$ is a distinguished set of vanishing cycles. This implies, that the class $g$ of the product $\tau_1 \circ \cdots \circ \tau_k$ of right Dehn twists can also be obtained
by extending the geometric monodromy of the $A_k$-singularity trivially to $\Sigma$. It therefore follows from Theorem 2.16 that
\[ \text{HF}_*(g) \cong H_*(\Sigma, N; \mathbb{Z}_2) \cong H_*(\Sigma, C_1 \cup \cdots \cup C_k; \mathbb{Z}_2). \]
This together with naturality proves the corollary.

Finally, we mention a further application of Theorem 2.17. Floer homology theory for two-manifolds with boundary assigns a pair of vector spaces $\text{HF}_*(g, +), \text{HF}_*(g, -)$ to every isotopy class $g$ of orientation preserving diffeomorphisms which are the identity near the boundary. The sign corresponds to two different perturbations of $\phi$ near the boundary. In Appendix C, Section 2.8, we give an outline of this version of Floer homology. The following result confirms a conjecture of Seidel in the case of isolated plane curve singularities [48, page 23].

**Theorem 2.19.** If $g$ denotes the geometric monodromy of an isolated plane curve singularity, then
\[ \text{HF}_*(g, +) = 0. \]

**Proof of Theorem 2.19.** Let $M$ be the Milnor fiber and $g$ be the geometric monodromy of an isolated plane curve singularity. Let $\phi \in \iota_* g$ be as in Theorem 2.17. Since $\phi$ only has positive twists, we can perturb it near the boundary to a diffeomorphism $\phi_+$ such that $\text{Fix}(\phi_+) = \emptyset$. Furthermore, it follows as in the proof of Proposition 2.6, that if $\omega$ is a $\phi$-invariant area form on $M$, then $[\omega_\phi] = 0$. Hence, $\phi$ is monotone in the sense of Definition 2.25. Therefore, $\text{HF}_*(g, +) = \text{HF}_*(\phi, +) = 0$. \(\square\)

### 2.6 Appendix A: Products of disjoint Dehn twists

The goal of this appendix is to prove Proposition 2.20, which was also stated in [47, Lemma 3]. This result is used for the proof of the fixed point as well as the action proposition in Section 2.2. Let $I$ denote either $[0, 1]$ or $S^1$. For every continuous map $\gamma : I \to \Sigma$ and compact 1-dimensional submanifold $C \subset \Sigma$ there is the geometric intersection number
\[ i(\gamma, C) = \min \{ \# \beta^{-1}(C) \mid \beta \text{ is homotopic rel endpoints to } \gamma \}. \]
If $I = S^1$, homotopic rel endpoints means freely homotopic.

**Proposition 2.20.** Let $C \subset \Sigma$ be a finite union of non-contractible disjoint circles. Let $\phi$ be a product of Dehn twists along $C$ which twists with the same sign along parallel components of $C$. Let $\gamma : I \to \Sigma$ be such that $\gamma(\partial I) \cap \text{supp}(\phi) = \emptyset$. If $\gamma$ is homotopic rel endpoints to $\phi \circ \gamma$, then $i(\gamma, C) = 0$.

An immediate consequence is the following
Corollary 2.21. Let $\phi$ be of finite type and $\ell > 0$ be as such that $\phi|\Sigma \setminus N = \text{id}$. Let $\gamma : I \to \Sigma$ be such that $\gamma(\partial I) \cap \text{supp}(\phi) = \emptyset$. If $\gamma$ is homotopic rel endpoints to $\phi \circ \gamma$, then there exists $\gamma' : I \to \text{Fix}(\phi')$ which is homotopic rel endpoints to $\gamma$.

Proof of Proposition 2.20. We begin with the case $I = [0,1]$. For every component of $C$, let $N'$ be the closed tubular neighborhood, where the Dehn twist is supported. For different components of $C$, these tubular neighborhoods are disjoint. Let $N$ be the union of the $N'$ and $\hat{C} := \partial N$. We prove in several steps that $i(\gamma, \hat{C}) = 0$, if $\gamma$ is homotopic rel endpoints to $\phi \circ \gamma$.

Let $\beta : [0,1] \to \Sigma$ and $u : [0,1]^2 \to \Sigma$ be such that

$$\#\beta^{-1}(\hat{C}) = i(\gamma, \hat{C})$$

and

$$u(s,0) = \beta(s), \quad u(s,1) = \phi(\beta(s)), \quad u(0,t) = \beta(0), \quad u(1,t) = \beta(1)$$

for all $s, t \in [0,1]$. We assume that $\beta(0), \beta(1) \in \Sigma \setminus N$. Without loss of generality we further assume that $u$ is transverse to $\hat{C}$. Hence,

$$B := u^{-1}(\hat{C}) \subset [0,1]^2$$

is a compact 1-dimensional submanifold with boundary

$$\partial B = \beta^{-1}(\hat{C}) \cup (\phi \circ \beta)^{-1}(\hat{C}).$$

Every component of $B$ is either a circle or an arc.

Claim 1. We may assume that no component of $B$ is a circle.

Let $S$ be a circle component of $B$. The interior of $S$, i.e. the region of $[0,1]^2$ bounded by $S$, is a disk. The restriction of $u$ to this disk induces an element of $\pi_2(\Sigma, C')$, where $C'$ is the component of $\hat{C}$ where $S$ is mapped to. Since $C'$ is non-contractible, $\pi_2(\Sigma, C') = 0$, and hence $u$ can be deformed in a neighborhood of the interior of $S$ in such a way that $B$ has less circle components. Repeating this argument finitely many times proves claim 1. From now on, we assume that every component of $B$ is an arc.

Claim 2. There is no component $B'$ of $B$ with $\partial B' \subset [0,1] \times 0$ or $\partial B' \subset [0,1] \times 1$. Assume that $B'$ is such that $\partial B' \subset [0,1] \times \{0\}$. Note that the boundary points of $B'$ are intersection points of $\beta$ with $\hat{C}$. Since $B'$ is mapped to some component of $\hat{C}$ under $u$, it follows that these intersection points can be removed by homotoping $\beta$. This however, contradicts the definition of $\beta$. Hence, every $B'$ with one boundary point on $[0,1] \times \{0\}$ has the other boundary point on $[0,1] \times \{1\}$. On the other hand, $B$ has the same number of boundary points on $[0,1] \times \{0\}$ as on $[0,1] \times \{1\}$, since $\beta$ and $\phi \circ \beta$ intersect $\hat{C}$ in the same number of points. This proves claim 2. The rest of the proof is devoted to

Claim 3. $B$ is empty, i.e. $\#(\text{im } \beta \cap \hat{C}) = 0$. 37
The proof is by contradiction; assume that $B \neq \emptyset$. First we define an integer valued function on $\pi_0(B)$. Let $B'$ be a component of $B$. By claim 1 and 2, $B'$ is an arc and $\partial B' = \{(b, 0), (b, 1)\}$, for some $b \in (0, 1)$. Let $C'$ be the component of $\tilde{C}$ where $B'$ is mapped to under $u$. Since $u(b, 1) = \phi(u(b, 0)) = u(b, 0)$, the restriction $u|B'$ has a well defined mapping degree $\deg u|B'$, once orientations of $B'$ and $C'$ are fixed. We orient $B'$ “from the bottom to the top” and choose an orientation of $\tilde{C}$ such that the orientations of two homotopic components match. We thus have the map

$$d : \pi_0(B) \longrightarrow \mathbb{Z}, \quad B' \longmapsto \deg u|B'.$$

Let $\pi_0(B) = \{B_1, \ldots, B_{2n}\}$ be ordered such that

$$b_i < b_j \iff i < j,$$

where $\partial B_i = \{(b_i, 0), (b_i, 1)\}$. Note that the cardinality of $\pi_0(B)$ is even, since $\beta(0)$ and $\beta(1)$ are both in the complement of $N$. Observe that the loops $u|B_1$ and $u|B_{2n}$ are both homotopic to the constant loop and hence

$$d(B_1) = d(B_{2n}) = 0.$$

To prove claim 3, we will now show that $d(B_{2n}) \neq 0$, which is a contradiction. In fact, we will prove by induction that

$$d(B_{2k}) \neq 0, \quad \text{sign}(b_{2k}) = \text{sign}(b_{2k-2}) \quad \text{and} \quad C_{2k} \sim C_{2k-2}, \quad (*)_k$$

for all $1 \leq k \leq n$. Here $\text{sign}(b_i)$ denotes the sign of $b_i$ as an intersection point of $\beta$ and $\tilde{C}$, and $C_i$ is the component of $\tilde{C}$ where $B_i$ is mapped to under $u$. We will use the following formula for the function $d$, which we prove later on:

$$d(B_{2k}) = -\sum_{i=1}^{k} \text{sign}(b_{2i}) \cdot \varepsilon_{2i}, \quad (12)$$

for all $1 \leq k \leq \ell$. Here $\varepsilon_i = \pm$ is the sign of the twist of $\phi$ along $C_i$. For $k = 1$, formula (12) gives $d(B_2) = \text{sign}(b_2) \cdot \varepsilon_2 \neq 0$. This proves the induction hypothesis $(*)_1$, since the other two statements are empty in this case. Assume that $(*)_k$ holds for all $1 \leq i \leq k < n$. To show that it also holds for $k + 1$ first note, that if

$$\text{sign}(b_{2k}) = \text{sign}(b_{2k+2}) \quad \text{and} \quad C_{2k} \sim C_{2k+2}, \quad (13)$$

then by formula (12), we get that

$$d(B_{2k+2}) = (k + 1) \cdot \text{sign}(b_{2k+2}) \cdot \varepsilon_{2k+2} \neq 0.$$

To prove (13), we consider the restriction of $u$ to the region $D \subset [0, 1]^2$ that is bounded by the arcs

$$B_{2k}, B_{2k+1} \quad \text{and} \quad [b_{2k}, b_{2k+1}] \times \{0\}, \ [b_{2k}, b_{2k+1}] \times \{1\}.$$
Since $u(D) \subset \Sigma \setminus \text{int}(N)$, we have that $u(s, 1) = u(s, 0)$ for all $s \in [b_{2k}, b_{2k+1}]$ and hence, $u|D$ gives a homotopy between the loops $d(B_{2k}) \cdot C_{2k}$ and $d(B_{2k+1}) \cdot C_{2k+1}$. This implies that $C_{2k} \sim C_{2k+1}$ and that $d(B_{2k}) = d(B_{2k+1})$. Since $C_{2k+1}$ and $C_{2k+2}$ are both contained in the boundary of one component of $N$, they are clearly homotopic and hence $C_{2k} \sim C_{2k+2}$. It thus remains to show that $\text{sign}(b_{2k}) = \text{sign}(b_{2k+2})$. For this we need

Claim 4. $C_{2k} \neq C_{2k+1}$ and $C_{2k+1} \neq C_{2k+2}$.

