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Quasiinvariants of Coxeter groups and m-harmonic polynomials

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Abstract

The space of m-harmonic polynomials related to a Coxeter group G and a multiplicity function m on its root system is defined as the joint kernel of the properly gauged invariant integrals of the corresponding generalised quantum Calogero-Moser problem. The relation between this space and the ring of all quantum integrals of this system (which is isomorphic to the ring of corresponding quasiinvariants) is investigated.

1 Introduction

Let G be any Coxeter group, i.e. a finite group generated by reflections with respect to some hyperplanes in a Euclidean space V of dimension n. Let Σ be a set of the hyperplanes $\Pi_\alpha : (\alpha, x) = 0$ corresponding to all the reflections $s_\alpha \in G$ and let A be a set of the corresponding (arbitrarily chosen) normals α. Let us also consider a G-invariant $\mathbb{Z}_{\geq 0}$ -valued function m on Σ which will be called *multiplicity*. In other words to any hyperplane $\Pi_\alpha \in \Sigma$ we prescribe a nonnegative integer m_α such that if $\Pi_\alpha = g(\Pi_\beta)$, $g \in G$ then $m_\alpha = m_\beta$.

Let $S = S(V)$ be the ring of all polynomials on V, S^G be the subring of G-invariant polynomials. According to classical Chevalley result [1] S^G is freely generated by some homogeneous polynomials $\sigma_1, \dots, \sigma_n$.

The main object of our investigation is the following subring $Q_m = Q_m(\Sigma) \subset S(V)$. It consists of the polynomials q which are invariant up to order $2m_\alpha$ with respect to any reflection s_α:

$$q(s_\alpha(x)) = q(x) + o\left((\alpha, x)^{2m_\alpha}\right) \qquad (1)$$

near the hyperplane $(\alpha, x) = 0$ for any $\alpha \in A$. Equivalently, for any $\alpha \in A$ the normal derivatives $\partial_\alpha^s q = (\alpha, \frac{\partial}{\partial x})^s q$ must vanish on Π_α for $s = 1, 3, 5, \dots, 2m_\alpha - 1$:

$$\partial_\alpha^s q|_{\Pi_\alpha} = 0.$$

We will call these polynomials *m–quasiinvariants* of the Coxeter group G or simply *quasiinvariants.*

The rings Q_m have been introduced in the theory of quantum Calogero–Moser systems by O.Chalykh and one of the authors [2]. It has been shown [2, 3] that for any Coxeter group G and any integer-valued multiplicity function m there exists a homomorphism

$$\varphi_m : Q_m \to D_\Sigma(V),$$

where $D_\Sigma(V)$ is the ring of all differential operators in V with rational coefficients from the algebra generated by $(\alpha, x)^{-1}, \alpha \in A$ and constant functions (see the next section for the details). In particular, for $q = x^2$ (which is obviously invariant and therefore quasiinvariant) the corresponding operator $\varphi_m(q)$ is the generalised Calogero–Moser operator

$$L = \Delta - \sum_{\alpha \in A} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \tag{2}$$

first introduced by Olshanetsky and Perelomov [4]. It will be more convenient for us to use the gauge transformation $\mathcal{L} = \hat{g} L \hat{g}^{-1}$, where $\hat{g}$ is the operator of multiplication by $g = \prod (\alpha, x)^{m_\alpha}$, after which the operator L takes the form

$$\mathcal{L} = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha. \tag{3}$$

This gauge is natural from the point of view of the theory of symmetric spaces where such operators appear as the radial parts of the Laplace–Beltrami operators (see e.g. [5]). Let

$$\chi_m : Q_m \to D_\Sigma(V)$$

be the corresponding gauged version of the homomorphism φ_m:

$$\chi_m(q) = \hat{g} \varphi_m(q) \hat{g}^{-1}.$$

We should mention that for a generic (not necessary integer-valued) multiplicity function there exists an isomorphism (sometimes called as Harish-Chandra isomorphism, see [6, 7]) between S^G and the ring D_m^G of G-invariant quantum integrals of the Calogero-Moser problem (2):

$$\gamma_m : S^G \cong D_m^G,$$

where

$$D_m^G = \{\mathcal{D} \in D_\Sigma(V) : [\mathcal{L}, \mathcal{D}] = 0, g(\mathcal{D}) = \mathcal{D} \text{ for all } g \in G\}.$$

As a corollary we have usual integrability for (3) with n commuting quantum integrals $\mathcal{L}_1 = \mathcal{L}, \mathcal{L}_2, ..., \mathcal{L}_n$ corresponding to some basic invariants $\sigma_1 = x^2, \sigma_2, ..., \sigma_n$.

An important novelty of [2] was the possibility of the extension of γ_m to a much bigger ring Q_m in the case when all m_α are integer, which implies the algebraic integrability of the corresponding quantum Calogero-Moser problem (see [3] for details).

The primary goal of the present paper is to explain that all the information about the additional quantum integrals of Calogero-Moser problem is actually contained in the joint kernel of its standard invariant integrals $\mathcal{L}_1, ..., \mathcal{L}_n$. More precisely, let us define the space H_m as the solutions of the following system

$$\begin{cases} \mathcal{L}_1 \psi = 0 \\ \dots\dots\dots \\ \mathcal{L}_n \psi = 0 \end{cases} \tag{4}$$

We will show that all the solutions of (4) are polynomial and that all these polynomials are m-quasiinvariant. We will call these polynomials *m-harmonic.* In the case $m = 0$ we have the space of the usual harmonic polynomials related to Coxeter group G (see e.g [5]).

The following linear map π_m from Q_m to H_m will play the central role in our considerations. Let us introduce m-discriminant

$$w_m = \prod_{\alpha \in A} (\alpha, x)^{2m_\alpha + 1}$$

which obviously is m-quasiinvariant. We will show that w_m is also m–harmonic. Let q be any quasiinvariant, $\mathcal{L}_q = \chi_m(q)$ be the corresponding differential operator. The map π_m is defined by the formula

$$\pi_m(q) = \mathcal{L}_q(w_m).$$

Since $\mathcal{L}_q$ commutes with $\mathcal{L}_i$ it preserves the space H_m, so $\pi_m(q) \in H_m$. The question now is what is the kernel of π_m. It is easy to show that the kernel of π_m contains the ideal $I_m \subset Q_m$ generated by the invariants $\sigma_1, \dots, \sigma_n$.

We conjecture that the following statements are true for any Coxeter group G and multiplicity function m.

Conjecture 1 *Kernel of π_m coincides with the ideal I_m.*

Consider the restriction of the map π_m onto the subspace $H_m \subset Q_m$.

Conjecture 2 *The linear map*

$$\pi_m|_{H_m} : H_m \to H_m$$

is an isomorphism.

As a corollary we have the following isomorphism

$$Q_m/I_m \cong H_m,$$

and the fact that Q_m is generated by H_m over S^G.

Conjecture 3 *The ring Q_m is a free module over S^G generated by any basis in H_m.*

In algebraic terminology this implies that Q_m is a Cohen-Macauley ring (see e.g. [8]). We believe that Q_m is actually a Gorenstein ring. This follows from the following Conjecture 2^* which can be considered as a stronger version of Conjecture 2.

Let us introduce the following bilinear form on the space H_m:

$$< p, q >= (\mathcal{L}_p \mathcal{L}_q w_m)(0),$$

where $\mathcal{L}_p = \chi_m(p), \mathcal{L}_q = \chi_m(q)$.

Conjecture 2* *The form $<,>$ on H_m is non-degenerate.*

This obviously implies Conjecture 2, but one can actually show that Conjectures 1 and 3 follow from Conjecture 2^* as well.

Notice that as a corollary we have the duality relation for the dimensions h_k of the spaces of m-harmonics of degree k:

$$h_k = h_{\mathcal{N}-k},$$

where $\mathcal{N} = \sum(2m_\alpha + 1)$ is the degree of w_m.

