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Author(s):
Feldman, Joel S.; Knörrer, Horst; Trubowitz, Eugene

Publication Date:
1996

Permanent Link:
https://doi.org/10.3929/ethz-a-004352755

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Riemann Surfaces of Infinite Genus
IV: The Kadomcev Petviashvilli Equation

Joel Feldman*
Department of Mathematics
University of British Columbia
Vancouver, B.C.
CANADA V6T 1Z2

Horst Knörrer†, Eugene Trubowitz
Mathematik
ETH-Zentrum
CH-8092 Zürich
SWITZERLAND

This work is dedicated to Henry P. McKean.

* Research supported in part by the Natural Sciences and Engineering Research Council of Canada, the Schweizerischer Nationalfonds zur Förderung der wissenschaftlichen Forschung and the Forschungsinstitut für Mathematik, ETH Zürich
† Research supported in part by the IHES and SFB288 Differentialgeometrie und Quantenphysik (Berlin)
Introduction

The Schrödinger spectral curve $S(q)$ associated to $q \in L^2_{\mathbb{R}}\left(\mathbb{R}/2\pi\mathbb{Z}\right)$ is the set of all points $({\xi}, {\lambda}) \in \mathbb{C}^* \times \mathbb{C}$ for which there is a nontrivial distributional solution $\psi(x)$ in $L^\infty_{\text{loc}}(\mathbb{R})$ of the Schrödinger equation

$$\frac{d^2}{dx^2} \psi + q(x) \psi = \lambda \psi$$

satisfying

$$\psi(x + 2\pi) = \xi \psi(x)$$

for all $x \in \mathbb{R}$. For generic $q \in L^2_{\mathbb{R}}\left(\mathbb{R}/2\pi\mathbb{Z}\right)$, the curve $S(q)$ is a Riemann surface of infinite genus and we showed ([FKT3, Theorem 14.1, Example 1]) that it satisfies the Geometric Hypotheses of [FKT2].

Suppose $u(x, t), -\infty < t < \infty$, is a solution to the initial value problem for the Korteweg-deVries equation

$$u_t = 3uu_x - \frac{1}{2}u_{xxxx}$$

with initial data $u(x, 0) = u_0(x) \in C^\infty_{\mathbb{R}}\left(\mathbb{R}/2\pi\mathbb{Z}\right)$. It is well known that

$$S(u(\cdot, t)) = S(u_0(\cdot))$$

as subsets of $\mathbb{C}^* \times \mathbb{C}$ for all $-\infty < t < \infty$ (see [McK1], or [FKT3, Theorem 15.14]). In [MT1], this fact was used to prove that every spatially periodic solution of the Korteweg-deVries equation propagates almost periodically in time. In [MT2], the theta function for $S(q)$ was used to give an “explicit” solution to the initial value problem. The technique of [MT1] and [MT2] relies on the explicit realization of $S(q)$, by projection onto the $\lambda$ plane, as a branched double cover of $\mathbb{C}$. Riemann surfaces of infinite genus that are finite sheeted branched covers of $\mathbb{C}$ have been used to study several other integrable $1+1$ dimensional partial differential equations. See, for example, [BKM], [EM], [McK2] and [Sch].

Let

$$\Gamma = (0, 2\pi)\mathbb{Z} \oplus (\omega_1, \omega_2)\mathbb{Z}$$

where $\omega_1 > 0$, $\omega_2 \in \mathbb{R}$. Recall that the heat curve $\mathcal{H}(q)$ associated to $q \in L^2(\mathbb{R}^2/\Gamma)$ is the set of all points $({\xi_1}, {\xi_2}) \in \mathbb{C}^* \times \mathbb{C}^*$ for which there is a nontrivial distributional solution
\( \psi(x_1, x_2) \) in \( L^\infty_{\text{loc}}(\mathbb{R}^2) \) of the “heat equation”

\[
\left( \frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2} \right) \psi + q(x_1, x_2) \psi = 0
\]

satisfying

\[
\psi(x_1 + \omega_1, x_2 + \omega_2) = \xi_1 \psi(x_1, x_2) \\
\psi(x_1, x_2 + 2\pi) = \xi_2 \psi(x_1, x_2)
\]

If \( q \in C^\infty(\mathbb{R}^2/\Gamma) \) and \( \mathcal{H}(q) \) is smooth, then, by Theorem 17.2 [FKT3], it satisfies the Geometric Hypotheses of [FKT2]. There is no natural realization of a heat curve as a branched finite cover of \( \mathfrak{C} \). For this reason heat curves are intrinsically more complicated than Schrödinger spectral curves.

For each \( u \in L^2(\mathbb{R}^2/\Gamma) \) define the function \( I(u) \) by

\[
I(u)(x_1, x_2) = \int_0^{x_2} u(x_1, s) \, ds - \frac{1}{2\pi} \int_0^{2\pi} dt \int_0^t u(x_1, s) \, ds
\]

The Kadomcev-Petviashvilli equation is

\[
u_t = 3uu_x - \frac{1}{2} u_{x_2} + \frac{3}{2} I(u_{x_1})
\]

(KP)

If one differentiates both sides of (KP) with respect to \( x_2 \) one recovers the standard KPII equation (see, for example, [K])

\[
\left( u_t - 3uu_x + \frac{1}{2} u_{x_2}x_2 \right)_{x_2} + \frac{3}{2} u_{x_1} = 0
\]

Suppose \( u = u(x_1, x_2, t) \) is a solution of the initial value problem for the (KP) equation with initial data \( u_0 \in C^\infty_{\mathfrak{R}}(\mathbb{R}^2/\Gamma) \). As above, there is an associated family \( \mathcal{H}(u(\cdot, t)) \), \(-\infty < t < \infty\), of heat curves. By Theorem 15.14 [FKT3],

\[
\mathcal{H}(u(\cdot, t)) = \mathcal{H}(u_0)
\]

as subsets of \( \mathfrak{C}^* \times \mathfrak{C}^* \) for all \(-\infty < t < \infty\). In this paper (Theorem 23.1), we use the theta function on \( \mathcal{H}(u_0) \), when \( u_0 \in C^\infty_{\mathfrak{R}}(\mathbb{R}^2/\Gamma) \), to give an explicit formula for the solution \( u(x_1, x_2, t) \). This formula is used to show (Corollary 23.3) that spatially periodic solutions of the Kadomcev-Petviashvilli equation propagate almost periodically in time.
§21. The Formula for the Solution

We return to the discussion of sections 15, 16, 17 and want to solve the initial value problem for the periodic KP-equation

\[ u_t = 3uu_{x_2} - \frac{1}{2}u_{x_2}x_2x_2 - \frac{3}{2}I(u_{x_1}, x_1) \quad \text{(KP)} \]

for given initial data \( q \). It has been shown by I. Krichever [K] that for \( q \in C^\omega(\mathbb{R}^2/\Gamma) \) the initial value problem is well posed and can be solved for all time. J. Bourgain [B] demonstrated the more difficult fact that the initial value problem is well posed on \( H^1(\mathbb{R}^2/\Gamma) \). Here we show that for real analytic \( q \) the solution is almost periodic in time. This is done by giving a formula for the solution in terms of theta functions associated to heat curves.

Such a formula is well known in the case that the normalization of the heat curve \( \mathcal{H}(q) \) has finite genus. Such potentials \( q \) are often called finite zone potentials. We recall the procedure to solve the initial value problem for the KP-equation for initial data \( q \) that are finite zone potentials (see [K, chap II] and [MII, chapt. IIIb, §4]).

In this case the normalization of \( \mathcal{H}(q) \) is the complement of one point \( P_\infty \) on a compact Riemann surface \( X(q) \). A local coordinate around \( P_\infty \) is \( \zeta = i/k_2 \) (see Theorem 16.1). Let \( A_1, B_1, \cdots A_g, B_g \) be a canonical homology basis for \( X(q) \), and let \( \omega_1, \cdots \omega_g \) be the holomorphic one forms on \( X(q) \) satisfying \( \int_{A_i} \omega_j = \delta_{ij} \). Furthermore denote by \( \theta \) the associated thetafunction. The expansions of the forms \( \omega_j \) at \( P_\infty \) define vectors \( U, V, W \in \mathbb{C}^g \) by

\[ \omega_j = U_j d\zeta + V_j \zeta d\zeta + \frac{1}{2}W_j \zeta^2 d\zeta + O(\zeta^3) \quad \text{near } P_\infty \]

Finally

\[ \{(\xi_1, \xi_2) \in \mathcal{H}(q) \mid (\xi_1, \xi_2) \text{ is a smooth point of } \mathcal{H}(q) \text{ and the nontrivial solution } \psi \text{ of } \left( \frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2} + q \right) \psi = 0 \text{ vanishes at the point } x_1 = x_2 = 0 \} \]

defines a divisor of \( D \) of degree \( g \) on \( X(q) \). This divisor is non-special, so by Riemann’s Vanishing Theorem there is \( Z \in \mathbb{C}^g \) such that \( D \) is the zero divisor of

\[ x \mapsto \theta(Z + \int_{P_\infty}^{x} \bar{\omega}) \]
on $X(q)$. Then there is a constant $c$ such that

$$u(x_1, x_2, t) = -2 \frac{\partial^2}{\partial x_2^2} \log \theta(Ux_2 + Vx_1 - \frac{1}{2} Wt + Z) + c \quad (21.1.a)$$

solves KP, and that

$$u(x_1, x_2, 0) = q(x_1, x_2) \quad (21.1.b)$$

In addition the constant $c$ is zero if $q \in \mathcal{U}(\Gamma)$, that is if $\int_0^{2\pi} q(x_1, x_2) dx_2 = 0$ (see Lemma 15.11).

Observe that for real valued $q$ the heat curve $\mathcal{H}(q)$ has an antiholomorphic involution induced by $(k_1, k_2) \mapsto (-k_1, -k_2)$. The local coordinate $i/k_2$ is real with respect to this involution. If the canonical homology basis is compatible with the antiholomorphic involution then the vectors $U, V, W$ are real. Also $Z$ is then invariant under the induced antiholomorphic involution on the Jacobian of $X(q)$

The purpose of this Chapter is to generalize formula (21.1) to initial data for which the heat curve has infinite genus. More precisely we assume from now on that $q \in C^\infty(\mathbb{R}^2/\Gamma)$ and that $\int_0^{2\pi} q(x_1, x_2) dx_2 = 0$. For simplicity we again assume that $\mathcal{H}(q)$ is smooth. The results of the Sections 7, 8, 9 and 17 show that $\mathcal{H}(q)$ has a canonical homology basis $A_b, B_b$ indexed by the set $\Gamma^\#_+$ of all $b \in \Gamma^#$ with $b_2 > 0$, a basis $(\omega_b)_{b \in \Gamma^#_+}$ of the Hilbert space of square integrable holomorphic one forms with

$$\int_{A_b} \omega_c = \delta_{bc}$$

such that the thetafunction associated to the Riemann period matrix

$$\mathcal{R}_{bc} = \int_{B_b} \omega_c$$

converges on the Banachspace

$$B = \{ (z_b)_{b \in \Gamma^#_+} \mid \sum \frac{|z_b|}{|\log t_b|} < \infty \}$$

Here, $t_b$ are positive constants satisfying (17.1).

To generalize formula (21.1) we have to define analogues of the vectors $U, V, W$ and the divisor $D$ (resp. the associated vector $Z$). First we discuss the vectors $U, V, W$. 

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Consider a marked Riemann surface $(X; A_1, B_1, \cdots)$ that satisfies the geometric hypotheses of §7 with $m = 1$. By Proposition 8.12 the restriction $w_j(z)dz$ of the differential $\omega_j$ to the regular piece can be written in the form

$$w_j(z) = w_{j,\text{com}}(z) + \sum_{s \in S} w_{j,s}(z)$$

where

$$w_{j,s}(z) = -\frac{1}{2\pi i} \int \frac{w_j(\zeta)}{\zeta - z} d\zeta$$
$$w_{j,\text{com}}(z) = -\frac{1}{2\pi i} \int_{\partial K} \frac{w_j(\zeta)}{\zeta - z} d\zeta$$

Furthermore each $w_{j,s}(z)$ and $w_{j,\text{com}}(z)$ is holomorphic outside a bounded subset of $\mathcal{C}$ and decays as $z \to \infty$. Therefore we can consider the expansions of the forms $w_{j,s}(z)dz$ and $w_{j,\text{com}}(z)dz$ at infinity with respect to the variable $i/z$

$$w_{j,s}(z)dz = w^{(1)}_{j,s} \left( -\frac{dz}{z} \right) + w^{(2)}_{j,s} \left( -\frac{i dz}{z^2} \right) + w^{(3)}_{j,s} \left( \frac{dz}{z^3} \right) + w^{(4)}_{j,s} \left( \frac{dz}{z^4} \right) + O(1/|z|^5)$$

$$w_{j,\text{com}}(z)dz = w^{(1)}_{j,\text{com}} \left( -\frac{dz}{z} \right) + w^{(2)}_{j,\text{com}} \left( -\frac{i dz}{z^2} \right) + w^{(3)}_{j,\text{com}} \left( \frac{dz}{z^3} \right) + w^{(4)}_{j,\text{com}} \left( \frac{dz}{z^4} \right) + O(1/|z|^5)$$

with constants $w^{(i)}_{j,s}, w^{(i)}_{j,\text{com}}$. By Proposition 8.12 $w_{j,s}$ decays quadratically if $s \neq s_1(j), s_2(j)$ so that in this case $w^{(1)}_{j,s} = 0$. Also by Proposition 8.12

$$w^{(1)}_{j,s_1(j)} + w^{(1)}_{j,s_2(j)} = 0 \quad \text{and} \quad w^{(1)}_{j,\text{com}} = 0$$

For the next terms we have

**Proposition 21.1** Let $(X; A_1, B_1, \cdots)$ be a marked Riemann surface that satisfies the geometric hypotheses (GH1)-(GH6) of Section 7 with $m = 1$. Then the sums

$$U_j = w^{(2)}_{j,\text{com}} + \sum_{s \in S} w^{(2)}_{j,s}$$
$$V_j = w^{(3)}_{j,\text{com}} + \sum_{s \in S} w^{(3)}_{j,s}$$
$$W_j = w^{(4)}_{j,\text{com}} + \sum_{s \in S} w^{(4)}_{j,s}$$
converges absolutely. Furthermore there is a numerical constant \( \const \) independent of \( j \) such that for all \( j \geq 1 \)

\[
|U_j + \frac{1}{2\pi} (s_1(j) - s_2(j))| \leq \const
\]

\[
|V_j + \frac{1}{2\pi i} (s_1(j)^2 - s_2(j)^2)| \leq \const
\]

\[
|W_j - \frac{1}{2\pi} (s_1(j)^3 - s_2(j)^3)| \leq \const
\]

**Proof:**

\[
w_{j,s}(z) = \frac{1}{2\pi i} \int_{|\zeta-s|=r(s)} \frac{w_j(\zeta)}{z-\zeta} d\zeta = \frac{1}{2\pi i} \int_{|\zeta-s|=r(s)} \frac{1}{z} w_j(\zeta) \frac{1}{1-\zeta/z} d\zeta
\]

\[
= \sum_{n=1}^{4} \frac{1}{2\pi i} \frac{1}{z^n} \int_{|\zeta-s|=r(s)} \zeta^{n-1} w_j(\zeta) d\zeta + O\left(\frac{1}{|s|^4}\right)
\]

Therefore

\[
w^{(n)}_{j,s} = \frac{(-i)^{1-n}}{2\pi i} \int_{|\zeta-s|=r(s)} \zeta^{n-1} w_j(\zeta) d\zeta
\]

(21.2)

Similarly

\[
w^{(n)}_{j,\text{com}} = \frac{(-i)^{1-n}}{2\pi i} \int_{\partial K} \zeta^{n-1} w_j(\zeta) d\zeta
\]

(21.3)

For \( s \neq s_1(j), s_2(j) \) we have

\[
w^{(n)}_{j,s} = \frac{(-i)^{1-n}}{2\pi i} \int_{|\zeta-s|=4r(s)} \left[ s^{n-1} + (n-1) s^{n-2} (\zeta-s) + \cdots + (\zeta-s)^{n-2} \right] w_j(\zeta) d\zeta
\]

\[
= \frac{(-i)^{1-n}}{2\pi i} \sum_{r=1}^{n-1} \frac{(n-1)}{s^{n-r}} \int_{|\zeta-s|=4r(s)} (\zeta-s)^r w_j(\zeta) d\zeta
\]

Therefore by the estimate of Proposition 8.12 we get for \( n \leq 4 \)

\[
|w^{(n)}_{j,s}| \leq \const \cdot \sum_{r=1}^{n-1} |s|^{n-r} r(s)^r \left| w_j dz \right|_{A(s)}
\]

So for \( n \leq 5 \) and \( s \neq s_1(j), s_2(j) \)

\[
|w^{(n)}_{j,s}| \leq \const r(s) |s|^{n-2} \left| w_j dz \right|_{A(s)}
\]

Similarly, for \( s = s_{\mu}(j) \)

\[
\left| w^{(n)}_{j,s} - \frac{(-i)^{1-n}}{2\pi i} \int_{|\zeta-s|=r(s)} \frac{(-1)^{\mu+1} \zeta^{n-1}}{2\pi i (\zeta-s)} d\zeta \right|
\]

\[
\leq \const r(s) |s|^{n-2} \left| \left( w_j - \frac{(-1)^{\mu+1}}{2\pi i} \frac{1}{z-s} \right) dz \right|_{A(s)}
\]
so that for \( n \geq 5 \)
\[
\left| w_{j,s_\mu}(n) + (i)^{1-n} \frac{(-1)^{\mu+1}}{2\pi i} s_\mu(j)^{n-1} \right| \leq \text{const} \ r(s) \ |s|^{n-2} \left( w_j - \frac{(-1)^{\mu+1}}{2\pi i} \frac{1}{z-s} \right) \ dz |A(s)|^2
\]

Therefore by (8.15b)
\[
\sum_{s \in S, s \neq s_1(j), s_2(j)} \left| w_{j,s}(n) + w_{j,s}(n) - (i)^{1-n} \left( \frac{1}{2\pi i} s_1(j)^{n-2} - \frac{1}{2\pi i} s_2(j)^{n-2} \right) \right|
\leq \text{const} \ \sum_{s, i} \left( r_1(s) |s_1(i)|^{n-1} + r_2(s) |s_2(i)|^{n-1} \right) \left( \Omega^j_i + \delta_{ij} \mathcal{R}_j \right) \tag{21.4}
\]
where, as in (8.18)
\[
\Omega^j_i = \left\{ \begin{array}{ll}
\| \omega_j \|_{Y'} & \text{for } i \neq j \\
\| \left( \omega_j - (\varphi_j)_* \left( \frac{dz_1}{2\pi i z_1} \right) \right) \|_{Y'} & \text{for } i = j
\end{array} \right.
\]

and \( \mathcal{R}_j \) is as in Lemma 8.3. In particular the sequence of the \( \mathcal{R}_j \) is bounded in \( j \). By Theorem 8.4 the norm of \( \left( \Omega^j_i \right)_{i,j=1}^{g+1} \) is bounded in \( j \). Thus the right hand side of (21.4) is finite and bounded uniformly in \( j \) if
\[
\sum_{s \in S} r(s)^2 |s|^{2n-4}
\]
is finite. By (GH 5ii)
\[
\sum_{s \in S} r(s)^2 |s|^{2n-4} \leq \sum_{s \in S} \frac{1}{|s|^{2d+4-2n}} < \infty
\]
for \( n \leq 4 \). In a similar way one sees that \( w_{j,\text{com}}(n) \) is bounded uniformly in \( j \) for all \( n \leq 4 \).

\[\square\]

**Remark 21.2:** Let \( q \in C^\infty(\mathbb{R}^2/\Gamma) \) and \( \mathcal{H}(q) \) be the normalization of the associated heat curve. If \( \mathcal{H}(q) \) has finite genus then the definition of \( U, V, W \) in Proposition 21.1 agrees with the one used in formula (21.1).

If \( \mathcal{H}(q) \) is smooth one has for every \( \gamma > 0 \)
\[
t_b \leq \text{const} \ \left( \| b^{\gamma+2} |\hat{\phi}(b)| \|_1 \right) \cdot \frac{1}{|b|^{2\gamma}}
\]

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(see Theorem 16.2 and 17.2). In particular \( \sum t_b^\beta < \infty \) for \( \beta > 1/\gamma \). In the formula for the solution of the periodic KP-equation we use the theta function on the subspace generated by \( U, V, W \). We will apply Proposition 4.15, so it is useful to note the estimate

\[
|U_b|^{\frac{1-2\beta'}{2k}} |V_b|^{\frac{1-2\beta'}{2k}} |W_b|^{\frac{1-2\beta'}{2k}} \leq \text{const} \left( \| b \|^{\gamma+2} |\hat{g}(b)| \|_1 \right) \cdot \frac{1}{|b|^{\frac{1-2\beta'}{k}-3}} \tag{21.5}
\]

which follows directly from the Proposition above.
In this section we formalize what it means for two marked Riemann surfaces that fulfill the geometric hypotheses of §7 to be close to each other. For notational simplicity, we consider only the single sheet case $m = 1$. The definition is such that the corresponding period matrices and theta functions are also close. Then we show that every surface fulfilling the hypotheses of §7 can be approximated by surfaces of finite genus. This approximation result is in turn used to show that $(21.1.a)$ is always a solution of the differentiated version (22.16) of the KP equation.

