Boundaries and random walks on finitely generated infinite groups

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Abstract

We prove that almost every path of a random walk on a finitely generated non-amenable group converges in the compactification of the group introduced by W.J. Floyd. In fact, we consider the more general setting of ergodic cocycles of some semigroup of 1-Lipschitz maps of a complete metric space with a boundary constructed following Gromov. We obtain in addition that when the Floyd boundary of a finitely generated group is non-trivial, then it is in fact maximal in the sense that it can be identified with the Poisson boundary of the group with reasonable measures. The proof relies on work of V. Kaimanovich together with visibility properties of Floyd boundaries. We also prove a related statement about convergence of certain sequences of points, for example quasi-geodesic rays or orbits of 1-Lipschitz maps.

1 Introduction

In several situations concerning an infinite group it has been useful to consider an auxiliary space which in some sense is a boundary. For a finitely generated group $\Gamma$ one could start with the Cayley graph $K(\Gamma, S)$ with respect to some finite set of generators $S$. The end-compactification of the group is the graph itself union the space of ends of this graph and was first defined by H. Freudenthal, see [S 71]. There are however certain groups with only one end, but for which one would like to have a non-trivial boundary. For example this is the case for fundamental groups of compact negatively
curved manifolds. A finer compactification \( \bar{\Gamma} = \Gamma \cup \partial \Gamma \) was first used by W.J. Floyd [Fl 80] and it is obtained by first rescaling the length 1 edges in a certain way so that the graph gets finite diameter, then one takes the completion of the graph as a metric space. Indeed, such a boundary \( \partial \Gamma \) of a fundamental group \( \Gamma \) of a compact surface of genus at least two is the circle. Starting with [Fl 80] this compactification has been used in Kleinian group theory. In this context, we wish to draw the reader’s attention to a conjecture stated in a recent paper by C.T. McMullen [McM 01] concerning the existence of a boundary map between the Floyd boundary of a fundamental group into the boundary of hyperbolic 3-space.

In this note we prove the convergence of quasigeodesics and paths of certain random walks in a geodesic space to points in a boundary which is constructed following M. Gromov [Gr 87] extending Floyd. In particular, when \( \Gamma \) is a finitely generated non-amenable group with a measure \( \mu \) whose support generates the group, we obtain that the Floyd boundary \( \partial \Gamma \) is a \( \mu \)-boundary in the sense of H. Furstenberg, see [Fu 73] and [Ka 00]. Using a different approach, by demonstrating certain visibility properties and then relying on work of V. Kaimanovich we obtain that this \( \mu \)-boundary is in fact either trivial or maximal, in the latter case it is the Poisson boundary. In general, it may happen that the boundary is trivial: the Floyd-boundary of two finitely generated infinite groups is one point.

It was previously known that the Poisson boundary (for reasonable measures) of a group with infinitely many ends can be identified with the space of ends with the hitting measure. See W. Woess [Wo 89], D.I. Cartwright and P.M. Soardi [CS 89], and Kaimanovich [Ka 00]. Furthermore, Kaimanovich obtained an identification of the Poisson boundary for hyperbolic groups, see [Ka 94] and [Ka 00]. See also the work by A. Ancona [A 90].

Note also the somewhat related results of Floyd [Fl 84] and C.W. Stark [St 92], which extend the result in the original paper [Fl 80] and concerns the comparison of \( \partial \Gamma \) with the Furstenberg boundary for rank one symmetric spaces.

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2 Preliminaries on metric spaces

The following material is standard; we borrow some notation from [BHK 01].
Let \((Y, d)\) be a metric space. The \textit{length} of a continuous curve \(\alpha : [a, b] \to Y\) is defined to be

\[
L(\alpha) = \sup \sum_{i=1}^{k} d(\alpha(t_{i-1}), \alpha(t_i))
\]

where the supremum is taken over all finite partitions \(a = t_0 < t_1 < ... < t_k = b\). When this supremum is finite, \(\alpha\) is said to be \textit{rectifiable}. For such \(\alpha\) we can define the \textit{arc length} \(s : [a, b] \to [0, \infty)\) by

\[
s(t) = L(\alpha|_{[a,t]}),
\]

which is a function of bounded variation.

A \textit{geodesic} is a curve \(\beta\) for which

\[
d(\beta(t), \beta(t')) = L(\beta|_{[t,t']}) = |t - t'|
\]

for any \(t, t'\). A metric space is called \textit{geodesic} if any two points can be joined by a geodesic segment.