In this paper we prove all these conjectures for all the Coxeter groups of rank 2 (i.e. for the dihedral groups $I_2(N)$) under additional assumption that all the multiplicities are equal. In this case we describe all the m-harmonic polynomials explicitly.

As a corollary we show that the Poincare series for the quasiinvariants of dihedral group $I_2(N)$ is given by

$$p(Q_m^{I_2(N)}, t) = \frac{1 + 2t^{(mN+1)} + \ldots + 2t^{(mN+N-1)} + t^{(2m+1)N}}{(1-t^2)(1-t^N)}.$$

Notice that Conjecture 3 and Chevalley theorem imply that the Poincare series of the quasiinvariants of any Coxeter group has the form

$$p(Q_m, t) = \frac{P(H_m, t)}{\prod_{i=1}^{n}(1 - t^{d_i})},$$

where $P(H_m, t)$ is the corresponding Poincare polynomial of the m-harmonics:

$$P(H_m, t) = \sum_{k=0}^{\mathcal{N}} h_k t^k.$$

Some formulas for the polynomials $P(H_m, t)$ have been recently found in [9][1].

2 Quasiinvariants and quantum integrals of Calogero–Moser systems

Let us first discuss the homomorphism χ_m in more details. We will need some facts from the theory of multidimensional Baker–Akhiezer functions related to a Coxeter configuration of hyperplanes (see [2], [3], [12]). Let us remind that we are using the gauge which is different from the chosen in these papers by $g(x) = \prod(\alpha, x)^{m_\alpha}$.

For any Coxeter group G and multiplicity function m there exists a Baker–Akhiezer function (BA function) of the form

$$\psi = P(k, x)e^{(k,x)} \tag{5}$$

with the following properties:

- $P(k, x)$ is a polynomial in $k \in V$ and $x \in V$ with the highest term

$$g(k)g(x) = \prod_{\alpha \in A}(\alpha, k)^{m_\alpha}(\alpha, x)^{m_\alpha}.$$

- ψ satisfies the quasiinvariance conditions in k-space

$$\psi(s_\alpha(k)) - \psi(k) = o((\alpha, k)^{2m_\alpha}) \text{ near } (\alpha, k) = 0.$$

It is known that such function does exist, and is unique and symmetric with respect to x and k:

$$\psi(k, x) = \psi(x, k)$$

[1] As we have recently learnt from I. Cherednik such formulas can be extracted from Opdam's papers [10, 11].

(see [2], [3]). As it has been explained in [2] for any quasiinvariant $q \in Q_m$ there exists a differential operator $\chi_m(q) = \mathcal{L}_q(x, \frac{\partial}{\partial x})$ such that

$$\mathcal{L}_q(x, \frac{\partial}{\partial x})\psi(x,k) = q(k)\psi(x,k).$$

The procedure of finding $\mathcal{L}_q$ is effective provided the formula for ψ is given. Since for $q = k^2$ we have the (gauged) Calogero-Moser operator (3) we have the following

Theorem 1 [2, 3] *For any Coxeter group G and integer-valued multiplicity m there exists a homomorphism $\chi_m : Q_m \to D_\Sigma(V)$ mapping the algebra of quasiinvariants Q_m into the commutative algebra of quantum integrals of generalised Calogero–Moser problem.*

One can write down the following explicit formula for this homomorphism suggested by Yu. Berest [13]:

$$\chi_m(q) = c(ad_{\mathcal{L}})^{d(q)}\hat{q}. \tag{6}$$

Here $\mathcal{L}$ is the gauged Calogero-Moser operator (3), $ad_L A = LA - AL$, $\hat{q}$ is the operator of multiplication by q, $d(q)$ is degree of polynomial q and the constant $c = c(q) = (2^{d(q)}d(q)!)^{-1}$.

Indeed because of the symmetry of ψ with respect to x and k we have

$$\mathcal{L}(k, \frac{\partial}{\partial k})\psi(x,k) = x^2\psi(x,k).$$

Thus ψ satisfies the so-called bispectral problem in the sense of Duistermaat and Grünbaum and one can use the general identity (1.8) from their paper [14] which states that

$$(ad_{\mathcal{L}})^r(q)[\psi] = (ad_{\hat{x}^2})^r(\mathcal{L}_q)[\psi].$$

For $r = \deg q$ we arrive at the formula (6).

As we have mentioned in the Introduction the restriction χ_m onto the subring of invariants S^G gives an isomorphism γ_m between S^G and the ring D_m^G of invariant integrals of the Calogero-Moser quantum problem in the gauge (3). The following result shows that the map χ_m defined in the Theorem 1 is in a certain sense the maximal extension of this map.

Let D_m be the maximal commutative ring of differential operators on V with rational coefficients which contains D_m^G as a subring.

Theorem 2 *The map χ_m is an isomorphism between the ring of m-quasiinvariants Q_m and the ring D_m.*

The proof follows from the following lemma. Let $\sigma_1 = k^2, \sigma_2, ..., \sigma_n$ be some generators of S^G, and let $\mathcal{L}_1 = \chi_m(\sigma_1) = \mathcal{L}, \ldots, \mathcal{L}_n = \chi_m(\sigma_n)$ be the corresponding invariant integrals of the Calogero-Moser problem.

Lemma 1 *Let A be a differential operator commuting with all $\mathcal{L}_i, i = 1, ..., n$. Then $A = \mathcal{L}_q = \chi_m(q)$ for some quasiinvariant $q \in Q_m$.*

To prove this let us notice that since A commutes with all $\mathcal{L}_i$ it preserves their joint eigenspace $V(k)$ consisting of the solutions of the system

$$\begin{cases} \mathcal{L}_1\psi = \sigma_1(k)\psi \\ \ldots\ldots\ldots\ldots\ldots \\ \mathcal{L}_n\psi = \sigma_n(k)\psi \end{cases} \tag{7}$$

where $k \in V$ is a "spectral" parameter. For generic k this space is spanned by the Baker-Akhiezer functions $\psi(x, g(k)), g \in G$. From the form (5) of this function it follows that ψ itself must be an eigenvector of A:

$$A\psi(x,k) = a(k)\psi(x,k)$$

for some polynomial $a(k)$. To show that $a(k)$ is a quasiinvariant let us notice that the left hand side of the last formula satisfies the quasiinvariance conditions in k (see the properties of the BA function) and therefore must the right hand side. A simple analysis shows that $a(k)$ must be a quasiinvariant in that case.

Another relation between quasiinvariants and quantum integrals $\mathcal{L}_q$ is given by the following

Theorem 3 *The space Q_m of all m-quasiinvariants is invariant under the action of all the operators $\mathcal{L}_q$, $q \in Q_m$.*

For the operator $\mathcal{L}$ (3) this can be proven by direct local considerations (c.f. [15] where a similar observation has been first made). The fact that the same is true for any $\mathcal{L}_q$ now follows from Berest's formula (6).

3 m-harmonic polynomials

Consider again the space $V(k)$ of the solutions of compatible system of equations (7). Let us put now k to be zero, i.e. consider the system

$$\begin{cases} \mathcal{L}_1\psi = 0 \\ \ldots\ldots\ldots \\ \mathcal{L}_n\psi = 0 \end{cases} \tag{8}$$

We claim that all the solutions of the system (8) are polynomials in x. More precisely we have the following

Theorem 4 *For any Coxeter group G and multiplicity function m all the solutions of the system (8) are polynomial. They form the space of dimension $|G|$ where the natural action of G is its regular representation.*

When all the multiplicities are zero this is the classical result (see [16, 5]) and the corresponding polynomials are called harmonic. For the general multiplicity m we will call the corresponding solutions of (8) as *m-harmonic polynomials* and denote the space $V(0)$ as H_m.