**Definition 22.1** Recall that $H(t) = \{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = t \text{ and } |z_1|, |z_2| \leq 1 \}$ is a model handle. Let $K \geq 2, 1 - \frac{28}{15 K} \geq \sqrt{t}$ and let $\hat{Y} \subset H(t)$ be diffeomorphic to an annulus and contain $\{ (z_1, z_2) \in H(t) \mid |z_1|, |z_2| \leq 1 - \frac{1}{K} \}$. Similarly, let $\hat{Y}' \subset H(t')$ be diffeomorphic to an annulus and contain $\{ (z_1, z_2) \in H(t') \mid |z_1|, |z_2| \leq 1 - \frac{1}{K} \}$. A diffeomorphism

$$f : \hat{Y} \rightarrow \hat{Y}'$$

is said to be $K$-quasiconformal with distortion at most $\epsilon$ if, firstly, $f$ is holomorphic on $|z_1| \geq 1 - \frac{28}{15 K}$ and on $|z_2| \geq 1 - \frac{28}{15 K}$, secondly, $\left| f\left(\frac{z_1, z_2}{\mu}\right) - 1 \right| < \min \left\{ \epsilon, \frac{1}{15 K} \right\}$ for at least one point of $\hat{Y}$ for each of $\mu = 1, 2$ and, thirdly, the pull-back

$$f^* \left( \frac{dz_1, dz_2}{z_1} \right) = a(z_1, z_2) \frac{dz_1}{z_1} + b(z_1, z_2) \frac{dz_2}{z_1}$$

obeys

$$|a(z_1, z_2) - 1| \leq \min \left\{ \epsilon, \frac{1}{150 K} \right\} (|z_1| + |z_2|)$$

$$|b(z_1, z_2)| \leq \min \left\{ \epsilon, \frac{1}{150 K} \right\} (|z_1| + |z_2|)$$

$$\left| \frac{\partial}{\partial z_1} a(z_1, z_2) \right| \leq \min \left\{ \epsilon, \frac{1}{150 K} \right\} (|z_1| + |z_2|)$$

The diagram shows a model handle $\hat{Y}$ and its approximation $\hat{Y}'$ through a diffeomorphism $f$. The inequalities ensure that the distortion is bounded by $\epsilon$ and that the pull-back of the differential forms is close to the identity.
A diffeomorphism $F$ between two marked Riemann surfaces $X$ and $X'$ of genus $g$ is called $K$–quasiconformal with distortion at most $\epsilon$ if there are $t_1, \cdots, t_r, t'_1, \cdots, t'_r > 0$, sets

$$\left\{ (z_1, z_2) \in H(t) \mid (1 - \frac{1}{K})^{-1} t \leq |z_1| \leq 1 - \frac{1}{K} \right\} \subset \hat{Y}_j \subset H(t_j)$$

$$\left\{ (z_1, z_2) \in H(t) \mid (1 - \frac{1}{K})^{-1} t \leq |z_1| \leq 1 - \frac{1}{K} \right\} \subset \hat{Y}'_j \subset H(t'_j)$$

and maps $\phi_j : H(t_j) \rightarrow X$, $\phi'_j : H(t'_j) \rightarrow X'$, $1 \leq j \leq r$ that are biholomorphic onto their images such that

- $\phi'^{-1}_j \circ F \circ \phi_j \mid \hat{Y}_j$ is $K$–quasiconformal with distortion at most $\epsilon$ for $1 \leq j \leq r$
- The sets $\phi_j(H(t_j))$ are pairwise disjoint and $F$ induces a biholomorphic map between

$$X \setminus \bigcup_{j=1}^r \phi_j(\hat{Y}_j) \quad \text{and} \quad X' \setminus \bigcup_{j=1}^r \phi'_j(\hat{Y}'_j)$$

- $\phi_j\left( \left\{ (z_1, z_2) \in H(t_j) \mid |z_1| = \sqrt{t_j} \right\} \right)$ is homologous to a linear combination of $A_i$, $1 \leq i \leq g$.

**Remark.** Recall that, by definition, a smooth map $U : S \rightarrow S'$ between Riemann surfaces is quasiconformal with Beltrami coefficient at most $\epsilon$ if, for every $x \in S$, there exists a holomorphic coordinate $z$ around $x$ and a holomorphic coordinate $u$ around $U(x)$ such that $u(z) = u^{-1} \circ U \circ z$ obeys

$$|u_z(z)| \leq \epsilon |u_z(z)|$$

Observe that an $K$–quasiconformal diffeomorphism with distortion at most $\epsilon < \frac{1}{3}$ is quasiconformal with Beltrami coefficient at most $4\epsilon$. We choose this stronger definition, which restricts the second as well as first derivatives of $F$, so as to be able to convert $L^2$ bounds on differential forms into pointwise bounds.

**Definition 22.2** Let $X = X^{\text{com}} \cup X^{\text{reg}} \cup X^{\text{han}}$ and $X' = X'^{\text{com}} \cup X'^{\text{reg}} \cup X'^{\text{han}}$ be marked Riemann surfaces fulfilling (GH1)-(GH5) with the number of sheets $m$ of $X^{\text{reg}}$ and $m'$ of $X'^{\text{reg}}$ both being one. In what follows we mark all the objects associated to $X^{\text{reg}}$ with a prime.
Furthermore let $X_0$ resp. $X'_0$ be such that

\[ X^\text{com} \subset X_0 \quad \quad X'^\text{com} \subset X'_0 \]
\[ \partial X_0 \subset X^\text{reg} \quad \quad \partial X'_0 \subset X'^\text{reg} \]
\[ Y_j \subset X_0 \text{ if } Y_j \cap X_0 \neq \emptyset \quad Y'_j \subset X'_0 \text{ if } Y'_j \cap X'_0 \neq \emptyset \]

Let $0 < \epsilon < \frac{1}{8}$, $K \geq 2$. We say that the pair $(X,X_0)$ is $(\epsilon,K)$–close to $(X',X'_0)$ if the following holds.

(i) The set $\mathcal{J} = \{ j \mid Y_j \subset X_0 \}$ agrees with $\{ j \mid Y'_j \subset X'_0 \}$. There are compact, simply connected sets $\hat{D}(s_\mu(j)), \ j \in \mathcal{J}, \ \mu = 1, 2$ obeying

\[ D(s_\mu(j)) \cup D'(s'_\mu(j)) \subset \hat{D}(s_\mu(j)) \subset \{ z \in \mathbb{C} \mid |z - s_\mu(j)| < r_\mu(j) \} \]
\[ \Phi(\partial \hat{D}(s_\mu(j))) \subset \phi_j \left( \{ (z_1, z_2) \in H(t_j) \mid \tau_\mu(j) \leq |z_2| \leq 2\tau_\mu(j) \} \right) \]
\[ \Phi'(\partial \hat{D}(s_\mu(j))) \subset \phi'_j \left( \{ (z_1, z_2) \in H(t'_j) \mid \tau'_\mu(j) \leq |z_2| \leq 2\tau'_\mu(j) \} \right) \]

and a diffeomorphism

\[ F : X_0 \to X'_0 \]

such that

\[ \Phi'^{-1} \circ F = \Phi^{-1} \quad \text{on } \left\{ x \in X_0 \bigg| \Phi^{-1}(x) \notin \bigcup_{\mu=1,2} \hat{D}(s_\mu(j)) \right\} \]

For each $j \in \mathcal{J}$

\[ F \circ \phi_j(Y_j^{(0)}) = \phi'_j(Y_j^{(0)'}), \]

and $\phi'^{-1}_j \circ F \circ \phi_j | Y_j^{(0)}$ is $2$–quasiconformal with distortion at most $\epsilon$. The restriction of $F$ to $X^\text{com}$ is a $K$–quasiconformal diffeomorphism between $X^\text{com}$ and $X'^\text{com}$ with distortion at most $\epsilon$. Furthermore

\[ F_*(A_j) = A'_j \]

for all $1 \leq j \leq g = g'$.

(ii) For all $j \in \mathcal{J}$ and $\mu = 1, 2$

\[ s_\mu(j) = s'_\mu(j) \quad r_\mu(j) = r'_\mu(j) \quad R_\mu(j) = R'_\mu(j) \]
(iii) \[ \| \mathfrak{A} \| \leq K \quad \| \mathfrak{A}' \| \leq K \]
\[ \sum_{s \in S} \frac{1}{|s|^{d-4\delta-2}} \leq K \]
\[ \sum_{s' \in S'} \frac{1}{|s'|^{d-4\delta-2}} \leq K \]
and, for all \( j \),
\[ \mathcal{O}^j \leq K^2 \quad \mathcal{O}'^j \leq K^2 \]
\[ \mathcal{N}_j \leq K \quad \mathcal{N}'_j \leq K \]
\[ (\tau_1(j)^2 + \tau_2(j)^2) \ln \frac{\tau_2(j)}{\tau_1(j)} \leq \frac{K}{|\ln\tau_j|^2} \]

Here, \( \mathfrak{A} \), \( \mathcal{O}^j \) and \( \mathcal{N}_j \) were defined just before Lemma 8.3. All the other data were defined in (GH1-5).

(iv) For \( j \notin \mathcal{J} \)
\[ \mathcal{O}^j \leq \epsilon^2 \quad \mathcal{O}'^j \leq \epsilon^2 \]
\[ t_j \leq \epsilon \quad t'_j \leq \epsilon \]
Furthermore,
\[ \left\| (\mathfrak{A}_{i,j})_{i \in \mathcal{J}} \right\| \leq \epsilon \quad \left\| (\mathfrak{A}'_{i,j})_{i \in \mathcal{J}} \right\| \leq \epsilon \]
\[ \left\| (\mathfrak{A}_{i,j})_{i \notin \mathcal{J}} \right\| \leq \epsilon \quad \left\| (\mathfrak{A}'_{i,j})_{i \notin \mathcal{J}} \right\| \leq \epsilon \]
and, for \( j \in \mathcal{J} \)
\[ \left\| (\tilde{\Omega}^j)_{i \in \mathcal{J}} \right\|_2 \leq \epsilon \quad \left\| (\tilde{\Omega}'^j)_{i \notin \mathcal{J}} \right\|_2 \leq \epsilon \]
where \( \tilde{\Omega} \) was defined in (8.13).

(v) There exists a \( \gamma > 0 \), a collar \( T \) of \( \Phi^{-1}(\partial X^{\text{com}}) \) that is contained in \( \Phi^{-1}(X_0 \cap X_{\text{reg}}) \) and a curve \( \Gamma \) in \( \Phi^{-1}(X_{\text{reg}} \cap X_0) \) of length at most \( K \) such that
- for every \( j > g \), the points \( s_1(j) \) and \( s_2(j) \), can be connected by a curve in \( \{ z \in G \mid \frac{\text{dist}(z,T)^2}{1+|z|^2} \geq \frac{\gamma}{K} \} \). So can the points \( s'_1(j) \) and \( s'_2(j) \).
- for any holomorphic function \( w \) on \( T \) obeying \( \int_{\Phi^{-1}(\partial X^{\text{com}})} w(\zeta) \, d\zeta = 0 \)
\[ \left\| \frac{1}{2\pi i} \int_{\Phi^{-1}(\partial X^{\text{com}})} \frac{w(\zeta)}{\zeta - z} \, d\zeta \right\| \leq \frac{\gamma}{\text{dist}(z,T)^2} \left\| w \right\|_{T} \]
- $\Phi(\Gamma)$ decomposes $X$ into a compact connected component $X(\Gamma)$ containing $X^{\text{com}} \cup \Phi(T)$ and a noncompact component such that for all $j \geq g + 1$ either $Y_j^{(\circ)} \subset X(\Gamma)$ or $Y_j^{(\circ)} \cap X(\Gamma) = \emptyset$.

- Let $J(\Gamma) = \{ \hat{i} \in J \mid Y_{\hat{i}}^{(\circ)} \subset X(\Gamma) \}$.

\[
\| (\mathcal{X}_{i,k})_{i \in J(\Gamma)} \|_{k \geq g+1} < \frac{1}{8}
\]

\[
\| (\mathcal{X}'_{i,k})_{i \in J(\Gamma)} \|_{k \geq g+1} < \frac{1}{8}
\]

\[
\sum_{\mu=1,2, i \in J(\Gamma)} \frac{R_{\mu}(\hat{i})^2}{\text{dist} (s_{\mu}(\hat{i}), T)^4} < \frac{1}{2^{10} \pi^2 \gamma^2}
\]

\[
\sum_{\mu=1,2, i \in J(\Gamma)} \frac{R'_{\mu}(\hat{i})^2}{\text{dist} (s'_{\mu}(\hat{i}), T)^4} < \frac{1}{2^{10} \pi^2 \gamma^2}
\]

- \[
\text{length}(\Gamma) \sup_{\hat{i} \in J(\Gamma)} \left( \sum_{\mu=1,2} \frac{3r_{\mu}(j)}{|z-s_{\mu}(j)|^2} N_j + \frac{1}{2\pi} \frac{1}{|z-s_1(j)|} + \frac{1}{2\pi} \frac{1}{|z-s_2(j)|} \right) \leq K
\]

- \[
\text{length}(\Gamma) \sup_{\hat{i} \in J(\Gamma)} \left( \sum_{\mu=1,2} \frac{3r'_{\mu}(j)}{|z-s'_{\mu}(j)|^2} N_j + \frac{1}{2\pi} \frac{1}{|z-s'_1(j)|} + \frac{1}{2\pi} \frac{1}{|z-s'_2(j)|} \right) \leq K
\]

- \[
\text{length}(\Gamma) \sup_{\hat{i} \in J(\Gamma)} \left( \sum_{\mu=1,2} \frac{3r_{\mu}(j)}{|z-s_{\mu}(j)|^2} N_j + \frac{1}{2\pi} \frac{1}{|z-s_1(j)|} + \frac{1}{2\pi} \frac{1}{|z-s_2(j)|} \right) \leq \epsilon
\]

- \[
\text{length}(\Gamma) \sup_{\hat{i} \in J(\Gamma)} \left( \sum_{\mu=1,2} \frac{3r'_{\mu}(j)}{|z-s'_{\mu}(j)|^2} N_j + \frac{1}{2\pi} \frac{1}{|z-s'_1(j)|} + \frac{1}{2\pi} \frac{1}{|z-s'_2(j)|} \right) \leq \epsilon
\]

- \[
\text{dist} (\Gamma, T)^2 \geq 4\gamma \text{length} (\Gamma)
\]

- \[
\text{length} (\Gamma)^2 \sup_{\hat{i} \in J(\Gamma)} 30r_{\mu}(k)^2 \sum_{\mu=1,2} \frac{30r_{\mu}(k)^2}{\text{dist} (s_{\mu}(k), T)^4} \leq \frac{1}{16}
\]

(vi) For each $1 \leq i \leq g$ there exist $L_i, \delta_i$ with $\frac{L_i}{\delta_i} \leq K^2$ and a quasiconformal diffeomorphism $u_i$, with Beltrami coefficient bounded by $\frac{1}{2}$, from $\mathcal{U} = \mathbb{R}/L_i \mathbb{Z} \times [0, \delta_i]$ into $X^{\text{com}}$ with $u_i(\mathbb{R}/L_i \mathbb{Z} \times \{0\}) = B_i$. 

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First we observe that every marked Riemann surface $X = X^\text{com} \cup X^\text{reg} \cup X^\text{han}$ is close to one of finite genus.

**Proposition 22.3** Let $X = X^\text{com} \cup X^\text{reg} \cup X^\text{han}$ be a marked Riemann surface with $m = 1$ fulfilling (GH1)-(GH5) such that

$$
\sup_j (\tau_1(j)^2 + \tau_2(j)^2) \left| \ln t_j \right|^2 \ln \frac{\tau_1(j)\tau_2(j)}{t_j} < \infty
$$

Then there is $K > 0$ such that for every compact subset $Z$ of $X$ and any $\epsilon > 0$ there is

- a submanifold $X_0$ of $X$ with boundary
- a marked Riemann surface $X'$ of genus genus $(X_0)$ and
- a compact submanifold $X'_0 \subset X'$

such that $X_0$ contains $Z$ and $(X', X'_0)$ is $(\epsilon, K)$–close to $(X, X_0)$ and the $K$–quasiconformal diffeomorphism $F$ is biholomorphic on $X^\text{com}$.

**Proof:** By Lemma 8.9b there is a collar $T$ of $\partial X^\text{com}$ and a constant $\gamma$ such that the second bullet of condition (v) in Definition 22.2 is satisfied. By Lemma 8.1, there exists a curve $\Gamma$ in $\phi^{-1}(X^{\text{reg}})$ satisfying the third, fourth, sixth and seventh bullets of (v). By Lemma 8.3 and (GH5ii) there is $K \geq 0$ such that the conditions (iii) in Definition 22.2 are fulfilled for $X$. It is clearly also possible to choose $K$ large enough that the first bullet and the first half of the fifth bullet of (v) are satisfied. Again possibly enlarging $K$, it is possible to choose $L_i, \delta_i, \mu_i > 0$, $1 \leq i \leq g$, and quasiconformal diffeomorphisms $u_i$, $1 \leq i \leq g$ satisfying condition (vi).

By Lemma 8.3 and (GH2iv) it is possible to choose $n$ so that condition (iv) and the second part of the fifth bullet of (v) are satisfied for $j > n$. Choose $X_0$ such that $Z \subset X_0$ and $Y_j \subset X_0$ for all $g < j \leq n$. This can be done by putting $X_0 = X(\Gamma')$ for a suitable $\Gamma'$ as in Lemma 8.1. Put $\mathcal{J} = \{ j \mid Y_j \subset X_0 \}$. We define $X'$ as the Riemann surface obtained by gluing $X_0$ to $\Phi^{-1}(\partial X^\text{com}) \setminus \left( K \cup \bigcup_{j \in \mathcal{J}} \text{int} D(s_{\mu}(j)) \right)$ along $X_0 \cap X^{\text{reg}}$ resp. $\Phi^{-1}(X_0 \cap X^{\text{reg}})$ using $\Phi$ as a gluing map. Furthermore, define $X'_0$ to be the part of $X'$ corresponding to $X_0$ in this construction and $F : X_0 \to X'_0$ to be the identity map. Conditions (i) and (ii) are trivially satisfied and the required bounds on primed quantities are inherited from the corresponding bounds on unprimed quantities.

\[\square\]
Theorem 22.4 Let $X = X^{\text{com}} \cup X^{\text{reg}} \cup X^{\text{han}}$ and $X' = X'^{\text{com}} \cup X'^{\text{reg}} \cup X'^{\text{han}}$ be marked Riemann surfaces with $m = m' = 1$ fulfilling (GH1)-(GH5) and let $X_0 \subset X$, $X'_0 \subset X'$ be compact submanifolds with boundary in $X$ and $X'$ respectively. Assume that $(X, X_0)$ and $(X', X'_0)$ are $(\epsilon, K)$-close, with $\epsilon$ smaller than some strictly positive universal constant. Let $F : X_0 \to X'_0$ be a diffeomorphism as in part (i) of Definition 22.2. Then there is a numerical constant $\text{const}$ independent of $\epsilon, K, X$ and $X'$ such that

a) For all $j \in J$

$$|t_j - t'_j| \leq \text{const} \epsilon t_j$$

b) Define $J_c = \{ i \mid Y_i \subset X_0 \} \cup \{ 1, \cdots, g \}$. For $i, j \in J_c$

$$\left| \int_{B_i} \omega_j - \int_{B'_i} \omega'_j \right| \leq \text{const} \epsilon K^4$$

c) For every compact subset $\mathcal{K}$ in the universal covering $\tilde{X}_0 \to X_0$ there is a constant $C_{\mathcal{K}}$ which depends only on $\mathcal{K}$, not on $X'$, such that for all $x_1, x_2 \in \mathcal{K}$ and all $j \in J_c = J$

$$\left| \int_{x_1}^{x_2} (\omega_j - F^* \omega'_j) \right| \leq \epsilon K^6 C_{\mathcal{K}}$$

d) Consider the quantities $U, V, W$ of Proposition 21.1. Then for $j \in J_c$

$$|U_j - U'_j| \leq \text{const} \epsilon K^5$$

$$|V_j - V'_j| \leq \text{const} \epsilon K^5$$

$$|W_j - W'_j| \leq \text{const} \epsilon K^6$$

and for $j \notin J_c$

$$|U_j + \frac{1}{2\pi} (s_1(j) - s_2(j))| \leq \text{const} \epsilon K^3$$

$$|V_j + \frac{1}{2\pi} (s_1(j)^2 - s_2(j)^2)| \leq \text{const} \epsilon K^3$$

$$|W_j - \frac{1}{2\pi} (s_1(j)^3 - s_2(j)^3)| \leq \text{const} \epsilon K^4$$

To prepare for the proof we first note the following

Lemma 22.5 (Modification of Lemma 8.8) Let $\sqrt{t} < a < A < 1$. Let $f$ be a differentiable function on a neighbourhood of the annulus $\{ z \in \mathbb{C} \mid t \leq |z| \leq 1 \}$. Let $C_1$ and $C_2$ be curves
(without self-intersection) of winding number one in the outer annulus \( \{ z \in \mathbb{C} \mid A \leq |z| \leq 1 \} \) and inner annulus \( \{ z \in \mathbb{C} \mid t \leq |z| \leq t/A \} \) respectively. Suppose that

\[
\int_{C_1} f(\zeta) \frac{d\zeta}{\zeta} = 0
\]

Then, for all \( t/a \leq |z| \leq a \)

\[
|f(z)| \leq \frac{|z|}{2\pi(A-a)} \left| \int_{C_1} f(\zeta) \frac{d\zeta}{\zeta} \right| + \frac{t}{2\pi(A-a)|z|} \left| \int_{C_2} f(\zeta) \frac{d\zeta}{\zeta} \right| + \frac{1}{2\pi} \left| \int_{R_{12}} \frac{f(\zeta)}{\zeta-t} d\zeta \wedge d\bar{\zeta} \right|
\]

where \( R_{12} \) is the region between \( C_1 \) and \( C_2 \).

