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Explicit equivariant compactification and Riemann-Roch for algebraic groups

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Ai miei genitori,
e a Raffaella.

Abstract

The main goal of this work is to get rid of the commutativity hypothesis in the study of an equivariant projective completion of a connected algebraic group initiated by J.-P. Serre in [29] and continued by other autors, such as J. Knop and H. Lange (see [14]) and G. Wüstholz (see [37]). In the first chapter, we start by describing Chevalley's Theorem on the structure of an algebraic group G as a principal fibre bundle over an abelian variety A. This is the starting point of our work. Successively, we show how Serre's equivariant completion (which consists in completing the fibres of G over A) can be extended to the noncommutative groups. In chapter 2, we show how to construct a very ample invertible sheaf on the completion $\overline{G}$, and so that $\overline{G}$ is in fact a projective variety. In the third chapter, we finally do some computations: in particular we show that, once the results of the first two chapters are at place, many properties which were up to now proved for the commutative groups hold also in the general case. The most important for us are a Vanishing Theorem and a Riemann-Roch Theorem for the cohomology of invertible sheaves (for the commutative version, see [37]). Both this results are seen to follow from the corresponding properties of abelian varieties, and the rest of the third chapter is based upon them. In the last chapter, we switch to a purely affine setting, and we show how the results on *linear* algebraic groups contained in V.L. Popov's work [24] mirror some of the results in the third chapter of this work.

Riassunto

L'obiettivo principale di questo lavoro è di sbarazzarsi dell'ipotesi della commutatività nello studio di un completamento equivariante e proiettivo di un gruppo algebrico connesso, studio iniziato da J.-P. Serre in [29] e proseguito da altri autori, quali ad esempio J. Knop e H. Lange (si veda [14]) e G. Wüstholz (si veda [37]). Nel primo capitolo iniziamo con la presentazione del Teorema di Chevalley sulla struttura di un gruppo algebrico G come fibrazione principale su di una varietà abeliana A. Questo Teorema è il punto di partenza del nostro lavoro. Successivamente, dimostriamo come il completamento equivariante di Serre (che consiste nel completare le fibre di G sopra A) può essere esteso ai gruppi non commutativi. Nel secondo capitolo mostriamo come costruire un fascio invertibile molto ampio sul completamento $\overline{G}$, che risulta quindi essere una varietà proiettiva. Nel terzo capitolo, eseguiamo finalmente qualche calcolo. In particolare dimostriamo che, non appena i risultati dei primi due capitoli sono disponibili, molte proprietà che fino ad ora erano state dimostrate solo nel caso commutativo possono essere estese al caso generale: le più importanti per noi sono un teorema di annullamento e un teorema di Riemann-Roch per la coomologia dei fasci (per il caso commutativo, si veda [37]). Tali risultati seguono entrambi dalle proprietà corrispondenti delle varietà abeliane, ed il resto del terzo capitolo è basato su di essi. Nell'ultimo capitolo passiamo a uno scenario puramente affine, e mostriamo come i risultati sui gruppi algebrici *lineari* contenuti nel lavoro [24] di V.L. Popov rispecchiano alcuni dei risultati contenuti nel terzo capitolo di questo lavoro.

Contents

Introduction

In 1950, in a talk at the C.N.R.S. *Colloque d' Algèbre et Théorie des Nombres* (see [34]), André Weil made an interesting remark on the algebraic groups which are obtained by extending an abelian variety with a product of additive and multiplicative groups, i.e., on those groups G which appear in a short exact sequence

$$0 \longrightarrow \mathbb{G}_a^{\ell_a} \times \mathbb{G}_m^{\ell_m} \longrightarrow G \longrightarrow A \longrightarrow 0$$

where ℓ_a resp. ℓ_m are positive constants and A is an abelian variety. He noticed that, since G is a fibering over the complete variety A, one can get a completion $\overline{G}$ of G by embedding the fibre $\mathbb{G}_a^{\ell_a} \times \mathbb{G}_m^{\ell_m}$ in a product $\left(\mathbb{P}^1\right)^{\ell_a+\ell_m}$ of projective spaces and glueing the completed fibers again over A. As we shall see in the following, the groups $\mathbb{G}_a$ and $\mathbb{G}_m$ act on $\mathbb{P}^1$, and this induces an action of G on $\overline{G}$, so that in fact one obtains an equivariant completion of G.
The idea of considering the algebraic groups as fibre bundles was already present in F. Severi's book [30], from which Weil took inspiration.
In the course of the same talk, Weil conjectured that an exact sequence as above exists for any commutative algebraic group, and so that one can complete any commutative algebraic group this way. Shortly after, this conjecture was proved to be right: first, by C. Chevalley, who proved it for commutative algebraic groups and then by M. Rosenlicht and I. Barsotti, who independently proved that for any connected algebraic group there exist an abelian variety A and a short exact sequence

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0 \quad ,$$

where L is the largest connected *linear* algebraic subgroup of G. This fact has the consequence that Weil's recipe for the construction of a completion $\overline{G}$ of an algebraic group G is valid for *any* connected group, as we shall see later.

But let us step back to the commutative groups. These objects have become a central tool in modern transcendental number theory, in particular due to the behaviour of their exponential functions. In transcendence proofs involving algebraic groups, one usually needs some data on the geometry, such as the dimension of a projective embedding or the degree of the translation operators, and it was probably J.-P. Serre who remarked how well-suited Weil's idea is in this context: it is not by chance that *Quelques proprietés des groupes algébriques commutatifs* ([29]) was published as an appendix in a book on transcendental number theory. In this note, Serre showed that $\overline{G}$ admits a very ample divisor, and so that is in fact an equivariant projective completion (or *equivariant compactification*) of G, i.e., a projective variety with an action of G which extends the translation on the group. Furthermore he showed that the expliciteness of the construction makes it possible to collect concrete data on the group.

After Serre, many other authors followed this idea for the completion of a group: in particular it played an important role in the multiplicity estimates of G. Wüstholz (see [35]), and successively in the proof of his Analytic Subgroup Theorem (see [36]), a very deep result on the exponential function of the commutative algebraic groups which has many applications in transcendental number theory.

Another important contribution in this context was given by G. Faltings and G. Wüstholz in [5], where the language of sheaves replaced Serre's use of the divisors. Their idea for the construction of a very ample invertible sheaf on the completion will be of great interest to us, since it admits an immediate application also in our setting. We shall also be inspired by F. Knop and H. Lange's paper [14], where the sheaves on the completion are constructed as quotients of geometric vector bundles.
In [37], among other things, Wüstholz computed the cohomology and the Euler characteristic of a very ample line bundle on $\overline{G}$. This made it possible to give explicit bounds for the projective embedding. Another work to which we shall refer is Lange's [17], where it is shown that translation on the completion $\overline{G}$ can be defined locally by quadratic forms.

The goal of this work will be to get rid of the commutativity hypothesis in the aforementioned completion (which we shall call "Serre's completion" from now on, although it has many fathers), in order to extend some of its applications to the noncommutative case. Let us briefly outline how we shall proceed.

In the first chapter, we see how Serre's equivariant completion can be extended to the general case of a connected algebraic group. We begin by introducing the theorem on the fibre bundle structure of a group (which is known today as Chevalley's Theorem, although Chevalley did not publish his proof). This result is so important for us that it seemed appropriate to devote a whole section to it. In the second section, after an excursus on quotients, we study the fibre bundle structure of an algebraic group from a local point of view. An algebraic group G is always a principal fibre bundle over a quotient G/H by an algebraic subgroup and so, as shown by Chevalley's Theorem, over an abelian variety A. But, unlike the commutative groups, the noncommutative algebraic groups cannot be assumed to be locally trivial over A. Luckily, they still possess some local structure of this kind. Indeed we shall see, following [27], that they satisfy local isotriviality, a weakening of local triviality which will be sufficient for our purposes. An immediate application of this fact will be a proof of the quasiprojectivity of the algebraic

groups.
Finally, in the last section of the first chapter, we construct the equivariant completion of a connected algebraic group G. That is, we show how to embed G in a complete G-variety $\overline{G}$ in a way which is compatible with the left translation of G on itself. This is obtained by means of an associated bundle. Hence, we spend some words on associated bundles, in a slightly more general fashion than what we shall need. Namely, we show how to construct a G-variety $G_H(X)$ over G/H with fibre X out of a variety X which admits an action of an algebraic subgroup H of G. If X is an equivariant completion $\overline{L}$ of the "linear part" L in Chevalley's fibering, its associated bundle $\overline{G} = G_L(\overline{L})$ will be an equivariant completion of G.
Since in our main existence proof we make use of Galois coverings, we include also some results on the action of a *finite* group on a variety. In particular we shall see how such an action can be characterized by means of Galois cohomology.

The aim of the second chapter is to prove the projectivity of the completion constructed in Chapter 1 (and, more in general, of a variety $G_L(X)$ constructed out of a projective L-variety X, where L is as above). For this purpose, in Section 2 we show how to construct a sheaf $G_L(\mathcal{F})$ on $G_L(X)$ out of an L-linearized sheaf $\mathcal{F}$ on X (and, as a special case, how a linear representation of L gives rise to a vector bundle on the abelian variety A). This construction, which generalizes both the taking of a quotient of a geometric vector bundle in [14] and the glueing procedure for sheaves adopted in [5], can be formalized in an elegant way by means of descent theory. The first section of this chapter is therefore devoted to a brief summary on the theory of faithfully flat descent of coherent modules. In particular, we shall see how a linearization of a sheaf on a principal fibre bundle gives rise to a descent datum (and vice versa).
In the third and final section of Chapter 2 we show how to obtain ample and very ample invertible sheaves on $G_L(X)$. As we prove at the beginning of the section, this leads to consider sheaves of the form $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$, where $\mathcal{L}$ is an L-linearized sheaf on the L-variety X, p is the projection $G_L(X) \to A$ and $\mathcal{L}_0$ is a (very) ample invertible sheaf on A. It is quite easy to prove the *existence* of an ample line bundle of this kind on $G_L(X)$, but in order to be able to control the dimension of the projective embedding some new ideas are needed. These are provided by G. Faltings and G. Wüstholz in [5], where they show that $G_L(X)$ can be embedded in the projective space bundle $\mathbb{P}(p_*G_L(\mathcal{L}))$, which they show to be a projective variety. The latter result is obtained by means of a filtration of the vector bundle $p_*G_L(\mathcal{L})$ on the abelian variety A. In [5], the filtration is constructed explicitely. But, as remarked by Knop and Lange in [14], this is not necessary: from a result of S. Mukai (see [20]) follows that a homogeneous vector bundle on an abelian variety always admits such a filtration. Hence, the proof of the projectivity of $G_L(X)$ can be reduced to the verification that $p_*G_L(\mathcal{L})$ is homogeneous. We shall adopt this approach, since it also works if G is not commutative.

In the third chapter, we obtain some more results on the structure of the variety $G_L(X)$ constructed in Chapter 2 out of a (projective) L-variety X, and so in particular on the equivariant projective completion $\overline{G} = G_L(\overline{L})$. We start, in Section 1, by computing the cohomology of a line bundle of the type $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$ (see above). The main result here is a vanishing theorem for the cohomology of such a sheaf, which extends Wüstholz's result from [37] to the noncommutative case. In its proof, we shall project the sheaf on the abelian variety $A = G/L$, and successively apply the corresponding result for abelian varieties (which can be found in [21]). Here, an important role is again played by the filtration of $p_*G_L(\mathcal{L})$ from Chapter 2. An analoguous method will allow us to compute the Euler characteristic of $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$, and so to give bounds for the dimension of a projective embedding of $G_L(X)$.

In Section 2, we illustrate two more applications of the vanishing theorem: combining it with some ideas of Mumford (see [22]), we find conditions under which a line bundle $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$ defines a projectively normal embedding of the variety $G_L(X)$, resp. a projective embedding where the homogeneous ideal of $G_L(X)$ is generated by polynomials of degree 2 (in this case, one says that $G_L(X)$ is *cut out by quadrics*).

In Section 3, we repeat some work of Lange, slightly adapted to our noncommutative setting, in order to show that on a connected algebraic group translation can be defined locally by quadratic forms. Also in this case, the proof makes use of the corresponding result on an abelian variety (taken from Lange and Ruppert's [18]).

Finally, in the last section, we resume some of the results obtained up to this point, and we relate them to some explicit compactifications of the linear part L of G.

The fourth chapter of this work is based on [24], a recent work by V.L. Popov (still a work in progress, according to the Author, who plans to extend his results to the groups which are defined over a field of positive characteristic). It can be read independently of the other three, since the methods here are completely different.

In [24], a conjecture of D. E. Flath and J. Towber (see [6]) on the structure of the affine coordinate ring of a *reductive* algebraic group is proved for the *semisimple* groups. As a consequence, Popov is able to give an explicit way for constructing a presentation of the affine coordinate ring of a semisimple algebraic group by generators and relations out of the monoid of its dominant weights, and so a new affine embedding of the group. The interesting fact for us is that the ideal corresponding to the group in this embedding is generated by (inhomogeneous) polynomials of degree 2, and so Popov's result mirrors what we showed for a projective embedding of a (completed) algebraic group in Chapter 3, Section 2. Hence, the material collected here provides an affine counterpart to the first three chapters of this work.

We begin, in Section 1, by giving a summary of Popov's results. The methods here come from the theory of the *linear* algebraic groups. In Section 2, we show by means of an example how Popov's methods can be put to use: by applying them to the special linear group SL_n, we recover the presentation given in [33].

Of particular interest will be the third section, where we show how Popov's results imply the existence of upper bounds for the affine embedding of simple groups which depend only on the type of the group, where the group is cut out by quadrics in the affine space.

Chapter 1

The equivariant completion

In the first chapter, we show how to construct an equivariant completion of a connected algebraic group. The starting point is Chevalley's Theorem, which describes an algebraic group as a fibering over an abelian variety. The completion is then constructed as an associated bundle out of this fibering.
Most of the results of this chapter are taken from [26] and [27].

1.1 The Theorem of Chevalley

Our main object of study in this Thesis are the *algebraic groups* (short for *algebraic group varieties*). These are the "group objects" in the category of algebraic varieties, i.e. algebraic varieties endowed with a group structure such that the product and inverse morphisms are defined in the category of varieties.
Let us briefly recall what an *abstract* algebraic variety is: a *prevariety* is a reduced scheme of finite type over an algebraically closed field k (or, in an equivalent way, a finite union of affine varieties), and a *variety* is a separated prevariety[1]. In particular, an abstract variety is not supposed a priori to be quasiprojective. In the course of this work we shall always assume that the algebraically closed field over which our objects are defined has characteristic zero.
In this short introductive section we describe an important result on the structure of a connected algebraic group (an algebraic group is said to be *connected* if it is irreducible as an algebraic variety; one avoids the word "irreducible" here since it has already a meaning in the context of group representations).

[1] This definition, taken from Borel's book [2], differs slightly from the one given in Hartshorne's [12], pg. 105, where "reduced" is replaced by "integral" (i.e. reduced and irreducible, see [12], Prop. 3.1, pg. 82). Indeed, we allow a non-irreducible scheme to be a variety: this has the advantage that an algebraic group can be defined as a variety (and not as a union of varieties, as Hartshorne's definition would require).

Two kinds of algebraic groups arise naturally as the most remarkable: those which are ”purely” affine and the projective ones.

Example 1.1.1. A closed subgroup of the *general linear group* $\mathrm{GL}_n(k)$ is called a *linear algebraic group*. It is affine, since $\mathrm{GL}_n(k)$ itself can be embedded as an open subvariety of $\mathbb{A}^{n^2}(k)$. On the other side, one can show that to an affine algebraic group always belongs a faithful algebraic representation, and so that it is isomorphic to a subgroup of $\mathrm{GL}_n(k)$ (see for instance [13], pg. 63). Hence, the linear algebraic groups are exactly the affine ones. Good references for the theory of linear algebraic groups are for example the books of A. Borel ([2]) and J. E. Humphreys ([13]).

Example 1.1.2. A complete algebraic group (i.e. one that is proper over the ground field k) is automatically commutative (see [21], Cor. 1, pg. 44). Therefore, it is called an *abelian variety*. Such an algebraic group is always a projective variety (see [21], pg. 163). The theory of abelian varieties is rich and interesting; the standard reference is D. Mumford's book [21].

Abelian varieties and subgroups of GL_n are not the only examples of algebraic groups, but in a certain sense the whole theory is built upon them. A connected algebraic group can always be realised as the extension of an abelian variety by a linear algebraic group:

Theorem 1.1.1 (Chevalley, Rosenlicht, Barsotti). *Let G be a connected algebraic group. Then, there exists a linear and connected normal algebraic subgroup L of G such that the quotient G/L is an abelian variety. L is unique and contains all other linear and connected algebraic subgroups of G.*

Proof: See [26], Thm. 16, pg. 439 or [1], Thm. 6.4, pg. 116. ■

By this theorem, one gets an exact sequence

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0 \tag{1.1}$$

where $A = G/L$: G is an extension of A by L. The fact that A is complete has an interesting consequence, namely that G is commutative if and only if L is (one implication is trivial, the other one follows from [26], Cor. 2, pg. 433). This means that in order to extend the existing results to noncommutative groups one has to take into account noncommutative L's.

