Asymptotics in a Riemann-Hilbert boundary problem for pseudoholomorphic curves

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0 Introduction

We will study the existence and regularity of the solution space of a Riemann-Hilbert problem for pseudoholomorphic curves of disc- and annulus-type in an almost-complex four-manifold \((M^4, J)\). The boundaries of the curves are constrained to lie in a fixed embedded real surface

\[
F : \Sigma^2 \to M^4.
\]

To formulate the geometric assumption on (1), we introduce as in [K] a complex structure \(j\) on \(\Sigma\) that is adapted to its embedding (1) in the following way. A point \(\sigma \in \Sigma\), \(m = F(\sigma)\), is called a complex point if \(T_m \Sigma \subset (T_m M, J(m))\) is a complex line. Set \(j(\sigma) := F^*(\sigma)J\) at an isolated complex point \(\sigma\) and choose an arbitrary smooth extension of \(j\) to a neighborhood of \(\sigma \in \Sigma\). Now define

\[
\partial F := 1/2(dF - JdF \circ j) \in \Omega^{10}(F^* T^{10} M) \simeq \Omega^{10}(\Sigma) \otimes F^* T^{10} M
\]

\[
\bar{\partial} F := 1/2(dF + JdF \circ j) \in \Omega^{01}(F^* T^{10} M) \simeq \Omega^{01}(\Sigma) \otimes F^* T^{10} M
\]

Then

\[
\partial F \wedge \bar{\partial} F \in \Omega^{10}(F^* T^{10} M) \wedge \Omega^{01}(F^* T^{10} M) \simeq \Omega^{11} \otimes \text{det} F^* T^{10} M \simeq \text{det} F^* T^{10} M, \tag{4}
\]

where we fix a volume element on \(\Sigma\) to achieve the last isomorphism. The section (3) vanishes precisely at the complex points \(\sigma \in \Sigma\). Such a point is called elliptic if \(\partial F \wedge \bar{\partial} F\) is transversal to the zero section in (4) and the intersection number \(\text{int}_\sigma (\partial F \wedge \bar{\partial} F, 0_{\partial F \wedge \bar{\partial} F})\) is one, see [K], and general elliptic is defined below in section 1. Our main result implies the following

**Corollary 2.** Let \(F : \Sigma^2 \to (M^4, J)\) be an embedded real surface in an almost complex 4-manifold and \(\sigma \in \Sigma^2\) a general elliptic point. Then, for a neighborhood \(U\) of \(\sigma\), there exists an embedded real hypersurface \((N^3, \Sigma^2 \cap U, \sigma) \to (M^4, \Sigma^2, \sigma)\) with

a) \(N^3\) is Levi-flat and foliated by \(J\)-holomorphic curves of disc-type in \((M, J)\) and boundary in \(\Sigma^2\).
b) if $\Sigma$ is $C^{k,\alpha}$-smooth, then $N^3$ is $C^{k+2,\alpha}$-smooth up to the boundary $\Sigma$ and if $\Sigma$ is $C^\omega$, then so is $N^3$.

The regularity statement in part b) is sharp. In case $M = \mathbb{C}^2$, the existence statement part a) is due to Bishop [B], who proved existence and $C^\alpha$-regularity of $N^3$ up to the elliptic complex point $\sigma$. This was improved by [KW] and [Hu], and in [G], Gromov, who coined the term ‘pseudoholomorphic’, discussed the case that the ambient manifold is almost-complex. The authors of [BER] give an overview of real submanifolds in complex space. Note that Corollary 2 is a local result on the existence and regularity of a certain real hypersurface $N^3$ in $M^4$ part of whose boundary lies in a prescribed real surface $\Sigma^2$ of codimension two given which is parametrized by (1). The hypersurface is constructed by way of a foliating one-parameter family of $J$-holomorphic discs emanating from the elliptic complex point. This boundary value problem is a Riemann-Hilbert boundary problem, see [V], and thereby strictly elliptic if $\Sigma$ is totally real. By elliptic regularity, the discs retain the same smoothness as that of $\Sigma$ here. The ellipticity degenerates as the discs approach the complex point $\sigma$. The question of precise regularity and asymptotics arises here. Our Theorem 1, which implies the Corollary, deals with this question. In their paper, the authors of [KW], who work in $\mathbb{C}^2$, apply a Picard-type iteration scheme to pass from the osculating quadric $Q_\sigma$, which bounds the family of discs (9) below, to the given surface $\Sigma$. In contrast, our proof of is geometric and employs a blow-up process that regularizes the CR-singularity $\sigma \in \Sigma$, obtaining a family of $J$'-holomorphic curves of annulus-type in a branched cover $(V', J')$ of $(M, J)$ whose boundary lies in a resulting totally real surface $\Sigma'$ which is the union of two embedded Moebius bands. These curves converge to an annulus in the exceptional fibre of the blow-up. The resulting loss of regularity is accounted for by this covering transformation. We point out that already in the case that $(M, J) = \mathbb{C}^2$, the regularity of $N^3$ is at most $C^{(k+1)/2}$ as the example [Hu, paragraph 3] shows. Normal forms of real analytic surfaces $\Sigma$ near complex points and associated geometric transformations are given in [MW]. The problem of finding a Levi-flat hypersurface with prescribed boundary is analogous to the well-known Plateau problem for minimal surfaces in Euclidean three-space, where Hildebrandt [Hi] proved that the surface retains the same Hölder regularity up to the boundary as the boundary condition.