**Proof:** By Cauchy’s integral formula

\[
f(z) = \frac{1}{2\pi i} \left( \int_{C_1} f(\zeta) \frac{d\zeta}{\zeta-z} - \int_{C_2} f(\zeta) \frac{d\zeta}{\zeta-z} + \int_{R_{12}} \frac{f(\zeta)}{\zeta-z} d\zeta \wedge d\bar{\zeta} \right)
\]

\[
= \frac{1}{2\pi i} \left( \int_{C_1} f(\zeta) \frac{d\zeta}{\zeta-z} + z \int_{C_1} \frac{1}{\zeta-z} f(\zeta) \frac{d\zeta}{\zeta} - \int_{C_2} \frac{1}{\zeta-z} f(\zeta) \frac{d\zeta}{\zeta} + \int_{R_{12}} \frac{f(\zeta)}{\zeta-z} d\zeta \wedge d\bar{\zeta} \right)
\]

For \( t/a \leq |z| \leq a \) we have

\[
\frac{1}{|z-t|} \leq \frac{1}{A-a} \quad \text{if } \zeta \in C_1
\]

\[
\left| \frac{\zeta}{z-t} \right| \leq \frac{t}{2\pi(A-a)} \quad \text{if } \zeta \in C_2
\]

As \( \int_{C_1} f(\zeta) \frac{d\zeta}{\zeta} = 0 \) we get the desired estimate. \( \blacksquare \)

For the proof of Theorem 22.4, we define

\[
\Omega^j_z = \left\| \left( \omega_j - \delta_{ij} \frac{1}{2\pi i} \left( \phi_j \right)_z \left( \frac{d\zeta}{\bar{z}_1} \right) \right) \right\|_{Y_i^{(\beta)}}^2
\]

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and we define $\Omega_{i}^{j}$ analogously. Recall that $Y_{i}^{(o)}$ is the cylinder in $Y_{i}$ bounded by the curves $\Phi \left( \{ z \in \mathbb{C} \mid |z-s_{\mu}(i)| = r_{\mu}(i) \} \right)$, $\mu = 1, 2$. Furthermore write

$$\Phi^{*}(\omega_{j}) = w_{j}(z)dz$$

$$\Phi'^{*}(\omega'_{j}) = w'_{j}(z)dz$$

We plan to mimic the proof of Theorem 8.4. We put, for $i \in \mathcal{J}$, $j \in \mathcal{J}_{c}$

$$D_{i}^{j} = 2\pi \max_{\mu=1,2} R_{\mu}(i) \sup_{|z-s_{\mu}(i)|=r_{\mu}(i)} |w_{j}(z) - w'_{j}(z)|$$

and define $Y_{i}^{(oo)}$ to be the cylinder in $Y_{i}$ bounded by the curves

$$\Phi \left( \{ z \in \mathbb{C} \mid |z-s_{\mu}(i)| = 4r_{\mu}(i) \} \right), \mu = 1, 2$$

**Lemma 22.6** There is a numerical constant $\text{const}$ independent of $X$ and $X'$ such that for $i \in \mathcal{J}$, $j \in \mathcal{J}_{c}$

$$\left\| (\omega_{j} - F^{*}\omega'_{j}) \right\|_{Y_{i}^{(oo)}} \leq D_{i}^{j} + \epsilon \text{const} K \left( \Omega_{i}^{j} + \delta_{ij} \right)$$

(22.1)

More precisely, if one writes on the model handle $H(t_{i})$

$$\phi_{i}^{*}(\omega_{j} - F^{*}\omega'_{j}) = \alpha_{j}^{(i)}(z_{1}, z_{2}) \frac{dz_{1}}{z_{1}} + \beta_{j}^{(i)}(z_{1}, z_{2}) \frac{dz_{2}}{z_{2}}$$

Then on $\phi_{i}^{-1}(Y_{i}^{(oo)})$

$$|\alpha_{j}^{(i)}(z_{1}, z_{2})| \leq \frac{3}{\pi} (|z_{1}| + |z_{2}|) D_{i}^{j} + \epsilon \text{const} \frac{\epsilon K}{\ln t_{i}} \left( \Omega_{i}^{j} + \delta_{ij} \right)$$

$$|\beta_{j}^{(i)}(z_{1}, z_{2})| \leq \epsilon \text{const} (|z_{1}| + |z_{2}|) \left( \Omega_{i}^{j} + \delta_{ij} \right)$$

(22.2)

$\alpha_{j}^{(i)}$ is holomorphic outside $\{ (z_{1}, z_{2}) \in H(t_{i}) \mid |z_{1}| \leq 2r_{i} \text{ and } |z_{2}| \leq 2r_{i} \}$ and $\beta_{j}^{(i)}$ is zero outside this set.
Proof: Clearly (22.1) follows from (22.2), so it suffices to prove (22.2). To simplify the notation we delete the sub and superscripts $i$ and $j$ whenever convenient, write $\tilde{F}$ for $\phi_i^{-1} \circ F \circ \phi_i$, $z$ for the variable $z_1$ on $H(t_i)$ and $u = u(z)$ for the first component of $\tilde{F}(z, t_i / z)$. With this notation

$$\frac{du}{u} = a(z) \frac{dz}{z} + b(z) \frac{d\bar{z}}{z} \quad (22.3)$$

On $\phi_i^{-1} \left( \{ x \in X^{reg} \mid \Phi^{-1}(x) \notin \tilde{D}(s_1(i)) \cup \tilde{D}(s_2(i)) \} \right)$ the function $a(z)$ is holomorphic and $b(z)$ is zero. By part (i) of Definition 22.2 and Definition 22.1

$$d \ln \frac{u}{z} = \frac{du}{u} - \frac{dz}{z} = \left[ a(z) - 1 \right] \frac{dz}{z} + b(z) \frac{d\bar{z}}{z}$$

and, on $Y^{(c)}$,

$$\left| \left| a(z) - 1 \right| \frac{dz}{z} \right| \leq \epsilon \left| dz_1 \right| + \left| dz_2 \right|$$

$$\left| b(z) \frac{d\bar{z}}{z} \right| \leq \epsilon \left| dz_1 \right| + \left| dz_2 \right|$$

Consequently $\ln \frac{u}{z}$ varies by at most $4\pi \epsilon$ over the handle. Since, by the second requirement of Definition 22.1, we have that $1 - \epsilon \leq \left| \frac{u}{z} \right| \leq 1 + \epsilon$ for at least one $z \in Y^{(c)}$ there is a $\varphi \in \mathbb{R}$ such that

$$\left| \frac{u}{z} - e^{i\varphi} \right| \leq \text{const} \epsilon \quad (22.4)$$

Part a) of Theorem 22.4 follows from (22.4) and its analogue, $\left| \frac{u_z}{z} - e^{-i\varphi} \right| \leq \text{const} \epsilon$ since

$$\left| t_i - t_i' \right| = \left| z_1 z_2 - u_1 u_2 \right| \leq \left| z_1 e^{i\varphi} - u_1 \right| \left| z_2 \right| + \left| u_1 \right| \left| z_2 e^{-i\varphi} - u_2 \right|$$

$$\leq \text{const} \epsilon \left| z_1 z_2 \right| + \text{const} \left| u_1 \epsilon \right| \left| z_2 \right| \leq \text{const} \epsilon \left| z_1 \right| \left| z_2 \right| = \text{const} \epsilon t_i \quad (22.5)$$

Write

$$\phi_i^* \omega_j' = g'(u) \frac{du}{u} \quad \phi_i^* \omega_j = g(z) \frac{dz}{z}$$

By Lemma 8.8, scaled to the handle $H(4t)$ as in Proposition 8.16,

$$\left| g'(u) - \frac{\delta_i}{2\pi} \right| \leq \text{const} \left( \left| u \right| + \frac{t_i'}{\left| u \right|} \right) \Omega_i^{t_j} \quad (22.6a)$$

$$\left| g(z) - \frac{\delta_i}{2\pi} \right| \leq \text{const} \left( \left| z \right| + \frac{t_i}{\left| z \right|} \right) \Omega_i^z \quad (22.6b)$$

for $3t' \leq \left| u \right| \leq \frac{1}{3}$ and $3t \leq \left| z \right| \leq \frac{1}{3}$. Now

$$\phi_i^* \omega_j - \phi_i^* F^* \omega_j' = g(z) \frac{dz}{z} - g'(u) \left( a(z) \frac{dz}{z} + b(z) \frac{d\bar{z}}{z} \right) = a(z) \frac{dz}{z} + \beta(z) \frac{d\bar{z}}{z}$$
with \( \alpha(z) = g(z) - g'(u)a(z) \)
\[ \beta(z) = -g'(u)b(z) \]

Clearly \( \beta(z) \) is zero on \( \{ (z_1, z_2) \in H(t_1) \mid |z_1| > 2\tau_1 \text{ or } |z_2| > 2\tau_2 \} \). By (22.6a), part (i) of Definition 22.2 and Definition 22.1

\[ |\beta(z)| \leq \text{const } \epsilon \left( |z| + \frac{t}{|z|} \right) \left( \Omega_{ij}^\ell + \delta_{ij} \right) \]

To bound \( |\alpha(z)| \) we apply Lemma 22.5 to \( f(z) = \alpha(z) \),

\[ C_1 = \{ z \in H(t_1) \mid |g_{s_1}(z_1) - s_1(i)| = R_1(i) \} \]
\[ C_2 = \{ z \in H(t_1) \mid |g_{s_2}(z_2) - s_2(i)| = R_2(i) \} \]

On \( C_1 \), \( \beta(z) = 0 \) so that the hypothesis \( \int_{C_1} \frac{\alpha(z)}{z} dz = \int_{C_1} \phi_i^* \omega_j - \phi_i^* F^i \omega_j = 0 \) is satisfied. The bounds on the first two terms in the conclusion of Lemma 22.5 are

\[ \frac{3|z|}{\pi} \int_{C_1} |\alpha(\zeta) \frac{d\zeta}{\zeta^2}| \leq \frac{3|z|}{\pi} \int_{|\zeta - s_1(i)| = R_1(i)} |w_j(\xi) - w_j'(\xi)||d\xi| \leq \frac{3|z|}{\pi} D_i^j \]
\[ \frac{3|z|}{\pi} \int_{C_2} |\alpha(\zeta) \frac{d\zeta}{\zeta^2}| \leq \frac{3\tau}{|z|} \int_{|\zeta - s_2(i)| = R_2(i)} |w_j(\xi) - w_j'(\xi)||d\xi| \leq \frac{3\tau}{|z|} D_i^j \]

To bound the third term, we observe that

\[ \alpha_z = -(g'_u u_z a + g' a_z) \]
\[ = -(\frac{u}{\pi} g'_u a b + g' a_z) \]

by (22.3), since \( g(z) \) and \( g'(u) \) are holomorphic. By Cauchy’s estimate and (22.6a)

\[ |g'_u| \leq \text{const } \left( 1 + \frac{t'}{|u|} \right) \Omega_{ij}^\ell \]

for \( 6t' \leq |u| \leq \frac{1}{6} \). By (GH3) and Definition 22.2(i), the support of \( b \) and \( a_z \) is contained in \( \{ z \in \mathbb{C} \mid \frac{t}{2\tau_2} \leq |z| \leq 2\tau_1 \} \subset \{ z \in \mathbb{C} \mid 6t' \leq |u| \leq \frac{1}{6} \} \). Therefore,

\[ \int_{R_1} \left| \frac{\alpha_z}{\zeta^2} d\zeta \right| \leq \epsilon \text{const } \left( \Omega_{ij}^\ell + \delta_{ij} \right) \int_{\frac{t}{2\tau_2} \leq |\zeta| \leq 2\tau_1} \left[ \left( |\zeta| + \frac{t}{|\zeta|} \right)^2 \frac{1}{|\zeta|} + \left( |\zeta| + \frac{t}{|\zeta|} \right) \right] |\frac{d\zeta}{\zeta^2}| \]
\[ \leq \epsilon \text{const } \left( \Omega_{ij}^\ell + \delta_{ij} \right) \int_{\frac{t}{2\tau_2}}^{2\tau_1} dr \left[ r + \frac{r^2}{2} + \frac{r^2}{2} \right] \int_{-\pi}^{\pi} d\phi \frac{r}{|z - r e^{i\phi}|} \]
Since
\[ \int_{-\pi}^{\pi} d\phi \frac{1}{|z/r - e^{-i\phi}|} \leq \text{const} \left( 1 + |\ln |z| - r| + |\ln r| \right) \]
we get, using Definition 22.2(iii)
\[ \int_{R_{12}} \left| \frac{\alpha_{ij}}{z^2} d\zeta \wedge d\bar{\zeta} \right| \leq \epsilon \text{const} \left( \Omega^{\prime \prime}_{ij} + \delta_{ij} \right) \left( \tau_1^2 + \tau_2^2 \right) \ln \frac{\pi t}{\delta} \]
\[ \leq \epsilon \text{const} \left( \Omega^{\prime \prime}_{ij} + \delta_{ij} \right) \frac{K}{|\ln t|} \]
\[
\]
We shall first prove Theorem 22.4 under the additional hypotheses that \( X^{\text{com}} = X^{\prime \prime \text{com}} = \emptyset \) and \( ||\mathcal{A}||, ||\mathcal{A}'|| < \frac{1}{4} \).

**Lemma 22.7** Assume that \( X^{\text{com}} = X^{\prime \prime \text{com}} = \emptyset \) and \( ||\mathcal{A}||, ||\mathcal{A}'|| < \frac{1}{4} \). Then
\[
\|\Omega^j\|_2, \|\Omega'^j\|_2 \leq 3K \quad \text{for all } j
\]
\[
\|\Omega^j\|_2, \|\Omega'^j\|_2 \leq 3\epsilon K \quad \text{for } j \notin \mathcal{J}
\]
\[
\left\| (\Omega^j_i)_{i \notin \mathcal{J}} \right\|_2, \left\| (\Omega'^j_i)_{i \notin \mathcal{J}} \right\|_2 \leq 9\epsilon K \quad \text{for all } j
\]
\[
\|D^j\|_2 \leq \text{const} \epsilon K^2 \quad \text{for } j \in \mathcal{J}
\]

**Proof:** By the inequality following (8.17)
\[
\|\Omega^j\|_2 \leq 2 \left( \sqrt{\mathcal{O}^j} + \| (\mathcal{A}_{ij} \mathcal{N}_j)_{i > j} \| \right)
\]
and one has the same inequality for the primed objects. So the assertions of the first two lines of the Lemma follow from parts (iii) and (iv) of Definition 22.2.

Inequality (8.16a) yields for \( j \in \mathcal{J} \)
\[
\left\| (\Omega^j_i)_{i \notin \mathcal{J}} \right\|_2 \leq \left\| (\Omega^j_i)_{i \notin \mathcal{J}} \right\|_2 + \| \mathcal{N}_j \| (\mathcal{A}_{i,j})_{i \notin \mathcal{J}} \|_2 + \left\| (\mathcal{A}_{i,k})_{i \notin \mathcal{J}} \|_2 + \| (\mathcal{A}_{i,k})_{i \notin \mathcal{J}} \|_2 \right\|_2
\]
So the third assertion of the Lemma follows from the preceding ones and part (iv) of Definition 22.2.
As in Proposition 8.5 we write

\[ w_j(z) = \sum_{s \in S} w_{j,s}(z) \]  

(22.7a)

with

\[ w_{j,s}(z) = -\frac{1}{2\pi i} \int_{|\zeta-s|=r(s)} \frac{w_{j}(\zeta)}{\zeta-z} d\zeta \]

Define \( w_{j,s'}^f \) in the same way so that

\[ w_j^f(z) = \sum_{s' \in S'} w_{j,s'}^f(z) \]  

(22.7b)

By estimate (8.9) in Proposition 8.5, we have for \( i, j \in \mathcal{J}, k \notin \mathcal{J} \)

\[
2\pi \max_{\mu=1,2} R_\mu(i) \sup_{|z-s_\mu(i)|=R_\mu(i)} \left| w_{j,s_1(k)}(z) + w_{j,s_2(k)}(z) \right|
\leq 24\pi \max_{\mu,\tau=1,2} R_\mu(i) r_\tau(k) \frac{1}{|s_\mu(i) - s_\tau(k)|^2} \Omega^j_k
\]

(22.8a)

and similarly

\[
2\pi \max_{\mu=1,2} R_\mu(i) \sup_{|z-s_\mu(i)|=R_\mu(i)} \left| w_{j,s_1(k)}^f(z) + w_{j,s_2(k)}^f(z) \right| \leq \mathcal{A}_{i,k} \Omega^{i,j}_k
\]

(22.8b)

Furthermore, by Lemma 8.9c, for \( k \in \mathcal{J} \)

\[
|w_{j,s_+(k)}(z) - w_{j,s_-(k)}(z)| \leq \frac{3r_\tau(k)}{|z-s_\tau(k)|^2} \left\| (\omega_j - F^*\omega_j)^{i,j} \right\|_{Y^\infty_k} \]

if \( |z-s_\mu(i)| = R_\mu(i) \) for some \( i \in \mathcal{J} \) and \( \mu = 1,2 \). Therefore

\[
2\pi \max_{\mu=1,2} R_\mu(i) \sup_{|z-s_\mu(i)|=R_\mu(i)} \left| \sum_{\tau=1}^{2} w_{j,s_+(k)}(z) - w_{j,s_-(k)}(z) \right| \leq \mathcal{A}_{i,k} \left\| (\omega_j - F^*\omega_j)^{i,j} \right\|_{Y^\infty_k} \]

(22.9)

Inserting (22.7), (22.8), (22.9) in the definition of \( D_i^j \) we get

\[
D_i^j \leq \sum_{k \in \mathcal{J}} \mathcal{A}_{i,k} \left\| (\omega_j - F^*\omega_j)^{i,j} \right\|_{Y^\infty_k} + \sum_{k \notin \mathcal{J}} \left( \mathcal{A}_{i,k} \Omega^{i,j}_k + \mathcal{A}_{i,k} \Omega^{i,j}_k \right)
\]

(22.10)

In this inequality we insert the first statement of Lemma 22.6 and get, for \( i, j \in \mathcal{J} \)

\[
D_i^j \leq \sum_{k \in \mathcal{J}} \mathcal{A}_{i,k} D_k^j + \sum_{k \notin \mathcal{J}} \left( \mathcal{A}_{i,k} \Omega^{i,j}_k + \mathcal{A}_{i,k} \Omega^{i,j}_k \right) + \epsilon \text{ const } K \sum_{k \in \mathcal{J}} \mathcal{A}_{i,k} \left( \Omega^{i,j}_k + \delta_{jk} \right)
\]

As \( \|\mathcal{A}\| < 1/4 \) we get

\[
\|D_i^j\| \leq 2\|\mathcal{A}\| \left\| (\Omega^{i,j}_k)_{k \notin \mathcal{J}} \right\|_2 + 2\|\mathcal{A}'\| \left\| (\Omega^{i,j}_k)_{k \notin \mathcal{J}} \right\|_2 + \epsilon \text{ const } K \|\mathcal{A}\| \left( \|\Omega^{i,j}\| + 1 \right)
\]

\[
\leq \text{ const } \epsilon K^2
\]

by the first and third lines of this Lemma.
Lemma 22.6 and Lemma 22.7 combined give pointwise bounds on \((\omega_j - F^*\omega'_j)|_Y_{\infty}\) for \(i, j \in J\). Pointwise bounds on the regular pieces are given by

**Lemma 22.8** Assume that \(X^{com} = X'^{com} = \emptyset\) and \(\|\alpha\|, \|\alpha'\| < \frac{1}{4}\).

a) Define

\[ U_R = \{ z \in \mathbb{C} \mid |z - s| \geq r(s), \ |z - s'| \geq r(s') \ \text{for all} \ s \in S, \ s' \in S' \} \]

\[ U_r = \{ z \in \mathbb{C} \mid |z - s| \geq 4r(s), \ |z - s'| \geq 4r(s') \ \text{for all} \ s \in S, \ s' \in S' \} \]

There is a universal constant independent of \(X, X'\) etc. such that for all \(j \in J, z \in U_R\)

\[ |w_j(z) - w'_j(z)| \leq \text{const} \frac{\epsilon K^3}{1 + |z|^2} \]

If \(z \in U_r \setminus U_R\) then there exists an \(s \in S\) and/or an \(s' \in S'\) such that \(|z - s| \leq r(s)\) and/or \(|z - s'| \leq r(s')\). Then, for all \(j \in J\)

\[ |w_j(z) - w'_j(z)| \leq \text{const} \epsilon K^2 \left( \frac{r(s)}{|z - s|^2} + \frac{r'(s')}{|z - s'|^2} \right) + \text{const} \frac{\epsilon K^3}{1 + |z|^2} \]

b) For \(j \notin J\)

\[ \left| \frac{w_j(z)}{2\pi i} - \frac{1}{z - s_1(j)} + \frac{1}{2\pi i} \frac{1}{z - s_2(j)} \right| \leq \text{const} \frac{\epsilon K^2}{1 + |z|^2} \]

if \(|z - s| \geq r(s)\) for all \(s \in S\). If for some \(s \in S, 4r(s) \leq |z - s| \leq r(s)\) then

\[ \left| \frac{w_j(z)}{2\pi i} - \frac{1}{z - s_1(j)} + \frac{1}{2\pi i} \frac{1}{z - s_2(j)} \right| \leq \text{const} \epsilon K \left( \frac{r(s)}{|z - s|^2} \right) + \text{const} \frac{\epsilon K^2}{1 + |z|^2} \]

Similar bounds apply to \(w'_j(z)\).

c) For \(i \in J, j \notin J\)

\[ \left| \frac{\phi_i^*(\omega_j)}{d\bar{z}_1/z_1} \right| \leq \text{const} \epsilon K (|z_1| + |z_2|) \quad \text{if} \ |z_1|, |z_2| \leq 1/4 \]

\[ \left| \frac{\phi_i^* F^*\omega'_j}{d\bar{z}_1/z_1} \right| \leq \text{const} \epsilon K (|z_1| + |z_2|) \quad \text{if} \ |z_1|, |z_2| \leq 1/4 \]
Proof: a) By Lemma 8.9.c and Proposition 8.5

\[ |w_j(z) - w'_j(z)| \leq \sum_{k \in J, \mu = 1, 2} |w_{j, \mu}(j)(z) - w'_{j, \mu}(j)(z)| + \sum_{k \in J, \mu = 1, 2} |w_{j, \mu}(j)(z)| + \sum_{k \in J, \mu = 1, 2} |w'_{j, \mu}(j)(z)| \]

\[ \leq \sum_{k \in J, \mu = 1, 2} \frac{3r(\mu)(k)}{|z - s_{\mu}(k)|^2} \|\omega_j - F^{*}\omega_j'|_{Y_{j,k}} \|^2 + \sum_{k \in J, \mu = 1, 2} \frac{3r'(\mu)(k)}{|z - s'_{\mu}(k)|^2} \Omega_j \]

\[ \leq \text{const} eK^2 \left( \sum_{s \in S} \frac{r(s)}{|z - s|^2} + \sum_{s' \in S'} \frac{r'(s')}{|z - s'|^2} \right) \]

by Lemmas 22.6 and 22.7. As in (8.3), but using part (iii) of Definition 22.2,

\[ \sum_{s \in S} \frac{r(s)}{|z - s|^2} + \sum_{s' \in S'} \frac{r'(s')}{|z - s'|^2} \leq \text{const} \frac{K}{1 + |z|^2} \]

The claim follows.

b) follows from Proposition 8.5 and Lemma 22.7.

c) The first line follows from Lemma 8.8, applied to the scaled handle \( H(4t_i) \), and Lemma 22.7. To prove the second line observe that \( \phi^*_i F^{*}\omega_j' = g'(u_1) \left( a(z_1) \frac{dz_1}{z_1} + b(z_1) \frac{dz_1}{z_1} \right) \) with \( g'(u_1) \) estimated in (22.6a). Definition 22.2 (i) and Definition 22.1 provide bounds on \( a(z_1) \) and \( b(z_1) \).

\[ \square \]

Proof of Theorem 22.4 - simple single sheet case: We now prove Theorem 22.4 under the additional hypotheses that \( X^{\text{com}} = X^{'\text{com}} = 0 \) and \( \|\Xi\|, \|\Xi'\| < \frac{1}{4} \). Part (a) has been proven in (22.5). For part (b), observe that, for each \( i \in J \), the cycle \( B_i \) can be represented as the union of

\[ h_i = \phi_i \left( \{ (z_1, z_2) \in H(t_i) \mid z_1 > 0 \} \right) \cap Y_i^{(o)} \]

and \( \Phi(b_i) \), where \( b_i \) is a path in \( \{ z \in \Phi \mid |z - s| \geq r(s) \text{ for } s \in S, |z' - s'| \geq r'(s') \text{ for } s' \in S' \} \) with the property that

\[ \text{length} \{ z \in b_i \mid |z| \leq \rho \} \leq \text{const } \rho \quad \text{for all } \rho > 0 \]
By the first statement of Lemma 22.8(a)
\[
\left| \int_{\Phi(b_i)} \omega_j - \int_{\Phi'(b_i)} \omega'_j \right| \leq \int_{b_i} \left| (w_j(z) - w'_j(z)) \right| dz \leq \text{const} K^3
\]

By the second statement of Lemma 22.8(a)
\[
\left| \int_{h_i \setminus Y_i^{(\infty)}} (\omega_j - F^* \omega'_j) \right| \leq \text{const} K^2 \sum_{\mu=1}^2 \int_{R^\mu(j)} t_{\mu}(j) dt t^2 + \text{const} K^3 \leq \text{const} K^3
\]

Furthermore, by the pointwise estimate of Lemma 22.6 and the estimate on $D_k^j$ of Lemma 22.7
\[
\left| \int_{h_i \cap Y_i^{(\infty)}} (\omega_j - F^* \omega'_j) \right| \leq \text{const} K^2
\]

This proves part (b) of the Theorem. Part (c) is proven in the same way.