1.2 Quotients and principal fibre bundles

A convenient structure which one can introduce in order to study the fibration of an algebraic group by an algebraic subgroup is the structure of a *principal bundle*. This is a fibering over a variety whose fibres are homeomorphic to some fixed algebraic group. In order to be able to work with such an object, one needs some further conditions on the local structure of the

bundle. An assumption which appears to be quite natural, and which is actually sufficient in the analytic context, is the so-called *local triviality*. This amounts to the fact that the bundle looks like a direct product over some open cover. But this turns out to be too restrictive for us: a homogeneous space G/H, the quotient of an algebraic group by an algebraic subgroup, is not locally trivial in general, even if H is normal in G. Therefore, one needs to find some condition which is less restrictive. In [27], Serre introduces the notion of *local isotriviality*, a weakening of local triviality that is satisfied by algebraic homogeneous spaces (this amounts to the local triviality with respect to a different topology, the *étale* topology).
The aim of this section is the description of some of Serre's results, which will be useful in the following.

Since a principal bundle is obtained by taking the quotient of an algebraic variety by the action of an algebraic group, we include here a brief excurs on the theory of quotients. The material collected here will be again useful in the next section.

We begin with a very general notion of quotient: let $\mathfrak{C}$ be a category, and G a group which acts on an object X of $\mathfrak{C}$. A pair (Y, π), where Y is an object in $\mathfrak{C}$ and $\pi : X \to Y$ is a morphism in $\mathfrak{C}$ is said to be a *categorical quotient* for the action of G on X in the category $\mathfrak{C}$ if the following universal property holds: for any G-invariant morphism $\alpha : X \to Z$ in $\mathfrak{C}$ there exists a unique morphism $\alpha' : Y \to Z$ in $\mathfrak{C}$ such that $\alpha = \alpha' \circ \pi$:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & Z \\ Y \arrow[ur, "\alpha'"'] & \end{tikzcd} \quad .$$

Because if its universal nature, this property characterizes Y up to isomorphism.
We now restrict ourselves to the category of *ringed spaces*: its objects are pairs $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, and a morphism between two objects $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ is a pair $(f, f^\sharp)$ where $f : X \to X'$ is a continuous map and $f^\sharp : \mathcal{O}_{X'} \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings (see [12], pg. 72). An action of a group G on a ringed space $(X, \mathcal{O}_X)$ is given by an automorphism $(\tau_g, \tau_g^\sharp)$ of ringed spaces for each $g \in G$ satisfying the axioms

$$(\tau_1, \tau_1^\sharp) = (\mathrm{Id}_X, \mathrm{Id}_{\mathcal{O}_X}) \quad ,$$

where $1 \in G$ denotes the neutral element, and

$$(\tau_{gh}, \tau_{gh}^\sharp) = (\tau_g \circ \tau_h, \tau_{g*}\tau_h^\sharp \circ \tau_g^\sharp)$$

for all $g, h \in G$. If such an action is given, we define a new ringed space $(X/G, \mathcal{O}_{X/G})$, and a map $(\pi, \pi^\sharp) : (X, \mathcal{O}_X) \to (X/G, \mathcal{O}_{X/G})$ as follows: X/G is the set of the G-orbits, together with the natural projection $\pi : X \to X/G$ and the quotient topology (i.e. the weakest topology for which π is continuous), and we let $\mathcal{O}_{X/G} := \pi_*\mathcal{O}_X^G$, the direct image of the sheaf of invariant sections on X. On an open set $V \subseteq X/G$ it is explicitely given by the rule $\mathcal{O}_{X/G}(V) = \mathcal{O}_X\left(\pi^{-1}(V)\right)^G$, the ring of G-invariant sections on the counterimage of V (which is an open, G-invariant subset of X). It follows that the map $\pi^\sharp : \mathcal{O}_{X/G} = \pi_*\mathcal{O}_X^G \to \pi_*\mathcal{O}_X$ is the map defined by the natural inclusion $\mathcal{O}_X^G(\pi^{-1}(V)) \to \mathcal{O}_X(\pi^{-1}(V))$ for each open subset V of X/G.

Lemma 1.2.1. *The pair $\left((X/G, \mathcal{O}_{X/G}), (\pi, \pi^\sharp)\right)$ is a categorical quotient for the action of G on X in the category of ringed spaces.*

Proof: $(X/G, \mathcal{O}_{X/G})$ is a ringed space. This follows from the preceding discussion. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a G-invariant morphism of ringed spaces. This means that

$$f \circ \tau_g = f : X \longrightarrow Y$$

and

$$(f \circ \tau_g)^\sharp = f_*\tau_g^\sharp \circ f^\sharp = f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

for all $g \in G$. We want to construct the map

$$(\overline{f}, \overline{f}^\sharp) : (X/G, \mathcal{O}_{X/G}) \longrightarrow (Y, \mathcal{O}_Y)$$

satisfying the universal property. It is immediately clear that $f : X \to Y$ defines a unique map $\overline{f} : X/G \to Y$ such that $\overline{f} \circ \pi = f$. This map is continuous, since the counterimage $(\overline{f})^{-1}(V)$ of an open set $V \subseteq Y$ is equal to $\pi\left(f^{-1}(V)\right)$, which is open in the quotient topology.
In order to construct the map $\overline{f}^\sharp$ consider, for an open subset $V \subseteq Y$, the ring homomorphism

$$f^\sharp(V) : \mathcal{O}_Y(V) \longrightarrow f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V)) \quad .$$

Since $f^\sharp$ is G-invariant by hypothesis it follows that, for a section $s \in \mathcal{O}_Y(V)$,

$$\left(\tau_g^\sharp\left(f^{-1}(V)\right) \circ f^\sharp(V)\right)(s) = \tau_g^\sharp\left(f^{-1}(V)\right)\left(f^\sharp(V)(s)\right) = f^\sharp(V)(s)$$

for all $g \in G$. This means that

$$\mathrm{Im}(f^\sharp(V)) \subseteq \mathcal{O}_X(f^{-1}(V))^G \quad .$$

Now, since $f = \overline{f} \circ \pi$,

$$\mathcal{O}_X(f^{-1}(V))^G = \mathcal{O}_X\left(\pi^{-1}\left((\overline{f})^{-1}(V)\right)\right)^G = \mathcal{O}_{X/G}\left((\overline{f})^{-1}(V)\right) \quad ,$$

and since this is equal to $\overline{f}_*\mathcal{O}_{X/G}(V)$, it follows that $f^\sharp$ induces a well-defined morphism

$$\overline{f}^\sharp : \mathcal{O}_Y \longrightarrow \overline{f}_*\mathcal{O}_{X/G}$$

of sheaves on X/G. ■

Let us now turn our attention to the category of algebraic varieties. This category can be considered in a natural way as a subcategory of the category of ringed spaces: a variety X possesses a topology, the Zariski topology, and a sheaf of rings, the structure sheaf $\mathcal{O}_X$.
A *left morphical action* (or, if there's no danger of confusion, just a *left action*) of an algebraic group G on a variety X is given by a morphism

$$\begin{array}{rcl} \tau : G \times X & \longrightarrow & X \\ (g, x) & \longmapsto & \tau(g, x) =: gx \end{array}$$

of varieties satisfying the usual axioms for a group action.

Remark 1.2.1. If we consider the restriction $\tau_g := \tau|_{\{g\}\times X}$ of τ to $\{g\} \times X$, and we identify canonically $\{g\} \times X$ with X, and furthermore we notice that here $\tau_g^\sharp : \mathcal{O}_X \to \tau_{g*}\mathcal{O}_X$ is given by the rule

$$\left(\tau_g^\sharp(U)(s)\right)(x) = s(\tau_g x)$$

for $U \subseteq X$ open, $s \in \mathcal{O}_X(U)$ and $x \in U$, we get for each $g \in G$ a morphism $(\tau_g, \tau_g^\sharp)$ of ringed spaces as above. This shows that a morphical action of an algebraic group is a special case for an action of a group on a ringed space.

In an analoguous way, one defines *right* morphical actions of an algebraic group on a variety. We shall denote by *action* both left and right actions.

For an action of an algebraic group G on a variety X, we define a new kind of quotient, the *geometric quotient*: it is a pair (Y, π) consisting of a variety Y and a morphism $\pi : X \to Y$ of varieties satisfying:

- **(GQ1)** π is surjective.
- **(GQ2)** π is open.
- **(GQ3)** The fibres of π are exactly the orbits of G.
- **(GQ4)** For every open subset $U \subseteq Y$, the ring homomorphism

$$\begin{array}{rcl} \mathcal{O}_Y(U) & \longrightarrow & \mathcal{O}_X\left(\pi^{-1}(U)\right)^G \\ \varphi & \longmapsto & \varphi \circ \pi \end{array}$$

  is an isomorphism.

The conditions **(GQ2)** and **(GQ4)** are often the most difficult to verify. Fortunately, in a lot of cases it is not necessary to do it, as shown by the next theorem. We recall that a variety X is said to be *normal* if all its local rings are integrally closed: an important class of normal varieties is given by the nonsingular ones, since a regular local ring is always integrally closed.

Theorem 1.2.2. *Let X be an irreducible variety with an action of an algebraic group G and let Y be a normal variety. Then, if $\pi : X \to Y$ is a morphism which satisfies* **(GQ1)** *and* **(GQ3)**, *the pair (Y, π) is a geometric quotient for the action of G on X.*

Proof: See [25], Thm. 4.2, pg. 187. ■

Consider an action of a *finite* group Γ on an *affine* variety $X = \operatorname{Spec} R$; the affine variety $X/\Gamma := \operatorname{Spec} R^{\Gamma}$, where R^{Γ} denotes the k-subalgebra of the Γ-invariants, together with the morphism $\pi : X \to X/\Gamma$ induced by the natural injection $R^{\Gamma} \hookrightarrow R$, is a geometric quotient of X by Γ. The following result relies on the fact that, under some mild hypotheses, this procedure can be applied locally on a variety:

Proposition 1.2.3. *The quotient of an algebraic variety with respect to the action of a finite group Γ is a geometric quotient if and only if any orbit of Γ is contained in an affine open subset, and in this case the quotient morphism is finite.*

Proof: See for instance [25], Thm. 4.4, pg. 191. ■

If the action of Γ on X is free, then X is said to be a *Galois covering* of $Y = X/\Gamma$. Γ is the *Galois group* of the covering, denoted $\operatorname{Gal}(X/Y)$.

We conclude this excursus with a proposition which relates the two notions of quotient which have been introduced so far:

Proposition 1.2.4. *Let X be a variety with an action of an algebraic group G, and suppose that (Y, π) is a geometric quotient of X by G. Then, (Y, π) is a categorical quotient of X by G in the category of algebraic varieties.*

Proof: See [23], Prop. 0.1, pg. 4. ■

Proposition 1.2.4 shows, in particular, that the axioms **(GQ1)-(GQ4)** characterize a geometric quotient up to isomorphism, and so that there is no danger of confusion by denoting a geometric quotient as above by X/G.

Remark 1.2.2. By definition, a variety is a ringed space. As such we can construct, for an action of an algebraic group G, a quotient X/G in the category of ringed spaces. If X/G together with the sheaf $\mathcal{O}_{X/G}$ is an algebraic variety, then **(GQ1)-(GQ4)** hold automatically; thus X/G is a geometric quotient so that that by Prop. 1.2.4 it is a categorical quotient in the category of algebraic varieties.

We proceed now to the theory of the principal bundles.
Let $\pi : Y \to X$ be a morphism of varieties. π is said to be an *étale covering* if it is finite and étale. We recall that "étale" means smooth of relative dimension zero (see [12], ex. 10.3, pg. 275).

Remark 1.2.3. This definition of étale covering corresponds to what J.-P. Serre calls, in [27], a *revêtement non ramifié*. Indeed, a *revêtement* ([27], Def. (E), pg. 1-02) is a finite morphism (see [12], pg. 84) and furthermore Serre's definition of *non ramifié*, namely the fact that for all $x \in X$ the morphism

$$\hat{\pi} : \hat{\mathcal{O}}_{\pi(x)} \longrightarrow \hat{\mathcal{O}}_x$$

induced on the completions of the local rings is an isomorphism is equivalent to the fact that π is étale if one works, as in our case, with nonsingular varieties over an algebraically closed field k (see [12], Ex. 10.4, pg. 275).

Étale and Galois coverings are closely related, as shown by the

Proposition 1.2.5.

1. *A Galois covering of algebraic varieties is étale.*

2. *Let X be an irreducible variety, and $\pi : Y \to X$ an étale covering. Then, there exists a Galois covering $\pi' : Z \to Y$ such that the composition $\pi \circ \pi' : Z \to X$ is a Galois covering.*

Proof:

1. See [27], pp. 1-05 and ff.

2. See [27], pg. 1-07. ■

A *G-bundle* (G, P, X) with *base space* X is given by an algebraic group G acting on a variety P and a geometric quotient X for this action. This implies that there exists a surjective morphism $\pi : P \to X$ of varieties compatible with the action, i.e. such that $\pi(pg) = \pi(p)$ for all $g \in G$ and all $p \in P$. G is the *structure group* of the bundle. G-bundles with fixed base space are objects in a category whose morphisms are the *G-equivariant morphisms* $P' \to P$ over X, that is morphisms which commute with the group action. By varying the base X, we obtain a fibered category, with a well-defined notion of *inverse image* $\phi^* P$ with respect to a morphism $\phi : X' \to X$ (a *base change*). A G-bundle (G, P, X) is said to be *trivial*, if it is isomorphic to $(G, X \times G, X)$ with the action $(x, g')g = (x, g'g)$ and the projection morphism $\pi = \mathrm{pr}_1 : X \times G \to X$, and *isotrivial*, if it becomes trivial over a finite and étale base change, i.e. if there is an étale covering $\phi : X' \to X$ such that $(G, \phi^* P, X')$ is isomorphic to $(G, X' \times G, X')$.

In the following, we shall need local versions of the definitions above: consider, for a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X by Zariski-open sets, the restricted bundles $\left(G, \pi^{-1}(U_\alpha), U_\alpha\right)$ for $\alpha \in I$. (G, P, X) is said to be *locally isotrivial* (resp. *locally trivial*), if the cover can be chosen in such a way that all restricted bundles are isotrivial (resp. trivial): for each $\alpha \in I$, there is an étale covering $\phi_\alpha : U'_\alpha \to U_\alpha$ and a cartesian diagram

$$\begin{tikzcd} U'_\alpha \times G \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & \pi^{-1}(U_\alpha) \arrow[d, "\pi"] \\ U'_\alpha \arrow[r, "\phi_\alpha"] & U_\alpha \end{tikzcd}$$

(resp. $\phi_\alpha = \mathrm{Id}_{U_\alpha}$ and $\pi^{-1}(U_\alpha) \cong U_\alpha \times G$, in the locally trivial case). In particular, such a bundle is locally trivial in the étale topology. We shall stick to the older terminology of ”locally isotrivial” in order to avoid confusion, since we work exclusively with the Zariski topology.

A bundle (G, P, X) as above is called a *(right) principal fibre bundle* (or just a *principal bundle*), if the following conditions are satisfied:

(PB1) The map

$$\begin{array}{rcl} \Psi : P \times G & \longrightarrow & P \times_X P \\ (p, g) & \longmapsto & (p, pg) \end{array}$$

is an isomorphism between the direct product $P \times G$ and the diagonal $P \times_X P$ of $P \times P$.

(PB2) $\pi : P \to X$ is a flat morphism.

Here, we consider a right action of G; if G acts on the left, the definition is analoguous.

Remark 1.2.4. A scheme S over a fixed scheme T (and so in particular a variety, which is a scheme over $T = \mathrm{Spec}(k)$) can be regarded as a contravariant functor $X \mapsto S(X)$ from the category of schemes over T to the category of sets: $S(\,\cdot\,)$ associates with a scheme X over T the set $S(X)$ of all morphisms $X \to S$ over T , the X*-valued points* of S (see [9], Vol. I, pg. 242). Under this functorial interpretation, the morphism Ψ defined just above has to be regarded as a natural transformation, and it induces a bijection

$$\begin{array}{rcl} \Psi(X) : \left(P \times_{\mathrm{Spec}(k)} G\right)(X) = P(X) \times G(X) & \longrightarrow & (P \times_X P)(X) \\ (p, g) & \longmapsto & (p, pg) \quad . \end{array}$$

Let $\Gamma(X, P) \subseteq P(X)$ be the set of *sections* $X \to P$, i.e.

$$\Gamma(X, P) = \{s \in P(X) | \pi \circ s = \mathrm{Id}_X\} \quad ,$$

and let s_1 and s_2 be two elements of $\Gamma(X, P)$. Then, since $\pi \circ s_1 = \mathrm{Id}_X = \pi \circ s_2$, it follows from the universal property of fibered products that

$$\begin{array}{rcl} (s_1, s_2) : X & \longrightarrow & P \times P \\ x & \longmapsto & (s_1(x), s_2(x)) \end{array}$$

defines an element of $(P \times_X P)(X)$. Hence, (s_1, s_2) lies in the image of $\Psi(X)$, and so $(s_1, s_2) = (s_1, s_1 g)$: there exists a uniquely determined element $g = g(s_1, s_2)$ of $G(X)$ such that $s_2 = s_1 g$. This shows that the set $\Gamma(X, P)$ of the sections of a principal fibre bundle (G, P, X) is a *principal homogeneous $G(X)$-space.*

In the following, we shall only work with bundles which are locally isotrivial. The next proposition shows that this condition is quite strong, since it already implies that a bundle is principal:

Proposition 1.2.6. *Let (G, P, X) be a locally isotrivial G-bundle. Then it is principal.*

Proof: We prove that, under the assumption of local isotriviality, the G-bundle (G, P, X) with the projection π satisfies **(PB1)** and **(PB2)**.
We prove the first axiom locally, on an open cover of X over which P becomes isotrivial. Let U be an open set in such a cover of X:

$$\begin{array}{ccc} U' \times G & \xrightarrow{\varphi'} & \pi^{-1}(U)\ ; \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ U' & \xrightarrow{\varphi} & U \end{array} \tag{1.2}$$

over U, the map $\Psi : P \times G \to P \times_X P$ lifts to

$$\begin{array}{rcl} \tilde{\Psi} : \pi^{-1}(U) \times G & \longrightarrow & \pi^{-1}(U) \times_U \pi^{-1}(U) \\ (p, g) & \longmapsto & (p, pg) \quad ; \end{array}$$

its inverse image over $\varphi : U' \to U$ is

$$\begin{array}{rcl} \varphi^* \tilde{\Psi} : (U' \times G) \times G & \longrightarrow & (U' \times G) \times_{U'} (U' \times G) \\ (p', g\, ; g') & \longmapsto & (p', g\, ; p', gg') \quad , \end{array}$$

which is an isomorphism (compatible with the action of Γ) since $(G, U' \times G, U')$ is the trivial principal G-bundle over U'. Taking the quotient[2] it follows that $\tilde{\Psi}$ is an isomorphism.
The axiom **(PB2)** holds for the following reason: consider again the cartesian square (1.2); since $U' \times G \to U'$ is flat, it follows from [9], IV, Seconde partie, Prop. (2.5.1), pg. 22 that $\pi^{-1}(U) \to U$ is flat and so, since flatness is a local property by definition ([12], pg. 254) that $P \to X$ is flat. ■

The proposition shows that the new axiom

(PB)' (G, P, X) is locally isotrivial

[2] This is a result from the theory of Galois-descent (see for instance [3], Example B, pg. 139); since we only make use of this theory once, it did not seem appropriate to include more material on this subject. In the next chapter we shall meet another aspect of descent theory.

is enough in order to characterize locally isotrivial, principal G-bundles; this is exactly Serre's older definition of principal fibre bundles given in [27], pg. 1-08.