To prove the theorem let us consider first the general system (7). Heckman and Opdam [6] showed that it is equivalent to a holonomic system of the first order of rank $|G|$. Components of this system are $\phi = (\phi_i)$, $\phi_i = q_i(\partial)\psi$, where q_i, $i = 1, \dots, |G|$ is a basis of harmonic polynomials of G (see [6]).[2]

For generic k (more precisely, if $\prod_\alpha (k, \alpha) \neq 0$) we can choose the functions $\psi_\sigma = \psi(\sigma(k), x), \sigma \in G$, where ψ is the Baker–Akhiezer function (5) from the previous section, as a basis of the correspondent space $V(k)$. Since BA function is regular everywhere as a function of x, the same is true for the solutions of (7) if $\prod_\alpha (k, \alpha) \neq 0$.

To prove that this is true for any k, in particular for $k = 0$, one can argue as follows. Let us consider the natural complex version of the system (7) by assuming simply that $x \in V^{\mathbb{C}}$ and ψ takes values in $\mathbb{C}$. Consider a point x_0 such that $\prod_\alpha (x, \alpha) \neq 0$ and fix the solution of (7) $\psi(k, x; x_0, a)$, $a \in \mathbb{C}^{|G|}$ by fixing the initial data in the corresponding holonomic system $\phi_i(x_0) = a_i$. Since the system (7) (and the corresponding holonomic system) is regular in k everywhere $\psi(k, x; x_0, a)$ is analytic in k everywhere for any x such that $\prod_\alpha (x, \alpha) \neq 0$. By Hartogs theorem (see e.g. [17]), $\psi(k, x; x_0, a)$ is analytic in k and x everywhere. In particular, $\psi(0, x; x_0, a)$ is analytic in x at $x = 0$. Since the system (8) is homogeneous, any component in the Taylor expansion of this function at $x = 0$ of a given degree d is as well a solution of this system. This proves that all the solutions of (8) are polynomial.

To prove the second statement of the theorem let us notice that a natural action of the group G on the space $V(k)$ for a generic k is regular. This immediately follows from the formula for a basis

$$\psi_\sigma(x, k) = \psi(x, \sigma(k)), \ \sigma \in G$$

in terms of BA function. Indeed, for any $\tau \in G$

$$\psi_\sigma(\tau^{-1}(x), k) = \psi(\tau^{-1}(x), \sigma(k)) = \psi(x, (\tau \circ \sigma)(k)) = \psi_{\tau\sigma}(x, k)$$

[2]Strictly speaking the authors of [6] considered the trigonometric analogues of (7), (8) related to Weyl groups; the systems (7), (8) for all Coxeter groups (with generic parameters m_α) have been discussed in details later by E. Opdam in [10]. Corresponding holonomic systems have been recently rewritten in explicit way as a version of Knizhnik–Zamolodchikov equation in [9].

since $\psi(\tau x, \tau k) = \psi(x, k)$. For arbitrary k this follows now by standard continuation arguments.

4 Quasiinvariants and m-harmonic polynomials

The first relation between the space H_m of m-harmonic polynomials and the space Q_m of m-quasiinvariants is given by the following

Theorem 5 *Any m-harmonic polynomial is m-quasiinvariant: $H_m \subset Q_m$.*

We will prove actually the following more general statement.

Proposition 1 *Any polynomial $p(x)$ belonging to the kernel of the operator*

$$\mathcal{L} = \Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha$$

is a quasiinvariant.

Let us deduce the quasiinvariance condition (1) for polynomial $p(x)$ at the hyperplane $(\alpha, x) = 0$. Choose an orthogonal coordinate system $(t, y_1, \dots, y_{n-1})$ such that the first axis is normal to the hyperplane. Then the operator $\mathcal{L}$ can be represented as

$$\mathcal{L} = \partial_t^2 + \Delta_y - \left(\frac{2m_\alpha}{t} + tf(t^2, y) \right) \partial_t + \sum_{i=1}^{n-1} g_i(t^2, y) \partial_{y_i},$$

where $\Delta_y = \partial_{y_1}^2 + \ldots + \partial_{y_{n-1}}^2$. The functions f and g_i are analytic at $t = 0$ and invariant under reflection $t \to -t$ with respect to $(\alpha, x) = 0$ due to invariance of the operator $\mathcal{L}$ (c.f. [3]). For a polynomial $p(x)$ we also have a similar expansion

$$p = \sum_{i=0}^{\deg p} p_i(y) t^i.$$

Substituting this into the equation $\mathcal{L}p = 0$ we have

$$\left(\partial_t^2 - \frac{2m_\alpha}{t} \partial_t + \Delta_y - tf(t^2, y)\partial_t + \sum_{i=1}^{n-1} g_i(t^2, y) \partial_{y_i} \right) \left(\sum_{i=0}^{\deg p} p_i(y) t^i \right) = 0.$$

Considering all possible terms at t^{-1} in the lefthand side we conclude that $p_1 \equiv 0$. Considering now the terms at t we come to

$$(6 - 6m_\alpha) p_3 \equiv 0$$

which implies that $p_3 \equiv 0$ if $m_\alpha > 1$. Continuing in this way we obtain

$$p_1 = p_3 = \ldots = p_{2m_\alpha - 1} \equiv 0$$

or equivalently

$$\partial_\alpha^{2s-1} p(x)|_{(\alpha,x)=0} = 0 \text{ for } 1 \leq s \leq m_\alpha$$

which are the quasiinvariance conditions. Thus the proposition (and therefore the theorem) is proven.

We know the classical fact (see [16]) that the space H_0 of usual harmonic polynomials related to a Coxeter group G is isomorphic to the quotient

$$H_0 \approx \mathbb{C}[x_1, \ldots, x_n]/I_0 \tag{9}$$

where I_0 is the ideal generated by the G-invariants of positive degree. We are going to present some arguments in favour of the following generalisation of (9):

$$H_m \approx Q_m/I_m, \tag{10}$$

where Q_m is the ring of all m-quasiinvariants, $I_m \subset Q_m$ is its ideal generated by the invariants of positive degree.

Following the classical scheme [16] let us define the map

$$\pi_m : Q_m \to H_m \tag{11}$$

by the formula

$$\pi_m(q) = \mathcal{L}_q(w_m) \tag{12}$$

where $w_m = \prod_\alpha (\alpha, x)^{2m_\alpha+1}$ and $\mathcal{L}_q = \chi_m(q)$ is defined by (6). To prove that $\mathcal{L}_q(w_m) \in H_m$ we will need the following lemma. Let $A_m \subset Q_m$ be the subspace of antiinvariants, i.e. the quasiinvariants q satisfying the property

$$q(s_\alpha(x)) = -q(x)$$

for any reflection $s_\alpha \in G$.

Lemma 2 *A_m is a one-dimensional module over S^G generated by w_m.*

It is easy to show that such an antiinvariant is divisible by w_m. Since the quotient is G-invariant this implies the lemma.

Lemma 3 *The quasiinvariant w_m is m-harmonic.*

Indeed, since all $\mathcal{L}_i$ are G-invariant and preserve the space Q_m (see theorem 3 above) the polynomials $\mathcal{L}_i(w_m)$ belong to A_m. Since they have degree less than the degree of w_m they must be zero.

Lemma 4 *The space H_m is invariant under the action of $\mathcal{L}_q$ for any $q \in Q_m$.*

This follows from commutativity of $\mathcal{L}_q$ and $\mathcal{L}_i$, $i = 1, \dots, n$. All this implies

Theorem 6 *The formula*

$$\pi_m(q) = \mathcal{L}_q(w_m)$$

defines a linear map from Q_m to H_m.

Let us discuss the properties of the map π_m.