We prove the bound of part (d) on $|W_j - W'_j|$, $j \in J$. The remaining bounds are proven similarly. Recall from Proposition 21.1 and (21.2) that
\[
W_j = \sum_{s \in S} w^{(4)}_{j,s}
\]

where
\[
w^{(n)}_{j,s} = -\frac{1}{2\pi} \int_{\zeta - s} r(s) \zeta^{n-1} w_j(\zeta) d\zeta
\]

For $j \in J$ write
\[
W_j - W'_j = \sum_{i \in J} \left[ w^{(4)}_{j,s_{\mu(i)}} - w^{(4)}_{j',s_{\mu(i)}} \right] + \sum_{i \notin J} \left[ w^{(4)}_{j,s_{\mu(i)}} - w^{(4)}_{j',s_{\mu(i)}} \right]
\]

If $s = s_{\mu}(i)$ for some $i \in J$ then
\[
\left| w^{(n)}_{j,s} - w^{(n)}_{j',s} \right| \leq \frac{1}{2\pi} \int_{|\zeta - s| = 4r(s)} \left| \zeta^{n-1} [w_j(\zeta) - w'_j(\zeta)] \right| d\zeta
\]
\[
= \frac{1}{2\pi} \int_{|\zeta - s| = 4r(s)} \left[ \zeta^{n-1} - s^{n-1} \right] \left| w_j(\zeta) - w'_j(\zeta) \right| d\zeta
\]
\[
\leq r(s) (n - 1) |s| + 4r(s) |s|^{n-2} \int_{|\zeta - s| = 4r(s)} \left| w_j(\zeta) - w'_j(\zeta) \right| d\zeta
\]
\[
\leq \text{const} \epsilon K^3 r(s) |s|^{n-2} \int_{|\zeta - s| = 4r(s)} \left( \frac{r(s)}{|\zeta - s|^2} + \frac{1}{1 + |\zeta|^2} \right) d\zeta
\]
\[
\leq \text{const} \epsilon K^3 r(s) |s|^{n-2} \left[ 1 + r(s) |s|^{-2} \right]
\]
\[
\leq \text{const} \epsilon K^3 r(s) |s|^{n-2}
\]
by Lemma 22.8(a). Similarly, by Lemma 8.9(a) followed by Lemma 22.7 if $s = s_\mu(i)$ and $s' = s'_\mu(i)$ for $i \notin J$ then

\[
\left| w_{j,s}^{(n)} \right| \leq \frac{1}{2\pi} \left| \int_{|\zeta - s| = r(s)} \zeta^{n-1} w_j(\zeta) d\zeta \right|
\]

\[
= \frac{1}{2\pi} \left| \int_{|\zeta - s| = r(s)} \left[ \zeta^{n-1} - s^{n-1} \right] w_j(\zeta) d\zeta \right|
\]

\[
\leq \left\| \left[ \zeta^{n-1} - s^{n-1} \right] w_j(\zeta) \right\|_{r < |\zeta - s| < 2r} \leq \text{const}_n r(s) |s|^{n-2} \Omega_j^2
\]

\[
\leq \text{const}_n \varepsilon K r(s) |s|^{n-2}
\]

and

\[
\left| w_{j,s'}^{(n)} \right| \leq \text{const}_n \varepsilon K r(s) |s|^{n-2}
\]

Consequently

\[
|W_j - W_j'| \leq \text{const} \varepsilon K^3 \left[ \sum r(s) |s|^2 + \sum r'(s') |s'|^2 \right] \leq \text{const} \varepsilon K^4
\]

by (GH5) part (ii) and Definition 22.2 part (iii).

We now wish to prove Theorem 22.4, allowing $X_{\text{com}}$ to be nonempty and deleting the simplifying assumption $\|\mathfrak{R}\| < 1/4$. We start with three general Lemmata.

**Lemma 22.9** Let $U : S \to S'$ be a quasiconformal diffeomorphism with Beltrami coefficient at most $\varepsilon < 1$. Let $\omega'$ be any form on $S'$. Then

\[
\|U^* \omega'\|_{L^2(S)} \leq \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \|\omega'\|_{L^2(S')}
\]

Write $U^* \omega' = \alpha + \beta$ with $\alpha$ of type $(1,0)$ and $\beta$ of type $(0,1)$. Then, if $\omega'$ is of type $(1,0)$,

\[
\|\beta\|_{L^2(S)} \leq \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} \|\omega'\|_{L^2(S')}
\]

**Proof:** Locally, let $\omega' = a(u) du + b(u) d\bar{u}$. Then

\[
U^* \omega' \wedge \overline{U^* \omega'} = i \left\{ |a|^2 + |b|^2 \right\} \left[ |u_z|^2 + |u_{\bar{z}}|^2 \right] + 4 \text{Re} \left[ a \bar{b} u_z u_{\bar{z}} \right] dz \wedge d\bar{z}
\]

\[
U^* \left( \omega' \wedge \overline{\omega'} \right) = i \left[ |a|^2 + |b|^2 \right] \left[ |u_z|^2 - |u_{\bar{z}}|^2 \right] dz \wedge d\bar{z}
\]

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The first claim follows from
\[
\left| |a|^2 + |b|^2 \right| \left| |u_z|^2 - |u_z|^2 \right| \geq \left| |a|^2 + |b|^2 \right| |u_z|^2 (1 - \varepsilon^2)
\]
\[
\left| |a|^2 + |b|^2 \right| \left| |u_z|^2 + |u_z|^2 \right| + 4 |a\overline{b}u_z u_{\overline{z}}| \leq \left| |a|^2 + |b|^2 \right| |u_z|^2 (1 + \varepsilon^2 + 2\varepsilon)
\]

For the second observe that
\[
\beta \wedge *\overline{\beta} = i |au_z|^2 dz \wedge d\overline{z} = \left| \frac{u_z}{u_{\overline{z}}} \right|^2 \left[ 1 - \left| \frac{u_z}{u_{\overline{z}}} \right|^2 \right]^{-1} U^* (\omega' \wedge *\overline{\omega'})
\]

\[\square\]

**Lemma 22.10** Let $S$ be a Riemann surface with boundary $\partial S$ and canonical homology basis $A_1, \cdots, A_g, B_1, \cdots, B_g$. Suppose that $\alpha$ and $\beta$ are differential forms of type $(1, 0)$ and $(0, 1)$ respectively on $S$ such that
\[
\begin{align*}
d(\alpha + \beta) &= 0 \\
\int_{A_i} (\alpha + \beta) &= 0 \quad \text{for } i = 1, \cdots, g \\
\int_{\Delta} (\alpha + \beta) &= 0 \quad \text{for all components } \Delta \text{ of } \partial S
\end{align*}
\]
Then
\[
\|\alpha + \beta\|_{L^2(S)} \leq \sqrt{2} \|\beta\|_{L^2(S)} + \int_{\partial S} |\alpha + \beta|
\]

**Proof:** Put
\[
\omega = \alpha + \beta
\]
We have
\[
*\omega = *\alpha + *\beta = -i\alpha + i\beta
\]
\[
*\overline{\omega} = i\overline{\alpha} - i\overline{\beta} = i\overline{\omega} - 2i\overline{\beta}
\]
Hence
\[
\|\omega\|_{L^2(S)}^2 = \int_S \omega \wedge *\overline{\omega} = i \int_S \omega \wedge \overline{\omega} - 2i \int_S \omega \wedge \overline{\beta}
\]
We apply Lemma 2.8, the Riemann period relations, with $\omega = \omega$ and $\eta = \overline{\omega}$ to bound the first term. Both $\omega$ and $\eta$ are closed. By hypothesis the integral of $\omega$ around each component
of $\partial S$ is zero. Therefore there is a single-valued $C^\infty$ function $f$, defined on a neighbourhood of $\partial S$, such that $\omega = df$ on that neighbourhood and $f$ has a zero on each component of $\partial S$.

Hence by Lemma 2.8 and the vanishing of the $A$ periods,

$$\int_S \omega \wedge \bar{\omega} = \sum_{k=1}^{g} \left( \int_{A_k} \omega \int_{B_k} \bar{\omega} - \int_{B_k} \omega \int_{A_k} \bar{\omega} \right) + \int_{\partial S} f \bar{\omega} = \int_{\partial S} f \bar{\omega}$$

Correspondingly

$$\left| \int_S \omega \wedge \bar{\omega} \right| \leq \left[ \int_{\partial S} |\omega| \right]^2$$

Since $\alpha \wedge \bar{\beta} = 0$ we have

$$-i \int_S \omega \wedge \bar{\beta} = -i \int_S \beta \wedge \bar{\alpha} = \int_S \beta \wedge \ast \bar{\beta} = \|\beta\|^2_{L^2(S)}$$

Therefore

$$\|\omega\|^2_{L^2(S)} \leq \left[ \int_{\partial S} |\omega| \right]^2 + 2 \|\beta\|^2_{L^2(S)}$$

which implies that

$$\|\omega\|_{L^2(S)} \leq \sqrt{2} \|\beta\|_{L^2(S)} + \int_{\partial S} |\omega|$$

\[\square\]

**Lemma 22.11** Let $\mathcal{U}$ be either $\mathbb{R}/L\mathbb{Z} \times [0, \delta]$ or $[0, L] \times [0, \delta]$ with the natural complex structures. Denote by $\mathcal{U}_t$ the subset of $\mathcal{U}$ consisting of those points whose second component is $t$. Let $u : \mathcal{U} \to X$ be a quasiconformal diffeomorphism into a Riemann surface $X$ whose Beltrami coefficient is bounded by $\mu < 1$. Let $\omega$ be a closed one form on $u(\mathcal{U})$. In the event that $\mathcal{U} = [0, L] \times [0, \delta]$, assume that $\frac{\omega(u(0,t))}{dt}$ and $\frac{\omega(u(L,t))}{dt}$ are bounded in absolute value by $C_B$. Then

$$\left| \int_{u(\mathcal{U}_t)} \omega \right| \leq \sqrt{\frac{L}{\delta}} \sqrt{\frac{1+\mu}{1-\mu}} \|\omega|_{u(\mathcal{U})}\|_2 + \begin{cases} 0 & \text{if } \mathcal{U} = \mathbb{R}/L\mathbb{Z} \times [0, \delta] \\ \delta C_B & \text{if } \mathcal{U} = [0, L] \times [0, \delta] \end{cases}$$

**Proof:** Let $\omega' = u^* \omega = S(s,t) ds + T(s,t) dt$ be the pull-back of $\omega$ by $u$. By Stokes’ Theorem,
for every $\tau \in [0, \delta]$

\[
\int_{\mathcal{U}(\mathcal{U}_0)} \omega = \int_{\mathcal{U}_0} \omega' \\
= \int_{\{0\} \times [0, \tau]} \omega' + \int_{\mathcal{U}, \tau} \omega' - \int_{\{L\} \times [0, \tau]} \omega' \\
= \int_0^L S(s, \tau) \, ds + \int_0^\tau T(0, t) \, dt - \int_0^\tau T(L, t) \, dt
\]

Averaging over $\tau$

\[
\int_{\mathcal{U}(\mathcal{U}_0)} \omega = \frac{1}{\delta} \int_0^\delta \int_0^L S(s, \tau) \, ds \, d\tau + \frac{1}{\delta} \int_0^\delta \left[ \int_0^\tau T(0, t) \, dt - \int_0^\tau T(L, t) \, dt \right] \, d\tau
\]

If $\mathcal{U} = \mathbb{R}/L \mathbb{Z} \times [0, \delta]$, the second term is exactly zero, while if $\mathcal{U} = [0, L] \times [0, \delta]$, it is bounded by

\[
\frac{1}{\delta} \int_0^\delta \int_0^\tau 2C_B \, dt \, d\tau = \frac{1}{\delta} \int_0^\delta 2C_B \tau \, d\tau = C_B \delta
\]

By the Cauchy-Schwarz inequality the first term is bounded by

\[
\frac{1}{\delta} \int_0^\delta \int_0^L S(s, \tau) \, ds \, d\tau \leq \frac{1}{\delta} \|S\|_{L^2(\mathcal{U})} \sqrt{L \delta} \\
\leq \sqrt{L/\delta} \|\omega'\|_{L^2(\mathcal{U})} \\
\leq \sqrt{L/\delta} \sqrt{\frac{1+\mu}{1-\mu}} \|\omega\|_{L^2(\mathcal{U}(\mathcal{U}))}
\]

by Lemma 22.9.

For $j \in \mathcal{J}_c = \mathcal{J} \cup \{1, \ldots, g\}$ and $i \in \mathcal{J}$, define

\[
M^j_i = \left\| (\omega^j - F^* \omega^j_i) \right\|_{Y^{\delta, \mu}} \\
M^j_{com} = \left\| (w^j - w^j) \, dz \right\|_T \\
D^j_i = \sup_{z \in \Gamma} |w^j(z) - w^j(z)|
\]
Lemma 22.12

\[
\|\Omega^j\|_2, \|\Omega'^j\|_2 \leq \text{const } K^2 \quad \text{for all } j \geq 1
\]

\[
\|\omega_j\|_{L^2(X^{com})}, \|\omega'_j\|_{L^2(X^{com})} \leq \text{const } K^2 \quad \text{for all } j \geq 1
\]

\[
\|\Omega^j\|_2, \|\Omega'^j\|_2 \leq \epsilon K \quad \text{for } j \notin J_c
\]

\[
\left\| \left( \Omega^j \right)_{i \notin J} \right\|_2, \left\| \left( \Omega'^j \right)_{i \notin J} \right\|_2 \leq \epsilon K^2 \quad \text{for all } j \geq 1
\]

\[
\|M^j\|_2 \leq \text{const } \epsilon K^3 \quad \text{for } j \in J_c
\]

\[
\left\| \left( \omega_j - F^* \omega'_j \right) \right\|_{X(\Gamma)} \leq \text{const } \epsilon K^3 \quad \text{for } j \in J_c
\]

Proof: We first prove the first two lines for \(1 \leq j \leq g\). Let \(u_j\) be the quasiconformal map of Definition 22.2 part (vi). By the Riemann period relations and Lemma 22.11

\[
\|\omega_j\|^2 = \text{Im} \int_{B_j} \omega_j \leq \sqrt{\frac{L_j}{k_j}} \sqrt{\frac{3/2}{1/2}} \|\omega_j\| \leq \sqrt{3} K \|\omega_j\|
\]

Therefore

\[
\|\omega_j\| \leq \sqrt{3} K
\]

which give the first two lines for \(\omega_j\). To get the first two lines for \(\omega'_j\), it suffices to replace \(\omega_j\) by \(\omega'_j\), \(B_j\) by \(B'_j\) and \(u_j\) by \(u_j \circ F\) in the above argument. Note that, since \(u_j\) and \(F\) are quasiconformal with Beltrami coefficients at most \(\frac{1}{2}\) and \(\frac{1}{4}\) respectively, the composition \(u_j \circ F\) is quasiconformal with Beltrami coefficient at most \(\frac{6}{7}\).

Next we prove the first three lines for \(j \geq g + 1\). The bounds on the first line follow from (8.24) and Definition 22.2 parts (iii,v) via

\[
\|\bar{\Omega}\| \leq |\bar{\Omega}_1^j| + 2 \left( \|\bar{\Omega}^j\| + \|\mathcal{A}_{i j} \mathcal{N}_{j} \|_{i > g} \right)
\]

\[
\leq K + 2(K + K^2) \leq 4K^2
\]

\[
\|\bar{\Omega}'\| \leq 3 |\bar{\Omega}_1^j| + 2 \left( \|\bar{\Omega}^j\| + \|\mathcal{A}_{i j} \mathcal{N}_{j} \|_{i > g} \right)
\]

\[
\leq 3K + 2(K + K^2) \leq 6K^2
\]

and the analogous bounds on the primed quantities. We have used \(\|\bar{\Omega}^j\| = \sqrt{\bar{\Omega^j}} \leq K\). The third line follows similarly using part (iv) of Definition 22.2.
To get the second line we apply (8.22) to give
\[ \|\omega_j\|_{L^2(X)} \leq \|\omega_j\|_{L^2(X)} \leq |\Omega^j_k| + \frac{1}{2} \|\Omega^j_k\| \leq K + 4K^2 \leq 5K^2 \]

We now move on to the fourth line for \( j \in \mathcal{J}_c \). By (8.18), for \( j \in \mathcal{J} \) and the analogous inequality for \( 1 \leq j \leq g \)
\[ \left\| \left( \Omega^j_i \right)_{i \notin \mathcal{J}} \right\|_{\Phi} \leq \left\| \left( \mathcal{A}_{i,k} \right)_{i \notin \mathcal{J}} \right\| \left\| \Omega^j_k \right\|_{\Phi} = \left\| \left( \Omega^j_k \right)_{k \notin \Phi} \right\|_2 + \left\| \left( \mathcal{A}_{i,k} \right)_{i \notin \mathcal{J}} \right\|_2 \]

\[ \leq 8\epsilon K^2 + 2 \left( \Omega^j_k \right)_{k \notin \Phi} + \epsilon + K \epsilon \]

Here \( i, k \notin \mathcal{J} \) includes \( i, k = \text{com} \). For the second term we used the fact that \( i \notin \mathcal{J} \) implies \( Y^{(o)}(\mathcal{J}) \cap X_\Gamma = \emptyset \). The desired result follows.

That just leaves the last two lines. Recall that, by (22.1) of Lemma 22.6
\[ M_i^j \leq D_i^j + \epsilon \text{ const } K \left( \Omega_i^j + \delta_{ij} \right) \quad (22.11) \]
for all \( i \in \mathcal{J}, j \in \mathcal{J}_c \). As in (8.16), (8.18) and (22.10)
\[ D_i^j \leq \mathcal{A}_{i,\text{com}} M_i^j + \sum_{k \in \mathcal{J}} \mathcal{A}_{i,k} M_i^j + \sum_{k \notin \mathcal{J}} \mathcal{A}_{i,k} \Omega_i^j + \sum_{k \notin \mathcal{J}} \mathcal{A}_{i,k} \Omega_i^j \quad (22.12) \]
\[ D_i^j \leq \mathcal{A}_{\mathcal{J}_c,\text{com}} M_i^j + \sum_{k \notin \mathcal{J}} \mathcal{A}_{\mathcal{J}_c,k} M_i^j + \sum_{k \notin \mathcal{J}} \mathcal{A}_{\mathcal{J}_c,k} \Omega_i^j + \sum_{k \notin \mathcal{J}} \mathcal{A}_{\mathcal{J}_c,k} \Omega_i^j \quad (22.13) \]

where we recall that
\[ \mathcal{A}_{i,\text{com}} = \sup_{\mu = 1,2} \frac{4\pi \gamma \rho^\mu(i)}{\text{dist}(s^\mu(i),\Gamma)^2} \]
and define
\[ \mathcal{A}_{\mathcal{J}_c,\text{com}} = \frac{4\gamma}{\text{dist}(\Gamma)^2} \]
\[ \mathcal{A}_{\mathcal{J}_c,k} = \sup_{\mu = 1,2} \frac{6\rho^\mu(k)}{\text{dist}(s^\mu(k),\Gamma)^2} \]

As in (8.20)
\[ (M_{i,\text{com}}^j)^2 + \sum_{i \geq g+1} \left( M_{i}^j \right)^2 \leq \left\| (\omega_j - F^* \omega_j) \right\|_{X_\Gamma}^2 \]
Let $\beta$ be the $(0,1)$ part of $\omega_j - F^*\omega'_j$. It is the same as the $(0,1)$ part of $-F^*\omega'_j$ and vanishes on $X_{\text{reg}} \cap X_0$. By Lemma 22.10, Lemma 22.9 and Lemma 22.6

$$
\left\| \left(\omega_j - F^*\omega'_j\right)\right\|_{2_{X(\Gamma)}} \leq 2\left\|\beta\right\|_{L^2(X(\Gamma))} + \int_{\Gamma} |w_j - w'_j| |dz|
\leq 2\left\|\beta\right\|_{L^2(X(\Gamma))}^2 + \sum_{y_{(1)}^{(1)} \subseteq X(\Gamma), y_{(1)}^{(1)} \subseteq X(\Gamma)} \left\|\beta\right\|_{L^2(Y_{(1)}^{(1)})}^2 \int_{\Gamma} |w_j - w'_j| |dz|
\leq \text{const} \epsilon \left[\left\|\omega'_j\right\|_{L^2(X_{\text{com}})}^2 + \sum_{y_{(1)}^{(1)} \subseteq X(\Gamma), y_{(1)}^{(1)} \subseteq X(\Gamma)} \left(\Omega_{y_{(1)}^{(1)}}^j + \delta_{ij}\right)^2 \right]^{1/2} \text{length }\Gamma D_i^j
\leq \text{length }\Gamma D_i^j + \text{const }\epsilon K^2
$$

(22.14) by the first two lines of the current Lemma.