The next lemma shows that locally isotrivial principal bundles behave well with respect to base change; this means in particular that the family of all locally isotrivial G-bundles for a fixed G gives rise to a fibered subcategory of the fibered category of all G-bundles:

Lemma 1.2.7. *Let $f : X' \to X$ be a morphism of varieties. Let (G, P, X) be a locally isotrivial G-bundle over X. Then, (G, f^*P, X') is a locally isotrivial principal G-bundle over X'.*

Proof: See [27], pg. 1-14. ■

As we already mentioned, our motivation for the introduction of the principal bundles is the study of the fibration of an algebraic group by an algebraic subgroup. The following proposition will be decisive in the next section:

Proposition 1.2.8. *Let G be an algebraic group, and H an algebraic subgroup of G. Let H operate on G by right translation. Then, the H-bundle $(H, G, G/H)$ is a locally isotrivial principal fibre bundle.*

Proof: See [27], Prop. 3, pg. 1-12. ■

Let G be a connected algebraic group. By what we saw in the first section, there is an exact sequence

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

where L is the largest connected linear subgroup of G and A is an abelian variety. As a consequence of Proposition 1.2.8, we see that there is an open cover $\mathcal{U} = \{U_i\}_{i=1}^n$ of A by open subsets such that the restriction $\pi^{-1}(U_i) \to U_i$ of the bundle is isotrivial for all i: there is an étale covering $U_i' \to U_i$ such that the fibering becomes trivial over U_i'. As we saw in Prop. 1.2.5, we can assume that this covering is Galois: this means that $\pi^{-1}(U_i)$ is isomorphic to a quotient $U_i' \times L/\Gamma_i$ for the action of some finite group Γ_i. This fact has many interesting consequences, as we shall see later on. The first one is the

Proposition 1.2.9. *A connected algebraic group is quasiprojective.*

Proof: Let, as above, L be the largest linear and connected algebraic subgroup of a connected algebraic group G, and A the abelian variety G/L. A is projective and so quasiprojective; this means that the structure morphism $\varphi_A : A \to \operatorname{Spec} k$ is a quasiprojective morphism ([9], II, Déf. 5.3.1, pg. 99). Furthermore, the morphism $\pi : G \to A$ is affine: if we choose the open cover $\mathcal{U}$ in such a way that U_i is affine for all i, we have that $\pi^{-1}(U_i) \cong U_i' \times L/\Gamma_i$ is affine (U_i' is affine since the morphism $U_i' \to U_i$ is finite, L is linear hence affine and the quotient of an affine variety by a finite group is affine). From [9], II, Cor. (5.3.4)(i), pg. 99 it follows that π is a quasiprojective morphism, and with part (ii) of the same Corollary that $\varphi_A \circ \pi : G \to \operatorname{Spec} k$ is quasiprojective, i.e. that G is a quasiprojective variety. ■

Remark 1.2.5. If one restricts oneself to the case of commutative groups, one has to deal only with locally trivial fibre bundles (i.e. such that $U_i' = U_i$ and $\pi^{-1}(U_i) \cong U_i \times L$). This is for instance the case in [5], [29] or [37]. Groups characterized by the fact that their actions give rise to locally trivial fibrations are called *special* in [27]. Their family does not contain only the commutative algebraic groups, but for instance all solvable groups and even the general linear group GL_n. Some results on the theory of special groups, such as the fact that they are all linear, can be found in [27]. We shall meet again the special groups in the third chapter of this work, where we shall be more precise in their description.

1.3 Galois coverings and associated fibre bundles

In the previous section, we showed that a connected algebraic group G can always be realized as a locally isotrivial fibration over an abelian variety, with a linear algebraic group L as fibre. The aim of this section will be the construction of a new fibre bundle on the abelian variety whose fibre will be a given space X on which L operates. This associated bundle will be very useful, since it will allow us to exhibit a quite explicit completion $\overline{G}$ of the group G, and later on a projective embedding of the proper variety $\overline{G}$.

We begin by constructing the associated bundle as a ringed space. Then, we shall show that it admits the structure of a variety.

Let G be an algebraic group, and H an algebraic subgroup of G. Let X be a variety, on which H acts on the left. Define an action of H on $G \times X$ as follows:

$$h(g, x) := (gh^{-1}, hx) \quad ,$$

and denote by $G_H(X)$ the categorical quotient[3] in the category of ringed spaces, as introduced in the previous section. The fact that $\pi \circ \mathrm{pr}_1$ is invariant under the action of H implies the existence of a commutative diagram

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & G_H(X) \arrow[d, "p"] \\ G \arrow[r, "\pi"] & G/H \end{tikzcd} \tag{1.3}$$

[3]This space is denoted by $G \times^H X$ in [27] and by $G \underset{H}{*} X$ in [25]; we have chosen the notation $G_H(X)$ in order to emphasize on the functorial nature of $G_H(\,\cdot\,)$.

which allows us to define a natural projection $p : G_H(X) \to G/H$ (in the category of ringed spaces). The fibres of the projection p are isomorphic to X. Furthermore, an H-equivariant morphism $\varphi : X \to Y$ between H-varieties (i.e. a morphism which is compatible with the action of H on X and Y) gives rise in a natural way to a morphism $G_H(\varphi) : G_H(X) \to G_H(Y)$ of ringed spaces.

Our next aim is to show that $G_H(X)$ carries the structure of an algebraic variety, i.e. that $G_H(X)$ is a geometric quotient for the action of H on $G \times X$. For this, we shall need some notions introduced in the course of the previous section. In particular, we shall reduce the problem to a question of existence for bundles which are trivialized over a Galois covering. Therefore, we need to investigate the action of the Galois group on an isotrivial principal bundle more in detail.

Let (G, P, X) be a principal bundle with the projection $\pi : P \to X$, and let $\varphi : X' \to X$ be a Galois covering. Let $P' := X' \times_X P$ be the fibered product:

$$\begin{tikzcd} P' \arrow[r, "\varphi'"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ X' \arrow[r, "\varphi"] & X \end{tikzcd} \quad .$$

As a set, it can be explicitely defined by the rule

$$P' = \{(x', p) \in X' \times P \mid \varphi(x') = \pi(p)\} \quad .$$

The actions of $\Gamma = \mathrm{Gal}(X'/X)$ on X' and of G on P induce commuting actions on $X' \times P$, given respectively by

$$(x', p) \cdot \sigma = (x' \cdot \sigma, p)$$

and

$$(x', p)g = (x', pg) \quad ,$$

with $x' \in X'$, $p \in P$, $g \in G$ and $\sigma \in \Gamma$, and so a (right) action of $\Gamma \times G$ on $X' \times P$.
Let p_0 be some point in P. Its fibre under $\varphi' : P' \to P$ is given by

$$(\varphi')^{-1}(p_0) = \{(x', p_0) \in X' \times P' \mid \varphi(x') = \pi(p_0)\} \quad ;$$

if we set $x_0 := \pi(p_0)$, we see that

$$(\varphi')^{-1}(p_0) = \varphi^{-1}(x_0) \times \{p_0\}$$

and so that $(\varphi')^{-1}(p_0)$ is an orbit for the action of Γ on P', since $\varphi^{-1}(x_0)$ is an orbit for the action of Γ on X. Therefore, the pair (P, φ') satisfies the axioms **(GQ1)** and **(GQ3)**; if we assume P to be normal (as will be the case in our applications, where we deal with nonsingular varieties), Theorem 1.2.2 implies that (P, φ') is a geometric quotient for the action of Γ on P'.

Assume now that the principal fibre bundle is trivialized over the Galois covering $X' \to X$. This can be expressed with a diagram as follows:

$$\begin{tikzcd} X' \times G \arrow[r, "\sim", "\Phi"'] \arrow[dr, "\mathrm{pr}_1"'] & P' \arrow[r] \arrow[d] & P \arrow[d] \\ & X' \arrow[r] & X \end{tikzcd} \quad .$$

The isomorphism $\Phi : X' \times G \to P'$ over X' (which is a *trivialization* of P') induces the trivial right action of G on $X' \times G$:

$$\begin{array}{ccc} X' \times G \times G & \longrightarrow & X' \times G \\ (x, g, g') & \longmapsto & (x, gg') \end{array} \quad ,$$

and a right action of Γ on $X' \times G$ which commutes with the projection $\mathrm{pr}_1 : X' \times G \to X'$, i.e. a Γ-*linearization* of the trivial principal G-bundle $(G, X' \times G, X')$. This discussion shows that P is isomorphic to the quotient of a trivial G-bundle by an action of Γ and so that, in order to get informations on the isotrivial principal bundle (G, P, X) a good starting point is the study of the linearizations of a trivial principal bundle. This is what we are now going to do.

The Γ-linearizations of a trivial principal G-bundle $(G, X \times G, X)$ (we drop the "$'$" in the notation) are objects in a category whose morphisms are the Γ-equivariant morphisms $X \times G \to X \times G$ over X. The following proposition gives an explicit description of the linearizations, which leads to a classification of their isomorphy classes in terms of Galois cohomology:

Proposition 1.3.1. *Let $(G, X \times G, X)$ be a trivial principal G-bundle, and let Γ be a finite group which operates on X.*

1. *Let $\varphi_\sigma \in Z^1(\Gamma, G(X))$ be a cocycle with coefficients in $G(X)$; then the rule*

$$\begin{array}{ccc} X \times G \times \Gamma & \longrightarrow & X \times G \\ (x, g, \sigma) & \longmapsto & (x, g) \cdot \sigma = (x \cdot \sigma, (\varphi_\sigma^{-1})(x) g) \end{array}$$

   defines a Γ-linearization of the bundle.

2. *All Γ-linearizations of $(G, X \times G, X)$ are isomorphic to a linearization of this kind, and two linearizations are isomorphic if and only if the corresponding cocycles are cohomologuous.*

The proof of the proposition requires the following, easy

Lemma 1.3.2. *All automorphisms of the trivial principal G-bundle $(G, X \times G, X)$ are of the form*

$$\begin{array}{ccc} \Phi : X \times G & \longrightarrow & X \times G \\ (x, g) & \longmapsto & (x, \phi(x) g) \end{array}$$

with $\phi \in G(X)$.

Proof: The requirement that $\mathrm{pr}_1 \circ \Phi((x,g)) = \mathrm{pr}_1((x,g)) = x$ implies that there exists a morphism $\tilde{\phi} : X \times G \to G$ such that $\Phi((x,g)) = (x, \tilde{\phi}(x,g))$; furthermore, by the invariance of Φ under the action of G one has $(x, \tilde{\phi}(x,g)) = (x, \tilde{\phi}(x, 1_G))g = (x, \tilde{\phi}(x, 1_G)g)$. If we set $\phi(x) := \tilde{\phi}(x, 1_G)$, the lemma is proved. ■

Proof of Proposition 1.3.1:

1. Let $(x,g) \in X \times G$ and $\sigma, \tau \in \Gamma$. Then we have

$$\begin{aligned}((x,g)\cdot\sigma)\cdot\tau &= \Big(x\cdot\sigma, (\varphi_\sigma)^{-1}(x)g\Big)\cdot\tau = \Big(x\cdot\sigma\cdot\tau, (\varphi_\tau)^{-1}(x\cdot\sigma)(\varphi_\sigma)^{-1}(x)g\Big) = \\ &= \Big(x\cdot\sigma\cdot\tau, \Big(\big(\varphi_\tau^\sigma\big)^{-1}(\varphi_\sigma)^{-1}\Big)(x)g\Big)\end{aligned}$$

   and from the cocycle relation $\varphi_{\sigma\tau} = \varphi_\sigma\varphi_\tau^\sigma$ we get $(\varphi_{\sigma\tau})^{-1} = \big(\varphi_\tau^\sigma\big)^{-1}(\varphi_\sigma)^{-1}$, and so

$$((x,g)\cdot\sigma)\cdot\tau = \Big(x\cdot\sigma\tau, (\varphi_{\sigma\tau})^{-1}(x)g\Big) = (x,g)\cdot\sigma\tau \quad ,$$

   where for $f \in G(X)$ we denote by $f^\sigma \in G(X)$ the image of f under right translation of functions, given by $f^\sigma(x) = f(x\cdot\sigma)$. This shows that the rule given above defines a right action of Γ on $G \times X$, and it is furthermore clear that this action commutes with the projection, i.e. that it is a linearization.

2. We proceed as follows: first, we show how to recover a cocycle $\{\varphi_\sigma\}_{\sigma\in\Gamma}$ out of a Γ-linearization; then, we check that to isomorphic linearizations belong cohomologuous cocycles, and finally we show that the cocycle corresponding to the action given in 1. really is φ_σ.
   Let $(x,g,\sigma) \mapsto (x,g)\cdot\sigma$ be a Γ-linearization; denote by $\Gamma(X, X\times G)$ the set of sections of the trivial principal bundle and define a map

$$\begin{aligned}\Gamma\times\Gamma(X, X\times G) &\longrightarrow \Gamma(X, X\times G)\\ (\sigma, s) &\longmapsto \sigma * s\end{aligned}$$

   by the rule $(\sigma * s)(x) := s(x\cdot\sigma)\cdot\sigma^{-1}$; it really maps to sections, since for a section $s \in \Gamma(X, X\times G)$ we have

$$\mathrm{pr}_1\circ(\sigma * s) = \mathrm{pr}_1(s(x\cdot\sigma)\cdot\sigma^{-1}) = \mathrm{pr}_1(s(x\cdot\sigma))\cdot\sigma^{-1} = \mathrm{pr}_1(x)$$

   by the Γ-equivariance of pr_1, and furthermore it defines a left action of Γ on $\Gamma(X, X\times G)$ since

$$(\sigma*(\tau*s))(x) = (\tau*s)(x\cdot\sigma)\cdot\sigma^{-1} = s(x\cdot\sigma\tau)\cdot\tau^{-1}\sigma^{-1} = (\sigma\tau*s)(x) \quad .$$

Let now $s_1 \in \Gamma(X, X \times G)$ be the 1-section, defined by

$$\begin{aligned} s_1 : X &\longrightarrow X \times G \\ x &\longmapsto (x, 1) \end{aligned}$$

(where we denote by 1 the neutral element of the group G). Since $\Gamma(X, X \times G)$ is a principal homogeneous $G(X)$-space (see Remark 1.2.4), for each $\sigma \in \Gamma$ there exists exactly one element φ_σ in $G(X)$ such that

$$\sigma * s_1 = s_1\, \varphi_\sigma \quad ,$$

i.e. $(\sigma * s_1)(x) = s_1(x)\, \varphi_\sigma(x)$ for all $x \in X$. This way, we associate with the linearization a collection $\{\varphi_\sigma\}_{\sigma\in\Gamma} \subseteq G(X)$. We now claim that this is a 1-cocycle. Namely, consider again the relation

$$(\sigma\tau) * s_1 = \sigma * (\tau * s_1) \quad ;$$

its left-hand side becomes $s_1\, \varphi_{\sigma\tau}$, while for the right-hand side we get

$$\begin{aligned} \sigma * (\tau * s_1)\,(x) &= (\tau * s_1)\,(x \cdot \sigma) \cdot \sigma^{-1} = s_1(x \cdot \sigma)\varphi_\tau(x \cdot \sigma) \cdot \sigma^{-1} = \\ &= (\sigma * s_1)\,(x)\,(\varphi_\tau)^\sigma\,(x) = s_1(x)\varphi_\sigma(x)\,(\varphi_\tau)^\sigma\,(x) \end{aligned}$$

and so the relation $\varphi_{\sigma\tau} = \varphi_\sigma\,(\varphi_\tau)^\sigma$, which shows that $\{\varphi_\sigma\}_{\sigma\in\Gamma} \in Z^1(\Gamma, G(X))$.
Assume now that $X \times G$ admits two isomorphic linearizations

$$X \times G \times \Gamma \longrightarrow X \times G$$

denoted respectively by

$$(x, g, \sigma) \longmapsto (x, g) \cdot \sigma$$

and

$$(x, g, \sigma) \longmapsto (x, g) \odot \sigma \quad .$$

This means that there exists a G-automorphism Φ of $X \times G$ with

$$\Phi((x, g) \cdot \sigma) = \Phi((x, g)) \odot \sigma$$

for all $(x, g) \in X \times G$ and all $\sigma \in \Gamma$. Denote by "$*$" resp. "$\star$" the actions on $\Gamma(X, X \times G)$ obtained out of "$\cdot$" resp. "$\odot$":

$$(\sigma * s)(x) = s(x \cdot \sigma) \cdot \sigma^{-1} \quad \text{resp.} \quad (\sigma \star s)(x) = s(x \cdot \sigma) \odot \sigma^{-1} \quad ,$$

and by $\{\varphi_\sigma\}_{\sigma\in\Gamma}$ resp. $\{\varphi'_\sigma\}_{\sigma\in\Gamma}$ the cocycles associated to the linearizations:

$$\sigma * s_1 = s_1\, \varphi_\sigma \quad \text{resp.} \quad \sigma \star s_1 = s_1\, \varphi'_\sigma \quad .$$