Theorem 7 *The kernel of π_m contains the ideal I_m.*

To prove this let us represent any element $q \in I_m$ as

$$q = \sum_s q_s p_s$$

where $q_s \in Q_m$, $p_s \in S^G$. We have

$$\mathcal{L}_q(w_m) = \sum_s \mathcal{L}_{q_s} \mathcal{L}_{p_s}(w_m) = 0$$

since $\mathcal{L}_{p_s}(w_m) = 0$ due to lemma 3.

Our first two conjectures (see the Introduction) claim that the kernel of π_m coincides with I_m and that the restriction of π_m onto H_m:

$$\pi_m|_{H_m} : H_m \to H_m$$

is an isomorphism. This implies that

$$Q_m / I_m \approx H_m$$

and that Q_m is generated by H_m as a module over S^G. Our third conjecture says that this module is actually free (like in the classical situation $m = 0$).

In the next section we are giving the proofs of all our conjectures for two-dimensional Coxeter groups with constant multiplicity function.

5 Proofs for the dihedral groups

Consider the dihedral group $G = I_2(N)$ which is a symmetry group of a regular N-gon. It is generated by the plane reflections with respect to the lines $\alpha_1^j x_1 + \alpha_2^j x_2 = 0$, $\alpha_1^j = -\sin\frac{2\pi j}{N}$, $\alpha_2^j = \cos\frac{2\pi j}{N}$, $j = 0, 1, \dots N-1$. We suppose that the multiplicity function m_{α^j} is equal to $m \in \mathbb{Z}_{\geq 0}$ for all $j = 0, \dots, N-1$. It will be convenient for us to use the complex coordinates $z = x_1 + ix_2, \bar{z} = x_1 - ix_2$. The generators of the ring of invariants can be chosen as $\sigma_1 = z\bar{z}, \sigma_2 = z^N + \bar{z}^N$.

We are going first to investigate the ring Q_m of m-quasiinvariants. For $m = 1$ this ring has been studied in the paper [18] where in particular the multiplicative generators of Q_1 have been found. We will use some observations from that paper to describe the ring Q_m with general m.

Let us consider the following $2(N-1)$ polynomials:

$$q_j = a_{j0} z^{mN+j} + a_{j1} \bar{z}^N z^{(m-1)N+j} + \dots + a_{jm} \bar{z}^{mN} z^j, \tag{13}$$

$$\bar{q}_j = \bar{a}_{j0} \bar{z}^{mN+j} + \bar{a}_{j1} z^N \bar{z}^{(m-1)N+j} + \dots + \bar{a}_{jm} z^{mN} \bar{z}^j, \tag{14}$$

$j = 1, \dots, N-1$, where the coefficients a_{js} are chosen to satisfy the system of equations

$$\begin{cases} (mN+j)a_{j0} + ((m-2)N+j)a_{j1} + \dots + (-mN+j)a_{jm} = 0 \\ (mN+j)^3 a_{j0} + ((m-2)N+j)^3 a_{j1} + \dots + (-mN+j)^3 a_{jm} = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (mN+j)^{2m-1} a_{j0} + ((m-2)N+j)^{2m-1} a_{j1} + \dots + (-mN+j)^{2m-1} a_{jm} = 0 \end{cases}$$

One can easily see that rank of this system is equal to m so for any j the coefficients a_{js} are defined uniquely up to proportionality.

Explicitly polynomial q_j can be given as determinant

$$q_j = detZ_j \tag{15}$$

of the following matrix

$$Z_j = \begin{pmatrix} (mN+j) & ((m-2)N+j) & \dots & (-mN+j) \\ (mN+j)^3 & ((m-2)N+j)^3 & \dots & (-mN+j)^3 \\ \vdots & \vdots & \ddots & \vdots \\ (mN+j)^{2m-1} & ((m-2)N+j)^{2m-1} & \dots & (-mN+j)^{2m-1} \\ z^{mN+j} & \bar{z}^N z^{(m-1)N+j} & \dots & \bar{z}^{mN} z^j \end{pmatrix}.$$

Proposition 2 *The polynomials $q_j, \bar{q}_j$ belong to the space Q_m of quasiinvariants.*

Proof. Let us introduce the polar coordinates $z = re^{i\varphi}$, $\bar{z} = re^{-i\varphi}$. Then from the system of equations defining coefficients a_{js} it obviously follows that $\partial_\varphi^{2s-1} q_j|_{\varphi=\frac{\pi k}{N}} = 0$, $\partial_\varphi^{2s-1} \bar{q}_j|_{\varphi=\frac{\pi k}{N}} = 0$, $k = 0, \dots, N-1$, $s = 1, \dots, m$. Now the statement follows from the following lemma.

Lemma 5 *For any polynomial $p(x_1, x_2)$, any vector $\alpha = (-\sin\varphi_0, \cos\varphi_0)$ and for arbitrary $m \in \mathbb{Z}_+$ the conditions*

$$\partial_\alpha^{2s-1} p|_{(\alpha,x)=0} = 0, \quad s = 1, \dots, m$$

are satisfied if and only if the following conditions in polar coordinates hold:

$$\partial_\varphi^{2s-1} p|_{\varphi=\varphi_0} = 0, \quad s = 1, \dots, m$$

Now let us introduce two more quasiinvariants

$$q_0 = 1, \; q_N = (z^N - \bar{z}^N)^{2m+1}. \tag{16}$$

The last polynomial q_N is actually a basic antiinvariant quasiinvariant (w_m in our previous notations).

Theorem 8 *The ring Q_m is a free finitely generated module over its subring $S^G \subset Q_m$ of invariant polynomials. One can choose the polynomials $q_0, q_1, \dots, q_{N-1}$, $\bar{q}_1, \dots, \bar{q}_{N-1}, q_N$ as a basis of Q_m over S^G.*

Proof. At first let us show that the polynomials $q_0, q_1, \dots, q_{N-1}, \bar{q}_1, \dots, \bar{q}_{N-1}, q_N$ do generate Q_m over S^G. To prove this we will use induction on a degree of a polynomial $q \in Q_m$. If $\deg q = 0$ then $q = const = c$, thus $q = cq_0$ so we have checked the base of induction. Suppose now that $\deg q = d$ and $q = Az^d + B\bar{z}^d + z\bar{z}p_{d-2}$ is an arbitrary quasiinavariant, $q \notin S^G$. We will use further the following two lemmas.

Lemma 6 *Any m-quasiinvariant of degree $d \leq mN$ is actually invariant.*

Proof. It is enough to prove the lemma for an arbitrary homogeneous polynomial. Let

$$q = a_0 z^{mN-\sigma} + a_1 z^{mN-\sigma-1}\bar{z} + a_2 z^{mN-\sigma-2}\bar{z}^2 + \ldots + a_{mN-\sigma}\bar{z}^{mN-\sigma} \in Q_m \tag{17}$$

for some $\sigma \geq 0$. According to lemma 5 the conditions of quasiinvariance in polar coordinates are $\partial_\varphi^{2s-1} q = 0$ for $\varphi = \frac{\pi k}{N}$, $0 \leq k \leq N-1$. We have

$$(mN-\sigma)^{2s-1} a_0 e^{i\frac{\pi}{N}(mN-\sigma)k} + (mN-\sigma-2)^{2s-1} a_1 e^{i\frac{\pi}{N}(mN-\sigma-2)k} +$$

$$(mN-\sigma-4)^{2s-1}a_2 e^{i\frac{\pi}{N}(mN-\sigma-4)k}+\ldots+(-mN+\sigma)^{2s-1}a_{mN-\sigma}e^{i\frac{\pi}{N}(-mN+\sigma)k}=0$$