As in §8, we use $\mathbf{V}$ to denote the vector having components $V_{\text{com}}$ and $V_i$ with $i \geq g+1$, $Y_{(1)}^{(1)} \subseteq X(\Gamma)$ and $\mathbf{V}$ to denote the vector having components $V_i$ with $i \in \mathcal{J}$, $Y_{(1)}^{(1)} \subseteq X(\Gamma) = \emptyset$. In this notation, the conclusion of the last paragraph is that

$$
\|M^j\| \leq \text{length }\Gamma D_i^j + \text{const }\epsilon K^2
$$

By (22.11) and the first line of the current Lemma

$$
\|\mathbf{M}^j\| \leq \|\mathbf{D}^j\| + \text{const }\epsilon K^3
$$

Definition 22.2 (v) implies that $\|\mathbf{A}V\| \leq \frac{1}{4}\|V\|$. So, by (22.12) and (22.13)

$$
\|\mathbf{D}^j\| \leq \frac{1}{4}\|M^j\| + \text{const }\epsilon K^3
$$

$$
\text{length }\Gamma D_i^j \leq \frac{1}{4}\|M^j\| + \text{const }\epsilon K^2
$$

Hence

$$
\|M^j\| \leq \text{const }\epsilon K^3
$$

$$
\left\|(\omega_j - F^*\omega'_j)\right\|_{2_{X(\Gamma)}} \leq \text{const }\epsilon K^3
$$

\[\square\]
Lemma 22.13

a) Define

\[ U_R = \{ z \in \Phi^{-1}(X^{\text{reg}}) \mid \frac{\text{dist}(z,T)^2}{1+|z|^2} \geq \frac{2}{K} \text{ and } |z-s| \geq r(s), \ |z-s'| \geq r(s') \ \forall \ s \in S, \ s' \in S' \} \]

\[ U_r = \{ z \in \Phi^{-1}(X^{\text{reg}}) \mid \frac{\text{dist}(z,T)^2}{1+|z|^2} \geq \frac{2}{K} \text{ and } |z-s| \geq 4r(s), \ |z-s'| \geq 4r(s') \ \forall \ s \in S, \ s' \in S' \} \]

There is a universal constant independent of \( X, X' \) etc. such that for all \( j \in \mathcal{J}_c, \ z \in U_R \)

\[ |w_j(z) - w'_j(z)| \leq \text{const} \frac{\epsilon K^4}{1+|z|^2} \]

If \( z \in U_r \setminus U_R \) then there exists an \( s \in S \) and/or an \( s' \in S' \) such that \( |z-s| \leq r(s) \) and/or \( |z-s'| \leq r(s') \). Then, for all \( j \in \mathcal{J}_c \)

\[ |w_j(z) - w'_j(z)| \leq \text{const} \epsilon K^3 \left( \frac{r(s)}{|z-s|^2} + \frac{r'(s')}{|z-s'|^2} \right) + \text{const} \frac{\epsilon K^4}{1+|z|^2} \]

b) For \( j \notin \mathcal{J}_c \)

\[ \left| w_j(z) - \frac{1}{2\pi i} \frac{1}{z-s_1(j)} + \frac{1}{2\pi i} \frac{1}{z-s_2(j)} \right| \leq \text{const} \frac{\epsilon K^2}{1+|z|^2} \]

if \( \frac{\text{dist}(z,T)^2}{1+|z|^2} \geq \frac{2}{K} \) and \( |z-s| \geq r(s) \) for all \( s \in S \). If for some \( s \in S \), \( 4r(s) \leq |z-s| \leq r(s) \) then

\[ \left| w_j(z) - \frac{1}{2\pi i} \frac{1}{z-s_1(j)} + \frac{1}{2\pi i} \frac{1}{z-s_2(j)} \right| \leq \text{const} \epsilon K \left( \frac{r(s)}{|z-s|^2} \right) + \text{const} \frac{\epsilon K^2}{1+|z|^2} \]

Similar bounds apply to \( w'_j(z) \).

c) For \( i \in \mathcal{J}, \ j \notin \mathcal{J}_c \)

\[ \left| \frac{\phi_i^*(w_j)}{dz_1/z_1} \right| \leq \text{const} \epsilon K (|z_1| + |z_2|) \quad \text{if } |z_1|, |z_2| \leq 1/4 \]

\[ \left| \frac{\phi_i^* F_\epsilon w'_j}{dz_1/z_1} \right| \leq \text{const} \epsilon K (|z_1| + |z_2|) \quad \text{if } |z_1|, |z_2| \leq 1/4 \]
\[ |w_j(z) - w'_j(z)| \leq |w_{j,\text{com}}(z) - w'_{j,\text{com}}(z)| + \sum_{k \in \mathcal{J}, \mu=1,2} |w_{j,s\mu(j)}(z) - w'_{j,s\mu(j)}(z)| \]
\[ + \sum_{k \in \mathcal{J}, \mu=1,2} |w_{j,s\mu(j)}(z)| + \sum_{k \in \mathcal{J}, \mu=1,2} |w'_{j,s\mu(j)}(z)| \]
\[ \leq \frac{\gamma}{\text{dist}(z, T)^2} M^j_{\text{com}} + \sum_{k \in \mathcal{J}, \mu=1,2} \frac{3r_{\mu}(k)}{|z - s\mu(k)|^2} M^j_k \]
\[ + \sum_{k \in \mathcal{J}, \mu=1,2} \frac{3r_{\mu}(k)}{|z - s\mu(k)|^2} \Omega^j + \sum_{k \in \mathcal{J}, \mu=1,2} \frac{3r'_{\mu}(k)}{|z - s'\mu(k)|^2} \Omega'^j \]
\[ \leq \text{const} \epsilon K^3 \left( \frac{\gamma}{\text{dist}(z, T)^2} + \sum_{s \in S} \frac{r(s)}{|z - s|^2} + \sum_{s' \in S'} \frac{r'(s')}{|z - s'|^2} \right) \]

by Lemma 22.12. As in (8.3)
\[ \frac{\gamma}{\text{dist}(z, T)^2} + \sum_{s \in S} \frac{r(s)}{|z - s|^2} + \sum_{s' \in S'} \frac{r'(s')}{|z - s'|^2} \leq \text{const} \frac{K}{1 + |z|^2} \]

The claim now follows from part (iii) of Definition 22.2.

b) follows from Proposition 8.12 and Lemma 22.12.

c) The first line follows from Lemma 8.8, applied to the scaled handle \( H(4t_i) \), and Lemma 22.12. To prove the second line observe that, in the notation of Lemma 22.6, \( \phi^*_i F^* \omega'_j = g'(u_1) \left( a(z_1) \frac{dz}{z_1} + b(z_1) \frac{dz}{z_1} \right) \) with \( g'(u_1) \) estimated in (22.6a) and (22.4). Definition 22.2 (i) and Definition 22.1 provide bounds on \( a(z_1) \) and \( b(z_1) \). \[ \Box \]

**Proof of Theorem 22.4:** Part (a) has been proven in (22.5). For part (b), first consider \( i \in \mathcal{J} \). The cycle \( B_i \) can be represented as the union of \( h_i \cup \Phi(b_i) \) with \( \sum_{i', \mu} c_{i',i} A_{i'} \), where
\[ h_i = \phi_i \left( \{ (z_1, z_2) \in H(t_i) \mid z_1 > 0 \} \right) \cap Y_{i}^{(s)} \]
and \( b_i \) is a path in \( U_R \) with the property that
\[ \text{length} \left\{ z \in b_i \mid |z| < \rho \right\} \leq \text{const} \rho \quad \text{for all} \quad \rho > 0 \]

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The integral of \( \omega_j - F^* \omega'_j \) over each \( A \)-cycle \( A_i \) is zero. By the first statement of Lemma 22.13(a)

\[
\left| \int_{\Phi(b_i)} \omega_j - \int_{\Phi'(b_i)} \omega'_j \right| \leq \int_{b_i} \left| \left( w_j(z) - w'_j(z) \right) dz \right| \leq \text{const} \epsilon K^4
\]

By the second statement of Lemma 22.13(a)

\[
\left| \int_{h_i \setminus \Omega_{\infty}^i} (\omega_j - F^* \omega'_j) \right| \leq \text{const} \epsilon K^3 \sum_{\mu=1}^{2} \int_{R_{\mu}(j)}^{2} \text{const} \epsilon K^4 \leq \text{const} \epsilon K^4
\]

Furthermore, by the pointwise estimate of Lemma 22.6, the bound on \( D_i^j \) arising from Lemma 22.13(a) and the estimate on \( \Omega_{\infty}^i \) of Lemma 22.12,

\[
\left| \int_{h_i \cap \Omega_{\infty}^i} (\omega_j - F^* \omega'_j) \right| \leq \text{const} \epsilon K^4
\]

This proves part (b) of the Theorem when \( i \in \mathcal{J} \).

Now fix any \( 1 \leq i \leq g \). By Definition 22.2 part (vi), there exist \( L_i, \delta_i \) obeying \( \sqrt{L_i/\delta_i} \leq K \) and a quasiconformal diffeomorphism \( u_i \), with Beltrami coefficient bounded by \( 1/2 \), from \( \mathcal{U} = \mathbb{R}/L_i \mathbb{Z} \times [0, \delta_i] \) into \( X^{\text{com}} \) with \( u_i(\mathcal{U}_0) = B_i \). Then, by Lemmas 22.11 and 22.12

\[
\left| \int_{B_i} \omega_j - F^* \omega'_j \right| \leq \sqrt{\frac{L_i}{\delta_i}} \sqrt{\frac{3/2}{1/2} \epsilon K^3} \leq \text{const} \epsilon K^4
\]

We now prove part (c). Recall that, by part (i) and the first bullet of part (v) of Definition 22.2 and by Definition 22.1, there is a cover

\[
X^{\text{com}} \cup \Phi \{ z \in G \mid \frac{\text{dist}(z,T)}{1+|z|} \leq \frac{2}{K} \} \subset \bigcup_{i=1}^{N} D_i \cup \bigcup_{\ell=1}^{r} H_{\ell} \subset X(\Gamma)
\]

sets

\[
\{ (z_1, z_2) \in H(t_\ell) \mid \left(1 - \frac{1}{K}\right)^{-1} t_\ell \leq |z_1| \leq 1 - \frac{1}{K} \} \subset \hat{Y}_\ell \subset H(t_\ell) \quad 1 \leq \ell \leq r
\]

\[
\{ (z_1, z_2) \in H(t'_\ell) \mid \left(1 - \frac{1}{K}\right)^{-1} t'_\ell \leq |z_1| \leq 1 - \frac{1}{K} \} \subset \hat{Y}'_\ell \subset H(t'_\ell) \quad 1 \leq \ell \leq r
\]

and biholomorphic maps

\[
\Phi_i : \{ z \in \mathbb{C} \mid |z| < 1 \} \rightarrow D_i \quad 1 \leq i \leq N
\]

\[
\psi_\ell : H(t_\ell) \rightarrow H_{\ell} \quad 1 \leq \ell \leq r
\]
By the Cauchy integral formula, for \(1 \leq i \leq N\)

\[
\Phi_i^{-1} \circ F \circ \Phi_i \text{ is biholomorphic}
\]

\[
\psi_{\ell}^{-1} \circ F \circ \psi_{\ell} \mid \hat{Y}_{\ell} \text{ is } K\text{-quasiconformal of distortion at most } \epsilon \quad 1 \leq \ell \leq r
\]

Furthermore we may choose this cover such that \(x_1\) is joined to by \(x_2\) by a curve that is a union of a finite number (depending only on \(K\)) of pieces with the image under the universal covering \(\pi\) of each piece being of one of the four following types:

- the image under some \(\phi_k\) of a line segment in \(Y_k\)
- the image under \(\Phi\) of a line segment in \(U_R\)
- the image under some \(\Phi_i\) of a line segment in \(\{ z \in \mathbb{C} \mid |z| \leq 1/2 \}\)
- the image under some \(\psi_{\ell}\) of a line segment in \(\{ (z_1, z_2) \in H(t_{\ell}) \mid |z_1|, |z_2| \leq 1 - \frac{20}{15K} \}\)

Pieces of the first two types were treated in part (b).

Pieces of the third type are bounded using Lemma 8.9(a). Because \(F \circ \Phi_i\) is holomorphic on the unit disk, the pullback

\[
\Phi_i^* \left( \omega_j - F^* \omega_j' \right) = w_{i,j}(z) dz
\]

is a holomorphic form with \(L^2\) norm bounded by \(\| (\omega_j - F^* \omega_j') \|_{X(\Gamma)} \leq \text{const} K^3\). Hence, by Lemma 8.9(a), \(|w_{i,j}(z)| \leq \text{const} K^3\) on \(\{ z \in \mathbb{C} \mid |z| \leq 1/2 \}\) and the integral of \(\Phi_i^* \left( \omega_j - F^* \omega_j' \right)\) along any line segment in \(\{ z \in \mathbb{C} \mid |z| \leq 1/2 \}\) obeys a similar bound.

Pieces of the fourth type are bounded using a variant of Lemma 22.6 similarly to pieces of the first type. By way of preparation we make some preliminary bounds. Because \(F \circ \psi_{\ell}\) is holomorphic on \(|z_1| > 1 - \frac{28}{15K}\) and on \(|z_2| > 1 - \frac{28}{15K}\), the pullback

\[
\psi_{\ell}^* \left( \omega_j - F^* \omega_j' \right) = \alpha_{\ell,j}(z_1) \frac{dz_1}{z_1} + \beta_{\ell,j}(z_1) \frac{dz_1}{z_1}
\]

restricted to these two neighbourhoods is a holomorphic form (that is, \(\beta_{\ell,j} = 0\) and \(\alpha_{\ell,j}\) is holomorphic) with \(L^2\) norm bounded by \(\| (\omega_j - F^* \omega_j') \|_{X(\Gamma)} \leq \text{const} K^3\). Define

\[
D_{\ell,j} \equiv \sup_{|z_1|=1-\frac{18}{15K}} |\alpha_{\ell,j}(z)| + \sup_{|z_2|=1-\frac{18}{15K}} |\alpha_{\ell,j}(z)|
\]

By the Cauchy integral formula, for \(|z_1| = 1 - \frac{17}{15K}\),

\[
\alpha_{\ell,j}(z_1) = \frac{1}{2\pi i} \int_{|\zeta|=1-\frac{18}{15K}} \frac{\alpha_{\ell,j}(\zeta)}{\zeta - z_1} d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=1-\frac{18}{15K}} \frac{\alpha_{\ell,j}(\zeta)}{\zeta - z_1} d\zeta
\]
By Lemma 8.9(a),

\[
\frac{1}{2\pi i}\int_{|\zeta|=1-\frac{18}{15K}} \frac{\alpha_{\ell,j}(\zeta)}{\zeta-z_1} d\zeta \leq \text{const } \epsilon K^{3/2} \quad \| (\omega_j - F^*\omega'_j) \|_X(\Gamma) \|_2 \leq \text{const } \epsilon K^{9/2}
\]

Applying a similar argument for the integral over \(|\zeta| = 1 - \frac{18}{15K}\) and then twice more for \(|z_2| = 1 - \frac{17}{15K}\)

\[D_{\ell,j} \leq \text{const } \epsilon K^{9/2}\]

As the image under \(\psi\_\ell\) of the circle \(|z_1| = \sqrt{t_\ell}\) is homologous to a finite linear combination of the cycles \(A_j, 1 \leq j \leq g\), we have that

\[
\begin{align*}
\int_{|z_1|=\sqrt{t_\ell}} \psi^*_\ell (\omega_j - F^*\omega'_j) & = 0 \quad \text{for all } j \geq 1 \\
\int_{|z_1|=\sqrt{t_\ell}} \psi^*_\ell \omega_j & = 0 \quad \text{for all } j > g \\
\left| \int_{|z_1|=\sqrt{t_\ell}} \psi^*_\ell \omega_j \right| & \leq C'_{K_c} \quad \text{for all } 1 \leq j \leq g
\end{align*}
\]

In Lemma 22.12 it was proven that

\[
\| \psi^*_\ell \omega_j |_{\tilde{Y}_\ell} \|_2 \leq \| \omega_j \|_{X(\Gamma)} \|_2 \leq \text{const } K^2
\]

Similarly,

\[
\| (\psi^*_\ell)^* \omega'_j |_{\tilde{Y}'_\ell} \|_2 \leq \text{const } K^2
\]

Using the above preliminary estimates, we have, as in Lemma 22.6,

\[
\begin{align*}
|\alpha_{\ell,j}(z_1)| & \leq K \left( |z_1| + \left| \frac{t_\ell}{z_1} \right| \right) D_{\ell,j} + \text{const } \epsilon K^{5/2} \frac{1}{t_\ell} \left( \| (\psi^*_\ell)^* \omega'_j \|_{\tilde{Y}'_\ell} \|_2 + C'_{K_c} \right) \\
|\beta_{\ell,j}(z_1)| & \leq \text{const } \epsilon K^{3/2} \left( |z_1| + \left| \frac{t_\ell}{z_1} \right| \right) \left( \| (\psi^*_\ell)^* \omega'_j \|_{\tilde{Y}'_\ell} \|_2 + C'_{K_c} \right)
\end{align*}
\]

for \(|z_1|, |\frac{t_\ell}{z_1}| \leq 1 - \frac{20}{15K}\). Thus, on this domain,

\[
\begin{align*}
|\alpha_{\ell,j}(z_1)| & \leq \text{const } K^{11/2} \\
|\beta_{\ell,j}(z_1)| & \leq \text{const } K^{5/2}
\end{align*}
\]

and part (c) follows.
We prove the bound of part (d) on $|W_j - W'_j|$, $j \in J$. The remaining bounds are proven similarly. From Proposition 21.1 and (21.2,3) and the Cauchy integral formula we have that

$$W_j = w_{j,\Gamma}^{(4)} + \sum_{s \in S} w_{j,s}^{(4)}$$

where

$$w_{j,s}^{(n)} = -\frac{i^{1-n}}{2\pi i} \int_{|z-s|=r(s)} \zeta^{n-1} w_j(\zeta) d\zeta$$

$$w_{j,\Gamma}^{(n)} = -\frac{i^{1-n}}{2\pi i} \int_{\Gamma} \zeta^{n-1} w_j(\zeta) d\zeta$$

The proof now continues as in the simple single sheet case.

Corollary 22.14 (Solutions of the KP equation) Let $X = X^{\text{com}} \cup X^{\text{reg}} \cup X^{\text{han}}$ be a marked Riemann surface that fulfils (GH1-GH5) with one regular sheet ($m = 1$). Assume that

$$\sup_j (\tau_1(j)^2 + \tau_2(j)^2) |\log t_j|^2 \ln \frac{\tau_1(j)\tau_2(j)}{t_j} < \infty$$

and that

$$\lim_{j \to \infty} \frac{1}{|\log t_j|} (s_1(j)^n - s_2(j)^n) = 0 \quad \text{for } n = 1, 2, 3$$

Then there is a constant $c$ such that for every $e \in B$

$$u(x_1, x_2, t) = -2 \frac{\partial^2}{\partial x_2^2} \ln \theta(Ux_2 + Vx_1 - \frac{1}{2} Wt + e) + c \quad (22.15)$$

solves the KP equation

$$(ut - 3uu_{x_2} + \frac{1}{2} u_{x_1}x_2x_2)_{x_2} + \frac{3}{2} u_{x_1}x_1 = 0 \quad (22.16)$$

whenever $\theta(Ux_2 + Vx_1 - \frac{1}{2} Wt + e) \neq 0$.

Proof: Fix $R > 0$. By the holomorphicity of $\theta$, it suffices to prove that there exists a constant $c$ such that the expression (22.15) satisfies the KP equation for all $e, t, x_1$ and $x_2$ obeying $\|Ux_2\|, \|Vx_1\|, \|Wt\|, \|e\| < R$.
We denote by $\mathcal{R}$ the period matrix $\mathcal{R}_{ij} = \int_{B_i} \omega_j$ of $X$, so $\theta(z) = \theta(z, \mathcal{R})$. For $\mathcal{J} \subset \{1, 2, \cdots\}$ let $\theta_{\mathcal{J}}(z, \mathcal{R})$ be the truncated theta function

$$
\theta_{\mathcal{J}}(z, \mathcal{R}) = \sum_{n \in \mathbb{Z}^\infty, n_j = 0 \text{ if } j \notin \mathcal{J}} e^{2\pi i(z,n)} e^{\pi \langle n, \mathcal{R} n \rangle}
$$

Now fix $\epsilon > 0$. Since the series for $\theta(z, \mathcal{R})$ converges uniformly on $B_{5R}$ there is $N > 0$ such that for all $\mathcal{J} \subset \mathbb{N}$ with $\{1, 2, \cdots, N\} \subset \mathcal{J}$

$$
|\theta(z, \mathcal{R}) - \theta_{\mathcal{J}}(z, \mathcal{R})| < \epsilon \quad \text{for } z \in B_{5R}
$$

Furthermore there is $\delta$ such that for $\{1, 2, \cdots, N\} \subset \mathcal{J}$

$i)$ $|\theta(z, \mathcal{R}) - \theta_{\mathcal{J}}(z, \mathcal{R}')| < \epsilon \quad \text{for } z \in B_{5R}$

if $|\mathcal{R}_{ij} - \mathcal{R}'_{ij}| < \delta \quad \text{for } i, j \in \mathcal{J}$

$ii)$ $|\theta(z, \mathcal{R}) - \theta_{\mathcal{J}}(z', \mathcal{R})| < \epsilon$

if $z, z' \in B_{5R}$ with $|z_j - z'_j| < \delta$ for $j \in \mathcal{J}$

Now, by Proposition 22.3 and Theorem 22.4, there exist

- a compact submanifold (with boundary) $X_0$ in $X$ containing the image of $\mathcal{K}$
- a marked Riemann surface $X' = X'_{\text{com}} \cup X'_{\text{reg}} \cup X'_{\text{han}}$ of genus $\text{genus}(X_0)$
- a compact submanifold $X'_0 \subset X'$ and a diffeomorphism $F : X_0 \to X'_0$

such that

$$
\mathcal{J} = \{ j \mid Y_j \subset X_0 \} \text{ contains } \{1, 2, \cdots, N\}
$$

the period matrix $\mathcal{R}'$ of $X'$ fulfills $|\mathcal{R}'_{ij} - \mathcal{R}_{ij}| < \delta$ for all $i, j \in \mathcal{J}$

$$
\|U_{x_2} + V_{x_1} - \frac{1}{2} W t - U'_{x_2} - V'_{x_1} + \frac{1}{2} W' t\| < \delta \quad \text{for all } t, x_1 \text{ and } x_2 \text{ obeying}
$$

$$
\|U_{x_2}\|, \|V_{x_1}\|, \|W t\| < R.
$$

Consequently, for any $e, t, x_1$ and $x_2$ obeying $\|U_{x_2}\|, \|V_{x_1}\|, \|W t\| < R$ and $\|e\| < 2R$

$$
|\theta(U_{x_2} + V_{x_1} - \frac{1}{2} W t + e; \mathcal{R}) - \theta(U'_{x_2} + V'_{x_1} - \frac{1}{2} W' t + e; \mathcal{R}')| < \epsilon
$$

By [MII, p.329], for each such $X'$ there is a constant $c'$ such that

$$
u'(x_1, x_2, t) = -2\frac{\partial^2}{\partial x_2^2} \ln \theta(U'_{x_2} + V'_{x_1} - \frac{1}{2} W' t + e; \mathcal{R}')$$

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solves

\[(u_t' - 3u'u_{x_2}'' + \frac{1}{2}u_{x_1x_2x_2}'))_{x_2} + \frac{3}{2}u_{x_1x_1}x_2 = 3\partial u'_{x_2x_2}
\]

We can choose a sequence of \(c\)'s and approximating Riemann surfaces \(X'\) such that the corresponding \(c\)'s converge to some value \(c\), which is possibly infinite. Then \(u' + c'\) converges to \(u\).