From the equivariance of the automorphism Φ we get the relation

$$\sigma \star (\Phi \circ s_1) = \Phi \circ (\sigma * s_1) \quad . \tag{1.4}$$

Let us look at the left-hand side of (1.4): writing $\Phi((x, g)) = (x, \phi(x)g)$ as in Lemma 1.3.2, we get

$$\begin{aligned} \sigma \star (\Phi \circ s_1) &= (x \cdot \sigma, \phi(x \cdot \sigma)) \odot \sigma^{-1} = (x \cdot \sigma, 1)\phi^{\sigma}(x) \odot \sigma^{-1} = \\ &= \left((x \cdot \sigma, 1) \odot \sigma^{-1}\right) \phi^{\sigma}(x) = (\sigma \star s_1)(x)\phi^{\sigma}(x) = s_1(x)\varphi'_{\sigma}(x)\phi^{\sigma}(x) \end{aligned}$$

(recall that the actions of G and Γ are supposed to commute). The right-hand side of (1.4) becomes

$$\begin{aligned} \Phi \circ (\sigma * s_1)(x) &= \Phi\left(s_1(x)\varphi_{\sigma}(x)\right) = \left(\Phi \circ s_1(x)\right)\varphi_{\sigma}(x) = \\ &= (x, \phi(x))\varphi_{\sigma}(x) = s_1(x)\,\phi(x)\,\varphi_{\sigma}(x) \quad . \end{aligned}$$

By the effectivity of the action, we get $\varphi'_{\sigma}\,\phi^{\sigma} = \phi\,\varphi_{\sigma}$, i.e.

$$\varphi_{\sigma} = \phi^{-1}\,\varphi'_{\sigma}\,\phi^{\sigma} \quad :$$

to isomorphic bundles correspond cohomologuous cocycles.
On the other side, if $\{\varphi_{\sigma}\}_{\sigma\in\Gamma}$ and $\{\varphi'_{\sigma}\}_{\sigma\in\Gamma}$ are cohomologuous cocycles, the morphism $\phi \in G(X)$ allows one to "reconstruct" the isomorphism Φ as in Lemma 1.3.2. This shows that there is a one-to-one correspondence between the isomorphic classes of Γ-linearized, trivial principal G-bundles and the pointed set $H^1(\Gamma, G(X))$.
Let now $\{\varphi_{\sigma}\}_{\sigma\in\Gamma} \in Z^1(\Gamma, G(X))$ be a representant for a class in $H^1(\Gamma, G(X))$ associated to some Γ-linearization, and consider the new linearization given by means of the rule $(x, g) \cdot \sigma := (x \cdot \sigma, (\varphi_{\sigma})^{-1}(x)g)$. We compute the cocycle associated to it:

$$\begin{aligned} (\sigma * s_1)(x) &= s_1(x \cdot \sigma) \cdot \sigma^{-1} = (x \cdot \sigma, 1) \cdot \sigma^{-1} = \left(x \cdot \sigma \cdot \sigma^{-1}, \left(\varphi_{\sigma^{-1}}\right)^{-1}(x \cdot \sigma)\right) = \\ &= \left(x, \left(\varphi^{\sigma}_{\sigma^{-1}}\right)^{-1}(x)\right) = (x, \varphi_{\sigma}(x)) = (x, 1)\varphi_{\sigma}(x) = \\ &= s_1(x)\varphi_{\sigma}(x) \quad , \end{aligned}$$

since from the cocycle relation it follows that $\left(\varphi^{\sigma}_{\sigma^{-1}}\right)^{-1} = \varphi_{\sigma}$. The cocycle obtained out of this action is again $\{\varphi_{\sigma}\}_{\sigma\in\Gamma}$: this shows that the new linearization is isomorphic to the original one and so, since this holds for any Γ-linearizations, that all Γ-linearizations are isomorphic to a linearization as in 1. ■

Recall from Proposition 1.2.5 that an étale covering can always be extended to a Galois one; this implies that an isotrivial principal G-bundle (G, P, X) can always be supposed to be trivialized over a Galois covering $X' \to X$. Together with Proposition 1.3.1 and the preceding discussion this implies the

Corollary 1.3.3. *Let (G, P, X) be an isotrivial principal fibre bundle. Then there is a Galois covering $X' \to X$ with Galois group Γ over which the bundle becomes trivial, and a cocycle $\{\varphi_\sigma\}_{\sigma\in\Gamma} \in Z^1(\Gamma, G(X'))$ such that P is the geometric quotient of $X' \times G$ for the action of Γ given by*

$$(x', g)\cdot\sigma = (x'\cdot\sigma, (\varphi_\sigma)^{-1}(x')g) \quad ,$$

with $(x', g) \in X' \times G$ and $\sigma \in \Gamma$.

Note that "our" $(\varphi_\sigma)^{-1}$'s take the place of Serre's φ_σ's in [27], Prop. 1, pg. 1-09.

In order to prove that $G_H(X)$ is an algebraic variety, we shall look at coverings of G/H over which the bundles become trivial; in particular, we shall work with Galois coverings. This implies that we must be able to take quotients with respect to actions of finite groups. Since the conditions of Proposition 1.2.3 of the orbits being affine is too troublesome in general, we replace it by the following, stronger condition, sufficient for our purposes:

(F) Any finite subset is contained in an affine open subset.

This condition is not "too" strong: it is satisfied for instance by quasiprojective varieties, since one can always find some hyperplane which does not intersect a finite set of points, or by algebraic groups, since here it is possible to move hyperplanes away from finite sets (see also [28], Ex. 1 and 2, pg. 59). Note that **(F)** would not make sense for a scheme over a finite field, as shown by the counterexample $\mathbb{P}^1(\mathbb{F}_2)$.

We are now ready to prove the central result of this section, namely that $G_H(X)$ is an algebraic variety, if we restrict ourselves to varieties X which satisfy the condition **(F)**.

Theorem 1.3.4. *Let G be a connected algebraic group and H an algebraic subgroup of G. Let H act on the left on an algebraic variety X which satisfies* **(F)**. *Then, $G_H(X)$ is a geometric quotient for the action of H on $G \times X$ and the H-bundle $(H, G \times X, G_H(X))$ is a locally isotrivial principal fibre bundle.*

Proof: (See [27], Prop. 4, pg. 1-15 or [25], Thm. 4.19, pg. 195) We already saw that $G_H(X)$ carries in a natural way the structure of a quotient ringed space of $G \times X$ by the action of H. We now show that $G_H(X)$ is a variety, which can be covered by (finitely many) subvarieties whose structure will make clear that the fibration is principal.
For this purpose, choose an open affine subset $U \subseteq G/H$ such that there is a Galois covering $U' \to U$ over which G becomes trivial: this gives rise to a commutative diagram

$$\begin{tikzcd}
U' \times H \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & \pi^{-1}(U) \arrow[r, hook] \arrow[d] & G \arrow[d, "\pi"] \\
U' \arrow[r] & U \arrow[r, hook] & G/H
\end{tikzcd}$$

where $\pi : G \to G/H$ denotes again the quotient morphism. By Cor. 1.3.3, there exists a cocycle $\varphi_\sigma \in Z^1(\Gamma, H(U'))$ such that $\pi^{-1}(U)$ is the geometric quotient of $U' \times H$ by the action of Γ given by $(u', h) \cdot \sigma = (u' \cdot \sigma, (\varphi_\sigma)^{-1}(u')h)$ with $u' \in U', h \in H, \sigma \in \Gamma$.
We consider the covering given by

$$U' \times H \times X \longrightarrow \pi^{-1}(U) \times X \hookrightarrow G \times X \quad ;$$

it is Galois, if the action of Γ on $U' \times H \times X$ is the lifting of the action on $U' \times H$:

$$(u', h, x) \cdot \sigma = (u' \cdot \sigma, (\varphi_\sigma)^{-1}(u')h, x) \quad ,$$

with $(u', h, x) \in U' \times H \times X$ and $\sigma \in \Gamma$. This induces an open embedding

$$(U' \times H \times X)/\Gamma \hookrightarrow G \times X$$

of varieties; the quotient $(U' \times H \times X)/\Gamma$ is a variety since $U' \times H \times X$ satisfies **(F)**: U' is affine, since it is finite over the affine variety U, H is an algebraic group and X satisfies **(F)** by hypothesis. Lifting the action of H on $G \times X$ to $U' \times H \times X$ gives the action

$$k(u', h, x) = (u', hk^{-1}, kx)$$

with (u', h, x) as above and $k \in H$. Note that the map

$$\begin{array}{ccc} U' \times H \times X & \longrightarrow & U' \times X \\ (u', h, x) & \longmapsto & (u', hx) \end{array}$$

defines a geometric quotient of $U' \times H \times X$ by H. The actions of Γ and H on $U' \times H \times X$ commute, as can be immediately verified (the left and right translations of H on itself commute). This implies that the action of H induces an action on the quotient $(U' \times H \times X)/\Gamma$, which is therefore an open, H-invariant subvariety of $G \times X$. Taking the quotient by H, we get the diagram

$$\begin{array}{ccc} (U' \times H \times X)/\Gamma & \hookrightarrow & G \times X \\ \downarrow & & \downarrow \\ (U' \times X)/\Gamma & \hookrightarrow & G_H(X) \end{array}$$

where the left vertical morphism can be explicitely described by the rule

$$(u', h, x)\Gamma \longmapsto (u', hx)\Gamma \quad .$$

The space $(U' \times X)/\Gamma$ is a variety, since $U' \times X$ satisfies **(F)**. Since G/H can be covered by (finitely many) open subsets like U, it follows that $G_H(X)$ admits a finite open cover of varieties like $(U' \times X)/\Gamma$. This shows that $G_H(X)$ is an algebraic prevariety, since it follows immediately that it can be covered by finitely many *affine* varieties.

In order to show that $G_H(X)$ is itself a variety, we have to check its separatedness over $\operatorname{Spec} k$. This is done as follows: by base extension ([12], Cor. 4.6(c), pg. 99), $U' \times X$ is separated over U'; from [9], IV, Seconde partie, Prop. (2.7.1), pg. 29 it follows that $U' \times X/\Gamma$ is separated over $U = U'/\Gamma$; since separatedness is a local condition ([12], Cor. 4.6(f), pg. 99), it follows that $G_H(X)$ is separated over G/H and with [12], Cor. 4.6(e), pg. 99 that it is separated over $\operatorname{Spec} k$, since G/H is a variety and therefore separated.
In order to prove that $G \times X$ is principal over $G_H(X)$, we just have to look at the diagram

$$\begin{tikzcd} U' \times H \times X \arrow[r] \arrow[d] & (U' \times H \times X)/\Gamma \arrow[d] \\ U' \times X \arrow[r] & (U' \times X)/\Gamma \end{tikzcd}$$

where the horizontal arrows denote quotients by Γ, while the vertical arrows denote quotients by H. This shows that over the étale Galois covering $U' \times X \to (U' \times X)/\Gamma$ the fibration $G \times X \to G_H(X)$ pulls back to $(H, U' \times H \times X, U' \times X)$. This means that $(H, G \times X, G_H(X))$ is a locally isotrivial principal bundle. ■

The theorem, together with Prop. 1.2.4, implies that $G_H(X)$ is a categorical quotient for the action of H on $G \times X$ in the category of algebraic varieties. If φ is an H-equivariant morphism between two H-varieties, and we consider the H-equivariant composition of morphisms

$$q \circ (\mathrm{Id}_G \times \varphi) : G \times X \longrightarrow G_H(Y) \quad ,$$

where $q : G \times Y \to G_H(Y)$ denotes the natural projection, the universal property of the categorical quotient implies the existence of a unique morphism $G_H(\varphi) : G_H(X) \to G_H(Y)$ which makes a commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{Id}_G \times \varphi"] \arrow[d] & G \times Y \arrow[d] \\ G_H(X) \arrow[r, "G_H(\varphi)"] & G_H(Y) \end{tikzcd} \quad ,$$

the *quotient* of $\mathrm{Id}_G \times \varphi$ by H.

Theorem 1.3.5. *$G_H(\cdot)$ defines an additive functor from the category of H-varieties which satisfy* **(F)** *to the category of varieties over G/H.*

Proof: By construction, there is a morphism $G_H(X) \to G/H$. This shows that $G_H(X)$ is a variety over G/H. The discussion just above shows that if $\varphi : X \to Y$ is H-invariant, then $G_H(\varphi)$ is a morphism over G/H.
Now let $\psi : Y \to Z$ be another H-invariant morphism between H-varieties; then, the morphism $G_H(\psi) \circ G_H(\varphi)$ satisfies the universal property for the quotient of $\mathrm{Id}_G \times (\psi \circ \varphi)$ by H,

and therefore it follows from the uniqueness of $G_H(\psi \circ \varphi)$ that $G_H(\psi) \circ G_H(\varphi) = G_H(\psi \circ \varphi)$. The additivity of $G_H(\cdot)$ is proven as follows: let again X and Y be H-varieties; for the trivial G-bundle $G \times (X \times Y)$ we have the canonical isomorphism

$$G \times (X \times Y) = (G \times X) \times_G (G \times Y) \quad ;$$

taking the quotient by the action of H, we get

$$G_H(X \times Y) = G_H(X) \times_{G/H} G_H(Y) \quad :$$

the image of the direct product of X and Y is the fibered product over G/H of $G_H(X)$ and $G_H(Y)$. This concludes the proof. ■

Example 1.3.1. Let the notations be as above. Then $G_H(H) \cong G$ and the structure morphism $p : G_H(H) \to G/H$ is the natural projection $\pi : G \to G/H$. Indeed, the map

$$\begin{array}{rcl} \psi : G \times H & \longrightarrow & G \\ (g, h) & \longmapsto & gh \end{array}$$

makes G into a geometric quotient of $G \times H$ by the action of H given by $k(g, h) = (gk^{-1}, kh)$: it is surjective, and its fibres are exactly the H-orbits, since

$$\psi(g, h) = \psi(g', h') \qquad \Longleftrightarrow \qquad gh = g'h' \quad ,$$

and this is true if and only if $(g', h') = (gk^{-1}, kh)$ with $k = h'h^{-1}$, i.e. if (g, h) and (g', h') lie in the same orbit. The claim follows then from Theorem 1.2.2.

Example 1.3.2. $G_H(G) \cong (G/H) \times G$ with the structure morphism $\mathrm{pr}_1 : (G/H) \times G \to G/H$. In order to see this, consider the map

$$\begin{array}{rcl} \psi : G \times G & \longrightarrow & (G/H) \times G \\ (g_1, g_2) & \longmapsto & (\pi(g_1), g_1 g_2) \quad , \end{array}$$

and proceed as above.

Example 1.3.3. Let X be a variety on which H acts trivially. Then we have the relation $G_H(X) \cong (G/H) \times X$, and the structure morphism of the variety $G_H(X)$ is the first projection $\mathrm{pr}_1 : (G/H) \times X \to G/H$. To prove this, consider the map

$$\begin{array}{rcl} \psi : G \times X & \longrightarrow & (G/H) \times X \\ (g, x) & \longmapsto & (\pi(g), x) \quad , \end{array}$$

and proceed as above.
As special cases, we see immediately that $G_H(G/H) \cong (G/H) \times (G/H)$, and $G_H(\mathrm{Spec}(k)) \cong (G/H) \times \mathrm{Spec}(k) \cong G/H$, if G is defined over k.

Before turning our attention to one further example, which will be the most important for us, we collect two more results on associated bundles. For their proofs, we need the

Lemma 1.3.6. *Let (G, P, X) and (G, P', X') be locally isotrivial principal fibre bundles, and let $f : X' \to X$ be a base change. Then, $P' \cong f^*P$ if and only if there exists a G-equivariant morphism $F : P' \to P$ such that the square*

$$\begin{tikzcd} P' \arrow[r, "F"] \arrow[d] & P \arrow[d] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is cartesian.

Proof: See [27], pg. 1-15. ■

This implies that the diagram (1.3) on page 23 is in fact a cartesian square

$$\begin{tikzcd} G \times X \arrow[r, "q"] \arrow[d, "\mathrm{pr}_1"'] & G_H(X) \arrow[d, "p"] \\ G \arrow[r, "\pi"] & G/H \end{tikzcd} \tag{1.5}$$

since now we know that both vertical arrows are locally isotrivial fibrations with fibre H.