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1 Asymptotics and main result

Choose a connection $\nabla$ on $(M, J)$ with $\nabla J = 0$, and let $\nabla = \nabla^{11} + \nabla^{01}$ be its decomposition into $J$-linear and $J$-antilinear parts. For the projection $TM \rightarrow N\Sigma := TM \mod T\Sigma$
and choice of embedding $N\Sigma \hookrightarrow TM$, we denote by $Q_\sigma$ the image of

$$
(dF, \pi \nabla dF)(\sigma) : T_\sigma \Sigma \to T_m M,
$$

which is a quadratic osculant of $\Sigma$ at $\sigma$. Analogously let

$$
\pi^+ : T^{10}M \to N^+ := T^{10}M \mod (\text{image } \partial F) \hookrightarrow T^{10}M.
$$

Then

$$
T^{10}M \approx (\text{image } \partial F) \oplus N^+
$$

and denote by $\pi^- : T^{10}M \to \text{image } \partial F$ the projection onto the first summand. We denote by $Q^+_{\sigma}$ be the image of the quadratic map

$$
(\partial F, \pi^+(\nabla dF - \pi_\beta \nabla \delta F))(\sigma) : T_\sigma \Sigma \to F^*T^{10}M.
$$

Here, as in [K, Definition 3], the map $\pi_\beta = \pi^- \oplus \pi^+ : TM \to TM$, is defined to be twice of the orthogonal projection onto the real line subbundles $\mathbb{R}^{\pi \pm \nabla \delta F}$ of $\pi^{\pm}T^{10}M$ in the decomposition (6) of $T^{10}M$. These maps may be written as $\pi^\pm_\beta v = e^{i\beta_+}v + v$ for all $v \in \pi^{\pm}T_m M$ and some functions $\beta_\pm : \Sigma \to \mathbb{R}$.

By [K, section 2], the map (7) is holomorphic and therefore $Q^+_{\sigma}$ is a quadratic holomorphic curve in $T_M M$. Let $\sigma$ be a complex point of $\Sigma$. Given the way we defined $j$ in section 0, this is a positive complex point in the terminology of [K]. The requirement that $\sigma$ be elliptic is equivalent to the condition that

$$
e^{-i\beta_+ / 2} \pi^+_{\beta} \nabla \delta F(\sigma) : T_\sigma \Sigma \otimes T_\sigma \Sigma \to \mathbb{R}
$$

is a positive definite quadratic form, whose level sets are ellipses in $T_\sigma \Sigma$, and in this case,

$$
[Q^+_{\sigma} + (0, e^{i\beta_+ / 2} \cdot t)] \cap \{e^{-i\beta_+ / 2} \pi^+_{\beta} \nabla \delta F(\sigma) \leq t\} \hookrightarrow (T^{10}_m M, Q_\sigma) ; \quad t \geq 0
$$

is, by the definitions of $Q$ in (5) and $Q^+$ in (7), a family of disc-type holomorphic curves in $(T_m M, J(m))$ with boundary in $Q_\sigma$.

**Definition.** An elliptic complex point is called general (special) if the eigenvalues of (8) don’t (do) coincide.

Let $D \subseteq \mathbb{C}$ be the unit disc, $\text{Aut}(D)$ be the Moebius group, and

$$
\mathcal{D}(M, \Sigma) := \{f : (D, \partial D) \to (M, \Sigma) : f \in W^{1,p}, \delta F = 0\}/\text{Aut } D,
$$

namely unparametrized $J$-holomorphic curves in $M$ of disc-type with boundary in $\Sigma$. Now $\mathcal{D}(M, \Sigma) \subset W^{1,p}(D, \partial \partial D, M, \Sigma)/\text{Diffeo}(D)$, the latter being the unparametrized smooth discs of class $W^{1,p}$. A point $\sigma \in \Sigma$ may be regarded as a (degenerate) element in $\mathcal{D}(M, \Sigma)$. Let $\mathcal{D}_\sigma(M, \Sigma)$ be the connected component of $\mathcal{D}(M, \Sigma)$ containing $\sigma$.
Example. Let \( \sigma \in \Sigma \) be an elliptic complex point, then
\[
\mathcal{D}_0(T_m M, Q_\sigma) = \text{family } (9)
\]
is a one dimensional submanifold of \( W^{1,p}(D, \partial D; M, \Sigma) \mod \text{Diff}(D) \) that is diffeomorphic to \( \mathbb{R} \), and \( \sigma = \partial \mathcal{D}_\sigma(T_m M, Q_\sigma) \). In addition, the elements of \( \mathcal{D}_0(T_m M, Q_\sigma) \) foliate a real hypersurface of \( T_m M \). Our main result is the following.

