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ISOPERIMETRIC INEQUALITIES AND RANDOM WALKS ON QUOTIENTS OF EUCLIDEAN BUILDINGS

ENRICO LEUZINGER

ABSTRACT. Let $\mathcal{B}$ be a Euclidean building and let $G \subset \operatorname{Aut} \mathcal{B}$ be a locally compact unimodular group, which acts strongly transitively on $\mathcal{B}$. We use graphs $\mathcal{G}$, quasi-isometric to $\mathcal{B}$, to study asymptotic properties of quotients $\Gamma\backslash\mathcal{B}$, where Γ is a discrete subgroup of G. If G has Kazhdan's property (T) we show that such quotients satisfy strong isoperimetric inequalities. This yields new examples of graphs with positive Cheeger constant. Moreover, simple random walks on such quotients are shown to be recurrent if and only if Γ is a lattice in G.

1. INTRODUCTION

In this paper we are mainly interested in certain asymptotic or large scale properties of Euclidean (or affine) buildings. These properties are deduced from the additional assumption that there exists a locally compact subgroup G of the automorphism group of the building $\mathcal{B}$, which has Kazhdan's property (T) and acts strongly transitively on $\mathcal{B}$ (see section 5 for definitions).

Since we are interested in the coarse geometry of the simplicial complex $\mathcal{B}$ we may replace the latter by any geodesic metric space (e.g. a graph) quasi-isometric to $\mathcal{B}$. In section 2 we therefore first work in a more general setting and study connected, locally finite graphs $\mathcal{G}$, whose vertex sets can be written as homogeneous spaces $V(\mathcal{G}) = G/B$ (where $B \subset G \subset \operatorname{Aut}\mathcal{G}$ is open and compact in G).

In section 3 we prove the main result (for graphs) under the assumption that G has Kazhdan's property (T). We show that the graph $\mathcal{G}$ and all its quotients by discrete subgroups $\Gamma \subset G$ satisfy strong isoperimetric inequalities. More precisely, we obtain uniform positive lower bounds for the Cheeger constants.

Such graphs are investigated in various contexts (see e.g. [3] for structural properties and [9], [1], [17] and [19] for application in molecular biology, computer science and statistical physics). We remark that many known examples of graphs with positive Cheeger constant are (Gromov) hyperbolic (see e.g. [8]). In contrast, the examples we shall describe below are not hyperbolic.

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A positive isoperimetric constant is equivalent to the existence of a spectral gap for the combinatorial Laplacian. For the above graphs $\Gamma\backslash\mathcal{G}$ the size of that gap is uniform (see section 4). Other applications include recurrence versus transience of simple random walks and the impossibility of bi-Lipschitz embedding of (quotients of) graphs into Hilbert space.

In the final section 5 we eventually discuss a main source of examples of graphs to which all the previous results can be applied. Namely, we show how the graphs studied before naturally occur as quasi-isometric approximations of Euclidean buildings associated to higher rank p-adic Lie groups.

Analogous results for real Lie groups have been obtained in [15].

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2. VERTEX TRANSITIVE GRAPHS

2.1. **General assumptions.** We consider a connected, locally finite graph $\mathcal{G}$ with graph metric $d_{\mathcal{G}}$. Let $V(\mathcal{G})$ resp. $E(\mathcal{G})$ be the set of vertices resp. the set of (undirected) edges of $\mathcal{G}$. Assume that a locally compact, unimodular topological group G acts transitively on $V(\mathcal{G})$. Hence, if $v_0 \in V(\mathcal{G})$ is a chosen base point with stabilizer $B := \mathrm{stab}(v_0)$, the vertex set is a homogeneous space $V(\mathcal{G}) = G/B$. We also assume that B is both open and compact in G.