Given a continuous, (strictly) positive function \(f\) on \(Y\), we define the \(f\)-\textit{length} of a rectifiable curve \(\alpha\) to be

\[
L_f(\alpha) = \int_a^b f \, ds = \int_a^b f(\alpha(t))ds(t).
\]

If \(f \equiv 1\), then \(L_f = L\).

Assume from now on that \((Y, d)\) is a geodesic space. A new distance \(d_f\) is defined by

\[
d_f(x, y) = \inf L_f(\alpha)
\]

where the infimum is taken over all rectifiable curves \(\alpha\) with \(\alpha(a) = x\) and \(\alpha(b) = y\). It is straightforward to verify that \((Y, d_f)\) indeed is a metric space and that the two metrics induce the same topology. Note that for a geodesic \(\beta\), we have the simple bound

\[
L_f(\beta) \leq \max f |_{\beta} \cdot L(\beta).
\]

## 3 Definition of \(f\)-boundaries

This section defines certain boundaries of a complete metric space. The construction here is a somewhat more restrictive version of the one given by Gromov [Gr 87] section 7.2.K “A conformal view on the boundary”, which
extends Floyd [Fl 80], which in turn is “based on an idea of Thurston’s and
inspired by a construction of Sullivan’s”.

Let \((Y,d)\) be a complete metric space which is geodesic and let \(f\) be a
continuous, (strictly) positive function on \(Y\). We will assume that for some
point \(y\) in \(Y\), \(f\) is bounded by a monotone real function \(F\) in the following
way:

\[
f(x) \leq F(d(y,x)),
\]

for every \(x\) and furthermore we require that this \(F\) is summable:

\[
\sum_{r=1}^{\infty} F(r) < \infty,
\]

and that for every \(c > 0\) there is a number \(N\) such that

\[
F(cr) \leq NF(r)
\]

for all \(r \geq 0\).

Let the \(f\)-boundary of \(Y\) be the space \(\partial f Y := Y_f - Y\), where \(Y_f\) denotes
the metric space completion of \((Y,d_f)\).

**Floyd’s boundary.** This is essentially the construction introduced in
[Fl 80]. Let \(\Gamma\) be a group generated by a finite set of elements \(S\). Associated
to \(S\) there is a left-invariant metric (word metric) \(d\) on \(\Gamma\) and a 1-complex
(Cayley graph) \(K(\Gamma, S)\). The vertices of this graph consists of the elements
of \(\Gamma\) and two vertices are connected by an (unoriented) edge if they differ
by an element of \(S\) on the left. When the edges are assigned to have length
1, the distance \(d\) on \(\Gamma\) is simply the geodesic distance in the graph.

Let \(F\) be a monic, summable function \(F : \mathbb{N} \rightarrow \mathbb{R}\), such that given \(k \in \mathbb{N}\)
there exists \(M, N > 0\) so that \(MF(kr) \leq F(kr) \leq NF(r)\) for all natural
numbers \(r\). (It is common to consider \(F(r - 1) := r^{-2}\).) We insist for conve-
nience that \(F\) is monotonically decreasing and let \(f(x) = F(d(x,e))\). Since
\(F\) is summable the graph now has finite diameter. The **group completion in the sense of Floyd** \(\overline{\Gamma} = \Gamma \cup \partial \Gamma\) is (just as above) the completion of the Cay-
ley graph with the new distance \(d_f\) as a metric space (i.e. the equivalence
classes of Cauchy sequences). The group \(\Gamma\) acts on \(\overline{\Gamma}\) by homeomorphisms.
If \(\partial \Gamma\) consists of only 0, 1, or 2 points, we say that the boundary is trivial.

**Examples.** It is easy to see that for groups with infinitely many ends
\(\partial \Gamma\) coincides with the space of ends. When \(\Gamma\) is a word hyperbolic group
then, under some conditions, the \(f\)-boundary \(\partial f \Gamma\) coincides with the stan-
dard hyperbolic boundary, see [Gr 87] or for an exposition with more details
The conjecture stated in [McM 01] predicts that the Floyd boundary of a finitely generated fundamental group of a hyperbolic 3-manifold is at least as large as the limit set. For geometrically finite Kleinian groups (in every dimension) this was already proven in [Fl 80] and [Tu 88]. Some calculations by G. Noskov and the author indicate that when $Y$ is a bounded convex domain equipped with Hilbert’s metric, the Floyd type boundary is the usual boundary but collapsing every face to points and also every two non-trivial adjacent faces are identified. It was suggested by Gromov that for a Hadamard space (CAT(0)-space) the points of the Floyd boundary will be the Tits components of the boundary.