Theorem 1. Let \( F : \Sigma^2 \rightarrow (M^4, J) \) be an embedded real surface and \( \sigma \in \Sigma \) be a general elliptic point. Then

a) if \( F \) is \( C^{k,\alpha} \) (resp \( C^\infty \))- smooth, then \( \mathcal{D}_\sigma(M, \Sigma) \) is a \( C^{\frac{k}{2},\frac{\alpha}{2}} \) (resp \( C^\infty \))- smooth one-dimensional submanifold of \( W^{1,p}(D, \partial D; M, \Sigma) / \text{Diff}(D) \) with boundary \( \sigma \in \partial \mathcal{D}_\sigma(M, \Sigma) \)

b) \( T_\sigma \mathcal{D}_\sigma(M, \Sigma) = \mathcal{D}_0(T_m M, Q_\sigma) \).

Remarks. a) By claim b), and since \( \mathcal{D}_0(T_m M, Q_\sigma) \) does, also \( \mathcal{D}_\sigma(M, \Sigma) \) induces near \( \sigma \) a foliation on a real hypersurface, which implies Corollary 2.

b) By elliptic boundary regularity, the individual discs in \( \mathcal{D}_\sigma(M, \Sigma) \setminus \{ \sigma \} \) are \( C^{k,\alpha} \)-smooth up to the boundary. The point of the Theorem is that this regularity persists up to the point \( \sigma \).

c) Inspection of our proof shows that in fact it suffices to assume \( C^{k,\alpha} \) (resp \( C^\infty \))-regularity of the complex structure \( J \) on \( M \).

2 Proper transform of admissible complex points

Let \( F : \Sigma \rightarrow (M, J) \) be a real surface and \( \sigma \in \Sigma \) a complex point with \( \partial F \cap \bar{F} \cap \sigma \setminus \partial F \neq \emptyset \). Such a complex point will be called admissible. The quadratic map (5) admits a complexification which we denote by \( \mathbb{C} \otimes (5) \). By composing with the projection onto \( T_m^0 M \), we obtain the following quadratic map:

\[
\mathbb{C} \otimes T_{\sigma} \Sigma \xrightarrow{\mathbb{C} \otimes (5)} \mathbb{C} \otimes T_m M \cong T_{m}^0 M \oplus T_m^0 M \xrightarrow{\mathbb{R}^{T_m^0 M}} T_{m}^0 M \cong T_m M. \tag{10}
\]

Let \( \omega := \pi^{-1} \partial F(\sigma) \in \Omega^0_{\sigma}(\Sigma) \), then we may represent
\[
\pi^+ \nabla \partial F(\sigma) = a \omega \otimes \omega + b \omega \otimes \overline{\omega} + c \overline{\omega} \otimes \overline{\omega}.
\]

Here, for \( v \in T_{\sigma}^0 \Sigma \) with \( \omega(v) = 1 \), we have \( a = (\pi^+ \nabla^0 \partial F)(v, v) \), \( b = 2(\pi^+ \nabla^1 \delta F)(v, \overline{v}) \), \( c = 2(\pi^+ \nabla^0 \delta F)(\overline{v}, \overline{v}) \). Setting \( \theta = \omega \otimes \overline{\omega} \), and making use of the decomposition (6), the map (10) may be written as

\[
(\omega, a \omega \otimes \omega + b \omega \otimes \theta + c \theta \otimes \theta) : \mathbb{C} \otimes T_{\sigma} \Sigma \rightarrow T_m M. \tag{11}
\]

Also, consider the quadratic map

\[
(dF, \pi^+ \nabla \delta F) : T_{\sigma} \Sigma \rightarrow T_m M. \tag{12}
\]
Its image we will denote by \( Q^\beta_\sigma \). The second component of this map may be written as follows.

\[
(13) \quad \pi^+_{\beta}(\nabla \mathcal{D} F)(\sigma) = e^{i\beta} \tau \omega \otimes \omega + b \omega \otimes \omega + c \omega \otimes \omega.
\]

The complexification of (12) analogous to (10) is represented by

\[
(14) \quad (\omega, e^{i\beta} \tau \omega \otimes \omega + b \omega \otimes \theta + c \theta \otimes \theta) : \mathbb{C} \otimes T_\sigma \Sigma \to T_{mM}.
\]

Also, the eigenvalues of the quadratic form (13) are given by \( 1 \pm |2c/b| \), and for \( b = 0 \), they are \( \pm |c| \). We see that \( \sigma \in \Sigma \) is general elliptic iff \( 0 < |2c/b| < 1 \) iff \( |b/c| > 2 \) since in this case, (13) is positive definite and the eigenvalues don’t coincide. In particular, we have \( c \neq 0 \) in this case, and both maps (11) and (14) are double branched coverings.

In case \( \sigma \in \Sigma \) is an admissible complex point that is not special elliptic, let

\[
(15) \quad \tau : \mathbb{C} \otimes T_\sigma \Sigma \to \mathbb{C} \otimes T_\sigma \Sigma , \quad \tau \circ \tau = \text{id}
\]

be the deck transformation associated to (11). This map is a linear holomorphic involution. With respect to the basis dual to \( \{ \omega, \theta \} \) of \( \mathbb{C} \otimes T_\sigma \Sigma \cong T_{10}^\sigma \oplus T_{01}^\sigma \), the map \( \tau \) may be represented by the matrix

\[
(16) \quad \tau = \begin{pmatrix} 1 & 0 \\ -b/c & -1 \end{pmatrix}.
\]

**Lemma 3.** Let \( \sigma \in \Sigma \) be an admissible complex point which is not special elliptic. Then

\( a \) the inverse image \( (9)^{-1}(Q_\sigma) = L^+_{\sigma} \cup L^-_{\sigma} \) is the union of two totally real planes \( L^\pm_{\sigma} \subset \mathbb{C} \otimes T_\sigma \Sigma \) that intersect transversally in the origin, and \( \tau L^+_{\sigma} = L^-_{\sigma} \)

\( b \) \( \tau \) is a covering map for (14)

\( c \) \( (12)^{-1}(Q^0_\sigma) = (9)^{-1}(Q_\sigma) \).

