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Abstract

We will describe three theorems (A,B, and C) which summarize what survives in three dimensions of Smale's proof of the higher dimensional Poincaré conjecture. The proofs require Smale's Cancellation Lemma and a lemma asserting the existence of a 2-gon (see [9]). Such 2-gons are the analogues in dimension two of Whitney disks in higher dimensions. They are also embedded lunes; a (immersed) lune is an index one connecting orbit in the Lagrangian Floer homology determined by two embedded loops in a 2-manifold.

1 Introduction

This is an expository paper. We wrote it to teach ourselves some low dimensional topology. Our objective was to understand the speculation of Hsiang [10] concerning Floer homology and the Poincaré conjecture.

Intersection numbers

For transverse embedded closed curves α, β in an orientable 2-manifold Σ there are three ways we can count the number of points in their intersection:

1. The **numerical intersection number** $\mathrm{num}\,(\alpha, \beta)$ is the actual number of intersection points.

2. The **geometric intersection number** $\text{geo}(\alpha, \beta)$ is defined as the minimum of the numbers $\text{num}(\alpha, \beta')$ over all embedded loops β' that are transverse to α and isotopic to β.

3. The **algebraic intersection number** $\text{alg}(\alpha, \beta)$ is the absolute value $\text{alg}(\alpha, \beta) = |\alpha \cdot \beta|$ of the sum $\alpha \cdot \beta = \sum_{x \in \alpha \cap \beta} \pm 1$ where the plus sign is chosen iff the two orientations of $T_x\Sigma = T_x\alpha \oplus T_x\beta$ match. This definition is independent of the choice of orientations of α, β, and Σ.

The inequalities

$$\text{alg}(\alpha, \beta) \leq \text{geo}(\alpha, \beta) \leq \text{num}(\alpha, \beta)$$

are immediate.

Remark 1.1. Two embedded loops in Σ are homotopic if and only if they are isotopic (see [4]). Hence, if in the definition of geometric intersection number the word *isotopic* is replaced by the word *homotopic*, the value of $\text{geo}(\alpha, \beta)$ remains unchanged.

Morse–Smale/Floer systems

Throughout this section M is a compact m-manifold, possibly with boundary. We assume throughout that ξ is a vector field on M, transverse to the boundary, and denote by φ^t the flow of ξ and by $P(\xi)$ denote the set of rest points. The stable and unstable manifolds of the rest point p are

$$W^s(p) := W^s(p; \xi) := \left\{ z \in M \mid \lim_{t \to \infty} \varphi(t, z) = p \right\},$$

$$W^u(p) := W^u(p; \xi) := \left\{ z \in M \mid \lim_{t \to -\infty} \varphi(t, z) = p \right\}.$$

The vector field ξ is called **gradient-like** if $P(\xi)$ is a finite set and there exists a smooth **height function** $h : M \to \mathbb{R}$ such that $dh(z)\xi(z) \leq 0$ for all $z \in M$ with equality if and only if $z \in P(\xi)$. It follows that

$$M = \bigcup_{p \in P(\xi)} W^s(p; \xi) = \bigcup_{p \in P(\xi)} W^u(p; \xi).$$

If ξ has only hyperbolic rest points we write

$$P(\xi) = \bigcup_{k=0}^{m} P_k(\xi),$$

where $P_k(\xi)$ denotes the set of rest points of Morse index k. A vector field ξ is called **Morse–Smale**[1] iff it is gradient-like and has only hyperbolic rest points such that $W^u(p;\xi)$ and $W^s(q;\xi)$ intersect transversally for all $p, q \in P(\xi)$. A gradient-like vector field ξ is called **Morse–Floer** if all its rest points are hyperbolic, if $W^u(q;\xi)$ and $W^s(p;\xi)$ intersect transversally for all $p \in P_k(\xi)$ and $q \in P_{k+1}(\xi)$, and if there exists a $z \in W^u(q;\xi) \cap W^s(p;\xi)$ with $W^u(q;\xi) \pitchfork_z W^s(p;\xi)$ whenever $W^u(q;\xi) \cap W^s(p;\xi) \neq \emptyset$. (Compare with "Axiom B" of [22].) Note that if M has dimension three then a Morse–Floer vector field is automatically Morse–Smale.

Remark 1.2. Every Morse–Floer vector field ξ on M admits a **self-indexing** height function $h : M \to \mathbb{R}$, i.e. one which satisfies $h(p) = k$ for $p \in P_k(\xi)$ and is constant on each boundary component (see [13]).

Define the **Smale order** on $P(\xi)$ by $p \preceq_\xi q$ iff there exists a sequence of rest points $p = p_0, p_1, \ldots, p_{n-1}, p_n = q$ such that $W^u(p_i;\xi) \cap W^s(p_{i-1};\xi) \neq \emptyset$ for $i = 1, \ldots, n$. If ξ is gradient-like this is a partial order. For a Morse–Floer vector field it is equivalent to take $n = 1$:

$$p \preceq_\xi q \qquad \Longleftrightarrow \qquad W^u(q;\xi) \cap W^s(p;\xi) \neq \emptyset.$$

(This is the "λ-Lemma" of Palis. See [22, 11].)

HMS structures

Henceforth Y is a **closed** (i.e. compact and without boundary) connected oriented smooth 3-manifold.

Definition 1.3. An **HMS structure** on Y is a triple (Y_0, Y_1, ξ) consisting of a Morse–Smale vector field ξ on Y and a decomposition $Y = Y_0 \cup Y_1$ of Y into two 3-submanifolds intersecting in their common boundary

$$Y = Y_0 \cup Y_1, \qquad Y_0 \cap Y_1 = \partial Y_0 = \partial Y_1, \tag{1}$$

such that

[1] Our terminology is nonstandard in that for us a Morse–Smale system has no periodic orbits.

(i) ξ has one rest point p_0 of index zero, one rest point q_0 of index three, g rest points $p_1, \dots, p_g$ of index one, and g rest points $q_1, \dots, q_g$ of index two;

(ii) $p_0, p_1, \dots, p_g \in Y_0$ and $q_0, q_1, \dots, q_g \in Y_1$;

(iii) ξ is transverse to Σ.

A **Heegard splitting** of Y is a decomposition $Y = Y_0 \cup Y_1$ as in (1) which arises from some HMS structure.

Remark 1.4. If a Morse–Smale vector field on Y has exactly one critical point of index zero and exactly one critical point of index three, then (by Theorem 2.1 below) the number of critical points of index one must equal the number of critical points of index two. In Corollary 2.3 we show that this number is equal to the genus of Σ; we call it the *genus* of the HMS structure.

Definition 1.5. Let $\alpha := \alpha_1 \cup \cdots \cup \alpha_g$ and $\beta := \beta_1 \cup \cdots \beta_g$ be the 1-submanifolds of $\Sigma := Y_0 \cap Y_1$ defined by

$$\alpha_i := W^s(p_i) \cap \Sigma, \qquad \beta_j := W^u(q_j) \cap \Sigma, \qquad i, j = 1, \dots, g$$

The pair (α, β) is called **the trace** of the HMS structure (Y_0, Y_1, ξ) and **a trace** of the Heegard splitting (Y, Y_0, Y_1). Each connecting orbit from q_j to p_i intersects Σ in an intersection point of α_i and β_j. We say that an HMS structure is

$$\left\{ \begin{array}{l} \textbf{algebraically} \\ \textbf{geometrically} \\ \textbf{numerically} \end{array} \right\} \textbf{ reduced } \text{iff} \left\{ \begin{array}{l} \mathrm{alg}\,(\alpha_i, \beta_j) = \delta_{ij} \\ \mathrm{geo}\,(\alpha_i, \beta_j) = \delta_{ij} \\ \mathrm{num}\,(\alpha_i, \beta_j) = \delta_{ij} \end{array} \right\}$$

for $i, j = 1, \dots, g$.

Remark 1.6. Let Σ be a closed connected oriented 2-manifold of genus g. A **trace** in Σ is a closed 1-submanifold $\alpha \subset \Sigma$ such that the complement $\Sigma \setminus \alpha$ is connected. In Appendix A we show that a 1-submanifold $\alpha \subset \Sigma$ is a trace if and only if it arises from an HMS structure as in definition 1.5. There we also explain how to reconstruct the HMS structure (Y_0, Y_1, ξ) from a transverse pair of traces $\alpha, \beta \subset \Sigma$. Indeed, up to an appropriate notion of equivalence, a closed connected oriented 3-manifold is the same as a 2-manifold equipped with a transverse pair of traces.

Main theorems

Theorem A. *Every closed connected oriented 3-manifold Y admits an HMS structure.*

Theorem B. *A closed connected oriented 3-manifold Y is an integral homology 3-sphere if and only if it admits an algebraically reduced HMS structure.*

Theorem C. *For every closed connected oriented 3-manifold Y the following are equivalent.*

(i) *Y is diffeomorphic to the 3-sphere.*

(ii) *Y admits an HMS structure of genus zero.*

(iii) *Y admits a numerically reduced HMS structure.*

(iv) *Y admits a geometrically reduced HMS structure.*

When we began to work on this project we hoped that the mere existence of an algebraically reduced HMS structure that is not geometrically reduced would imply that the homology 3-sphere Y has nontrivial Floer homology and is therefore not simply connected (and that the difficulty in establishing the Poincaré conjecture lay in proving nontriviality of Floer homology under this hypothesis). However, there is an algebraically reduced HMS structure on S^3 which is not geometrically reduced. See Example D.1.

Roadmap

Except for the implication (iv) $\Longrightarrow$ (iii) in Theorem C, the proofs of these theorems are the same as, or refinements of, the proofs used in the higher dimensional Poincaré conjecture. (The standard exposition is [13].)

Theorem A is explicitly stated in [20]. Its proof uses the Cancellation Lemma (see Theorem 3.1) and the "Morse homology theory" described below. We give a proof of Theorem A in Section 3.

Theorem B also uses this Morse homology theory and a "handle sliding argument"; the proof is the same as in higher dimensions and is carried out in Section 2.

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) in Theorem C are obvious.

The implication (ii) $\Longrightarrow$ (i) is essentially a smooth version of Reeb's Theorem [16]. It follows easily from that fact that the group $\mathrm{Diff}_+(S^2)$ of orientation preserving diffeomorphisms of the 2-sphere is connected. We give a proof of this well-known fact as well as the details of the proof of (ii)$\Longrightarrow$(i) in Appendix B.

To prove (iii) $\Longrightarrow$ (ii) we cancel critical points as in the higher dimensional case. This only requires an alteration of the vector field in an arbitrarily small neighbourhood of the connecting orbit. Hence the cancellation of critical points can be carried out on a numerically reduced HMS structure so as to leave another numerically reduced HMS structure. The proof of the Cancellation Lemma is given in Appendix C and the proof of (iii)$\Longrightarrow$(ii) in Section 3.

The implication (iv) $\Longrightarrow$ (iii) is proved in Section 4. Here is where the existence of a 2-gon is used.

Floer homology

The traces α and β of an HMS structure (Y, Y_0, Y_1, ξ) can be interpreted as Lagrangian submanifolds of $\Sigma := Y_0 \cap Y_1$ (with respect to any area form). The connecting orbits of the Morse complex (2) below are intersections points of α and β and hence can be interpreted as the critical points in Floer homology. The 2-gons appear as connecting orbits of index one in the Floer complex. In general, the Floer connecting orbits of index one need not be embedded, but are immersed half discs with boundary arcs in α and β respectively (see Section 5).

2 Morse Homology

Let M be a compact m-manifold with boundary

$$\partial M = \Sigma_0 \cup \Sigma_1$$

and ξ be a Morse–Floer vector field on M that points in on Σ_1 and points out on Σ_0. When the index difference of q and p is not one let $n(q,p) := n(q,p;\xi) := 0$; for $p \in P_k(\xi)$ and $q \in P_{k+1}(\xi)$ we denote the number of connecting orbits by

$$n(q,p) := n(q,p;\xi) := \#\left(W^u(q;\xi) \cap W^s(p;\xi)\right)/\mathbb{R}.$$

Similarly, we define the algebraic number $\nu(q,p)=\nu(q,p;\xi)$ of connecting orbits to be zero when the index difference of q and p is not one; for $p\in P_k(\xi)$ and $q\in P_{k+1}(\xi)$ this number is defined as follows. Orient each $W^u(p)$ arbitrarily. For every integral curve $u:\mathbb{R}\to M$ of ξ running from q to p choose an invariant complement E_t to $\mathbb{R}\xi(u(t))$ in $T_{u(t)}W^u(q)$. This complement inherits an orientation from $W^u(q)$ and, as t tends to infinity, converges to $\pm T_pW^u(p)$ in the Grassmann bundle of oriented k-planes in TM. Denote the sign by $\varepsilon(u)$ and define

$$\nu(q,p):=\sum_{[u]}\varepsilon(u)$$

where the sum runs over the equivalence classes $[u]$ of integral curves of ξ from q to p; the equivalenve relation is given by time translation. If M is oriented then $W^s(p)$ can be oriented so that the product orientation of $T_pM\cong T_pW^u(p)\oplus T_pW^s(p)$ is the orientation of T_pM. In this case $\nu(q,p)$ is the algebraic intersection number of $W^u(q)\cap h^{-1}(k+\frac{1}{2})$ with $W^s(p)\cap h^{-1}(k+\frac{1}{2})$ for $q\in P_{k+1}$ and $p\in P_k$, where h is a self-indexing height function. Define $\partial:C_{*+1}\to C_*$ by

$$C_k:=\bigoplus_{p\in P_k}\mathbb{Z}p,\qquad \partial q:=\sum_{p\in P_k}\nu(q,p)p,\quad q\in P_{k+1},\tag{2}$$

This chain complex is usually ascribed to Witten [23] and Floer [7], but the following theorem is older: a proof may be found in [12]. Other proofs can be found in [18] and [19].