4 Ergodic cocycles and $\mu$-boundaries

Let $(Y, d), y, f$, and $\partial Y := \partial_f Y$ be as in section 3.

Let $(X, \nu)$ be a measure space with $\nu(X) = 1$ and let $L : X \to X$ be an ergodic measure preserving transformation. Let $w : X \to S$ be a measurable map into a semigroup $S$ of selfmaps of $Y$ which does not increase $d$-distances. Assume that the following integrability condition

$$\int_X d(y, w(x)y)d\nu(x) < \infty$$

holds. Define the associated ergodic cocycle (or random product)

$$u(n, x) = w(x)w(Lx) \cdots w(L^{n-1}x)$$

and

$$A := \lim_{n \to \infty} \frac{1}{n} \int_X d(e, u(n, x))d\mu(x).$$

A basic observation is that $a(n, x) := d(y, u(n, x)y)$ is a subadditive cocycle. The following purely subadditive ergodic statement was proved in [KM 99]:

**Lemma.** For each $\epsilon > 0$, let $E_\epsilon$ be the set of $x$ for which there exist an integer $K = K(x)$ and infinitely many $n$ such that

$$a(n, x) - a(n - k, L^kx) \geq (A - \epsilon)k$$

for all $k, K \leq k \leq n$. Then $\nu(\cap_{\epsilon > 0} E_\epsilon) = 1$.

Using this lemma we can prove:

**Theorem.** Assume that $A > 0$. Then for almost every $x$ the trajectory $u(n, x)y$ converges to a point $\xi = \xi(x) \in \partial Y$. 

5
Proof. Fix an \( x \in E_\epsilon \) for some \( \epsilon < A \). For each \( m \) denote by \( y_m \) the point \( u(m,x)y \). Consider an \( n_i \) and \( k \), \( K \leq k \leq n_i \), such that (†) in the lemma holds, and let \( \beta \) be a geodesic joining \( y_{n_i} \) and \( y_k \). Let \( J \) be the smallest integer larger than \( d(y_{n_i}, y_k) \) and \( j_0 \) be the smallest integer larger than \( 1/\min\{ A - \epsilon, 1/2 \} \). If \( J = 1 \), then similarly to the proof of the proposition above we get that

\[
d_f(y_{n_i}, y_k) \leq N F(k) \leq N \sum_{r=k-j_0}^{\infty} F(r)
\]

In the case \( J > 1 \), note that \( 1/2 \leq d(y_{n_i}, y_k)/J \leq 1 \) and let

\[
t_j = \frac{j}{J} d(y_{n_i}, y_k).
\]

Using the monotonicity of \( F \), (F1), the triangle inequality, the inequality (†), and (F3) we have:

\[
d_f(y_{n_i}, y_k) \leq L_f(\beta)
\]

\[
\leq \sum_{j=1}^{J} \max_{t_{j-1} \leq t \leq t_j} f(\beta(t))d(\beta(t_{j-1}), \beta(t_j))
\]

\[
\leq \sum_{j=1}^{J} F(d(y, \beta(t_i))) - 1
\]

\[
\leq \sum_{j=1}^{J} F(a(n_i, x) - d(y_{n_i}, y_k) - jd(y_{n_i}, y_k)/J - 1)
\]

\[
\leq \sum_{j=1}^{J} F(a(n_i, x) - a(n_i - k, L^k x) + j/2 - 1)
\]

\[
\leq \sum_{j=1}^{J} F((A - \epsilon)k + j/2 - 1)
\]

\[
\leq N \sum_{j=1}^{J} F(j + k - j_0)
\]

\[
\leq N \sum_{r=k-j_0}^{\infty} F(r).
\]
Since $N$ and $j_0$ depend only on $A - \epsilon$, and because of \( F2 \), it follows that $u(n_i, x)y$ is a Cauchy sequence and moreover that the whole sequence $u(m, x)y$ hence converges to a point in $\partial Y$. \qed

We also have:

**Corollary.** Let $\Gamma$ be a finitely generated group and $\mu$ be a probability measure with finite first moment such that the support of $\mu$ generate $\Gamma$ as a semigroup. Assume that the Poisson boundary of $(\Gamma, \mu)$ is nontrivial (which is the case if $\Gamma$ is nonamenable). Then almost every sample path of the random walk $(\Gamma, \mu)$ converges to a (random) point of $\partial \Gamma$ (which is defined as in the example in section 3). The space $\partial \Gamma$ with the resulting limit measure is a $\mu$-boundary.