**Proof.** The embedding \( \mathbb{R} \hookrightarrow \mathbb{C} \) gives rise to a natural embedding \( T_\sigma \Sigma \cong \mathbb{R} \otimes T_\sigma \Sigma \hookrightarrow \mathbb{C} \otimes T_\sigma \Sigma \cong T_{10}^\sigma \Sigma \oplus T_{01}^\sigma \Sigma \) whose image is totally real and by definition, it is contained in \( (9)^{-1}(Q_\sigma) \). Denote this plane by \( L^+_{\sigma} \). Since \( \tau \) is holomorphic, \( L^-_{\sigma} := \tau L^+_{\sigma} \subset (9)^{-1}(Q_\sigma) \) is also totally real. By assumption we have \( |b/c| \neq 2, 0 \), and therefore \( \tau \) has only one eigenspace \( 0 \oplus T_{01}^\sigma \Sigma \). This is transversal to the above embedding of \( T_\sigma \Sigma \), and therefore \( \tau L^+_{\sigma} \cap L^-_{\sigma} \). This proves a). Denote by \( \mathbb{C} \otimes \pi^+ \nabla dF, \mathbb{C} \otimes \pi^+ \nabla \delta F \) the second components of (11), (14). Then we have

\[
(\mathbb{C} \otimes \pi^+ \nabla dF - \mathbb{C} \otimes \pi^+ \nabla \delta F) \circ \tau = \mathbb{C} \otimes \pi^+ \nabla dF - \mathbb{C} \otimes \pi^+ \nabla \delta F.
\]

Since \( \tau \) is a deck transformation for (9), this implies \( \mathbb{C} \otimes \pi^+ \nabla \delta F \circ \tau = \mathbb{C} \otimes \pi^+ \nabla \delta F \). The claim b) now follows from the form (16) of \( \tau \) and (11) above. Part c) follows from part b) and since \( L^+_{\sigma} \subset (12)^{-1}(Q^0_\sigma) \).

\[
\text{QED}
\]
The preceding Proposition describes the linear version of the transformation to which we now subject our surface \( \Sigma \) near \( \sigma \) and which gives rise to an immersed totally real surface \( \Sigma' \). Namely consider

\[(17) \quad (\mathbb{C} \otimes \tau_\sigma \Sigma, 0) \xrightarrow{[\text{10}]} (T_m M, 0) \supset (U, 0) \xrightarrow{\exp_\Sigma} (M, m) \xrightarrow{F} (\Sigma, \sigma).\]

**Proposition 4.** Assume that \( \sigma \in \Sigma \) is an admissible complex point which is not special \( elliptic \). Denote the pullback via (17) of \( F : (\Sigma \cap \exp U, \sigma) \to (M, m) \) by \( F' : (\Sigma', \sigma') \to (\mathbb{C} \otimes \tau_\sigma \Sigma, 0) \) and the pullback of \( (\exp U, J) \subseteq (M, J) \) by \( (V', J') \), where \( V' \subseteq \mathbb{C} \otimes \tau_\sigma \Sigma \) is a neighborhood of \( \Sigma \).

Then

a) \( (V', J') \) is almost complex, and \( \tau \) is \( J' \)-holomorphic: \( d\tau J' = J' d\tau \).

b) If \( U \) is chosen sufficiently small and \( \exp U \cap \Sigma \) is of disc-type, then \( (\Sigma', \sigma') = (\Sigma^+, \sigma^+) \cup (\Sigma^-, \sigma^-) \) is a union of two disc-type surfaces \( \Sigma^\pm \) with distinguished points \( \sigma^\pm \), and \( F^\pm : F|\Sigma^\pm : (\Sigma^\pm, \sigma^\pm) \to (V', 0) \) are totally real embeddings into \( (V', J') \).

In addition, \( \tau \circ F^+ = F^- \), \( I_{\sigma^\pm}^\pm = dF^\pm(T_{\sigma^\pm} \Sigma^\pm) \).

c) If \( F \in C^{\kappa, \alpha} \) (or \( C^\omega \)) then \( F^\pm \in C^{\frac{\kappa}{\kappa + 1}, \frac{\alpha}{\alpha + 1}} \) (or \( C^\omega \)).

**Proof.** Part a) holds since \( J' \) is a pullback by (11), and \( \tau \) a covering transformation thereof. Part b) follows from Lemma 3 and the smooth Weierstrass Preparation Theorem [GG], which also gives the regularity claimed in c), since the problem involves a quadratic singularity.