Proposition 1.3.7. *Let H be an algebraic subgroup of an algebraic group G, and let X and Y be varieties satisfying* **(F)** *with an action of H.*

1. *If $i : X \hookrightarrow Y$ is an H-equivariant, open embedding, then*

$$G_H(i) : G_H(X) \longrightarrow G_H(Y)$$

   is an open embedding.

2. *If X is complete (i.e. proper over the ground field k), then $G_H(X)$ is proper over G/H.*

Proof:

1. The embedding i induces an open, H-equivariant embedding

$$\mathrm{Id}_G \times i : G \times X \hookrightarrow G \times Y \quad .$$

   over G. Since the surjective morphism $\pi : G \to G/H$ is flat, it follows from [9], IV, Seconde partie, Prop. (2.7.1)(x), pg. 29 that $G_H(i) : G_H(X) \to G_H(Y)$ is an open embedding.

2. If X is proper over $\operatorname{Spec} k$, it follows by base extension that $G \times X$ is proper over G ([12], Cor. 4.8(c), pg. 102). Then, by [9], IV, Seconde partie, Prop. (2.7.1)(vii), pg. 29, $p : G_H(X) \to G/H$ is a proper morphism. ■

Let G be a connected algebraic group. Recall that, by Chevalley's theorem, to G belongs an exact sequence

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

where L is the largest connected linear subgroup of G and A is an abelian variety. In the following, we shall only be interested in the case $H = L$, i.e. we shall study only associated bundles of the form $G_L(X)$. Note that, since L is uniquely determined, $G_L(X)$ (resp. $G_L(\varphi)$) depends only on G and X (resp. G and φ). Because of that, Knop and Lange in [14] use the notation "$G(X)$" (resp. "$G(\varphi)$"). We do not use this simplified notation here, in order to avoid confusion with the "functor of points" $G(\,\cdot\,)$, which we used on page 20.
An *equivariant completion* of an algebraic group G is a complete variety $\overline{G}$ on which G operates, together with an open, G-equivariant embedding $G \hookrightarrow \overline{G}$. Here, the group G acts on itself by left translation.
If $\overline{L}$ is an equivariant completion of the linear algebraic group L, the embedding $i : L \hookrightarrow \overline{L}$, which is open, induces an open embedding

$$G_L(i) : G \cong G_L(L) \hookrightarrow \overline{G} := G_L(\overline{L})$$

(if we assume that $\overline{L}$ satisfies **(F)**). The variety $\overline{G}$ is complete, since it is proper over A (see the proposition above), and the abelian variety A is projective, and so in particular proper over k (see also [12], Cor. 4.8, pg. 102). Together with the observation that the action of G on $G \times \overline{L}$ defined by $g(g', x) = (gg', x)$ induces an action of G on $\overline{G}$, which leaves the open subvariety $G \cong G_L(L)$ invariant, we get the

Corollary 1.3.8. *Let $\overline{L}$ be an equivariant completion of L, satisfying* **(F)**. *Then, $\overline{G} = G_L(\overline{L})$ is an equivariant completion of G.*

If the completion of L is a *compactification* (i.e. if $\overline{L}$ is projective), the condition **(F)** holds automatically. In §3.4, we shall show that an equivariant compactification $\overline{L}$ of L exists, and we shall give some hints on how it can be obtained explicitely.

Chapter 2

The projective embedding

In this chapter, we show how the equivariant completion constructed in Chapter 1 can be made to be a projective variety. In particular, we show how ample and very ample line bundles can be obtained on the completion.
The methods in this chapter are similar to [5] and [14], with the difference that we make intensively use of the theory of faithfully flat descent.

2.1 Faithfully flat descent and linearization

In this section, we collect some results on the topic of faithfully flat descent which will be useful later. Our exposition, especially at the beginning, is based on [3], §6.1 (which is essentially a translation of [10]). We begin with a brief outline of the main ideas behind the descent theory of coherent sheaves, and successively we take a look at the situation in the case of principal fibre bundles. Finally, at the end of the section we collect some useful results on faithfully flat morphisms.

The main problem of descent theory is the study of the inverse image functor (i.e. the "pullback") in a fibered category (see [10], pp. 190-02 and ff.). In our context, it translates as follows: let $\pi : X \to Y$ be a morphism of varieties, and consider the functor $\mathcal{F} \mapsto \pi^*\mathcal{F}$ which maps a quasicoherent $\mathcal{O}_Y$-module to its inverse image under π (see [12], pp. 109-110); then, one wants to characterize the image of π^*: an $\mathcal{O}_X$-module (resp. a morphism of $\mathcal{O}_X$-modules) which lies in the image of π^* is said to *descend* from X to Y.

Consider, for X and Y as above, the fibered product $X \times_Y X$, with the natural projections $p_1, p_2 : X \times_Y X \to X$. For any quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$, we call an isomorphism $\phi : p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ a *covering datum* on $\mathcal{F}$. The set of all pairs $(\mathcal{F}, \phi)$ where ϕ and $\mathcal{F}$ are as above gives rise in a natural way to a category $\mathfrak{F}_{\mathrm{cd}}(X, Y)$: a morphisms between two such objects $(\mathcal{F}, \phi)$ and $(\mathcal{F}', \phi')$ consists of a morphism $\varphi : \mathcal{F} \to \mathcal{F}'$ of $\mathcal{O}_X$-modules which is *compatible* *with the covering data*, i.e. such that the diagram

$$\begin{tikzcd} p_1^*\mathcal{F} \arrow[r, "\phi"] \arrow[d, "p_1^*\varphi"'] & p_2^*\mathcal{F} \arrow[d, "p_2^*\varphi"] \\ p_1^*\mathcal{F}' \arrow[r, "\phi'"] & p_2^*\mathcal{F}' \end{tikzcd} \tag{2.1}$$

commutes.
The pull-back $\pi^*\mathcal{G}$ of a quasicoherent $\mathcal{O}_Y$-module $\mathcal{G}$ with respect to the morphism π admits a covering datum in a natural way: it is given by the canonical isomorphism

$$p_1^*\left(\pi^*\mathcal{G}\right) = (\pi \circ p_1)^*\mathcal{G} = (\pi \circ p_2)^*\mathcal{G} = p_2^*\left(\pi^*\mathcal{G}\right) \quad .$$

Hence, we can consider π^* as a functor from the category of quasicoherent $\mathcal{O}_X$-modules to the category $\mathfrak{F}_{\mathrm{cd}}(X, Y)$.
If the morphism $\pi : X \to Y$ is *faithfully flat* (i.e. flat and surjective), we have the following result:

Proposition 2.1.1. *Assume that $\pi : X \to Y$ is faithfully flat. Then, the functor $\mathcal{G} \to \pi^*\mathcal{G}$ from quasi-coherent $\mathcal{O}_Y$-modules to quasi-coherent $\mathcal{O}_X$-modules with covering data is fully faithful.*

Proof: See [3], Prop. 1, pg. 130. Note that the assumption of quasi-compactness is superfluos here, since it is always satisfied by a morphism of varieties. ■

The proposition states that, for any two quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{G}$ and $\mathcal{G}'$, π^* describes a bijective map between the set of morphisms between $\mathcal{G}$ and $\mathcal{G}'$ and the set of morphisms between $(\pi^*\mathcal{G}, \phi_{\mathcal{G}})$ and $(\pi^*\mathcal{G}', \phi_{\mathcal{G}'})$ in $\mathfrak{F}_{\mathrm{cd}}(X, Y)$, where we denote by $\phi_{\mathcal{G}}$ resp. $\phi_{\mathcal{G}'}$ the canonical covering datum for $\pi^*\mathcal{G}$ resp. $\pi^*\mathcal{G}'$.
It remains to find those objects $(\mathcal{F}, \phi)$ in $\mathrm{Ob}\left(\mathfrak{F}_{\mathrm{cd}}(\mathrm{X}, \mathrm{Y})\right)$ which are in the image of π^*. A necessary (but not sufficient) condition is the commutativity of

$$\begin{tikzcd} p_{12}^*p_1^*\mathcal{F} \arrow[rr, "p_{12}^*\phi"] \arrow[d, equal] & & p_{12}^*p_2^*\mathcal{F} \arrow[d, equal] \\ p_{13}^*p_1^*\mathcal{F} \arrow[dr, "p_{13}^*\phi"'] & & p_{23}^*p_1^*\mathcal{F} \arrow[dl, "p_{23}^*\phi"] \\ & p_{13}^*p_2^*\mathcal{F} = p_{23}^*p_2^*\mathcal{F} & \end{tikzcd} \tag{2.2}$$

(where the unspecified identities are canonical isomorphisms of $\mathcal{O}_X$-modules). Let us namely consider the diagram

$$X \times_Y X \times_Y X \overset{p_{ij}}{\Rrightarrow} X \times_Y X \overset{p_k}{\rightrightarrows} X \xrightarrow{\pi} Y \tag{2.3}$$

where p_{ij}, $(i, j) = (1, 2), (1, 3), (2, 3)$ denote the projections $X \times_Y X \times_Y X \to X \times_Y X$ on the i-th and j-th factors; if we set $\mathcal{F} := \pi^* \mathcal{G}$ for a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, the diagram (2.2) is commutative because all appearing isomorphisms are canonical: the canonical isomorphism

$$\phi : p_1^* \mathcal{F} = p_1^* \pi^* \mathcal{G} \xrightarrow{\sim} p_2^* \pi^* \mathcal{G} = p_2^* \mathcal{F}$$

plays the role of the covering datum, and it follows immediately that $p_{ij}^* p_1^* \mathcal{F} \cong p_{ij}^* p_2^* \mathcal{F}$ canonically.

A covering datum ϕ on a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ for which (2.2) is commutative is said to be a *descent datum* on $\mathcal{F}$. The relation

$$p_{13}^* \phi = p_{23}^* \phi \circ p_{12}^* \phi$$

satisfied by ϕ is called the *cocycle condition*. We denote by $\mathfrak{F}_{\mathrm{dd}}(X, Y)$ the category of the pairs $(\mathcal{F}, \phi)$ where $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module and ϕ a descent datum on $\mathcal{F}$, together with the morphisms which are compatible with the data. The discussion above implies that π^* can be seen in a natural way as a functor to $\mathfrak{F}_{\mathrm{dd}}(X, Y)$.
A descent datum ϕ on $\mathcal{F}$ is said to be *effective* if $(\mathcal{F}, \phi) \in \mathrm{Ob}\,(\mathfrak{F}_{\mathrm{dd}}(X, Y))$ is isomorphic to $\pi^* \mathcal{G}$ together with its canonical descent datum for some quasicoherent $\mathcal{O}_X$-module $\mathcal{G}$. If every descent datum is effective, the morphism π is said to be a *strict descent morphism*.

Theorem 2.1.2 (Grothendieck). *Let $\pi : X \to Y$ be a faithfully flat morphism of varieties. Then, π is a strict descent morphism for quasi-coherent $\mathcal{O}_X$-modules.*

Proof: See [3], Thm. 4, pg. 134. ■

The theorem shows that, if $\pi : X \to Y$ is faithfully flat, the category of quasi-coherent $\mathcal{O}_Y$-modules is equivalent to $\mathfrak{F}_{\mathrm{dd}}(X, Y)$: it amounts to the same to give a quasi-coherent $\mathcal{O}_Y$-module or a quasi-coherent $\mathcal{O}_X$-module equipped with a descent datum.

We reformulate now the descent problem in a special setting. Namely, we let (G, X, Y) be a (left) principal fibre bundle. Then, the projection $\pi : X \to Y$ is faithfully flat, and one could ask oneself whether the isomorphism

$$\begin{aligned} f : G \times X &\longrightarrow X \times_Y X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

makes it possible to express descent data and cocycle conditions in terms of sheaves on $G \times X$. As we are now going to show, this leads in a natural way to consider *G-linearized* $\mathcal{O}_X$-modules. Denote by $\sigma : G \times X \to X$ the action of G on X and by $q_2 : G \times X \to X$ the second projection. A *linearization* of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is an isomorphism

$$\psi : \sigma^* \mathcal{F} \xrightarrow{\sim} q_2^* \mathcal{F}$$

of sheaves on $G \times X$ which satisfies a condition which is expressed by the commutativity of

$$\begin{tikzcd}
{[\sigma \circ (\mathrm{Id}_G \times \sigma)]^* \mathcal{F}} \arrow[rr, "(\mathrm{Id}_G \times \sigma)^* \psi"] \arrow[d, equal] & & {[q_2 \circ (\mathrm{Id}_G \times \sigma)]^* \mathcal{F}} \arrow[d, equal] \\
{[\sigma \circ (\mu \times \mathrm{Id}_X)]^* \mathcal{F}} \arrow[dr, "(\mu \times \mathrm{Id}_X)^* \psi"'] & & {[\sigma \circ q_{23}]^* \mathcal{F}} \arrow[dl, "q_{23}^* \psi"] \\
& {[q_2 \circ (\mu \times \mathrm{Id}_X)]^* \mathcal{F} = [q_2 \circ q_{23}]^* \mathcal{F}} &
\end{tikzcd} \tag{2.4}$$

where $q_1 : G \times X \to G$, $q_2 : G \times X \to X$, $q_{12} : G \times G \times X \to G \times G$, $q_{13} : G \times G \times X \to G \times X$, $q_{23} : G \times G \times X \to G \times X$ are the usual projection maps in the different components, $\mu : G \times G \to G$ is the product morphism and the equalities denote canonical isomorphisms.

Remark 2.1.1. Let g be an element of G. If we restrict the isomorphism ψ to the subvariety $\{g\} \times X$ (which is canonically identified with X), and we denote by $\sigma_g : X \to X$ the automorphism of X given by $x \mapsto \sigma(g, x)$, we get an isomorphism

$$\psi_g : \sigma_g^* \mathcal{F} \xrightarrow{\sim} \mathcal{F} \quad ,$$

and the cocycle condition translates into the commutativity of

$$\begin{tikzcd}
\sigma_{gh}^* \mathcal{F} \arrow[rr, "\psi_{gh}"] \arrow[dr, "\sigma_h^* \psi_g"'] & & \mathcal{F} \\
& \sigma_g^* \mathcal{F} \arrow[ur, "\psi_h"'] &
\end{tikzcd} \tag{2.5}$$

for each $g, h \in G$, i.e. $\psi_{gh} = \psi_h \circ \sigma_h^* \psi_g$ (see also [23], pp. 30/31).
If $\mathcal{F} = \mathcal{O}_X$ and σ is the trivial linearization of $\mathcal{O}_X$, ψ_g is just the transposed homomorphism (see [9], I, 4.4.3, pg. 100) to the morphism $\tau_g^\sharp$ described in Remark 1.2.1.

The family consisting of all pairs $(\mathcal{F}, \psi)$ where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and ψ is a linearization of $\mathcal{F}$ builds a category whose morphisms are given by morphisms $\varphi : \mathcal{F} \to \mathcal{F}'$ between G-linearized $\mathcal{O}_X$-modules which are *compatible with the linearization*, i.e. which make the diagram

$$\begin{tikzcd}
\sigma^* \mathcal{F} \arrow[r, "\phi"] \arrow[d, "\sigma^* \varphi"'] & q_2^* \mathcal{F} \arrow[d, "q_2^* \varphi"] \\
\sigma^* \mathcal{F}' \arrow[r, "\phi'"] & q_2 \mathcal{F}'
\end{tikzcd} \tag{2.6}$$

commute, where ϕ' is a G-linearization for $\mathcal{F}'$. We denote this category by $\mathfrak{F}_G(X)$.

