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DIOPHANTINE INEQUALITIES ON PROJECTIVE VARIETIES

JAN-HENDRIK EVERTSE AND ROBERTO G. FERRETTI

1. Introduction

1.1. Let S be a finite set of places of a number field K. For $v \in S$, let $l_{0v}, \dots, l_{nv}$ be linearly independent linear forms in the variables $y_0, \dots, y_n$ with coefficients in K. Consider the system of inequalities

$$(1.1) \qquad \log\left(\frac{|l_{iv}(\mathbf{y})|_v}{\|\mathbf{y}\|_v}\right) \leqslant -d_{iv}h(\mathbf{y}) \quad (v \in S, i = 0, \dots, n) \quad \text{in } \mathbf{y} \in \mathbb{P}^n(K)$$

with reals $d_{iv} \geqslant 0$, where $|\ |_v, \|\ \|_v$ $(v \in S)$ are normalized absolute values and norms (see §2), and where $h(\mathbf{y})$ denotes the absolute logarithmic height. Schmidt's Subspace Theorem states that if

$$(1.2) \qquad \sum_{v \in S} \sum_{i=0}^{n} d_{iv} > n + 1,$$

then the set of solutions of (1.1) lies in the union of finitely many proper linear subspaces of $\mathbb{P}^n$.

We give an equivalent formulation on which we shall focus in this paper. Let $\{l_0, \dots, l_N\}$ be the union of the sets $\{l_{0v}, \dots, l_{nv}\}$ $(v \in S)$. Define the map $\varphi : \mathbb{P}^n \to \mathbb{P}^N$ by $\mathbf{y} \mapsto \big(l_0(\mathbf{y}) : \cdots : l_N(\mathbf{y})\big)$. Put $X := \varphi(\mathbb{P}^n)$; then X is a linear subvariety of $\mathbb{P}^N$ of dimension n defined over K. Write $x_i = l_i(\mathbf{y})$ $(i = 0, \dots, N)$, $\mathbf{x} = (x_0 : \cdots : x_N) = \varphi(\mathbf{y})$. For $v \in S$, let I_v be the set of indices given by $\{l_i : i \in I_v\} = \{l_{0v}, \dots, l_{nv}\}$, put $c_{iv} := d_{jv}$ if $l_i = l_{jv}$ and $c_{iv} = 0$ if $i \notin I_v$. Then (apart from some modifications in the norms and the height) we can rewrite (1.1) as

$$(1.3) \qquad \log\left(\frac{|x_i|_v}{\|\mathbf{x}\|_v}\right) \leqslant -c_{iv}h(\mathbf{x}) \quad (v \in S, i = 0, \dots, N) \quad \text{in } \mathbf{x} \in X(K)$$

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and (1.2) as

$$\sum_{v\in S}\sum_{i\in I_v} c_{iv} > n+1. \tag{1.4}$$

A set of indices $\{i_0,\dots,i_m\}\subseteq\{0,\dots,N\}$ is called *independent* with respect to X if there is no tuple $(a_{i_0},\dots,a_{i_m})\in\overline{\mathbb{Q}}^{m+1}\backslash\{\mathbf{0}\}$ such that $a_{i_0}x_{i_0}+\cdots+a_{i_m}x_{i_m}$ vanishes identically on X. In particular, the sets I_v from (1.4) are independent and of cardinality $n+1$.

Thus the Subspace Theorem can be stated alternatively as follows: if the reals c_{iv} satisfy (1.4) for certain independent sets I_v ($v\in S$) of cardinality $n+1$, then the set of solutions of (1.3) is contained in the union of finitely many proper linear subspaces of X.

1.2. Starting with Schmidt [17], much work has been done to obtain good quantitative versions of the Subspace Theorem. The sharpest such version to date is due to Evertse and Schlickewei ([4], Thm. 2.1). From their result we will deduce the following for (1.3). Let again $X\subseteq\mathbb{P}^N$ be an $n-$dimensional linear subvariety defined over a number field K. Denote by $h(X)$ the logarithmic height of X (cf. §2). Suppose that there are independent sets I_v ($v\in S$) of cardinality $n+1$ such that

$$\frac{1}{n+1}\sum_{v\in S}\sum_{i\in I_v} c_{iv} > 1+\delta \quad \text{with } \delta>0. \tag{1.5}$$

Then there are explicitly computable constants c_1, c_2, depending only on N, n, δ, such that the set of solutions $\mathbf{x}\in X(K)$ of (1.3) with $h(\mathbf{x})\geqslant c_1(1+h(X))$ is contained in the union of at most c_2 proper linear subspaces of X. It has turned out to be crucial for applications that c_1, c_2 are independent of K and S. More generally, the result of Evertse and Schlickewei allows to deduce a similar result for an "absolute" generalization of (1.3) dealing with points in $X(\overline{\mathbb{Q}})$ rather than in $X(K)$. For the precise statement we refer to Theorem 2.5 in §2.

1.3. Faltings and Wüstholz [6] proved a generalization of the Subspace Theorem, dealing with systems of inequalities (1.3) where X is an arbitrary projective subvariety of $\mathbb{P}^N$ rather than just a linear subvariety. They developed a method of proof totally different from Schmidt's, based on Faltings' Product Theorem (cf. [5], Theorem 3.1, 3.3).

Using the method of Faltings and Wüstholz, Ferretti [7] obtained a quantitative version of their result, an equivalent version of which reads as follows. Assume X has dimension n and degree d. Recall that the Chow form of X is a polynomial $F_X(h_{00},\dots,h_{0N};\dots;h_{n0},\dots,h_{nN})$ in $(N+1)(n+1)$ variables which is homogeneous of degree d in $(h_{i0},\dots,h_{iN})$ for $i=0,\dots,n$. For $v\in S$, let $\mathbf{c}_v=(c_{0v},\dots,c_{Nv})$ and define the *Chow weight* $e_X(\mathbf{c}_v)$ of X with respect to $\mathbf{c}_v$ to be the highest power of t occurring in $F_X(t^{c_{0v}}h_{00},\dots,t^{c_{Nv}}h_{0N};\dots;t^{c_{0v}}h_{n0},\dots,t^{c_{Nv}}h_{nN})$. Assume that

$$\frac{1}{(n+1)d}\sum_{v\in S}e_X(\mathbf{c}_v)>1+\delta \quad \text{with } \delta>0. \tag{1.6}$$

Then there are explicitly computable constants c_1,c_2,c_3, depending on N,n,δ,K,S and some geometric invariants of X, such that the set of solutions of (1.3) with $h(\mathbf{x})\geqslant c_1(1+h(X))$ lies in the union of at most c_2 proper subvarieties of X, each of degree $\leqslant c_3$. It can be shown that for linear varieties X, condition (1.6) is equivalent to (1.5).

1.4. In the present paper we prove another quantitative version of the result of Faltings and Wüstholz, in which the constants c_1,c_2,c_3 depend only on N,n,δ and the degree of X. Further, like for linear varieties, we prove a similar quantitative version for an absolute generalization of (1.3), dealing with points in $X(\overline{\mathbb{Q}})$. For the precise statement see Theorem 2.7 in §2.

We sketch our method which is very different from that of Faltings and Wüstholz. Let $\varphi_m:\mathbb{P}^N\hookrightarrow\mathbb{P}^R$ with $R=\binom{N+m}{m}-1$ denote the Veronese embedding, which maps $\mathbf{x}\in\mathbb{P}^N$ to the point consisting of all monomials in $\mathbf{x}$ of degree m. Let X_m denotes the smallest linear subvariety of $\mathbb{P}^R$ containing $\varphi_m(X)$. We construct from (1.3) a new system of a similar shape, with solutions taken from X_m, which is such that if $\mathbf{x}$ is a solution of (1.3) then $\varphi_m(\mathbf{x})$ is a solution of the new system. The hard core of our paper is to find an explicit value for m such that the analogue of condition (1.5) for the new system is satisfied. Having achieved this, we obtain our result for the original system (1.3) by applying our result for linear varieties to the new system.

In order to find a suitable value for m, we prove a result which gives, in some well-defined sense, an explicit lower bound of the m-th normalized Hilbert weight of X with respect to a tuple of reals $\mathbf{c}$ in terms of the normalized Chow weight of X with respect to $\mathbf{c}$ (cf. §3 for the definitions and the statement of the result). Our result may be viewed as a one-sided explicit version of a result of Mumford ([15], p. 61, Proposition 2.11) which states that the normalized Chow weight of X is the limit of the sequence of its normalized Hilbert weights.

As a by-product of our investigations we obtain that the result of Faltings and Wüstholz, which at a first glance seems to be a generalization of the Subspace Theorem, is in fact equivalent to the Subspace Theorem.

1.5. In §2 we give the precise statements of the results mentioned above related to (1.3) (Theorem 2.5 and Theorem 2.7). In §3 we recall the definition of the Hilbert weights and Chow weight of X, and state our result concerning these (Theorem 3.6). In §§4, 5 we prove Theorem 3.6. In §6 we prove Theorem 2.5 (the result for linear varieties). In §7 we prove an auxiliary result about heights. Finally, in §8 we prove Theorem 2.7 (the result for arbitrary varieties).

2. Statements of the results

2.1. We introduce the necessary absolute values and heights. Let K be a number field. Denote by M_K its set of places. For $v \in M_K$ we define a normalized absolute value $|.|_v$ on K by requiring that for $x \in \mathbb{Q}$:

$$\begin{aligned} |x|_v &= |x|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]} && \text{if } v \text{ is archimedean,} \\ |x|_v &= |x|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]} && \text{if } v \text{ lies above a prime number } p, \end{aligned}$$

where $\mathbb{Q}_p, K_v$ denote the respective completions. These absolute values satisfy the product formula $\prod_{v\in M_K} |x|_v = 1$ for $x \in K^*$.

Given a finite extension L of K we write $w|v$ to denote that a place w of M_L lies above $v \in M_K$. Further, we denote the completion of L at w by L_w. Then if we define normalized absolute values in the same manner for L, we get the extension formulas

$$|x|_w = |x|_v^{[L_w:K_v]/[L:K]} \quad \text{for } x \in K,\ w \in M_L,\ v \in M_K \text{ with } w|v. \tag{2.1}$$

2.2. For $\mathbf{x} = (x_0, \dots, x_N)$ with coordinates in a number field K and for $v \in M_K$ put

$$\|\mathbf{x}\|_v := \max\{|x_0|_v, \dots, |x_N|_v\}.$$

We then define the *absolute logarithmic height* of $\mathbf{x} \in \overline{\mathbb{Q}}^{N+1}$ by taking a number field K with $\mathbf{x} \in K^{N+1}$ and putting

$$h(\mathbf{x}) := \sum_{v \in M_K} \log \|\mathbf{x}\|_v.$$

By the product formula we have $h(\lambda\mathbf{x}) = h(\mathbf{x})$ for $\lambda \in K^*$ and by the extension formula, this height is independent of the choice of K. Therefore, h defines a height on $\mathbb{P}^N(\overline{\mathbb{Q}})$. For a polynomial P with coefficients in $\overline{\mathbb{Q}}$, we denote by $h(P)$ the absolute logarithmic height of the vector of coefficients of P.

2.3. We define the height of a projective variety. For vectors $\mathbf{a} = (a_0, \ldots, a_r)$, $\mathbf{b} = (b_0, \ldots, b_r)$ we define the usual scalar product $\mathbf{a} \cdot \mathbf{b} = a_0 b_0 + \cdots + a_r b_r$.

Let $X \subseteq \mathbb{P}^N$ be a projective variety of dimension n and degree d, defined over $\overline{\mathbb{Q}}$, where $1 \leqslant n < N$. To X we can associate an up to a constant factor unique polynomial $F_X = F_X(\mathbf{h}_0, \ldots, \mathbf{h}_n)$ in $n+1$ blocks of variables $\mathbf{h}_i = (h_{i0}, \ldots, h_{iN})$ $(i = 0, \ldots, n)$ which is irreducible in $\overline{\mathbb{Q}}[h_{00}, \ldots, h_{nN}]$ and which is homogeneous of degree d in each block $\mathbf{h}_i$, with the property that $F_X(\mathbf{h}_0, \ldots, \mathbf{h}_n) = 0$ if and only if X and the hyperplanes given by $\mathbf{h}_i \cdot \mathbf{x} = h_{i0}x_0 + \cdots + h_{iN}x_N = 0$ $(i = 0, \ldots, n)$ have a point in common. F_X is called the *(Cayley-Bertini-van der Waerden-)Chow form* of X. We define the height of X by

$$h(X) := h(F_X). \tag{2.2}$$

For instance, suppose that X is an n-dimensional linear subvariety of $\mathbb{P}^N$ over $\overline{\mathbb{Q}}$. Let $\{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$ be any basis of $X(\overline{\mathbb{Q}})$ considered as a vector space. Denote by $\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n$ the usual exterior product. Then

$$F_X(\mathbf{h}_0, \ldots, \mathbf{h}_n) = (\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n) \cdot (\mathbf{h}_0 \wedge \cdots \wedge \mathbf{h}_n), \tag{2.3}$$

and so

$$h(X) = h(\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n). \tag{2.4}$$

Faltings ([5], pp. 552, 553) defined another height for projective varieties by means of arithmetic intersection theory. Let $h_{\text{Falt}}(X)$ denote $\frac{1}{[K:\mathbb{Q}]}$ times the height introduced by Faltings where K is any number field over which X is defined. The quantity $h_{\text{Falt}}(X)$ is independent of K and by [1], Theorem 4.3.8, there is an explicitly computable constant $c(N)$ such that $|h(X) - h_{\text{Falt}}(X)| \leqslant c(N) \cdot \deg X$.