Assume that $C_{2k} = C_{2k+1}$ and consider the path $s \mapsto u(s, 0), s \in [b_{2k}, b_{2k+1}]$. We claim that since $d(B_{2k}) = d(B_{2k+1}) \neq 0$, the path can be deformed into $C_{2k}$, which contradicts minimality of $\beta$. For the proof of the claim, we refer to Corollary 3.4 in Section 3.1. Similarly, $C_{2k+1} = C_{2k+2}$ is also a contradiction to minimality of $\beta$.

Claim 5. $\text{sign}(b_{2k}) = \text{sign}(b_{2k+1}) = \text{sign}(b_{2k+2})$.

Since $C_{2k}$ and $C_{2k+1}$ are homotopic and disjoint, there exists an embedded annulus $A \subset \Sigma$ which bounds $C_{2k}$ and $C_{2k+1}$. Since $d(B_{2k}) = d(B_{2k+1}) \neq 0$, it follows that $u(D) = A$, where $D$ is defined as above. Otherwise, we could find a map from the torus to $\Sigma$ with nonzero degree. Such a map does not exist since the genus of $\Sigma$ is $> 1$. Thus, we know that the path $\beta$ enters $A$ at $b_{2k}$ and leaves at $b_{2k+1}$. The intersection points therefore have the same sign, so $\text{sign}(b_{2k}) = \text{sign}(b_{2k+1})$.

The proof that $\text{sign}(b_{2k+1}) = \text{sign}(b_{2k+2})$ is similar. In this case however, we know that the region $D'$ of $[0, 1]^2$ between $B_{2k+1}$ and $B_{2k+2}$ is mapped to a component $N'$ of $N$ under $u$. Since $d(B_{2k+1}) \neq 0$, it follows that $u(D') = N'$ and hence that $b_{2k+1}$ and $b_{2k+2}$ have the same sign, as above. This proves claim 5 and ends the proof of (13). It remains to prove formula (12).

Formula (12) is the consequence of the following properties of the function $d$. First of all, as was shown above, we have that

$$d(B_{2k}) = d(B_{2k+1}),$$

for all $1 \leq k \leq n - 1$. We now show that

$$d(B_{2k}) = d(B_{2k-1}) - \text{sign}(b_{2k}) \cdot \varepsilon_{2k},$$

for all $2 \leq k \leq \ell$. The idea is to look again at the region $D' \subset [0, 1]^2$ which is bounded by the arcs

$$B_{2k-1}, B_{2k} \text{ and } [b_{2k-1}, b_{2k}] \times \{0\}, [b_{2k-1}, b_{2k}] \times \{1\}.$$ 

Let $N'$ be the component of $N$ where $D'$ is mapped to. As in claim 4, we conclude that $\partial N' = C_{2k-1} \cup C_{2k}$. Choose an orientation preserving diffeomorphism $[0, 1] \times S^1 \cong N'$ such that $C_{2k-1} \cong 0 \times S^1$ and $C_{2k} \cong 1 \times S^1$. Assume for the moment that the orientations of $C_{2k-1}$ and $C_{2k}$ are compatible with these diffeomorphisms. This is equivalent to saying that $\text{sign}(b_{2k-1}) = 1$. Let $\text{pr} : [0, 1] \times S^1 \to S^1$ denote the projection onto the second factor. The map $\text{pr} \circ u : \partial D \to S^1$ is the composition of four loops, denoted by $v$,
$u|B_{2k}$, the inverse of $w$ and the inverse of $u|B_{2k-1}$. Here $v, w$ are the loops $s \mapsto \text{pr}(u(s, 0)), s \mapsto \text{pr}(u(s, 1))$ for $s \in [b_{2k-1}, b_{2k}]$. Since the mapping degree of $\text{pr} \circ u|\partial D$ vanishes, it follows that

$$\deg v + d(B_{2k}) - \deg w - d(B_{2k-1}) = 0.$$ 

However, $\deg v - \deg w = \varepsilon_{2k}$, and this proves equation (15) in the case that $\text{sign}(b_{2k}) = 1$. If $\text{sign}(b_{2k}) = -1$, $\text{pr}$ induces orientation reversing maps on $C_{2k-1}$ and $C_{2k}$. Hence, the same argument as above gives

$$\deg v - d(B_{2k}) - \deg w + d(B_{2k-1}) = 0,$$

which again implies (15). Formula (12) is now an immediate consequence of (14),(15) and the fact that $d(B_1) = 0$. This ends the proof of the proposition in the case $I = [0, 1]$.

The proof in the case $I = S^1$ follows the same line of arguments as above. In this case, however, $B = u^{-1}(\partial C)$ is a subset of $S^1 \times [0, 1]$ instead of $[0, 1]^2$. This does not affect the proofs of claims 1 and 2 above and hence, we can assume that every component $B'$ of $B$ is an arc with $\partial B' = \{(b, 0), (b', 1)\}$. Note that $b, b' \in S^1$ are not necessarily equal. We show how the proof of claim 3 extends to this case. The idea is to consider the $\ell$-fold catenation $v : S^1 \times [0, \ell] \to \Sigma$, defined by

$$v(s, t) := \phi^j(u(s, t - j)) \quad \text{for} \quad (s, t) \in S^1 \times [j, j + 1], j < \ell,$$

where $\ell > 0$ is chosen such that every component $A'$ of $A := v^{-1}(\partial C)$ satisfies $\partial A' = \{(a, 0), (a, \ell)\}$ for some $a \in S^1$. The rest of the argument is analogous to the above. Define the function $d : \pi_0(A) \to \mathbb{Z}$ and prove that it satisfies

$$d(A_{2k}) = d(A_{1}) - k \cdot \ell \cdot \text{sign}(a_{2k}) \cdot \varepsilon_{2k}. \quad (16)$$

Here, the signs $\text{sign}(a_i)$ and $\varepsilon_i$ are defined as above. The additional factor $\ell$ appears since $\phi^j$ twists with multiplicity $\ell$ along every component of $C$. The order on $\pi_0(A)$ is a cyclic order induced from the boundary points as above. Formula (16) is therefore a contradiction, if $\pi_0(A) \neq \emptyset$. This ends the proof of the proposition.

\begin{flushright} $\Box$ \end{flushright}

### 2.7 Appendix B: Decomposition of the monodromy

In this appendix we prove Theorem 2.17. For this we use the theory of splice diagrams which was developed by Eisenbud and Neumann [14], [38] to compute invariants of plane curve singularities. In the following we summarize their results on the geometric monodromy. We would like to emphasis here, that our discussion is far from being self-contained. Instead, we only concentrate on the facts which are useful for our purpose. For details, proofs as well as the general picture, we refer to the excellent monograph [14]. Let us introduce some terminology.
Definition 2.22. Denote by $M$ a compact oriented 2-manifold with boundary.

(i) Let $\phi : M \to M$ be a diffeomorphism and $C \subset M$ be a $\phi$-invariant union of disjoint non-contractible circles. A $\phi$-component of $(M, C)$ is a a union of connected components of $M \setminus C$ which are cyclically permuted by $\phi$. The topological type of a $\phi$-component $M'$ is the triple $(\chi, d, h)$, where $\chi$ is the Euler characteristic, $d$ the number of connected components and $h$ the number of ends of $\text{int}(M')$.

(ii) An orientation preserving diffeomorphism $\phi : M \to M$ is called an admissible twist map if $M$ is a union of annuli and:

(1) $\phi$ cyclically permutes the connected components of $M$.

(2) If $n > 0$ denotes the number of connected components of $M$, then $n$ is given by the local model

$$[0,1] \times S^1 \ni (q,p) \mapsto (q,p - f(q)),$$

where $f : [0,1] \to \mathbb{R}$ is monotone.

(3) There exists $q > 0$ such that $\phi^q | \partial M = \text{id}$.

If $\phi : M \to M$ is an admissible twist map, then its twist number is defined by

$$\ell := \frac{1}{q} \cdot \text{var}(\phi^q) \cdot \xi \in \mathbb{Q}.$$

Here $\xi$ is a generator of $H_*(M', \partial M')$, with $M'$ a connected component of $M$, $\text{var}(\phi^q) : H_*(M', \partial M') \to H_*(M')$ is the variation homomorphism of $\phi^q$ and the dot stands for the intersection pairing.

(iii) An admissible triple $(\phi, M, C)$ consists of an orientation preserving diffeomorphism $\phi : M \to M$ and a $\phi$-invariant finite union $C \subset M$ of disjoint non-contractible circles, such that the following holds. Let $M'$ be a $\phi$-component of $(M, C)$ and $\chi$ denote its Euler characteristic.

(1) If $\chi = 0$, then $\phi | \text{cl}(M')$ is an admissible twist map.

(2) If $\chi < 0$, then $\phi | \text{cl}(M')$ is a periodic map.

By the period of a periodic diffeomorphism $\phi$, we mean the smallest integer $\ell > 0$ such that $\phi^\ell = \text{id}$.

(iv) Set

$$T := \{ (\chi, d, h; \ell) \in \mathbb{Z}^3 \times \mathbb{Q} : d, h > 0; \chi \leq 0; \chi < 0 \Rightarrow \ell \in \mathbb{N}_{>0} \}.$$

Let $(\phi, M, C)$ be an admissible triple. For every $\phi$-component $M'$ of $(M, C)$, set $t(M') := (\chi, d, h; \ell) \in T$, where $(\chi, d, h)$ denotes the topological type of $M'$ and $\ell$ either the period of $\phi | M'$ if $\chi < 0$ or the twist number of $\phi | M'$ if $\chi = 0$. The characteristic set of $(\phi, M, C)$ is defined by

$$t(\phi, M, C) := \{ t(M') : M' \text{ is a } \phi \text{-component of } (M, C) \} \subset T.$$
Recall that an isolated plane curve singularity, or simply isolated singularity, is a germ \([f]\) of holomorphic functions \(f : (U, 0) \to (\mathbb{C}, 0)\), where \(U \subset \mathbb{C}^2\) is a neighborhood of 0, with \((df)^{-1}(0) = \{0\}\). The Milnor fiber of \([f]\) is a compact connected oriented 2-manifold \(M\) with boundary. The geometric monodromy \(g \) of \([f]\) is an isotopy class of orientation preserving diffeomorphisms of the Milnor fiber which are the identity near the boundary.