Theorem 2.1. *The operator ∂ defined in equation (2) satisfies $\partial\circ\partial=0$ and its (co)homology is isomorphic to the singular (co)homology of the pair (M,Σ_0). Namely, for every abelian group Λ we have*

$$\frac{\mathrm{Kernel}\,(\partial:C_k\otimes\Lambda\to C_{k-1}\otimes\Lambda)}{\mathrm{Image}\,(\partial:C_{k+1}\otimes\Lambda\to C_k\otimes\Lambda)}\cong H_k(M,\Sigma_0;\Lambda)$$

and

$$\frac{\mathrm{Kernel}\,(\partial^*:\mathrm{Hom}(C_k,\Lambda)\to\mathrm{Hom}(C_{k+1},\Lambda))}{\mathrm{Image}\,(\partial^*:\mathrm{Hom}(C_{k-1},\Lambda)\to\mathrm{Hom}(C_k,\Lambda))}\cong H^k(M,\Sigma_0;\Lambda).$$

Corollary 2.2 (Poincaré duality).

$$H^k(M,\Sigma_0;\Lambda)\cong H_{m-k}(M,\Sigma_1;\Lambda).$$

Hence, if Λ is a field,

$$H_k(M, \Sigma_0; \Lambda) \cong H_{m-k}(M, \Sigma_1; \Lambda).$$

Proof. Reverse the flow and use Theorem 2.1. $\square$

Corollary 2.3. *Let Y_0 be a compact connected oriented smooth 3-manifold with boundary and ξ be a Morse–Smale vector field on Y_0 that points in on the boundary and has only rest points of index zero and one. Then the 2-manifold $\Sigma = \partial Y_0$ is connected and has genus*

$$g := 1 - \#P_0(\xi) + \#P_1(\xi). \tag{3}$$

Proof. Take $\Lambda := \mathbb{Q}$. By Theorem 2.1, we have

$$H_2(Y_0) = \{0\}, \qquad H_1(Y_0, \Sigma) = \{0\}.$$

(The latter is proved by reversing the flow.) Hence, since the Euler characteristic of the chain complex agrees with the Euler characteristic of its homology, we have

$$\dim H_1(Y_0) - \dim H_0(Y_0) = \#P_1(Y_0) - \#P_0(Y_0).$$

Since Y_0 is connected it follows that

$$\dim H_1(Y_0) = g, \qquad \dim H_2(Y_0, \Sigma) = g.$$

(The latter is proved by reversing the flow.) Hence the homology exact sequence of the pair (Y, Σ) has the form

$$0 \to H_2(Y, \Sigma) \to H_1(\Sigma) \to H_1(Y) \to 0.$$

So $\dim H_2(\Sigma) = 2g$ as claimed. $\square$

Proof of Theorem B (assuming Theorem A). Take $M = Y$ and ξ the vector field of an HMS structure. Then equation (2) is

$$\partial q_0 = 0, \qquad \partial q_j = \sum_{i=1}^{g} (\alpha_i \cdot \beta_j) p_i, \qquad \partial p_i = 0.$$

Thus Y is an integral homology sphere if and only if the intersection matrix with entries

$$\nu_{ij} := \alpha_i \cdot \beta_j$$

is unimodular. This is certainly the case if the HMS structure is algebraically reduced.

For the converse assume that Y is an integral homology 3-sphere. By Theorem A, there exists an HMS structure (Y_0, Y_1, ξ) on Y. Let (ν_{ij}) be the corresponding intersection matrix. By Theorem 2.1, the matrix (ν_{ij}) is unimodular. Any integer matrix may be diagonalized by elementary row and column operations: scale, swap, and shear. The scale operation reverses the sign of a row or column, the swap operation interchanges two rows or columns, and the shear operation adds a row or column to a different one. Each operation may be realized by a corresponding operation on the HMS structure. Reversing the sign of the jth column corresponds to reversing the orientation of $W^u(q_j)$ and hence of β_j. Interchanging rows or columns corresponds to relabeling the components of α or β. To perform the shear which adds column i to column j we shall replace β_i by the connected sum

$$\beta_i' \cong \beta_i \# \beta_j.$$

To construct β_i' choose an embedding $\gamma : [0,1] \to \Sigma$ such that

$$\gamma(0) \in \beta_i, \qquad \gamma(1) \in \beta_j, \qquad \gamma((0,1)) \cap \beta = \emptyset,$$

and γ intersects β_i and β_j with opposite signs. This is possible because $\Sigma \setminus \beta$ is connected. Use this path as a guide to construct β_i' as an embedded path near one that traces out β_i, γ, β_j, and γ^{-1}. We construct a Morse–Smale vector field ξ' with trace (α, β'), where

$$\beta' := \beta_1 \cup \cdots \cup \beta_{i-1} \cup \beta_i' \cup \beta_{i+1} \cup \cdots \cup \beta_g,$$

as follows. Let $h : Y \to \mathbb{R}$ be a height function for ξ, i.e. $dh \cdot \xi$ is negative on the complement of the rest points. We assume

$$\max_{\nu} h(p_\nu) < h(\Sigma) < \min_{\nu \neq i,j} h(q_\nu) \leq \max_{\nu \neq i,j} h(q_\nu) < h(q_j) < h(q_i).$$

Then the level set $h^{-1}(c)$ is diffeomorphic to the 2-torus for $h(q_j) < c < h(q_i)$. Choose c and c' such that

$$h(q_j) < c' < c < h(q_i).$$

Let b_i be the intersection of the backwards orbit of β_i with $h^{-1}(c)$ and b_i' be the intersection of the backwards orbit of β_i' with $h^{-1}(c')$. Then $b_i = W^u(q_i) \cap h^{-1}(c)$ and b_i' is isotopic to $W^u(q_i) \cap h^{-1}(c')$ (see Figure 1). By familiar arguments $h^{-1}([c', c])$ is diffeomorphic to $\mathbb{T}^2 \times [c', c]$ with orbits $\{\mathrm{pt}\} \times [c', c]$ (see [13]). Modify the flow in $h^{-1}([c', c])$ so that it carries b_i to b_i'. □

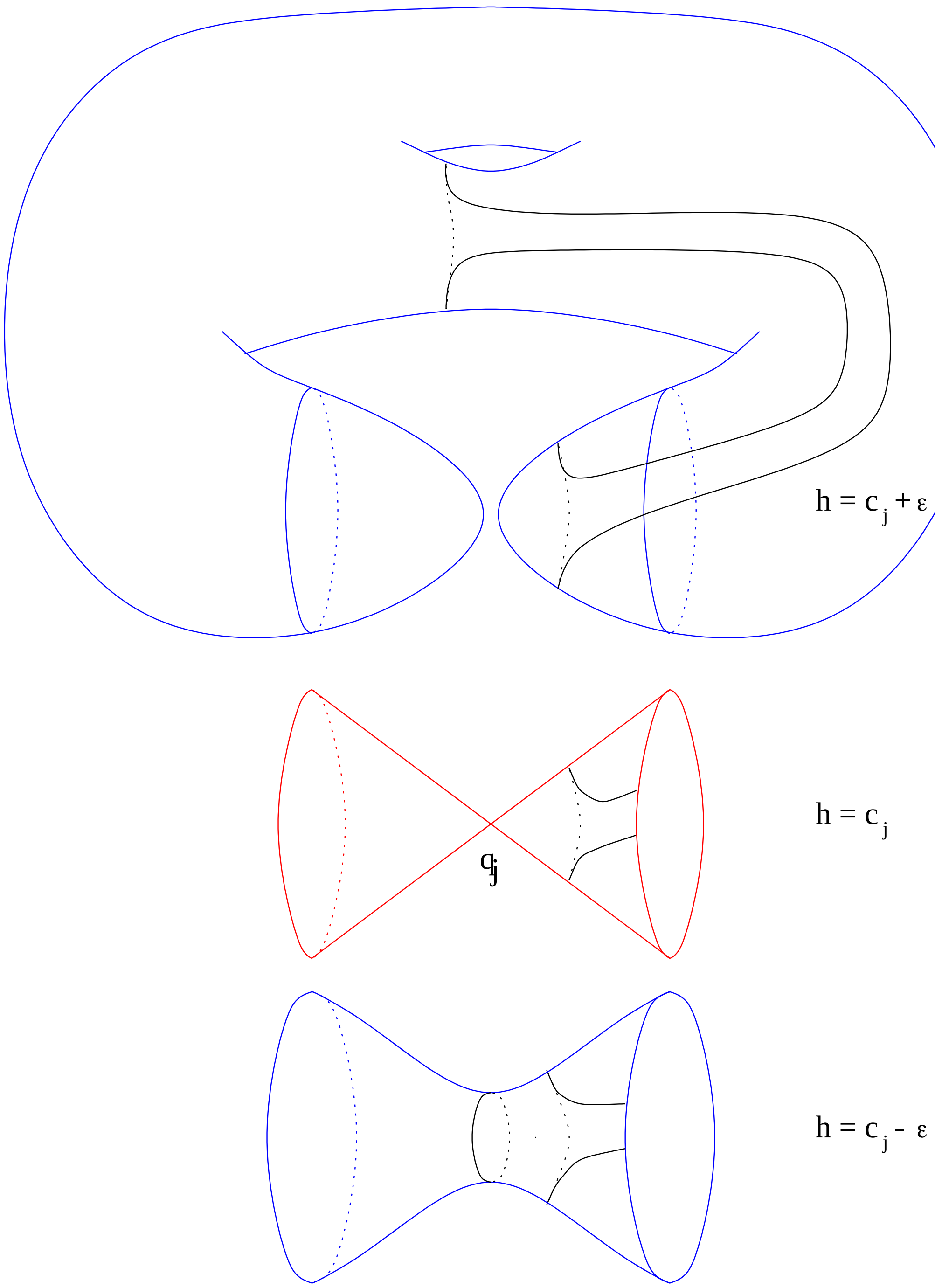

Figure 1: The backward orbit of $\beta_i \# \beta_j$ near q_j.

3 The Cancellation Lemma

The following is an improved form of Smale's Cancellation Lemma with essentially the same proof (see Appendix C).

Theorem 3.1 (Cancellation Lemma). *Suppose that ξ is a Morse-Floer vector field on M and let $\bar{p}, \bar{q} \in P(\xi)$ be such that*

$$n(\bar{q}, \bar{p}; \xi) = 1.$$

Let Γ denote the closure of the connecting orbit. Then, for every neighborhood U of Γ, there exists a Morse–Floer vector field η on M which agrees with ξ on the complement of U and satisfies

$$P(\eta) = P(\xi) \setminus \{\bar{p}, \bar{q}\}, \tag{4}$$

$$p \preceq_\eta q \qquad \Longleftrightarrow \qquad \begin{cases} \textit{either} & p \preceq_\xi q, \\ \textit{or} & p \preceq_\xi \bar{q} \textit{ and } \bar{p} \preceq_\xi q, \end{cases} \tag{5}$$

and

$$n(q, p; \eta) = n(q, p; \xi) + n(q, \bar{p}; \xi) n(\bar{q}, p; \xi) \tag{6}$$

for $p, q \in P(\eta)$.

Remark 3.2. If $n(q, \bar{p}; \xi) = 0$ then the closure of $W^u(q; \xi)$ does not intersect the closure of the connecting orbit from $\bar{q}$ to $\bar{p}$. Hence $W^u(q; \eta) = W^u(q; \xi)$ for every vector field η which agrees with ξ outside of a sufficiently small neighbourhood of the connecting orbit from $\bar{q}$ to $\bar{p}$. In this case the formula (6) holds trivially. A similar argument deals with the case $n(\bar{q}, p; \xi) = 0$.

Remark 3.3. If $n(\bar{q}, \bar{p}; \xi) = \nu(\bar{q}, \bar{p}; \xi) = 1$ then the algebraic numbers of connecting orbits of η are given by

$$\nu(q, p; \eta) = \nu(q, p; \xi) - \nu(q, \bar{p}; \xi) \nu(\bar{q}, p; \xi). \tag{7}$$

This follows from a refinement of the proof of Theorem 3.1 which we shall not discuss in this paper. Using equation (7) one can use standard arguments (see [6]) to construct a chain homotopy equivalence from the Morse complex of ξ to the Morse complex of η. This argument gives rise to an alternative proof of the fact that the Morse homology is independent of the Morse–Floer vector field ξ used to define it. Namely, in a generic one-parameter family of Morse–Floer vector fields the boundary operator changes only through cancellation of critical points of index difference one.