2.4. We state our quantitative result for (1.3) if X is a linear variety.

Let $X \subset \mathbb{P}^N$ be a linear subvariety of dimension n defined over a number field K, where $1 \leqslant n < N$. Recall that a set of indices $\{i_0, \dots, i_n\}$ is called independent if there is no tuple $(a_{i_0}, \dots, a_{i_n}) \in \overline{\mathbb{Q}}^{n+1} \backslash \{\mathbf{0}\}$ such that $a_{i_0}x_{i_0} + \cdots + a_{i_n}x_{i_n}$ vanishes identically on X. Denote by $\mathcal{I}_X$ the collection of all independent subsets of $\{0, \dots, N\}$ of cardinality $n+1$.

We consider the system of inequalities

$$\log\left(\frac{|x_i|_v}{\|\mathbf{x}\|_v}\right) \leqslant -c_{iv}h(\mathbf{x}) \quad (v \in S,\ i = 0, \dots, N) \quad \text{in } \mathbf{x} \in X(K)$$

with reals $c_{iv} \geqslant 0$, where as before, $(x_0 : \cdots : x_N)$ are the homogeneous coordinates of $\mathbf{x}$. More generally, for every finite extension L of K we consider

$$\log\left(\frac{|x_i|_w}{\|\mathbf{x}\|_w}\right) \;\leqslant\; -c_{iw}h(\mathbf{x}) \quad (w \in S_L,\ i = 0, \dots, N) \quad \text{in } \mathbf{x} \in X(L) \tag{2.5}$$

where S_L is the set of places of L lying above the places in S and where

$$c_{iw} = c_{iv} \cdot \tfrac{[L_w:K_v]}{[L:K]} \quad \text{for } i = 0, \dots, N,\ w \in S_L,\ v \in S \text{ with } w|v. \tag{2.6}$$

For a given finite extension L of K denote by $\mathcal{S}_X(L)$ the set of solutions of (2.5). Extension formula (2.1) implies that if $K \subset L_1 \subset L_2$ are number fields, then $\mathcal{S}_X(L_2) \cap X(L_1) = \mathcal{S}_X(L_1)$. We put

$$\mathcal{S}_X(\overline{\mathbb{Q}}) = \bigcup_{L \supseteq K} \mathcal{S}_X(L),$$

where the union is taken over all finite extensions L of K.

Theorem 2.5. *Let $X \subset \mathbb{P}^N$ be a linear subvariety of dimension n defined over K, where $1 \leqslant n < N$. Let S be a finite set of places of K. Further, let $\delta > 0$ and let c_{iv} $(v \in S,\ i = 0, \dots, N)$ be reals $\geqslant 0$ such that*

$$\frac{1}{n+1} \sum_{v \in S} \max_{\{i_0, \dots, i_n\} \in \mathcal{I}_X} (c_{i_0,v} + \cdots + c_{i_n,v}) \geqslant 1 + \delta. \tag{2.7}$$

Then there are proper linear subspaces $Y_1, \dots, Y_t$ of X, all defined over K, with

$$t \;\leqslant\; 4^{(n+10)^2}(1+\delta^{-1})^{n+5} \log(3N) \log\log(3N), \tag{2.8}$$

such that the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ *with*

$$h(\mathbf{x}) \geqslant (1+\delta^{-1})(N+1)^{n+1} \cdot \big(1+h(X)\big) \tag{2.9}$$

is contained in $Y_1 \cup \cdots \cup Y_t$.

2.6. We state a result similar to Theorem 2.5 for arbitrary projective subvarieties (i.e. geometrically irreducible Zariski closed subsets) of $\mathbb{P}^N$.

Let $X \subset \mathbb{P}^N$ be an arbitrary projective variety of dimension n and degree d which is defined over a number field K. We assume again $1 \leqslant n < N$. Let $\mathbf{c} = (c_0, \ldots, c_N) \in \mathbb{R}^N$ and let t be a parameter. By substituting $t^{c_j} h_{ij}$ for h_{ij} (the j-th variable of the i-th block) in the Chow form F_X for $i = 0, \ldots, n$, $j = 0, \ldots, N$ we obtain

$$F_X(t^{c_0}h_{00}, \ldots, t^{c_N}h_{0N}; \ldots; t^{c_0}h_{n0}, \ldots, t^{c_N}h_{nN}) = \sum_{k=0}^{T} t^{e_j} F_j \tag{2.10}$$

$$\text{with } F_0, \ldots, F_T \in K[h_{00}, \ldots, h_{nN}],\ \ e_0 > e_1 > \cdots > e_T.$$

Now we define the Chow weight of X with respect to $\mathbf{c}$ by

$$e_X(\mathbf{c}) := e_0. \tag{2.11}$$

If we write monomials in the variables h_{ij} $(i = 0, \ldots, n,\ j = 0, \ldots, N)$ as $\mathbf{h}_0^{\mathbf{a}_0} \cdots \mathbf{h}_n^{\mathbf{a}_n}$, where $\mathbf{a}_i = (a_{i0}, \ldots, a_{iN})$ is a vector with non-negative coordinates and $\mathbf{h}_i^{\mathbf{a}_i} := h_{i0}^{a_{i0}} \cdots h_{iN}^{a_{iN}}$ for $i = 0, \ldots, n$, then $e_X(\mathbf{c})$ is the maximum of the scalar products $(\mathbf{a}_0 + \cdots + \mathbf{a}_n) \cdot \mathbf{c}$, taken over all monomials $\mathbf{h}_0^{\mathbf{a}_0} \cdots \mathbf{h}_n^{\mathbf{a}_n}$ occurring with non-zero coefficient in F_X.

Let again S be a finite set of places of K, and c_{iv} $(v \in S,\ i = 0, \ldots, N)$ non-negative reals. For a finite extension L of K, let S_L be the set of places of L lying above those in S, and let c_{iw} $(w \in S_L,\ i = 0, \ldots, N)$ be defined by (2.6). Denote by $\mathcal{S}_X(L)$ the set of solutions of

$$\log\left(\frac{|x_i|_w}{\|x\|_w}\right) \leqslant -c_{iw} h(\mathbf{x}) \ \ (w \in S_L, i = 0, \ldots, N) \quad \text{in } \mathbf{x} \in X(L)$$

and let

$$\mathcal{S}_X(\overline{\mathbb{Q}}) = \bigcup_{L \supseteq K} \mathcal{S}_X(L),$$

where the union is taken over all finite extensions L of K.

By a *K-subvariety* of X we mean a Zariski closed subset which is defined over K and not the union of two strictly smaller Zariski closed subsets defined over K. Then we have:

Theorem 2.7. *Let $X \subset \mathbb{P}^N$ be a projective subvariety of dimension n and degree d defined over a number field K, where $1 \leqslant n < N$. Let S be a finite set of places of K. Further, let $\delta > 0$ and let $\mathbf{c}_v = (c_{0v}, \dots, c_{Nv})$ $(v \in S)$ be tuples of non-negative reals with*

$$\frac{1}{(n+1)d} \sum_{v \in S} e_X(\mathbf{c}_v) \;\geqslant\; 1 + \delta. \tag{2.12}$$

Put

$$\begin{cases} c_1(N,n,d,\delta) := \exp\Big((10n)^{4n} d^{4n+2} (1+\delta^{-1})^{2n}\Big) \cdot \log(3N) \log\log(3N), \\ c_2(N,n,d,\delta) := (8n+5)(1+\delta^{-1}) d^2 \min\big((n+1)d, N+1\big), \\ c_3(N,n,d,\delta) := \exp\Big((10n)^{2n+2} d^{2n+3} (1+\delta^{-1})^{n+1} \cdot \log(3N)\Big). \end{cases} \tag{2.13}$$

Then there are proper K-subvarieties $Y_1, \dots, Y_t$ of X with

$$t \;\leqslant\; c_1(N,n,d,\delta), \tag{2.14}$$

$$\deg Y_i \;\leqslant\; c_2(N,n,d,\delta) \quad \text{for } i = 1, \dots, t, \tag{2.15}$$

such that the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant c_3(N,n,d,\delta) \cdot \big(1 + h(X)\big) \tag{2.16}$$

is contained in $Y_1 \cup \cdots \cup Y_t$.

2.8. Let again $\mathbf{h}_i = (h_{i0}, \dots, h_{iN})$ $(i = 0, \dots, n)$ be the blocks of variables occurring in the Chow form F_X of an n-dimensional variety X. We define for each subset $I = \{j_0, \dots, j_n\}$ of $\{0, \dots, N\}$ with $j_0 < \dots < j_n$ the *bracket* $[I] = [j_0 \cdots j_n] = \det(h_{i,j_k})_{i,k=0,\dots,n}$. From [12], p. 41, Thm. IV it follows that F_X can be expressed as a polynomial in terms of such brackets. For given $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$, the substitution $t^{c_j} h_{ij}$ for h_{ij} $(i = 0, \dots, n,\ j = 0, \dots, N)$ transforms $[I]$ into $t^{\sum_{j \in I} c_j}[I]$.

For instance, let $X \subset \mathbb{P}^N$ be a linear subvariety of dimension n. Let again $\mathcal{I}_X$ denote the collection of independent subsets of $\{0, \dots, N\}$ of cardinality $n+1$. Then from (2.3) it follows that $F_X = \sum_{I \in \mathcal{I}_X} \gamma_I [I]$ with $\gamma_I \neq 0$ and therefore,

$$e_X(\mathbf{c}) = \max_{\{i_0, \dots, i_n\} \in \mathcal{I}_X} c_{i_0} + \cdots + c_{i_n}.$$

So for linear varieties X, (2.12) is equivalent to (2.7).

2.9. Now let $X \subset \mathbb{P}^N$ be the hypersurface given by $f = 0$, where

$$f = \sum_{\mathbf{a} \in A} \beta(\mathbf{a}) x_0^{a_0} \dots x_N^{a_N} \in K[x_0, \dots, x_N] \tag{2.17}$$

is a homogeneous polynomial of degree d which is irreducible over $\overline{\mathbb{Q}}$. Here A is a finite set of tuples of non-negative integers $\mathbf{a} = (a_0, \dots, a_N)$ with $a_0 + \cdots + a_N = d$, and $\beta(\mathbf{a}) \neq 0$ for $\mathbf{a} \in A$.

The variety X has dimension $n = N - 1$ and degree d. The Chow form of X is equal to

$$\begin{aligned} F_X &= f([1\ 2 \cdots N], -[0\ 2 \cdots N], \dots, (-1)^{N-1}[0\ 1 \cdots N-1]) \\ &= \sum_{\mathbf{a} \in A} \pm \beta(\mathbf{a}) [1\ 2 \cdots N]^{a_0} \cdots [0\ 1 \cdots N-1]^{a_N}. \end{aligned} \tag{2.18}$$

It follows that for $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}$ we have

$$e_X(\mathbf{c}) = d(c_0 + \cdots + c_N) - \min_{\mathbf{a} \in A} (a_0 c_0 + \cdots + a_N c_N).$$

From (2.18) we deduce, after some elementary height computations,

$$h(X) \leqslant h(f) + 3Nd \log N.$$

By combining these two facts with Theorem 2.7 we obtain at once:

Corollary 2.10. *Let $X \subset \mathbb{P}^N$ be the irreducible hypersurface given by $f = 0$, where f is given by (2.17). Let S, δ be as in Theorem 2.7, and let $\mathbf{c}_v = (c_{0v}, \dots, c_{Nv})$ $(v \in S)$ be tuples of non-negative reals with*

$$\frac{1}{N} \sum_{v \in S} \sum_{i=0}^{N} c_{iv} - \frac{1}{Nd} \sum_{v \in S} \min_{\mathbf{a} \in A} (a_0 c_{0v} + \cdots + a_N c_{Nv}) \geqslant 1 + \delta.$$

Further, let $c_1(N,n,d,\delta)$, $c_2(N,n,d,\delta)$, $c_3(N,n,d,\delta)$ *be the constants defined by (2.13).*

Then there are proper K*-subvarieties* $Y_1,\dots,Y_t$ *of* X *with*

$$t \leqslant c_1(N,N-1,d,\delta), \quad \deg Y_i \leqslant c_2(N,N-1,d,\delta) \text{ for } i=1,\dots,t$$

such that the set of $\mathbf{x}\in\mathcal{S}_X(\overline{\mathbb{Q}})$ *with*

$$h(\mathbf{x}) \geqslant \{c_3(N,N-1,d,\delta)\}^2(1+h(f))$$

is contained in $Y_1\cup\cdots\cup Y_t$.

2.11. We mention that computing the Chow weights $e_X(\mathbf{c})$ for arbitrary projective varieties X is in general quite difficult. In [8], [9] Ferretti discussed various methods to compute these quantities, and computed them for certain varieties other than linear varieties or hypersurfaces.

By approximating a given tuple of non-negative reals $\mathbf{c}=(c_0,\dots,c_N)$ by a tuple of rationals and then clearing denominators, we see that $e_X(\mathbf{c})$ is very close to a constant factor depending only on $\mathbf{c}$ times $e_X(\mathbf{c}')$ where $\mathbf{c}'$ is a tuple of non-negative integers. Therefore, it suffices to compute $e_X(\mathbf{c})$ if $\mathbf{c}$ consists of integers. Further, after a permutation of coordinates we may assume that $c_0\leqslant c_1\leqslant\cdots\leqslant c_N$.