We now proceed as follows: we explain in 2.7.1 how to associate a diagram \(\Gamma\), called splice diagram, to \([f]\) and in 2.7.2, how to associate a set \(T \subset T\) to such a diagram \(\Gamma\). The significance of these constructions is expressed by the following result from [14], which we state precisely in Theorem 2.23: There exists an admissible triple \((\phi, M, C)\) with characteristic set \(T\) and such that \(\phi \in \iota_* g\). Moreover, we have a formula for the twist map components of \(\phi\). This is used in 2.7.3 to prove Proposition 2.17.

As the title of the monograph [14] indicates, Eisenbud and Neumann’s point of view is that of link theory. Our discussion of splice diagrams, however, is without any reference to link theory. We simply consider them as a tool for encoding some algebraic data. This has the advantage that the reader is not assumed to be familiar with 3-manifold theory. The downside is that these diagrams seem to lack intrinsic geometric relevance and that it is not clear, how the stated results are actually proven. For this we refer to the original literature [14], [38].

### 2.7.1 Puiseux data and splice diagrams

Let \([f]\) be an isolated singularity. In the following, we define the diagram \(\Gamma[f]\) as shown in [14, Appendix 1]. The construction can be divided into 4 steps.

**Step 1.** The first step uses the so-called Newton method for solving the equation \(f(x, y) = 0\) for \(y\) in terms of \(x\) in a neighborhood of 0. We only state the result without going into detail and refer the reader to [5, Chapter 8.3]. Recall that a fractional power series is a pair \((P; d)\), where

\[
P(x) = \sum_{i=1}^{r} a_i x^{n_i}, \quad r \in \mathbb{N} \cup \{\infty\}, \quad a_i \in \mathbb{C}, a_i \neq 0, n_i \in \mathbb{N}_{>0}, n_i < n_{i+1},
\]

is a power series converging in a neighborhood of 0, and \(d\) is a positive integer which is relatively prime to the set \(\{n_i : i \in \mathbb{N}, i \leq r\}\). Two fractional power series \((P, d), (\tilde{P}, \tilde{d})\) are called equivalent, we write \((P, d) \sim (\tilde{P}, \tilde{d})\), if \(d = \tilde{d}\) and there exists \(\theta \in \mathbb{C}\) such that \(\theta^d = 1\) and \(P(x) = P(\theta \cdot x)\).

Now if \([f]\) is an isolated singularity, there is a collection \(\{(P_1, d_1), \ldots, (P_K, d_K)\}\) of pairwise non-equivalent fractional power series, called Puiseux series, such that

\[
f(x, y) = 0 \iff \exists z \in \mathbb{C}, \exists j \in \{1, \ldots, K\} : x = z^{d_j}, y = P_j(z).
\]

---

\[\text{Here, } \iota : \text{Diff}^+(M) \to \text{Diff}^+(M, \partial M) \text{ is the inclusion.}\]
The Puiseux series are uniquely determined up to equivalence; \( \kappa \) is the number of branches of \([f]\).

**Step 2.** Starting with a collection \( \{(P_1, d_1), \ldots, (P_\kappa, d_\kappa)\} \) of pairwise non-equivalent fractional power series, we now define another such collection
\[
\{(P'_1, d_1), \ldots, (P'_\kappa, d_\kappa)\}
\]
with the property that each \( P'_j \) is a finite series. For this, we need the following notation. Let \( \Pi = (P, d) \) be a fractional power series. For \( s \geq N \), set
\[
d_s(\Pi) := \min\{d' \in \mathbb{N} : 1 \leq i \leq \min(s, r) \Rightarrow d' \cdot n_i \in d \cdot \mathbb{N}\}.
\]
and
\[
P^{(s)}(x) := \sum_{i=1}^{\min(s, r)} a_i x^{m_i}, \quad m_i := n_i d_s(\Pi)/d.
\]
Note that \( \Pi^{(s)} := (P^{(s)}, d_s(\Pi)) \) is a fractional power series and that \( d_s(P, d) \) is increasing in \( s \) and eventually equals \( d \). For every \( j = 1, \ldots, \kappa \), set \( \Pi_j := (P_j, d_j) \) and define
\[
r_j := \min\{s \in \mathbb{N} : d_s(\Pi_j) = d_j \text{ and } j \neq j' \Rightarrow \Pi^{(s)}_j \neq \Pi^{(s)}_{j'}\}.
\]
Note that \( \Pi^{(r_j)}_j \neq \Pi^{(r_j)}_{j'} \) if \( j \neq i \). Now let \( \{\Pi_1, \ldots, \Pi_\kappa\} \) be the Puiseux series of \([f]\). We call
\[
\{\Pi^{(r_1)}_1, \ldots, \Pi^{(r_\kappa)}_\kappa\}
\]
the **Puiseux data** of \([f]\).

**Step 3.** Using the Puiseux data of \([f]\), one can now define the diagram \( \tilde{\Gamma}[f] \) from which the diagram \( \Gamma[f] \) is obtained in the next step. For the sake of simplicity, we give the precise definition of \( \tilde{\Gamma}[f] \) only for the cases \( \kappa = 1, 2 \). Assume that \( \kappa = 1 \) and let \( (P, d) \) denote the Puiseux data. Define the integers \( q_1, \ldots, q_r, p_1, \ldots, p_r > 0 \) such that \( \gcd(q_i, p_i) = 1 \) and
\[
P(x^{1/d}) = x^{\frac{q_1}{p_1}} (a_1 + x^{\frac{q_2}{p_2}} (a_2 + \ldots (a_{r-1} + a_r x^{-\frac{q_r}{p_r}}) \ldots ))).
\]
Define the integers \( \alpha_1, \ldots, \alpha_r \) recursively by
\[
\alpha_1 = q_1, \quad \alpha_{i+1} = p_i \alpha_i + q_i.
\]
Note that \( \gcd(\alpha_i, p_i) = 1 \) for all \( i = 1, \ldots, r \). The graph \( \tilde{\Gamma}[f] \) is given by

![Diagram](attachment:image.png)

The coefficients \( a_i \) only enter the description of \( \tilde{\Gamma}[f] \) if \( \kappa > 1 \). Let \( \{\Pi, \tilde{\Pi}\} \) be the Puiseux data of \([f]\). To give the diagram \( \Gamma[f] \) one distinguishes three cases. Set
\[
t := \min\{s \geq 0 : \Pi^{(s+1)} \neq \tilde{\Pi}^{(s+1)}\}.
\]

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and let $\tilde{r}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_r, \tilde{p}_1, \ldots, \tilde{p}_r$ denote the integers associated to $\tilde{\Pi}$.

(i) Assume that $t < r, \tilde{r}$ and $q_{t+1} = \tilde{q}_{t+1}, p_{t+1} = \tilde{p}_{t+1}$.

(ii) Assume that $t < r, \tilde{r}$ and $\frac{q_{t+1}}{p_{t+1}} < \frac{\tilde{q}_{t+1}}{\tilde{p}_{t+1}}$.

(iii) Assume that $t = \tilde{r} < r$. In this case, the diagram is obtained from the diagram in case (ii) by terminating the edge with weight $p_{t+1}$ by an arrowhead.

By interchanging $\Pi$ and $\tilde{\Pi}$ if necessary, (i–iii) define $\tilde{\Gamma}[f]$ in the case $\kappa = 2$. The general case is obtained by induction on $\kappa$. The induction step involves operations of the kind (i–iii). Instead of giving the precise definition, we give a list of properties of the diagram $\tilde{\Gamma} = \tilde{\Gamma}[f]$ for an arbitrary singularity $[f]$. In the case $\kappa = 1, 2$, these properties are easily verified from the definitions above.

(A1) $\tilde{\Gamma}$ has the structure of a weighted tree. All weights are positive integers.

(A2) $\tilde{\Gamma}$ has three kinds of vertices: arrowhead, knob and box vertices. The number of arrowhead vertices equals the number of branches of $[f]$. The arrowhead and knob vertices have 1 incoming edge, the boxed ones at least 3. A box vertex has at most 2 neighboring knob vertices.

(A3) An edge of $\tilde{\Gamma}$ carries a weight at each ending box vertex. The edge-weights at a box vertex are pairwise relatively prime.

(A4) Let $b$ denote a box vertex of $\tilde{\Gamma}$. Let $E$ be the set of edges which connect $b$ to its neighboring box vertices. The graph $\tilde{\Gamma} \setminus E$ has $\# E + 1$ connected components. There is at most one component which does neither contain $b$ nor any arrowhead vertex. Assume that there is such a component. Let $e$ be the edge which connects $b$ to that component and let $b'$ denote the other vertex of $e$. Then $e$ has weight 1 at $b'$.

(A5) Let $b, b'$ denote connected box vertices of $\tilde{\Gamma}$. Let $a_1, \ldots, a_k$ be the weights at $b$ of the edges connecting $b$ and its neighboring vertices. Similarly, let $a'_1, \ldots, a'_k$. 

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be the weights at \( b' \). Assume that \( a_1, a'_1 \) are the weights of the edge connecting \( b \) and \( b' \). Then
\[
 a_1 a'_1 - a_2 \cdots a_k a_2' \cdots a_k' > 0.
\]

Step 4. The diagram \( \Gamma = \Gamma'[^f] \) is obtained from \( \tilde{\Gamma} = \tilde{\Gamma}'[^f] \) by the following algorithm. Set \( \Gamma_0 := \tilde{\Gamma} \).

STEP: Let \( e \) be an edge of \( \Gamma_i \) connecting a box vertex \( b \) and a knob vertex \( v \) and having weight 1. If \( e \) does not exist, set \( \Gamma := \Gamma_i \) and STOP. Otherwise, let \( k \geq 3 \) denote the number of incoming edges at \( b \) and \( n \leq 2 \) the number of neighboring box vertices of \( b \). If \( k = 3 \) and \( n = 0 \), set \( \Gamma := \Gamma_* := \) and STOP.