2.2. **Metric and measure on $\Gamma\backslash G$.** The graph metric $d_{\mathcal{G}}$ defines a left-invariant pseudo-metric on the group G:

$$d_G(g_1, g_2) := d_{\mathcal{G}}(g_1 v_0, g_2 v_0) \qquad (g_1, g_2 \in G)$$

where v_0 is the vertex in $\mathcal{G}$ with stabilizer $\mathrm{stab}(v_0) = B$. Recall that B is assumed to be open and compact in G. Observe that

$$d_G(g_1, g_2) = 0 \iff d_{\mathcal{G}}(g_1 x_0, g_2 x_0) = 0 \iff d_{\mathcal{G}}(x_0, g_1^{-1} g_2 x_0) = 0 \iff g_1^{-1} g_2 \in B.$$

Further let μ be the right invariant Haar measure on the locally compact (unimodular) group G normalized such that $\mu(B) = 1$. Note that μ induces the counting measure on the vertex set $V(\mathcal{G}) = G/B$ of the graph $\mathcal{G}$.

Let $\Gamma \subset G \subset \mathrm{Aut}\,\mathcal{G}$ be a discrete subgroup of G. Then d_G (resp. μ) induces a pseudo-metric d (resp. a measure, also denoted by μ) on the set of Γ–orbits $\Gamma\backslash G$.

The action of the group G on $\Gamma\backslash G$ from the right is transitive and mesure preserving. The *right regular representation $\mathcal{R}$ of G* on $(L^2(\Gamma\backslash G), \mu)$ is defined as

$$(\mathcal{R}(g)f)(x) := f(xg), \quad (x \in \Gamma\backslash G).$$

By the previous observations $\mathcal{R}$ is a unitary representation whose invariant vectors (i.e., $f \in L^2(\Gamma\backslash G)$ with $\mathcal{R}(g)f = f$ for all $g \in G$) are precisely the constants. Thus if $\mu(\Gamma\backslash G) = \infty$ or, equivalently, if the quotient graph has infinitely many vertices, then $\mathcal{R}$ does not have non-zero invariant vectors.

3. Isoperimetric inequalities

3.1. **Kazhdan's property (T) and an isoperimetic inequality for $\Gamma\backslash G$.** Certain group-theoretic properties of locally compact groups are reflected by the so-called property (T) introduced by D. Kazhdan. We refer to [24] for an overview. Property (T) can be defined in many equivalent ways. We shall use the following definition.

Consider a locally compact topological group G which is compactly generated. Choose a compact neighbourhood H of the identity of G. Then G has *Kazdan's property (T)*, if there exists a number $\varepsilon = \varepsilon(G, H) > 0$, depending only on G and H such that the following assertion holds:

If ρ is a unitary representation of G on a Hilbert space $\mathcal{H}$ which does not have non-zero invariant vectors, then, given any $v \in \mathcal{H}$, there is an $h \in H$ such that

$$\varepsilon\|v\| \leq \|v - \rho(h)v\|.$$

(A vector $w \in \mathcal{H}$ is called *invariant* if $\rho(g)w = w$ for all $g \in G$.)

Remark 1. From the point of view of the present paper a main class of examples of groups with property (T) is the following (see section 5 below). Let $\mathbf{G}$ be simple, simply connected linear algebraic group defined over a non-archimedean local field $\mathbb{F}$ and let $G = \mathbf{G}(\mathbb{F})$ be the group of $\mathbb{F}$–rational points of $\mathbf{G}$. Then G has property (T) iff the $\mathbb{F}$–rank of $\mathbf{G}$ is ≥ 2.

Let $\mathcal{G}$, G, d_G, $\Gamma \subset G$, d and μ be as in section 2. In the metric measure space $(\Gamma\backslash G, d, \mu)$ we consider a measurable subset Ω. For any positive real number $M > 0$ we define the *M–boundary* of Ω by

$$\partial_M\Omega := \{x \in \Omega \mid \exists\, y \in \Omega^C,\ d(x,y) \leq M\} \cup \{x \in \Omega^C \mid \exists\, y \in \Omega,\ d(x,y) \leq M\},$$

here Ω^C denotes the complement of Ω in $\Gamma\backslash G$.

We shall also use the following notation. Let H be a compact subset of G. For any $h \in H$ we set $|h| := d_G(e,h)$ and $|H| := \max_{h\in H} |h|$.