Proposition 2.1.3. *Let (G, X, Y) be a principal fibre bundle. Then the two categories $\mathfrak{F}_{\mathrm{dd}}(X, Y)$ and $\mathfrak{F}_G(X)$ are equivalent: to each descent datum on an $\mathcal{O}_X$-module $\mathcal{F}$ corresponds exactly a G-linearization of $\mathcal{F}$, and a morphism of $\mathcal{O}_X$-modules is compatible with the descent datum if and only if it is compatible with the corresponding linearization.*

Proof: Let $f : G \times X \to X \times_Y X$ be the isomorphism cited above, and let $(\mathcal{F}, \phi) \in \mathrm{Ob}(\mathfrak{F}_{\mathrm{dd}}(X, Y))$ be a quasi-coherent $\mathcal{O}_X$-module equipped with a descent datum. We prove first that $\psi := f^*\phi$ is an isomorphism of sheaves on $G \times X$ which satisfies the condition for a linearization, and then that a morphism $\varphi : \mathcal{F} \to \mathcal{F}'$ which is compatible with the descent datum ϕ is also compatible with the linearization ψ.
Since $f : G \times X \to X \times_Y X$ is an isomorphism, the morphism

$$\begin{aligned} g : G \times G \times X &\longrightarrow X \times_Y X \times_Y X \\ (g_1, g_2, x) &\longmapsto (g_1 g_2 x, g_2 x, x) \end{aligned}$$

is an isomorphism, too. Consider the diagrams

$$\begin{tikzcd} G \times G \times X \arrow[r, "r_{ij}", shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "g"] & G \times X \arrow[r, "r_k", shift left] \arrow[r, shift right] \arrow[d, "f"] & X \arrow[r, "\pi"] \arrow[d, equal] & Y \arrow[d, equal] \\ X \times_Y X \times_Y X \arrow[r, "p_{ij}", shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times_Y X \arrow[r, "p_k", shift left] \arrow[r, shift right] & X \arrow[r, "\pi"] & Y \end{tikzcd} \tag{2.7}$$

where p_{ij} resp. p_k are the natural projections on the (i, j)-th resp. k-th factor $X \times_Y X$ resp. X. We want to determine morphisms

$$r_{ij} : G \times G \times X \longrightarrow G \times X \quad ,$$

for $(i, j) = (1, 2), (1, 3), (2, 3)$ resp. morphisms

$$r_k : G \times X \longrightarrow X$$

for $k = 1, 2$ which make the diagrams commutative for each pair (i, j) as above and for $k = 1, 2$. These are easily found: we set

$$\begin{aligned} r_{12}(g_1, g_2, x) &:= (g_1, g_2 x) \quad , \quad \text{i.e.} \quad r_{12} = \mathrm{Id}_G \times \sigma \\ r_{13}(g_1, g_2, x) &:= (g_1 g_2, x) \quad , \quad \text{i.e.} \quad r_{13} = \mu \times \mathrm{Id}_X \\ r_{23}(g_1, g_2, x) &:= (g_2, x) \quad , \quad \text{i.e.} \quad r_{23} = q_{23} \end{aligned}$$

resp.

$$\begin{aligned} r_1(g, x) &:= gx \quad , \quad \text{i.e.} \quad r_1 = \sigma \\ r_2(g, x) &:= x \quad , \quad \text{i.e.} \quad r_2 = q_2 \quad . \end{aligned}$$

The commutativity of the diagrams (2.7) will allow us to prove all required relations for ψ. First of all, from the canonical isomorphisms

$$f^* p_1^* \mathcal{F} \cong r_1^* \mathcal{F} = \sigma^* \mathcal{F}$$

and

$$f^* p_2^* \mathcal{F} \cong r_2^* \mathcal{F} = q_2^* \mathcal{F}$$

it follows that

$$\psi = f^* \phi : f^* p_1^* \mathcal{F} \longrightarrow f^* p_2^* \mathcal{F}$$

defines in a natural way an isomorphism $\sigma^* \mathcal{F} \xrightarrow{\psi} q_2^* \mathcal{F}$; in order to show that ψ is a linearization of $\mathcal{F}$, we apply the functor g^* (inverse image of modules) to the commutative diagram (2.2), and we use the relations between the different morphisms given by (2.7). We get the commutative diagram

$$\begin{tikzcd}
(r_1 \circ r_{12})^* \mathcal{F} \arrow[rr, "r_{12}^* f^* \phi"] \arrow[d, equal] & & (r_2 \circ r_{12})^* \mathcal{F} \arrow[d, equal] \\
(r_1 \circ r_{13})^* \mathcal{F} \arrow[d, "r_{13}^* f^* \phi"'] & & (r_1 \circ r_{23})^* \mathcal{F} \arrow[d, "r_{23}^* f^* \phi"] \\
(r_2 \circ r_{13})^* \mathcal{F} \arrow[rr, equal] & & (r_2 \circ r_{23})^* \mathcal{F}
\end{tikzcd} .$$

Now, if write out explicitely all morphisms r_{ij} and r_k, we see immediately that this diagram is nothing else than (2.4). This means that ψ is a linearization of $\mathcal{F}$.
If $\varphi : \mathcal{F} \to \mathcal{F}'$ is a morphism of $\mathcal{O}_X$-modules which is compatible with descent data ϕ resp. ϕ' for $\mathcal{F}$ resp. $\mathcal{F}'$, applying the functor f^* to the diagram (2.1) gives us the commutative diagram

$$\begin{tikzcd}
r_1^* \mathcal{F} \arrow[r, "f^* \phi"] \arrow[d, "r_1^* \varphi"'] & r_2^* \mathcal{F} \arrow[d, "r_2^* \varphi"] \\
r_1^* \mathcal{F}' \arrow[r, "f^* \phi'"] & r_2^* \mathcal{F}'
\end{tikzcd}$$

which is exactly (2.6), since $f^* \phi = \psi$, $r_1 = \sigma$ and $r_2 = q_2$. This shows that φ is also compatible with the linearization.
The proof works also the other way around; namely, if $(\mathcal{F}, \psi)$ is a G-linearized sheaf and if $h : X \times_Y X \to G \times X$ is the inverse isomorphism to f, one shows exactly as above (it is enough to reverse the vertical arrows in (2.7)) that $h^* \psi$ is a descent datum for $\mathcal{F}$, and that a morphism which is compatible with the linearizations is also compatible with the descent datum.
This proves that $\mathfrak{F}_{\mathrm{dd}}(X, Y)$ and $\mathfrak{F}_G(X)$ are equivalent categories. ■

Applying Theorem 2.1.2 here gives the result that the G-linearized bundles on X are exactly those which ”descend” to Y, i.e. we have the

Corollary 2.1.4. *Let (G, X, Y) be a principal bundle with projection morphism $\pi : X \to Y$. Then, a coherent $\mathcal{O}_X$-module $\mathcal{F}$ admits a G-linearization if and only if $\mathcal{F} \cong \pi^*\mathcal{G}$ for a quasicoherent $\mathcal{O}_Y$-module $\mathcal{G}$, and a morphism φ between G-linearized $\mathcal{O}_X$-modules is compatible with the linearizations if and only if $\varphi = \pi^*\phi$ for a morphism ϕ between the corresponding $\mathcal{O}_Y$-modules.*

As announced, we conclude this section with some more results on the faithfully flat morphisms. In particular, we are interested in those sheaf-theoretical properties which descend by means of such a map. The first one, which is maybe the most important for us, is the fact of being a vector bundle:

Lemma 2.1.5. *Let $\pi : X \to Y$ be a faithfully flat morphism of varieties. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_Y$-module. Then, $\mathcal{F}$ is locally free of rank n if and only if $\pi^*\mathcal{F}$ is.*

Proof: See [9], IV, Seconde Partie, Prop. (2.5.2)(iv), pg. 22. ■

The lemma implies, for instance, that it is possible to obtain invertible sheaves on Y from invertible sheaves on X by means of a descent datum.

Remark 2.1.2. Consider the category whose objects are *invertible* sheaves on X equipped with a G-linearization, and whose morphisms are those which are compatible with the linearizations. Proposition 2.1.3 shows that, for a principal bundle (G, X, Y), this category is equivalent to the category of invertible sheaves with a descent datum. If we denote by $\mathrm{Pic}^G(X)$ the group of isomorphic classes of G-linearized line bundles, descent theory shows that $\mathrm{Pic}(Y) \cong \mathrm{Pic}^G(X)$ (i.e. the linearized line bundles are those which descend to the quotient), as one could expect (see also [21], §1.3). Note that, since a line bundle might admit more than just one linearization, the natural map $\mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$ is in general *not* injective. This means that we cannot consider $\mathrm{Pic}^G(X)$ as a subgroup of $\mathrm{Pic}(X)$. In [21], §1.3, a criterium is given for this map to be injective.

Now that we know that line bundles descend to line bundles, we could ask ourselves how well do ampleness and very ampleness behave with respect to to faithfully flat morphisms; the answer is given by the following

Lemma 2.1.6. *Let $f : X \to Y$ be a faithfully flat morphism of varieties. Let $g : Z \to Y$ be another morphism of varieties. Consider the pull-back diagram*

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd} .$$

An invertible sheaf $\mathcal{L}$ on Z is ample (resp. very ample) relatively to g if and only if its inverse image on $Z \times_Y X$ is ample (resp. very ample) relatively to g'.

Proof: See [9], IV, Seconde partie, Cor. (2.7.2), pg. 32. ■

2.2 Constructing sheaves on an associated bundle

We have already mentioned the fact that a variety $G_L(X)$ constructed out of a projective variety X can be shown to be itself projective. In order to obtain this result, we have to construct a very ample invertible sheaf on $G_L(X)$. It is therefore of great importance for us to know how sheaves on $G_L(X)$ can be obtained. In this section, starting with a quasi-coherent sheaf $\mathcal{L}$ on an L-variety X, equipped with a linearization with respect to the action of the linear algebraic group L, we show how to construct a sheaf $G_L(\mathcal{L})$ on the associated bundle $G_L(X)$. If the sheaf $\mathcal{L}$ is locally free (i.e. a *vector bundle* over X), F. Knop and H. Lange in [14] solve the problem as follows: they consider $\mathcal{L}$ as a *geometric* vector bundle (i.e. a variety V over X with ”linear coordinate change”, see [12], Ex. 5.18, pg. 128) and they construct the associated bundle $G_L(V)$, which they show to be in a natural way a geometric vector bundle on $G_L(X)$. This amounts to take the quotient of $G \times V = p_2^* V$ by an action of L, where $p_2 : G \times X \to X$ is the second projection.
Our approach here is slightly different (although fully equivalent, for vector bundles): we show that the inverse image $p_2^*\mathcal{L}$ on $G \times X$ of an L-linearized sheaf on X admits in a natural way an L-linearization, and so that it ”descends” naturally to a sheaf $G_L(\mathcal{L})$ on $G_L(X)$. In the next section we shall then show how to use $G_L(\mathcal{L})$ in order to construct a projective embedding of $G_L(X)$.

The core of this section is already contained in the next proposition, which is a direct application of the descent theory of the previous section.

Proposition 2.2.1. *Let G be a connected algebraic group, and L its largest connected linear subgroup. Let X be a quasiprojective L-variety, and $\mathcal{L}$ an L-linearized quasi-coherent sheaf on X. Denote by $q : G \times X \to G_L(X)$ the quotient map, and by $p_2 : G \times X \to X$ the second projection. Then, there exists a quasi-coherent sheaf $G_L(\mathcal{L})$ on $G_L(X)$ such that*

$$q^* G_L(\mathcal{L}) \cong p_2^* \mathcal{L} \quad .$$

Furthermore, let $\mathcal{M}$ be another L-linearized quasi-coherent sheaf on X, and $f : \mathcal{L} \to \mathcal{M}$ a morphism of sheaves, compatible with the linearizations. Then, there exists a morphism $G_L(f) : G_L(\mathcal{L}) \to G_L(\mathcal{M})$ such that $q^* G_L(f) \cong p_2^* f$.

Proof: The morphism $q : G \times X \to G_L(X)$ is a strict descent morphism for the category of quasi-coherent sheaves on $G \times X$, since it is surjective and flat ($G \times X \to G_L(X)$ is a principal fibre bundle, as shown in Theorem 1.3.4). Hence, in order to ”descend” from $G \times X$ to $G_L(X)$ we shall need a descent datum, or, as we showed in the previous section, an L-linearization of $p_2^*\mathcal{L}$. Recall that $G_L(X)$ is the quotient of $G \times X$ by the action of L given by

$$\begin{array}{rcl} L \times G \times X & \longrightarrow & G \times X \\ (l, g, x) & \longmapsto & (gl^{-1}, lx) \end{array} \quad ;$$

furthermore, the second projection

$$p_2 : G \times X \longrightarrow X$$

is equivariant with respect to this action. Hence, the proposition follows from the more general lemma that we prove just below. ■

Lemma 2.2.2. *Let L be an algebraic group, and $f : Z \to X$ be an L-equivariant morphism between L-varieties. Let $\mathcal{L}$ be a quasicoherent sheaf on X equipped with an L-linearization ψ. Then, the $\mathcal{O}_Z$-module $f^*\mathcal{L}$ admits in a natural way an L-linearization. Furthermore, a morphism $\varphi : \mathcal{L} \to \mathcal{M}$ between linearized sheaves which is compatible with the linearizations pulls back to a morphism $f^*\varphi$ which is compatible with the corresponding linearizations for $f^*\mathcal{L}$ and $f^*\mathcal{M}$.*

Proof: Let us denote by

$$\sigma : L \times X \longrightarrow X$$

and

$$\mu : L \times Z \longrightarrow Z$$

the actions of L on X and Z respectively. The equivariance of f is then expressed by the relation

$$f \circ \mu = \sigma \circ (\mathrm{Id_L} \times f) : L \times Z \longrightarrow X \quad ,$$

where Id_L is the identity map on L. We want to show that the isomorphism of sheaves on $L \times Z$ given by

$$\overline{\psi} := (\mathrm{Id_L} \times f)^*\psi : (\mathrm{Id_L} \times f)^*\sigma^*\mathcal{L} = \mu^*\left(f^*\mathcal{L}\right) \xrightarrow{\sim} q_2^*\left(f^*\mathcal{L}\right) = (\mathrm{Id_L} \times f)^* p_2^*\mathcal{L}$$

is a linearization of $f^*\mathcal{L}$ (where $q_2 : L \times Z \to Z$ is the projection on the second factor and we denote by "=" the canonical isomorphisms). In order to do this, we have to show that it satisfies the cocycle condition, which in this case amounts to the relation

$$(m \times \mathrm{Id_Z})^*\overline{\psi} = q_{23}^*\overline{\psi} \circ (\mathrm{Id_L} \times \mu)^*\overline{\psi}$$

(where $q_{23} : L \times L \times Z \to L \times Z$ is the projection on the second and third factors, $m : L \times L \to L$ is the product morphism on L and Id_Z is the identity map on Z). But straightforward calculations show that

$$\begin{array}{rcl} (m \times \mathrm{Id_Z})^*\overline{\psi} & = & (\mathrm{Id_L} \times \mathrm{Id_L} \times f)^*(m \times \mathrm{Id_X})^*\psi \\ q_{23}^*\overline{\psi} & = & (\mathrm{Id_L} \times \mathrm{Id_L} \times f)^* p_{23}^*\psi \\ (\mathrm{Id_L} \times \mu)^*\overline{\psi} & = & (\mathrm{Id_L} \times \mathrm{Id_L} \times f)^*(\mathrm{Id_L} \times \sigma)^*\psi \end{array}$$

(where $p_{23} : L \times L \times X \to L \times X$ is the projection on the second and third factors), and so that the cocycle condition for $\overline{\psi}$ follows from the cocycle condition for ψ by functoriality.
Now let $\mathcal{L}$ and $\mathcal{M}$ be L-linearized quasi-coherent sheaves on X, and $\overline{\psi}_{\mathcal{L}}$ resp. $\overline{\psi}_{\mathcal{M}}$ be linearizations for $f^*\mathcal{L}$ resp. $f^*\mathcal{M}$ as above. Then, the sheaf morphism $f^*\varphi : f^*\mathcal{L} \to f^*\mathcal{M}$ is compatible with the linearization if and only if the square diagram

$$\begin{tikzcd}
\mu^*(f^*\mathcal{L}) \arrow[r, "\overline{\psi}_{\mathcal{L}}"] \arrow[d, "\mu^*(f^*\varphi)"'] & q_2^*(f^*\mathcal{L}) \arrow[d, "q_2^*(f^*\varphi)"] \\
\mu^*(f^*\mathcal{M}) \arrow[r, "\overline{\psi}_{\mathcal{M}}"] & q_2^*(f^*\mathcal{M})
\end{tikzcd}$$

is commutative. But this follows by functoriality from the corresponding diagram for the morphism φ, together with the relations $\mu^* f^* = (\mathrm{Id}_L \times f)^*\sigma^*$ and $q_2^* f^* = (\mathrm{Id}_L \times f)^* p_2^*$. ■

The map $G_L(\,\cdot\,)$ admits also a functorial interpretation:

Corollary 2.2.3. *Let the notations be as in Prop. 2.2.1. Then, the map $\mathcal{L} \mapsto G_L(\mathcal{L})$ is an exact (covariant) functor between the category of L-linearized, quasi-coherent sheaves on X and the category of quasi-coherent sheaves on $G_L(X)$. Furthermore, if $\mathcal{L}$ and $\mathcal{M}$ are L-linearized, quasi-coherent sheaves on X, $\mathcal{L} \otimes \mathcal{M}$ is linearized in a natural way, and*

$$G_L(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}) \cong G_L(\mathcal{L}) \otimes_{\mathcal{O}_{G_L(X)}} G_L(\mathcal{M}) \quad .$$

Proof: Let $f : \mathcal{L}_1 \to \mathcal{L}_2$ and $g : \mathcal{L}_2 \to \mathcal{L}_3$ be morphisms between L-linearized $\mathcal{O}_X$-modules which are compatible with the linearizations. Then, the same holds for $g \circ f : \mathcal{L}_1 \to \mathcal{L}_3$, and from $p_2^*(g \circ f) = p_2^* g \circ p_2^* f$ it follows that $G_L(g \circ f) = G_L(g) \circ G_L(f)$.
Now let

$$0 \longrightarrow \mathcal{L} \xrightarrow{\varphi} \mathcal{L}' \xrightarrow{\varphi'} \mathcal{L}'' \longrightarrow 0$$

be an exact sequence of L-linearized, quasi-coherent sheaves on X; since the second projection $p_2 : G \times X \to X$ is a flat morphism, this induces an exact sequence

$$0 \longrightarrow p_2^*\mathcal{L} \xrightarrow{p_2^*\varphi} p_2^*\mathcal{L}' \xrightarrow{p_2^*\varphi'} p_2^*\mathcal{L}'' \longrightarrow 0$$

of L-linearized sheaves and L-morphisms on $G \times X$. Now, since taking inverse images with respect to $q : G \times X \to G_L(X)$ defines an equivalence of categories between quasi-coherent sheaves on $G_L(X)$ and L-linearized, quasi-coherent sheaves on $G \times X$, there are sheaves $G_L(\mathcal{L})$, $G_L(\mathcal{L}')$ and $G_L(\mathcal{L}'')$ on $G_L(X)$ with $q^*G_L(\mathcal{L}) \cong p_2^*\mathcal{L}$, $q^*G_L(\mathcal{L}') \cong p_2^*\mathcal{L}'$ and $q^*G_L(\mathcal{L}'') \cong p_2^*\mathcal{L}''$ and an exact sequence

$$0 \longrightarrow G_L(\mathcal{L}) \xrightarrow{G_L(\varphi)} G_L(\mathcal{L}') \xrightarrow{G_L(\varphi')} G_L(\mathcal{L}'') \longrightarrow 0$$

on X.
Finally, let $\mathcal{L}$ and $\mathcal{M}$ be quasicoherent $\mathcal{O}_X$-modules, with linearizations $\psi_{\mathcal{L}}$ resp. $\psi_{\mathcal{M}}$. A natural linearization for $\mathcal{L} \otimes \mathcal{M}$ is then given by

$$\psi_{\mathcal{L}} \otimes \psi_{\mathcal{M}} : \sigma^*\mathcal{L} \otimes \sigma^*\mathcal{M} = \sigma^*(\mathcal{L} \otimes \mathcal{M}) \longrightarrow p_2^*(\mathcal{L} \otimes \mathcal{M}) = p_2^*\mathcal{L} \otimes p_2^*\mathcal{M} \quad .$$