Proof of Theorem A. By transversality, Y admits a Morse–Smale vector field ξ. For $q \in P_1(\xi)$ and $p \in P_0(\xi)$ we have $n(q,p) = 0,1,2$ and $\nu(q,p) = 0$ if $n(q,p) = 0,2$. Hence by Theorem 2.1 there must be a pair with $n(q,p) = 1$ if $P_0(\xi)$ has more than one element. Then by Theorem 3.1 we may find another Morse–Smale vector field η with $P_0(\eta)$ of smaller size than $P_0(\xi)$. The same argument works to reduce $P_3(\xi)$. □

Proof of Theorem C (iii) $\Longrightarrow$ (ii). The proof uses the Cancellation Lemma only under the hypothesis $n(q,\bar{p};\xi) = n(\bar{q},p;\xi) = 0$ (see Remark 3.2). In this case Theorem 3.1 says that we can modify a numerically reduced HMS structure so as to produce another numerically reduced HMS structure of genus one less. The result now follows by induction. □

4 Isotopy

Lemma 4.1 (Isotopy Lemma). *Let (Y_0, Y_1, ξ) be an HMS structure on Y with $\Sigma := Y_0 \cap Y_1$ and trace*

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g, \qquad \beta = \beta_1 \cup \cdots \cup \beta_g.$$

Suppose that $f : \Sigma \to \Sigma$ is a diffeomorphism isotopic to the identity such that $f(\beta)$ is transverse to α. Then there is an HMS structure (Y_0, Y_1, ξ') on Y with trace

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g, \qquad f(\beta) = f(\beta_1) \cup \cdots \cup f(\beta_g).$$

Proof. Use the graph of the isotopy to modify the flow. □

Lemma 4.1 does not suffice to prove (iv) $\Longrightarrow$ (iii) in Theorem C. If the HMS structure is geometrically reduced but not algebraically reduced there is a pair of indices (i_0, j_0) and a diffeomorphism f isotopic to the identity with

$$\delta_{i_0,j_0} = \mathrm{geo}\,(\alpha_{i_0}, \beta_{j_0}) = \mathrm{num}\,(\alpha_{i_0}, f(\beta_{j_0})) < \mathrm{num}\,(\alpha_{i_0}, \beta_{j_0});$$

This does not prove (iv) $\Longrightarrow$ (iii) because we do not know that

$$\mathrm{num}\,(\alpha_i, f(\beta_j)) \le \mathrm{num}\,(\alpha_i, \beta_j)$$

for all $i, j = 1, 2, \ldots, g$. We need to choose f more carefully. For this we require the following lemma which is proved as Lemma 3.1 on page 108 in [9]. The formulation here has additional hypotheses (which hold in our application) but our proof is the same as the proof in [9].

Lemma 4.2. *Let Σ be a closed oriented 2-manifold and $\alpha, \beta \subset \Sigma$ be two noncontractible tranverse embedded loops. Assume that*

$$\mathrm{geo}\,(\alpha, \beta) < \mathrm{num}\,(\alpha, \beta).$$

Then there exists a smooth orientation preserving embedding $u : \mathbb{D} \to \Sigma$ of the half disk

$$\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| \le 1,\, \mathrm{Im}\, z \ge 0\}$$

such that

$$u(\mathbb{D} \cap \mathbb{R}) \subset \alpha, \qquad u(\mathbb{D} \cap S^1) \subset \beta.$$

A subset L of an oriented 2-manifold Σ is called a 2-**gon** if it is the image of an orientation preserving embedding $u : \mathbb{D} \to \Sigma$. The points $u(-1)$ and $u(1)$ are called the **corner points** of L, respectively, and the arcs $u(\mathbb{D} \cap \mathbb{R})$ and $u(\mathbb{D} \cap S^1)$ are called the **boundary arcs** of L, respectively.

Lemma 4.3. *Let $A, B \subset \mathbb{R}^2$ be embedded arcs intersecting only in their endpoints x and y. Let U denote the bounded component of $\mathbb{R}^2 \setminus (A \cup B)$. Then the following are equivalent.*

(i) *The closure L of U is a 2-gon.*

(ii) *The interior angles of U at the two corners are less than π.*

Proof. That (i) implies (ii) is obvious. To prove the converse, construct the diffeomorphism $u : \mathbb{D} \to L$ near the corners, extend it to a collar neighbourhood of the boundary, and, by Morse theory, extend it to all of $\mathbb{D}$. □

Lemma 4.4. *Let Σ, α, and β be as in Lemma 4.2. Let $\pi : \tilde{\Sigma} \to \Sigma$ be a covering. Call two intersection points $x, y \in \alpha \cap \beta$ π-equivalent if there exist lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β, respectively, and points $\tilde{x}, \tilde{y} \in \tilde{\alpha} \cap \tilde{\beta}$ such that $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$. If $\mathrm{num}\,(\alpha, \beta) > \mathrm{geo}\,(\alpha, \beta)$ then there exists a pair of distinct, but equivalent, intersection points.*

Proof. Let $[0, 1] \times S^1 \to \Sigma : (t, \theta) \mapsto b(t, \theta) = b_t(\theta)$ be an isotopy such that $b_0(S^1) = \beta$, b and b_1 are transverse to α, and $\mathrm{num}\,(\alpha, b_1(S^1)) = \mathrm{geo}\,(\alpha, \beta)$. Since $\mathrm{num}\,(\alpha, b_0(S^1)) > \mathrm{num}\,(\alpha, b_1(S^1))$ there must be a component of the 1-manifold $b^{-1}(\alpha)$ with both endpoints in $\{0\} \times S^1$. The images of these endpoints under b_0 are distinct intersection points of α and β. By covering space theory, they are equivalent □

Proof of Lemma 4.2. Let $\pi : \mathbb{R}^2 = \tilde{\Sigma} \to \Sigma$ be the universal cover. A 2-gon $\tilde{L} \subset \tilde{\Sigma}$ is called *admissable* if

$$\partial\tilde{L} \subset \pi^{-1}(\alpha) \cup \pi^{-1}(\beta).$$

It follows that one of the boundary arcs is contained in $\pi^{-1}(\alpha)$ and the other in $\pi^{-1}(\beta)$. The set $\mathcal{L}$ of admissable 2-gons is partially ordered by inclusion.

By Lemma 4.4, there exists a pair of distinct, but π-equivalent, intersection points of α and β. Hence there exist lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β, respectively, and intersection points $\tilde{x}, \tilde{y} \in \tilde{\alpha} \cap \tilde{\beta}$ such that $\pi(\tilde{x}) \neq \pi(\tilde{y})$. Changing $\tilde{y}$, if necessary, we may assume that the arc $\tilde{B} \subset \tilde{\beta}$ from $\tilde{x}$ to $\tilde{y}$ lies on one side of $\tilde{\alpha}$. Let $\tilde{A}$ be the arc in $\tilde{\alpha}$ from $\tilde{x}$ to $\tilde{y}$. Then, by Lemma 4.3, $\tilde{A}$ and $\tilde{B}$ bound an admissable 2-gon. Hence $\mathcal{L} \neq \emptyset$, and hence $\mathcal{L}$ contains a minimal element $\tilde{L}$. Every such minimal 2-gon satisfies

$$\mathrm{int}(\tilde{L}) \cap \pi^{-1}(\alpha) = \mathrm{int}(\tilde{L}) \cap \pi^{-1}(\beta) = \emptyset.$$

This is because no component of $\pi^{-1}(\alpha)$ or $\pi^{-1}(\beta)$ can lie entirely inside a bounded open set; hence any such component which intersects the interior would have to exit and therefore cut off a smaller admissable 2-gon.

Let $\tilde{L}$ be a minimal admissable 2-gon with corner points $\tilde{x}, \tilde{y} \in \pi^{-1}(\alpha) \cap \pi^{-1}(\beta)$ and boundary arcs $\tilde{A} \subset \pi^{-1}(\alpha)$ and $\tilde{B} \subset \pi^{-1}(\beta)$. It remains to show that $\pi|_{\tilde{L}} : \tilde{L} \to \Sigma$ is injective. To see this, let $g : \tilde{\Sigma} \to \tilde{\Sigma}$ be a deck transformation other than the identity. Then

$$g(\mathrm{int}(\tilde{L})) \cap \mathrm{int}(\tilde{L}) = \emptyset.$$

Otherwise, $g(\mathrm{int}(\tilde{L})) = \mathrm{int}(\tilde{L})$, so $g(\tilde{L}) = \mathrm{cl}(\tilde{L})$ and hence g has a fixed point, a contradiction. Moreover, $g(\tilde{x}) \neq \tilde{y}$ and $g(\tilde{y}) \neq \tilde{x}$ because g is orientation preserving and the intersection numbers of $\tilde{A}$ and $\tilde{B}$ at $\tilde{x}$ and $\tilde{y}$ are opposite. It follows that $g(\tilde{x}) \notin \tilde{A}$ and $g(\tilde{y}) \notin \tilde{A}$ and hence

$$g(\tilde{A}) \cap \tilde{A} = \emptyset = g(\tilde{B}) \cap \tilde{B}.$$

Thus $g(\tilde{L}) \cap \tilde{L} = \emptyset$ for every nontrivial deck transformation g and so $\pi|_{\tilde{L}}$ is injective as claimed. □

Proof of Theorem C (iv) $\Longrightarrow$ (iii). Let (Y_0, Y_1, ξ) be a geometrically reduced HMS structure on Y with $\Sigma := Y_0 \cap Y_1$ and trace

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g, \qquad \beta = \beta_1 \cup \cdots \cup \beta_g.$$

Assume that this HMS structure is not numerically reduced so that

$$\mathrm{geo}\,(\alpha_{i_0}, \beta_{j_0}) < \mathrm{num}\,(\alpha_{i_0}, \beta_{j_0})$$

for some pair (i_0, j_0). As in Definition A.6, the homology classes of $\alpha_1, \ldots, \beta_g$ form an integral basis of $H_1(\Sigma; \mathbb{Z})$. In particular, α_{i_0} and β_{j_0} are not contractible.

By Lemma 4.2, there is a smooth embedding $u : \mathbb{D} \to \Sigma$ with $u(\mathbb{D} \cap \mathbb{R}) \subset \alpha_{i_0}$ and $u(\mathbb{D} \cap S^1) \subset \beta_{j_0}$. We shall use this embedding to deform β_{j_0} by an ambient isotopy to remove the two intersections between α_{i_0} and β_{j_0} at the corners of the 2-gon. Under this isotopy none of the numbers $\mathrm{num}\,(\alpha_i, \beta_j)$ increases. More precisely, extend u to an embedding (still denoted by u) of the open set

$$\mathbb{D}_\varepsilon := \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > -\varepsilon, |z| < 1 + \varepsilon\}$$

for $\varepsilon > 0$ sufficiently small such that

$$u(\mathbb{D}_\varepsilon) \cap \beta_{j_0} = u(\mathbb{D}_\varepsilon \cap S^1), \qquad u(\mathbb{D}_\varepsilon) \cap \alpha_{i_0} = u(\mathbb{D}_\varepsilon \cap \mathbb{R}),$$

and

$$u(\{z \in \mathbb{D}_\varepsilon \,|\, |z| > 1\}) \cap \beta_j = \emptyset, \quad u(\{z \in \mathbb{D}_\varepsilon \,|\, \mathrm{Re}\, z < 0\}) \cap \alpha_i = \emptyset,$$

for all i and j. Choose an isotopy $\psi_t : \Sigma \to \Sigma$ supported in $u(\mathbb{D}_\varepsilon)$ such that $\psi_0 = \mathrm{id}$ and

$$\psi_1(\mathbb{D}) \subset \{z \in \mathbb{D}_\varepsilon \,|\, \mathrm{Im}\, z < 0\}$$

(see Figure 2).

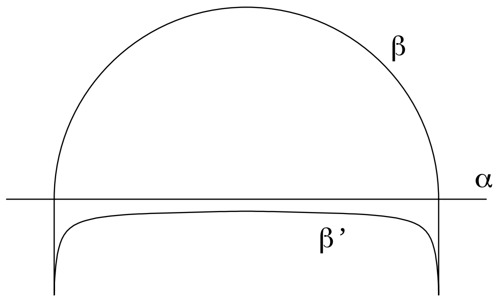

Figure 2: Removing a 2-gon.

Now replace β_j by

$$\beta_j' := \psi_1(\beta_j).$$

Then

$$\text{num}\left(\alpha_{i_0}, \beta'_{j_0}\right) \le \text{num}\left(\alpha_{i_0}, \beta_{j_0}\right) - 2$$

and $\text{num}\left(\alpha_i, \beta'_j\right) \le \text{num}\left(\alpha_i, \beta_j\right)$ for all i and j. □

5 Floer homology

The Lagrangian Floer homology $\text{HF}(\alpha, \beta)$ for pairs of loops α, β on a Riemann surface Σ can be viewed as an infinite dimensional analogue of the Morse homology described in Section 2: the manifold M is replaced by the space of paths in Σ from α to β and the "critical points" are the constant paths, i.e. the points of $\alpha \cap \beta$. To define an operator as in equation (2) we require a notion of "connecting orbit of index (difference) one" and a way of counting these connecting orbits. In the present (two dimensional case) the connecting orbits can be defined combinatorially, following Vin de Silva [1], rather than analytically as in Floer's original approach [6]. In this section we descibe this combinatorial definition; the proof of Theorem 5.2 is given in [2].