QED

Next we study the proper transform of \( (\Sigma', \sigma) \). To this effect consider

\[(18) \quad (\hat{V}, \hat{V}_0, \hat{J}) \xrightarrow{\pi} (V', 0, J') \xrightarrow{\hat{F}} (\Sigma', \sigma') \],

where \( \pi \) denotes the blow-up of \( (V', J') \) at \( 0 \in V' \) with the exceptional fibre \( \hat{V}_0 := \pi^{-1}(0) \simeq \mathbb{C}P(1) \). The existence of a blow-up in a not necessarily integrable complex manifold follows from the local existence Theorem III in [NN] for \( J' \)-holomorphic curves in \( (V', J') \) tangent to given complex lines in \( T_0 V', J'(0) \). This result is a direct generalization of the classical Korn-Lichtenstein-Bers-Chern Theorem on existence of isothermal parameters on surfaces. These curves, as is shown in [NN 4.5], admit smooth dependence on the directional datum and may be chosen to fibre a punctured neighborhood of \( 0 \) in \( V' \) and give rise to a blow-up map in (18).

**Lemma 5.** Let \( (\sigma', \Sigma', F', \tau, J') \) be as in Lemma 4. These lift through through the blow-up (18) to \( (\hat{\Sigma}, \hat{F}, \hat{\tau}, \hat{J}) \), where

a) \( (\hat{V}, \hat{J}) \) is almost-complex, and \( \hat{\tau} : \hat{V} \to \hat{V} \) is a \( \hat{J} \)-holomorphic involution: \( d\hat{\tau} \hat{J} = \hat{J} d\hat{\tau} \).

b) \( \hat{\Sigma} = \hat{\Sigma}^+ \cup \hat{\Sigma}^- \) is a union of two surfaces of type \( \Sigma^\pm \# \mathbb{R}P(2) \), namely the Möbius band, with distinguished closed curves \( \Sigma^\pm \subset \Sigma^\pm \) in the class of the soul of the Möbius band, and \( \hat{F}^\pm = \hat{F}|\Sigma^\pm : (\Sigma^\pm, \Sigma^\pm) \to (\hat{V}, \hat{V}_0) \) are totally real embeddings into \( (\hat{V}, \hat{J}) \) with \( \hat{\tau} \hat{F}^\pm = \hat{F}^- \).
c) the complex point $\sigma \in \Sigma$ is elliptic (parabolic, hyperbolic) according to the definition to the definition in [B], [K] iff $C^+ \cap C^- = \emptyset$ (one point, two points), if and only if $|b/c| > 2 (=, <)$.

d) if $F \in C^k,\alpha$ (or $C^\omega$), then $\tilde{F} \in C^{\frac{k}{4},\frac{\alpha}{2}}$ (or $C^\omega$).

**Proof.** a) follows from Proposition 4a). In part b), $\hat{\Sigma}^+$ is the total transform of $\hat{\Sigma} \to V'$ under the blow-up (18). Part c) follows easily from the definitions in [B], [K], which correspond to the cases that there exist no (one, two) complex lines in $\mathbb{C} \otimes T_{\sigma} \Sigma$ which intersect both planes $L^e_{\sigma}$ in (neccesarily distinct) real lines. To see the distinction in terms of $|b/c|$, we introduce coordinates $(z, w) \in T_{\sigma}^{10} \Sigma \oplus T_{\sigma}^{01} \Sigma$ on $\mathbb{C} \otimes T_{\sigma} \Sigma$ and a homogeneous coordinate $[w : z]$ on $P(\mathbb{C} \otimes T_{\sigma} \Sigma) \approx \mathbb{C} P(1) \simeq V_0$. Then by (16), $\hat{\tau}$ acts on $V_0$ as $[w : z]' = -b/c - [w : z]$. In addition, the total transform of the image of

$$T_{\sigma} \Sigma \leftrightarrow \mathbb{C} \otimes T_{\sigma} \Sigma \simeq T_{\sigma}^{10} \Sigma \oplus T_{\sigma}^{01} \Sigma : \zeta \to (\zeta, \bar{\zeta})$$

is given by the unit circle $C = \{[w : z] | = 1\}$. Now clearly $C \cap \hat{\tau} C = \emptyset$ (one point, two points) iff $|b/c| > 2 (= 2, < 2)$. We saw in section 2 that the first case holds iff the point $\sigma$ is elliptic. Part d) follows from the fact that the blow-up $\hat{F}$ of $F'$ retains the regularity of $F'$, which by Proposition 4c) is in $C^{\frac{k}{4},\frac{\alpha}{2}}$. QED

3 An equivariant Riemann-Hilbert boundary problem

Assume now that $\sigma \in \Sigma$ is a general elliptic point and denote by $A \subseteq \hat{V}_0$ the annulus bounded by the distinct simple curves $\partial^+ A$ and $\partial^- A$ in the exceptional fibre $\hat{V}_0$ of $\pi$. It is embedded in $\hat{V}$:

$$\hat{V} = (A, \partial^+ A) \leftrightarrow (\hat{V}, \hat{\Sigma}^\pm).$$

**Proposition 6.** In the above situation, we have the following:

a) The normal Maslov index $\mu(A, \Sigma)$ vanishes

b) $\hat{\tau}(A, \partial^\pm A) = (A, \partial^\mp A)$

c) the quotient $(A, \partial^\pm A) \to (A, \partial^\pm A)/\{1, \hat{\tau}\}$ is diffeomorphic to the disk, and the quotient map is a double branched cover with two branching points, the fixed points of $\hat{\tau}$.