Otherwise, apply the operation \( \Gamma_i \mapsto \Gamma_{i+1} \) defined as follows

\[
\begin{align*}
\text{if } k = 3, n = 1, 2 & \quad \cdots a_1 a_2 a_3 v \quad \rightarrow \quad \cdots a_3 v \\
\text{if } k > 3 & \quad v 1 \quad \rightarrow \quad \leftarrow
\end{align*}
\]

Finally repeat the STEP.

Among all the possible diagrams \( \Gamma \) which are obtained by this algorithm, \( \Gamma_* \) is exceptional in the sense that it does not have box vertices. It is important to note, that in all other cases the properties (A1–5) still hold if \( \tilde{\Gamma} \) is replaced by \( \Gamma \). Additionally, we have

(A6) Every edge of \( \Gamma \) connecting a box and a knob vertex has weight \( > 1 \).

We end the discussion of splice diagrams with the remark that the diagram associated to the quadratic singularity \( (x, y) \mapsto x^2 + y^2 \) is \( \Gamma_* \).

2.7.2 Characteristic set

Let \( \Gamma \) be the splice diagram of an isolated singularity. In this section, we define the set \( t \subset T \). In the exceptional case \( \Gamma = \Gamma_* \), this is simply the set \( \{ (0, 1, 2; 1) \} \) which is the characteristic set of a positive Dehn twist. Assume from now on that \( \Gamma \neq \Gamma_* \). Denote by \( A \) respectively \( B \) the set of arrowhead respectively box vertices of \( \Gamma \). Moreover, denote by \( E \) the set of edges of \( \Gamma \) which connect \( B \) to \( A \cup B \).

In the following, we define (i) for each \( b \in B \) an element \( t_b \in T \) and (ii) for each \( e \in E \) an element \( t_e \in T \). We then set
\[
t_\Gamma := \{ t_e : e \in B \cup E \}.
\]

Finally we define (iii) for each \( e \in E \) an admissible twist map \( \phi_e \).

Let \( V \) denote the set of ordered pairs of connected vertices of \( \Gamma \). We start by introducing the function \( m : V \to \mathbb{N} \). Let \( (v, v') \in V \) and let \( e \) be the edge connecting \( v \) and \( v' \). The graph \( \Gamma \setminus \{ e \} \) has two components. Denote by \( \Gamma' \) that component which contains \( v' \). Let \( A' \) denote the set of arrowhead vertices of \( \Gamma' \).
For each \( a \in A' \) there is a unique path in \( \Gamma' \), denoted by \( \gamma_a \), which connects \( v' \) and \( a \). Define \( \sigma_a > 0 \) to be the product of all edge-weights adjacent to \( \gamma_a \), but not on \( \gamma_a \). For examples see [14, Page 84]. If \( \gamma_a \) is the constant path, set \( \sigma_a := 1 \).

Define
\[
m(v, v') := \sum_{a \in A'} \sigma_a.
\]

Note that \( n(v, v') = 0 \) if and only if \( A' = \emptyset \). Together with (A4), this has the following consequences:

(i) If \( n(v, v') = 0 \), then \( n(v', v) \neq 0 \).

(ii) If \( b, b' \in B \) and \( m(b, b') = 0 \), then the edge connecting \( b \) and \( b' \) has weight 1 at \( b' \).

(iii) For each \( b \in B \), there exists at most one neighboring vertex \( b' \in B \) such that \( m(b, b') = 0 \).

(iv) If \( b \in B \) and \( a \in A \), then \( m(b, a) = 1 \).

Let \( b \in B \) and denote by \( \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_k\}, 1 \leq n \leq k \), the set of vertices which are connected to \( b \), ordered such that \( v_i \in A \cup B \) if and only if \( i \leq n \). Note that \( k \geq 3 \) and that \( k - n \in \{0, 1, 2\} \). Further denote by \( a_i > 0, i = 1, \ldots, k \), the weight at \( b \) of the edge connecting \( b \) and \( v_i \). Define the numbers
\[
\begin{align*}
  d_b &:= \gcd (m(b, v_1), \ldots, m(b, v_n)), \\
  h_b &:= \sum_{i=1}^n \gcd (m(b, v_i), m(v_i, b)), \\
  \ell_b &:= \sum_{i=1}^n m(b, v_i) \cdot a_1 \cdots \hat{a}_i \cdots a_k, \\
  \chi_b &:= \ell_b \cdot (2 - k + \sum_{i=n+1}^k \frac{1}{a_i}).
\end{align*}
\]

The hat means that the underlying factor is omitted. Note that (B3), (B4) respectively (B1) imply that \( d_b \) respectively \( h_b \) is well defined and therefore \( > 0 \). Similarly, it follows from (B3), (B4) that the integer \( \ell_b \) is \( > 0 \). We furthermore claim, that \( \chi_b < 0 \). This is because \( k \geq 3 \) and \( a_{n+1}, \ldots, a_k > 1 \) are pairwise relatively prime. Hence, we can set
\[
t_b := (\chi_b, d_b, h_b; \ell_b).
\]

Finally note, that from the definition of \( m \), it follows that for each \( i = 1, \ldots, n \),
\[
\ell_b = m(b, v_i) \cdot a_1 \cdots \hat{a}_i \cdots a_k + m(v_i, b) \cdot a_i.
\]
let \( a_1', \ldots, a_{k'} \) be the weights at \( b' \). Assume that \( a_1, a'_1 \) are the weights of \( e \). Define the numbers

\[
\begin{align*}
d_e & := \gcd(m(b, b'), m(b', b)), \\
\Delta_e & := a_1 a'_1 - a_2 \cdots a_k a'_2 \cdots a_{k'} , \\
\ell_e & := \frac{d_e \cdot \Delta_e}{\ell_b \cdot \ell_{b'}} .
\end{align*}
\]  

(19)

By (B1), \( d_e \) is well defined and therefore \( > 0 \). Set

\[ t_e := (0, d_e, 2d_e; \ell_e) \in T. \]

Now let \( e \in E \) be an edge which connects the vertex \( b \in B \) to an arrowhead vertex. In this case set

\[ t_e := (0, d_e, 2d_e; \ell_e) := (0, 1, 2; 1/\ell_b) \in T. \]

(iii) Let \( e \) be an edge which connects \( b \in B \) and \( b' \in A \cup B \). By (B1), (B4) we can assume without loss of generality that \( m(b, b') \neq 0 \). Now choose integers \( n, n' \) with \( m(b, b') \cdot n' + m(b', b) \cdot n = d_e \). Denote by \( a \) the weight of \( e \) at \( b \) and set \( m := m(b, b') \). Define \( \phi_e \) to be the admissible twist map which cyclically permutes \( d_e \) annuli and such that

\[
\phi^d_e(q, p) = \left( q, p - q \cdot d_e \ell_e - \frac{d_e}{m} \left( n - \frac{d_e}{\ell_b} \cdot a \right) \right),
\]

for all \( (q, p) \in [0, 1] \times S^1 \).

2.7.3 Proof of Theorem 2.17

We first state the precise results that are needed for the proof.

**Theorem 2.23 (Eisenbud, Neumann).** Let \( [f] \) be an isolated plane curve singularity with Milnor fiber \( M \) and geometric monodromy \( g \). There exists an admissible triple \((\phi, M, C)\) such that \( \phi \in \iota_* g \) and

\[ t((\phi, M, C)) = t_{(f)}. \]

Moreover, the twist map components of \( \phi \) are given by the model (20).

**Remarks.** (i) This theorem is a summary of results from Sections 9, 10, 11 and 13 of [14]. The periodic components of the monodromy are described in Lemma 11.4 and the twist map components in Theorems 13.1, 13.5. The positive twist property \( (A5) \) is contained in Theorem 9.4. We remark that our notation differs at some points from that of [14].

(ii) Eisenbud-Neumann give a detailed description of the periodic components of the monodromy in terms of cyclic branched coverings in [14, Lemma 11.4].

(iii) The graph \( \Gamma[f] \) contains more information than the characteristic set \( t_{(f)}. \) It also shows how the \( \phi \)-components are pieced together to give the Milnor fiber.
The second main ingredient for our proof is the following result of [1] and [31]. We would like to point out that this result holds in much greater generality than we use it here, namely for holomorphic hypersurface singularities, isolated or non-isolated, in any dimension.

**Theorem 2.24 (A’Campo, Lê).** Let $g$ be the geometric monodromy of an isolated plane curve singularity. Then

$$
\Lambda(g) = 0,
$$

where $\Lambda$ denotes the Lefschetz number.

**Proof of Theorem 2.17.** Let $[f]$ be an isolated plane curve singularity. Let $M$ denote the Milnor fiber and $g$ the geometric monodromy of $[f]$. By Theorem 2.23, there exists a representative $\phi \in \pi_1 M$ and a finite $\phi$-invariant union of circles $C \subset M$, such that if $M'$ is a $\phi$-component of $M \setminus C$, then either $M'$ has negative Euler characteristic and $\phi|M'$ is periodic, or $M'$ is a union of annuli and $\phi|M'$ is an admissible twist map. If $(\chi, d, h)$ denotes the topological type of $M'$ and $\ell$ the order/twist number of $\phi|M'$, then $(d, \chi, h; \ell) \in t_{\Gamma}[f]$. Moreover, if $M'$ is a union of annuli, then (20) is a model for $\phi|M'$.

The strategy of the proof is as follows. We will prove below that

**Claim 1.** If $\chi = 0$, then $\text{Fix}(\phi) \cap \text{int}(M') = \emptyset$.

**Claim 2.** If $\chi < 0$, then $\ell > 1$.

**Claim 3.** $\phi$ only has positive twists.

Claim 1 implies that $\phi$ is a diffeomorphism of finite type. From Proposition 2.5 about the fixed point classes of a diffeomorphism of finite type and claim 2, it follows that $\text{Fix}(\phi) \cap \text{int}(M)$ is a discrete set of fixed points with fixed point index 1. The Lefschetz fixed point theorem therefore implies that

$$
\Lambda(\phi) = \# \left( \text{Fix}(\phi) \cap \text{int}(M) \right).
$$

Note that $\partial M$ has fixed point index 0. From Theorem 2.24, it hence follows that $\text{Fix}(\phi) \cap \text{int}(M) = \emptyset$. Together with claim 3, this proves Proposition 2.17, up to claim 1, 2 and 3.