**Proof.** We will apply Theorem 4 to $(Y, d)$ being the group $\Gamma$ equipped with a left-invariant word-distance. Furthermore, we let $S = \Gamma$, which acts on $Y = \Gamma$ by translations preserving the word-metric. Let $(X, \nu)$ be the infinite product of $(\Gamma, \mu)$ and $L$ be the shift. It is a known fact that $L$ is an ergodic measure preserving transformation and the finiteness of the first moment simply translates into the integrability assumed above.

It is known that when the Poisson boundary is nontrivial the word length of the trajectory growths linearly (so $A > 0$) for almost every trajectory, see [Gu 80] or Theorem 5.5 in [Ka 00]. Hence by Theorem 4 almost every sample path converges to a point in $\partial \Gamma$.

The resulting measure space, $\partial \Gamma$ with the hitting measure, is a $\mu$-boundary, see [Fu 71] or for example [Ka 00] p. 660. \qed

**Remark.** Note that it does not seem to be clear whether the $\mu$-boundary obtained is nontrivial or not. In the next sections however, combining some observations about the Floyd boundary with works of Kaimanovich, we obtain that whenever the Floyd boundary is non-trivial, then it is indeed maximal.

## 5 Visibility of Floyd’s boundary

Let $\Gamma$ be a finitely generated infinite group with a boundary $\partial \Gamma$ of Floyd type, see the example in section 3. We start with a lemma:

**Lemma.** Let $z, w$ be two points in $\Gamma$ and let $[z, w]$ be a geodesic segment connecting $z$ and $w$. Then

$$d'(z, w) \leq 4rF(r) + 2 \sum_{j=r}^{\infty} F(j),$$

7
where \( r = d(e, [z, w]) \).

**Proof.** Let \( a \) denote the distance to \( z \) from a point \( m \) on \([z, w]\) closest to \( e \). The triangle inequality imply that \( a \leq r + R \). Let \( x_j, j = 0, ..., a \) be the points (vertices) of the geodesic segment \([m, z] \subset [w, z]\). Because of the minimality of \( r \) and the triangle inequality we have the following estimates:

\[
\begin{align*}
    d(e, x_j) &\geq r \\
    d(e, x_j) &\geq R - (a - j) \geq j - r.
\end{align*}
\]

For the usual reasons, we hence get

\[
    d'(m, z) \leq \sum_{j=0}^{a} F(\min\{d(e, x_j), d(e, x_{j+1})\})
\]

\[
    \leq \sum_{j=0}^{2r-1} F(r) + \sum_{j=2r}^{a} F(j - r)
\]

\[
    \leq 2r F(r) + \sum_{j=r}^{\infty} F(j).
\]

By the same consideration with \( w \) instead of \( z \), the lemma is proved. \( \square \)

Two boundary points \( \gamma, \xi \) are said to be *connected by a geodesic line* if there is a \( d \)-geodesic \( \alpha \) such that \( \alpha(n) \to \gamma \) and \( \alpha(-n) \to \xi \) as \( n \to \infty \).

**Proposition.** Every two points in \( \partial \Gamma \) can be connected by a geodesic line.

**Proof.** It is known and easy to show (see [Fl 80]) that every point \( \gamma \in \partial \Gamma \) can be represented as an endpoint of a geodesic ray from \( e \). Consider the geodesic segments \([\gamma(n), \xi(n)]\) for any two distinct boundary points. It follows from the lemma that the distance \( r_n \) from these curve segments to \( e \) must bounded (due to the summability of \( F \) and since \( \gamma(\infty) \neq \xi(\infty) \)). Thanks to the local finiteness of \( \Gamma \) (the Cayley graph) we may now extract a desired geodesic line using Cantor’s diagonal argument. \( \square \)

### 6 Poisson boundaries

The results in the previous section lead to (in notation and definitions as in [Ka 00]):

**Theorem.** If \( \partial \Gamma \) contains at least three points, then the compactification \( \bar{\Gamma} \) of Floyd type satisfies Kaimanovich’s conditions (CP), (CS), and (CG).
Proof. The action of $\Gamma$ by left translation on itself extends to an action on $\bar{\Gamma}$ by homeomorphisms. Since $\bar{\Gamma}$ is a compact metric space it is separable. Any two sequences on bounded distance from each other clearly cannot converge to different boundary points; this is (CP).