Proposition 1. *Assume that the locally compact group G is unimodular and has property (T). Let H be a compact neighbourhood of the identity in G and let $\varepsilon =$*

$\varepsilon(G,H) > 0$ be the corresponding Kazhdan constant and set $\varepsilon_* := \min\{\varepsilon, \frac{1}{2}\varepsilon^2\} > 0$. Then, for any discrete subgroup $\Gamma \subset G$ and any measurable subset Ω of $\Gamma\backslash G$ with $\mu(\Omega) \leq \frac{1}{2}\mu(\Gamma\backslash G)$ the following isoperimetric inequality holds:

$$\varepsilon_* \mu(\Omega) \leq \mu(\partial_{|H|}\Omega).$$

3.1.1. *Proof of Proposition 1. (a): the case of infinite measure, $\mu(\Gamma\backslash G) = \infty$.*
We consider a measurable subset Ω in $\Gamma\backslash G$. Invariant vectors for the right regular representation $\mathcal{R}$ are precisely the constant L^2–functions on $\Gamma\backslash G$. Consequently, since $\mu(\Gamma\backslash G) = \infty$, $\mathcal{R}$ does not have non-zero invariant vectors. Thus the inequality given in the above definition of property (T) applies in particular to the characteristic function χ_Ω of Ω in $L^2(\Gamma\backslash G)$: there is $h \in H$ such that

$$\varepsilon\mu(\Omega) = \varepsilon\|\chi_\Omega\| \leq \|\chi_\Omega - \mathcal{R}(h)\chi_\Omega\| = \int_{\Gamma\backslash G} |\chi_\Omega(x) - \chi_\Omega(xh)| d\mu(x) = \mu(E \cup F),$$

where $E := \{x \in \Gamma\backslash G \mid x \in \Omega, xh \notin \Omega\}$ and $F := \{x \in \Gamma\backslash G \mid x \notin \Omega, xh \in \Omega\}$.

We claim that the sets E and F are contained in the $|h|$–boundary $\partial_{|h|}\Omega$ of Ω. To see this for E pick $x = \Gamma g \in E$, i.e., $x \in \Omega$ and $xh \notin \Omega$. Assume that $d(x, \Omega^C) \geq |h| + 2a$ for some $a > 0$. Then the ball $B_{|h|+a}(x)$ is contained in Ω. Since

$$d(x, xh) = d(\Gamma g, \Gamma gh) = \inf_{\gamma\in\Gamma} d_G(\gamma g, gh) \leq d_G(g, gh) = d_G(e, h) = |h|$$

we have $xh \in \Omega$, which is a contradiction. Hence $E \subset \partial_{|h|}\Omega \subset \partial_{|H|}\Omega$. The proof for F is analogous. This yields $\mu(E \cup F) \leq \mu(\partial_{|H|}\Omega)$ which proves the proposition in the case of infinite measure.

3.1.2. *Proof of Proposition 1. (b): the case of lattices, $\mu(\Gamma\backslash G) < \infty$.*
The idea of the proof is the same as in (a) with the following two modifications. Firstly, the constants are (the only) invariant vectors in $L^2(\Gamma\backslash G)$ for the right regular representation. We thus consider the subspace $L^2_0(\Gamma\backslash G)$ orthogonal to the constant functions. Secondly, we must take the condition $\mu(\Omega) \leq \frac{1}{2}\mu(\Gamma\backslash G) < \infty$ into account.

We now proceed as in (a). We set

$$v := \mu(\Gamma\backslash G) < \infty, \quad a := \mu(\Omega), \qquad b := v - a,$$

and define

$$f : \Gamma\backslash G \longrightarrow \mathbb{R}; \quad f(x) := \begin{cases} b & \text{if } x \in \Omega \\ -a & \text{if } x \notin \Omega. \end{cases}$$