**Proof.** a) The orientation of the two boundary components of $A$ is opposite and modulo sign they contribute the same number to the total Maslov index. The claims b), c) follow from the construction of $\tau$ and $A$, and the fact that the fixed points of $\hat{\tau}$ on $\hat{V}_0$ are $-b/2c$ and $\infty$, both of which lie in $A$ by Lemma 5c). QED

By its construction as a blow-up of $(V', J')$, the almost complex four-manifold $\hat{V}$ is endowed with the real structure of a real plane bundle (namely the tautological bundle) over $\hat{V}_0 \simeq \mathbb{C} P(1)$. Denote this bundle by

$$\hat{V} \xrightarrow{p} \hat{V}_0.$$
We have $T\hat{V}|\hat{V}_0 = $ tautological bundle over $\mathbb{C}P(1)$. We may assume that $p^{-1}(\hat{\Sigma}) \subset p^{-1}(\partial A)$. The connection $V$ induces a decomposition

\begin{equation}
T\hat{V} = T\hat{V}_0 \oplus \hat{V},
\end{equation}

where here $\hat{V}$ stands for the fibres of (20). In addition, the complex structure $J$ has the form

\begin{equation}
J|\hat{V}_0 \times 0 = \begin{pmatrix}
  j & 0 \\
  0 & i
\end{pmatrix}.
\end{equation}

along the base of the bundle (20). Here, $j$ denotes the complex structure on $\hat{V}_0$. Now restricting the bundle (20) to $A$ by setting $\hat{V} := p^{-1}(A)$, we obtain a plane bundle over $A$ with a fibre-subbundle with real one dimensional fibres over $\partial A$:

\begin{equation}
(\hat{V}, \hat{\Sigma}) \rightarrow (A, \partial A).
\end{equation}

The graph of a section $\varphi \in \Gamma(\hat{V}, \hat{\Sigma})$ describes a real surface

\begin{equation}
\text{graph} \, \varphi = \{(a, \varphi(a)); a \in A \} \subset \hat{V}
\end{equation}

whose boundary lies in $\hat{\Sigma} : \partial \text{graph} \, \varphi \subset \hat{\Sigma}$. This surface is $J$-holomorphic if its tangent bundle

\begin{equation}
\text{graph} \nabla \varphi = \{(v, \nabla \varphi \cdot v); v \in TA\}
\end{equation}

is $J$-invariant:

\begin{equation}
\text{graph} \nabla \varphi = J \text{graph} \nabla \varphi \equiv \{(J_{11} + J_{12} \nabla \varphi) v, (J_{21} + J_{22} \nabla \varphi) v); v \in TA\},
\end{equation}

where $J_{ij}$ denote the entries of the matrix of $J$ with respect to the the decomposition (21). We now define the differential operator

\begin{equation}
\overline{\partial} : \Gamma(\hat{V}, \hat{\Sigma} \cap \hat{V}|\partial A) \rightarrow \Omega^{01}(\hat{V}), \quad \overline{\partial} \varphi = \nabla \varphi - K \nabla \varphi,
\end{equation}

where $K \nabla \varphi = (J_{11} + J_{12} \nabla \varphi) \circ (J_{11} + J_{12} \nabla \varphi)^{-1}$. The graphs in $(\hat{V}, J)$ of solutions of $\overline{\partial} \varphi = 0$ are $\hat{J}$-holomorphic curves on which $\hat{J}$ induces the orientation compatible with that of $(A, j)$. We are interested in $\hat{\tau}$-invariant holomorphic curves of annulus-type that are sections of $(\hat{V}, J)$ with boundary in $\hat{\Sigma} = \hat{\Sigma}^{+} \cup \hat{\Sigma}^{-}$ that are graphs over $A$. That is we seek solutions of

\begin{equation}
\overline{\partial} \varphi = 0, \quad \varphi \in \Gamma(\hat{V}, \hat{\Sigma} \cap \hat{V}|\partial A)
\end{equation}

To this end we linearize the problem. Let \[ R := \hat{V}|\partial A \cap T\hat{\Sigma} \hookrightarrow \hat{V}|\partial A, \]
which is a real line-subbundle of \( \overline{V} | \partial A \). Note that the linearization along \( A \times 0 \) of \( \tilde{\tau} \) acts on the pair \( \tilde{\tau} : (\hat{V}, R) \rightarrow (\hat{V}, R) \) and denote the invariant sections of \( \Gamma(\hat{V}, R), \Omega^{\mathfrak{g}_1}(\hat{V}) \) by \( \Gamma_{\mathfrak{g}_1}(\hat{V}, R), \Omega^{\mathfrak{g}_1}(\hat{V}) \). Also let for \( p > 2 \),

\[
L : W^{1,p}(\hat{V}, R) \rightarrow L^p(\Omega^{\mathfrak{g}_1}(\hat{V}))
\]

be the linearization of \( \tilde{\partial} \) along the zero-section \( 0_{\mathfrak{g}} \) of \( \hat{V} \) given by \( L(\psi) = d/dt|_{t=0}\tilde{\partial}(0_{\mathfrak{g}} + t\psi) \).