If \(c\) is finite, then, by the Cauchy integral formula, we get the desired equation. Assume now that \(c\) is infinite. Then \(u_{x_2x_2} = 0\). That is

\[\frac{\partial^4}{\partial x^4} \ln \theta(e + UX_2) = 0 \quad \text{for all } e \in B\]

Therefore, for each \(e \in B\), the line

\[\{ e + x_2U \mid x_2 \in \mathcal{C} \}\]

is either completely contained in the theta-divisor \(\Theta\), or does not meet it at all. So, for all smooth points \(e\) of \(\Theta\) the vector \(U\) lies in the tangent space \(T_e \Theta\) of \(\Theta\) at \(e\). Therefore \(U\) also lies in the kernel of the second derivative of \(\theta\) at any such point. That is,

\[U \in \ker H(e) \quad \text{for all } e \in \Theta_{\text{reg}}\]

As \(\ker H(z) = 0\) for \(z\) in a dense subset of \(\Theta_{\text{reg}}\) (Corollary 11.9 and Lemma 11.4a), this implies that \(U = 0\). Then \(u\) is a constant and the KP equation is trivially satisfied. ■

**Remark.** In a similar way one can show that for any marked Riemann surface \(X = X^{\text{com}} \cup X^{\text{reg}} \cup X^{\text{han}}\) that fulfills (GH1-GH5) with one regular sheet \((m = 1)\) and with

\[
\sup_j \left( \tau_1(j)^2 + \tau_2(j)^2 \right) \ln |t_j|^2 \ln \frac{\tau_1(j)\tau_2(j)}{t_j} < \infty
\]

any point \(x\) and any local coordinate \(\zeta\) about this point, formula (22.15) gives a solution of the KP equation for \(U, V, W\) defined by

\[
\omega_j = U_j d\zeta + V_j \zeta d\zeta + \frac{1}{2}W_j \zeta^2 d\zeta + O(\zeta^3) \quad \text{near } x
\]

as in the beginning of §21.
The main result in this section is

**Theorem 23.1** Let \( q \in L^2(\mathbb{R}^2/\Gamma) \) be a real analytic potential satisfying
\[
\int_0^{2\pi} q(x_1, x_2) \, dx_2 = 0 \quad \text{for all } x_1 \in \mathbb{R}.
\]
Assume that its associated heat curve \( \mathcal{H}(q) \) is smooth. Let \( \theta \) be the theta function of \( \mathcal{H}(q) \) on the torus
\[
\mathcal{T} = (\mathbb{R}/\mathbb{Z})^\infty
\]
with metric
\[
d(x, y) = \inf_{n \in \mathbb{Z}^\infty} \|x - y - n\|
\]
Then there exist \( e \in \mathcal{T} \) and \( U, V, W \in \mathbb{R}^\infty \) such that
\[
u(x_1, x_2, t) = -2\frac{\partial^3}{\partial x_2^3} \ln \theta(e + Ux_2 + Vx_1 - \frac{1}{2}Wt)
\]
is a \( C^\infty \) function on \( \mathbb{R}^3 \) which solves the KP equation
\[
u_t = 3\nu_{x_2} - \frac{1}{3}u_{x_2}x_2x_2 - \frac{3}{2}I(u_{x_1}, x_1) \quad (KP)
\]
and satisfies the initial condition
\[
u(x_1, x_2, 0) = q(x_1, x_2)
\]
Here, as before,
\[
I(u)(x_1, x_2) = \int_0^{x_2} u(x_1, s) \, ds - \frac{1}{\pi} \int_0^{2\pi} dt \int_0^t u(x_1, s) \, ds
\]

**Remark 23.2** In the event that \( \mathcal{H}(q) \) is singular, the results of the Theorem apply to the normalization of \( \mathcal{H}(q) \).

Theorem 23.1 and Remark 23.2 show that the initial value problem for periodic (KP) with real analytic initial data can be solved for all time. This had been shown by Krichever [K] and refined by Bourgain [B]. In addition the Theorem gives qualitative information on the solution. Namely, by Proposition 4.16,
Corollary 23.3 Let \( q \in L^2(\mathbb{R}^2/\Gamma) \) be a real analytic potential for which \( \int_0^{2\pi} q(x_1, x_2) \, dx_2 = 0 \) for all \( x_1 \in \mathbb{R} \). Then the solution of \((KP)\) with initial data \( q \) is almost periodic in time.

In preparation for the proof of Theorem 23.1, we investigate the structure of \( \mathcal{H}(q) \) for real potentials \( q \). Clearly \( \mathcal{H}(q) \) is invariant under the antiholomorphic involution

\[
\sigma : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^* \\
(\xi_1, \xi_2) \mapsto (\bar{\xi}_1^{-1}, \bar{\xi}_2^{-1})
\]

In \([K, \text{chI \S} 2]\), Krichever describes the structure of \( \mathcal{H}(q) \) in more detail.

Theorem 23.4 ([K, chI \S 2]) Let \( q \in L^2(\mathbb{R}^2/\Gamma) \) be a real analytic potential. Let

\[
\mathcal{H}_A(q) = \{ (\xi_1, \xi_2) \in \mathcal{H}(q) \mid \xi_1, \xi_2 \in \mathbb{R} \}
\]

Then \( \mathcal{H}_A(q) \) is the disjoint union of connected components \( a_0 \) and \( a_b, \ b \in \Gamma^#, \ b_2 > 0 \). For each \( b \in \Gamma^#, \ b_2 > 0 \), either \( a_b \) is diffeomorphic to a circle or \( a_b \) is a point. In the latter case \( a_b \) is an ordinary double point of \( \mathcal{H}(q) \). There is holomorphic map

\[
K_2 : \mathcal{H}(q) \setminus \left( \bigcup_{\substack{b \in \Gamma^# \\ b_2 > 0}} a_b \right) \to \mathbb{C}
\]

such that

\[
\xymatrix{ \mathcal{H}(q) \ar[rr]^{K_2} \ar[dr]_{(\xi_1, \xi_2)} & & \mathbb{C} \ar[dl]^{\Phi} \\
& \mathbb{C}^* \ar[dr]^{e^{2\pi i k_2}} & \\
& \xi_2 & \\
& \xi_2 \ar[ur]_{k_2} &}
\]

commutes. \( K_2 \) is biholomorphic to its image and \( K_2(a_0) \) is the imaginary axis. The image of \( K_2 \) is the complement of a set of disjoint “cuts” \( c_b(q), \ b \in \Gamma^#, \ b_2 \neq 0 \). Each cut \( c_b(q) \) is either a compact interval or a point on the line \( \{ k_2 \in \mathbb{C} \mid \text{Re } k_2 = -\frac{1}{2} b_2 \} \). The cut \( c_b(q) \) is the reflection of \( c_{-b}(q) \) across the imaginary axis. For each fixed \( b_2 \), the cuts \( c_b(q) \) are ordered along the line \( \{ k_2 \in \mathbb{C} \mid \text{Re } k_2 = -\frac{1}{2} b_2 \} \) according to \( b_1 \).
Let $\mathcal{M}(q)$ be the Riemann surface obtained from $\mathcal{C}$ by gluing, for each $b \in \Gamma^*$, $b_2 > 0$, the cut $c_b$ to the cut $c_{-b}$ using translation by $b_2$. Then $K_2$ induces a biholomorphic map from $\mathcal{H}(q)$ to $\mathcal{M}(q)$ that maps $a_b$ to the circle defined by $c_b$.

Let

$$D = \left\{ (\xi_1, \xi_2) \in \mathcal{H}(q) \mid \exists \psi(x_1, x_2) \neq 0 \text{ obeying } \psi(x_1 + \omega_1, x_2 + \omega_2) = \xi_1 \psi(x_1, x_2) \right\}$$

Then $D \subset \bigcup_{b \in \Gamma^*} a_b$ and each $a_b$ contains exactly one point of $D$, counted with multiplicity.

**Lemma 23.5** Let $\beta \geq 4$. Assume that $q \in L^2(\mathbb{R}^d/\Gamma)$ obeys $\hat{q}(0) = 0$ and $\|b\|^3 \hat{q}(b)\|_1 < \infty$. Then there is a constant $\text{const}$, depending only on $\|b\|^3 \hat{q}(b)\|_1$ such that for every $d \in \Gamma^*$ with $d_2 \neq 0$ and $|d| > \text{const}$

$$|v_d - s_d| \leq \frac{\text{const}}{|d|^{\beta}}$$

$$|w_d - s_d| \leq \frac{\text{const}}{|d|^{\beta}}$$

where $s_d = \hat{\phi}_d(0, 0)$ was defined in Theorem 16.2.

**Proof:** Define, as in the proof of Theorem 16.2,

$$x_1(k_1, k_2) = P_0(k) - \mathcal{D}_{1,1}(k_1, k_2) = ik_1 + k_2^2 - \mathcal{D}_{1,1}(k_1, k_2)$$

$$x_2(k_1, k_2) = P_d(k) - \mathcal{D}_{2,2}(k_1, k_2) = i(k_1 + d_1) + (k_2 + d_2)^2 - \mathcal{D}_{2,2}(k_1, k_2)$$
The functions $\mathcal{D}(k)_{i,j}$ were given in Proposition 16.5, where it was also shown that $k \in T_0 \cap T_d$ is on $\mathcal{H}(q)$ if and only if

$$x_1(k)x_2(k) = h(k)$$

where $h(k) = (\hat{q}(d) - \mathcal{D}_{2,1}(k))(\hat{q}(d) - \mathcal{D}_{1,2}(k))$

We shall shortly show that, for $|d| \geq \text{const}$, the equation

$$\frac{\partial}{\partial k_1}(x_1(k)x_2(k) - h(k)) = 0$$

has precisely two solutions in $T_0 \cap T_d$. These are the points $v_d$ and $w_d$.

In Proposition 16.5 and Lemmas 16.6 and 16.8, it was proven that, for $k \in T_0 \cap T_d$ and $m, n \geq 0,

$$\left| \frac{\partial^{n+m}}{\partial^n k_1 \partial^m k_2} \mathcal{D}_{i,j}(k) \right| \leq \begin{cases} \text{const} & \text{if } i = j \\ \frac{1 + |z_d|^{2\beta - m}}{|d|^m} \text{const} & \text{if } i \neq j \end{cases}$$

with the constant depending only on $n, m$ and $\|b^\beta \hat{q}(b)\|_1$. Consequently,

$$\frac{\partial x_1}{\partial k_1} = i - \frac{\partial \mathcal{D}_{1,1}}{\partial k_1} = i + O(|z_d|^{-2})$$

$$\frac{\partial x_2}{\partial k_1} = i - \frac{\partial \mathcal{D}_{2,2}}{\partial k_1} = i + O(|z_d|^{-2})$$

and

$$\frac{\partial}{\partial k_1}(x_1x_2 - h) = \left( i - \frac{\partial \mathcal{D}_{2,2}}{\partial k_1} \right) x_1 + \left( i - \frac{\partial \mathcal{D}_{1,1}}{\partial k_1} \right) x_2 - \frac{\partial h}{\partial k_1}$$

$$= (i + O(|z_d|^{-2})) x_1 + (i + O(|z_d|^{-2})) x_2 + O(|d|^{-2\beta})$$

First, substitute $k_1 = k_1(x_1, x_2)$ and $k_2 = k_2(x_1, x_2)$ in $h(k)$, $\frac{\partial \mathcal{D}_{2,2}}{\partial k_1}$, $\frac{\partial \mathcal{D}_{2,2}}{\partial k_1}$ and $\frac{\partial h}{\partial k_1}$ and think of

$$x_1x_2 = h$$

$$(i - \frac{\partial \mathcal{D}_{2,2}}{\partial k_1}) x_1 + (i - \frac{\partial \mathcal{D}_{1,1}}{\partial k_1}) x_2 - \frac{\partial h}{\partial k_1} = 0$$

as two equations in the two unknowns $x_1$ and $x_2$. As in the proof of Theorem 16.2

$$\left| \frac{\partial}{\partial x_1} \frac{\partial \mathcal{D}_{1,1}}{\partial k_1} \right| \leq \frac{\text{const}}{1 + |d_2z_d|}$$

$$\left| \frac{\partial}{\partial x_1} \frac{\partial \mathcal{D}_{2,2}}{\partial k_1} \right| \leq \frac{\text{const}}{1 + |d_2z_d|}$$

$$\left| \frac{\partial h}{\partial x_1} \right| \leq \frac{\text{const}}{1 + |d|^{2\beta - 1}}$$

$$\left| \frac{\partial}{\partial x_1} \frac{\partial h}{\partial k_1} \right| \leq \frac{\text{const}}{1 + |d|^{2\beta - 1}}$$
so that, by the implicit function theorem, the second equation has a unique solution \( x_2 = x_2(x_1) \) and this solution obeys

\[
x_2 = -\frac{i - \frac{\partial D_{x_2}}{\partial k_1}}{i - \frac{\partial D_{x_1}}{\partial k_1}} x_1 + \frac{1}{i - \frac{\partial D_{x_1}}{\partial k_1}} \frac{\partial h}{\partial k_1}
\]

Substituting this in the first equation gives

\[
x_1 (ax_1 + b) - h = 0
\]

with

\[
a = -\frac{i - \frac{\partial D_{x_2}}{\partial k_1}}{i - \frac{\partial D_{x_1}}{\partial k_1}} = -1 + O(|z_d|^{-2})
\]

\[
b = \frac{1}{i - \frac{\partial D_{x_1}}{\partial k_1}} \frac{\partial h}{\partial k_1} = O(|d|^{-2\beta})
\]

\[
h = O(|d|^{-2\beta})
\]

By Rouchés Theorem, this has the same number of solutions as \(-x^2 = 0\), namely two. The solutions obey

\[
x_1 = \frac{1}{2a} \left\{ -b \pm \sqrt{b^2 + 4ah} \right\}
\]

and hence

\[
|x_1|, |x_2| \leq \frac{\text{const}}{1 + |d|}\]

In the proof of Theorem 16.2 we showed that the point \( s_d \) obeyed

\[
|x_1|, |x_2| \leq \frac{1}{(1 + |d|)^{2\beta - 1}}
\]

Hence, in terms of the coordinates \((x_1, x_2)\), the points \( v_d, w_d \) are at most a distance \( \frac{\text{const}}{(1 + |d|)^\beta} \) away from \( s_d \). Since we also showed in Theorem 16.2 that

\[
\frac{\partial k_2}{\partial x_1} = -\frac{1}{2d_2} (1 + O(|d_2 z_d|^{-1}))
\]

\[
\frac{\partial k_2}{\partial x_2} = \frac{1}{2d_2} (1 + O(|d_2 z_d|^{-1}))
\]

the corresponding distance in terms of \( k_2 \) is at most \( \frac{\text{const}}{(1 + |d|)^\beta} \). \( \blacksquare \)
We give a decomposition of $\mathcal{H}(q)$ based on Theorem 23.4 that is slightly different from the one used in §17. Define, as in §17, for $d \in \Gamma^\#$, $d_2 > 0$,

$$\tau_d = \frac{1}{|z_d|^{13}} \quad r_d = \frac{2}{|d_2 z_d^{14}|} \quad R_d = \frac{1}{6 |d_2 z_d|}$$

and also set

$$\hat{t}_d = \frac{|v_d - w_d|}{9 R_d^2}$$

Fix $\beta$ sufficiently large. Then there is a constant $\rho$ depending only on $|||b|\beta|q||_1$ such that, for $d \in \Gamma^\#$, $d_2 > 0$, $|d| > \rho$, the circle of radius $R_d$ and centre $\frac{v_d + w_d}{2}$ lies completely between the ellipse with foci $v_d, w_d$ and semiaxes $\frac{|v_d - w_d|}{4} \left( \frac{1}{\sqrt{\hat{t}_d}} \pm \sqrt{\hat{t}_d} \right)$ and the ellipse with foci $v_d, w_d$ and semiaxes $\frac{|v_d - w_d|}{4} \left( \frac{1}{2 \sqrt{\hat{t}_d}} \pm 2 \sqrt{\hat{t}_d} \right)$. Also, if $\rho$ is chosen large enough (depending only on $|||b|\beta|q||_1$), then for $|d| > \rho$

- no two of the above ellipses intersect and
- the conditions of (GH2) are fulfilled.

Choose a closed $\sigma$–invariant curve around the origin with inner and outer radii $\rho$ and $2\rho$ that avoids all of the ellipses above. Define $K_2^{\text{com}}$ to be the interior of this curve and $\mathcal{H}(q)^{\text{com}}$ to be the closure of the inverse image of $K_2^{\text{com}}$ under the map $K_2$. Furthermore, let $G$ be the complement of the interior of $K_2^{\text{com}}$ and the union over all $d$ with $\frac{v_d + w_d}{2} \notin K_2^{\text{com}}$ of the interior of the ellipse with foci $v_d, w_d$ and semiaxes $\frac{|v_d - w_d|}{4} \left( \frac{\tau_d}{\sqrt{\hat{t}_d}} \pm \tau_d \sqrt{\hat{t}_d} \right)$. Define $\Phi : G \to \mathcal{H}(q)$ as the inverse of the map $K_2$.

We set

$$t_d = \hat{t}_d \quad \text{if } \frac{v_d + w_d}{2} \notin K_2^{\text{com}}$$

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If \( \frac{v_d + w_d}{2} \in K_2^\text{com} \) choose \( t_d \) close to 1 such that the ellipses with with focii \( v_d, w_d \) and semiaxes \( \frac{v_d - w_d}{4} \left( \frac{1}{\sqrt{t_d}} \pm \sqrt{t_d} \right) \) do not overlap. Define

\[
P_{d,1} : H(t_d) \to \mathcal{C} \\
(z_1, z_2) \mapsto \frac{v_d + w_d}{2} + \frac{v_d - w_d}{4\sqrt{t_d}} (z_1 + z_2)
\]

\[
P_{d,2} : H(t_d) \to \mathcal{C} \\
(z_1, z_2) \mapsto \frac{v_d - w_d}{2} + \frac{v_d - w_d}{4\sqrt{t_d}} (z_1 + z_2)
\]

Then \( P_{d,1} \) maps the “centre of the handle”

\[
\{ (z_1, z_2) \in H(t_d) \mid |z_1| = |z_2| = \sqrt{t_d} \}
\]

as a two-fold cover on the line segment \([v_d, w_d]\) and it maps the “edge of the handle”

\[
\{ (z_1, z_2) \in H(t_d) \mid |z_1| = 1 \}
\]

to the ellipse with focii \( v_d \) and \( w_d \) and semiaxes of lengths \( \frac{v_d - w_d}{4} \left( \frac{1}{\sqrt{t_d}} \pm \sqrt{t_d} \right) \). Similarly, \( P_{d,2} \) maps the “centre of the handle” as a two-fold cover on the line segment \([v_d, w_d]\) and maps the “edge of the handle” to the ellipse with focii \( v_d \) and \( w_d \) and semiaxes of lengths \( \frac{v_d - w_d}{4} \left( \frac{1}{\sqrt{t_d}} \pm \sqrt{t_d} \right) \).

Let \( \tilde{\sigma} : \mathcal{C} \to \mathcal{C}, \ k_2 \mapsto -k_2 \) be reflection in the imaginary axis. Then for each \((z_1, z_2) \in H(t)\)

\[
P_{d,2}(z_2, z_1) = \tilde{\sigma} \circ P_{d,1}(z_1, z_2)
\]

We define

\[
P_d : H(t_d) \setminus \{ (z_1, z_2) \in H(t_d) \mid |z_1| = |z_2| = \sqrt{t_d} \} \to \mathcal{C}
\]

\[
(z_1, z_2) \mapsto \begin{cases} P_{d,1}(z_1, z_2) & \text{if } |z_1| > \sqrt{t_d} \\ P_{d,2}(z_1, z_2) & \text{if } |z_2| > \sqrt{t_d} \end{cases}
\]

There is a unique holomorphic

\[
\phi_d : H(t_d) \to \mathcal{H}(q)
\]

such that

\[
H(t_d) \setminus \{ (z_1, z_2) \in H(t_d) \mid |z_1| = |z_2| = \sqrt{t_d} \} \xrightarrow{\phi_d} \mathcal{H}(q) \setminus \bigcup_c a_c \xrightarrow{P_d} \mathcal{K}_2
\]

commutes. As in §17 one verifies that this decomposition satisfies (GH1-6).
Theorem 23.4 also shows that the heat curves for real potentials all have the same topological structure. To make this more precise, we fix a ball \( \mathcal{P} \) in the space of real analytic potentials \( q \in L^{2}(\mathbb{R}^{2}/\Gamma) \) with \( x_2 \)-average zero and construct the “universal family” \( \mathcal{H} \) of heat curves over \( \mathcal{P} \). More precisely, put

\[
\mathcal{H} = \{ \,(\xi_1, \xi_2); q) \in \mathbb{C}^{n} \times \mathbb{C}^{n} \times \mathcal{P} \mid (\xi_1, \xi_2) \in \mathcal{H}(q) \,\}
\]

By Lemma 15.6 and Theorem 15.8, \( \mathcal{H} \) is an analytic subvariety of \( \mathbb{C}^{n} \times \mathbb{C}^{n} \times \mathcal{P} \). Denote by \( h : \mathcal{H} \to \mathcal{P} \) the projection \( (\xi_1, \xi_2); q) \mapsto q \). The fiber \( h^{-1}(q) \) is the heat curve \( \mathcal{H}(q) \).