The last claim then follows, since $q^*G\,(\mathcal{L} \otimes \mathcal{M}) \cong p_2^*(\mathcal{L} \otimes \mathcal{M})$ and $q^*\,(G_L(\mathcal{L}) \otimes G_L(\mathcal{M})) = q^*G_L(\mathcal{L}) \otimes q^*G_L(\mathcal{M}) \cong p_2^*\mathcal{L} \otimes p_2^*\mathcal{M}$, both compatibly with the linearizations, and $p_2^*(\mathcal{L} \otimes \mathcal{M})$ is canonically isomorphic to $p_2^*\mathcal{L} \otimes p_2^*\mathcal{M}$. ■

A consequence of the exactness of the functor $G_L(\,\cdot\,)$ is that, for a quotient $\mathcal{L}_1/\mathcal{L}_2$ of L-linearized sheaves, one has $G_L(\mathcal{L}_1/\mathcal{L}_2) \cong G_L(\mathcal{L}_1)/G_L(\mathcal{L}_2)$. In particular, if an L-linearized sheaf $\mathcal{L}$ is generated by its global sections, it is a quotient of the free sheaf $\mathcal{O}_X^{\oplus n}$ (see [12], pg. 121), which carries in a natural way a linearization. Followingly, $G_L(\mathcal{L})$ is a quotient of the free sheaf $\mathcal{O}_{G_L(X)}^{\oplus n}$, and so generated by its global sections.

Let V be a vector space over k. Then, via the functor $V \mapsto V^{\sim}$ (which is an equivalence of categories, see [12], pp. 110 and ff.), V gives rise in a natural way to a vector bundle over $\mathrm{Spec}(k)$. We let L act trivially on $\mathrm{Spec}(k)$, and we consider a linearization ψ of $V^{\sim}$. Then, for each $\ell \in L$, to the automorphism $\psi_\ell : V^{\sim} \to V^{\sim}$ of $\mathcal{O}_{\mathrm{Spec}(k)}$-modules corresponds in a unique way an automorphism $\sigma(\ell) : V \to V$ of k-vector spaces. The commutative diagram (2.5), pg. 38 translates then into the diagram

$$\begin{tikzcd} V \arrow[rr, "\sigma(\ell_1\ell_2)"] \arrow[dr, "\sigma(\ell_2)"'] & & V \\ & V \arrow[ur, "\sigma(\ell_1)"'] & \end{tikzcd}$$

(recall that the transition from V to $V^{\sim}$ is *contravariant*, so that we have to reverse the arrows). This means that the rule $(\ell, x) \mapsto \sigma(\ell)x$ defines a *linear action* of L on V. Hence, a linearization of $V^{\sim}$ is equivalent to a linear representation of V.
For such a vector bundle $V^{\sim}$, the associated bundle $G_L(V^{\sim})$ is a sheaf on $G_L(\mathrm{Spec}(k))$, which is canonically identified with A. This shows how it is possible to produce vector bundles on the abelian variety A out of linear representations of L.

The following proposition illustrates the meaning of the vector bundles constructed out of the representation of L in the cohomology of an L-linearized invertible sheaf (as in [5], Beispiel 4, pg. 180 and [14], Prop. 1.8, pg. 558):

Proposition 2.2.4. *Let X be a projective L-variety, and $\mathcal{L}$ an L-linearized invertible sheaf on X. Denote as usual by $p : G_L(X) \to A$ the projection. Then, there is a natural isomorphism*

$$R^i p_* G_L(\mathcal{L}) \cong G_L(H^i(X, \mathcal{L})^{\sim}) \tag{2.8}$$

of vector bundles on A. In particular, for $i = 0$,

$$p_* G_L(\mathcal{L}) \cong G_L(H^0(X, \mathcal{L})^\sim) \quad .$$

Hence, the direct image of $G_L(\mathcal{L})$ on the abelian variety A is a vector bundle of rank equal to $\dim(H^0(X, \mathcal{L}))$.

Proof: We show that both sides of (2.8) pull back under the natural projection $\pi : G \to A$ to the same L-linearized vector bundle on G. The claim then follows from the theory of faithfully flat descent, since (L, G, A) is a principal fibre bundle.
Let us consider first the right-hand side of (2.8): we see immediately, from the definition of $G_L(\,\cdot\,)$, that

$$\pi^* \left(G_L(H^i(X, \mathcal{L})^\sim) \right) \cong pr_2^* H^i(X, \mathcal{L})^\sim \quad ,$$

where $pr_2 : G \times \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is the second projection. Now, since under the canonical identification $G \times \mathrm{Spec}(k) \xrightarrow{\sim} G$ the map pr_2 coincides with the structure morphism φ_G of G over $\mathrm{Spec}(k)$, it follows that

$$\pi^* \left(G_L(H^i(X, \mathcal{L})^\sim) \right) \cong \varphi_G^* H^i(X, \mathcal{L})^\sim \quad ,$$

i.e. that $G_L(H^i(X, \mathcal{L})^\sim)$ pulls back to the trivial $H^i(X, \mathcal{L})$-bundle over G together with the L-linearization induced by the action of L on the cohomology.
In order to deal with the left-hand side, we consider the square

$$\begin{tikzcd} G \times X \arrow[r, "q"] \arrow[d, "p_1"'] & G_L(X) \arrow[d, "p"] \\ G \arrow[r, "\pi"] & A \end{tikzcd} \quad ;$$

together with [12], Prop. 9.3, pg. 255 ("cohomology commutes with flat base change"), it shows that there is a natural isomorphism

$$\pi^* R^i p_* G_L(\mathcal{L}) \cong R^i p_{1*} \left(q^* G_L(\mathcal{L}) \right) \quad .$$

Now, since $q^* G_L(\mathcal{L}) \cong p_2^* \mathcal{L}$ by definition of $G_L(\,\cdot\,)$, where $p_2 : G \times X \to X$ is the second projection, we get

$$R^i p_{1*} \left(q^* G_L(\mathcal{L}) \right) \cong R^i p_{1*} \left(p_2^* \mathcal{L} \right) \quad ;$$

by another application of [12], Prop. 9.3, this time to the square

$$\begin{tikzcd} G \times X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "\varphi_X"] \\ G \arrow[r, "\varphi_G"] & \mathrm{Spec}(k) \end{tikzcd}$$

we obtain the relation

$$\pi^* R^i p_* G_L(\mathcal{L}) \cong \varphi_G^* \left(R^i \varphi_{X*} \mathcal{L} \right) = \varphi_G^* H^i(X, \mathcal{L})^\sim \quad .$$

This shows that both $\pi^* R^i p_* G_L(\mathcal{L})$ and $\pi^* G_L(H^i(X, \mathcal{L})^\sim)$ amount to the same L-linearized (trivial) vector bundle on G, and with that the proof is concluded. ■

Remark 2.2.1. Proposition 2.2.4 can be restated in terms of the functor $G_L(\cdot)$: recalling that $p = G_L(\varphi_X)$, (2.8) translates into

$$R^i G_L(\varphi_X)_* G_L(\mathcal{L}) \cong G_L(R^i \varphi_{X*} \mathcal{L}) \quad .$$

2.3 Ample and very ample line bundles

The aim of this section will be the study of sheaf-theoretical properties of a line bundle $G_L(\mathcal{L})$ constructed as above out of an L-linearized line bundle on the L-variety X. In particular, we shall show how to obtain ample and very ample line bundles on the variety $G_L(X)$ out of it. We shall begin with some ineffective results, obtained from general facts in algebraic geometry, and then improve them using the methods of [5].

Let us briefly recall some definitions. Let $q : X \to Y$ be a morphism of varieties, and $\mathcal{L}$ an invertible sheaf on X. The sheaf $\mathcal{L}$ is said to be *very ample with respect to* q (or *very ample over* Y), if there exist a quasi-coherent $\mathcal{O}_Y$-module ξ and an immersion $i : X \hookrightarrow \mathbb{P}(\xi)$ over Y such that $\mathcal{L}$ is isomorphic to $i^*\mathcal{O}_{\mathbb{P}(\xi)}(1)$. The sheaf $\mathcal{L}$ is said to be *ample with respect to* q (or *ample over* Y) if $\mathcal{L}^{\otimes n}$ is very ample with respect to q for some positive integer n. This definition of relative very ampleness, taken from EGA (see [9], II, pg. 79), is not equivalent to the one that can be found in Hartshorne's book ([12], pg. 120), which requires $\xi = \mathcal{O}_Y^{\oplus n}$ for some n (i.e. $\mathbb{P}(\xi) = \mathbb{P}^n_Y$). Here we must adopt Grothendieck's technically more complicated definition in order to be able to apply his results from [9], IV and [10] (which would not be available otherwise). Note that Hartshorne's and Grothendieck's notions of "very ample over k" coincide, since a quasi-coherent $\mathcal{O}_{\mathrm{Spec}k}$-module ξ corresponds to a k-vector space, and so $\mathbb{P}(\xi) \cong \mathbb{P}^n_k$ for some n.
The following technical result will be useful later:

Lemma 2.3.1. *Let $q : X \to Y$ be a morphism of varieties, and $\mathcal{L}$ an invertible sheaf on X, very ample with respect to q. Then, $q_*\mathcal{L}$ is a quasi-coherent $\mathcal{O}_Y$-module and there exists a closed immersion $i : X \hookrightarrow \mathbb{P}(q_*\mathcal{L})$ over Y such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}(q_*\mathcal{L})}(1)$.*

Proof: See [9], II, pp. 79-80. ■

The lemma states that (in our case of varieties) very ampleness can be defined by means of the sheaf $\xi = q_*\mathcal{L}$.

We are now ready to state the first result of this section. Our notations are the same as in the previous one: G is a connected algebraic group, L its largest linear and connected algebraic subgroup, A the abelian variety G/L and $G_L(X)$ resp. $G_L(\mathcal{L})$ the fibre bundle over A associated to an L-variety X resp. the line bundle on $G_L(X)$ associated to an L-linearized line bundle $\mathcal{L}$ on X.

Lemma 2.3.2. *Assume that $\mathcal{L}$ is ample (resp. very ample) on X. Then, the line bundle $G_L(\mathcal{L})$ is relatively ample (resp. relatively very ample) with respect to the natural projection $p : G_L(X) \to A$.*

Proof: Let $\mathcal{L}$ be ample on X, i.e. ample relatively to the structure morphism $X \to \mathrm{Spec}(k)$. Since relative ampleness is stable under base change (see [9], Vol. II, Prop. (4.6.13)(iii), pg. 91), it follows that $p_2^*\mathcal{L}$ is ample relatively to $p_1 : G \times X \to G$. From Lemma 2.1.6, together with the cartesian diagram

$$\begin{array}{ccc} G \times X & \longrightarrow & G_L(X) \\ \downarrow & & \downarrow \\ G & \longrightarrow & A \end{array}$$

it follows that $G_L(\mathcal{L})$ is ample relatively to $p : G_L(X) \to A$. If $\mathcal{L}$ is very ample, the proof is the same, with [9], Vol. II, Prop. (4.4.10) instead of [9], Vol. II, Prop. (4.6.13). ∎

The lemma shows that, in general, the line bundle $G_L(\mathcal{L})$ is not enough for a projective embedding of $G_L(X)$, since it is only very ample over A. In order to get a very ample line bundle, we have to consider sheaves of the type $p^*\mathcal{L}_0 \otimes G_L(\mathcal{L})$, where $\mathcal{L}_0$ is a sheaf on A. That is, we shall consider the map

$$\begin{array}{rcl} \mathrm{Pic}(A) \times \mathrm{Pic}^L(X) & \longrightarrow & \mathrm{Pic}(G_L(X)) \\ (\mathcal{L}_0, \mathcal{L}) & \longmapsto & p^*\mathcal{L}_0 \otimes G_L(\mathcal{L}) \end{array} .$$

Lemma 2.3.3.

1. *Let $\mathcal{L}_0$ be ample on A, and let $\mathcal{L}$ be ample on X. Then, there is a natural number n_0 such that*
$$p^*\mathcal{L}_0^{\otimes n} \otimes G_L(\mathcal{L})$$
is ample on $G_L(X)$ for all $n \geq n_0$.

2. *Let $\mathcal{L}_0$ be very ample on A, and let $\mathcal{L}$ be very ample on X. Then, there is a natural number n_0 such that*
$$p^*\mathcal{L}_0^{\otimes n} \otimes G_L(\mathcal{L})$$
is very ample on $G_L(X)$ for all $n \geq n_0$.

Proof:

1. Since $G_L(\mathcal{L})$ is ample relatively to $p : G_L(X) \to A$, it follows from [9], Vol. II, Prop. (4.6.13)(ii), pg. 91 that $p^*\mathcal{L}_0^{\otimes n} \otimes G_L(\mathcal{L})$ is ample relatively to $\mathrm{Spec}(k)$, i.e. ample, if n is large enough.

2. The statement is again a consequence of Lemma 2.3.2, together with [9], Vol. II, Prop. (4.4.10). ■

Lemma 2.3.3 is enough in order to show that $G_L(X)$ is a projective variety, if X is projective and $\mathcal{L}$ is an ample, L-linearized bundle. But it does not give any information on n_0: our next aim is to show that this number can be chosen to be equal to one. This is proven for commutative algebraic groups in [5], but the commutativity does not play a special role in the proof. The idea is to embed $G_L(X)$ in the projective space bundle $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is the direct image of $G_L(\mathcal{L})$ on the abelian variety A, and to use the properties of $\mathcal{F}$ to construct a very ample line bundle on $\mathbb{P}(\mathcal{F})$.
We begin by recalling some definitions. A vector bundle $\mathcal{V}$ on an abelian variety A is said to be *homogeneous*, if $T_x^*\mathcal{V} \cong \mathcal{V}$ for all $x \in A$. In particular, $\mathrm{Pic}^0(A)$ consists of the homogeneous line bundles (see [21]). A vector bundle $\mathcal{U}$ is said to be *unipotent*, if it admits a filtration

$$0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \subset \mathcal{U}_n = \mathcal{U}$$

such that $\mathcal{U}_i/\mathcal{U}_{i-1} \cong \mathcal{O}_A$ for $i = 1, \ldots, n$ (clearly, $n = \mathrm{rank}(\mathcal{U})$). We will denote by $\mathrm{U}(A)$ the set of all unipotent vector bundles on A.
All homogeneous vector bundles on an abelian variety can be constructed out of homogeneous line bundles and unipotent vector bundles as follows:

Theorem 2.3.4 (Matsushima, Morimoto, Miyanishi, Mukai). *Let $\mathcal{V}$ be a vector bundle on an abelian variety A. Then, the following conditions are equivalent:*

1. *$\mathcal{V}$ is homogeneous ;*

2. *there exist line bundles $\mathcal{P}_i$ in $\mathrm{Pic}^0(A)$ and vector bundles $\mathcal{U}_i \in \mathrm{U}(A)$, $i = 1, \ldots, m$ such that*

$$\mathcal{V} \cong \bigoplus_{i=1}^{m} (\mathcal{P}_i \otimes \mathcal{U}_i) \quad .$$

Proof: See [20], Thm. 4.17, pg. 256. ■

Remark 2.3.1. Keeping the notation of the theorem, it is clear that

$$\mathrm{rank}(\mathcal{V}) = \sum_{i=1}^{m} \mathrm{rank}(\mathcal{U}_i) \quad .$$

By the following proposition, Theorem 2.3.4 applies to the vector bundle $p_* G_L(\mathcal{L})$:

Proposition 2.3.5. *Let $\mathcal{L}$ be an L-linearized, invertible sheaf on the projective L-variety X. Then the vector bundle $p_* G_L(\mathcal{L})$ on A is homogeneous.*