Definition 5.1. Throughout α and β are transverse embedded loops in a closed orientable 2-manifold Σ. A **smooth** (α, β)**-lune** is an equivalence class of orientation preserving immersions $u : \mathbb{D} \to \Sigma$ such that

$$u(\mathbb{D} \cap \mathbb{R}) \subset \alpha, \qquad u(\mathbb{D} \cap S^1) \subset \beta.$$

The equivalence relation is defined by

$$[u] = [u']$$

iff there is an orientation preserving diffeomorphism $\varphi : \mathbb{D} \to \mathbb{D}$ such that

$$\varphi(-1) = -1, \qquad \varphi(1) = 1, \qquad u' = u \circ \varphi.$$

That u is an immersion means that u is smooth and du is injective in all of $\mathbb{D}$, even at the corners ± 1. The **endpoints** of the lune are intersection points

$$u(-1), u(1) \in \alpha \cap \beta$$

of α and β. When $x = u(-1)$ and $y = u(1)$ we say the lune is **from** x **to** y. The image of an embedded lune is a 2-gon as defined in Section 4. These notions are clearly independent of the choice of the immersion u representing the smooth lune.

In the remainder of this section Σ is a closed connected oriented 2-manifold of positive genus. For each pair α and β of transverse noncontractible embedded loops which are not istopic to each other we define

$$\mathrm{CF}(\alpha,\beta) = \bigoplus_{x\in\alpha\cap\beta} \mathbb{Z}_2 x,$$

and a linear map $\partial : \mathrm{CF}(\alpha,\beta) \to \mathrm{CF}(\alpha,\beta)$, called the **Floer boundary operator**, by

$$\partial x = \sum_y \big(n(x,y) \bmod 2\big)\, y, \tag{8}$$

where $n(x,y)$ denotes the number of smooth (α,β)-lunes from x to y.

Theorem 5.2. (a) *$n(x,y)\in\{0,1\}$ for all $x,y\in\alpha\cap\beta$.*
(b) *The operator $\partial : \mathrm{CF}(\alpha,\beta)\to\mathrm{CF}(\alpha,\beta)$ is a chain complex, i.e. $\partial\circ\partial = 0$. Its homology will be denoted by*

$$\mathrm{HF}(\alpha,\beta) := \ker\partial/\mathrm{im}\partial$$

and is called the **Floer homology** *of the pair (α,β).*
(c) *If $\alpha',\beta'\subset\Sigma$ are transverse embedded loops such that α is isotopic to α' and β is isotopic to β' then*

$$\mathrm{HF}(\alpha,\beta)\cong\mathrm{HF}(\alpha',\beta').$$

(d) *If the Floer boundary operator $\partial : \mathrm{CF}(\alpha,\beta)\to\mathrm{CF}(\alpha,\beta)$ is nonzero then there exists an embedded (α,β)-lune.*

Corollary 5.3.

$$\dim\mathrm{CF}(\alpha,\beta) = \mathrm{num}\,(\alpha,\beta),\qquad \dim\mathrm{HF}(\alpha,\beta) = \mathrm{geo}\,(\alpha,\beta).$$

Proof. The first statement follows from the definition of $\mathrm{CF}(\alpha,\beta)$. To prove the second statement choose β' isotopic to β so that β' is transverse to α and $\mathrm{num}\,(\alpha,\beta') = \mathrm{geo}\,(\alpha,\beta)$. Then the boundary operator of the pair (α,β') is zero: if not then, by (d), there is an embedded (α,β')-lune and hence, as in the proof of (iv)$\Longrightarrow$(iii) in Theorem C, there exists an embedded loop β'' isotopic to β' with $\mathrm{num}\,(\alpha,\beta'') < \mathrm{num}\,(\alpha,\beta')$, a contradiction. Hence, by (c),

$$\dim\mathrm{HF}(\alpha,\beta) = \dim\mathrm{HF}(\alpha,\beta') = \mathrm{num}\,(\alpha,\beta') = \mathrm{geo}\,(\alpha,\beta),$$

as claimed. $\square$

Remark 5.4. It is easy to show that if there is a lune, then there is an embedded lune. Hence Corollary 5.3 provides another proof of Lemma 4.2.

Remark 5.5. The proof of (a) in [2] is based on a combinatorial characterization of smooth lunes which shows that a smooth lune is uniquely determined by its boundary arcs. In contrast, there exists an immersion of the circle into the plane with transverse self intersections which extends in nonequivalent ways to an immersion of the disc (see [15]).

Remark 5.6. If $x, y \in \alpha \cap \beta$ such that $n(x, y) = 1$ then α and β have opposite intersection numbers at x and y. In particular, $n(x, x) = 0$. This shows that the Floer homology groups have a mod 2 grading. Namely, orient α and β and write

$$\mathrm{CF}(\alpha, \beta) = \mathrm{CF}_0(\alpha, \beta) \oplus \mathrm{CF}_1(\alpha, \beta),$$

where $\mathrm{CF}_i(\alpha, \beta)$ is generated by those intersection points where the intersection number is $(-1)^i$. Then the Floer boundary operator interchanges CF_0 and CF_1.

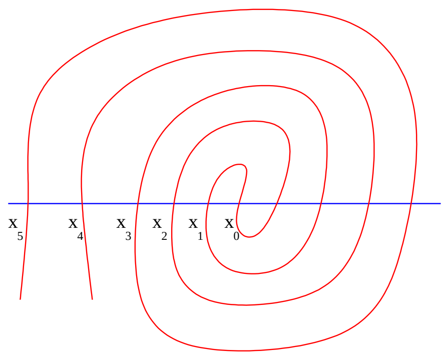

Figure 3: Lunes from x_i to x_{i-1}.

Remark 5.7. Define a relation $x \preceq y$ on $\alpha \cap \beta$ by $x \preceq y$ if and only if there is a sequence $x = x_0, \ldots, x_k = y$ in $\alpha \cap \beta$ with $k \geq 0$ such that $n(x_i, x_{i-1}) \neq 0$ for each $i > 0$ (see Figure 3). Then $x \preceq y$ is a partial order. To prove this let $\Omega_{\alpha,\beta}$ denote the space of all smooth curves $z : [0, 1] \to \Sigma$ satisfying the boundary conditions $z(0) \in \alpha$ and $z(1) \in \beta$. The intersection points of $\alpha \cap \beta$ are the constant curves in $\Omega_{\alpha,\beta}$. Each component of the space $\Omega_{\alpha,\beta}$ is simply connected and hence, for every area form on Σ, the symplectic action is single valued. It is monotone with repect the relation $x \preceq y$. This means that there is a function $\mathcal{A} : \Omega_{\alpha,\beta} \to \mathbb{R}$ (the "action functional") such that for any curve $\{z_s\}_{0\le s\le 1}$ in $\Omega_{\alpha,\beta}$ the number $\mathcal{A}(z_0) - \mathcal{A}(z_1)$ is the area of the region swept out. This function satisfies $\mathcal{A}(x_{i-1}) < \mathcal{A}(x_i)$ for every $i > 0$ and hence, by induction,

$$x \preceq y \implies \mathcal{A}(x) \le \mathcal{A}(y).$$

The relation $x \preceq y$ is called the **Smale order** determined by (α, β).

Remark 5.8. The proof of (c) in [2] establishes the following analog of the Cancellation Lemma 3.1. Suppose that the isotopy is elementary in the sense that

$$\alpha' \cap \beta' = \alpha \cap \beta \setminus \{x, y\}$$

and the change in the number of intersection points occurs just at one parameter value and in the manner suggested by Figure 2. Then for $x', y' \in \alpha' \cap \beta'$, we have

$$x' \preceq' y' \qquad \iff \qquad \begin{cases} \text{either} & x' \preceq y', \\ \text{or} & x' \preceq y \text{ and } x \preceq y', \end{cases}$$

and

$$n'(x', y') = n(x', y') + n(x', y)n(x, y'),$$

where $n(x', y')$ denotes the number of (α, β)-lunes from x' to y', $n'(x', y')$ denotes the number of (α', β')-lunes from x' to y', and $x' \preceq' y'$ is the Smale order of (α', β').

Remark 5.9. In Floer's original theory the number $n(x, y)$ is defined as the (oriented) number of index one holomorphic strips from x to y. To relate this definition to the above one must show the following.

(i) The linearized Fredholm operator is surjective for every holomorphic strip. It follows that the number of index one holomorphic strips from x to y (modulo time shift) is finite and is indepedent of the complex structure on Σ.

(ii) The Fredholm index is one if and only if the holomorphic strip factors through an (α, β)-lune.

(iii) The correspondence between index one holomorphic strips and lunes in (ii) is bijective.

These assertions are specific to the two dimensional case. The proof of (ii) follows from the asymptotic analysis established in [17] and an identity relating the Maslov index to the number of branch points. This approach leads to another proof of Theorem 5.2. Details will appear elsewhere.

Remark 5.10. Without the assumptions that α and β are not contractible and not isotopic to each other it can happen that $\partial \circ \partial \neq 0$ (so there is no homology theory) or that $\partial \circ \partial = 0$ but the resulting homology theory is not invariant under isotopy. As an example of the former take $\alpha := S^1 \times \{\mathrm{pt}\} \subset \mathbb{T}^2$ and β to be a small circle intersecting α transversely in two points. As an example of the latter take $\alpha := S^1 \times \{\mathrm{pt}\} \subset \mathbb{T}^2$ and β to be the graph of a smooth map $f : S^1 \to S^1$. If α and β don't intersect then $\mathrm{HF}(\alpha, \beta) = 0$ and if they do then $\mathrm{HF}(\alpha, \beta) \cong H_*(S^1)$. Floer's original theory is invariant only under Hamiltonian isotopy and only applies to the case where α and β are not contractible and are Hamiltonian isotopic to each other. In their recent work [8] Fukaya, Oh, Ohta, and Ono develop an obstruction theory for Floer homology of Lagrangian intersections which allows the construction of Floer homology groups in some cases where $\partial \circ \partial \neq 0$.

A Handlebodies

Definition A.1. Let Y_0 be a compact connected oriented 3-manifold with boundary ∂Y_0. A **handlebody structure** on Y_0 is a Morse–Smale vector field that points in on the boundary, has a single rest point p_0 of index zero, rest points $p_1, \dots, p_g$ of index one, and no other rest point. The **trace** of the handlebody structure is the 1-submanifold

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_g$$

of ∂Y_0 defined by

$$\alpha_i = W^s(p_i) \cap \partial Y_0;$$

we say that α is **the trace** of (Y_0, ξ) and **a trace** of Y_0. It follows that ∂Y_0 is a closed connected oriented 2-manifold of genus g (see Corollary 2.3). A **handlebody** is a compact connected oriented 3-manifold Y_0 which admits a handlebody structure.

Remark A.2. A compact connected oriented 3-manifold Y_0 is a handlebody if and only if it admits a Morse–Smale vector field ξ which points in on the boundary and has only rest points of index zero and one, i.e. excess rest points of index zero can be cancelled. Namely, if $\#P_0(\xi) > 1$ then, as $H_0(Y_0;\mathbb{Q}) = \mathbb{Q}$, there must exist a pair of rest points $p \in P_0(\xi)$ and $q \in P_1(\xi)$ with $n(q,p) = 1$. Use the Cancellation Lemma repeatedly to reduce $\#P_0(\xi)$.

Theorem A.3. *Two handlebodies whose boundaries have the same genus are diffeomorphic. More precisely, let Y_0 and $\tilde{Y}_0$ be handlebodies with traces α and $\tilde{\alpha}$, respectively. Suppose that ∂Y_0 and $\partial \tilde{Y}_0$ have the same genus g. Then there exists a diffeomorphism $\varphi : \partial Y_0 \to \partial \tilde{Y}_0$ such that $\varphi(\alpha) = \tilde{\alpha}$ and any such φ extends to a diffeomorphism $\psi_0 : Y_0 \to \tilde{Y}_0$.*

Definition A.4. Let Σ be a closed connected oriented 2-manifold and $\alpha \subset \Sigma$ be a compact 1-submanifold, i.e.

$$\alpha = \alpha_1 \cup \cdots \cup \alpha_n$$

where $\alpha_1, \ldots, \alpha_n$ are disjoint embedded loops. (We do not assume here that n is the genus of Σ.) There is a compact oriented 2-manifold Σ_α (with boundary) and a smooth map $f_\alpha : \Sigma_\alpha \to \Sigma$ such that f_α has an invertible derivative at every point, f_α restricts to a diffeomorphism from the interior of Σ_α to $\Sigma \setminus \alpha$, and f_α restricts to a trivial orientation preserving double covering $\partial\Sigma_\alpha \to \alpha$. The manifold Σ_α is unique in the sense that if $f'_\alpha : \Sigma'_\alpha \to \Sigma$ is another such map, then there is a unique diffeomorphism $\varphi : \Sigma'_\alpha \to \Sigma_\alpha$ with $f_\alpha \circ \varphi = f'_\alpha$. We say that Σ_α results by **cutting** Σ **along** α (see Figure 4).