Note that if $\Gamma[f] = \Gamma_\ast$, then $M$ is an annulus and $\phi$ is a positive Dehn twist. Claim 1, 2 and 3 obviously hold in this case and we assume from now on that $\Gamma[f] \neq \Gamma_\ast$. We begin by proving claim 2. Recall that $t_{\Gamma}[f] = \{ t_x : x \in B \cup E \}$ and that if $\chi < 0$, there exists $b \in B$ such that $(\chi, d, h; \ell) = (\chi_b, d_b, h_b; \ell_b)$. Consider equation (17) and assume that $\ell_b = 1$. In this case, only one summand of $\ell_b$ is non-zero, which, by (B3), is only possible if $k \leq 2$. Since $n \geq 3$, however, it follows from (A6) that $a_n > 1$ and hence that $1 = \ell_b \geq a_n > 1$, a contradiction.

To proof claim 1, let $e \in E$ be such that $(\chi, d, h; \ell) = (0, d_e, 2d_e; \ell_e)$. We can assume that $d_e = 1$, otherwise the claim is obviously true. Hence, $M'$ is
an annulus and $\phi|\Sigma = \phi_c$. Consider equation (20) and assume that $\phi_c(q,p) = (q,p)$ for some $(q,p) \in (0,1) \times S^1$. This is only possible if

$$-q \cdot \ell_e - \frac{1}{m}(n - \frac{a}{\ell_b}) \in \mathbb{Z},$$

where we use the same notation as in (20). This in turn implies that

$$a - qm\ell_b \ell_e \in \ell_b \mathbb{Z}. (21)$$

Assume for the moment that $0 < m \ell_b \ell_e \leq a$. Since $0 < q < 1$, it follows from (21) that $0 < \ell_b < a$. To show that this is a contradiction, recall from (18) that $\ell_b = m a_2 \cdots a_r + m'a$, where $m' := m(b', b)$. If $m' \neq 0$, it follows that $\ell_b \geq a$ and hence $m' = 0$. By (B2) however, this implies that $a = 1 > \ell_b$, which is a contradiction.

It remains to prove that $0 < m \ell_b \ell_e \leq a$. First note, that $m \ell_b \ell_e > 0$ if $\ell_e > 0$. If $b' \in \mathcal{A}$, then $\ell_b > 0$ by definition. If $b \in \mathcal{B}$, then (A5) is exactly the statement that $\ell_e > 0$. This in fact proves claim 3. To prove that $m \ell_b \ell_e \leq a$, first assume that $b' \in \mathcal{A}$. Then $m = 1, a = 1, \ell_b \ell_e = 1$ and we are finished. If $b' \in \mathcal{B}$, then it follows from (18) and (19) that

$$\ell_{b'} \geq m a', \quad \Delta_e \leq a a',$$

where $a'$ denotes the weight of $e$ at $b'$. This implies that

$$\ell_e \leq \frac{a}{\ell_b m},$$

which proves the required inequality. This ends the proof of the proposition. □

### 2.8 Appendix C: Floer homology on surfaces with boundary

Let $M$ be a compact connected oriented 2-manifold with boundary $\partial M \neq \emptyset$. Recall that $\text{Diff}_+^c(M)$ denotes the group of diffeomorphisms which preserve orientation and equal the identity near $\partial M$. This appendix addresses Floer homology theory for elements of $\text{Diff}_+^c(M)$. In higher dimensions, this is known as Floer homology theory for exact symplectomorphisms of exact symplectic manifolds with contact type boundary and was used in [48, Section 4], see also [6]. Due to the dimensional restriction, the theory exhibits auxiliary structure, namely isotopy invariance, as in the closed case. The central notion around this issue is that of monotonicity. We start by defining monotonicity and show that it has naturality, isotopy and inclusion properties similar to the ones discussed in Section 2.1 for the closed case.

Let $\omega$ be an area form on $M$ and denote by $\text{Symp}(M, \omega)$ the group of diffeomorphisms which preserve $\omega$ and equal the identity near $\partial M$. If $\phi \in \text{Symp}(M, \omega)$, $\omega$ induces a closed 2-form $\omega_\phi$ on the mapping torus $T_\phi$. 

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Definition 2.25. $\phi \in \text{Symp}_c(M, \omega)$ is called monotone, if

$$[\omega_\phi] = 0$$

in $H^2(T_\phi; \mathbb{R})$; $\text{Symp}_c^m(\Sigma, \omega)$ denotes the set of monotone symplectomorphisms.

As in the closed case it is useful to look at the short exact sequence

$$0 \to H^1(M; \mathbb{R}) \to H^2(T_\phi; \mathbb{R}) \to H^2(M; \mathbb{R}) = 0 \to 0$$

and define the class $m(\phi) \in H^1(M; \mathbb{R})/\text{im}(\text{id} - \phi^*)$ satisfying

$$\delta m(\phi) = [\omega_\phi].$$

Naturality, isotopy and inclusion properties discussed on page 12 in the closed case, carry over word by word to the current situation with the addition of a subscript $c$ to all diffeomorphism groups. For the first two properties this is straightforward to check. The inclusion property needs separate consideration.

Recall:

(Inclusion) The inclusion $\text{Symp}_c^m(M, \omega) \hookrightarrow \text{Diff}_c^+(M)$ is a homotopy equivalence. In particular, every connected component of $\text{Symp}_c^m(\Sigma, \omega)$ is contractible.

The proof is analogous to the closed case and uses the following three facts.

Firstly, the inclusion $\text{Symp}_c^m(M, \omega) \hookrightarrow \text{Diff}_c^+(M)$ is a homotopy equivalence. This follows from an extension of Moser’s theorem, see [35, Exercise 3.18]. Secondly, every connected component of $\text{Diff}_c^+(M)$ is contractible. If the genus of $M$ is $6 \neq 0$, this is shown using the Earl-Eells Theorem [13]. If $g = 0$, it follows from the corresponding result for the disk, which is due to Smale [52, Theorem B]. Thirdly, we have

Lemma 2.26. If $\phi \in \text{Symp}_c(M, \omega)$, there exists a closed 1-form $\theta \in m(\phi)$ such that $\text{supp}(\theta) \subset \text{int}(M)$. The flow $\psi_t \in \text{Symp}_c(M, \omega)$ and $m(\phi \circ \psi_t) = 0$.

Proof. The second part of the statement follows immediately from the first one and the isotopy property:

$$m(\phi \circ \psi_1) = m(\phi) + [\text{Flux}(\psi_1)]$$

in $H^1(M; \mathbb{R})/\text{im}(\text{id} - \phi^*)$. To prove the first statement, let $\beta \in m(\phi)$ be a closed 1-form. Let $S_1, \ldots, S_n$ denote the connected components of $\partial M$ and choose for each $i$ a collar neighborhood $N_i \subset M$ of $S_i$ and a closed 1-form $\theta_i$ on $N_i$ such that $[\theta_i], [S_i] = 1$. There exist $f : M \to \mathbb{R}$ smooth and $t_1, \ldots, t_n \in \mathbb{R}$ such that

$$(\beta + df)|_{N_i} = t_i \cdot \theta_i, \quad \forall i = 1, \ldots, n. \quad (22)$$

We claim that $\theta := \beta + df$ is the required 1-form. Recall from page 12, that $\delta : H^1(M; \mathbb{R}) \to H^2(T_\phi; \mathbb{R})$ is given by $\delta[\theta] = [\rho \cdot \theta \wedge dt]$, with $\rho : [0, 1] \to \mathbb{R}$ a
smooth function vanishing near 0 and 1, and satisfying \( \int_0^1 \rho \, dt = 1 \). Furthermore note, that \( S^1 \times S_i \subset T_\phi \) is an embedded 2-torus for each \( i = 1, \ldots, n \). Using (22), it follows that

\[
\langle [\omega_\phi], [S^1 \times S_i] \rangle = \langle [\alpha], [S^1 \times S_i] \rangle = -\langle [\rho \cdot dt], [S^1] \rangle \cdot \langle [\alpha], [S_i] \rangle = -t_i \cdot \langle [\theta_i], [S_i] \rangle = -t_i.
\]

On the other hand,

\[
\langle [\omega_\phi], [S^1 \times S_i] \rangle = 0,
\]

since \( \omega_\phi \) has no \( dt \)-component. Hence, \( t_i = 0 \) for all \( i = 1, \ldots, n \), which proves the claim.

Recall that in the closed case monotonicity (i) guarantees compactness of the space of Floer connecting orbits and (ii) is used to prove invariance. The same holds in the current situation.

(Floer homology) To every \( \phi \in \text{Symp}_c^m(\Sigma, \omega) \) symplectic Floer homology theory assigns a pair of \( \mathbb{Z}_2 \)-graded vector spaces \( HF_\ast(\phi, \pm) \) over \( \mathbb{Z}_2 \), with multiplicative structures \( HF_\ast(M; \mathbb{Z}_2) \otimes HF_\ast(\phi, \pm) \longrightarrow HF_\ast(\phi, \pm) \).

(Invariance) If \( \phi, \phi' \in \text{Symp}_c^m(\Sigma, \omega) \) are isotopic, then \( HF_\ast(\phi, \pm) \) and \( HF_\ast(\phi', \pm) \) are naturally isomorphic as modules over \( H^\ast(M; \mathbb{Z}_2) \).

The appearance of the sign in the Floer homology corresponds to two ways of perturbing \( \phi \in \text{Symp}_c^m(\Sigma, \omega) \) near \( \partial M \). To be more precise, let \( j : \bigcup\{[-\varepsilon, 0] \times S^1 \} \to M \) be a collar neighborhood of \( \partial M \), such that \( j^*\omega = dq \wedge dp \) with \((q,p) \in (-\varepsilon,0] \times S^1 \). Choose \( H : M \to \mathbb{R} \) with support near \( \partial M \) and such that \( j^*H(q,p) = -q \). Let \( (\psi_t)^{t \in \mathbb{R}} \) denote the Hamiltonian flow generated by \( H \), choose \( 0 < \delta < 1 \) and set

\[
\phi_+ := \phi \circ \psi_\delta, \quad \phi_- := \phi \circ \psi_{-\delta}.
\]

The definition of the Floer complex for \( \phi_\pm \) is along the same line as that in the closed case [51], with the usual modifications that are needed in the presence of a contact type boundary.