Collecting the terms in this sum with equal exponents, we get

$$\sum_{j=0}^{N-1}\sum_{\substack{t\geq 0\\ j+Nt\leq mN-\sigma}}(mN-\sigma-2(j+Nt))^{2s-1}a_{j+Nt}e^{i\frac{\pi}{N}(mN-\sigma-2(j+Nt))k}=$$

$$\sum_{j=0}^{N-1}e^{i\frac{\pi}{N}(mN-\sigma-2j)k}\sum_{\substack{t\geq 0\\ j+Nt\leq mN-\sigma}}(mN-\sigma-2(j+Nt))^{2s-1}a_{j+Nt}=0$$

Now let us consider these conditions for all possible $k=0,1,\ldots,N-1$. We arrive at the Vandermonde-type system with different exponents $e^{i\frac{\pi}{N}(mN-\sigma-2j)}$, $0\leq j\leq N-1$. Hence, for all $j=0,\ldots,N-1$ the following property is satisfied

$$\sum_{\substack{t\geq 0\\ j+Nt\leq mN-\sigma}}(mN-\sigma-2(j+Nt))^{2s-1}a_{j+Nt}=0 \tag{18}$$

Let us analyze the conditions (18) for all possible s, $1\leq s\leq m$. We again have a system of Vandermonde type with the exponents $(mN-\sigma-2(j+Nt))^2$, $0\leq t\leq [\frac{mN-\sigma-j}{N}]\leq m$. Notice that the exponents corresponding to different t may coincide only in pairs. The condition for that is

$$mN-\sigma-(j+Nt_1)=j+Nt_2. \tag{19}$$

In the case $j=\sigma=0$ the number of equations in (18) is less than number of unknown coefficients a_{j+Nt}. But after collecting the terms in (18) corresponding to equal exponents the number of equations becomes not less than the number of unknowns. We conclude that all nonzero coefficients a_{j+Nt} can be divided into pairs so that $a_{j+Nt_1}-a_{j+Nt_2}=0$ and also the condition (19) is satisfied. In terms of polynomial q this means that it can be represented in the form

$$q=\sum_{j=0}^{N-1}\sum_{(t_1,t_2)}a_{j+Nt_1}z^{mN-\sigma-(j+Nt_1)}\bar{z}^{j+Nt_1}+a_{j+Nt_2}z^{mN-\sigma-(j+Nt_2)}\bar{z}^{j+Nt_2}=$$

$$\sum_{j=0}^{N-1}\sum_{(t_1,t_2)}a_{j+Nt_1}(z\bar{z})^{j+Nt_2}(z^{N(t_1-t_2)}+\bar{z}^{N(t_1-t_2)}),$$

where the pairs (t_1,t_2) satisfy (19) and also we suppose that $t_1>t_2$. Hence polynomial q is an invariant.

Lemma 7 *If* $q = Az^d + B\bar{z}^d + z\bar{z}p_{d-2}$ *is* m*-quasiinavariant of degree* $d = mN + lN$, $1 \le l \le m$ *then* $A = B$.

Proof. We will use the notations and scheme of proof of lemma 6. Let us consider q of the form (17), now we have $\sigma = -lN$. As above in lemma 6 the conditions (18) for $j = 0, \dots, N-1$ should be satisfied. Let us fix $j = 0$, we get

$$\sum_{t=0}^{m+l} ((m+l-2t)N)^{2s-1} a_{Nt} = 0$$

or equivalently

$$\sum_{t=0}^{[\frac{m+l}{2}]} ((m+l-2t)N)^{2s-1} (a_{Nt} - a_{N(m+l-t)}) = 0.$$

We have got a system of Vandermonde type with different exponents. For $l \le m$ from that it follows that $a_{Nt} = a_{N(m+l-t)}$. If $t = 0$ we get $a_0 = a_{N(m+l)}$ which completes the proof of the lemma.

To continue the proof of the theorem let us represent d in the form $d = mN + jN + k$, where $0 \le k < N$. We have to consider few different cases.

a) If $k \ne 0$ then $q - \frac{A}{a_{k0}} q_k (z^N + \bar{z}^N)^j - \frac{B}{\bar{a}_{k0}} \bar{q}_k (z^N + \bar{z}^N)^j = z\bar{z}p_1$ for some polynomial $p_1 \in Q_m$, here constants $a_{k0}, \bar{a}_{k0}$ are defined by the form (13), (14) of the polynomials q_k, $\bar{q}_k$. Since $\deg p_1 = d - 2 < d$ we have done the induction step.

b) If $k = 0$, $1 \le j \le m$ then lemma 7 states that $A = B$, hence $q - A(z^N + \bar{z}^N)^{m+j} = z\bar{z}p_2$, where $p_2 \in Q_m$ and it can be represented as a linear combination of the polynomials $q_i, \bar{q}_i$ with invariant coefficients.

c) If $k = 0$, $j \ge m+1$ then $q - \frac{A-B}{2} q_N (z^N + \bar{z}^N)^{j-m-1} - \frac{A+B}{2} (z^N + \bar{z}^N)^{m+j} = z\bar{z}p_3$, where $p_3 \in Q_m$ and one can apply the induction hypothesis.

Thus we have proved that polynomials $q_i, \bar{q}_i$ generate Q_m as an S^G–module. Now we are going to show that this module is free.

To see this let us consider arbitrary nontrivial combination of the polynomials $q_i, \bar{q}_i$ with invariant coefficients. Since the ring of invariants for the dihedral group is the ring freely generated by two polynomials $\sigma_1 = z\bar{z}$ and $\sigma_2 = z^N + \bar{z}^N$, the linear combination takes the form

$$p_0^1(\sigma_1, \sigma_2) q_0 + p_1^1(\sigma_1, \sigma_2) q_1 + p_1^2(\sigma_1, \sigma_2) \bar{q}_1 + \dots +$$

$$p_{N-1}^1(\sigma_1, \sigma_2) q_{N-1} + p_{N-1}^2(\sigma_1, \sigma_2) \bar{q}_{N-1} + p_N^1(\sigma_1, \sigma_2) q_N = 0,$$

where for some s, ϵ we have $p_s^\epsilon \ne 0$. Also we can suppose that p_s^ϵ is not divisible by σ_2. Further, let us represent polynomials p_j as combinations of monomials in σ_1, σ_2 and let us move monomials containing σ_1 into righthand side. We have then that

$$r_0^1(\sigma_2) q_0 + r_1^1(\sigma_2) q_1 + r_1^2(\sigma_2) \bar{q}_1 + \dots\dots + r_{N-1}^1(\sigma_2) q_{N-1} + r_{N-1}^2(\sigma_2) \bar{q}_{N-1} + r_N^1(\sigma_2) q_N$$

is divisible by σ_1 and some polynomial $r_s^\epsilon \neq 0$. Let us consider monomials having degree which is equal to s modulo N. If $1 \leq s \leq N-1$ then $r_s^1(\sigma_2)q_s + r_s^2(\sigma_2)\bar{q}_s$ must be divisible by $z\bar{z}$, which is impossible as $r_s^1(\sigma_2)q_s$ contains monomial of the form $\lambda_1 z^{\mu_1}$ and does not contain degrees of $\bar{z}$, and $r_s^2(\sigma_2)\bar{q}_s$ contains monomial of the form $\lambda_2 \bar{z}^{\mu_2}$ and does not contain degrees of z. If $s = 0$ or $s = N$ then $r_0^1(\sigma_2) + r_N^1(\sigma_2)(z^{(2m+1)N} - \bar{z}^{(2m+1)N})$ must be divisible by $z\bar{z}$, which is possible only if $r_0^1 = r_N^1 = 0$ but this is not the case. Thus the theorem is proven.

Corollary *The Poincare series for the m-quasiinvariants of dihedral group $I_2(N)$ is*

$$p(Q_m, t) = \frac{1 + 2t^{(mN+1)} + \ldots + 2t^{(mN+N-1)} + t^{(2m+1)N}}{(1-t^2)(1-t^N)}.$$

Now we are going to show that polynomials $q_i, \bar{q}_i$ (13), (14), (16) are actually m-harmonic. This will complete the proof of conjecture 3.