We compute

$$\int_{\Gamma\backslash G} f d\mu = b\mu(\Omega) + (-a)(v - \mu(\Omega)) = 0 \quad \text{and} \quad \int_{\Gamma\backslash G} f^2 d\mu = (a+b)ab = vab < \infty,$$

i.e., $f \in L^2_0(\Gamma\backslash G)$. Since the right regular representation $\mathcal{R}$ of G on $L^2_0(\Gamma\backslash G)$ has no non-zero invariant vectors, property (T) asserts that there is $h \in H$ such that

$$\varepsilon\|f\| \leq \|\mathcal{R}(h)f - f\|.$$

But $\|\mathcal{R}(h)f - f\|^2 = (a+b)^2\mu(E \cup F) = v^2\mu(E \cup F)$ where

$$E := \{x \in \Gamma\backslash G \mid x \in \Omega, xh \notin \Omega\}, \quad F := \{x \in \Gamma\backslash G \mid x \notin \Omega, xh \in \Omega\}$$

and $E \cup F \subset \partial_{|h|}(\Omega) \subset \partial_{|H|}(\Omega)$ as in (a). Thogether with $\frac{1}{2}v \leq b$ the last inequality thus yields

$$\varepsilon^2\mu(\Omega)\frac{1}{2}v^2 \leq \varepsilon^2\mu(\Omega)bv = \varepsilon^2 abv \leq v^2\mu(E \cup F) \leq v^2\mu(\partial_{|H|}\Omega).$$

This completes the proof also in the case of lattices.

3.2. **Cheeger constants for $\Gamma\backslash\mathcal{G}$.** In this section we provide new examples of locally finite, connected graphs with positive Cheeger constants. Such graphs have been investigated for instance by Alon, Milman, Benjamini, Schramm and Cao (see [1], [3] and [8]).

We will use the following notation in connection with graphs. Let $\mathcal{X}$ be a locally finite, connected graph and denote by $V(\mathcal{X})$ (resp. $E(\mathcal{X})$) the vertex set (resp. the edge set) of $\mathcal{X}$. For a subset Ω of $V(\mathcal{X})$ its *edge-boundary* is defined as $\partial\Omega := \{[xy] \in E(\mathcal{X}) \mid x \in \Omega, y \notin \Omega\}$. The cardinality of a subset L of $\mathcal{X}$ is denoted by $|L|$. The *Cheeger* or *isoperimetric constant* of the graph $\mathcal{X}$ is

$$h(\mathcal{X}) := \inf_{|\Omega| \leq \frac{1}{2}|V(\mathcal{X})|} \frac{|\partial\Omega|}{|\Omega|},$$

where Ω runs through all non-empty finite subsets of $V(\mathcal{X})$ with $|\Omega| \leq \frac{1}{2}|V(\mathcal{X})| \leq \infty$.

The following lemma compares M-boundaries and edge-boundaries of graphs.

Lemma 1. *Let $\mathcal{X}$ be a graph with bounded geometry (i.e., there exists a positive constant $D > 0$ such that for every vertex $x \in V(\mathcal{X})$, $\deg(x) \leq D$). Then for any $M > 0$ there is a constant,* const (D, M), *depending only on D and M such that for any finite subset $F \subset V(\mathcal{X})$ holds*

$$|\partial_M F| \leq \text{const}\,(D, M)|\partial F|.$$

Proof. We claim that for any edge $[zy] \in \partial F$ (i.e. $z \in F$, $y \in F^C = \mathcal{X} \setminus F$) the number of vertices $x \in F$ with $d_\Lambda(x, y) \leq M$ is uniformly bounded. In fact, let $x \in F$ be such a vertex of $\mathcal{X}$, then there is a shortest path from z to x of length $d(z, x) = d(y, x) - 1 \leq M - 1$. But the number of such path is bounded by D^{M-1} which proves the lemma. $\square$

Theorem 1. *Let $\mathcal{G}$ be a connected, locally finite graph. Assume that a locally compact, unimodular topological group G acts transitively on the vertex set $V(\mathcal{G})$ with open and compact stabilizers. If G has property (T), then there exists a universal constant $c(G) > 0$ which depends only on G such that for any discrete subgroup $\Gamma \subset G$ the isoperimetric constant of the quotient graph $\Gamma\backslash\mathcal{G}$ satisfies*