Definition A.5. Let (Y_0, ξ) be a handlebody structure with rest points $p_0, \ldots, p_g$ and let

$$A := \bigcup_{i=1}^{g} A_i, \qquad A_i := W^s(p_i).$$

There is compact oriented 3-manifold Y_A with corners and a smooth map

$$F_A : Y_A \to Y_0$$

such that F_A has an invertible derivative at every point, F_A restricts to a diffeomorphism from $Y_A \setminus {F_A}^{-1}(A)$ to $Y \setminus A$, and F_A restricts to a trivial orientation preserving double covering from ${F_A}^{-1}(A)$ to A. The manifold Y_A is unique in the sense that if $F'_A : Y'_A \to Y_0$ is another such map then there is a unique diffeomorphism $\Phi : Y'_A \to Y_A$ such that $F_A \circ \Phi = F'_A$.

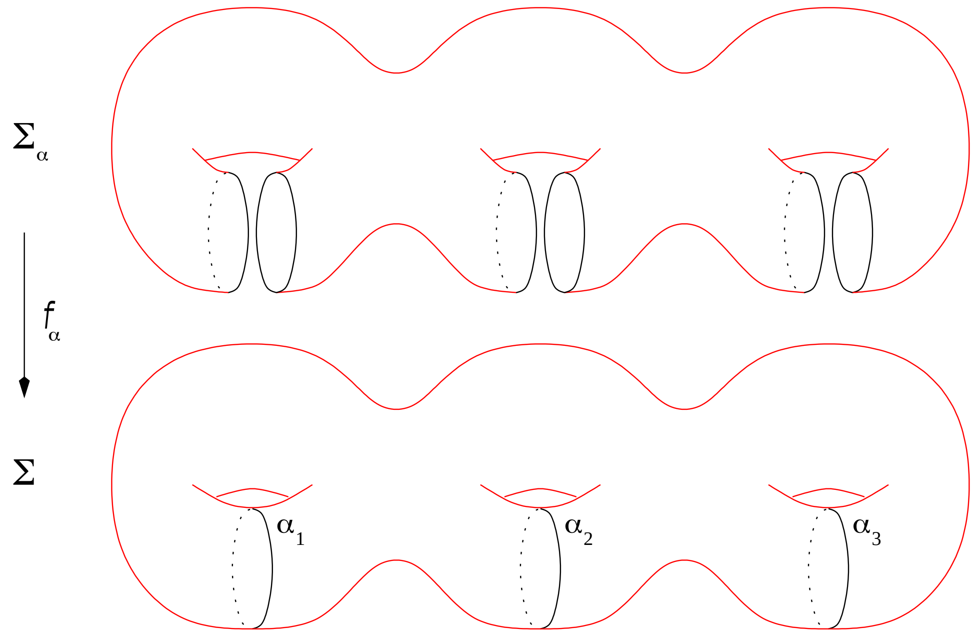

Figure 4: Cutting Σ along α.

We say that Y_A is the 3-manifold with corners that results by **cutting** Y_0 **along** A. As a topological manifold Y_A is homeomorphic to the 3-ball. As a smooth manifold Y_A is diffeomorphic to a 3-ball with $2g$ spherical caps sliced off. To prove this, cut out tubular neighborhood of the discs A_i to obtain a submanifold with corners $Y' \subset Y_0 \setminus A$ that is diffeomorphic to Y_A. Choose a smooth submanifold with boundary $Y'' \subset Y_0 \setminus A$ that contains Y'. The orbits of ξ define a diffeomorphism from the 3-ball centered at p_0 to Y''. The preimage of Y' under this diffeomorphism is the required 3-ball with the caps sliced off. The vector field ξ on Y_0 pulls back under F_A to a vector field ξ_A on Y_A which is tangent to the $2g$ disks that form the preimage of A and otherwise points in on the boundary. It has a critical point of index one on each of these disks and a unique critical point of index zero in the interior.

Definition A.6. Let (Σ, α) be as in Definition A.4 and assume that $n = g$, i.e. the number of components of α is the genus of Σ. Another embedded 1-submanifold β is said to be **dual** to α if it also has g components, say

$$\beta = \beta_1 \cup \cdots \cup \beta_g$$

where $\beta_1, \ldots, \beta_g$ are disjoint embedded loops, and (for a suitable choice of orientations)

$$\alpha_i \cdot \beta_j = \delta_{ij}$$

for all i and j. It follows that the homology classes of $\alpha_1, \ldots, \beta_g$ form an integral basis of $H_1(\Sigma; \mathbb{Z})$. To see this express $\alpha_1, \ldots, \beta_g$ in terms of a symplectic integral basis of $H_1(\Sigma; \mathbb{Z})$. Since

$$\alpha_i \cdot \beta_j = \delta_{ij}, \qquad \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$$

for all i and j, the matrix of coefficients is symplectic and hence unimodular.

Theorem A.7. *Let (Σ, α) be as in Definition A.4 and assume $n = g$. Then the following are equivalent.*

(i) *There exists a handlebody Y_0 and a diffeomorphism $\iota : \Sigma \to \partial Y_0$ such that $\iota(\alpha)$ is a trace of Y_0.*

(ii) *The manifold Σ_α has genus zero.*

(iii) *The open set $\Sigma \setminus \alpha$ is connected.*

(iv) *The homology classes of $\alpha_1, \ldots, \alpha_g$ are linearly independent in $H_1(\Sigma; \mathbb{Q})$.*

(v) *The homology classes of $\alpha_1, \ldots, \alpha_g$ extend to a free basis of $H_1(\Sigma; \mathbb{Z})$.*

(vi) *There exists a 1-manifold β dual to α.*

If these equivalent conditions are satisfied then α is called a **trace in** Σ.

Proof. The pattern of proof is (ii) $\Longrightarrow$ (vi) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) and (ii)$\Longrightarrow$(i)$\Longrightarrow$(iii). Let $f_\alpha : \Sigma_\alpha \to \Sigma$ be as in Definition A.4 and write

$$\partial \Sigma_\alpha = \alpha_1' \cup \cdots \cup \alpha_g'', \qquad f_\alpha(\alpha_i') = f_\alpha(\alpha_i'') = \alpha_i.$$

We prove that (ii) implies (vi). Since Σ_α has genus zero it embeds in a 2-sphere, i.e.

$$\Sigma_\alpha = S^2 \setminus \bigcup_{i=1}^{g} (C_i' \cup C_i''), \qquad \alpha_i' = \partial \bar{C}_i', \quad \alpha_i'' = \partial \bar{C}_i'', \tag{9}$$

where $\bar{C}_i'$ and $\bar{C}_i''$ are embedded closed disks with interiors C_i' and C_i'' respectively. Connect α_j' to α_j'' with an arc $b_j \subset \Sigma_\alpha$; do this in such a way that the

b_j are disjoint, that b_j intersects $\partial\Sigma_\alpha$ only in the endpoints, that f_α maps the two endpoints of b_j to the same point in Σ, and that, for $j = 1, \ldots, g$, the image $\beta_j := f_\alpha(b_j)$ is a smooth submanifold of Σ transverse to α_j. Then $\beta = \beta_1 \cup \cdots \cup \beta_g$ is dual to α as required.

We prove (vi) implies (v) implies (iv). Let $\beta = \beta_1 \cup \cdots \cup \beta_g$ be dual to α. As in Definition A.6, the homology classes of $\alpha_1, \ldots, \beta_g$ form an integral basis of $H_1(\Sigma; \mathbb{Z})$. This proves (v). That (v) implies (iv) is trivial.

We prove that (iv) implies (iii). Assume (iii) fails. Let C be the closure of a connected component of $\Sigma \setminus \alpha$. Then $C \neq \Sigma$. Hence the boundary of C is homologous to zero and gives rise to a nontrivial relation among the homology classes of the α_i. Hence (iv) fails.

We prove that (iii) implies (ii). Assume $\Sigma \setminus \alpha$ is connected. Then Σ_α is also connected. Each identification $f(\alpha_i') = f(\alpha_i'')$ contributes one to the genus so Σ_α must have genus zero. Also note that (ii) implies (iii) is obvious.

We prove that (ii) implies (i) implies (iii). To prove that (ii) implies (i) reverse the construction of Definition A.5. Now assume (i) and let ξ be a handlebody structure on Y_0 with trace $\iota(\alpha)$. Choose points $x, y \in \Sigma \setminus \alpha$. The forward orbits of $\iota(x)$ and $\iota(y)$ get close to p_0 and hence may be connected by an arc in Y_0 which, by transversality, misses $\bigcup_{i=1}^g W^u(p_i)$. Now let this arc flow backwards out of Y_0. The exit points trace out an arc in $\partial Y_0 \setminus \iota(\alpha)$ connecting $\iota(x)$ to $\iota(y)$. □

Proof of Theorem A.3. The existence of φ follows from item (ii) in Theorem A.7. Namely, let $\Sigma := \partial Y_0$ and $\tilde{\Sigma} := \partial\tilde{Y}_0$ and choose a diffeomorphism $\Sigma_\alpha \to \tilde{\Sigma}_{\tilde{\alpha}}$ which maps pairs of equivalent boundary circles to pairs of equivalent boundary circles. Then isotop so that the diffeomorphism descends to the quotient. Given φ, exend it to a diffeomorphism $U \to \tilde{U}$, where U is a neighborhood of $\partial Y_0 \cup A$, $\tilde{U}$ is a neighborhood of $\partial\tilde{Y}_0 \cup \tilde{A}$, $A = \bigcup_{i=1}^g W^s(p_i) \subset Y_0$, and $\tilde{A} = \bigcup_{i=1}^g W^s(\tilde{p}_i) \subset \tilde{Y}_0$. The argument in Definition A.5 shows that these neighborhoods can be chosen such that the complements $Y_0 \setminus U$ and $\tilde{Y}_0 \setminus \tilde{U}$ are smooth submanifolds with boundary, each diffeomorphic to the 3-ball. Since the group of orientation preserving diffeomorphisms of the 2-sphere is connected φ extends to a diffeomorphism $\psi_0 : Y_0 \to \tilde{Y}_0$ as required. □

Definition A.8. Let Σ be a closed oriented 2-manifold. Two traces $\alpha, \beta \subset \Sigma$ are called **equivalent** if there exists a handlebody Y_0 and a diffeomorphism $\iota : \Sigma \to \partial Y_0$ such that both $\iota(\alpha)$ and $\iota(\beta)$ are traces of Y_0. By Theorem A.3, two traces $\alpha, \beta \subset \Sigma$ are equivalent if and only if, for every handlebody Y_0

and every diffeomorphism $\iota : \Sigma \to \partial Y_0$, we have that $\iota(\alpha)$ is a trace of Y_0 if and only if $\iota(\beta)$ is a trace of Y_0. Hence equivalence of traces is an equivalence relation.

Remark A.9. Equivalent traces generate the same Lagrangian subspace of $H_1(\Sigma; \mathbb{Z})$, namely the kernel of the map $\iota_* : H_1(\Sigma; \mathbb{Z}) \to H_1(Y_0; \mathbb{Z})$.

Remark A.10. Isotopic traces are equivalent (Lemma 4.1). Proof: Use the isotopy to modify the flow on a collar neighborhood of the boundary.

Remark A.11. Let Σ be a closed connected oriented 2-manifold, Y_0 be a handlebody, and $\iota : \Sigma \to Y_0$ be a diffeomorphism. Let $\mathrm{Diff}(\Sigma, \iota) \subset \mathrm{Diff}(\Sigma)$ denote the subgroup of all diffeomorphisms $\varphi : \Sigma \to \Sigma$ that extend to Y_0 in the sense that there exists a diffeomorphism $\psi_0 : Y_0 \to Y_0$ such that

$$\psi_0 \circ \iota = \iota \circ \varphi.$$

Let $\alpha \subset \Sigma$ be a trace such that $\iota(\alpha)$ is a trace of Y_0. Then, by Theorem A.3, a trace $\beta \subset \Sigma$ is equivalent to α if and only if there exists a diffeomorphism $\varphi \in \mathrm{Diff}(\Sigma, \iota)$ such that $\varphi(\alpha) = \beta$.

Example A.12. A trace on a surface of genus one is a noncontractible embedded loop. Two such loops are equivalent as traces if and only if they are isotopic. For an example of two nonisotopic, but equivalent, traces on a surface of genus two see Example D.1 below.

An HMS structure (Y_0, Y_1, ξ) on a closed connected oriented 3-manifold Y determines two handlebody structures $(Y_0, \xi|_{Y_0})$ and $(Y_1, -\xi|_{Y_1})$. Recall that the trace of (Y_0, Y_1, ξ) is the pair of 1-submanifolds $\alpha, \beta \subset Y_0 \cap Y_1$ where α is the trace of $(Y_0, \xi|_{Y_0})$ and β is the trace of $(Y_1, -\xi|_{Y_1})$. The operation

$$(Y, Y_0, Y_1, \xi) \mapsto (Y_0 \cap Y_1, \alpha, \beta)$$

is bijective in the sense of the following two propositions.