Under these hypotheses, there is a description of $e_X(\mathbf{c})$ in terms of Segre classes. This means that $e_X(\mathbf{c})$ can be interpreted as a multiplicity of some sort. Define the line bundle $\mathcal{O}_{X\times\mathbb{A}^1}(1)=\mathcal{O}_X(1)\otimes\mathcal{O}_{\mathbb{A}^1}$, and denote by t the coordinate of $\mathbb{A}^1$. The set $\{t^{c_N-c_i}x_i:\ i=0,\dots,N\}$ generates a $K[t]$-submodule I of $H^0(X\times\mathbb{A}^1,\mathcal{O}_{X\times\mathbb{A}^1}(1))$. Let $J\subseteq\mathcal{O}_{X\times\mathbb{A}^1}$ be the ideal sheaf given by

$$J\cdot\mathcal{O}_{X\times\mathbb{A}^1}(1) = \text{sheaf generated by } I \text{ in } \mathcal{O}_{X\times\mathbb{A}^1}(1).$$

Choose a compactification Y of $X\times\mathbb{A}^1$ on which $\mathcal{O}_{X\times\mathbb{A}^1}(1)$ extends to a line bundle $\mathcal{L}$, and let $\pi: B\to Y$ be the blow-up along the subscheme Z of Y defined by the ideal sheaf J. Then

$$c_N(n+1)d - e_X(\mathbf{c}) = c_1(\pi^*\mathcal{L})^n - \Big(c_1(\pi^*\mathcal{L}) - c_1(\mathcal{O}_B(E))\Big)^n, \tag{2.19}$$

where E is the exceptional divisor, ([14], Proposition 2.4; [14], Proposition 3.2). According to [10] §4 (cf. [10], Corollary 4.2.2, and the projection formula) one may rewrite the right-hand side in terms of Segre classes:

$$c_N(n+1)d - e_X(\mathbf{c}) = \Big(1 + c_1(\mathcal{L})\Big)^n \cap s(Z, Y).$$

If Z is set-theoretically a point then $c_N(n+1)d - e_X(\mathbf{c})$ is the multiplicity of X at Z ([15], §2, Examples *ii*) p.55). Using this interpretation of $e_X(\mathbf{c})$, Ferretti ([9], Proposition 2.14) showed that if X is a $K3$-surface whose Picard group has rank 1 then for a suitable choice of coordinates $x_0, \dots, x_N$ one has

$$e_X(\mathbf{c}) \geqslant -8c_N - 4c_0 + 6\sum_{i=1}^{N} c_i.$$

Another method to compute $e_X(\mathbf{c})$ is by means of Gröbner basis theory. Assume that X is defined by a homogeneous ideal $I \subseteq K[x_0, \dots, x_N]$. For $f \in I$ and a variable t, write $f(t^{c_0}x_0, \dots, t^{c_N}x_N) = \sum_{i=0}^{h} t^{b_i} f_i$ with $b_0 < t_1 < \cdots < b_h$ and with $f_0, \dots, f_h$ polynomials independent of t and define the *initial part* of f with respect to $\mathbf{c}$ by $in_{\mathbf{c}}(f) := f_0$. Further, let $in_{\mathbf{c}}(I)$ be the ideal generated by the polynomials $in_{\mathbf{c}}(f)$ $(f \in I)$. Assume (as is often the case) that $in_{\mathbf{c}}(I)$ is a monomial ideal, i.e., generated by monomials. Then by means of Buchberger's algorithm (cf. [3], §15.3) one may compute from a given set of generators of I a set of generators of $in_{\mathbf{c}}(I)$, and from that the prime ideals of maximal dimension associated to $in_{\mathbf{c}}(I)$ together with their multiplicities (cf. [18], Proposition 3.4 or Lemma 4.3 of the present paper). For a subset W of $\{0, \dots, N\}$, let P_W denote the ideal of polynomials vanishing identically on the coordinate plane given by $x_j = 0$ for $j \in W$. Then the prime ideals of maximal dimension associated to $in_{\mathbf{c}}(I)$ are $P_{W_1}, \dots, P_{W_g}$ where each W_i is a subset of $\{0, \dots, N\}$ of cardinality $N - \dim X$, and according to (4.7) and Lemma 5.5 of the present paper or [9], Lemma 3.3, one has

$$e_X(\mathbf{c}) = \sum_{k=1}^{g} \mu_k \cdot \Big(\sum_{j \notin W_k} c_j\Big),$$

where μ_k is the multiplicity of P_{W_k} in $in_{\mathbf{c}}(I)$.

3. Hilbert weights and Chow weights

3.1. Denote by $\mathbb{Z}_{\geqslant 0}^{N+1}$, $\mathbb{R}_{\geqslant 0}^{N+1}$ the sets of $(N+1)$-tuples consisting of non-negative integers, non-negative reals, respectively. For $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{Z}_{\geqslant 0}^{N+1}$ we write $\mathbf{x}^{\mathbf{a}}$

for the monomial $x_0^{a_0}\cdots x_N^{a_N}$. In this section, K is an algebraically closed field of characteristic 0. A homogeneous ideal I of $K[x_0,\dots,x_N]$ is said to be relevant if $I\neq(0)$ and if there is no positive integer k such that $x_0^k,\dots,x_N^k\in I$.

3.2. For a positive integer m, let $K[x_0,\dots,x_N]_m$ denote the space of homogeneous polynomials in $K[x_0,\dots,x_N]$ of degree m (including 0). Let I be a relevant homogeneous ideal of $K[x_0,\dots,x_N]$. Put $I_m:=K[x_0,\dots,x_N]_m\cap I$ and define the Hilbert function H_I of I by

$$H_I(m):=\dim_K\Big(K[x_0,\dots,x_N]_m/I_m\Big)\quad\text{for } m=1,2,\dots. \tag{3.1}$$

We define the dimension $\dim I$ to be the maximum of the dimensions of the prime ideals (i.e., of the varieties defined by these prime ideals) associated to I. Thus, if $\dim I=n$ we have

$$H_I(m)=d\cdot\frac{m^n}{n!}+O(m^{n-1})\quad\text{as } m\to\infty, \tag{3.2}$$

where d is a positive integer, called the *degree* of I, notation $\deg I$.

Let $P_1,\dots,P_g$ be the prime ideals of maximal dimension associated to I. For $i=1,\dots,g$, let $O_{P_i,I}$ be the localization of $K[x_0,\dots,x_N]/I$ at P_i and let $\mu_{P_i,I}:=l_{O_{P_i,I}}(O_{P_i,I})$ be the length of $O_{P_i,I}$ as a $O_{P_i,I}$-module. This quantity is known to be finite. We call $\mu_{P_i,I}$ the multiplicity of I with respect to P_i. Then

$$\deg I=\sum_{i=1}^{g}\mu_{P_i,I}\deg P_i. \tag{3.3}$$

3.3. Let again I be a relevant homogeneous ideal of $K[x_0,\dots,x_N]$, and let $\mathbf{c}=(c_0,\dots,c_N)\in\mathbb{R}_{\geqslant 0}^{N+1}$. We define the *$m$-th Hilbert weight $s_I(m)$* of I with respect to $\mathbf{c}$ by

$$s_I(m,\mathbf{c})=\max(\mathbf{a}_1+\cdots+\mathbf{a}_{H_I(m)})\cdot\mathbf{c}, \tag{3.4}$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}$,…,$\mathbf{x}^{\mathbf{a}_{H_I(m)}}$ whose residue classes modulo I form a basis of the K-vector space $K[x_0,\dots,x_N]_m/I_m$.

3.4. We define the Chow form of a homogeneous prime ideal P of $K[x_0,\dots,x_N]$ by $F_P:=F_X$, where X is the variety defined by P and F_X is the Chow form of X as defined in **2.3**. Let again I be an arbitrary relevant homogeneous ideal I of

$K[x_0, \dots, x_N]$ with $\dim I = n$, $\deg I = d$. We define the Chow form of I by

$$F_I := \prod_{i=1}^{g} F_{P_i}^{\mu_{P_i,I}}, \tag{3.5}$$

where $P_1, \dots, P_g$ are the prime ideals of maximal dimension associated to I and where $\mu_{P_i,I}$ is the multiplicity of I with respect to P_i. By **2.3** and (3.2), F_I is a polynomial in $n+1$ blocks of $N+1$ variables $\mathbf{h}_i = (h_{i0}, \dots, h_{iN})$ $(i = 0, \dots, n)$ such that F_I is homogeneous of degree d in each block $\mathbf{h}_i$. Let again $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}_{\geqslant 0}^{N+1}$. Then similarly as in (2.10) we write

$$F_I(t^{c_0}h_{00}, \dots, t^{c_N}h_{0N}; \dots; t^{c_0}h_{n0}, \dots, t^{c_N}h_{nN}) = \sum_{k=0}^{T} t^{e_j} F_j$$

with $F_0, \dots, F_T \in K[h_{00}, \dots, h_{nN}]$, $e_0 > e_1 > \cdots > e_T$ and define the Chow weight of I with respect to $\mathbf{c}$ by

$$e_I(\mathbf{c}) = e_0. \tag{3.6}$$

3.5. According to Mumford [15], p.61, Proposition 2.11 we have

$$s_I(m, \mathbf{c}) = e_I(\mathbf{c}) \cdot \frac{m^{n+1}}{(n+1)!} + O(m^n) \quad \text{as } m \to \infty$$

where $n = \dim I$, and together with (3.2) this gives

$$\lim_{m\to\infty} \frac{1}{mH_I(m)} \cdot s_I(m, \mathbf{c}) = \frac{1}{(n+1)d} \cdot e_I(\mathbf{c}).$$

We call $\frac{1}{mH_I(m)} \cdot s_I(m, \mathbf{c})$ the m-th normalized Hilbert weight and $\frac{1}{(n+1)d} \cdot e_I(\mathbf{c})$ the normalized Chow weight of I.

We deduce an explicit lower bound for the m-th normalized Hilbert weight in terms of the normalized Chow weight in the special case that $I = P$ is a prime ideal. In this case we write $H_X(m)$, $\deg X$, $s_X(m, \mathbf{c})$, $e_X(\mathbf{c})$ for $H_P(m)$, $\deg P$, $s_P(m, \mathbf{c})$, $e_P(\mathbf{c})$, where X is the variety defined by P.

Theorem 3.6. *Let X be a subvariety of $\mathbb{P}^N$ of dimension n and degree d, defined over an algebraically closed field K of characteristic 0. Let $m > d$ be an integer. Further, let $\mathbf{c} = (c_0, \dots, c_N)$ be a tuple of non-negative reals. Then*

$$\frac{1}{mH_X(m)}\, s_X(m, \mathbf{c}) \geqslant \frac{1}{(n+1)d}\, e_X(\mathbf{c}) - \frac{(2n+1)d}{m} \cdot \Big(\max_{i=0,\dots,N} c_i\Big). \tag{3.7}$$

Inequality (3.7) is sufficient for our purposes. It is probably more difficult to prove an inequality in the other direction. In the proof of Theorem 3.6 we follow some ideas of Kapranov, Sturmfels and Zelevinsky [13] which were also implicit in Mumford's paper [15]: in Section 4 we deduce an auxiliary result for monomial ideals and in Section 5 we deduce from this Theorem 3.6.

4. Monomial ideals

4.1. We keep the notation introduced in the previous section, so in particular K is an algebraically closed field of characteristic 0. In addition, for $\mathbf{a} = (a_0, \dots, a_N)$, $\mathbf{b} = (b_0, \dots, b_N) \in \mathbb{R}^{N+1}$ we write $\mathbf{a} \leqslant \mathbf{b}$ or $\mathbf{b} \geqslant \mathbf{a}$ if $a_i \leqslant b_i$ for all $i = 0, \dots, N$. We define the norm of $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{R}^{N+1}$ by $\|\mathbf{a}\| := \sum_{i=0}^{N} |a_i|$. For $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{R}^{N+1}$ we define the *support* $\operatorname{supp} \mathbf{a} = \{i : 0 \leqslant i \leqslant N, a_i \neq 0\}$. For $\mathbf{a} = (a_0, \dots, a_N) \in \mathbb{R}^{N+1}$ and $W \subset \{0, \dots, N\}$ we denote by $\mathbf{a}_W$ the vector obtained by setting the coordinates of $\mathbf{a}$ with indices outside W to 0, i.e., $\mathbf{a}_W := (b_0, \dots, b_N)$ with $b_i = a_i$ for $i \in W$, $b_i = 0$ for $i \notin W$.

We denote by $(f_1, \dots, f_T)$ the ideal in $K[x_0, \dots, x_N]$ generated by $f_1, \dots, f_T$. For $W \subset \{0, \dots, N\}$ let $P_W = (x_i : i \in W)$ denote the ideal in $K[x_0, \dots, x_N]$ generated by x_i $(i \in W)$.

4.2. Throughout this section, let

$$I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_T}) \tag{4.1}$$

be the ideal generated by the monomials $\mathbf{x}^{\mathbf{a}_i}$ $(i = 1, \dots, T)$, where $\mathbf{a}_i = (a_{i0}, \dots, a_{iN}) \in \mathbb{Z}_{\geqslant 0}^{N+1}$. We assume that I is relevant. Note that $\mathbf{x}^{\mathbf{a}} \in I$ if and only if $\mathbf{a} \geqslant \mathbf{a}_i$ for some $i \in \{1, \dots, T\}$. Let $S(I)$ be the collection of sets $W \subseteq \{0, \dots, N\}$ with the property that for every $i \in \{1, \dots, T\}$ there is a $j \in W$ with $a_{ij} > 0$. Given $W \in S(I)$, let

$$A_W(I) := \{\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{N+1} : \ \operatorname{supp} \mathbf{a} \subseteq W, \ \mathbf{a} \not\geqslant \mathbf{a}_{i,W} \text{ for all } i = 1, \dots, T\}. \tag{4.2}$$

We have included a proof of the following simple lemma (see also [18], Proposition 3.4).