$$h(\Gamma\backslash\mathcal{G}) \geq c(G) > 0.$$

3.2.1. *Proof of Theorem 1.* We pick a finite (test)set of vertices, $\tilde{\Omega} \subset V(\Gamma\backslash\mathcal{G})$ (with $|\tilde{\Omega}| \leq \frac{1}{2}|V(\Gamma\backslash\mathcal{G})|$). Recall that the vertex set of $\Gamma\backslash\mathcal{G}$ can be identified with the locally homogeneous space $\Gamma\backslash G/B$. Let $\pi : \Gamma\backslash G \longrightarrow \Gamma\backslash G/B$ be the canonical projection. If we write $\tilde{\Omega} = \{\Gamma g_1 B, \ldots, \Gamma g_n B\}$ then $\Omega := \pi^{-1}(\tilde{\Omega}) = \bigcup_{i=1}^{n} \Gamma g_i B$ is a measurable subset of $\Gamma\backslash G$. By the definition of the measure μ and the normalization $\mu(B) = 1$ we have $\mu(\Omega) = n = |\tilde{\Omega}|$.

We also have $\pi(\partial_{|H|}\Omega) \subseteq \partial_{|H|}\tilde{\Omega}$. To see this recall that

$$\partial_{|H|}\Omega = \{x \in \Omega \mid \exists y \in \Omega^C, d(x,y) \leq |H|\} \cup \{x \in \Omega^C \mid \exists y \in \Omega, d(x,y) \leq |H|\}$$

Hence given $\pi(x) \in \pi(\partial_{|H|}\Omega)$ with, say, $x \in \Omega$ there is $y \in \Omega^C$ such that $d(x,y) \leq |H|$. Thus $\pi(y) \in \pi((\pi^{-1}(\tilde{\Omega}))^C) \subseteq \tilde{\Omega}^C$ and by definition of the pseudo-metric $d(\pi(x), \pi(y)) \leq d(x,y) \leq |H|$.

Together with Proposition 1 and Lemma 1 the above arguments yield, for $\varepsilon = \varepsilon(G,H)$,

$$\begin{aligned} \varepsilon|\tilde{\Omega}| = \varepsilon\mu(\Omega) \leq \mu(\partial_{|H|}\Omega) \leq \mu(\pi^{-1}(\pi(\partial_{|H|}\Omega))) \leq \\ \leq \mu(\pi^{-1}(\partial_{|H|}\tilde{\Omega})) = |\partial_{|H|}\tilde{\Omega}| \leq \mathrm{const}|\partial\tilde{\Omega}|. \end{aligned}$$

Since the set $\tilde{\Omega}$ has been chosen arbitrarily we conclude from the above inequality and the definition of the isoperimetric constant that

$$h(\Gamma\backslash\mathcal{G}) \geq \frac{\varepsilon(G,H)}{\mathrm{const}(G,|H|)} > 0,$$

which finishes the proof. □

Remark 2. Notice that for graphs with bounded geometry "to have a positive Cheeger constant" is a property that is invariant under quasi-isometries (see e.g. [25], Theorem 4.7).

4. Applications

4.1. **Spectral gaps.** Let $\mathcal{X}$ be a locally finite, connected graph with bounded geometry (i.e., the vertex degrees are uniformly bounded). We define the *Laplace* *operator* of $\mathcal{X}$ on L^2-functions on the vertex set of $\mathcal{X}$ by the following formula: if $f : V(\mathcal{X}) \longrightarrow \mathbb{R}$ then

$$\Delta f(x) := \deg(x) f(x) - \sum_{y \sim x} f(y).$$

We also set $\lambda_0(\mathcal{X}) := \inf \operatorname{spec} \Delta$ and $\lambda_1(\mathcal{X}) := \inf(\operatorname{spec} \Delta \setminus \{0\})$ (see also [10], [14] and [17]).

Theorem 2. *Let $\mathcal{G}$ be a connected, locally finite graph. Assume that a locally compact, unimodular group G acts transitively on the vertex set $V(\mathcal{G})$ with open and compact stabilizer. If G has property (T), there is a constant $c_*(G) > 0$ (which depends only on G) such that for any discrete subgroup $\Gamma \subset G$ the following dichotomy holds:*

(a) Γ *is a lattice in* $G \iff \lambda_0(\Gamma\backslash\mathcal{G}) = 0 < c_*(G) \leq \lambda_1(\Gamma\backslash\mathcal{G})$;

(b) Γ *is not a lattice in* $G \iff 0 < c_*(G) \leq \lambda_0(\Gamma\backslash\mathcal{G})$.