Proposition A.13. *Let (α, β) be a transverse pair of traces in a closed connected oriented 2-manifold Σ. Then there is an HMS structure (Y_0, Y_1, ξ) on a closed connected oriented 3-manifold Y and a diffeomorphism $\iota : \Sigma \to Y_0 \cap Y_1$ such that $\iota(\alpha)$ is the trace of $(Y_0, \xi|_{Y_0})$ and $\iota(\beta)$ is the trace of $(Y_1, -\xi|_{Y_1})$.*

Proof. By definition of trace, there exist handlebody structures (Y_0, ξ_0) and (Y_1, ξ_1) and diffeomorphisms $\iota_0 : \Sigma \to \partial Y_0$ and $\iota_1 : \Sigma \to \partial Y_1$ such that $\iota_0(\alpha)$ is the trace of (Y_0, ξ_0) and $\iota_1(\beta)$ is the trace of (Y_1, ξ_1). Suppose, without loss of generality, that ι_0 is orientation preserving and ι_1 is orientation reversing. Then the flow of ξ_0 and the embedding ι_0 define an orientation preserving diffeomorphism from $(-\varepsilon, 0] \times \Sigma$ to a tubular neighborhood $U_0 \subset Y_0$ of the boundary. Likewise the flow of ξ_1 and the embedding ι_1 define an orientation preserving diffeomorphism from $[0, \varepsilon) \times \Sigma$ to a tubular neighborhood $U_1 \subset Y_1$ of the boundary. There is a unique manifold structure on the union

$$Y := Y_0 \cup_\Sigma Y_1$$

such that the map $(-\varepsilon, \varepsilon) \times \Sigma \to U_0 \cup_\Sigma U_1$ is a diffeomorphism and the inclusions of Y_0 and of Y_1 are embeddings. □

Proposition A.14. *Let Σ be a closed connected oriented 2-manifold and suppose that $(Y, Y_0, Y_1, \xi, \iota, \alpha, \beta)$ and $(\tilde{Y}, \tilde{Y}_0, \tilde{Y}_1, \tilde{\xi}, \tilde{\iota}, \tilde{\alpha}, \tilde{\beta})$ are as in the statement of Proposition A.13. Then the following are equivalent.*

(i) *α is equivalent to $\tilde{\alpha}$ and β is equivalent to $\tilde{\beta}$.*

(ii) *There exists a diffeomorphism $\psi : Y \to \tilde{Y}$ such that $\psi \circ \iota = \tilde{\iota}$.*

Proof. If (ii) holds then the pullback $\psi^*(\tilde{Y}_0, \tilde{Y}_1, \tilde{\xi})$ is an HMS structure on Y with traces $\iota(\tilde{\alpha})$ and $\iota(\tilde{\beta})$. Hence (i) holds. Conversely, assume (i). Since α is equivalent to $\tilde{\alpha}$ we have that $\iota(\tilde{\alpha})$ is a trace of Y_0. Hence the diffeomorphism $\varphi := \tilde{\iota} \circ \iota^{-1} : \partial Y_0 \to \partial \tilde{Y}_0$ maps a trace of Y_0 to a trace of $\tilde{Y}_0$. Hence, by Theorem A.3, there exists a diffeomorphism $\psi_0 : Y_0 \to \tilde{Y}_0$ such that $\psi_0 \circ \iota = \tilde{\iota}$. Similarly for Y_1 and this proves the proposition. □

B Diffeomorphisms of the two sphere

Let $\mathrm{Diff}_+(S^2)$ denote the group of orientation preserving diffeomorphisms of the two sphere and let $\mathrm{PSL}_2(\mathbb{C})$ denote the subgroup of fractional linear transformations.

Theorem B.1 (Smale). *The subgroup $\mathrm{PSL}_2(\mathbb{C})$ is a deformation retract of $\mathrm{Diff}_+(S^2)$.*

Proof. Our proof is inspired by [3] but uses a different PDE. Let $\omega \in \Omega^2(S^2)$ be the standard volume form and denote by $\mathcal{J}_+(S^2)$ the space of complex structures on S^2 that are compatible with ω. We prove that there is a fibration

$$\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{C}) & \longrightarrow & \mathrm{Diff}_+(S^2) \\ & & \downarrow \\ & & \mathcal{J}_+(S^2) \end{array}.$$

The projection $\mathrm{Diff}_+(S^2) \to \mathcal{J}_+(S^2)$ is given by $\psi \mapsto \psi^* J_0$, where $J_0 \in \mathcal{J}_+(S^2)$ denotes the standard complex structure. We prove that it has the path lifting property. Let $[0,1] \to \mathcal{J}_+(S^2) : t \mapsto J_t$ be a smooth path in $\mathcal{J}_+(S^2)$. We must prove that there is an isotopy $t \mapsto \psi_t$ of S^2 such that

$$\psi_t{}^* J_t = J_0. \tag{10}$$

Suppose that the isotopy ψ_t is generated by a smooth family of vector fields $X_t \in \mathrm{Vect}(S^2)$ via

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \qquad \psi_0 = \mathrm{id}.$$

Then (10) is equivalent to

$$\mathcal{L}_{X_t} J_t + \dot{J}_t = 0, \tag{11}$$

where $\dot{J}_t := d/dt J_t \in C^\infty(\mathrm{End}(TS^2))$. Since $J_t{}^2 = -1\!\!1$ we have

$$\dot{J}_t J_t + J_t \dot{J}_t = 0.$$

This means that $\dot{J}_t : TS^2 \to TS^2$ is complex anti-linear with respect to J_t. Hence we can think of $\dot{J}_t$ as a $(0,1)$-form on S^2 with values in the complex line bundle

$$E_t := (TS^2, J_t).$$

The vector field X_t is a section of this line bundle. Let

$$\bar{\partial}_{J_t} : C^\infty(E_t) \to \Omega^{0,1}(E_t)$$

denote the Cauchy-Riemann operator associated to the the metric $\omega(\cdot, J_t \cdot)$ on S^2 and the Levi-Civita connection of this metric on E_t. Thus

$$\bar{\partial}_{J_t} X = \frac{1}{2}\left(\nabla X + J_t \circ \nabla X \circ J_t\right).$$

Now, for every vector field $Y \in \mathrm{Vect}(S^2)$, we have

$$\begin{aligned}(\mathcal{L}_{X_t} J_t)Y &= \mathcal{L}_{X_t}(J_t Y) - J_t \mathcal{L}_{X_t} Y \\ &= [J_t Y, X_t] - J_t [Y, X_t] \\ &= \nabla_{X_t}(J_t Y) - \nabla_{J_t Y} X_t - J_t \nabla_{X_t} Y + J_t \nabla_Y X_t \\ &= J_t \nabla_Y X_t - \nabla_{J_t Y} X_t \\ &= 2 J_t (\bar{\partial}_{J_t} X_t)(Y).\end{aligned}$$

The penultimate equality uses the fact that J_t is integrable and so $\nabla J_t = 0$. Hence equation (11) can be expressed in the form

$$\bar{\partial}_{J_t} X_t = -\frac{1}{2} J_t \dot{J}_t. \tag{12}$$

Now the line bundle E_t has Chern number $c_1(E_t) = 2$ and hence, by the Riemann-Roch theorem, the Cauchy-Riemann operator $\bar{\partial}_{J_t}$ has real Fredholm index six and is surjective for every t. Denote by

$$\bar{\partial}_{J_t}^* : \Omega^{0,1}(E_t) \to C^\infty(E_t)$$

the formal L^2-adjoint operator of $\bar{\partial}_{J_t}$. By elliptic regularity, the formula

$$X_t := -\frac{1}{2} \bar{\partial}_{J_t}^* \left(\bar{\partial}_{J_t} \bar{\partial}_{J_t}^* \right)^{-1} \left(J_t \dot{J}_t \right)$$

defines a smooth family of vector fields on S^2 and this family obviously satisfies (12). Hence the isotopy ψ_t generated by X_t satisfies (10).

Thus we have prove that the projection $\mathrm{Diff}_+(S^2) \to \mathcal{J}_+(S^2)$ is a fibration and, in particular, is surjective. Since the space $\mathcal{J}^+(S^2)$ is contractible (it is the space of sections of a bundle over S^2 with contractible fibres) it follows that $\mathrm{Diff}_+(S^2)$ is homotopy equivalent to $\mathrm{PSL}(2, \mathbb{C})$. □

Corollary B.2. *The group* $\mathrm{Diff}_+(S^2)$ *is connected.*

We emphasize that our proof of Theorem B.1 uses the integrability of almost complex structures in dimension two, the Riemann-Roch theorem, and elliptic regularity.

Proof of Theorem C (ii)$\Longrightarrow$(i). Choose an HMS structure (Y, Y_0, Y_1, ξ) so that $\Sigma := Y_0 \cap Y_1$ is a 2-sphere. Then ξ is a Morse–Smale vector field on Y with exactly two critical points, p_0 of index zero and q_0 of index three, in particular,

$$W^s(p_0, \xi) = Y \setminus \{q_0\}, \qquad W^u(q_0, \xi) = Y \setminus \{p_0\}.$$

Let φ denote the flow of ξ. After modifying ξ near p_0 and q_0 we may assume that there are diffeomorphisms $u : \mathbb{R}^3 \to Y \setminus \{q_0\}$ and $v : \mathbb{R}^3 \to Y \setminus \{p_0\}$ so that

$$u(e^{-t}x) = \varphi^t(u(x)), \qquad v(e^t y) = \varphi^t(v(y)).$$

After a further modification of ξ away from p_0 and q_0 we may assume that $u(S^2) = v(S^2)$. It follows that

$$|u^{-1}(v(x))| = |x|^{-1}$$

for $x \in \mathbb{R}^3 \setminus \{0\}$. We may assume that $u^{-1} \circ v|S^2$ is orientation preserving. (If not replace v by v composed with with a reflection.) As $\mathrm{Diff}_+(S^2)$ is connected (see Corollary B.2), there is a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$|f(x)| = |x|$$

and

$$f(x) = \begin{cases} x, & \text{for } |x| \leq 1, \\ |x|^2 u^{-1}(v(x)), & \text{for } |x| \geq 2. \end{cases}$$

Define $g : \mathbb{R}^3 \to Y$ by

$$g(x) := u(f(x)).$$

Let $y \in \mathbb{R}^3$ with $|y| \leq 1/2$ and denote $T := -\ln |y|^2$ so that $e^T = |y|^{-2}$. Then

$$g(|y|^{-2}y) = u(e^T u^{-1}(v(e^T y)) = \varphi^{-T}(u(u^{-1}(v(e^T y)))) = v(y)$$

This shows that $g \circ \sigma$ extends to a diffeomorphism $S^3 \to Y$, where $\sigma : S^2 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ is the stereographic projection. □

C Proof of the Cancellation Lemma

Before giving the proof we give some preliminary definitions and lemmas. Let $(P, \preceq)$ be a finite poset. An ordered pair $(p, q) \in P \times P$ is called **adjacent** if $p \preceq q$, $p \neq q$, and

$$p \preceq r \preceq q \qquad \Longrightarrow \qquad r \in \{p, q\}.$$

Fix an adjacent pair $(\bar{p},\bar{q}) \in P \times P$ and consider the relation $\preceq'$ on $P' = P \setminus \{\bar{p},\bar{q}\}$ defined by

$$p \preceq' q \qquad \Longleftrightarrow \qquad \begin{cases} \text{either} & p \preceq q, \\ \text{or} & \bar{p} \preceq q \text{ and } p \preceq \bar{q}. \end{cases} \tag{13}$$

Lemma C.1. $(P', \preceq')$ *is a poset.*

Proof. We prove that the relation $\preceq'$ is transitive. Let $p, q, r \in P'$ such that $p \preceq' q$ and $q \preceq' r$. There are four cases. If $p \preceq q$ and $q \preceq r$ then $p \preceq r$ and hence $p \preceq' r$. The second case is $p \not\preceq q$ and $q \preceq r$. In this case $\bar{p} \preceq q \preceq r$ and $p \preceq \bar{q}$, and hence $p \preceq' r$. The third case is $p \preceq q$ and $q \not\preceq r$, and the argument is as in the second case. The fourth case is $p \not\preceq q$ and $q \not\preceq r$. In this case it follows that $p \preceq \bar{q}$ and $\bar{p} \preceq r$, and hence $p \preceq' r$.

Next we prove that the relation $\preceq'$ is anti-symmetric. Hence assume that $p, q \in P'$ such that $p \preceq' q$ and $q \preceq' p$. We claim that $p \preceq q$ and $q \preceq p$. Assume otherwise that $p \not\preceq q$. Then $\bar{p} \preceq q$ and $p \preceq \bar{q}$. Since $q \preceq' p$, it follows that $\bar{p} \preceq p \preceq \bar{q}$ and $\bar{p} \preceq q \preceq \bar{q}$, and hence $\{p, q\} \subset \{\bar{p}, \bar{q}\}$, a contradiction. Thus we have shown that $p \preceq q$. Similarly, $q \preceq p$ and hence $p = q$. □

Two Morse–Floer vector fields are called **MF-equivalent** if there exists a diffeomorphism $\psi : M \to M$ such that

$$P_k(\xi') = P_k(\psi^*\xi)$$

for $k = 0, \ldots, m$ and

$$\psi(p) \preceq_{\xi'} \psi(q) \iff p \preceq_\xi q, \qquad n(\psi(q), \psi(p); \xi') = n(q, p; \xi)$$

for all $p, q \in P(\xi)$. Morse–Floer vector fields are stable in the sense that equivalence classes are open in the C^1-topology. Moreover, Morse–Floer vector fields are stable under certain C^0-perturbations as we explain next. Let ξ be a Morse–Floer vector field on M and $r \in P(\xi)$. A compact set $U \subset M$ is called **ξ-unrevisited** if no ξ-orbit exits U and then returns to U. A neighborhood U_r of $r \in P(\xi)$ is called **ξ-admissible** iff it is ξ-unrevisited and satisfies the following conditions.