As shown in the proposition, any two boundary points can be joined by $d$-geodesics. Therefore the sets (strips) $S(\gamma, \xi)$ equal the union of all joining geodesic lines are non-empty and constitute an equivariant family of borel maps. Furthermore, for any three distinct boundary points $\xi_i$ we may find small neighbourhoods $U_i$ in $\bar{\Gamma}$ of each three points so that

$$S(U_1, U_2) \cap U_3 = \emptyset.$$ 

To see this, take for $U_i$ small disjoint $\varepsilon$ neighborhoods in metric $d'$. Now assume that we can find geodesic lines $\gamma_k$ in $S(U_1, U_2)$ and indices $n_k \to \infty$ (because every geodesic line in $S$ intersects a ball around $e$) such that $\gamma_k(n_k) \to \xi$ some point in $U_3 \cap \partial\Gamma$. But since the $d'$ length of every $d$-geodesic ray is uniformly bounded it follows $d'(\gamma_k(n_k), \gamma_k(\infty)) \to 0$, which cannot happen since the neighborhoods were disjoint. This proves (CS).

Finally the condition (CG): $d$ is a left-invariant metric on $\Gamma$ and the corresponding gauge is temperate (just meaning in this context that $\Gamma$ is finitely generated). Finally, as already discussed, for any two boundary points every joining geodesic line intersects the same $d$-finite radius ball around $e$ (a consequence of the lemma in section 5).

We can now invoke the nice arguments (in particular, the strip approximation criterion) in Kaimanovich [Ka 00], sections 2, 3, 4, and 6 to conclude as he does in Theorem 6.6:

Corollary. Let $\Gamma$ be a finitely generated group and assume that a Floyd type boundary $\partial\Gamma$ contains more than three points. Let $\mu$ be a probability measure on $\Gamma$ with finite entropy and finite first logarithmic moment, and whose support generates a subgroup which is nonelementary with respect to $\bar{\Gamma}$, that is, it does not fix a finite subset of $\partial\Gamma$. Then the compactification $\bar{\Gamma}$ is $\mu$-maximal.

This implies that the Floyd-boundary of a finitely generated amenable group can only contain at most two points, since it is known that there always exists non-degenerate measures for which the Poisson boundary is trivial for such a group. For more related results, we refer to [K 01b]. In particular, it is shown that a group with non-trivial Floyd boundary contains a non-commutative free subgroup and its Floyd boundary is a boundary in the sense of [Fu 73].
If the Floyd boundary is non-trivial for a finitely generated fundamental group of a hyperbolic 3-manifold, then by knowing that it can be identified with the Poisson boundary as we now do, we hence get an (almost everywhere) unique $\Gamma$-equivariant measurable map onto the limit set. Is this map continuous? This is now the content of the conjecture stated in [McM 01].

7 Convergence of certain sequences

Let $(Y,d)$, $y$, $f$, and $\partial Y := \partial_f Y$ be as in the previous section.

**Proposition.** Let $y_n$ be a sequence of points in $Y$ for which there exists two positive constants $A$ and $C$ such that $d(y_n, y_{n+1}) < C$ and $d(y_n, y) > An$ for all large $n$. Then $y_n$ converges to a point in $\partial Y$.

**Proof.** Let $\beta$ be a geodesic joining $y_n$ and $y_{n+1}$. For every large $n$ we have:

$$d_f(y_n, y_{n+1}) \leq \inf L_f(\alpha) \leq L_f(\beta) \leq \max_t f(\beta(t)) d(y_n, y_{n+1}) \leq F(An - C)C \leq CNF(n).$$

Hence the sequence of points in question is a $d'$-Cauchy sequence, because $F$ is summable, and $N$ and $C$ are independent of $n$. As $y_n \to \infty$ in $Y$, the sequence therefore converges to a point in the boundary of $Y$.

As a corollary of the proposition we have that any quasigeodesic ray converges. If $F(r-1) = r^{-(2+\epsilon)}$, for some positive $\epsilon$, then a similar argument shows that also any regular sequence in the sense of [Ka 00] converges.

Using the argument in the proof of Proposition 5.1. in [K 01a], the lemma in section 5 and the inequality

$$(z|w)_y \leq d(y, [z,w])$$

one can prove:

**Proposition.** Let $\phi$ be a 1-Lipschitz (or non-expanding) map of a complete metric space $Y$ to itself. Assume that $\partial_f Y$ is an $f$-boundary such that the lemma in section 5 holds. If $d(\phi^n(y), y) \to \infty$, then $\phi^n(y)$ converges to a point in $\partial_f Y$ as $n \to \infty$. 

10
References