Lemma 4.3. *Let $W_1, \dots, W_g$ be the non-empty sets in $S(I)$ of minimal cardinality. Then $P_{W_1}, \dots P_{W_g}$ are the prime ideals of maximal dimension associated to I. Further, for $i = 1, \dots, g$, the multiplicity $\mu_{P_{W_i},I}$ of I with respect to P_{W_i} is equal to the cardinality of $A_{W_i}(I)$.*

Proof. For points $\mathbf{x} = (x_0 : \cdots : x_N) \in \mathbb{P}^N(K)$ we have $\mathbf{x}^{\mathbf{a}_i} = 0$ if and only if $a_{i,j_i} > 0$ and $x_{j_i} = 0$ for some j_i, and so $\mathbf{x}^{\mathbf{a}_i} = 0$ for $i = 1, \dots, T$ if and only if there is a set $W \in S(I)$ such that $x_j = 0$ for $j \in W$. This implies that the minimal prime ideals containing I are precisely the ideals P_W where W is a minimal set in $S(I)$. Since $\dim P_W = N - \#W$, the prime ideals of maximal dimension containing I are $P_{W_1}, \dots, P_{W_g}$.

Let W be one of the sets $W_1, \dots, W_g$. We first show that $A_W(I)$ is finite. Let $\mathbf{a} = (a_0, \dots, a_N) \in A_W(I)$. Then for $i = 1, \dots, T$ there is an index $j_i \in W$ such that $a_{j_i} < a_{i,j_i}$. The set of $j_1, \dots, j_T$ must be equal to W since otherwise $S(I)$ would contain a set strictly smaller than W which is impossible by the minimal cardinality of W. Therefore, the coordinates of $\mathbf{a} \in A_W(I)$ are bounded.

Suppose that $W = \{0, \dots, r\}$. Let $K' = K(x_{r+1}, \dots, x_n)$, $R = K'[x_0, \dots, x_r]$, $I' = (\mathbf{x}^{\mathbf{a}_{1,W}}, \dots, \mathbf{x}^{\mathbf{a}_{T,W}})$. Then $O_{P_W,I} = R/I'$. Note that the latter is a K'-vector space with basis $\{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in A_W(I)\}$. Therefore, $\mu_{P_W,I} = l_{O_{P_W,I}}(O_{P_W,I}) = \dim_{K'} R/I'$ is equal to the cardinality of $A_W(I)$. □

We make some further observations. Let I be as in (4.1) and let $W_1, \dots, W_g$ be the sets from Lemma 4.3. Let $n = \dim I$, $d = \deg I$, $\mu_i = \mu_{P_{W_i},I}$ $(i = 1, \dots, g)$. Then

$$\#W_i = N - n \quad (i = 1, \dots, g). \tag{4.3}$$

Further, by (3.3) we have

$$\sum_{i=1}^{g} \mu_i = d. \tag{4.4}$$

Lastly,

$$\|\mathbf{a}\| \leqslant \mu_i \quad \text{for } \mathbf{a} \in A_{W_i}(I),\ i = 1, \dots, g. \tag{4.5}$$

Indeed, let $\mathbf{a} \in A_{W_i}(I)$. Then every $\mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1}$ with $\mathbf{b} \leqslant \mathbf{a}$ belongs to $A_{W_i}(I)$. The number of $\mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1}$ with $\mathbf{b} \leqslant \mathbf{a}$ is at least $\|\mathbf{a}\|$, and so $\|\mathbf{a}\|$ is at most the cardinality of $A_{W_i}(I)$.

4.4. Let $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}_{\geqslant 0}^{N+1}$. Note that $K[x_0, \dots, x_n]_m / I_m$ has a unique monomial basis consisting of those monomials $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a} \ngeqslant \mathbf{a}_i$ for $i = 1, \dots, T$ and $\|\mathbf{a}\| = m$. This implies

$$s_I(m, \mathbf{c}) := \sum_{\substack{\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{N+1},\ \|\mathbf{a}\| = m, \\ \mathbf{a} \ngeqslant \mathbf{a}_i \text{ for } i = 1, \dots, T}} \mathbf{a} \cdot \mathbf{c}. \tag{4.6}$$

Let $W_k^c = \{0, \dots, N\} \backslash W_k$ for $k = 1, \dots, g$. Let $\mathbf{h}_i = (h_{i0}, \dots, h_{iN})$ $(i = 0, \dots, n)$ be the blocks of variables occurring in the Chow form F_I of I. Then by (3.5) we have, with the bracket notation from **2.8**,

$$F_I = \prod_{k=1}^{g} F_{P_{W_k}}^{\mu_k} = \prod_{k=1}^{g} [W_k^c]^{\mu_k}.$$

Hence

$$e_I(\mathbf{c}) = \sum_{k=1}^{g} \mu_k \Big(\sum_{j \in W_k^c} c_j \Big). \tag{4.7}$$

We prove:

Lemma 4.5. *Let m be an integer $> d$ and $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}_{\geqslant 0}^{N+1}$. Then*

$$s_I(m, \mathbf{c}) \geqslant \tfrac{m-d}{n+1} \tbinom{m-d+n}{n} \cdot e_I(\mathbf{c}) - d^2 m \tbinom{m+n-1}{n-1} \cdot \Big(\max_{0 \leqslant i \leqslant N} c_i \Big). \tag{4.8}$$

Proof. For a finite subset S of $\mathbb{Z}_{\geqslant 0}^{N+1}$ we write $\Sigma_{\mathbf{c}}(S) := \sum_{\mathbf{a} \in S} \mathbf{a} \cdot \mathbf{c}$. Write A_k for the set $A_{W_k}(I)$ given by (4.2). For $k = 1, \dots, g$, $\mathbf{a} \in A_k$, let $S_k(\mathbf{a})$ be the set of vectors $\mathbf{r}$ such that

$$\begin{cases} \mathbf{r} = \mathbf{a} + \mathbf{b} & \text{for some } \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1} \text{ with } \operatorname{supp} \mathbf{b} \subseteq W_k^c, \\ \|\mathbf{r}\| = m. & \end{cases}$$

Notice that for $\mathbf{r} \in S_k(\mathbf{a})$ we have $\mathbf{r} \ngeqslant \mathbf{a}_i$ for $i = 1, \dots, T$. Together with (4.6) and the principle of inclusion and exclusion this implies

$$\begin{aligned} s_I(m, \mathbf{c}) &\geqslant \Sigma_{\mathbf{c}} \Big(\bigcup_{k=1}^{g} \bigcup_{\mathbf{a} \in A_k} S_k(\mathbf{a}) \Big) \\ &\geqslant \sum_{k=1}^{g} \sum_{\mathbf{a} \in A_k} \Sigma_{\mathbf{c}}(S_k(\mathbf{a})) \; - \sum_{(k, \mathbf{a}') \neq (l, \mathbf{a}'')} \Sigma_{\mathbf{c}}(S_k(\mathbf{a}') \cap S_l(\mathbf{a}'')), \end{aligned} \tag{4.9}$$

where the last summation is over all quadruples $(k, l, \mathbf{a}', \mathbf{a}'')$ with $k, l = 1, \ldots, g$, $\mathbf{a}' \in A_k$, $\mathbf{a}'' \in A_l$ and $(k, \mathbf{a}') \neq (l, \mathbf{a}'')$.

For $k = 1, \ldots, g$, $\mathbf{a} \in A_k$ we have, noticing that $\#W_k^c = n+1$ by (4.3) and $\|\mathbf{a}\| \leqslant d$ by (4.4), (4.5),

$$\begin{aligned}
\Sigma_{\mathbf{c}}(S_k(\mathbf{a})) &\geqslant \Sigma_{\mathbf{c}}(\{\mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1} : \operatorname{supp} \mathbf{b} \subseteq W_k^c,\ \|\mathbf{b}\| = m - \|\mathbf{a}\|\}) \\
&= \binom{m - \|\mathbf{a}\| + \#W_k^c - 1}{\#W_k^c - 1} \cdot \frac{m - \|\mathbf{a}\|}{\#W_k^c} \sum_{j \in W_k^c} c_j \\
&\geqslant \binom{m - d + n}{n} \cdot \frac{m - d}{n+1} \sum_{j \in W_k^c} c_j .
\end{aligned}$$

Summing over $k = 1, \ldots, g$, $\mathbf{a} \in W_k$, using that by Lemma 4.3 the cardinality of A_k is μ_k and using (4.7), we obtain

$$\begin{aligned}
\sum_{k=1}^{g} \sum_{\mathbf{a} \in W_k} \Sigma_{\mathbf{c}}(S_k(\mathbf{a})) &\geqslant \frac{m-d}{n+1} \binom{m - d + n}{n} \sum_{k=1}^{g} \mu_k \Big(\sum_{j \in W_k^c} c_j \Big) \\
&= \frac{m-d}{n+1} \binom{m - d + n}{n} \cdot e_I(\mathbf{c}).
\end{aligned} \tag{4.10}$$

Let $(k, l, \mathbf{a}', \mathbf{a}'')$ be a quadruple with $k, l = 1, \ldots, g$, $\mathbf{a}' \in W_k$, $\mathbf{a}'' \in W_l$ and $(k, \mathbf{a}') \neq (l, \mathbf{a}'')$. If $k = l$ then $S_k(\mathbf{a}') \cap S_l(\mathbf{a}'') = \emptyset$. Assume $k \neq l$. Write $\mathbf{a}' = (a_0', \ldots, a_N')$, $\mathbf{a}'' = (a_0'', \ldots, a_N'')$ and put $\max(\mathbf{a}', \mathbf{a}'') := (\max(a_0', a_0''), \ldots, \max(a_0', a_0''))$. Then $S_k(\mathbf{a}') \cap S_l(\mathbf{a}'')$ consists of all vectors $\mathbf{r}$ such that

$$\begin{cases} \mathbf{r} = \max(\mathbf{a}', \mathbf{a}'') + \mathbf{b} \quad \text{for some } \mathbf{b} \in \mathbb{Z}_{\geqslant 0}^{N+1} \text{ with } \operatorname{supp} \mathbf{b} \subseteq W_k^c \cap W_l^c, \\ \|\mathbf{r}\| = m. \end{cases}$$

Since by (4.3) $W_k^c \cap W_l^c$ has cardinality $\leqslant n$, the number of vectors in this set is at most

$$\binom{m - \|\max(\mathbf{a}', \mathbf{a}'')\| + n - 1}{n - 1} \leqslant \binom{m + n - 1}{n - 1}.$$

Further, for each vector $\mathbf{r}$ in this set we have $\mathbf{r} \cdot \mathbf{c} \leqslant m \cdot \big(\max_{0 \leqslant i \leqslant N} c_i\big)$. Hence

$$\|\Sigma_{\mathbf{c}}(S_k(\mathbf{a}') \cap S_l(\mathbf{a}''))\| \leqslant m \binom{m + n - 1}{n - 1} \cdot \Big(\max_{0 \leqslant i \leqslant N} c_i \Big).$$

By Lemma 4.3, (4.4), the number of pairs $(k, \mathbf{a})$ with $k = 1, \ldots, g$, $\mathbf{a} \in A_k$ is equal to $\mu_1 + \cdots + \mu_g = d$. Therefore,

$$\| \sum_{(k,\mathbf{a}') \neq (l,\mathbf{a}'')} \Sigma_{\mathbf{c}}(S_k(\mathbf{a}') \cap S_l(\mathbf{a}'')) \| \leqslant d^2 m \binom{m+n-1}{n-1} \cdot \Big(\max_{0 \leqslant i \leqslant N} c_i \Big) .$$

By inserting this and (4.10) into (4.9) we obtain (4.8). □

5. Proof of Theorem 3.6

5.1. Much of the material in this section can be found in bits in pieces in the literature, in particular in [13], [3, Chapter 15], [16]. For convenience of the unspecialized reader we have worked out more details. We keep the previously introduced notation; in particular K is an algebraically closed field of characteristic 0.

We recall some properties of Chow forms. Let $A \in GL(N+1, K)$. For $f \in K[x_0, \ldots, x_N]$ define f_A by $f_A(\mathbf{x}) := f(A\mathbf{x})$, and for a relevant homogeneous ideal I of $K[x_0, \ldots, x_N]$, let $I_A = (f_A : f \in A)$. Then the Chow forms of I and I_A are related by

$$F_{I_A}(\mathbf{h}_0, \ldots, \mathbf{h}_n) = F_I((A^{-1})^T \mathbf{h}_0, \ldots, (A^{-1})^T \mathbf{h}_n), \tag{5.1}$$

where $n = \dim I$ and where $(A^{-1})^T$ is the transpose of the inverse of A.

There is a surjective map from the collection of relevant homogeneous ideals of $K[x_0, \ldots, x_N]$ to the collection of closed subschemes of $\mathbb{P}^N$ defined over K, given by $I \mapsto Z(I) = \mathrm{Proj}(K[x_0, \ldots, x_N]/I)$. We have $Z(I_1) = Z(I_2)$ if and only if there is an integer m_0 such that $(I_1)_m = (I_2)_m$ for all $m \geqslant m_0$. For such ideals I_1, I_2, the dimensions, degrees and Chow forms are equal. Thus we may define the dimension, degree and Chow form of a closed subscheme Z of $\mathbb{P}$ by $\dim Z := \dim I$, $\deg Z := \deg I$ and $F_Z := F_I$ where I is any homogeneous polynomial with $Z = Z(I)$. Assuming that $\dim Z = n$, F_Z is a multihomogeneous polynomial in $n+1$ blocks of $N+1$ variables, and therefore, F_Z defines a closed subscheme $ch(Z)$ of $(\mathbb{P}^N)^{n+1} = \mathbb{P}^N \times \cdots \times \mathbb{P}^N$ ($n+1$ times) of codimension 1. Thus, the map $I \mapsto F_I$ from an ideal to its Chow form gives rise to a map

$$ch : \ Z \mapsto ch(Z) \tag{5.2}$$

from the collection of closed subschemes of $\mathbb{P}^N$ of dimension n to the collection of closed subschemes of $(\mathbb{P}^N)^{n+1}$ of codimension 1. From the arguments in [16], Sections 5.2, 5.4 it follows that if Z runs through a family of schemes flat over some Noetherian scheme S, then $ch(Z)$ also runs through a family flat over S.