(i) If $r \not\preceq_\xi q$ then $\overline{W}^u(q) \cap U_r = \emptyset$.

(ii) If $p \not\preceq_\xi r$ then $\overline{W}^s(p) \cap U_r = \emptyset$.

(iii) If $p, q \in P(\xi)\setminus\{r\}$ such that $p \preceq_\xi q$ then there is a transverse connecting orbit from q to p that misses U_r.

Call a vector field ξ' on M an **admissible perturbation of** ξ (supported near $r \in P(\xi)$) iff it satisfies the following conditions.

(iv) $\xi = \xi'$ outside of some ξ-admissible neighborhood U_r of r.

(v) $U_r \cap P(\xi) = U_r \cap P(\xi') = \{r\}$, r is a hyperbolic rest point of ξ', and

$$W^u(r;\xi') \cap U_r = W^u(r;\xi) \cap U_r, \quad W^s(r;\xi') \cap U_r = W^s(r;\xi) \cap U_r.$$

(vi) Every ξ'-orbit that stays in U_r in positive time lies in $W^s(r;\xi')$, and every ξ'-orbit that stays in U_r in negative time lies in $W^u(r;\xi')$.

Lemma C.2. *Let ξ be a Morse–Floer vector field. Then every admissible perturbation of ξ is a Morse–Floer vector field and is MF-equivalent to ξ.*

Proof. Let ξ' be a vector field on M that satisfies (iv), (v), and (vi). From (vi) and the unrevisitedness of U_r we conclude that

$$M = \bigcup_{p\in P(\xi)} W^s(p;\xi') = \bigcup_{p\in P(\xi)} W^u(p;\xi'). \tag{14}$$

We prove the assertion in three Steps.

Step 1. *For all $p, q \in P(\xi)$,*

$$W^u(q;\xi) \cap W^s(p;\xi) = \emptyset \qquad \Longrightarrow \qquad W^u(q;\xi') \cap W^s(p;\xi') = \emptyset.$$

To see this note that if $W^u(q;\xi)\cap W^s(p;\xi) = \emptyset$ then $p \not\preceq_\xi q$ and hence either $r \not\preceq_\xi q$ or $p \not\preceq_\xi r$. Assume, without loss of generality, that $r \not\preceq_\xi q$. Write $P(\xi)$ as a disjoint union of a lower set Q containing q and an upper set R containing p:

$$Q := \{q' \in P(\xi) \,|\, q' \preceq_\xi q\}, \qquad R := P(\xi) \setminus Q.$$

Then the set

$$A = \bigcup_{q'\in Q} W^u(q';\xi)$$

is an attractor for ξ and, in particular, is a compact subset of M. By the assumption that $r \not\preceq_\xi q$ we have that $r \in R$. Hence $r \not\preceq_\xi q'$ for every $q' \in Q$ and hence, by (i),

$$U_r \cap A = \emptyset.$$

Now A is a ξ-attractor and ξ and ξ' agree near A so A is a ξ'-attractor. Since $p \notin A$ and $q \in A$ it follows that there is no ξ'-orbit connecting q to p. Hence $W^u(q;\xi') \cap W^s(p;\xi') = \emptyset$ as claimed. This proves Step 1. It follows from Step 1 and (14) that ξ' is a Morse–Floer vector field.

Step 2. *$p \preceq_\xi q$ if and only if $p \preceq_{\xi'} q$ for all $p, q \in P(\xi)$.*

It follows from Step 1 that $p \preceq_{\xi'} q$ implies $p \preceq_\xi q$. The converse follows immediately from condition (iii) on U_r.

Step 3. *$n(q,p;\xi') = n(q,p;\xi)$ for all $p, q \in P(\xi)$.*

Suppose that q and p have index difference one (otherwise the assertion is obvious). Assume first that $q, p \in P(\xi) \setminus \{r\}$. Then either $p \not\preceq_\xi r$ or $r \not\preceq_\xi q$ and, by (i) and (ii),

$$W^s(p;\xi) \cap U_r = \emptyset \qquad \text{or} \qquad W^u(q;\xi) \cap U_r = \emptyset.$$

Hence no ξ-orbit from q to p passes through U_r, hence the ξ-orbits from q to p survive as ξ'-orbits, and hence $n(q,p;\xi) \le n(q,p;\xi')$. Suppose, by contradiction, that $n(q,p;\xi) < n(q,p;\xi')$. Then there exists a ξ'-orbit from q to p that passes through U_r. Hence

$$W^u(q;\xi) \cap U_r \neq \emptyset \qquad \text{and} \qquad W^s(p;\xi) \cap U_r \neq \emptyset,$$

contradicting the above. This proves Step 3 in the case $p, q \in P(\xi) \setminus \{r\}$. Now it follows from (iv) and (v) that $W^s(r;\xi') = W^s(r;\xi)$ and $W^u(r;\xi') = W^u(r;\xi)$. Hence $n(q,r;\xi') = n(q,r;\xi)$ and $n(r,p;\xi') = n(r,p;\xi)$ for all $p, q \in P(\xi)$. This proves the lemma. □

Proposition C.3 (Normal form). *Let ξ be a Morse–Floer vector field and $\bar{p} \in P_k(\xi)$, $\bar{q} \in P_{k+1}(\xi)$. Let Γ denote the closure of a connecting orbit from $\bar{q}$ to $\bar{p}$. Then, for every neighborhood U of Γ, there exist a compact neighborhood $N \subset U$ of Γ, a diffeomorphism*

$$f : D^k \times D^{m-k-1} \times [-1, 2] \to N,$$

a Morse–Floer vector field $\tilde{\xi}$ *on* M*, and a smooth function* $w : [-1,2] \to \mathbb{R}$ *such that* $f^*\tilde{\xi}$ *has the form*

$$f^*\tilde{\xi}(x,y,z) = (x, -y, w(z)),$$

$\tilde{\xi}$ *agrees with* ξ *outside of* U*,* $\tilde{\xi}$ *is MF-equivalent to* ξ*, and*

$$w^{-1}(0) = \{0,1\}, \qquad w'(0) = -1, \qquad w'(1) = 1, \tag{15}$$

Proof. The proof consists five steps.

Step 1. *There is an admissible perturbation* ξ' *of* ξ *supported near* $\bar{p}$ *and coordinates* $x_1 \in \mathbb{R}^k$*,* $y_1 \in \mathbb{R}^{m-k-1}$*,* $z_1 \in \mathbb{R}$ *near* $\bar{p}$ *such that* ξ' *is given by the equations* $\dot{x}_1 = x_1$*,* $\dot{y}_1 = -y_1$*,* $\dot{z}_1 = -z_1$*. Moreover, the connecting orbit* Γ' *is defined by* $x_1 = 0$*,* $y_1 = 0$*,* $z_1 \geq 0$ *and the unstable manifold* $W^u(\bar{q};\xi')$ *is defined by* $y_1 = 0$ *and* $z_1 > 0$*.*

Let B^u be a small ball in the unstable subspace $T_{\bar{p}}W^u(\bar{p})$ and B^s be a small ball in the stable subspace $T_{\bar{p}}W^s(\bar{p})$. Use the exponential map to identify the product $B^u \times B^s$ with a neighborhood of $\bar{p}$. We may assume that the balls B^u and B^s and the exponential map have been chosen such that $B^u \times \{0\}$ is a subset of $W^u(\bar{p};\xi)$, $\{0\} \times B^s$ is a subset of $W^s(\bar{p};\xi)$, ξ points in on $B^u \times \partial B^s$, and ξ points out on $\partial B^u \times B^s$. Let ζ be a product vector field on $B^u \times B^s$ which is the radial vector field on the first factor and the negative of the radial vector field on the second. Consider the vector field $\xi' := \beta\xi + (1-\beta)\zeta$, where $\beta : B^u \times B^s \to [0,1]$ is a cutoff function which is zero near $\bar{p}$ and identically one near the boundary of $B^u \times B^s$. If $\beta^{-1}((0,1])$ is contained in a sufficiently small neighborhood of the boundary of $B^u \times B^s$ then ξ' satisfies the requirements of Lemma C.2. In any linear coordinates x in B^u and (y,z) in B^s the vector field ξ' has the required form. Choose these coordinates such that Γ has the required form. By transversality and invariance under the flow the unstable manifold $W^u(\bar{q};\xi')$ has an equation of the form

$$y = zg(zx).$$

Make the further change of variables

$$(x_1, y_1, z_1) = (x, y - zg(zx), z).$$

to achieve the required equation for $W^u(\bar{q};\xi')$.

Step 2. *There is an admissible perturbation ξ'' of ξ' supported near $\bar{q}$ and coordinates $x_2 \in \mathbb{R}^k$, $y_2 \in \mathbb{R}^{m-k-1}$, $z_2 \in \mathbb{R}$ near $\bar{q}$ such that ξ'' is given by the equations $\dot{x}_2 = x_2$, $\dot{y}_2 = -y_2$, $\dot{z}_2 = z_2$. Moreover, the connecting orbit Γ'' is defined by $x_2 = 0$, $y_2 = 0$, $z_2 \leq 0$, and the stable manifold $W^s(\bar{p};\xi)$ is defined by $x_2 = 0$ and $z_2 < 0$.*

The proof is the same as for Step 1. Henceforth we drop the primes and assume that ξ satisfies the conclusions of Steps 1 and 2.

Let $L \subset M$ be the smooth (non-compact) one dimensional submanifold determined by the conditions that it contains Γ in its interior and $L \setminus \{p, q\}$ consists of three orbits of φ. Thus L intersects each of the coordinate systems of Steps 1 and 2 in the z-axis. Choose a diffeomorphism $\ell : \mathbb{R} \to L$ such that $\ell(0) = \bar{p}$ and $\ell(1) = \bar{q}$ and the pull back vector field

$$w(z) := \ell^* \xi(z)$$

satisfies the following strengthened form of (15):

$$w(z) = -z \text{ for } z \approx 0, \qquad w(z) = z - 1 \text{ for } z \approx 1.$$

Step 3. *The restriction $T_L M$ of the tangent bundle TM of M to the curve L admits a smooth direct sum decomposition*

$$T_L M = E^u \oplus E^s \oplus TL$$

which is invariant in the sense that

$$d\varphi^t(z)E^u_z = E^u_{\varphi^t(z)}, \qquad d\varphi^t(z)E^s_z = E^s_{\varphi^t(z)}, \qquad d\varphi^t(z)T_z L = T_{\varphi^t(z)}L$$

for $z \in L$ and satisfies

$$T_z W^u(q) = E^u_z \oplus T_z L \qquad \text{for} \quad z \in W^u(q) \cap L,$$

$$T_z W^s(q) = E^s_z \oplus T_z L \qquad \text{for} \quad z \in W^s(p) \cap L,$$

and

$$T_p W^u(p) = E^u_p, \qquad T_q W^s(q) = E^s_q.$$

To construct E^u choose E^u_z to agree with the x_2-subspace for z near $\bar{q}$ in the coordinates of Step 2. Extend by invariance. Then by transversality E^u_z has the form

$$E^u_z = \mathrm{graph}\,(\Lambda(z)) \times \{0\}, \qquad \Lambda(z) = {z_1}^2\Lambda_0$$

for $z = (0, 0, z_1) \in \Gamma$ near $\bar{p}$ where $\Lambda_0 : \mathbb{R}^k \to \mathbb{R}^{m-k-1}$ is linear. Extend to $L \setminus \Gamma$ using the same formula (and invariance). The construction of E^s is similar.

Step 4. *There exists a diffeomorphism* $f : B^k \times B^{m-k-1} \times [-1, 2] \to N$, *where* B^n *denotes the closed unit ball in* $\mathbb{R}^n$ *and* $N \subset U$ *is a neighborhood of* Γ *in* M, *such that* $f^*\xi'$ *has the form*

$$\xi(x, y, z) = \big(\hat{a}(z)x, \hat{b}(z)y, w(z)\big) + O(\|x\|^2 + \|y\|^2). \tag{16}$$

where $\hat{a}(z) \in \mathbb{R}^{k\times k}$ *and* $\hat{b}(z) \in \mathbb{R}^{(m-k-1)\times(m-k-1)}$ *satisfy*

$$\hat{a}(z) = \mathbb{1}, \qquad \hat{b}(z) = -\mathbb{1} \tag{17}$$

for z *near* 0 *and* 1.