Fix \( \rho > 0 \) such that the decomposition constructed above works for all \( q \in \mathcal{P} \). We can choose the decomposition of the heat curves \( \mathcal{H}(q), q \in \mathcal{P} \) in such a way that

i) the restriction of \( h \) to

\[
\mathcal{H}^{\text{com}} = \{ \,(\xi_1, \xi_2); q) \in \mathcal{H} \mid (\xi_1, \xi_2) \in \mathcal{H}(q)^{\text{com}} \,\}
\]

is a locally trivial differentiable fibre bundle over the complement of

\[
\Delta = \{ q \in \mathcal{P} \mid \mathcal{H}(q)^{\text{com}} \text{ is singular} \,\}
\]

The arithmetic genus of \( \mathcal{H}(q)^{\text{com}} \) is a constant over \( \mathcal{P} \). We denote this number by \( g \).

ii) the restriction of \( h \) to \( \partial \mathcal{H}^{\text{com}} \) is a trivial fibre bundle with fibres \( S^{1} \).

iii) for all real valued \( q \in \mathcal{P} \)

- \( \sigma \) maps each of the pieces \( \mathcal{H}(q)^{\text{com}}, \mathcal{H}(q)^{\text{reg}}, \mathcal{H}(q)^{\text{han}} \) onto itself
- the fixed point set of \( \sigma \) on \( \mathcal{H}(q)^{\text{com}} \) consists of one interval \( a_0 \cap \mathcal{H}(q)^{\text{com}} \) and \( g \) ovals \( a_1, \ldots, a_g \) which represent the cycles \( A_1, \ldots, A_g \).
- \( \Phi^{-1} \circ \sigma \circ \Phi \) is the map \( z \mapsto -\overline{z} \)
- for \( b \in \Gamma^{d} \) obeying \( b_2 > 0 \), \( \frac{w_t}{2} \neq K^{\text{com}} \) the involution on the model handle \( H(t_b) \) is given by

\[
\phi_b^{-1} \circ \sigma \circ \phi_b : H(t_b) \longrightarrow H(t_b)
\]

\[
(z_1, z_2) \mapsto \left( \frac{t_1}{z_1}, \frac{t_2}{z_2} \right) = (\overline{z}_2, \overline{z}_1)
\]

The fixed point set of \( \sigma \) on \( \mathcal{H}(q) \) consists of the curve \( a_0 \), the ovals \( a_1, \ldots, a_g \) and the ovals

\[
a_b = \phi_b \{ \,(z_1, z_2) \in H(t_b) \mid |z_1| = |z_2| = t_b^{1/2} \,\}
\]
and is illustrated in the figures below.

**Corollary 23.6** The Riemann period matrix $\mathcal{R}$, for the heat curve $\mathcal{H}(q)$, is pure imaginary. The vectors $U, V, W$ are real.

**Proof:** The $A$-cycles are invariant under the antiholomorphic involution $\sigma$. This implies that $\sigma^* \omega_b = \bar{\omega}_b$. As $\sigma$ is orientation reversing, $\sigma B_c = -B_c$. Hence

$$\mathcal{R}_{c,b} = \int_{B_c} \omega_b = \int_{-B_c} \bar{\omega}_b = -\overline{\mathcal{R}_{c,b}}$$

By (21.2,3) and the fact that $w_b(-\xi) = \overline{w_b(\xi)}$ we have that $w_{b,a}^{(n)} + w_{b,-a}^{(n)}$ and $w_{b,\epsilon,\text{com}}^{(n)}$ are real. The reality of $U, V, W$ then follows from Proposition 21.1.

We denote by $\mathcal{P}_{\mathbb{R}}$ the set of real valued potentials in $\mathcal{P}$ and set

$$\mathcal{H}_r = \{ ((\xi_1, \xi_2); q) \in \mathcal{H} \mid (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}, q \in \mathcal{P}_{\mathbb{R}} \}$$
The maps $K_2$ of Theorem 23.4 define a map

$$K_2 : h^{-1}(\mathcal{P}_{\mathbb{R}}) \setminus \mathcal{H}_r \to \mathbb{C} \times \mathcal{P}_{\mathbb{R}}$$

Furthermore, there are real analytic maps

$$v_b, w_b : \mathcal{P}_{\mathbb{R}} \to \mathbb{C}$$

such that $\text{Re} \, v_b = \text{Re} \, w_b = -\frac{1}{2}b_2$, $\text{Im} \, v_b \leq \text{Im} \, w_b$ and $c_b(q)$ is the line segment joining $v_b(q)$ and $w_b(q)$. Observe that, for $q \in \mathcal{P}_{\mathbb{R}}$, the heat curve $\mathcal{H}(q)$ is singular if and only if $v_b(q) = w_b(q)$ for some $b$. In this case $v_b(q)$ corresponds to an ordinary double point of $\mathcal{H}(q)$.

We define, for each $b \in \Gamma^\#$ with $b_2 > 0$,

$$A_b = \bigcup_{q \in \mathcal{P}_{\mathbb{R}}} a_b(q) \times \{q\}$$

This is a connected component of $\mathcal{H}_r$ and $A_b \cap \mathcal{H}(q) = a_b$.

Now fix $\hat{e} = (\hat{e}_b)_{b \in \Gamma^\#, b_2 > 0}$ such that $\hat{e}_b \in \mathbb{R}$ for all $b$ and

$$\lim_{b \to \infty} \frac{\hat{e}_b}{\ln |b|} = 0$$

For each $q \in \mathcal{P}$ for which $\mathcal{H}(q)$ is smooth $\hat{e}$ lies in the Banach space associated to $\mathcal{H}(q)$. If $q \in \mathcal{P}_{\mathbb{R}}$ and $\mathcal{H}(q)$ has an ordinary double point at $a_c$ then we consider $\hat{e}$ as a vector in the Banach space with the variable $e_c$ deleted. Put

$$\mathcal{N} = \left\{ (\xi_1, \xi_2) ; q \in \mathcal{H} \mid \theta \left( \hat{e} + \int_{\infty}^{(\xi_1, \xi_2)} \overline{\omega} \right) = 0 \right\}$$

By Theorem 9.11 the restriction of $h$ to $\mathcal{N} \cap \mathcal{H}^{\text{corn}}$ is a (possibly ramified) covering of degree $g$ over $\mathcal{P} \setminus \Delta$. We use this picture to prove

**Proposition 23.7** Let $q \in C^\omega(\mathbb{R}/\Gamma)$ be a real valued potential with $\int_0^{2\pi} q(x_1, x_2) \, dx_2 = 0$ and let $\hat{e} \in B$ be a real vector in the Banach space

$$B = \left\{ (e_b)_{b \in \Gamma^\#} \mid \lim_{b \to \infty} \frac{e_b}{\ln |b|} = 0 \right\}$$

Then

$$\theta \left( \hat{e} + \int_{\infty}^{x} \overline{\omega} \right)$$

has exactly one zero $\hat{x}_b$ on $a_b$, $b \in \Gamma^\#$, $b_2 > 0$ and no other zeroes.
Proof: Observe that \( \theta(\hat{c}) \neq 0 \) by Corollary 23.6 and Proposition 4.14. We may assume that \( q \) is in the ball \( \mathcal{P} \) discussed above. Put

\[
S = \left\{ q \in \mathcal{P}_{\mathbb{R}} \setminus \Delta \left| \mathcal{N} \cap A_b \cap \mathcal{H}(q) \text{ consists of exactly one point for each } b \in \Gamma^\#, b_2 > 0 \text{ with } a_b \subset \mathcal{H}^{\text{com}} \right. \right\}
\]

If \( q \in \mathcal{P} \) is small, then by Theorem 9.11 (choosing \( \mathcal{H}(q)^{\text{com}} = \emptyset \))

\[
\mathcal{N} \cap \mathcal{H}(q) = \bigcup_{b \in \Gamma^\#} \big\{ q \in \mathcal{N} \cap Y_b \big\}
\]

and, for each \( b \in \Gamma^\# \), \( b_2 > 0 \), \( \mathcal{N} \cap Y_b \) consists of one point. If \( q \) is real-valued, the anti-holomorphic involution \( \sigma \) maps \( \mathcal{N} \) and \( Y_b \) to themselves. Thus, if \( q \) is small and real-valued, \( \mathcal{N} \cap Y_b \) is a fixed point of \( \sigma \), i.e. \( \mathcal{N} \cap Y_b \subset a_b \). This shows that all sufficiently small \( q \in \mathcal{P}_{\mathbb{R}} \setminus \Delta \) are contained in \( S \).

Now fix \( q \in \mathcal{P}_{\mathbb{R}} \setminus \Delta \). Let \( 0 = t_0 < t_1 < t_2 < \cdots < t_m \) be the points in \([0, 1]\) for which \( tq \in \Delta \). Put \( t_{m+1} = 1 \).

Next we claim that if \( tq \in S \) for some \( t \in (t_{n-1}, t_n) \), \( 1 \leq n \leq m + 1 \) then \( tq \in S \) for all \( t \in (t_{n-1}, t_n) \). The points of \( \mathcal{N} \cap \mathcal{H}(tq)^{\text{com}} \) move continuously with \( t \). Suppose that \( t \in (t_{n-1}, t_n) \) and \( b \in \Gamma^\# \), \( b_2 > 0 \) are such that the cardinality of \( A_b \cap \mathcal{H}(tq) \cap \mathcal{N} \) jumps. In other words, suppose that as \( t' \) passes through \( t \), a point \( z(t') \in \mathcal{H}(t'q) \cap \mathcal{N} \) leaves \( A_b \). If \( t' \) is such that \( z(t') \notin A_b \) then, by \( \sigma \)-invariance, \( \sigma z(t') \) is a second point of \( \mathcal{H}(t'q) \cap \mathcal{N} \) near \( A_b \). So there must be at least one point of multiplicity two in \( A_b \cap \mathcal{H}(tq) \cap \mathcal{N} \). This is impossible at any point \( t \) with \( tq \in S \) because then all the \( g \) points of \( \mathcal{H}(tq)^{\text{com}} \cap \mathcal{N} \) lie in different connected components of \( \mathcal{H}_r(tq) \). This shows that \( \left\{ t \in (t_{n-1}, t_n) \mid tq \in S \right\} \) is both open and closed in \( (t_{n-1}, t_n) \).

Next we show that if \( tq \in S \) for all \( t \in (t_{n-1}, t_n) \) for some \( 1 \leq n \leq m \) then there is \( t' \in (t_n, t_{n+1}) \) with \( t'q \in S \). Since \( h : \mathcal{N} \cap \mathcal{H}^{\text{com}} \to \mathcal{P} \) is a \( g \)-fold covering over \( \mathcal{P} \setminus \Delta \) there exist holomorphic maps from the punctured disk \( \hat{D} = \{ z \in \mathbb{C} \mid 0 < |z| < \epsilon \} \)

\[
f_i : \hat{D} \longrightarrow \mathcal{N} \cap \mathcal{H}^{\text{com}} \quad i = 1, \ldots, k
\]

and \( \alpha_i \in \mathbb{Z} \), \( \alpha_i \geq 1 \) such that the diagrams
commute, \( \alpha_1 + \cdots + \alpha_k = g \) and \( \mathcal{N} \cap \mathcal{H}^{\text{com}} \cap h^{-1}(t_n + \dot{D})q \) is the union of the images of the \( f_i \). The \( f_i \) have removable singularities at \( z = 0 \) and hence can be analytically continued to maps

\[
\tilde{f}_i : \{ z \in \Phi \mid |z| < \varepsilon \} \longrightarrow \mathcal{H}^{\text{com}}
\]

If \( \alpha_i \geq 2 \) for some \( i \) then there would be \( b \neq c \in \Gamma^\# \) such that the points of \( A_b \cap \mathcal{H}(tq) \cap \mathcal{N} \) and \( A_c \cap \mathcal{H}(tq) \cap \mathcal{N} \) would have the same limit as \( t \rightarrow t_n - \). This is impossible since \( A_b \cap \mathcal{H}(t_nq) \) and \( A_c \cap \mathcal{H}(t_nq) \) are separated. So \( k = g \) and \( \alpha_1 = \cdots = \alpha_k = 1 \). Since the construction is continuous and \( \sigma \)-invariant \( f_i((-\varepsilon, \varepsilon)) \) is completely contained in some \( A_{b_i} \). By hypothesis, for \( i \neq j \), \( f_i(-\varepsilon) \) and \( f_j(-\varepsilon) \) lie in different components of \( \mathcal{H}_r((t_n - \varepsilon)q) \). Thus \( b_i \neq b_j \) for \( i \neq j \) and \( (t_n + \varepsilon)q \in \mathcal{S} \).

Combining the last three paragraphs, it follows that \( tq \) lies in \( \mathcal{S} \) for \( 0 < t \leq 1 \) and \( t \neq t_1, \ldots, t_m \). In particular \( q \in \mathcal{S} \).

So \( \theta \left( \hat{\varepsilon} + \int_{\infty}^{\hat{x}} \omega \right) \) has exactly one zero \( \hat{x}_b \) on each \( a_b \) such that \( a_b \subset \mathcal{H}(q)^{\text{com}} \) and no other zeroes on \( \mathcal{H}(q)^{\text{com}} \). For \( \mathcal{H}(q) \setminus \mathcal{H}(q)^{\text{com}} \) the statement follows immediately from the \( \sigma \)-invariance of \( \mathcal{N} \) and Theorem 9.11.

Now, let \( q \in C^{\omega}(\mathbb{R}^2 / \Gamma) \) be a real potential with \( \int_0^{2\pi} q(x_1, x_2)dx_2 = 0 \) for which the associated heat curve \( \mathcal{H}(q) \) is smooth. Choose a real vector \( \hat{e} \) in the Banach space

\[
B = \{ z \in \Phi^\infty \mid \lim_{b \rightarrow \infty} \frac{z_b}{\log t_b} = 0 \}
\]

of \( \mathcal{H}(q) \) and let \( \hat{x}_b \) be the zero of \( \theta \left( \hat{\varepsilon} + \int_{\infty}^{\hat{x}} \omega \right) \) on the oval \( a_b \) as in Proposition 23.7. Furthermore, let \( y_b \in a_b \) be the unique point of \( D \cap a_b \) specified in Theorem 23.4. Then, for \( j \geq g + 1 \),

\[
\left| \int_{\hat{x}_b}^{y_b} (\phi_b)_* \left( \frac{1}{2\pi i} \frac{dz}{z_l} \right) \right| \leq 1
\]
so by Lemma 8.8 and Theorem 8.4, the sequence \( \left( \int_{x_k}^{y_k} \omega_b \right) \) is bounded. Therefore, \((y_1, y_2, \cdots)\) is a divisor of index zero in the sense of §10. We put

\[
e = \hat{e} - \sum_{b \in \Gamma^*} \int_{x_b}^{y_b} \tilde{\omega}
\]

Observe that, again by Corollary 23.6 and Proposition 4.14, \( \theta(e) \neq 0 \). Then, by Proposition 10.5a, the points \( y_1, y_2, \cdots \) are the zeroes of \( \theta(e + \int_{x}^{x_b} \omega) \) on \( \mathcal{H}(q) \).

Now define \( U, V, W \) as in §21. It follows from Theorem 4.6, Corollary 23.6, Proposition 4.14 and (21.5) that

\[
(x_1, x_2, t) \mapsto \theta(e + x_2 U + x_1 V - \frac{1}{2} t W)
\]

is a nowhere vanishing \( C^\infty \) function of \((x_1, x_2, t)\). Hence

\[
u(x_1, x_2, t) = -2 \frac{d^2}{dx_2^2} \ln \theta(e + U x_2 + V x_1 - \frac{1}{2} W t)
\]

is well-defined everywhere. We can make our main result more precise.

**Theorem 23.1’** *In the situation above, \( u(x_1, x_2, t) \) solves \((KP)\) and \[
u(x_1, x_2, 0) = q(x_1, x_2)
\]

In §22, we showed that there is a constant \( c \) such that \( u(x_1, x_2, t) + c \) is a solution of the \((KP)\) equation. It remains to verify the initial condition and \( c = 0 \).

We first verify that if \( u + c \) obeys the initial condition then \( c = 0 \). So assume that \( u + c \) obeys the initial condition. Then, as \( \int_{0}^{2\pi} q(x_1, x_2) \, dx_2 = 0 \) we have

\[
2\pi c = 2 \int_{\xi}^{\xi + 2\pi} \frac{d}{dx_2} \ln \theta(e + U x_2) \, dx_2
\]

\[
= 2 \frac{d}{dx_2} \ln \theta(e + U x_2) \bigg|_{x_2 = \xi + 2\pi} - 2 \frac{d}{dx_2} \ln \theta(e + U x_2) \bigg|_{x_2 = \xi}
\]

\[
= 2 \frac{d}{dx_2} \ln \frac{\theta(e + 2\pi U + U x_2)}{\theta(e + U x_2)} \bigg|_{x_2 = \xi}
\]

Hence

\[
\theta(e + 2\pi U + U \xi) = e^{\pi i \xi} \theta(e + U \xi)
\]
Since, for real $\xi$, $\theta$ is real and bounded we have $c = 0$.

We have chosen the vector $e$ by an algorithm that, in the finite genus case, ensures that the initial condition is satisfied. To prove it in the general case, we approximate $\mathcal{H}(q)$ by heat curves of finite genus. By convention, if $\theta'$ is the $\theta$ function for an approximating finite genus heat curve $\mathcal{H}(q')$ and $z \in B$, then $\theta'(z)$ is evaluated by ignoring all components $z_b$ of $z$ for which $a'_b$ is a point. The following two theorems prove that approximation by finite genus heat curves is possible.

**Theorem 23.8 ([K, chI §3])** Let $q \in C^\omega(\mathbb{R}^2/\Gamma)$ be a real potential with $\int_0^{2\pi} q(x_1, x_2) dx_2 = 0$. Then, for any $n > 0$, $\epsilon > 0$, there exists a real-valued finite zone potential $q'$ with

$$
\sum_{b \in \Gamma^\#} (1 + |b|^n) |\tilde{q}(b) - \tilde{q'}(b)| < \epsilon
$$

**Theorem 23.9** Fix a real $q \in C^\omega(\mathbb{R}^2/\Gamma)$. There is a constant $K$ such that the following holds. Let $Z \subset \mathcal{H}(q)$ be a compact subset containing $\mathcal{H}(q)^{\text{com}}$ and let $\epsilon > 0$. Then, there is $\delta > 0$ such that for all $q'$ with $\sum_b |b|^4 |\tilde{q}(b) - \tilde{q'}(b)| < \delta$ there are compact submanifolds with boundary $X_0 \subset \mathcal{H}(q)$, $X'_0 \subset \mathcal{H}(q')$ with $Z \subset X_0$ and a diffeomorphism $F : X_0 \to X'_0$ such that

(i) $(\mathcal{H}(q), X_0)$ is $(\epsilon, K)$-close to $(\mathcal{H}(q'), X'_0)$ via $F$

(ii) the antiholomorphic involutions preserve $X_0$ and $X'_0$ and $F$ is compatible with the antiholomorphic involutions. In formulae: $\sigma(X_0) = X_0$, $\sigma'(X'_0) = X'_0$ and $F \circ \sigma = \sigma' \circ F$

(iii) Let $z'_b \in a'_b$ be the zeroes of $\theta'\left( e + \int_{z'_b}^{z'_b} \bar{\omega}' \right)$ on $\mathcal{H}(q')$. Then

$$
|\phi_b^{-1}(z_b) - \phi_b^{-1}(F^{-1}(z'_b))| \leq \epsilon \sqrt{t_b} \quad \text{for all } b \text{ such that } Y_b \subset X_0
$$

(iv) Let $y'_b \in a'_b$ be the unique point of $D' \cap a'_b$ of Theorem 23.4. Then

$$
|\phi_b^{-1}(y_b) - \phi_b^{-1}(F^{-1}(y'_b))| \leq \epsilon \sqrt{t_b} \quad \text{for all } b \text{ such that } Y_b \subset X_0
$$
Before we prove Theorem 23.9, we use it to give the

**Proof of Theorem 23.1**: Approximate \( q \) by finite zone potentials with respect to the norm \( \sum_b |b|^4 |\hat{q}(b)| \). If \( q' \) is an approximating potential as in Theorem 23.8, then

\[
q'(x_1, x_2) = -\frac{\partial^2}{\partial x_1^2} \log \theta'(e' + U'x_2 + V'x_1)
\]

where

\[
e' = e - \sum_b \int_{z_b'}^{y_b'} \tilde{\omega}'
\]

since

\[
\theta' \left( e' + \int_{-\infty}^{y_b'} \tilde{\omega}' \right)
\]

vanishes precisely on \( \{y_b'\} \). By Proposition 10.1, Proposition 8.16 and Theorem 23.9, \( e' \) converges to \( e \) in \( B \). Furthermore by Theorem 23.9 (i) and Theorem 22.4 the theta function

\[
\theta'(e' + U'x_2 + V'x_1)
\]

converges to

\[
\theta(e + Ux_2 + Vx_1)
\]

uniformly for all \( x_1, x_2 \) in any bounded set by the method of Corollary 22.14 (solutions of the KP equation). By analyticity this proves that \( u(x_1, x_2, 0) = q(x_1, x_2) \). That \( u \) obeys the KP equation was proven before.

For the proof of Theorem 23.9 we need

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Lemma 23.10 Let \( v, w, v', w' \in \mathbb{C}, 0 < t, t' < 1 \) and let
\[
P_1 : H(t) \rightarrow \mathbb{C}
\]
\[
(z_1, z_2) \mapsto \frac{v+u}{2} + \frac{1}{4\sqrt{t}} (v-w)(z_1 + z_2)
\]
\[
P'_1 : H(t') \rightarrow \mathbb{C}
\]
\[
(z_1, z_2) \mapsto \frac{v'+u'}{2} + \frac{1}{4\sqrt{t'}} (v'-w')(z_1 + z_2)
\]
If \( t \leq \left( \frac{4}{15x^3} \right)^2 \) set \( K = 2 \). Otherwise, let \( K > 2 \) obey \( t < (1 - \frac{28}{15xK})^2 (1 + \frac{1}{K})^{-4} \). Assume that \( |v-v'| < \varepsilon |v-w|, |w-w'| < \varepsilon |v-w| \) and \( |t-t'| < c \varepsilon \). There is a universal constant \( \text{const} \) such that, if \( \varepsilon < \frac{1}{\text{const}K^5} \), then all of the following hold. There are
\[
\{ (z_1, z_2) \in H(t) \mid \frac{t}{1-\text{const}K^5 \varepsilon} \leq |z_1| \leq 1 - \text{const}K^5 \varepsilon \} \subset \hat{Y} \subset H(t)
\]
\[
\{ (z_1, z_2) \in H(t') \mid \frac{t'}{1-\text{const}K^5 \varepsilon} \leq |z_1| \leq 1 - \text{const}K^5 \varepsilon \} \subset \hat{Y}' \subset H(t')
\]
and a \( K \text{-quasiconformal diffeomorphism } f : \hat{Y} \rightarrow \hat{Y}' \) of distortion at most \( \text{const}K^5 \varepsilon \) such that
\[
P'_1 \circ f (z_1, z_2) = P_1 (z_1, z_2)
\]
for \( (z_1, z_2) \in \hat{Y} \) with \( |z_1| \geq 1 - \frac{28}{15K} \) or \( |z_2| \geq 1 - \frac{28}{15K} \) (\( = \frac{1}{15} \) for \( K = 2 \)) and
\[
f(z_1, z_2) = \sqrt{\frac{t'}{t}} (z_1, z_2)
\]
for \( (z_1, z_2) \in \hat{Y} \) with \( |z_1|, |z_2| \leq (1 - \frac{28}{15K})(1 + \frac{1}{K})^{-1} \) (\( = \frac{2}{45} \) for \( K = 2 \)). Furthermore
\[
|f(z_1, z_2) - (z_1, z_2)| \leq \text{const}K\varepsilon
\]
If the line through \( v \) and \( w \) coincides with that through \( v' \) and \( w' \) then \( f \) commutes with the antiholomorphic involutions \( (z_1, z_2) \mapsto (\bar{z}_2, \bar{z}_1) \) on \( H(t) \) and \( H(t') \).