First let us rewrite the operator $\mathcal{L}$ in the complex coordinates. The set of vectors α for the group $I_2(N)$ has the form $\alpha = (-\sin\varphi_k, \cos\varphi_k)$, where $\varphi_k = \frac{\pi k}{N}$, $k = 0, \ldots, N-1$. Substituting $\partial_x = \partial_z + \partial_{\bar{z}}$, $\partial_y = i(\partial_z - \partial_{\bar{z}})$ we get

$$\mathcal{L} = \Delta - 2m \sum_\alpha \frac{\partial_\alpha}{(\alpha, x)} = \Delta - 2m \sum_\alpha \frac{-\sin\varphi_k \partial_x + \cos\varphi_k \partial_y}{-\sin\varphi_k x + \cos\varphi_k y} =$$

$$4\partial_z\partial_{\bar{z}} - 2m \sum_{k=0}^{N-1} \frac{(-\sin\varphi_k + i\cos\varphi_k)\partial_z + (-i\cos\varphi_k - \sin\varphi_k)\partial_{\bar{z}}}{\frac{1}{2}z(-i\cos\varphi_k - \sin\varphi_k) + \frac{1}{2}\bar{z}(i\cos\varphi_k - \sin\varphi_k)} =$$

$$4\left(\partial_z\partial_{\bar{z}} - m\sum_{k=0}^{N-1} \frac{e^{i\varphi_k}\partial_z - e^{-i\varphi_k}\partial_{\bar{z}}}{-e^{-i\varphi_k}z + e^{i\varphi_k}\bar{z}}\right).$$

The operator $\mathcal{L} = \mathcal{L}_1$ has a commuting operator $\mathcal{L}_2$ which is also invariant under dihedral group, it is homogeneous of degree N and has the form $\mathcal{L}_2 = \partial_z^N + \partial_{\bar{z}}^N +$ lower order terms.

Theorem 9 *The polynomials* (13), (14), (16) *belong to the common kernel of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, i.e. they are m-harmonic.*

Proof. From theorem 3 and degree consideration it follows immediately that $\mathcal{L}_1(q_0) = \mathcal{L}_2(q_0) = 0$. As q_N is an antiinvariant quasiinvariant of the smallest possible degree and due to invariance of the operators $\mathcal{L}_1, \mathcal{L}_2$, we have $\mathcal{L}_1(q_N) = \mathcal{L}_2(q_N) = 0$.

Let us show now that $\mathcal{L}_2(q_s) = 0$, $1 \leq s \leq N-1$. Let $\mathcal{L}_2(q_s) = r_s, \mathcal{L}_2(\bar{q}_s) = \bar{r}_s$. Notice that two dimensional space $V_s = < q_s, \bar{q}_s >$ is an irreducible representation for the group $G = I_2(N)$. Since operator $\mathcal{L}_2$ being invariant commutes with the action of G, then by Schur lemma the kernel of $\mathcal{L}_2|_{V_s}$ is either V_s or 0. In the last case the space $< r_s, \bar{r}_s >$ is an irreducible representation for G. But since $\deg r_s = \deg q_s - N < mN$ the polynomials $r_s, \bar{r}_s$ should be invariant according to lemma 6. The contradiction means that $\mathcal{L}_2|_{V_s} = 0$, i.e. $\mathcal{L}_2(q_s) = \mathcal{L}_2(\bar{q}_s) = 0$.

Now let us show that $\mathcal{L}_1(q_s) = \mathcal{L}_1(\bar{q}_s) = 0$. Let $p_s = \mathcal{L}_1(q_s), \bar{p}_s = \mathcal{L}_1(\bar{q}_s)$. As above by Schur lemma either $\mathcal{L}_1|_{V_s} = 0$ or $\mathcal{L}_1|_{V_s}$ is an isomorphism. In the second case representation $< p_s, \bar{p}_s >$ is isomorphic to irreducible representation V_s. It is easy to see from the formulas (13), (14) that among all the representations $V_t, 1 \leq t \leq N-1, t \neq s$ only V_{N-s} is isomorphic to V_s, so that q_{N-s} corresponds to $\bar{q}_s$, and $\bar{q}_{N-s}$ corresponds to q_s. This implies that $p_s = P(\sigma_1, \sigma_2)\bar{q}_{N-s}$, $\bar{p}_s = P(\sigma_1, \sigma_2) q_{N-s}$ for some polynomial $P(x, y)$. But since

$$p_s = 4\left(\partial_z\partial_{\bar{z}} - m\sum_{k=0}^{N-1}\frac{e^{i\varphi_k}\partial_z - e^{-i\varphi_k}\partial_{\bar{z}}}{-e^{-i\varphi_k}z + e^{i\varphi_k}\bar{z}}\right)(a_{s0}z^{mN+s} + a_{s1}\bar{z}^N z^{(m-1)N+s} + \ldots + a_{sm}\bar{z}^{mN}z^s),$$

the degree of p_s in $\bar{z}$ is less or equal then mN while $\deg_{\bar{z}} P(\sigma_1, \sigma_2)\bar{q}_{N-s} > mN$. This means that $\mathcal{L}_1|_{V_s} = 0$ so $\mathcal{L}_1(q_s) = \mathcal{L}_1(\bar{q}_s) = 0$. The theorem is proven.

The theorems 8 and 9 imply that our conjecture 3 is true for any dihedral group and constant multiplicity function. Let us now prove the first two conjectures.

The following lemma will be essential for us. Let

$$w = w_m = \prod_\alpha (\alpha, x)^{2m+1} = \prod_{j=0}^{N-1}(-\sin\frac{2\pi j}{N}x_1 + \cos\frac{2\pi j}{N}x_2)^{2m+1}$$

be m-discriminant. Let us introduce the corresponding quantum integral $\mathcal{L}_w = \chi_m(w)$. We claim that the constant $\mathcal{L}_w w$ is non-zero for any m.

Lemma 8 *The constant $\mathcal{L}_w w$ is determined by the formula*

$$\mathcal{L}_w(w) = \frac{N^{2m+1}}{2^{(2m+1)(N-1)}}\prod_{j=1}^{2m+1}(2j - 2m - 1)\prod_{\substack{d=1 \\ d \not\equiv 0 (mod N)}}^{(2m+1)N}(d - mN),$$

and in particular, is non-zero for any $m \in \mathbb{Z}_{\geq 0}$.

Remark When $m = 0$ $\mathcal{L}_w = \prod_\alpha \partial_\alpha$ and the lemma claims that

$$(\prod_\alpha \partial_\alpha)\prod_\alpha(\alpha, x) = \frac{N!}{2^{N-1}}$$

which is a particular case of the following Macdonald identity:

$$(\prod_\alpha \partial_\alpha)\prod_\alpha(\alpha, x) = \prod_i d_i!$$

where d_i are the degrees of generators σ_i of the invariants S^G of a Coxeter group G and all the roots are supposed to be normalised as $(\alpha, \alpha) = 2$.