We have

\[
\tilde{\tau}\text{graph } \varphi = \text{graph } s\varphi,
\]

where \( \tilde{\tau} = (s_1, s_2) \), \( s\varphi = s_2 \circ s_1^{-1} \varphi \), and \( \nabla s\varphi = S\nabla \varphi \). Therefore

\[
d\tilde{\tau}\text{graph } \nabla \varphi = \text{graph } S\nabla \varphi.
\]

Since \( \tilde{\tau} \) is holomorphic, we have

\[
\hat{J}d\tilde{\tau} = d\tilde{\tau}\hat{J}, \text{ and } KS = SK, \text{ therefore } \tilde{\partial}s\varphi = S\tilde{\partial}\varphi.
\]

**Proposition 7.** For \( \psi \in W^{1,p}(\hat{V}, R) \), we have

\begin{enumerate}
  \item[\( a \)] \( L\psi = \nabla \psi + i\nabla \psi \circ j + dJ_{21}(0_{\mathfrak{g}})\psi \circ j \)
  \item[\( b \)] \( L \) is Fredholm
  \item[\( c \)] \( L \) is \( S \) - equivariant : \( LS = dS L \)
\end{enumerate}

**Proof.** Part a) follows from (22) and (25), and part b) follows since \( L \) has the same symbol as the Cauchy-Riemann operator. Part c) is seen as follows.

\[
LS\psi = d/dt|_{t=0} \tilde{\partial}s\varphi = d/dt|_{t=0} (1 - K)\nabla s\varphi = (1 - dK)dS\nabla \psi = dS L \psi.
\]

By the last claim, \( L \) induces a map on the \( \tilde{\tau} \)- invariant sections :

\( L_\tau : W^{1,p}(\hat{V}, R) \rightarrow L^p(\Omega^{\mathfrak{g}_1}(\hat{V})) \). For this operator we have the following.

**Proposition 8.** For the \( \tilde{\tau} \)-equivariant operator \( L_\tau \) we have: \( L_\tau \) is Fredholm, index \( L_\tau = 1 \), and \( L_\tau \) is surjective.

**Proof.** The first claim follows from Proposition 7 c). To verify the second claim, we need to study the following elliptic complex \( E \):

\[
0 \longrightarrow W^{1,p}(\hat{V}, R) \longrightarrow L^p(\Omega^{\mathfrak{g}_1}(\hat{V})) \longrightarrow 0.
\]
By the preceding result, the group generated by \( \{id, \tau\} \) acts on this complex \( E \) and the index of the quotient operator \( L_\tau \) is given by

\[
\text{Index } L_\tau = \frac{1}{2} \sum_{g \in \{1, \tau\}} \text{Lef}(g, E),
\]

where \( \text{Lef}(g, E) \) indicates the Lefschetz number of \( g \):

\[
\text{Lef}(g, E) = \sum_{i=0,1} \text{trace}(g|H^i(E)).
\]

As \( H^0(E) = \ker L \), \( H^1(E) = \text{coker}L \), we have \( \text{Lef}(1, E) = \text{index}L \), and by [HLS, (2)], we have index \( L = \mu + 2 - \sigma - 2g = 0 \), where by Proposition 6 the total Maslov index \( \mu = 0 \), and \( \sigma \) denotes the number of boundary components, which is two, and the genus vanishes. Next note that the fixed points of \( \tau \) are \( \{-1/2\gamma, \infty\} \). By [AS, Theorem 3.1], the Lefschetz number is given by \( \text{Lef}(\tau, E) = \nu(-\frac{1}{2\gamma}) + \nu(\infty) \), where the \( \nu \) involve alternating sums of the traces of the \( \tau \) and are evaluated as follows:

\[
\begin{align*}
\nu(-\frac{1}{2\gamma}) &= \frac{1}{2} (\text{tr} \ \tau | W^{1,p}_{-1/2\gamma}) - \text{tr} \ \tau | \Omega^0_{-1/2\gamma}) = \frac{1}{2} (1 - (-1)) = 1, \\
\nu(\infty) &= \frac{1}{2} (\text{tr} \ \tau | W^{1,p}_{\infty}) - \text{tr} \ \tau | \Omega^0_{\infty}) = \frac{1}{2} (1 - (-1)) = 1.
\end{align*}
\]

This proves the second claim. To prove the third part, we construct a double \( A^d = A \cup \bar{A} \) by identifying the corresponding points on the boundary of \( A \). The double of the bundle \( (V, R) \to (A, \partial A) \) is given by \( V^d = (V, R) \cup (\bar{V}, R) \), where the fibres of the two components over \( \partial A \) are identified via the antiholomorphic reflection in the fibre fixing \( R \). This gives rise to a line bundle \( V^d \to A^d \). The holomorphic involution \( \tau \) extends to a holomorphic bundle endomorphism preserving \( R \): \( \tau^d : (V^d, R) \to (\bar{V}^d, R) \). We denote the quotient bundle by

\[
(V^d_\tau, \tau) := (V^d, R) / \{1, \tau^d\},
\]