5.2. Let $\mathbf{c} = (c_0, \dots, c_N) \neq \mathbf{0}$ be a tuple of reals. Let t be a parameter. For $f \in K[x_0, \dots, x_N]$ write

$$f(t^{c_0}x_0, \dots, t^{c_N}x_N) = \sum_{i=0}^{h} t^{b_i} f_i$$

where $f_0, \dots, f_h \in K[x_0, \dots, x_N]$ and $b_0 < b_1 < \cdots < b_h$, and define

$$in_{\mathbf{c}}(f) := f_0.$$

Alternatively, if we write $f = \sum_{\mathbf{a}\in A} \beta(\mathbf{a})\mathbf{x}^{\mathbf{a}}$ with $\beta(\mathbf{a}) \neq 0$ for $\mathbf{a} \in A$ then $in_{\mathbf{c}}(f) = \sum_{\mathbf{a}\in A'} \beta(\mathbf{a})\mathbf{x}^{\mathbf{a}}$, where A' consists of those $\mathbf{a} \in A$ for which $\mathbf{a}\cdot\mathbf{c}$ is minimal among all values $\mathbf{a}'\cdot\mathbf{c}$, $\mathbf{a}' \in A$. For a relevant ideal I of $K[x_0, \dots, x_N]$ we denote by $in_{\mathbf{c}}(I)$ the ideal generated by $in_{\mathbf{c}}(f)$ $(f \in I)$. The following lemma is a slight generalization of [3], Theorem 15.26.

Lemma 5.3. *Let I be a relevant homogeneous ideal and $m \geqslant 1$ an integer. Let $\{\mathbf{x}^{\mathbf{a}_1}, \dots \mathbf{x}^{\mathbf{a}_{H_I(m)}}\}$ be a basis of $K[x_0, \dots, x_N]_m/I_m$ for which $(\mathbf{a}_1 + \cdots + \mathbf{a}_{H_I(m)})\cdot\mathbf{c}$ is maximal. Then $\{\mathbf{x}^{\mathbf{a}_1}, \dots \mathbf{x}^{\mathbf{a}_{H_I(m)}}\}$ is a basis of $K[x_0, \dots, x_N]_m/in_{\mathbf{c}}(I)_m$. Consequently, $in_{\mathbf{c}}(I)$ has the same Hilbert function as I.*

Proof. Write $H := H_I(m)$, $R = \binom{N+m}{N}$. Let $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_R}$ be all monomials of degree m in $x_0, \dots, x_N$, ordered such that $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_H}$ are the monomials from the statement of the lemma. Then I_m is generated by

$$f_i = \mathbf{x}^{\mathbf{a}_i} - \sum_{j\in B_i} \beta_{ij}\mathbf{x}^{\mathbf{a}_j} \quad (i = H+1, \dots, R),$$

where $B_i \subseteq \{1, \dots, H\}$ and $\beta_{ij} \neq 0$ for $j \in B_i$. For $i \in \{H+1, \dots, R\}$, $j \in B_i$ we can make a new basis of $K[x_0, \dots, x_N]_m/I_m$ by replacing $\mathbf{x}^{\mathbf{a}_j}$ by $\mathbf{x}^{\mathbf{a}_i}$ in $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_H}\}$, therefore $\mathbf{a}_i\cdot\mathbf{c} \leqslant \mathbf{a}_j\cdot\mathbf{c}$. For $i = H+1, \dots, R$ let B_i' be the set of indices $j \in B_i$ for which $\mathbf{a}_j\cdot\mathbf{c} = \mathbf{a}_i\cdot\mathbf{c}$. We claim that $f_i' = \mathbf{x}^{\mathbf{a}_i} - \sum_{j\in B_i'} \beta_{ij}\mathbf{x}^{\mathbf{a}_j}$ $(i = H+1, \dots, R)$ generate $in_{\mathbf{c}}(I)_m$. Since clearly, $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_H}$, $f_{H+1}', \dots, f_R'$ form a basis of $K[x_0, \dots, x_N]$, the claim implies Lemma 5.3.

Pick $f \in in_{\mathbf{c}}(I)_m$. Then for certain monomials $\mathbf{x}^{\mathbf{b}_k}$ and certain polynomials $g_k \in I$ we have $f = \sum_{k=1}^{M} \mathbf{x}^{\mathbf{b}_k} in_{\mathbf{c}}(g_k)$. Writing $h_k = \mathbf{x}^{\mathbf{b}_k} g_k$ we have $f = \sum_{k=1}^{M} in_{\mathbf{c}}(h_k)$ with $h_k \in I_m$. Then $h_k = \sum_{i=H+1}^{R} \gamma_i f_i$ with $\gamma_i \in K$, hence $in_{\mathbf{c}}(h_k) = \sum_{i \in C} \gamma_i in_{\mathbf{c}}(f_k) = \sum_{i \in C} \gamma_i f_i'$, where C consists of those i for which $\gamma_i \neq 0$ and $\mathbf{a}_i \cdot \mathbf{c} \leqslant \mathbf{a}_k \cdot \mathbf{c}$ for all $k \in \{H+1, \ldots, R\}$ with $\gamma_k \neq 0$. This proves our claim, hence our lemma. □

If for the Chow form F_I of a given relevant homogeneous polynomial I we write

$$F_I(t^{c_0} h_{00}, \ldots, t^{c_N} h_{0N}; \ldots; t^{c_0} h_{n0}, \ldots, t^{c_N} h_{nN}) = \sum_{j=1}^{T} t^{e_j} F_j \tag{5.3}$$

with $e_0 > e_1 > \cdots > e_T$ and $F_0, \ldots, F_T \in K[h_{00}, \ldots, h_{nN}]$ then we define

$$in_{\mathbf{c}}(F_I) := F_0 \,. \tag{5.4}$$

Lemma 5.4. *Let I be a relevant homogeneous ideal of $K[x_0, \ldots, x_N]$. Then, apart from a constant factor, $F_{in_{\mathbf{c}}(I)} = in_{\mathbf{c}}(F_I)$.*

Proof. We first show that it suffices to prove Lemma 5.4 for $\mathbf{c} \in \mathbb{Z}^{N+1}$. Let $n = \dim I$, $d = \deg I$. Choose polynomials $f_1, \ldots, f_T \in K[x_0, \ldots, x_N]$ such that $I = (f_1, \ldots, f_T)$ and $in_{\mathbf{c}}(I) = (in_{\mathbf{c}}(f_1), \ldots, in_{\mathbf{c}}(f_T))$. Let $M > \max(\deg f_1, \ldots, \deg f_T, (n+1)d)$ be an integer. Let V be the smallest linear subspace of $\mathbb{R}^{N+1}$ defined over $\mathbb{Q}$ containing $\mathbf{c}$. By choosing a vector in V with coordinates in $\mathbb{Q}$ very close to $\mathbf{c}$ and then clearing denominators, we find $\mathbf{c}' \in V \cap \mathbb{Z}^{N+1}$ such that $\mathbf{b}_1 \cdot \mathbf{c}' < \mathbf{b}_2 \cdot \mathbf{c}'$ for any two vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{Z}^{N+1}_{\geqslant 0}$ such that both $\mathbf{b}_1$, $\mathbf{b}_2$ have sum of coordinates $\leqslant M$ and such that $\mathbf{b}_1 \cdot \mathbf{c} < \mathbf{b}_2 \cdot \mathbf{c}$. Further, $\mathbf{c}$, $\mathbf{c}' \in V$, hence $\mathbf{b}_1 \cdot \mathbf{c} = \mathbf{b}_2 \cdot \mathbf{c}$ implies $\mathbf{b}_1 \cdot \mathbf{c}' = \mathbf{b}_2 \cdot \mathbf{c}'$. Since $M > (n+1)d$ this implies $in_{\mathbf{c}'}(F_I) = in_{\mathbf{c}}(F_I)$. Further, since $M > \max_i \deg f_i$, we have $in_{\mathbf{c}'}(I) \supseteq in_{\mathbf{c}}(I)$. But by Lemma 5.3, these two ideals have the same Hilbert function, hence they are equal. Therefore, it suffices to show $F_{in_{\mathbf{c}'}(I)} = in_{\mathbf{c}'}(F_I)$.

So suppose $\mathbf{c} \in \mathbb{Z}^{N+1}$. For $f \in K[x_0, \ldots, x_N]$ and a variable t, define $f_t = t^{-b_f} f(t^{c_0} x_0, \ldots, t^{c_N} x_N)$, where b_f has been chosen such that the smallest power of t occurring in f_t is 0, and let $\overline{I}$ be the ideal in $K[t][x_0, \ldots, x_N]$ generated by the polynomials f_t ($f \in I$). For $t_0 \in K$, let I_{t_0} be the ideal in $K[x_0, \ldots, x_N]$ obtained from $\overline{I}$ by substituting t_0 for t. It is easy to verify that $I_0 = in_{\mathbf{c}}(I)$. Let $S = \mathrm{Spec}(K[t])$ and let Z be the subscheme of $\mathbb{P}^N \times_K S$ defined by $\overline{I}$. Then Z is a scheme over S and the fiber of $Z \to S$ over $t_0 \in K$ is $Z_{t_0} = Z(I_{t_0})$. From e.g. [3], p. 343, Theorem 15.17 it follows that $Z \to S$ is flat (i.e. the fibers Z_{t_0} $(t_0 \in K)$ form a flat family). By [16], sections 5.2, 5.4, the scheme $ch(Z)$, defined by taking as fiber over t_0 the scheme $ch(Z_{t_0})$ determined by the Chow form of Z_{t_0}, is also flat over S.

Now let $F_I = F_I(h_{00}, \dots, h_{0N}; \dots; h_{n0}, \dots, h_{nN})$ be the Chow form of I and define

$$F_{I,t} = t^{c_I} F_I(t^{-c_0}h_{00}, \dots, t^{-c_N}h_{0N}; \dots; t^{-c_0}h_{n0}, \dots, t^{-c_N}h_{nN}),$$

where c_I has been chosen such that the smallest power of t occurring in $F_{I,t}$ is 0. Let C be the closed subscheme of $(\mathbb{P}^N)^{n+1} \times_K S$ defined by $F_{I,t}$. Then, again [3], p. 343, Theorem 15.17 implies that C is flat over S. Denote by F_{I,t_0} the polynomial obtained by substituting t_0 for t in $F_{I,t}$ and by C_{t_0} the corresponding scheme, i.e. the fiber of $C \to S$ over t_0. One easily verifies that $F_{I,0} = in_{\mathbf{c}}(F_I)$. If $t_0 \neq 0$, we obtain by applying (5.1) with the linear transformation A given by $x_i \mapsto t_0^{c_i} x_i$ $(i = 0, \dots, N)$, that (apart from a constant factor) $F_{I,t_0} = F_{I_{t_0}}$ and therefore, $C_{t_0} = ch(Z_{t_0})$. Using that both C and $ch(Z)$ are flat over S, it follows from [11], p. 258, Prop. 9.8 that also $C_0 = ch(Z_0)$. This implies Lemma 5.4. □

Lemma 5.5. *Let I be a relevant homogeneous ideal of $K[x_0, \dots, x_N]$ and let $\mathbf{c} = (c_0, \dots, c_N) \in \mathbb{R}^{N+1}_{\geqslant 0}$. Then $\dim in_{\mathbf{c}}(I) = \dim I$, $\deg in_{\mathbf{c}}(I) = \deg I$, $s_{in_{\mathbf{c}}(I)}(m, \mathbf{c}) = s_I(m, \mathbf{c})$, $e_{in_{\mathbf{c}}(I)}(\mathbf{c}) = e_I(\mathbf{c})$.*

Proof. The first two identities follow at once from Lemma 5.3. To prove the other identities assume first that $c_0, \dots, c_N$ are linearly independent over $\mathbb{Q}$. Then for any two different integer vectors $\mathbf{a}$, $\mathbf{b}$ we have $\mathbf{a} \cdot \mathbf{c} \neq \mathbf{b} \cdot \mathbf{c}$. This implies that for each non-zero $f \in K[x_0, \dots, x_N]$, $in_{\mathbf{c}}(f)$ is a monomial, and so $in_{\mathbf{c}}(I)$ is a monomial ideal. Further, there is a unique vector $\mathbf{a}_1 + \cdots + \mathbf{a}_H$ such that $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_H}\}$ is a basis of $K[x_0, \dots, x_N]_m / I_m$ and $(\mathbf{a}_1 + \cdots + \mathbf{a}_m) \cdot \mathbf{c}$ is maximal. By Lemma 5.1 the latter basis is the only monomial basis of $K[x_0, \dots, x_N]_m / in_{\mathbf{c}}(I)_m$. This proves the third identity. By (5.3), (5.4), the highest power of t occurring in the left-hand side of (5.3) is equal to the single power of t occurring in the expression obtained by substituting $t^{c_j} h_{ij}$ for h_{ij} in $in_{\mathbf{c}}(F_I)$ for $i = 0, \dots, n$, $j = 0, \dots, N$. Together with Lemma 5.4 and definition (3.6) this implies the fourth identity. One proves the third and fourth identity for arbitrary tuples $\mathbf{c}$ by approximating $\mathbf{c}$ by a tuple with coordinates which are linearly independent over $\mathbb{Q}$ and using continuity arguments. □

We are now ready to prove the following result:

Lemma 5.6. *let I be a homogeneous ideal of $K[x_0,\dots,x_N]$ of dimension n and degree d. Let $m>d$ be an integer, and let $\mathbf{c}\in\mathbb{R}^{N+1}_{\geqslant 0}$. Then*

$$s_I(m,\mathbf{c})\geqslant \tfrac{m-d}{n+1}\tbinom{m+n-d}{n}\cdot e_I(\mathbf{c})-d^2m\tbinom{m+n-1}{n-1}\cdot\Big(\max_{0\leqslant i\leqslant N}c_i\Big). \tag{5.5}$$

Proof. Similarly as in the end of the proof of Lemma 5.5 it suffices to prove Lemma 5.6 for tuples of reals $\mathbf{c}$ whose coordinates are linearly independent over $\mathbb{Q}$, so that $in_{\mathbf{c}}(I)$ is a monomial ideal. Then Lemma 5.6 is an immediate consequence of Lemma 4.5 and Lemma 5.5. □

Our last auxiliary result is an upper bound for the Hilbert function of a projective variety, due to Chardin [2], Théorème 1. In what follows, X is a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d defined over K.