Choose a Riemannian metric on T_LM which agrees with the standard metric in the (x_1, y_1, z_1) coordinates near $\bar{p}$ and agrees with the standard metric in the (x_2, y_2, z_2) coordinates near $\bar{q}$. The coordinate systems of Steps 1 and 2 determine trivializations of $E^u \oplus E^s$ near $\bar{p}$ and $\bar{q}$; extend to a vector bundle trivialization

$$\mathbb{R}^k \times \mathbb{R}^{m-k-1} \times \mathbb{R} \to E^u \oplus E^s$$

that covers the diffeomorphism $\ell : \mathbb{R} \to L$. (It may be necessary to reverse the sign of one component of x_1 and/or of one component of y_1 to match orientations.) Compose with the exponential map to obtain a tubular neigborhood

$$\mathbb{R}^k \times \mathbb{R}^{m-k-1} \times \mathbb{R} \to M$$

of L. This gives coordinates (x, y, z) on a neigbourhood of Γ. We use the same letters φ and ξ to represent the flow and vector field in these coordinates. Thus $\bar{p} = (0, 0, 0)$ and $\bar{q} = (0, 0, 1)$ and Γ is the set of points $(0, 0, z)$ where $0 \le z \le 1$. Since $L = \{0\} \times \{0\} \times \mathbb{R}$ is invariant by φ the restriction has the form

$$\varphi^t(0, 0, z) = (0, 0, \psi^t(z)). \tag{18}$$

By invariance of the splitting, $d\varphi^t(0,0,z)$ has the form

$$d\varphi^t(0,0,z) = a^t(z) \oplus b^t(z) \oplus c^t(z) \tag{19}$$

where $a^t(z) \in \mathrm{GL}_k(\mathbb{R})$, $b^t(z) \in \mathrm{GL}_{m-k-1}(\mathbb{R})$, and $c^t(z) > 0$. Differentiate (18) and (19) to deduce that the vector field ξ satisfies

$$\xi(0,0,z) = \big(0,0,w(z)\big), \qquad d\xi(0,0,z) = \hat{a}(z) \oplus \hat{b}(z) \oplus w'(z)$$

where

$$\hat{a}(z) = \frac{\partial}{\partial t} a^t(z)\bigg|_{t=0}, \qquad \hat{b}(z) = \frac{\partial}{\partial t} b^t(z)\bigg|_{t=0}.$$

The construction of E^u and E^s shows that $\hat{a}$ and $\hat{b}$ satisfy (17). Use Taylor's formula in (x,y) to obtain (16). Rescale (x,y) so that the coordinates are defined for $\|x\|$, $\|y\| \le 1$ and $-1 \le z \le 2$.

Step 5. *We prove Proposition C.3.*

Construct a C^1-perturbation $\tilde{\xi}$ of ξ near Γ using a cutoff function to eliminate the higher order terms in (16), Then $\tilde{\xi}$ is a Morse–Floer vector field with $P(\tilde{\xi}) = P(\xi)$, $\tilde{\xi}$ is MF-equivalent to ξ, and $f^*\tilde{\xi}$ has the form

$$f^*\tilde{\xi}(x,y,z) = \big(\hat{a}(z)x, \hat{b}(z)y, w(z)\big)$$

in some neighborhood of Γ. Consider the coordinate change

$$(x,y,z) = g(\tilde{x},\tilde{y},z) := (\Phi(z)\tilde{x}, \Psi(z)\tilde{y}, z),$$

where

$$w(z)\partial_z\Phi(z) = \Phi(z)(\mathbb{1} - \hat{a}(z)), \qquad \Phi(0) = \mathbb{1},$$

and

$$w(z)\partial_z\Psi(z) = \Psi(z)(\mathbb{1} + \hat{b}(z)), \qquad \Psi(0) = \mathbb{1}.$$

By (17), we have $\partial_z\Phi(z) = 0$ and $\partial_z\Psi(z) = 0$ for z near 0 and 1. It follows that

$$g^*f^*\tilde{\xi}(x,y,z) = \big(x, -y, w(z)\big).$$

Now read $f \circ g$ for f, rescale in (x,y) and restrict the domain as required. This proves Proposition C.3. □

Proof of the Cancellation Lemma 3.1. Choose a finite set S of ξ-orbits which contains all the orbits between pairs of index difference one and also at least one orbit of transverse intersection of $W^u(p,\xi) \cap E^s(q,\xi)$ for any pair of rest points $p, q \in P(\xi) \setminus \{\bar{p}, \bar{q}\}$ with $p \preceq_\xi q$. Let $U_{\bar{p}}$ be a ξ-admissible neighborhood of $\bar{p}$ and $U_{\bar{q}}$ be a ξ-admissible neighborhood of $\bar{q}$. Suppose, without loss of generality, that the neighborhood U of Γ is so small that

$$U \cap S = \emptyset \tag{20}$$

and every ξ-orbit that enters U must first pass through $U_{\bar{p}} \cup U_{\bar{q}}$ and every ξ-orbit that leaves U passes afterwards through $U_{\bar{p}} \cup U_{\bar{q}}$. Thus, for every ξ-orbit $\gamma : \mathbb{R}^\pm \to M$,

$$\gamma(\mathbb{R}^\pm) \cap U = \{\gamma(0)\} \qquad \Longrightarrow \qquad \gamma(\mathbb{R}^\pm) \cap (U_{\bar{p}} \cup U_{\bar{q}}) \neq \emptyset, \tag{21}$$

where $\mathbb{R}^+ := [0,\infty)$ and $\mathbb{R}^- := (-\infty, 0]$. By Proposition C.3 we may assume without loss of generality that ξ is in normal form near Γ, i.e. there exist $N \subset U$, f, and w such that the conclusions of Proposition C.3 hold with $\tilde{\xi}$ replaced by ξ. Define the vector field η by $\eta = \xi$ on $M \setminus N$ and

$$f^*\eta(x,y,z) = (x, -y, v(x,y,z)),$$

where

$$v(x,y,z) = \beta(r)w(z) + (1-\beta(r))\Big(\rho(z)w(z) + (1-\rho(z))\varepsilon\Big).$$

Here $r = \sqrt{x^2+y^2}$, $\beta : [0,1] \to [0,1]$ satisfies

$$\beta(r) = \begin{cases} 0, & \text{if } r \leq 1/3, \\ 1, & \text{if } r \geq 2/3, \end{cases}$$

and $\rho : [-1,2] \to \mathbb{R}$ is chosen such that

$$\rho(z) = \begin{cases} 1, & \text{if } z \leq -2\varepsilon, \\ 0, & \text{if } -\varepsilon \leq z \leq 1+\varepsilon, \\ 1, & \text{if } z \geq 1+2\varepsilon. \end{cases}$$

By construction, the vector field η has only hyperbolic rest points,

$$P(\eta) = P(\xi) \setminus \{\bar{p}, \bar{q}\},$$

it agrees with ξ outside N (and hence with ξ outside U), and

$$M = \bigcup_{p \in P(\eta)} W^s(p;\eta) = \bigcup_{p \in P(\eta)} W^u(p;\eta). \tag{22}$$

We must show that for $p, q \in P(\eta)$ we have

(a) $p \preceq_\xi q \Longrightarrow W^s(p;\eta) \cap W^u(q;\eta) \neq \emptyset$;

(b) $p \preceq_\xi \bar{q}$ and $\bar{p} \preceq_\xi q \Longrightarrow W^s(p;\eta) \cap W^u(q;\eta) \neq \emptyset$;

(c) $W^s(p;\eta) \cap W^u(q;\eta) \neq \emptyset$ and $p \not\preceq_\xi q \Longrightarrow p \preceq_\xi \bar{q}$ and $\bar{p} \preceq_\xi q$.

By Lemma C.1, the right hand side of formula (5) defines a partial order on $P(\eta)$ whenever $\bar{p}, \bar{q}$ are an adjacent pair in $P(\xi)$. Hence it follows from (c) and (22), that η is gradient-like and that the Smale order $\preceq_\eta$ is given by (5).

Assume that $p \preceq_\xi q$. Then there is a ξ-orbit from the set S which runs from q to p. By (20), the set U misses this orbit and $\eta - \xi$ is supported in U; hence this orbit is an η-orbit. Hence $W^s(p;\eta) \cap W^u(q;\eta) \neq \emptyset$. This proves (a). Next assume that $W^s(p;\eta) \cap W^u(q;\eta) \neq \emptyset$ and $p \not\preceq_\xi q$. Then there exists an η-orbit from q to p that passes through U. By (21), this orbit must pass through $U_{\bar{p}} \cup U_{\bar{q}}$ before entering U and must pass again through $U_{\bar{p}} \cup U_{\bar{q}}$ after leaving U. Since $U_{\bar{p}}$ and $U_{\bar{q}}$ are ξ-admissible it follows that there is a ξ-orbit from q to either $\bar{p}$ or $\bar{q}$ and another ξ-orbit from either $\bar{p}$ or $\bar{q}$ to p. Since $p \not\preceq_\xi q$ it follows that $p \preceq_\xi \bar{q}$ and $\bar{p} \preceq_\xi q$, as claimed. This proves (c). Assertion (b) follows from a gluing argument. Namely, if there exists a ξ-orbit from q to $\bar{p}$ then $W^u(q;\xi)$ intersects N in a slice along the x-plane near $z = 1$, provided that $\varepsilon > 0$ is chosen sufficiently small. Likewise, if there exists a ξ-orbit from $\bar{q}$ to p, then $W^s(p;\xi)$ intersects N in a slice along the y-plane near $z = 0$. The orbits of η connect these two transverse slices. Moreover, the resulting η-orbit from q to p is transverse. The same argument shows, in the case where $\bar{q}$ and q have the same index and $\bar{p}$ and p have the same index, that every pair of connecting orbits from q to $\bar{p}$ and from $\bar{q}$ to p gives rise to a transverse η-orbit from q to p. Hence η is a Morse–Floer vector field that satisfies (6). □

D An example

Example D.1. Francois Laudenbach and Denis Auroux showed us the following example from [5] of an algebraically reduced HMS structure on S^3

which is not geometrically reduced. Let $\Sigma = \partial Y_0 = \partial Y_1$ have genus two and let the embedded loops α_1, α_2, β_1, β_2 form a standard basis of $H_1(\Sigma)$. The embedded loop $\gamma \subset \Sigma$ is homologous to zero in Σ and contractible in both handlebodies Y_0 and Y_1 (see Figure 5). Hence the Dehn twist $\varphi : \Sigma \to \Sigma$ along

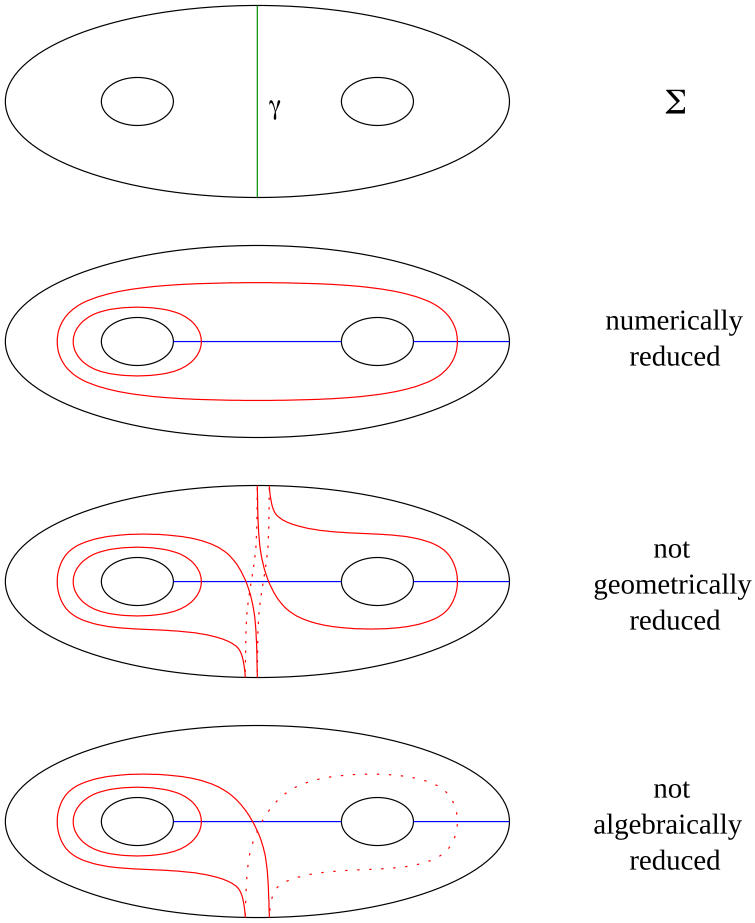

Figure 5: Three HMS structures

γ extends to a diffeomorphism of Y_1 and hence, by Remark A.11, the trace $\beta' := \varphi(\beta)$ is equivalent to β. Hence, by Proposition A.14, the pair (α, β') is a trace of the same Heegard splitting of S^3. It is algebraically reduced, but not geometrically reduced. Replacing φ by a diffeomorphism which rotates Σ by a half turn on one side of γ (i.e. a square root of φ) we obtain a trace (α, β'') of the same Heegard splitting, which is not algebraically reduced.

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