The modifications include a condition on the path \( J = (J_t)^{t \in \mathbb{R}} \) of \( \omega \)-compatible complex structures that is used to define the Floer connecting orbits; namely that \( j^*J_t \) is the standard complex structure on \( \bigcup\{[-\varepsilon, 0] \times S^1 \} \), for all \( t \in \mathbb{R} \). We briefly recall the use of this condition. Assume without loss of generality that \( \phi| \operatorname{im} j = \text{id} \). Now let \( u : \mathbb{R}^2 \to M \) be a smooth map satisfying

\[
\left\{ \begin{array}{l}
  u(s,t) = \phi_+(u(s,t+1)), \\
  \partial_s u + J_t(u) \partial_t u = 0, \\
  \lim_{s \to \pm \infty} u(s,t) \in \text{Fix}(\phi_+). 
\end{array} \right.
\] (23)
We claim that $\text{im } u \subset M \setminus \text{im } j$. Assume by contradiction that $u^{-1}(\text{im } j)$ is non-empty and let $u_q : u^{-1}(\text{im } j) \to \mathbb{R}$ denote the $q$-component of $j^{-1} \circ u$. By construction, $u_q$ is smooth and not locally constant. Using the first and third equation in (23), one can now show that $u_q$ has a global maximum. From the second equation in (23) together with the above assumption on $J_t$, it furthermore follows that $u_q$ is a harmonic function. This contradicts the maximum principle and hence proves the claim, which assures that the Floer connecting orbits are contained in a compact subset of $\text{int}(M)$.

We close this section by remarking that $HF_+^*(\phi, +), HF_-^*(\phi, -)$ are independent of the choices of the local chart $j$ and perturbation data $H, \delta$. They are invariants of the isotopy class of $\phi$ in $\text{Diff}_c^+(M)$. 

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3 Freiheitssatz for surface groups

In this chapter we prove two versions of the Freiheitssatz for fundamental groups of closed two-manifolds and discuss some applications. The organization is as follows. In Section 3.1 we begin with a short introduction to the Freiheitssatz and then state our results. In 3.2 we recall the relevant background material on fundamental groups of compact two-manifolds. Section 3.3 is devoted to the proofs of the results stated in 3.1.

Chapter 3 is independent of Chapter 2. However, Corollary 3.4 is used in the proof of Proposition 2.20.

3.1 Statement of the results

The Freiheitssatz is a classical result in combinatorial group theory and was proven by W. Magnus in [33]. We introduce the basic terminology.

In the following, $G$ always denotes a finitely generated group. The rank of $G$ is the cardinality of a minimal set of generators of $G$. Now let $\Gamma \subseteq G$ be a finite subset. We denote by $\Gamma \mathbb{G}$ the free group on the set $\Gamma$ and by $f_\Gamma : \Gamma \mathbb{G} \to G$ the homomorphism which sends a word in the elements of $\Gamma$ and its formal inverses to the corresponding product in $G$. Let $W \subseteq \Gamma \mathbb{G}$ be a finite set of cyclically reduced elements and denote by $N_W \subseteq \Gamma \mathbb{G}$ the smallest normal subgroup containing $W$. The pair $\langle \Gamma, W \rangle$ is called a presentation of $G$, we write $G = \langle \Gamma, W \rangle$, if

$$\text{im } f_\Gamma = G \quad \text{and} \quad \text{ker } f_\Gamma = N_W.$$ 

A presentation $\langle \Gamma, W \rangle$ is called a one-relator presentation, if

$$\# W = 1.$$ 

A group $G$ is called a one-relator group, if there exists a one-relator presentation of $G$.

The statement of the Freiheitssatz is the following. If $G = \langle \Gamma, W \rangle$ is a one-relator presentation and $\Gamma' \subseteq \Gamma$ omits a generator appearing in $W$, then $\Gamma'$ generates a free subgroup of rank $\# \Gamma'$ of $G$.

The classical examples of one-relator groups are surface groups, i.e. fundamental groups of closed surfaces of positive genus. By a surface, we always mean a connected smooth 2-manifold with possible non-empty boundary and by a closed surface a compact surface without boundary. The following theorem is closely related to the Freiheitssatz for surface groups. However, its statement and proof do not involve any presentations of groups. We explain below how the Freiheitssatz for minimal presentations, Theorem 3.6, follows from Theorem 3.1.
Theorem 3.1. Let \( \Sigma \) be a closed surface and \( G \subset \pi_1(\Sigma) \) be a subgroup with 
\[
\text{rank } G < \text{rank } \pi_1(\Sigma).
\]
Then \( G \) is a free group.

It is well known that the fundamental group of a closed surface of positive genus is not a free group. The converse, that \( \Sigma \) is necessarily closed if \( \pi_1(\Sigma) \) is (finitely generated and) not free, is the key step in the proof of Theorem 3.1. Its proof relies on the classification of compact surfaces and a lemma by D. Epstein [15] which reduces the non-compact case to the compact one. We summarize this in

Lemma 3.2. Let \( \Sigma \) be a surface, not diffeomorphic to \( S^2 \), such that \( \pi_1(\Sigma) \) is finitely generated. Then \( \Sigma \) is closed if and only if \( \pi_1(\Sigma) \) is not free.

We remark that this lemma also holds without the finiteness assumption; see [3, Page 102]. However, this is not used here.

Theorem 3.1 is useful for the study of possible relations that can occur between a (small) number of elements of \( \pi_1(\Sigma) \). The first corollary settles the case of two loops in an orientable surface.

Corollary 3.3. Let \( \Sigma \) be a closed orientable surface of genus \( > 1 \). If \( \alpha, \beta \in \pi_1(\Sigma) \) satisfy
\[
\alpha^{m_1} \beta^{n_1} \alpha^{m_2} \beta^{n_2} \cdots \alpha^{m_\ell} \beta^{n_\ell} = 1,
\]
where \( m_1, n_1, \ldots, m_\ell, n_\ell \) are non-zero integers, then there exists a class \( \sigma \in \pi_1(\Sigma) \) and integers \( q \) and \( r \) such that
\[
\alpha = \sigma^q, \quad \beta = \sigma^r.
\]
If in addition, \( \alpha \) is assumed to be represented by a non-contractible embedded circle, then \( \sigma = \alpha \) satisfies (2) with \( q = 1 \).

The proof of the additional assertion uses a characterisation result for the cylinder which is proven in Appendix D, that is Section 3.4.

An example of a relation of the form (1) is provided by a continuous map from the torus to \( \Sigma \). Corollary 3.3 has the following consequence for the homotopy class of such a map. Note that \( S^1 \) is identified with \( \mathbb{R}/\mathbb{Z} \).

Corollary 3.4. Let \( \Sigma \) be a closed orientable surface of genus \( > 1 \) and \( f : S^1 \times S^1 \to \Sigma \) a continuous map. Assume that the loop \( f(\cdot, 0) \) is non-contractible and contained in an embedded circle \( C \subset \Sigma \). Then, there exists a continuous map \( f' : S^1 \times S^1 \to C \) which is homotopic to \( f \) relative to \( (0, 0) \in S^1 \times S^1 \).

We remark that a similar assertion was used, without proof, in a preliminary version of [47].

Another application of Theorem 3.1 is the proof of a version of the Freiheitssatz for what we call minimal presentations of surface groups.
Definition 3.5. Let $\Sigma$ be a closed surface of genus $g > 0$. A presentation $(\Gamma, W)$ of $\pi_1(\Sigma)$ is called **minimal** if
\[
\# \Gamma = \begin{cases} 
2g, & \text{if } \Sigma \text{ is orientable} \\
g, & \text{if } \Sigma \text{ is non-orientable.}
\end{cases}
\]

Theorem 3.6. Let $\Sigma$ be a closed surface of genus $g > 0$ and $(\Gamma, W)$ be a minimal presentation of $\pi_1(\Sigma)$. If $\Gamma' \subseteq \Gamma$, then $\Gamma'$ generates a free subgroup of rank $\# \Gamma'$ of $\pi_1(\Sigma)$.

Since the rank of $\pi_1(\Sigma)$ is $\# \Gamma$, see the next section, Theorem 3.1 implies that the subgroup generated by $\Gamma'$ is free. The additional assertion about the rank follows from a simple algebraic argument, Lemma 3.9. Note that $(\Gamma, W)$ is not assumed to be a one-relator presentation. We do not know, however, if every minimal presentation is in fact a one-relator presentation.

3.2 Background on surface groups

The material presented in this section can be found in most textbooks on algebraic topology, e.g. see [34].

We first recall the classification theorem of compact surfaces. Assume that $\Sigma$ is a closed surface. Then it is diffeomorphic either to the 2-sphere, to the $g$-fold connected sum of tori or to the $g$-fold connected sum of projective planes, for some $g > 0$. If $\Sigma$ is compact and $k > 0$ is the number of boundary components, then $\Sigma$ is diffeomorphic to one of the above with $k$ disjoint open disks removed. The integer $g$ (which is set to 0 if $\Sigma$ is a sphere) is called the genus of $\Sigma$ and is related to the Euler characteristic by the formula
\[
\chi = 2 - 2g - k,
\]
in the sphere/connected sum of tori case (the orientable case), and by
\[
\chi = 2 - g - k,
\]
in the connected sum of projective plane case (the non-orientable case). It is convenient to have explicit models for the manifolds listed above. Let $g > 0$ be the genus of $\Sigma$ and $k \geq 0$ its number of boundary components. Following the standard theory, $\Sigma$ is diffeomorphic to the quotient manifold of either a plane $4g$-gon $O_{g,k}$ with $k$ disjoint open disks removed or a plane $2g$-gon $N_{g,k}$ with $k$ disjoint open disks removed. The identifications for defining the quotients are indicated in Figures 1 and 2. We write
\[
\Sigma \cong O_{g,k}/ \sim \quad \text{respectively} \quad \Sigma \cong N_{g,k}/ \sim .
\]
If $k > 0$, $O_{g,k}/\sim$ can be deformed to a wedge of $2g + k - 1$ circles. Hence, $\pi_1(O_{g,k}/\sim)$ is a free group on $2g + k - 1$ generators. Similarly, $N_{g,k}/\sim$ can be deformed to a wedge of $g + k - 1$ circles and hence, $\pi_1(N_{g,k}/\sim)$ is a free group on $g + k - 1$ generators.