Proof. Set $D := \max_{x \in \mathcal{G}} \deg(x)$. Then one has the following inequalities (see [1], [10], [14], [17]):

$$\frac{h(\Gamma\backslash\mathcal{G})^2}{2D} \leq \lambda_i(\Gamma\backslash\mathcal{G}) \leq 2h(\Gamma\backslash\mathcal{G}),$$

where $i = 0$ if $\mu(\Gamma\backslash\mathcal{G}) = \infty$ and $i = 1$ if $\mu(\Gamma\backslash\mathcal{G}) < \infty$. Theorem 2 is thus a consequence of Theorem 1. □

4.2. **Simple random walks on quotients of graphs.** Let $\mathcal{X}$ be a connected, locally finite graph. The *simple random walk* on $\mathcal{X}$ is the Markov chain with transition probabilities

$$p(x, y) = \begin{cases} 1/\deg(x), & \text{if } y \sim x \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, the simple random walk is *recurrent* (resp. *transient*), if the expected number of visits to y of the walk starting at x is infinite (resp. finite). We refer to [25] for precise definitions and a detailed discussion.

Theorem 3. *Let $\mathcal{G}$ be a connected, locally finite graph. Assume that a locally compact, unimodular topological group G acts transitively on the vertex set $V(\mathcal{G})$ with open and compact stabilizer and that G has property (T). Let Γ be a discrete subgroup of G. Then the simple random walk on $\Gamma\backslash\mathcal{G}$ is recurrent if and only if Γ is a lattice in G.*

Proof. If Γ is *not* a lattice the graph $\Gamma\backslash\mathcal{G}$ is infinite, has bounded geometry and by Theorem 1 its isoperimetric constant satisfies $h(\Gamma\backslash\mathcal{G}) \geq c > 0$. It then follows from [14], Theorem 2.4 that the simple random walk on $\Gamma\backslash\mathcal{G}$ is transient (i.e., not recurrent).

If Γ is a lattice, i.e., $\mu(\Gamma\backslash\mathcal{G}) < \infty$, then the quotient graph $\Gamma\backslash\mathcal{G}$ is finite and consequently the simple random walk is recurrent (see e.g. [25], 1.17 (b).) □

4.3. **No bi-Lipschitz embeddings into Hilbert spaces.** A positive isoperimetric constant of a graph implies exponential growth (see e.g. [25], p.112). The volume growth in a Euclidean space is polynomial. Hence there is no map $\Gamma\backslash\mathcal{G} \longrightarrow \mathbb{R}^n, n \in \mathbb{N}$, of finite distorsion. The following theorem asserts that this remains true even if one allows $n = \infty$.

Recall that a *bi-Lipschitz embedding* of a metric space (X, d_X) into a metric space (Y, d_Y) is a map $f : X \longrightarrow Y$ such that for some $C > 0$ and all $x, y \in X$,

$$C^{-1} d_X(x, y) \leq d_Y(f(x), f(y)) \leq C d_X(x, y).$$

Theorem 4. *Let $\mathcal{G}$ be a locally finite graph which satisfies the assumptions of Theorem 1. Let Γ be a discrete subgroup of G which is* not *a lattice (i.e., the graph $\Gamma\backslash\mathcal{G}$ is an infinite graph). Then there is no bi-Lipschitz embedding of $\Gamma\backslash\mathcal{G}$ into a Hilbert space.*

Proof. From Theorem 1 we know that $h(\Gamma\backslash\mathcal{G}) > 0$. A result of Benjamini and Schramm (Theorem 1.5 in [3]) then asserts that the infinite graph $\Gamma\backslash\mathcal{G}$ contains a binary tree which is bi-Lipschitz embedded. Bourgain [5] proved that a binary tree does not have a bi-Lipschitz embedding into a Hilbert space. □

5. Bruhat-Tits buildings and graphs

In this section we discuss what we consider as the most interesting class of examples to which the results of the previous sections apply.

5.1. **Euclidean buildings.** For convenience we briefly review in an informal way some properties of buildings. We follow the book of Brown [7] to which we also refer for more details.