Lemma 5.7. $H_X(m)\leqslant d\binom{m+n}{n}$ *for* $m\geqslant 1$.

5.8. *Proof of Theorem 3.6.*
Let $m>d$. Put $C:=\max_{0\leqslant i\leqslant N}c_i$. By Lemma 5.6, Lemma 5.7 we have

$$\begin{aligned}\tfrac{1}{mH_X(m)}\cdot s_X(m,\mathbf{c}) &\geqslant \max\Big\{0,\ \tfrac{1}{mH_X(m)}\cdot\Big(\tfrac{m-d}{n+1}\tbinom{m+n-d}{n}\cdot e_X(\mathbf{c})-d^2m\tbinom{m+n-1}{n-1}\cdot C\Big)\Big\}\\ &\geqslant \tfrac{(m-d)\binom{m+n-d}{n}}{m\binom{m+n}{n}}\cdot\tfrac{1}{(n+1)d}\,e_X(\mathbf{c})\ -\ d\cdot\tfrac{n}{m+n}\cdot C.\end{aligned}$$

Together with

$$\frac{(m-d)\binom{m+n-d}{n}}{m\binom{m+n}{n}}=\prod_{i=0}^{n}\frac{m+i-d}{m+i}\geqslant\Big(1-\frac{d}{m}\Big)^{n+1}\geqslant 1-\frac{(n+1)d}{m}$$

and $\frac{1}{(n+1)d}e_X(\mathbf{c})\leqslant C$ (which follows from the remark following (2.11)) this implies

$$\begin{aligned}\tfrac{1}{mH_X(m)}\cdot s_X(m,\mathbf{c}) &\geqslant \tfrac{1}{(n+1)d}\cdot e_X(\mathbf{c})-\Big(\tfrac{(n+1)d}{m}+\tfrac{nd}{m+n}\Big)\cdot C\\ &\geqslant \tfrac{1}{(n+1)d}\cdot e_X(\mathbf{c})-\tfrac{(2n+1)d}{m}\cdot C.\end{aligned}$$

□

6. Proof of Theorem 2.5 (linear case)

6.1. We recall Theorem 2.1 of Evertse and Schlickewei [4] which is the main tool in the proof of our Theorem 2.5.

Let K be an algebraic number field. Let $N > n \geqslant 1$ be integers. Let $\mathcal{L} = \{l_0, \dots, l_N\}$ be a family (i.e., an unordered tuple with possibly repetitions) of linear forms in $K[x_0, \dots, x_n]$. Suppose that $\mathcal{L}$ has rank $n+1$. For every place $v \in M_K$, let I_v be a subset of $\{0, \dots, N\}$ of cardinality $n+1$ such that $\{l_i : i \in I_v\}$ is linearly independent. Let d_{iv} $(v \in M_K, i \in I_v)$ be reals such that for some finite subset T of M_K we have

$$d_{iv} = 0 \quad \text{for } v \in M_K \backslash T,\ i \in I_v. \tag{6.1}$$

For $Q \geqslant 1$ and for $\mathbf{y} \in K^{n+1}$ we define

$$H_Q(\mathbf{y}) = \prod_{v \in M_K} \max_{i \in I_v} \big(|l_i(\mathbf{y})|_v \cdot Q^{-d_{iv}} \big). \tag{6.2}$$

We will refer to H_Q as a twisted (exponential) height. By the product formula we have $H_Q(\lambda \mathbf{y}) = H_Q(\mathbf{y})$ for $\lambda \in K^*$, therefore, H_Q may be viewed as a twisted height on $\mathbb{P}^n(K)$.

We extend H_Q to $\mathbb{P}^n(\overline{\mathbb{Q}})$ as follows. Let $\mathbf{y} \in \mathbb{P}^n(\overline{\mathbb{Q}})$. Pick a finite extension L of K such that $\mathbf{y} \in \mathbb{P}^n(L)$. For a place $w \in M_L$ put

$$I_w = I_v, \qquad d_{iw} = \tfrac{[L_w:K_v]}{[L:K]} d_{iv}, \tag{6.3}$$

where $v \in M_K$ is the place lying below w. Then we put

$$H_Q(\mathbf{y}) = \prod_{w \in M_L} \max_{i \in I_w} \big(|l_i(\mathbf{y})|_w \cdot Q^{-d_{iw}} \big). \tag{6.4}$$

By (2.1) this is well-defined, i.e., independent of the choice of L.

6.2. Put

$$\Delta := \prod_{v \in M_K} |\det(l_i : i \in I_v)|_v, \tag{6.5}$$

where for any subset I of $\{0,\dots,N\}$ of cardinality $n+1$, $\det(l_i : i\in I)$ denotes the coefficient determinant of the linear forms l_i $(i\in I)$. Further, let

$$\mathcal{H}_{\mathcal{L}} := \prod_{v\in M_K}\Big(\max_I |\det(l_i : i\in I)|_v\Big), \tag{6.6}$$

where the maxima are taken over all subsets I of $\{0,\dots,N\}$ of cardinality $n+1$. We may view $\mathcal{H}_{\mathcal{L}}$ as a height of the family $\mathcal{L}=\{l_0,\dots,l_N\}$. We assume that the reals d_{iv} satisfy, apart from (6.1),

$$\sum_{v\in M_K}\sum_{i\in I_v} d_{iv}=0,\qquad \sum_{v\in M_K}\max_{i\in I_v} d_i \leqslant 1. \tag{6.7}$$

Then Theorem 2.1 of [4] can be stated as follows:

Proposition 6.3. *Let $0<\varepsilon<1$. Let H_Q be defined by (6.2)–(6.4). Then there are proper linear subspaces $T_1,\dots,T_t$ of $\mathbb{P}^n$, defined over K, with*

$$t \;\leqslant\; 4^{(n+9)^2}\varepsilon^{-n-5}\log(3N)\log\log(3N) \tag{6.8}$$

for which the following holds:
For every real Q with

$$Q \;\geqslant\; \max\Big(\mathcal{H}_{\mathcal{L}}^{1/\binom{N+1}{n+1}}, (n+1)^{2/\varepsilon}\Big) \tag{6.9}$$

there is a space $T_i\in\{T_1,\dots,T_t\}$ such that

$$\Big\{\mathbf{y}\in\mathbb{P}^n(\overline{\mathbb{Q}}) :\ H_Q(\mathbf{y})\leqslant \Delta^{1/(n+1)}\cdot Q^{-\varepsilon}\Big\}\subset T_i. \tag{6.10}$$

In addition, we need the following estimate for Δ:

Lemma 6.4. $\Delta\geqslant \mathcal{H}_{\mathcal{L}}^{1-\binom{N+1}{n+1}}$.

Proof. Let $I_1,\dots,I_R$ be the subsets I of cardinality $n+1$ of $\{0,\dots,N\}$ such that $\{l_i :\ i\in I\}$ is linearly independent and put $a_k := \det(l_i :\ i\in I_k)$ for $k=1,\dots,R$.

Then $\Delta = \prod_{v\in M_K} |a_{i_v}|_v$, where $i_v \in \{1,\dots,R\}$ for $v \in M_K$. With the product formula and $R \leqslant \binom{N+1}{n+1}$ this gives

$$\Delta = \prod_{v\in M_K} \frac{|\prod_{k=1}^{R} a_k|_v}{\prod_{k\neq i_v} |a_k|_v} \geqslant \prod_{v\in M_K} \Big(\max_{1\leqslant k\leqslant R} |a_i|_v\Big)^{1-R} = \mathcal{H}_{\mathcal{L}}^{1-R} \geqslant \mathcal{H}_{\mathcal{L}}^{1-\binom{N+1}{n+1}}.$$

□

6.5. *Proof of Theorem 2.5.* Let $X \subset \mathbb{P}^N$ be the linear variety from Theorem 2.5, defined over a number field K. Choose a basis $\mathbf{a}_0 = (a_{00},\dots,a_{0N}),\dots,\mathbf{a}_n = (a_{n0},\dots,a_{nN})$ of $X(\overline{\mathbb{Q}})$ (considered as a vector space) with $\mathbf{a}_0,\dots,\mathbf{a}_n \in K^{N+1}$. Define the family of linear forms

$$\text{(6.11)}\quad \mathcal{L} = \{l_0,\dots,l_N\} \quad \text{with } l_j = a_{0j}x_0 + \cdots + a_{Nj}x_N \quad (j = 0,\dots,N).$$

For $v \in S$, let I_v be an independent subset of cardinality $n+1$ of $\{0,\dots,N\}$ for which $\sum_{i\in I_v} c_{iv}$ is maximal. For $v \in M_K\backslash S$, let I_v be any independent subset of cardinality $n+1$ of $\{0,\dots,N\}$. Thus, for $v \in M_K$, $\{l_i : i \in I_v\}$ is a set of $n+1$ linearly independent linear forms. Notice that the quantity $\mathcal{H}_{\mathcal{L}}$ defined by (6.6) satisfies

$$\mathcal{H}_{\mathcal{L}} = \exp(h(X)), \tag{6.12}$$

where $h(X)$ is the logarithmic height of X. Further, by (6.12) and Lemma 6.4 we have for the quantity Δ defined by (6.5):

$$\Delta \geqslant \exp\Big(-\big\{\tbinom{N+1}{n+1} - 1\big\}h(X)\Big). \tag{6.13}$$

Put

$$\begin{cases} d_{iv} := E^{-1}\cdot\big(E_v - c_{iv}\big) & (v \in S,\ i \in I_v), \\ \\ d_{iv} := 0 & (v \in M_K\backslash S,\ i \in I_v), \end{cases} \tag{6.14}$$

where

$$E_v := \frac{1}{n+1}\sum_{i\in I_v} c_{iv} \quad (v \in S), \quad E := \sum_{v\in S} E_v. \tag{6.15}$$

It is clear that the numbers d_{iv} satisfy (6.1). Further, using that the numbers c_{iv} are $\geqslant 0$, it follows easily that the numbers d_{iv} satisfy (6.7).

Let $\varphi : \mathbb{P}^n \to X$ be the bijective linear map given by $\mathbf{y} = (y_0 : \cdots : y_n) \mapsto \sum_{i=0}^n y_i \mathbf{a}_i$. Let $\mathbf{x} = (x_0 : \cdots : x_N) \in \mathcal{S}_X(\overline{\mathbb{Q}})$ be a point with (2.9). This means that for some finite extension L of K, $\mathbf{x} \in X(L)$ and $\mathbf{x}$ satisfies (2.5). Let $\mathbf{y} = \varphi^{-1}(\mathbf{x})$. Then $\mathbf{y} \in \mathbb{P}^n(L)$ and by (6.11),

$$x_i = l_i(\mathbf{y}) \quad \text{for } i = 0, \dots, N. \tag{6.16}$$

Put

$$Q := \exp\big(E \cdot h(\mathbf{x})\big). \tag{6.17}$$

We estimate from above $H_Q(\mathbf{y})$, where H_Q is defined by (6.2)–(6.4).