Proof of the lemma. Since by formula (6) $\mathcal{L}_w = \frac{1}{2^M M!} ad_{\mathcal{L}}^M w$, $M = \deg w = (2m+1)N$ and $\mathcal{L}(w) = 0$, we have that $\mathcal{L}_w(w) = \frac{1}{2^M M!}\mathcal{L}^M(w^2)$. Now let us rewrite operator $\mathcal{L}$ in polar coordinates. We have

$$\Delta = \partial_r^2 + \frac{1}{r^2}\partial_\varphi^2 + \frac{1}{r}\partial_r.$$

If $\alpha = (\cos\varphi_0, \sin\varphi_0)$ then

$$\frac{\partial_\alpha}{(\alpha, x)} = \frac{\cos\varphi_0(\cos\varphi\partial_r - \frac{1}{r}\sin\varphi\partial_\varphi) + \sin\varphi_0(\sin\varphi\partial_r + \frac{1}{r}\cos\varphi\partial_\varphi)}{r(\cos\varphi_0\cos\varphi + \sin\varphi_0\sin\varphi)} = \frac{1}{r}\partial_r - \frac{1}{r^2}\tan(\varphi-\varphi_0)\partial_\varphi$$

and

$$\mathcal{L} = \Delta - 2m\sum\frac{\partial_\alpha}{(\alpha, x)} =$$
$$\partial_r^2 + \frac{1}{r^2}\partial_\varphi^2 + \frac{1}{r}\partial_r - \frac{2mN}{r}\partial_r - \frac{2mN}{r^2}\cot N\varphi\partial_\varphi.$$

Let us notice that $\mathcal{L}$ maps the space V^d of homogeneous functions of degree d to V^{d-2}, and the restriction of $\mathcal{L}$ onto V^d takes the form

$$\mathcal{L}|_{V^d} = \frac{1}{r^2}(d^2 - 2mNd + \partial_\varphi^2 - 2mN\cot N\varphi\partial_\varphi).$$

Now we are going to calculate $\mathcal{L}^M w^2$. In the polar coordinates we have

$$w^2 = \frac{r^{(2m+1)2N}(\sin N\varphi)^{4m+2}}{2^{(N-1)(4m+2)}}$$

and

$$\mathcal{L}^M(w^2) = \frac{1}{2^{(N-1)(4m+2)}}\left[\prod_{d=1}^{(2m+1)N}(4d^2 - 4mNd + \partial_\varphi^2 - 2mN\cot(N\varphi)\partial_\varphi)\right]\sin^{4m+2}N\varphi.$$

Let us introduce new variable $\phi = N\varphi$. Also due to commutativity we may order the terms in the previous product as follows

$$\mathcal{L}^M(w^2) = \frac{N^{(4m+2)N}}{2^{(N-1)(4m+2)}}\prod_{\substack{d=1\\ d\not\equiv 0 (mod N)}}^{(2m+1)N}(4(\frac{d}{N})^2 - 4m\frac{d}{N} + \partial_\phi^2 - 2m\cot\phi\partial_\phi)\times$$
$$\prod_{j=1}^{2m+1}(4j^2 - 4mj + \partial_\phi^2 - 2m\cot\phi\partial_\phi)\sin^{4m+2}\phi. \quad (20)$$

Now direct calculation shows that

$$(4j^2 - 4mj + \partial_\phi^2 - 2m\cot\phi\partial_\phi)\sin^{2j}\phi = 2j(2j-2m-1)\sin^{2j-1}\phi.$$

Thus we can simplify (20) as

$$\frac{N^{(4m+2)N}}{2^{(N-1)(4m+2)}}\prod_{\substack{d=1\\ d\not\equiv 0(modN)}}^{(2m+1)N}(4(\frac{d}{N})^2 - 4m\frac{d}{N} + \partial_\phi^2 - 2m\cot\phi\partial_\phi)c,$$

where $c = \prod_{j=1}^{2m+1} 2j(2j-2m-1)$. Finally we get

$$\mathcal{L}^M(w^2) = \frac{N^{(4m+2)N}}{2^{(N-1)(4m+2)}}\prod_{j=1}^{2m+1} 2j(2j-2m-1)\prod_{\substack{d=1\\ d\not\equiv 0(modN)}}^{(2m+1)N}(4(\frac{d}{N})^2 - 4m\frac{d}{N}) =$$

$$N^{4m+2}\prod_{j=1}^{2m+1} 2j(2j-2m-1)\prod_{\substack{d=1\\ d\not\equiv 0(modN)}}^{(2m+1)N} d(d-mN) = N^{4m+2}\prod_{j=1}^{2m+1} 2j(2j-2m-1)\times$$

$$\frac{M!}{N^{2m+1}(2m+1)!}\prod_{\substack{d=1\\ d\not\equiv 0(modN)}}^{(2m+1)N}(d-mN) = M!(2N)^{2m+1}\prod_{j=1}^{2m+1}(2j-2m-1)\prod_{\substack{d=1\\ d\not\equiv 0(modN)}}^{(2m+1)N}(d-mN)$$

Since $\mathcal{L}_w w = \frac{1}{2^M M!}\mathcal{L}^M(w^2)$ we have

$$\mathcal{L}_w w = \frac{(2N)^{2m+1}}{2^{(2m+1)N}}\prod_{j=1}^{2m+1}(2j-2m-1)\prod_{\substack{d=1\\ d\not\equiv 0(modN)}}^{(2m+1)N}(d-mN).$$

This proves the lemma.

Now we are ready to prove Conjecture 1.

Theorem 10 *For any dihedral group $I_2(N)$*

$$Ker\pi_m = I_m,$$

where π_m is defined by (12) and I_m is the ideal in Q_m generated by basic invariants σ_1, σ_2.

Proof. Let us represent an arbitrary quasiinvariant in the form

$$q = s_0 q_0 + \sum_{j=1}^{N-1}(s_j q_j + \bar{s}_j \bar{q}_j) + s_N q_N \tag{21}$$

where $s_j, \bar{s}_j$ are invariants and $q_j, \bar{q}_j$ are defined by (13), (14). Suppose that $q \in Ker\pi_m$, which is $\mathcal{L}_q w = 0$. As

$$\mathcal{L}_q = \mathcal{L}_{s_0} + \sum_{j=1}^{N-1}(\mathcal{L}_{s_j}\mathcal{L}_{q_j} + \mathcal{L}_{\bar{s}_j}\mathcal{L}_{\bar{q}_j}) + \mathcal{L}_{s_N}\mathcal{L}_{q_N},$$

the condition $\mathcal{L}_q w = 0$ is equivalent to $\mathcal{L}_{q^H} w = 0$, where $q^H \in H$ is a subsum in (21) corresponding to those j that s_j (or $\bar{s}_j$) are constants. Since $q - q^H \in I_m$ it is sufficient to prove that $\mathcal{L}_h w \neq 0$ for any $h \in H$.

It is sufficient to consider only the homogeneous h. If $h = const$ then the statement obviously holds. When $h = \mathrm{const}\, w$ it follows from lemma 8. Suppose now that $h = \lambda_1 q_j + \lambda_2 \bar{q}_j$ and $\mathcal{L}_h w = 0$. Let us consider

$$\mathcal{L}_{q_{N-j}}\mathcal{L}_h w = 0 = \mathcal{L}_{q_{N-j}h} w = \mathcal{L}_{\lambda_1 q_{N-j}q_j + \lambda_2 q_{N-j}\bar{q}_j} w$$

The formulas (13), (14) show that

$$\lambda_1 q_{N-j}q_j + \lambda_2 q_{N-j}\bar{q}_j = \lambda_1 a_{N-j0}a_{j0}z^{(2m+1)N} + (z\bar{z})p, \tag{22}$$

where p is some polynomial in $z, \bar{z}$. On the other hand, we should have a general representation (21) for some invariants s_i, $\bar{s}_i$

$$\lambda_1 q_{N-j}q_j + \lambda_2 q_{N-j}\bar{q}_j = \sum(s_i q_i + \bar{s}_i \bar{q}_i) + s_0 q_0 + s_N q_N \tag{23}$$

In the last expression the sum $\sum(s_i q_i + \bar{s}_i \bar{q}_i)$ cannot contain monomials $z^{(2m+1)N}$, $\bar{z}^{(2m+1)N}$ as $s_i, \bar{s}_i$ are nontrivial polynomials of $z\bar{z}$ which follows from degree considerations. Suppose that $\lambda_1 \neq 0$, then the lefthand side of (23) contains z^{2m+1} and it does not contain $\bar{z}^{(2m+1)N}$ (see (22)). Hence s_N must be a nonzero constant c. Now

$$\mathcal{L}_{q_{N-j}}\mathcal{L}_h w = \mathcal{L}_{\sum(s_i q_i + \bar{s}_i \bar{q}_i) + s_0} w + c\mathcal{L}_{q_N} w = c\mathcal{L}_{q_N} w$$

since $\sum(s_i q_i + \bar{s}_i \bar{q}_i) + s_0 \in I_m$. Due to lemma 8 $\mathcal{L}_{q_N} w \neq 0$ so $\mathcal{L}_{q_{N-j}}\mathcal{L}_h w \neq 0$ which contradicts the assumption that $\mathcal{L}_h w = 0$. This implies that $\lambda_1 = 0$. Similarly multiplying $\mathcal{L}_h w = 0$ by $\mathcal{L}_{\bar{q}_{N-j}}$ we derive that $\lambda_2 = 0$ which means that $h = 0$. This proves the theorem and conjecture 1 in this case.