giving a line bundle \( (V^d_\tau, \tau) \to (A^d, \partial A) \). The sections of \( V^d_\tau \) correspond to \( \tau^d \)-invariant sections of \( V^d \). Smooth sections of the doubled bundle \( V^d_\tau \) may be identified with pairs of sections of \( V_\tau \), which match up properly in the subbundle \( R \):

\[
W^{1,p}(\tau^d) = \{(u_1, u_2) \in W^{1,p}(V_\tau) \times W^{1,p}(\bar{V}_\tau) : u_1 + u_2 \in \Gamma(\bar{V}, R), i(u_1 - u_2) \in \Gamma(V, R)\},
\]

while integrable sections of the double don’t require any compatibility: \( L^p(\Omega^{0i}(\bar{V}_\tau)) = \{(u_1, u_2) \in L^p(\Omega^{0i}(\bar{V}_\tau)) \times L^p(\Omega^{0i}(\bar{V}_\tau))\} \). Then we have an induced operator

\[
L^d_\tau : W^{1,p}(\tau^d) \longrightarrow L^p(\Omega^{0i}(\bar{V}^d_\tau)).
\]

This admits an \( L^2 \)-adjoint operator

\[
L^{ad}_\tau : W^{1,p}(\tau^d) \longrightarrow L^p(\Omega^{0i}(\bar{V}^d_\tau))
\]
on the adjoint bundles given by

\[ \hat{V}_\tau^{ad} = \Omega(\hat{V}^d) \otimes \Omega(\mathcal{A}_\tau^d) \cong \Omega(\mathcal{V}_\tau^{ad}) \otimes \Omega^{10}(\mathcal{A}_\tau^d). \]

One sees that \( L_\tau^{ad} \) is a \( \bar{\partial} \) - operator and in order to prove that it is injective we compute the Chern class of \( \hat{V}_\tau^{ad} \):

\[ c_1(\hat{V}_\tau^{ad}) = -c_1(\hat{V}^{d}) - c_1(\mathcal{A}_\tau^d). \]

We have

\[ c_1(\hat{V}^{d}) = \mu(\hat{V}, \mathcal{A}_\tau) = \mu(\hat{V}, R) + \mu(\hat{V}, R|\mathcal{A}_\tau) = 0 \]

by Proposition 6 a). Secondly, \( \mathcal{A}_\tau^d \simeq S^2 \), and therefore \( c_1(\mathcal{A}_\tau^d) = 2 \). We conclude that \( c_1(\hat{V}_\tau^{ad}) < 0 \), and therefore \( L_\tau^{ad} \) is injective, which implies that \( L_\tau^{d} \) is surjective, and so is the operator \( L_\tau \). QED

Proof of Theorem 1: By Proposition 8, \( L_\tau \) is surjective and has a one-dimensional kernel in \( W^{1,p} \). The implicit function Theorem [MS, 3.3] implies that the equation (26) admits a one dimensional family of \( \bar{\tau} \)-invariant solutions passing through the solution \( \phi = 0 \), which corresponds to the surface \( (\mathcal{A}_\tau, \partial \mathcal{A}_\tau) \rightarrow (\mathcal{V}, \Sigma) \). Since \( \Sigma \) is totally real, the boundary problem (25) is strictly elliptic, see [V]. The boundary condition \( \Sigma \) being of class \( C^\infty \), we conclude that the solutions of (26), which are of annulus-type, retain this regularity. We now study the blow-down via (18) of this family, which gives rise to a one-dimensional submanifold of what we will now define as \( \mathcal{A}_\tau(V', \Sigma') \). For \( r \in (0, 1) \), let \( D_r \subset D \) denote the disc of radius \( r \) and \( A_r := D \setminus D_r \). Then every domain of annulus-type is conformally equivalent to one \( A_r \), and the moduli space of \( J' \)-holomorphic curves of annulus-type in \( (V', \Sigma') \) is

\[ \mathcal{A}(V', \Sigma') = \{ f \text{ mod } \text{Aut}A_r : f \in W^{1,p}(A_r, \partial A_r ; V', \Sigma'), \bar{\partial} f = 0, r \in (0, 1) \} \]

Let \( \mathcal{A}_\tau(V', \Sigma') \) be those elements of \( \mathcal{A}(V', \Sigma') \) which are invariant under the \( J' \)-holomorphic involution \( \tau \). They descend to \( J \)-holomorphic discs in \( \mathcal{D}(M, \Sigma) \) by way of composing with (10):

\[ \mathcal{A}_\tau(V', \Sigma') \xrightarrow{(10)} \mathcal{D}(V, \Sigma), \]

and its linearization

\[ \mathcal{A}_{d\tau}(T_0V', L') \xrightarrow{d(10)} \mathcal{D}(T_0V, Q_\sigma), \]

where on the left hand side we have \( J'(0) \)-holomorphic annuli which are invariant \( d\tau(0) \). Therefore

\[ T_\tau \mathcal{D}(V, \Sigma) = T_0(\mathcal{A}_\tau(V', \Sigma')) = d(10)\mathcal{A}_{d\tau}(T_0V', L') = \mathcal{D}(T_0V, Q_\sigma). \]

The latter is given by the family (9), and this proves claim b). The claim a) follows from the remarks on regularity above, and the fact that it is preserved under the trunsformations (10) and (18). QED
References


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