Put $I_w = I_v$, $d_{iw} := d_{iv} \cdot \frac{[L_w:K_v]}{[L:K]}$ for $w \in M_L$, $i \in I_w$. Further, let S_L be the set of places of L lying above the places in S, and put $E_w := \frac{1}{n+1} \sum_{i \in I_w} c_{iw}$ for $w \in S_L$. Then by (2.6) and (6.14) we have

$$\begin{cases} d_{iw} := E^{-1} \cdot \big(E_w - c_{iw}\big) & (w \in S_L,\ i \in I_w), \\ \\ d_{iw} := 0 & (w \in M_L \backslash S_L,\ i \in I_w), \end{cases} \tag{6.18}$$

Further, by (2.6), (6.15), (2.7) and the choices of the sets I_v we have

$$\sum_{w \in S_L} E_w = E \geqslant 1 + \delta. \tag{6.19}$$

For $w \in S_L$ we have by (6.16), (6.17), (6.18), (2.5),

$$\begin{aligned} \max_{i \in I_w} \big(|l_i(\mathbf{y})|_w Q^{-d_{iw}}\big) &= \max_{i \in I_w} \big(|x_i|_w \exp((c_{iw} - E_w) h(\mathbf{x}))\big) \\ &\leqslant \|\mathbf{x}\|_w \exp(-E_w h(\mathbf{x})), \end{aligned}$$

while for $w \in M_L \backslash S_L$ we have by (6.16), (6.18),

$$\max_{i \in I_w} \big(|l_i(\mathbf{y})|_w Q^{-d_{iw}}\big) = \max_{i \in I_w} |x_i|_w \leqslant \|x\|_w.$$

By taking the product over $w \in M_L$, invoking (6.17), (6.19), we obtain

$$H_Q(\mathbf{y}) \leqslant \exp\big(-(E-1)h(\mathbf{x})\big) = Q^{-(E-1)/E} \leqslant Q^{-(1+\delta^{-1})^{-1}}.$$

From (6.17) and (6.19), our assumption (2.9) and (6.13) it follows

$$\begin{aligned} \log Q &\geqslant h(\mathbf{x}) \geqslant (1+\delta^{-1})(N+1)^{n+1}(1 + h(X)) \\ &\geqslant 2(1+\delta^{-1}) \binom{N+1}{n+1} h(X) \geqslant 2(1+\delta^{-1}) \log \Delta^{-1/(n+1)}, \end{aligned}$$

and so

$$H_Q(\mathbf{y}) \leqslant \Delta^{1/(n+1)} Q^{-(2(1+\delta^{-1}))^{-1}}.$$

Thus we are in a position to apply Proposition 6.3 with $\varepsilon = (2(1+\delta^{-1}))^{-1}$. Our assumption (2.9), in combination with (6.17), (6.19), (6.12), implies that

$$\log Q \geqslant \log \max \left(\mathcal{H}_{\mathcal{L}}^{1/\binom{N+1}{n+1}}, (n+1)^{4(1+\delta^{-1})} \right),$$

i.e., that condition (6.9) of Proposition 6.3 is satisfied with our choice of ε. It follows that there are proper linear subspaces $T_1, \ldots, T_t$ of $\mathbb{P}^n$ defined over K, with

$$\begin{aligned} t &\leqslant 4^{(n+9)^2} \big(2(1+\delta^{-1})\big)^{n+5} \log(3N) \log\log(3N) \\ &\leqslant 4^{(n+10)^2} (1+\delta^{-1})^{n+5} \log(3N) \log\log(3N), \end{aligned}$$

such that $\mathbf{y} \in T_1 \cup \cdots \cup T_t$. Then $\mathbf{x} \in Y_1 \cup \cdots \cup Y_t$, where $Y_i = \varphi(T_i)$ $(i = 1, \ldots, t)$ are proper linear subspaces of X defined over K which do not depend on $\mathbf{x}$. This completes the proof of Theorem 2.5. $\square$

7. Heights

7.1. Let K be a number field. Denote by $M^\infty(K)$ the set of archimedean places and by M_K^0 the set of non-archimedean places of K. For each $v \in M^\infty(K)$, there is an isomorphic embedding $\sigma_v : K \hookrightarrow \mathbb{C}$ such that $|x|_v = |\sigma_v(x)|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ for $x \in K$. For $\mathbf{x} = (x_0, \ldots, x_N) \in K^{N+1}$, $v \in M^\infty(K)$ we put

$$\|\mathbf{x}\|_{v,1} = \Big(\sum_{i=0}^{N} |\sigma_v(x_i)| \Big)^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}, \quad \|\mathbf{x}\|_{v,2} = \Big(\sum_{i=0}^{N} |\sigma_v(x_i)|^2 \Big)^{[K_v:\mathbb{R}]/2[K:\mathbb{Q}]}.$$

We define heights $h_1(\mathbf{x})$, $h_2(\mathbf{x})$ for $\mathbf{x} \in \overline{\mathbb{Q}}^{N+1} \backslash \{\mathbf{0}\}$ by choosing a number field K with $\mathbf{x} \in K^{N+1}$ and putting

$$\begin{aligned} h_1(\mathbf{x}) &= \sum_{v \in M^\infty(K)} \log \|\mathbf{x}\|_{v,1} + \sum_{v \in M^0(K)} \log \|\mathbf{x}\|_v, \\ h_2(\mathbf{x}) &= \sum_{v \in M^\infty(K)} \log \|\mathbf{x}\|_{v,2} + \sum_{v \in M^0(K)} \log \|\mathbf{x}\|_v; \end{aligned}$$

these quantities are independent of the choice of K. By the product formula, h_1, h_2 define heights on $\mathbb{P}^N(\overline{\mathbb{Q}})$. We have

$$\text{(7.1)} \qquad \begin{cases} h(\mathbf{x}) \leqslant h_2(\mathbf{x}) \leqslant h_1(\mathbf{x}), \\ h_1(\mathbf{x}) \leqslant h(\mathbf{x}) + \log(N+1), \quad h_2(\mathbf{x}) \leqslant h(\mathbf{x}) + \frac{1}{2}\log(N+1) \end{cases}$$

for $\mathbf{x} \in \overline{\mathbb{Q}}^{N+1}$ and

$$\text{(7.2)} \qquad h_2(\mathbf{x}_0 \wedge \cdots \wedge \mathbf{x}_n) \leqslant \sum_{i=0}^{n} h_2(\mathbf{x}_i)$$

for $\mathbf{x}_0, \dots, \mathbf{x}_n \in \overline{\mathbb{Q}}^{N+1}$. Given a polynomial P with coefficients in $\overline{\mathbb{Q}}$, we define $h_1(P)$, $h_2(P)$ to be the respective heights of the coefficient vector of P.

In what follows, X is a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d, defined over $\overline{\mathbb{Q}}$. Let $P_X \subset \overline{\mathbb{Q}}[x_0, \dots, x_N]$ denote the prime ideal of X. Given any number field K such that X is defined over K, denote by $\mathcal{X}$ its Zariski closure over $\mathrm{Spec}(O_K)$, i.e. $\mathcal{X} = \mathrm{Proj}(R/P_X \cap R)$ where $R = O_K[x_0, \dots, x_N]$. Let $\tilde{h}(\mathcal{X})$ be the logarithmic height of $\mathcal{X}$ as defined by Faltings [5], pp. 552, 553. We then define the absolute Faltings height of X by $h_{\mathrm{Falt}}(X) := \frac{1}{[K:\mathbb{Q}]}\tilde{h}(\mathcal{X})$. By [1], p. 948 this is independent of the choice of K.

Lemma 7.2. $h_{\mathrm{Falt}}(X) \leqslant h(X) + d(n+1)(1 + 2\log(N+1))$.

Proof. From [1], Theorem 4.3.8, pp. 989, 990, formulas (4.3.31), (4.3.32), it follows that

$$\text{(7.3)} \qquad h_{\mathrm{Falt}}(X) \leqslant h_1(F_X) + d(n+1)\log(N+1).$$

Since the Chow form F_X is homogeneous of degree d in each of the $n+1$ blocks of $N+1$ variables, its number of coefficients is at most $\binom{N+d}{N}^{n+1} \leqslant (e(N+1))^{d(n+1)}$ with $e = 2.71\dots$, where the latter inequality follows from

$$\text{(7.4)} \qquad \binom{x+y}{x} \leqslant \frac{(x+y)^{x+y}}{x^x y^y} = (1 + x/y)^y (1 + y/x)^x \leqslant (e(1 + y/x))^x$$

for positive integers x, y. So by (7.1) we have

$$h_1(F_X) \leqslant h(F_X) + \log\left((e(N+1))^{d(n+1)}\right) = h(X) + d(n+1)(1 + \log(N+1)).$$

By combining this with (7.3) we obtain the lemma. $\square$

Lemma 7.3. *For every $\varepsilon > 0$, the set*

$$X(\varepsilon) := \{\mathbf{x} \in X(\overline{\mathbb{Q}}) : h_2(\mathbf{x}) \leqslant d^{-1} h_{\mathrm{Falt}}(X) + \varepsilon\}$$

is Zariski dense in X.

Proof. This follows from Zhang [19], p. 208, Theorem 5.2. □

Let m be a positive integer, and put $R := \binom{N+m}{N} - 1$. Let $\mathbf{x}^{\mathbf{a}_0}, \dots, \mathbf{x}^{\mathbf{a}_R}$ denote the monomials of degree m. Consider the Veronese embedding

$$\varphi_m : \mathbb{P}^N \hookrightarrow \mathbb{P}^R : \mathbf{x} \mapsto (\mathbf{x}^{\mathbf{a}_0} : \cdots : \mathbf{x}^{\mathbf{a}_R}). \tag{7.5}$$

Let X_m denote the smallest linear subvariety of $\mathbb{P}^R$ containing $\varphi_m(X)$.

Lemma 7.4. *(i) If X is defined over a number field K then X_m is defined over K.*
(ii) $\dim X_m = H_X(m) - 1 \leqslant d\binom{m+n}{n} - 1$.
(iii) $h(X_m) \leqslant dm\binom{m+n}{n} \cdot \Big(d^{-1}h(X) + (3n+4)\log(N+1)\Big)$.

Proof. Denote by $y_0, \dots, y_R$ the coordinates on $\mathbb{P}^R$. Let $P \subset \overline{\mathbb{Q}}[x_0, \dots, x_N]$ be the prime ideal of polynomials vanishing identically on X. The map given by $\mathbf{x}^{\mathbf{a}_i} \mapsto y_i$ ($i = 0, \dots, R$) defines an isomorphism from the vector space $P_m \subset \overline{\mathbb{Q}}[x_0, \dots, x_N]$ of homogeneous polynomials of degree m vanishing identically on X to the vector space of linear forms in $\overline{\mathbb{Q}}[y_0, \dots, y_R]$ vanishing identically on X_m. If X is defined over K then P_m is generated by polynomials in K. This implies (i). Further we have $\dim X_m = R - \dim P_m - 1 = H_X(m) - 1$ and together with Lemma 5.7 this implies (ii).

In order to prove (iii), let $\varepsilon > 0$ and let X'_m be the smallest linear subspace of $\mathbb{P}^R$ containing $\varphi_m(X(\varepsilon))$. We claim that $X'_m = X_m$. For assume the contrary: then there is a linear form vanishing identically on X'_m but not on X_m. Hence there is a polynomial of degree m vanishing identically on $X(\varepsilon)$ but not on X, which contradicts Lemma 7.3.

Therefore, $X_m(\overline{\mathbb{Q}})$ (considered as a vector space) has a basis of the shape $\{\varphi_m(\mathbf{x}_i) : i = 1, \dots, H\}$, with $H = \dim X_m + 1 = H_X(m)$ and $\mathbf{x}_i \in X(\varepsilon)$ for

$i = 1, \dots, H$. By (2.4), (7.1), (7.2) we have

$$h(X_m) \leqslant h_2(\varphi_m(\mathbf{x}_1) \wedge \cdots \wedge \varphi_m(\mathbf{x}_H)) \leqslant \sum_{i=1}^{H} h_2(\varphi_m(\mathbf{x}_i)).$$

Further, by (7.1), (7.4) we have for $i = 1, \dots, H$,

$$\begin{aligned} h_2(\varphi_m(\mathbf{x}_i)) &\leqslant \tfrac{1}{2}\log\tbinom{m+N}{N} + h(\varphi_m(\mathbf{x}_i)) \\ &\leqslant \tfrac{1}{2}m(1+\log(N+1)) + mh(\mathbf{x}_i) \leqslant m\big(\tfrac{1}{2}(1+\log(N+1)) + h_2(\mathbf{x}_i)\big) \\ &\leqslant m\Big(\tfrac{1}{2}(1+\log(N+1)) + d^{-1}h_{\mathrm{Falt}}(X) + \varepsilon\Big). \end{aligned}$$

Hence

$$h(X_m) \leqslant mH \cdot \Big(\tfrac{1}{2}(1+\log(N+1)) + d^{-1}h_{\mathrm{Falt}}(X) + \varepsilon\Big).$$

By inserting Lemma 5.7, Lemma 7.2 and using $N \geqslant 2$, we obtain

$$\begin{aligned} h(X_m) &\leqslant dm\binom{m+n}{n} \cdot \Big(\tfrac{1}{2}(1+\log(N+1)) + \\ &\qquad\qquad + d^{-1}h(X) + (n+1)(1+2\log(N+1)) + \varepsilon\Big) \\ &\leqslant dm\binom{m+n}{n} \cdot \Big(d^{-1}h(X) + (3n+4)\log(N+1) + \varepsilon\Big). \end{aligned}$$

Since we may choose ε arbitrarily small, this implies (iii). $\square$

8. Proof of Theorem 2.7 (the general case)

8.1. We keep the notation from §2. In particular, X is a projective subvariety of $\mathbb{P}^N$ of dimension n and degree d defined over a number field K, where $1 \leqslant n < N$. We assume that $c_{jv} \geqslant 0$ for $v \in S$, $j = 0, \dots, N$ which is no restriction since the left-hand sides of the inequalities in (2.5) are non-positive. Further we assume that none of the coordinates x_j ($j = 0, \dots, N$) vanishes identically on X which is also no loss of generality. Indeed, suppose for instance that $x_{M+1}, \dots, x_N$ vanish identically on X whereas the other coordinates do not vanish identically on X. Let $X' = \pi(X)$ where π is the projection $(x_0 : \cdots : x_N) \mapsto (x_0 : \cdots : x_M)$ and consider the system of inequalities with solutions in X' obtained from (2.5) by removing all inequalities involving x_i ($i = M+1, \dots, N$). Since the Chow form F_X of X is independent of the variables h_{ij} ($i = 0, \dots, n$, $j = M+1, \dots, N$), the Chow weight $e_X(\mathbf{c}_v)$ is independent of c_{jv} ($j = M+1, \ldots, N$) for $v \in S$. From this one easily deduces that the new system also satisfies (2.12).

In the remainder of the proof we distinguish two cases.