An *affine Coxeter group* is a group of isometries of a Euclidean space generated by reflections in affine hyperplanes belonging to an invariant, locally finite set $\mathcal{H}$ of affine hyperplanes. The complex obtained by the partition of the space by the elements of $\mathcal{H}$ is called an *affine Coxeter complex.* The maximal cells are called *chambers.* They all have the same dimension, and any two of them can be joined by a gallery (i.e., a sequence of chambers such that any two consecutive ones have a common codimension one face). Every codimension one face belongs to exacly two chambers (i.e., the Coxeter complex is *thin*).

An *affine Bruhat-Tits building* is a locally finite (poly-) complex $\mathcal{B}$ together with a collection of subcomplexes called *appartments* satisfying the following properties:

(1) Every appartment is an affine Coxeter complex.

(2) For each pair of cells $c_1, c_2 \in \mathcal{B}$ there exists an appartment containing it.

(3) If $\mathcal{A}_1, \mathcal{A}_2$ are two appartments containing c_1 and c_2 then there exists an isomorphism $\phi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ which stabilizes c_1 and c_2 pointwise.

A group G of automorphisms of a Euclidean building $\mathcal{B}$ acts *strongly transitively* if, given any appartment $\mathcal{A}_1$ and a chamber $c_1 \in \mathcal{A}_1$ and an appartment $\mathcal{A}_2$ and a chamber $c_2 \in \mathcal{A}_2$, there exist $g \in G$ such that $g\mathcal{A}_1 = \mathcal{A}_2$ and $gc_1 = c_2$.

The building $\mathcal{B}$ can be endowed with a metric $d_{\mathcal{B}}$ in such a way that each appartment is isometric to a Euclidean space and such that $(\mathcal{B}, d_{\mathcal{B}})$ is a CAT(0) space, i.e., $\mathcal{B}$ non-positively curved in the sense of Alexandrov (see e.g. [7],[6]). For that reason $\mathcal{B}$ is also called a *Euclidean building.*

5.2. **The dual graph of $\mathcal{B}$.** In this section we introduce a graph which is quasi-isometric to the building $\mathcal{B}$. More precisely, let the *dual graph* $\mathcal{B}^*$ of the building $\mathcal{B}$ be the graph whose vertices are the maximal simplices (chambers) of $\mathcal{B}$ and whose edges are defined as follows: Two vertices c_1 and c_2 are joined by an edge iff the chambers c_1 and c_2 are adjacent in the complex $\mathcal{B}$. We endow $\mathcal{B}^*$ with the graph metric $d_{\mathcal{B}^*}$. One can realize $\mathcal{B}^*$ geometrically in $\mathcal{B}$ by taking the barycenters of the chambers and the (minimal) geodesic segements joining them in $\mathcal{B}$ as edges. Note that $C^{-1}d_{\mathcal{B}} \mid_{\mathcal{B}^*} \leq d_{\mathcal{B}^*} \leq C d_{\mathcal{B}} \mid_{\mathcal{B}^*}$ for some constant $C > 0$.

Lemma 2. *Let G be a group acting strongly transitively on a building $\mathcal{B}$. Fix an appartment $\mathcal{A}$ and a chamber $c_0 \in \mathcal{A}$. We set $B := \{g \in G \mid gc_0 = c_0 \text{ pointwise}\}$. The graph $\mathcal{B}^*$ has bounded geometry (i.e., the degree of any vertex is uniformly bounded). The vertex set $V(\mathcal{B}^*)$ is the homogeneous space G/B*

Proof. By assumption the group G acts strongly transitively on $\mathcal{B}$ and $\mathcal{B}$ is a locally finite (poly-)simplicial complex. By definition the stabilizer of a vertex is the same as the stabilizer of a chamber and hence the lemma. □

Remark 3. As another graph quasi-isometric to $\mathcal{B}$ one may take the 1-skeleton $\mathcal{B}^{(1)}$. The vertex set of $\mathcal{B}^{(1)}$ is the set of zero-simplices. It is known that the only buildings which admit a vertex transitive action of a locally compact group G are those of type $\tilde{A}_n$ (see [11]). For $n \geq 2$ such a group G has property (T) (see [12], [20], [26] for $n = 2$ and [13], [23] for $n \geq 3$).