Conjecture 2 now simply follows from the previous arguments. Indeed we have shown in the proof of the previous theorem that if $\mathcal{L}_h w = 0$ for some $h \in H_m$ then $h = 0$. This implies the following

Theorem 11 *For any dihedral group the linear map*

$$\pi_m : H_m \to H_m$$

is an isomorphism.

Let us finally show that conjecture 2^* also holds. For that we fix normalisation of basic quasiinvariants as

$$q_j = z^{mN+j} + z\bar{z}p_j,$$
$$\bar{q}_j = \bar{z}^{mN+j} + z\bar{z}\bar{p}_j.$$

Since $q_{j_1}\bar{q}_{j_2}$ is divisible by $z\bar{z}$ it is obvious that $< q_{j_1}, \bar{q}_{j_2} >= 0$. The consideration of the degrees shows that $< q_{j_1}, q_{j_2} >= 0$ if $j_1 + j_2 \neq N$. Let us calculate $< q_j, q_{N-j} >$. We have

$$q_j q_{N-j} = z^{(2m+1)N} + (z\bar{z})\hat{p}_j = \frac{1}{2}(q_N + \sigma_2^{2m+1}) + (z\bar{z})Q_j,$$

where $\hat{p}_j$ is some polynomial and Q_j is a quasiinvariant. Hence

$$< q_j, q_{N-j} >= (\mathcal{L}_{\frac{1}{2}q_N} + \mathcal{L}_{\frac{1}{2}\sigma_2^{2m+1}+z\bar{z}Q_j})w = \frac{1}{2}\mathcal{L}_{q_N} w$$

which is non-zero by lemma 8. This implies the nondegeneracy of the form $<,>$ and conjecture 2^*.

6 Concluding remarks.

Besides the proof of the conjectures for all Coxeter groups one of the most interesting open problems is the description of the space of all m-harmonic polynomials. In the classical case $m = 0$ this space can be described as the result of the differential operators with constant coefficients applied to the discriminant $w = \prod_{\alpha \in A}(\alpha, x)$ (see [16]). In the general case one can use the operators $\mathcal{L}_q$ which correspond to the quasiinvariants but the quasiinvariants themselves need an effective description.

One of the alternative ideas is to use the relation between the system (7) and Knizhnik-Zamolodchikov (KZ) equation discovered by Matsuo and Cherednik (see [19]). The problem with this is that both Matsuo and Cherednik maps are degenerate when the spectral parameter is zero. Fortunately one can modify the KZ equation and define a map which is an isomorphism for all values of the spectral parameters (see [9]). The possibility to use these modified KZ equations for our problems is now under investigation.

Another interesting direction is to develop a similar approach for the rings of integrals of other algebraically integrable quantum problems, in particular for the trigonometric and difference versions of Calogero-Moser problem related to root systems. In trigonometric case the corresponding polynomials satisfy certain difference relations which in the rational limit coincide with the quasiinvariance relations (1) (see [2],[3]). It would be also very interesting to investigate possible analogues of harmonic polynomials in relation with the ring of quantum integrals for the generalised Calogero-Moser problems related to the deformed root systems discovered in [20] (see also [12]).

Finally we would like to mention that for the Weyl groups G the space of usual harmonic polynomials can be interpreted as cohomology of the generalised flag varieties [21]. It would be interesting to look at the space of m-harmonic polynomials from this point of view. This is also related to an important question about the multiplication structure on H_m induced by the isomorphism with Q_m/I_m.

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References

[1] Chevalley C. *Invariants of finite groups generated by reflections.* //Amer. J. Math, 1955, V. 77, 778-782.

[2] Chalykh O.A., Veselov A.P. *Commutative rings of partial differential operators and Lie alrebras* // Commun. Math. Phys., 1990, V. 126, P. 597–611.

[3] Veselov A.P., Styrkas K.L., Chalykh O.A. *Algebraic integrability for Schrodinger equation and finite reflection groups.* // Theor. Math. Phys., 1993, V. 94, N.2, P. 253–275.

[4] Olshanetsky M.A., Perelomov A.M. *Quantum integrable systems related to Lie algebras* // Phys. Rep., 1983, V. 94, P. 313–404.

[5] Helgason S. *Groups and Geometric Analysis.* Academic Press, New York, 1984.

[6] Heckman G.J., Opdam E.M. *Root systems and hypergeometric functions I* // Compositio Math., 1987, V. 64, P. 329–352.

[7] Heckman G.J. *A remark on the Dunkl differential-difference operators* // Prog. in Math., 1991, V. 101, P. 181–191.

[8] Benson D.J. *Polynomial invariants of finite groups.* LMS Lecture Notes Series,V. 190. Cambridge University Press, 1993.

[9] Felder G., Veselov A.P. *Action of Coxeter groups on m-harmonic polynomials and KZ equations.* //In preparation.

[10] Opdam E.M. *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group.* // Compositio Math. , 1993, V. 85, P. 333–373.

[11] Opdam E.M. *A remark on the irreducible characters and fake degrees of finite real reflection groups.* // Invent. Math., 1995, V. 120, P. 447–454.

[12] Chalykh O.A., Feigin M.V., Veselov A.P. *Multidimensional Baker–Akhiezer Functions and Huygens' Principle* // Commun. Math. Phys., 1999, V. 206, P. 533–566.

[13] Berest Yu. *Huygens' principle and the bispectral problem* // CRM Proceedings and Lecture Notes, 1998, V. 14, P. 11–30.

[14] Duistermaat J.J., Grünbaum F.A. *Differential equations with the spectral parameter* // Comm. Math. Phys., 1986, V. 103, P. 177–240.

[15] Chalykh O.A. *Darboux transformation in many dimensions.* // Uspekhi mat. nauk, 1998, V. 53, n.2, 167–168.

[16] Steinberg R. *Differential equations invariant under finite reflection groups* //Trans. Amer. Math. Soc.,V.112 (1964), 392-400.

[17] Shabat B.V. *Introduction to Complex Analysis. Part II. Functions of several variables.* // AMS Trans. of Math. Monographs, V. 110, 1992.

[18] Volchenko K., Kozachko A., Mishachev K. *The ring of quasiinvariants for the dihedral groups.* // Vestnik Moskov. Univ., Ser. 1, 1999, n.1, 48-51.

[19] Matsuo A. *Integrable connections related to zonal spherical functions.* // Invent. Math., 1992, vol. 110, 95-121.

[20] Veselov A.P., Feigin M.V., Chalykh O.A. *New integrable deformations of quantum Calogero-Moser problem.* Russian Math. Surv., 1996, 51(3), 185-186.

[21] Bernstein I.N., Gelfand I.M. and Gelfand S.I. *Schubert cells and the cohomology of G/P.* // Russ. Math. Surveys, 1973, v.28, 1-26.