In the closed case, i.e. $k = 0$, the fundamental groups are not free: the disks $\text{int}(O_{g,0})$ and $\text{int}(N_{g,0})$ give rise to nontrivial relations. More precisely, it follows from the Seifert-van Kampen theorem that the fundamental groups are given by

$$
O_g = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] \rangle \quad \text{(in the orientable case),} \tag{5}
$$

$$
N_g = \langle \gamma_1, \ldots, \gamma_g \mid \prod_{i=1}^{g} \gamma_i^2 \rangle \quad \text{(in the non-orientable case),} \tag{6}
$$

where the generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are represented by $a_1, b_1, \ldots, a_g, b_g$ and the generators $\gamma_1, \ldots, \gamma_g$ by $c_1, \ldots, c_g$. The presentations (5) and (6) are minimal presentations, we call them the standard presentations.

**Proposition 3.7.** The groups $O_g$ and $N_g$ are not free, for any $g > 0$. The ranks are given by

$$
\text{rank } O_g = 2g \quad \text{and} \quad \text{rank } N_g = g. \tag{7}
$$

**Proof.** To prove (7) it is enough to prove that

$$
\text{rank } O_g \geq 2g \quad \text{and} \quad \text{rank } N_g \geq g.
$$

For $O_g$, this is seen by looking at its abelianization

$$
H(O_g) := O_g/[O_g, O_g].
$$

It follows from (5) that $H(O_g) \cong \mathbb{Z}^{2g}$, see the remark below, and hence, that

$$
\text{rank } O_g \geq \text{rank } H(O_g) = 2g.
$$

For $N_g$, consider the tensor product

$$
H(N_g; \mathbb{Z}_2) := H(N_g) \otimes \mathbb{Z}_2.
$$
It follows from (6) that \( H(N_g; \mathbb{Z}_2) \cong \mathbb{Z}_2^g \) and hence that
\[
\text{rank} \ N_g \geq \dim H(N_g; \mathbb{Z}_2) = g.
\]
To prove the first part of the proposition, assume that \( O_g \) is free. By (7), \( O_g \) is a free group on \( 2g \) generators. But it is also the non-trivial quotient of a free group on \( 2g \) generators, by (5). This is a contradiction by the remark below. The argument for \( N_g \) is analogous. \( \square \)

Remarks. (i) A group is called Hopfian, if it is not isomorphic to any of its proper factor groups. Every finitely generated free group is Hopfian; for an elementary proof see [41, Theorem 6.1.12]. As a consequence, a free group on \( n \) generators has rank \( n \).

(ii) In the proof of Proposition 3.7 we used at several points that the abelianization of a free group on \( n \) generators is isomorphic to \( \mathbb{Z}^n \).

3.3 Proofs

We begin with the proof of Lemma 3.2. For this, we use a result of Epstein, see [15, Lemma 2.2].

Lemma 3.8. Let \( \Sigma \) be a surface such that \( \pi_1(\Sigma) \) is finitely generated. There is a compact subsurface \( \Pi \subset \Sigma \) such that the embedding induces an isomorphism of fundamental groups, \( \pi_1(\Pi) \cong \pi_1(\Sigma) \).

Proof of Lemma 3.2. The “only if” part is the first part of Proposition 3.7. Now assume that \( \Sigma \) is not closed. Let \( \Pi \subset \Sigma \) be a compact subsurface such that \( \pi_1(\Pi) \cong \pi_1(\Sigma) \). Since \( \Sigma \) is connected, \( \partial \Pi \neq \emptyset \) and hence, by our discussion in the last section, \( \pi_1(\Pi) \) is free. This proves the lemma. \( \square \)

For the proof of Theorem 3.1, we use the following fundamental fact from covering space theory. Let \( \Sigma \) be a surface and \( G \subset \pi_1(\Sigma) \) be any subgroup. There exists a surface \( \bar{\Sigma} \) with \( \pi(\bar{\Sigma}) \cong G \) and a covering map \( p : \bar{\Sigma} \to \Sigma \). Moreover, \( p \) is a \( k \)-fold covering, where \( k \) is the cardinality of the set \( \pi_1(\Sigma)/G \). Note that this also holds if \( \pi_1(\Sigma)/G \) is an infinite set. We call \( p \) a covering associated to the subgroup \( G \).

Proof of Theorem 3.1. The assertion of the theorem is obvious if \( \Sigma \) is the 2-sphere or the projective plane. We assume from now on that \( \text{rank} \pi_1(\Sigma) \geq 2 \). The proof of the theorem relies on the formula
\[
\chi(\Sigma) = 2 - \text{rank} \pi_1(\Sigma), \quad (8)
\]
which holds for every closed surface \( \Sigma \), orientable or non-orientable, and follows from (3), (4) and (7). Let \( G \subset \pi_1(\Sigma) \) be a finitely generated subgroup which is
not free and let \( p : \tilde{\Sigma} \to \Sigma \) be a covering associated to \( G \). By Lemma 3.2, \( \tilde{\Sigma} \) is closed, and hence, \( p \) is a \( k \)-fold covering with \( k < \infty \). Using the combinatorial definition of the Euler characteristic one can prove that \( \chi(\tilde{\Sigma}) = k\chi(\Sigma) \). Together with (8) we therefore get that

\[
\text{rank } G = \text{rank } \pi_1(\Sigma) + (k - 1)(\text{rank } \pi_1(\Sigma) - 2). 
\]

Since \( k \geq 1 \) and \( \text{rank } \pi_1(\Sigma) \geq 2 \), it follows that \( \text{rank } G \geq \text{rank } \pi_1(\Sigma) \), which proves the theorem.

**Remarks.** (i) A further look at (9) shows: If \( \text{rank } G = \text{rank } \pi_1(\Sigma) \neq 2 \), then \( k = 1 \), i.e. \( p \) is a diffeomorphism and \( G = \pi_1(\Sigma) \). In particular, it follows that any covering map \( \Sigma \to \tilde{\Sigma} \) of a closed surface \( \Sigma \) with rank \( \pi_1(\Sigma) \neq 2 \) is in fact a diffeomorphism.

(ii) As a special case of Theorem 3.1 we obtain the following well known result. Let \( \Sigma \) be a closed surface with rank \( \pi_1(\Sigma) > 1 \), i.e. \( \Sigma \) is not diffeomorphic to the 2-sphere or the projective plane. If \( \alpha \in \pi_1(\Sigma) \) is a non-trivial class and \( k \neq 0 \), then \( \alpha^k \) is non-trivial.

(iii) Let \( \Sigma \) be as in (ii) and \( C \subset \Sigma \) be an embedded non-contractible circle. Using the relative homotopy exact sequence of the pair \((\Sigma, C)\) and the fact that \( \pi_2(\Sigma) = 0 \), remark (ii) implies that \( \pi_2(\Sigma, C) = 0 \).

**Proof of Corollary 3.3.** Let \( \alpha, \beta \in \pi_1(\Sigma) \) and \( m_1, n_1, \ldots, m_\ell, n_\ell \) be non-zero integers such that

\[
a^{m_1} \beta^{n_1} a^{m_2} \beta^{n_2} \cdots a^{m_\ell} \beta^{n_\ell} = 1. 
\]

We assume without loss of generality, that \( \alpha \neq 1 \). Denote by

\[
\text{Sub}(\alpha, \beta) \subset \pi_1(\Sigma)
\]

the subgroup generated by \( \alpha \) and \( \beta \). Then

\[
0 < \text{rank } \text{Sub}(\alpha, \beta) \leq 2 < \text{rank } \pi_1(\Sigma)
\]

and it follows from Theorem 3.1, that \( \text{Sub}(\alpha, \beta) \) is a free group of rank 1 or 2. However, the left hand side of (10) is a non-trivial element of the group freely generated by \( \alpha \) and \( \beta \) and thus \( \text{Sub}(\alpha, \beta) \) is not free of rank 2. It is therefore infinite cyclic, which proves the first part of the corollary.

Let \( \tilde{\Sigma} \) be a covering associated to \( \text{Sub}(\alpha, \beta) \); \( \tilde{\Sigma} \) is orientable and its fundamental group is infinite cyclic. Hence, by Proposition 3.10 in Appendix D, \( \tilde{\Sigma} \cong \mathbb{R} \times S^1 \).

Now let \( \gamma : S^1 \to \tilde{\Sigma} \) be the lift of an embedded loop representing \( \alpha \). Note, that \( \gamma \) is embedded and non-contractible. We claim that the class of \( \gamma \) is a generator of \( \pi_1(\tilde{\Sigma}) \), which proves the second statement of the corollary.

To prove the claim, let \( \gamma' : S^1 \to \tilde{\Sigma} \cong \mathbb{R} \times S^1 \) denote the standard loop \( \gamma'(s) = (0, s) \), which represents a generator of \( \pi_1(\tilde{\Sigma}) \). Assume without loss of generality that \( \gamma \) and \( \gamma' \) are disjoint and note that the connected component of \( \tilde{\Sigma} \setminus (\text{im } \gamma \cup \text{im } \gamma') \) which is bounded by \( \gamma \) and \( \gamma' \) is diffeomorphic to an annulus. This finishes the proof. \( \square \)
Proof of Corollary 3.4. Let \( f : [0,1]^2 \to \Sigma \) be a continuous map with
\[
f(s,0) = f(s,1) \quad \text{and} \quad f(0,t) = f(1,t)
\]
for all \( s, t \in [0,1] \). Denote by \( \alpha \) respectively \( \beta \) the class of the loop \( f(\cdot,0) \) respectively \( f(0,\cdot) \). By assumption, \( \alpha \) is represented by a multiple of a non-contractible embedding \( S^1 \to C \). Moreover, \( \alpha \) and \( \beta \) satisfy
\[
\alpha \beta \alpha^{-1} \beta^{-1} = 1.
\]
Hence by Corollary 3.3, \( \alpha \) and \( \beta \) are both represented by loops in \( C \). Without loss of generality, we can therefore assume that \( f(s,t) \in C \) for all \( (s,t) \in \partial[0,1]^2 \).