Remark. Be careful to distinguish between \( \varepsilon \) and \( \varepsilon \). We shall use Lemma 23.10 to construct \( K \text{-quasiconformal diffeomorphisms of distortion at most } \varepsilon \) in the proof of Theorem 23.9. There, we first pick \( K \). Then, for any desired \( \varepsilon > 0 \), we pick \( \varepsilon \) such that \( \text{const}K^5 \varepsilon < \varepsilon \) and \( \text{const}K\varepsilon < \frac{1}{K} \).
Proof: Let \( \hat{F}_{t,K} : \mathbb{C} \to [0,1] \) be a \( C^\infty \) function which takes the value one inside the ellipse with focii \( \pm \sqrt{t} \) and semiaxes \( \frac{1}{2} \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \pm \frac{t}{2} \left( 1 - \frac{28}{15K} \right)^{-1} \left( 1 + \frac{1}{K} \right) \), which vanishes outside the ellipse with focii \( \pm \sqrt{t} \) and semiaxes \( \frac{1}{2} \left( 1 - \frac{28}{15K} \right) \pm \frac{t}{2} \left( 1 - \frac{28}{15K} \right)^{-1} \), and which is invariant under reflection about the real axis. Hence \( \hat{F}_{t,K}(\overline{z}) = \hat{F}_{t,K}(z) \). Because the distance between the two ellipses is

\[
\frac{1}{2} \left( 1 - \frac{28}{15K} \right) + \frac{t}{2} \left( 1 - \frac{28}{15K} \right)^{-1} - \frac{1}{2} \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} - \frac{t}{2} \left( 1 - \frac{28}{15K} \right)^{-1} \left( 1 + \frac{1}{K} \right) \\
= \frac{1}{2} \left( 1 - \frac{28}{15K} \right) \frac{1}{K} - \frac{t}{2} \left( 1 - \frac{28}{15K} \right)^{-1} \frac{1}{K}
\]

\[
\geq \frac{1}{2} \left( 1 - \frac{28}{15K} \right) \frac{1}{K} - \frac{t}{2} \left( 1 - \frac{28}{15K} \right)^{-1} \frac{1}{K}
\]

\[
\geq \frac{1}{2} \frac{2}{2} \frac{1}{2} \frac{1}{K} \left[ 1 - \left( 1 + \frac{1}{K} \right)^{-3} \right]
\]

\[
\geq \frac{\text{const}}{K^2}
\]

it is possible to construct \( \hat{F}_{t,K} \) so that

\[
\sup_{x,y} \left| \partial^a \hat{F}_{t,K}(x,y) \right| \leq \text{const} \left| \alpha \right| \left( \text{const} K^2 \right)^{|\alpha|}
\]

Define \( F : \mathbb{C} \to \mathbb{C} \) by

\[
F(z) = z + \left[ \frac{v' + w' - v - w}{2} + \frac{2z - v - w}{v - w} \frac{v' - w' - v + w}{2} \right] \hat{F}_{t,K} \left( \sqrt{t} \frac{2z - v - w}{v - w} \right)
\]

We will define \( f \) in terms of \( F \). Before doing so, we derive some properties of \( F \). Observe that, for each \( a > \sqrt{t} \), the image under the map

\[
(z_1, z_2) \in H(t) \longmapsto \sqrt{t} \frac{2P_1(z_1, z_2) - v - w}{v - w} = \frac{z_1 + z_2}{2}
\]

of the circle \( \left| z_1 \right| = a \) and of the circle \( \left| z_2 \right| = a \) is the ellipse with focii \( \pm \sqrt{t} \) and semiaxes \( \frac{1}{2} \left( a \pm \frac{t}{a} \right) \). Whenever \( \hat{F} \left( \sqrt{t} \frac{2z - v - w}{v - w} \right) = 1 \), and this includes all \( z = P_1(z_1, z_2) \) with \( \left| z_1 \right|, \left| z_2 \right| = \frac{t}{|z_1|} \leq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \),

\[
\frac{2F(z) - v' - w'}{v' - w'} = \frac{2z - v' - w'}{v' - w'} + \frac{v' + w' - v - w}{v' - w'} + \frac{2z - v - w}{v - w} \frac{v' - w'}{v' - w'}
\]

\[
= \frac{2z - v - w}{v - w} + \frac{2z - v - w}{v - w} \frac{v' - w'}{v' - w'} + \frac{2z - v' - w'}{v' - w'}
\]

\[
= \frac{2z - v - w}{v - w}
\]
Consequently, denoting \( \frac{2z - v - w}{v - w} = s \),

\[ F \left( \frac{v + w}{2} + s \frac{v - w}{2} \right) = \frac{v' + w'}{2} + s \frac{v' - w'}{2} \]

for all \( \sqrt{t}|s| \) inside the ellipse with focii \( \pm \sqrt{t} \) and semiaxes \( \frac{1}{2} \left( \frac{1 - \frac{28}{15K}}{1 + \frac{1}{K}} \right)^{\frac{1}{2}} \pm \frac{t}{2} \left( \frac{1 + \frac{1}{K}}{1 - \frac{28}{15K}} \right)^{\frac{1}{2}} \). This includes \( s = \pm 1 \), so

\[ F(v) = v' \]

\[ F(w) = w' \]

and \( F \) intertwines reflection about the line through \( v \) and \( w \) with reflection about the line through \( v' \) and \( w' \), at least in a neighbourhood of the line segment joining \( v \) and \( w \).

For general \( s \)

\[ F \left( \frac{v + w}{2} + s \frac{v - w}{2} \right) = \frac{v' + w'}{2} + s \frac{v' - w'}{2} \left( s + \left[ \frac{v' + w' - v - w}{v' - w'} + s \left( 1 - \frac{v - w}{v' - w'} \right) \right] \left[ F(t', s) - 1 \right] \right) \]

Note that, by hypothesis, \( \left| \frac{v' + w' - v - w}{v' - w'} \right| \leq 2\varepsilon \) and that, by construction, \( \left| \frac{d}{ds} \hat{F}_{t', K}(\sqrt{t}s) \right| \leq \text{const} K^2 \). Hence, as long as we have chosen the constant \( \varepsilon \leq \frac{1}{\text{const} K^2} \) sufficiently large, the map from \( s \) to \( s + \left[ \frac{v' + w' - v - w}{v' - w'} + s \left( 1 - \frac{v - w}{v' - w'} \right) \right] \left[ F(t, K)(\sqrt{t}s) - 1 \right] \), and consequently the map from \( z \) to \( F(z) \), is globally bijective. Furthermore, if the line through \( v \) and \( w \) coincides with the line through \( v' \) and \( w' \) then \( \frac{v - w}{v' - w'} \) and \( \frac{v' + w' - v - w}{v' - w'} \) are real so that

\[ F \left( \frac{v + w}{2} + s \frac{v - w}{2} \right) = \frac{v' + w'}{2} + s \frac{v' - w'}{2} \left( s + \left[ \frac{v' + w' - v - w}{v' - w'} + s \left( 1 - \frac{v - w}{v' - w'} \right) \right] \left[ F(t, K)(\sqrt{t}s) - 1 \right] \right) \]

and \( F \) commutes with reflection in the line joining \( v \) and \( w \).

Now define \( f : \hat{Y} = \{ (z_1, z_2) \in H(t) \mid F \circ P_1(z_1, z_2) \in \text{range} \ P' \} \rightarrow H(t') \) so that

\[ P'_1 \circ f = F \circ P_1 \]

\( P'_1 \) is a 2 to 1 map, so for each \( (z_1, z_2) \) in the domain of \( f \) we have two possible values of \( (u_1, u_2) = f(z_1, z_2) \). Choose \( |u_1| > |u_2| \) if and only if \( |z_1| > |z_2| \). Note the argument of \( \hat{F}_{t, K} \) in \( F \circ P_1 \) is \( \sqrt{\frac{2P_1(z_1, z_2) - v - w}{v - w}} = \frac{1}{2}(z_1 + z_2) \) and that for \( |z_1|, |z_2| \leq \left( 1 - \frac{28}{15K} \right)^{\frac{1}{2}} \left( 1 + \frac{1}{K} \right)^{-1} \) we have \( \hat{F}_{t, K}(\frac{z_1 + z_2}{2}) = 1 \). For all such \( (z_1, z_2) \)

\[ F \circ P_1(z_1, z_2) = \frac{v' + w'}{2} + \frac{z_1 + z_2}{2} \frac{v' - w'}{2} \]

\[ = \left( \frac{\sqrt{t'}}{t'} z_1, \sqrt{\frac{t'}{t}} z_2 \right) \]
Thus
\[ f(z_1, z_2) = \sqrt{\frac{v'}{t}}(z_1, z_2) \]
on \( \{ (z_1, z_2) \in H(t) \mid |z_1|, |z_2| \leq \left( 1 - \frac{2\delta}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \} \). On the other hand, in the event that \( \max\{|z_1|, |z_2|\} \geq \left( 1 - \frac{2\delta}{15K} \right) \), we have \( \hat{H}_{t,K}(\frac{z_1 + z_2}{2}) = 0 \). For all such \((z_1, z_2)\)
\[ P'_1 \circ f = F \circ P_1 = P_1 \]

Now write \( f(z_1, z_2) = (u_1, u_2) \). By definition
\[ \frac{v' + u'}{2} + \frac{v' - u'}{2} u_1 + u_2 = \frac{v' + u'}{2} + \frac{v' - u'}{2} \left\{ \frac{z_1 + z_2}{2\sqrt{t}} + \left[ \frac{v' + u' - u - w}{v'-w} + \frac{z_1 + z_2}{2\sqrt{t}} \frac{v'-u'-w+w}{v'-w} \right] \hat{H}(\frac{z_1 + z_2}{2}) - 1 \right\} \]
so that
\[ u_1 + u_2 = \sqrt{\frac{v'}{t}} \left\{ \frac{z_1 + z_2}{2} + 2\sqrt{t} \left[ \frac{v' + u' - u - w}{v'-w} + \frac{z_1 + z_2}{2\sqrt{t}} \frac{v'-u'-w+w}{v'-w} \right] \hat{H}(\frac{z_1 + z_2}{2}) - 1 \right\} \]
with
\[ U = 2\sqrt{t} \left[ \frac{v' + u' - u - w}{v'-w} + \frac{z_1 + z_2}{2\sqrt{t}} \frac{v'-u'-w+w}{v'-w} \right] \hat{H}(\frac{z_1 + z_2}{2}) - 1 \]
bounded by
\[ |U| \leq 6\sqrt{t} \varepsilon \left( 1 + \frac{z_1 + z_2}{2\sqrt{t}} \right) \leq 12\varepsilon \]
and, more generally, obeying
\[ |\partial^\alpha U| \leq \text{const} |\alpha| \varepsilon (\text{const } K^2)^{|\alpha|} \]

We now solve for \( u_1 \) in terms of \( z_1 \)
\[ u_1 + \frac{u'}{u_1} = \sqrt{\frac{v'}{t}} \left( \frac{z_1}{z_1} + U \right) \]
\[ \Rightarrow u_1^2 - \frac{v'}{t} \left( \frac{z_1}{z_1} + U \right) u_1 + u' = 0 \]
\[ \Rightarrow u_1 = \frac{1}{2} \left\{ \sqrt{\frac{v'}{t} \left( \frac{z_1}{z_1} + U \right)^2} \pm \sqrt{\frac{v'}{t} \left( \frac{z_1}{z_1} + U \right)^2 - 4u'} \right\} \]
\[ \Rightarrow u_1 = \frac{1}{2} \left\{ \sqrt{\frac{v'}{t} \left( \frac{z_1}{z_1} + U \right)} \pm \sqrt{\frac{v'}{t} \left( \frac{z_1}{z_1} + U \right)^2 + 2U (z_1 + \frac{t}{z_1}) + U^2} \right\} \]

To satisfy \(|u_1| > |u_2|\) for \(|z_1| > |z_2|\), we take the + sign, so
\[ u_1 = \sqrt{\frac{v'}{t} \left( \frac{z_1 + U}{2} \right)} + \sqrt{\frac{v'}{t} \left( \frac{z_1 - t}{z_1} \right)} \left[ \sqrt{1 + \left[ 2U (z_1 + \frac{t}{z_1}) + U^2 \right] \left( \frac{z_1 - t}{z_1} \right)^2 - 1} \right] \]

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We already know that \( U = 0 \) if \(|z_1|, |z_2| \leq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \). Otherwise,

\[
|z_1 - \frac{t}{z_1}| = |z_1 - z_2| \geq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} - \sqrt{t}
\]

\[
\geq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-2} \frac{1}{K} \geq \frac{4}{15K}
\]

and

\[
|u_1 - z_1| \leq \text{const } \varepsilon + \text{const } KU \leq \text{const } K\varepsilon
\]

Similarly \(|u_2 - z_2| \leq \text{const } K\varepsilon\), so we have verified

\[
|f(z_1, z_2) - (z_1, z_2)| \leq \text{const } K\varepsilon
\]

Finally, we must show that \( f^* \frac{du_1}{u_1} \) satisfies the \( K\)-quasiconformal bounds corresponding to distortion at most \( \text{const } K^5\varepsilon \). If \(|z_1|, |z_2| \leq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \)

\[
f^* \frac{du_1}{u_1} = \frac{dz_1}{z_1}
\]

On the other hand, if, for example, \(|z_1| \geq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \)
then

\[
f^* \frac{du_1}{u_1} = f^* \frac{d(u_1 + u_2)}{u_1 - u_2}
\]

\[
= \sqrt{\frac{t'}{t} \frac{dz_1 + z_2 + dU}{u_1 - u_2}}
\]

\[
= \sqrt{\frac{t'}{t} \frac{z_1 - z_2 + d(z_1 + z_2)}{z_1 - z_2}} + \sqrt{\frac{t'}{t} \frac{du}{u_1 - u_2}}
\]

\[
= a \frac{dz_1}{z_1} + b \frac{dz_1}{z_1}
\]

where

\[
a = \sqrt{\frac{t'}{t} \frac{z_1 - z_2}{u_1 - u_2}} + \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{u_1 - u_2}}
\]

\[
b = \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{u_1 - u_2}}
\]

Hence it remains only to show that, when \(|z_1| \geq \left( 1 - \frac{28}{15K} \right) \left( 1 + \frac{1}{K} \right)^{-1} \), each of

\[
a - 1 = \sqrt{\frac{t'}{t} - 1} + \sqrt{\frac{t'}{t} \frac{z_1 - u_1 - z_2 + u_2}{u_1 - u_2}} + \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{u_1 - u_2}}
\]

\[
b = \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{u_1 - u_2}}
\]

\[
\frac{\partial \Phi}{\partial z_1} = \sqrt{\frac{t'}{t} \frac{z_1 - z_2}{u_1 - u_2}} \frac{\partial}{\partial z_1} \frac{1}{u_1 - u_2} + \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{u_1 - u_2}} \frac{\partial}{\partial z_1} \frac{U_{z_1}}{u_1 - u_2}
\]

\[
= -\sqrt{\frac{t'}{t} \frac{z_1 - z_2}{(u_1 - u_2)^2}} \frac{\partial}{\partial z_1} (u_1 - u_2) - \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{(u_1 - u_2)^2}} \frac{\partial}{\partial z_1} (u_1 - u_2) + \sqrt{\frac{t'}{t} \frac{z_1 U_{z_1}}{u_1 - u_2}} U_{z_1 z_1}
\]

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is bounded by \( K^5 \varepsilon (|z_1| + |z_2|) \). In fact, as

\[
|z_1|, |z_2| \leq 1
\]

\[
|z_1| \geq (1 - \frac{28}{15K}) (1 + \frac{1}{K})^{-1} \geq \frac{1}{15} \frac{2}{3}
\]

\[
|z_1 - z_2| \geq |z_1 - \sqrt{t}| \geq \begin{cases} \frac{1}{15} \left( \frac{2}{3} - \frac{4}{9} \right) & \text{if } t \leq \left( \frac{4}{15 \times 9} \right)^2 \\ \frac{1}{15K} \left( 1 - \frac{28}{15K} \right) - \frac{1}{15) \left( \frac{2}{3} - \frac{4}{9} \right) & \text{if } t \geq \left( \frac{4}{15 \times 9} \right)^2 \end{cases}
\]

\[
|u_1 - z_1| \leq \text{const } K \varepsilon
\]

\[
|u_2 - z_2| \leq \text{const } K \varepsilon
\]

\[
|u_1 - u_2| \geq |z_1 - z_2| - \text{const } K \varepsilon \geq \frac{\text{const}}{K}
\]

\[
|\sqrt{\frac{u'}{u}} - 1| \leq \text{const } \varepsilon
\]

we have

\[
|a - 1| \leq \text{const } K^2 \varepsilon + \text{const } K |U_{z_1}|
\]

\[
|b| \leq \text{const } K |U_{z_1}|
\]

\[
\left| \frac{\partial a}{\partial z_1} \right| \leq \text{const } K^2 \left( 1 + |U_{z_1}| \right) \left| \frac{\partial}{\partial z_1} (u_1 - u_2) \right| + \text{const } K |U_{z_1} z_1|
\]

Applying \(|\partial^\alpha U| \leq \text{const } |\alpha| \varepsilon (\text{const } K^2)^{|\alpha|}\) yields

\[
|a - 1| \leq \text{const } K^2 \varepsilon + \text{const } K^3 \varepsilon
\]

\[
|b| \leq \text{const } K^3 \varepsilon
\]

\[
\left| \frac{\partial a}{\partial z_1} \right| \leq \text{const } K^2 \left| \frac{\partial}{\partial z_1} (u_1 - u_2) \right| + \text{const } K^5 \varepsilon
\]

We used \(|U_{z_1}| \leq \text{const } K^2 \varepsilon \leq \text{const } \) in the last line. Finally

\[
\left| \frac{\partial}{\partial z_1} (u_1 - u_2) \right| = \left| (1 + \frac{t'}{u_1}) \frac{\partial u_1}{\partial z_1} \right| = \left| (1 + \frac{t'}{u_1}) \frac{u_1}{z_1} b \right| = \left| (u_1 + u_2) \frac{b}{z_1} \right| \leq \text{const } K^3 \varepsilon
\]

\[
\n
\text{Proof of Theorem 23.9: } \text{Fix } q \in C^\infty (\mathbb{R}^2/\Gamma) \text{ and } \rho > 0 \text{ such that the decomposition after Theorem 23.4 works for all } q' \in C^\infty (\mathbb{R}^2/\Gamma) \text{ with } |||b|^{\beta} q'||_1 < 2|||b|^{\beta} q'||_1. \text{ Then there exist}
\]

\[
\[
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\]
$K > 0$ and $\gamma > 0$, a tubular neighbourhood $T$ and a contour $\Gamma$, such that all of Definition 22.2 parts (iii) and (vi) as well as all parts of (v) except the second half of the fifth bullet, are satisfied for all $q' \in C^\infty(\mathbb{R}^2/\Gamma)$ with $\|b|^\beta \tilde{q}'\|_1 < 2\|b|^\beta \tilde{q}\|_1$. Further increase $K$, if necessary, so that $t_b < \left(1 - \frac{2\varepsilon}{15K}\right)^2 \left(1 + \frac{1}{K}\right)^{-4}$ for all $b \in \Gamma^\sharp$, $b_2 > 0$ and all $q' \in C^\infty(\mathbb{R}^2/\Gamma)$ with $\|b|^\beta \tilde{q}'\|_1 < 2\|b|^\beta \tilde{q}\|_1$. Now let $0 < \varepsilon < \frac{1}{150}$ and pick $\varepsilon$ such that $const K^6 \varepsilon < \varepsilon$ with the const being that of Lemma 23.10. Choose a contour $\partial G_0$ in $\Phi$ such that (iv) and the second half of the fifth bullet of (v) are satisfied. If $q'$ is sufficiently close to $q$ then $v'_b$ can be made arbitrarily close to $v_b$ and $w'_b$ can be made arbitrarily close to $w_b$ for all $b$ with $\frac{u_b + w_b}{2}$ inside $\partial G_0$. In particular, $|v - v'| < \varepsilon|v_b - w_b|$, $|w_b - w'_b| < \varepsilon|v_b - w_b|$ and $|t_b - t'_b| < \varepsilon t_b$ can be satisfied for all $b$ with $\frac{u_b + w_b}{2}$ inside $\partial G_0$. Define $\mathcal{H}(q')_0$ to be the compact part of $\mathcal{H}(q')$ bounded by $\Phi'(\partial G_0)$. If $q'$ is close to $q$ we define

$$F : \mathcal{H}(q)_0 \rightarrow \mathcal{H}(q')_0$$

by

$$F(x) = K'_2^{-1} \circ K_2(x) \quad \text{for} \quad x \notin \phi_b\{ (z_1, z_2) \in H(t_b) \mid |z_1|, |z_2| \leq \frac{1}{2} \} \quad \forall b \in \Gamma^\sharp, \ b_2 > 0$$

$$F(x) = \phi_b^* \circ f_b \circ \phi_b(x) \quad \text{for} \quad x \in \phi_b\{ (z_1, z_2) \in H(t_b) \mid |z_1|, |z_2| \leq \frac{1}{2} \}$$

where $K_2$ was defined in the discussion following Lemma 23.5 and $f_b$ is defined in Lemma 23.10. Then, if $q'$ is sufficiently close to $q$ (i),(ii) hold.

Parts (iii) and (iv) of Theorem 23.9 are obvious as $z_b, y_b$ depend continuously on $q$. \hfill \blacksquare

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References