5.3. **Buildings and Kazhdan's property (T).** Let $\mathbb{F}$ be a non-archimedean local field (i.e., a finite extension either of the the p-adic numbers $\mathbb{Q}_p$ or of a formal power series field over a finite field). Let $\mathbf{G}$ be a simple, simply connected linear algebraic group defined over $\mathbb{F}$ and let $G = \mathbf{G}(\mathbb{F})$ be the group of $\mathbb{F}$–rational points of $\mathbf{G}$. Let $\mathbf{S} \subset \mathbf{G}$ be a maximal $\mathbb{F}$–split torus, i.e., an algebraic subgroup which is $\mathbb{F}$–isomorphic to $(\mathbb{F}^{\times})^r$ for some $r \in \mathbb{N}$, where $\mathbb{F}^{\times}$ is the multiplicative group of non-zero elements of $\mathbb{F}$. Any two such tori are conjugate and r is called the $\mathbb{F}$*–rank* of $\mathbf{G}$ (or G). It is well-known (see [7]) that G has a so-called BN-pair so that the associated building is an affine Bruhat-Tits building of dimension r. Moreover G acts strongly transitively on (the complete appartment system of) the building. As an excellent introduction to the example $\mathbf{G} = \mathbf{PGL}_n(\mathbb{Q}_p)$ we recommend [22].

The crucial fact now is that these groups G have property (T) if they are of higher rank. In fact we have (see e.g. [13]) the

Proposition 2. *Let* $\mathbf{G}$ *be a simple simply connected linear algebraic group defined over* $\mathbb{F}$ *and let* $G = \mathbf{G}(\mathbb{F})$ *be the group of* $\mathbb{F}$*–rational points of* $\mathbf{G}$*. Then* G *has Kazdan's property (T) if and only if* $\mathrm{rank}_{\mathbb{F}}(G) \geq 2$.

Remark 4. A group G as in Proposition 2 always contains cocompact lattices (see [4]). For a class of interesting discrete subgroups (generalized Schottky groups), which are not lattices, see for instance [2].

Theorem 5. *Let* $\mathbf{G}$ *be a simple simply connected linear algebraic group defined over* $\mathbb{F}$ *and let* $G = \mathbf{G}(\mathbb{F})$ *be the group of* $\mathbb{F}$*–rational points of* $\mathbf{G}$*. Assume that* $\mathrm{rank}_{\mathbb{F}}(G) \geq 2$*. Let* $\mathcal{B}$ *be the Euclidean building associated to* G *and let* $\mathcal{B}^*$ *be the dual graph of* $\mathcal{B}$*. Then there are constants* $c(G) > 0, c^*(G) > 0$*, such that for any discrete subgroup* $\Gamma \subset G$ *the following assertions hold.*

(1) *The isoperimetric constant satisfies* $h(\Gamma\backslash\mathcal{B}^*) \geq c(G) > 0$.

(2) *The bottom* λ_0 *of the spectrum of the Laplace operator on* $\Gamma\backslash\mathcal{B}^*$ *is zero iff* Γ *is a lattice. If* Γ *is not a lattice, then there is a uniform gap:* $\lambda_0(\Gamma\backslash\mathcal{B}^*) \geq c^*(G) > 0$.

(3) *The simple random walk on* $\Gamma\backslash\mathcal{B}^*$ *is recurrent iff* Γ *is a lattice.*

(4) *If* Γ *is not a lattice, there is no bi-Lipschitz embedding of* $\Gamma\backslash\mathcal{B}^*$ *into a Hilbert space.*

Proof. Assertions (1), (2), (3) and (4) are immediate consequences of Proposition 2 and Theorem 1, 2 ,3 and 4 respectively. $\square$

Remark 5.

1. Analoguous results for locally symmetric spaces associated to real semisimple Lie groups with property (T) are proved in [15].

2. Explicit examples of Bi-lipschitz embeddings of locally finite trees into buildings are constructed in [16].

3. Spectral properties of random walks on $\mathcal{B}^*$ and $\mathcal{B}^{(1)}$ for $\tilde{A}_n$ buildings are also studied in [21].

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