Since \( C \) is non-contractible, \( \pi_2(\Sigma, C) = 0 \) by remark (iii). Thus, there exists a homotopy \( f_r : [0,1]^2 \to \Sigma, r \in [0,1] \), such that \( f_0 = f \) and \( \text{im}(f_1) \subset C \). Moreover, \( f_t \) can be chosen such that \( f_t(s,t) = f(s,t) \) for all \( (s,t) \in \partial[0,1]^2 \) and \( f_r(0,0) = f(0,0) \) for all \( r \in [0,1] \). This means that \( f_r \) descends to a homotopy of maps on the torus relative to \( (0,0) \).

To prove Theorem 3.6, we use an algebraic lemma. If \( G \) is a group, we denote by \( h : G \to H(G) \) the natural projection, where \( H(G) \) is the abelianization of \( G \).

**Lemma 3.9.** Let \( N \subset G \) be a normal subgroup. There exists a natural isomorphism
\[
H(G/N) \cong H(G)/h_G(N).
\]

**Proof.** Let \( n : G \to G/N \) and \( \bar{n} : H(G) \to H(G)/h_G(N) \) denote the natural projections. There is a unique homomorphism
\[
\phi : G/N \to H(G)/h_G(N),
\]
such that \( \phi \circ n = \bar{n} \circ h_G \). The kernel of \( \phi \) is given by
\[
\ker \phi = n(h_G^{-1}(h_G(N))) = n(N \cdot [G,G]) = n([G,G]) = [G/N,G/N].
\]
Hence, \( \phi \) induces an isomorphism \( H(G/N) \cong H(G)/h_G(N) \).

**Proof of Theorem 3.6.** Let \( \langle \Gamma, W \rangle \) be a minimal presentation of \( \pi_1(\Sigma) \) and \( \Gamma' \subset \Gamma \). Let \( G \subset \pi_1(\Sigma) \) be the subgroup generated by \( \Gamma' \). Since
\[
\text{rank} \ G \leq \# \Gamma' < \text{rank} \ \pi_1(\Sigma),
\]
it follows from Theorem 3.1, that \( G \) is a free group. It remains to show that
\[
\text{rank} \ G = \# \Gamma'. \tag{11}
\]
Assume that $\Sigma$ is orientable. We prove in three steps that
\[
\text{rank } H(G) = \# \Gamma',
\] (12)
which is equivalent to (11), since we know that $G$ is free. Recall, that $F\Gamma$ respectively $F\Gamma'$ denotes the free group on the set $\Gamma$ respectively $\Gamma'$, and $N_W \subset F\Gamma$ denotes the smallest normal subgroup containing $W$.

**Claim 1.** $h_{F\Gamma}(N_W) = 0$.

To prove this, apply Lemma 3.9 to conclude that
\[
H(\pi_1(\Sigma)) \cong H(F\Gamma'/N_W) \cong H(F\Gamma)/h_{F\Gamma}(N_W).
\]
But $H(\pi_1(\Sigma))$ and $H(F\Gamma)$ are both free abelian of rank $\# \Gamma$. Hence $h_{F\Gamma}(N_W) = 0$, since every finitely generated free abelian group is Hopfian. This proves claim 1.

**Claim 2.** $h_{F\Gamma'}(N_W') = 0$, where $N_W' := N_W \cap F\Gamma'$.

By claim 1, we know: if $x \in N_W$ is expressed as a word in the elements of $\Gamma$ and its formal inverses, then the total number of times a particular element of $\Gamma$ is contained in that word (counted with sign) is zero. Note that $N_W'$ is the subgroup of those elements of $N_W$ which can be expressed in terms of elements of $\Gamma'$ only. From this, claim 2 obviously follows.

**Claim 3.** $\text{rank } H(G) = \# \Gamma'$

Note the $G \cong F\Gamma'/N_W'$. Apply again Lemma 3.9 and use claim 2:
\[
H(G) \cong H(F\Gamma')/h_{F\Gamma'}(N_W') \cong H(F\Gamma').
\]
This proves claim 3 and hence the theorem in the orientable case.

The proof of (11) in the case where $\Sigma$ is non-orientable is word by word the same as in the orientable case, if one replaces $\mathbb{Z}$-coefficients by $\mathbb{Z}_2$-coefficients.

This means that $H(G)$ is replaced by $H(G; \mathbb{Z}_2)$ and (12) by
\[
\dim H(G; \mathbb{Z}_2) = \# \Gamma'.
\]

Lemma 3.9 carries over to $\mathbb{Z}_2$-coefficients, since
\[
(A/B) \otimes \mathbb{Z}_2 \cong (A \otimes \mathbb{Z}_2)/(B \otimes \mathbb{Z}_2)
\]
for any pair $B \subset A$ of Abelian groups.

### 3.4 Appendix D: Characterization of the cylinder

By an open surface we mean a non-compact surface without boundary. The following proposition is a consequence of the classification of compact surfaces and the classification of open simply connected surfaces [15, Corollary 1.8]. The latter simply states that any such surface is diffeomorphic to $\mathbb{R}^2$. Besides that, we use Epstein’s Lemma 3.8.
Proposition 3.10. Let $\Sigma$ be an open orientable surface with infinite cyclic fundamental group. Then $\Sigma$ is diffeomorphic to $\mathbb{R} \times S^1$.

Proof. By Lemma 3.8, there exists a compact oriented subsurface $\Pi \subset \Sigma$ such that the inclusion induces and isomorphism $\pi_1(\Pi) \cong \pi_1(\Sigma)$. By the classification of compact surfaces, see Section 3.2, it follows that $\Pi$ is diffeomorphic to the 2-sphere with two open disks removed.

Claim 1. $\Sigma \setminus \Pi$ has two connected components.

It is clear that $\Sigma \setminus \Pi$ has either one or two connected components. Assume that it is connected. Let $\gamma : S^1 \to \Pi$ be the parameterization of a boundary component of $\Pi$. Then there exists an embedding $\delta : S^1 \to \Sigma$ which intersects the curve $\gamma$ transversally and exactly once. Let $\mathbb{R}^2$ be the universal cover of $\Sigma$ and $\tilde{\gamma}, \tilde{\delta} : \mathbb{R} \to \mathbb{R}^2$ be lifts of $\gamma, \delta$ respectively, chosen such that

$$\tilde{\gamma}(0) = \tilde{\delta}(0) = (0, 0).$$

Since $\delta$ is homotopic to a multiple of $\gamma$, there exists $t > 0$ such that $\tilde{\gamma}(t) = \tilde{\delta}(t)$. Choose the minimal $t > 0$ with this property. Since $\gamma$ and $\delta$ intersect only once, $\tilde{\gamma}(0)$ and $\tilde{\gamma}(t)$ project to the same point. On the other hand, $\tilde{\gamma}$ and $\tilde{\delta}$ intersect with different signs at $(0, 0)$ and $\tilde{\gamma}(t)$. This is a contradiction and hence proves the claim. From now on, we denote by $M$ the closure of a connected component of $\Sigma \setminus \Pi$.

Claim 2. $\pi_2(\Sigma) = 0$.

The universal cover of $\Sigma$ is diffeomorphic to $\mathbb{R}^2$.

Claim 3. $\pi_2(\Sigma, \partial M) = 0$.

To prove this, we use the following part of the relative homotopy exact sequence of the pair $(\Sigma, \partial M)$:

$$\pi_2(\Sigma) \to \pi_2(\Sigma, \partial M) \xrightarrow{\partial} \pi_1(\partial M) \xrightarrow{\iota_*} \pi_1(\Sigma).$$

From claim 2, it follows that $\partial$ is injective. On the other hand, $\iota_* : \pi_1(\partial M) \to \pi_1(\Sigma)$ is an isomorphism, since both groups are infinite cyclic and generated by the class of $\partial$. Hence, $\iota_*$ is injective and

$$\pi_2(\Sigma, \partial M) \cong \text{im} \partial = \ker \iota_* = 0.$$

Claim 4. The inclusion $\partial M \hookrightarrow M$ induces an isomorphism $\pi_1(\partial M) \cong \pi_1(M)$.

First note, that any loop in $\Sigma$ can by homotoped to a loop in $\partial M$. In particular, if $\eta : S^1 \to M$ is smooth and such that $\gamma(0) \in \partial M$, then there is a smooth map $u : [0, 1] \times S^1 \to \Sigma$ such that

$$u([0] \times S^1) = \eta, \quad u([1] \times S^1) \subset \partial M \quad \text{and} \quad u(t, 0) = \eta(0)$$

for all $t \in [0, 1]$. We show that $u$ can be chosen such that $\text{im}(u) \subset M'$, where $M'$ is the union of $M$ with a closed tubular neighborhood of $\partial M$. This implies
claim 4.
First, by making a $C^1$-small perturbation, we may assume that $u$ is transverse to $\partial M'$. This implies that the set

$$B := u^{-1}(\partial M') \subset [0, 1] \times S^1$$

is a compact 1-manifold with boundary

$$\partial B = u^{-1}(\partial M') \cap \partial([0, 1] \times S^1) = \emptyset.$$ 

Hence, $B$ is a finite union of circles. Assume that $B$ is non-empty and let $D$ be the interior of a component of $B$. Note, that the map $u|D$ represents an element of $\pi_2(\Sigma, \partial M')$. It follows from claim 3, however, that such a map can be deformed to the constant map. Using this argument inductively, it follows that $u$ can be deformed in the interior of $[0, 1] \times S^1$ such that $B$ disappears, which proves the claim.

Claim 5. $\Sigma$ is diffeomorphic to $\mathbb{R} \times S^1$.
Let $\mathbb{D}$ denote the unit disk in $\mathbb{R}^2$ and consider the open surface

$$\Sigma' = M \cup S^1 \mathbb{D}.$$ 

From claim 4 it follows, that $\Sigma'$ is simply connected and hence, diffeomorphic to $\mathbb{R}^2$. This in turn implies, that $\Sigma'$ is diffeomorphic to $\mathbb{R}^2 \setminus \mathbb{D}$. It follows, that $\Sigma$ is obtained from $\Pi$ by attaching two copies of $\mathbb{R}^2 \setminus \mathbb{D}$ along the boundaries and hence, is diffeomorphic to $\mathbb{R} \times S^1$. \qed
References