8.2. First assume that

$$\sum_{v\in S} \max_{0\leqslant j\leqslant N} c_{jv} \geqslant 2\min\big((n+1)d, N+1\big). \tag{8.1}$$

For $v \in S$, choose $j_v \in \{0, \ldots, N\}$ such that $c_{j_v,v} = \max_{0\leqslant j\leqslant N} c_{jv}$ and put

$$d_{j_v,v} = c_{j_v,v}, \quad d_{jv} = 0 \text{ for } j = 0, \ldots, N,\ j \neq j_v. \tag{8.2}$$

Let X_1 be the smallest linear subspace of $\mathbb{P}^N$ which contains X. Put $H := \dim X_1$. By Lemma 7.4, (i) with $m = 1$, X_1 is defined over K. For $v \in S$, let I_v be a subset of $\{0, \ldots, N\}$ of cardinality $H+1$ containing j_v which is independent with respect to X_1, i.e., no non-trivial linear combination of the variables x_j ($j \in I_v$) vanishes identically on X_1; such a set exists since x_{j_v} does not vanish identically on X, hence not on X_1. By Lemma 7.4, (ii) with $m = 1$, we have $H \leqslant \min((n+1)d - 1, N)$. Together with (8.2), (8.1) this implies

$$\frac{1}{H+1}\sum_{v\in S}\sum_{j\in I_v} d_{jv} \geqslant 2. \tag{8.3}$$

Invoking (2.6) we obtain that for every $\mathbf{x} \in X(L)$ satisfying (2.5) for some finite extension L we have

$$\log\left(\frac{|x_j|_w}{\|\mathbf{x}\|_w}\right) \leqslant -d_{jw}h(\mathbf{x}) \quad \text{for } w \in S_L,\ j = 0, \ldots, N, \tag{8.4}$$

where $d_{jw} = \frac{[L_w:K_v]}{[L:K]} d_{jv}$ with $v \in S$ being the place below w. So we are in a position to apply Theorem 2.5 with X_1, H, 1 in place of X, n, δ. It follows that the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x}) \geqslant 2(N+1)^{H+1}(1 + h(X_1)) \tag{8.5}$$

is contained in the union of at most

$$t_0 = 4^{(H+10)^2} 2^{H+5} \log(3N) \log\log(3N) \tag{8.6}$$

proper linear subspaces of X_1 which are all defined over K.

Note that by Lemma 7.4,(ii),(iii) with $m = 1$ the right-hand side of (8.5) is at most

$$\begin{aligned} &2(N+1)^{d(n+1)}\Big(1+(n+1)h(X)+d(3n+4)\log(N+1)\Big)\\ &\qquad\leqslant c_3(N,n,d,\delta)(1+h(X)), \end{aligned}$$

hence (8.5) is implied by (2.16). The intersection of X with a proper linear subspace of X_1 defined over K is a proper Zariski closed subset of X, and by Bézout's theorem, it is the union of at most d proper K-subvarieties of X, each of degree $\leqslant d$. Hence the set of $\mathbf{x} \in \mathcal{S}_X(\overline{\mathbb{Q}})$ with (2.16) is contained in the union of at most $t = dt_0$ proper K-subvarieties of X of degree at most d. Inserting Lemma 7.4, (ii) with $m = 1$ into (8.6) we obtain

$$t \leqslant d \cdot 4^{((n+1)d+9)^2} 2^{(n+1)d+4} \log 3N \log\log 3N \leqslant c_1(N,n,d,\delta).$$

Further, $d \leqslant c_2(N,n,d,\delta)$. This shows (2.14) and (2.15). Thus under assumption (8.1), Theorem 2.7 follows.

8.3. Now assume that

$$\sum_{v\in S} \max_{0\leqslant j\leqslant N} c_{jv} < 2\min\big((n+1)d, N+1\big). \tag{8.7}$$

Choose

$$m = 1 + \big[(8n+4)(1+\delta^{-1})d\min\big((n+1)d, N+1\big)\big]\ . \tag{8.8}$$

Put $R := \binom{N+m}{N} - 1$. Let $\varphi_m : \mathbb{P}^N \hookrightarrow \mathbb{P}^R$ be the Veronese embedding defined by (7.5), and let X_m be the smallest linear subvariety of $\mathbb{P}^R$ containing $\varphi_m(X)$. Recall that by Lemma 7.4, X_m is defined over K and $\dim X_m = H_X(m) - 1$.

For $\mathbf{x} \in X(\overline{\mathbb{Q}})$, put $y_i = \mathbf{x}^{\mathbf{a}_i}$ $(i = 0,\ldots,R)$, $\mathbf{y} = (y_0 : \cdots : y_R) = \varphi_m(\mathbf{x})$. Further, put

$$d_{iv} := \frac{1}{m}\mathbf{a}_i \cdot \mathbf{c}_v \quad (v \in S,\ i = 0,\ldots,R). \tag{8.9}$$

Write $\mathbf{a}_i = (a_{i0},\ldots,a_{iN})$ for $i = 0,\ldots,R$. Then using $\|\mathbf{y}\|_w = \|\mathbf{x}\|_w^m$, $h(\mathbf{y}) = mh(\mathbf{x})$, and invoking (2.6), we have for $\mathbf{x} \in X(L)$ satisfying (2.5) for some finite extension

L of K, that

$$
\begin{aligned}
\text{(8.10)} \qquad \log\Big(\frac{|y_i|_w}{\|\mathbf{y}\|_w}\Big) &= \sum_{k=0}^{N} a_{ik}\log\Big(\frac{|x_k|_w}{\|\mathbf{x}\|_w}\Big) \leqslant -\big(\sum_{k=0}^{N} a_{ik}c_{kw}\big)h(\mathbf{x}) \\
&\leqslant -d_{iw}h(\mathbf{y}) \quad \text{for } w \in S_L,\ j=0,\dots,R,
\end{aligned}
$$

where $d_{iw} = \frac{[L_w:K_v]}{[L:K]}d_{iv}$ with $v \in S$ being the place below w. We consider system (8.10) with solutions $\mathbf{y} \in X_m$. We show that the analogue of (2.7) for this system is satisfied.

Denote by $\mathcal{I}_{X_m}$ the collection of subsets of $\{0,\dots,R\}$ of cardinality $\dim X_m + 1 = H_X(m)$ which are independent with respect to X_m. Let $P_X \subset \overline{\mathbb{Q}}[x_0,\dots,x_N]$ be the prime ideal of X. A set I is independent with respect to X_m if and only if there is no non-trivial linear combination of the variables y_i $(i \in I)$ vanishing identically on X_m. This is equivalent to the statement that no such linear combination vanishes identically on $\varphi_m(X)$. The latter means precisely that no non-trivial linear combination of the monomials $\mathbf{x}^{\mathbf{a}_i}$ $(i \in I)$ vanishes identically on X, in other words, that the monomials $\mathbf{x}^{\mathbf{a}_i}$ $(i \in I)$ are linearly independent modulo $(P_X)_m$. Therefore,

$$
I \in \mathcal{I}_{X_m} \iff \{\mathbf{x}^{\mathbf{a}_i} : i \in I\} \text{ is a basis of } \overline{\mathbb{Q}}[x_0,\dots,x_N]_m/(P_X)_m.
$$

Together with (8.9) this implies

$$
\tfrac{1}{\dim X_m+1}\cdot \max_{I\in\mathcal{I}_{X_m}} \sum_{i\in I} d_{iv} = \tfrac{1}{mH_X(m)}\cdot s_X(m,\mathbf{c}_v),
$$

where $s_X(m,\mathbf{c}_v)$ is given by (3.4). Now from Theorem 3.6, (2.12), (8.7), (8.8), it follows

$$
\begin{aligned}
\tfrac{1}{mH_X(m)}\cdot\sum_{v\in S} s_X(m,\mathbf{c}_v) &\geqslant \tfrac{1}{(n+1)d}\cdot\sum_{v\in S} e_X(\mathbf{c}_v) - \tfrac{(2n+1)d}{m}\cdot\sum_{v\in S}\max_{0\leqslant j\leqslant N} c_{jv} \\
&\geqslant 1+\delta - \tfrac{(2n+1)d\cdot 2\min((n+1)d,N+1))}{m} \\
&\geqslant 1+\delta/2\,.
\end{aligned}
$$

Hence

$$
\text{(8.11)} \qquad \tfrac{1}{\dim X_m+1}\sum_{v\in S}\left(\max_{I\in\mathcal{I}_{X_m}}\big(\sum_{i\in I} d_{iv}\big)\right) \geqslant 1+\delta/2,
$$

which is the analogue of (2.7) for system (8.10) with $\delta/2$ replacing δ.

Now the conditions of Theorem 2.5 are satisfied with X_m, $R=\binom{N+m}{N}-1$, $H_X(m)-1$, $\delta/2$ in place of X, N, n, δ. It follows that there are proper linear subspaces $Z_1,\dots,Z_{t_0}$ of X_m, all defined over K, with

$$t_0=4^{(H_X(m)+9)^2}(1+2\delta^{-1})^{H_X(m)+4}\log\left(3\binom{N+m}{N}\right)\log\log\left(3\binom{N+m}{N}\right)$$

such that for every finite extension L of K the set of solutions $\mathbf{y}\in X_m(L)$ of (8.10) with

$$h(\mathbf{y})\geqslant h_0=\binom{N+m}{N}^{H_X(m)}(1+2\delta^{-1})(1+h(X_m))$$

is contained in $Z_1\cup\cdots\cup Z_{t_0}$.

For $i=1,\dots,t_0$, the intersection $X\cap\varphi_m^{-1}(Z_i)$ is contained in $X\cap Z(f_i)$, where $Z(f_i)$ is the zero locus of a homogeneous polynomial $f_i\in K[x_0,\dots,x_N]$ of degree m not vanishing identically on X. By Bézout's Theorem, $X\cap Z(f_i)$ is equal to the union of at most dm K-subvarieties, each of degree $\leqslant dm$. Using that $h(\varphi_m(\mathbf{x}))=mh(\mathbf{x})$, it follows that the set of $\mathbf{x}\in\mathcal{S}_X(\overline{\mathbb{Q}})$ with

$$h(\mathbf{x})\geqslant m^{-1}h_0=m^{-1}\binom{N+m}{N}^{H_X(m)}(1+2\delta^{-1})(1+h(X_m)) \tag{8.12}$$

is contained in the union of at most

$$t=dmt_0=dm\cdot 4^{(H_X(m)+9)^2}(1+2\delta^{-1})^{H_X(m)+4}\log\left(3\binom{N+m}{N}\right)\log\log\left(3\binom{N+m}{N}\right) \tag{8.13}$$

proper K-subvarieties of X, each of degree $\leqslant dm$.

Using Lemma 5.7, (7.4), (8.8), $n\geqslant 1$, $N\geqslant 2$, we obtain

$$\begin{aligned}
H_X(m) &\leqslant d\binom{m+n}{n}\leqslant d\big(e(m+1)\big)^n\\
&\leqslant d\Big(e(8n+5)(n+1)d^2(1+\delta^{-1})\Big)^n\leqslant d\Big(71n^2d^2(1+\delta^{-1})\Big)^n,\\
\binom{N+m}{N} &\leqslant \big(e(N+1)\big)^m\leqslant\big(e(N+1)\big)^{26n^2d^2(1+\delta^{-1})}.
\end{aligned}$$

Together with Lemma 7.4, (iii), this implies that the right-hand side of (8.12) is at most

$$\begin{aligned}
&m^{-1}\big(e(N+1)\big)^{dm(e(m+1))^n}\big(1+2\delta^{-1}\big)\cdot \\
&\qquad\cdot\Big(1+m\tbinom{n+m}{n}\big(h(X)+d(3n+4)\log(N+1)\big)\Big) \\
&\leqslant \big(e(N+1)\big)^{dm(e(m+1))^n}\big(1+2\delta^{-1}\big)\cdot \\
&\qquad\cdot m\tbinom{n+m}{n}\cdot\big(1+d(3n+4)\log(N+1)\big)\cdot\big(1+h(X)\big) \\
&\leqslant \big(e(N+1)\big)^{d(e(m+1))^{n+1}}\cdot(1+h(X)) \\
&\leqslant \big(e(N+1)\big)^{d(71n^2d^2(1+\delta^{-1}))^{n+1}}\cdot(1+h(X)) \\
&\leqslant (3N)^{(10n)^{2n+2}d^{2n+3}(1+\delta^{-1})^{n+1}}\cdot(1+h(X)) = c_3(N,n,d,\delta)\cdot(1+h(X)),
\end{aligned}$$

hence (8.12) is implied by (2.16).

In order to estimate from above the upper bound t for the number of subvarieties from (8.13), we first observe that

$$\begin{aligned}
\log\big(3\tbinom{N+m}{N}\big)\log\log\big(3\tbinom{N+m}{N}\big) &\leqslant \log\big(3(e(N+1))^m\big)\log\log\big(3(e(N+1))^m\big) \\
&\leqslant \log\big((3N)^{2m}\big)\log\log\big((3N)^{2m}\big) \\
&\leqslant 2m^2\log(3N)\log\log(3N).
\end{aligned}$$

Therefore,

$$\begin{aligned}
t &\leqslant dm\cdot 4^{(d\binom{m+n}{n}+9)^2}\cdot(1+2\delta^{-1})^{d\binom{m+n}{n}+4}\cdot 2m^2\log(3N)\log\log(3N) \\
&\leqslant \big(4e^{1/71}\big)^{\left((71n^2)^n d^{2n+1}(1+\delta^{-1})^n+10\right)^2}\log(3N)\log\log(3N) \\
&\leqslant \exp\Big((10n)^{4n}d^{4n+2}(1+\delta^{-1})^{2n}\Big)\cdot\log(3N)\log\log(3N) = c_1(N,n,d,\delta).
\end{aligned}$$

Finally, by (8.8), we have $md \leqslant (8n+5)(1+\delta^{-1})d^2\min((n+1)d, N+1) = c_2(N,n,d,\delta)$. Hence (2.14), (2.15) hold true. This completes the proof of Theorem 2.7. $\square$

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