Proof: Let $T_x : A \to A$ denote translation on the abelian variety A by $x \in A$. Our aim is to show that $T_x^* p_* G_L(\mathcal{L}) \cong p_* G_L(\mathcal{L})$ for all $x \in A$.
First of all, let us fix some notations. Denote by $\tau_g : G \to G$ left translation on G by an element g; this map induces the translation map $\tau_g^{G\times X} := \tau_g \times \mathrm{Id}_X : G \times X \to G \times X$, which is compatible with the action of L, and so a map $\tau_g^{G_L(X)} : G_L(X) \to G_L(X)$: the commutative square

$$\begin{tikzcd} G \times X \arrow[r, "\tau_g^{G\times X}"] \arrow[d, "q"'] & G \times X \arrow[d, "q"] \\ G_L(X) \arrow[r, "\tau_g^{G_L(X)}"] & G_L(X) \end{tikzcd}$$

shows that the relation

$$\tau_g^{G_L(X)} \circ q = q \circ \tau_g^{G\times X} \tag{2.9}$$

holds, where $q : G \times X \to G_L(X)$ denotes the quotient map. Consider furthermore the commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, "p_1"] \arrow[d, "q"'] & G \arrow[r, "\pi"] & A \\ G_L(X) \arrow[urr, "p"'] & & \end{tikzcd} \quad ;$$

the morphisms

$$T_{\pi(g)} \circ p : G_L(X) \longrightarrow A$$

and

$$p \circ \tau_g^{G_L(X)} : G_L(X) \longrightarrow A$$

pull back by means of $q : G \times X \to G_L(X)$ to the same L-equivariant morphism $G \times X \to A$:

$$\begin{aligned} \big(T_{\pi(g)} \circ p\big) \circ q &= T_{\pi(g)} \circ (\pi \circ p_1) = (\pi \circ p_1) \circ \tau_g^{G\times X} = \\ &= p \circ \Big(q \circ \tau_g^{G\times X}\Big) = \Big(p \circ \tau_g^{G_L(X)}\Big) \circ q \end{aligned}$$

and by the universal property of the quotient $G \times X \to G_L(X)$ it follows that they are the same: we get the relation

$$T_{\pi(g)} \circ p = p \circ \tau_g^{G_L(X)} \quad . \tag{2.10}$$

Let $\mathcal{L}$ be as above, and $\mathcal{L}' := p_2^* \mathcal{L}$ be its inverse image on $G \times X$, with the natural linearization obtained out of the L-linearization of $\mathcal{L}$ as in Prop. 2.2.1. As we have already seen, there is a natural isomorphism

$$\mathcal{L}' \cong q^* G_L(\mathcal{L}) \tag{2.11}$$

of L-linearized bundles on $G \times X$. This gives rise, by means of Lemma 2.2.2, to an isomorphism

$$\left(\tau_g^{G\times X}\right)^* \mathcal{L}' \cong \left(\tau_g^{G\times X}\right)^* q^* G_L(\mathcal{L}) \quad , \tag{2.12}$$

again compatibly with linearizations. Since $p_2 \circ \tau_g^{G\times X} = p_2$, the left-hand side of (2.12) becomes

$$\left(\tau_g^{G\times X}\right)^* \mathcal{L}' \cong \left(\tau_g^{G\times X}\right)^* p_2^* \mathcal{L} \cong p_2^* \mathcal{L} = \mathcal{L}'$$

canonically, and so it is L-isomorphic to $q^* G_L(\mathcal{L})$ by (2.11). Furthermore, an application of (2.9) to the right-hand side of (2.12) yields the canonical homomorphism

$$\left(\tau_g^{G\times X}\right)^* q^* G_L(\mathcal{L}) \cong q^* \left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L}) \quad ,$$

so that (2.12) becomes an isomorphism

$$q^* G_L(\mathcal{L}) \cong q^* \left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L})$$

of L-linearized sheaves on $G \times X$, which descends in a natural way to to an isomorphism

$$G_L(\mathcal{L}) \cong \left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L}) \quad . \tag{2.13}$$

Let us now consider the relation (2.10): it amounts to the commutativity of the square

$$\begin{tikzcd} G_L(X) \arrow[r, "\tau_g^{G_L(X)}"] \arrow[d, "p"'] & G_L(X) \arrow[d, "p"] \\ A \arrow[r, "T_{\pi(g)}"] & A \end{tikzcd} \quad . \tag{2.14}$$

From the relation (2.13), we get $p_* \left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L}) \cong p_* G_L(\mathcal{L})$. Proposition 9.3, pg. 255 of [12] states that the higher direct image functor $R^i p_*$ commutes with a flat morphism of the base; in particular, since the direct image functor p_* coincides with $R^0 p_*$, it follows from the commutative square (2.14) that

$$p_* \left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L}) \cong T_{\pi(g)}^* p_* G_L(\mathcal{L})$$

holds, and so that

$$T_{\pi(g)}^* p_* G_L(\mathcal{L}) \cong p_* G_L(\mathcal{L})$$

holds for all $g \in G$; since $\pi : G \to A$ is surjective, this has the consequence that

$$T_x^* p_* G_L(\mathcal{L}) \cong p_* G_L(\mathcal{L})$$

for all $x \in A$, i.e. $p_* G_L(\mathcal{L})$ is homogeneous. ■

Together with Theorem 2.3.4, the lemma gives an additive decomposition

$$p_* G_L(\mathcal{L}) \cong \bigoplus_i (\mathcal{P}_i \otimes \mathcal{W}_i) \quad , \quad \mathcal{P}_i \in \mathrm{Pic}^0(A) \quad , \quad \mathcal{W}_i \in \mathrm{U}(A) \quad .$$

This fact can be restated as follows (see also [14], §1 and [5], §II.1):

Corollary 2.3.6. *Let $\mathcal{L}$ be an L-linearized line bundle on the projective L-variety X. Then, the vector bundle $p_* G_L(\mathcal{L})$ on A admits a filtration*

$$\{0\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = p_* G_L(\mathcal{L}) \quad , \quad n = \dim H^0(X, \mathcal{L}) \tag{2.15}$$

such that $\mathcal{F}_i / \mathcal{F}_{i-1} \in \mathrm{Pic}^0(A)$, $i = 1, \ldots, n$.

Proof: We write again

$$p_* G_L(\mathcal{L}) = \bigoplus_{i=1}^m (\mathcal{P}_i \otimes \mathcal{W}_i)$$

with $\mathcal{P}_i \in \mathrm{Pic}^0(A)$ and $\mathcal{W}_i$ unipotent, $i = 1, \ldots, m$. To the $\mathcal{W}_i$ belong filtrations

$$\{0\} = \mathcal{W}_i^{(0)} \subset \mathcal{W}_i^{(1)} \subset \ldots \subset \mathcal{W}_i^{(n_i)} = \mathcal{W}_i$$

such that $\mathcal{W}_i^{(j+1)} / \mathcal{W}_i^{(j)} \cong \mathcal{O}_A$, $j = 1, \ldots, n_i$, $i = 1, \ldots, m$.
Consider the filtration given by

$$\begin{aligned}
\{0\} = \mathcal{P}_1 \otimes \mathcal{W}_1^{(0)} \subset \mathcal{P}_1 \otimes \mathcal{W}_1^{(1)} \subset \quad & \ldots \quad \subset \mathcal{P}_1 \otimes \mathcal{W}_1^{(n_1)} = \mathcal{P}_1 \otimes \mathcal{W}_1 \subset \\
\subset (\mathcal{P}_1 \otimes \mathcal{W}_1) \oplus \left(\mathcal{P}_2 \otimes \mathcal{W}_2^{(0)}\right) \subset \quad & \ldots \quad \subset (\mathcal{P}_1 \otimes \mathcal{W}_1) \oplus \left(\mathcal{P}_2 \otimes \mathcal{W}_2^{(n_2)}\right) \subset \ldots \\
& \ldots \quad \subset \bigoplus_{i=1}^m (\mathcal{P}_i \otimes \mathcal{W}_i) = p_* G_L(\mathcal{L}) \quad .
\end{aligned}$$

It has the properties stated in the corollary, since the quotient of two successive terms is isomorphic to some $\mathcal{P}_j$, and therefore it lies in $\mathrm{Pic}^0(A)$.
The fact that $n = \dim H^0(X, \mathcal{L})$ follows from Proposition 2.2.4. ■

Remark 2.3.2. Assuming L solvable, we could have recovered the filtration from an application of the Lie-Kolchin Theorem to the representation of L on $H^0(X, \mathcal{L})$, and using the functorial properties of $G_L(\,\cdot\,)$ as in [14], Prop. 1.9, since $p_* G_L(\mathcal{L}) \cong G_L(H^0(X, \mathcal{L})^{\sim})$.

The filtration in Corollary 2.3.6 is required in order to apply the following, crucial result from [5]:

Lemma 2.3.7 (Faltings-Wüstholz). *Let A be an abelian variety, and let $\mathcal{F}$ be a vector bundle on A, such that there is a filtration*

$$\{0\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = \mathcal{F}$$

with $\mathcal{F}_i/\mathcal{F}_{i-1} \in \mathrm{Pic}^0(A)$ for $i = 1, \ldots, r$. Then, if $\mathcal{L}_0$ is a very ample line bundle on A, the line bundle $\mathcal{M} = q^*\mathcal{L}_0 \otimes \mathcal{O}(1)$ is very ample on the projective space bundle $\mathbb{P}(\mathcal{F})$, where $q : \mathbb{P}(\mathcal{F}) \to A$ denotes the canonical projection.

Proof: See [5], Lemma 2, pg. 183. ■

At this point, we have collected all elements for the proof of the announced strengthening of Lemma 2.3.3:

Theorem 2.3.8. *Let*

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

be an extension of an abelian variety with a connected linear algebraic group. Let X be a projective L-variety and $\mathcal{L}$ an L-linearized, very ample line bundle on X. Let $\mathcal{L}_0$ be a very ample line bundle on A. Then, the line bundle

$$p^*\mathcal{L}_0 \otimes G_L(\mathcal{L})$$

is very ample on $G_L(X)$.

Proof: Lemma 2.3.2 shows that $G_L(\mathcal{L})$ is very ample with respect to the map $p : G_L(X) \to A$. With Lemma 2.3.1 it follows that there is a natural immersion

$$i : G_L(X) \hookrightarrow \mathbb{P}(p_*G_L(\mathcal{L}))$$

over A such that

$$i^*\mathcal{O}_{\mathbb{P}(p_*G_L(\mathcal{L}))}(1) \cong G_L(\mathcal{L}) \quad .$$

Lemma 2.3.7 and Corollary 2.3.6 show that the line bundle

$$\mathcal{M} := q^*\mathcal{L}_0 \otimes \mathcal{O}(1)$$

(where $q : \mathbb{P}(p_*G_L(\mathcal{L})) \to A$ is the natural projection) is very ample on $\mathbb{P}(p_*G_L(\mathcal{L}))$; together with the commutative diagram

$$\begin{tikzcd} G_L(X) \arrow[rr, hook, "i"] \arrow[dr, "p"'] & & \mathbb{P}(p_*G_L(\mathcal{L})) \arrow[dl, "q"] \\ & A & \end{tikzcd}$$

this shows that

$$i^*\mathcal{M} \cong i^*q^*\mathcal{L}_0 \otimes i^*\mathcal{O}(1) \cong p^*\mathcal{L}_0 \otimes G_L(\mathcal{L})$$

is very ample on $G_L(X)$. ■

An immediate consequence of the Theorem is the following

Corollary 2.3.9. *With the same notations as in Theorem 2.3.8, assume that $\mathcal{L}$ (resp. $\mathcal{L}_0$) is an ample, L-linearized line bundle on X (resp. an ample line bundle on A). Then, $p^*\mathcal{L}_0 \otimes G_L(\mathcal{L})$ is ample on $G_L(X)$.*

Proof: Let n_1 and n_2 be positive integers such that $\mathcal{L}^{\otimes m}$ resp. $\mathcal{L}_0^{\otimes m}$ is very ample on X resp. on A for $m \geq n_1$ resp. $m \geq n_2$ (see [12], Ex. 7.5, pg. 169). Let $n := \max(n_1, n_2)$. Then,

$$p^*\mathcal{L}_0^{\otimes n} \otimes G_L(\mathcal{L}^{\otimes n}) \cong \left(p^*\mathcal{L}_0 \otimes G_L(\mathcal{L})\right)^{\otimes n}$$

is very ample on $G_L(X)$ by Thm. 2.3.8, and this shows that $p^*\mathcal{L}_0 \otimes G_L(\mathcal{L})$ is ample (see [12], Thm. 7.6, pg. 154). ■

The above results show that, provided X is a projective variety with an ample, L-linearized bundle $\mathcal{L}$, $G_L(X)$ is projective. In §3.4, we shall prove the existence of an L-linearized ample line bundle on the completion $\overline{L}$ of the linear algebraic group L, and so that $\overline{G} = G_L(\overline{L})$ is not only proper (see §1.3), but also projective (i.e. that if $\overline{L}$ is a compactification of L, $\overline{G}$ is a compactification of G).

Chapter 3

Further results

In this chapter we draw some consequences from the results of Chapter 2. We begin with some cohomological computations on the associated bundle (which are essentially the same as in [37]), and successively we show how they imply results on the projective embeddings of the group. Then we extend a work of Lange's (see [17]) on the translation formulas on an algebraic group, and in the final section we resume the most important results which we have obtained so far.

3.1 Cohomology and Riemann-Roch

In this section, we study the sheaf $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$ constructed in the previous chapter from the cohomological point of view. For this purpose, we project the sheaf on the abelian variety A by means of the natural projection p, and use the decomposition of the sheaf $p_*G_L(\mathcal{L})$ provided by Mukai's Theorem 2.3.4, in order to be able to apply the well-known results on the cohomology of a line bundle on an abelian variety.

Let us recall some more facts about abelian varieties. An invertible sheaf $\mathcal{L}$ on an abelian variety is said to be *nondegenerated*, if the group

$$\mathrm{K}(\mathcal{L}) := \{x \in A \mid T_x^*\mathcal{L} \cong \mathcal{L}\}$$

is finite. In Mumford's book, the following result on the cohomology of a nondegenerate invertible sheaf is proved:

Theorem 3.1.1. *If for a line bundle $\mathcal{L}$ on an abelian variety A, $\mathrm{K}(\mathcal{L})$ is finite, there is a unique integer $i = i(\mathcal{L})$, $0 \leq i \leq \dim(A)$, such that $H^p(A, \mathcal{L}) = 0$ for $p \neq i$ and $H^i(A, \mathcal{L}) \neq 0$.*

Proof: See [21], pg. 150. ■

The number i is the *index* of the nondegenerate sheaf $\mathcal{L}$.
As an application of this vanishing theorem, we get the following result on the cohomology of the sheaf $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$:

Theorem 3.1.2. *Let $\mathcal{L}_0$ be a nondegenerate, invertible sheaf on the abelian variety A. Let $\mathcal{L}$ be an L-linearized invertible sheaf on the projective L-variety X, and $G_L(\mathcal{L})$ the sheaf on $G_L(X)$ constructed previously. Assume that $H^d(X, \mathcal{L}) = 0$ for $d > 0$ and let $i := i(\mathcal{L}_0)$ be the index of $\mathcal{L}_0$. Then, $H^d(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) \neq 0$ if and only if $d = i$.*

Proof: (see also [37], Thm. 6.1). We first show that, in the situation of the theorem, one can compute the cohomology on the direct image, i.e.

$$H^d(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) \cong H^d\left(A, p_*\left(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0\right)\right) \quad .$$

A sufficient condition (see [12], Ex. 8.1, pg. 252) is the vanishing of the the direct images $R^i p_*(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0)$ for $i > 0$. Since, by the projection formula,

$$R^i p_*\left(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0\right) \cong R^i p_* G_L(\mathcal{L}) \otimes \mathcal{L}_0$$

(see [12], Ex. 8.3, pg. 253), it will be sufficient that $R^i p_* G_L(\mathcal{L}) = 0$ for $i > 0$. But this follows from the condition $H^i(X, \mathcal{L}) = 0$ since, by Prop. 2.2.4,

$$R^i p_* G_L(\mathcal{L}) \cong G_L(H^i(X, \mathcal{L})^\sim) \quad .$$

This proves that

$$H^d(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) \cong H^d\left(A, p_*\left(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0\right)\right) \cong H^d(A, p_* G_L(\mathcal{L}) \otimes \mathcal{L}_0)$$

(the latter again by the projection formula).
As we showed above, $p_* G_L(\mathcal{L})$ is an homogeneous vector bundle on A, and so it can be written in the form

$$p_* G_L(\mathcal{L}) \cong \bigoplus_{j=1}^{k} \left(\mathcal{P}_j \otimes \mathcal{U}_j\right) \quad ,$$

with $\mathcal{P}_j \in \mathrm{Pic}^0(A)$ and $\mathcal{U}_j \in \mathrm{U}(A)$. Hence,

$$H^d(A, p_* G_L(\mathcal{L}) \otimes \mathcal{L}_0) \cong \bigoplus_{j=1}^{k} H^d(A, \mathcal{P}_j \otimes \mathcal{U}_j \otimes \mathcal{L}_0) \quad .$$

By [16], Cor. 2, pg. 100, we can find for each j an $x_j \in A$ such that $\mathcal{P}_j \cong T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{L}_0^{\otimes -1}$, since $\mathcal{P}_j \in \mathrm{Pic}^0(A)$. From this, it follows that $\mathcal{P}_j \otimes \mathcal{U}_j \otimes \mathcal{L}_0 \cong T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j$, and so that

$$H^d(A, p_* G_L(\mathcal{L}) \otimes \mathcal{L}_0) \cong \bigoplus_{j=1}^{k} H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j) \quad .$$

Recall that each $\mathcal{U}_j$, being unipotent, admits a filtration

$$0 = \mathcal{U}_j^{(0)} \subset \mathcal{U}_j^{(1)} \subset \ldots \subset \mathcal{U}_j^{(n_j-1)} \subset \mathcal{U}_j^{(n_j)} = \mathcal{U}$$

with trivial quotients. Proceeding by induction on l, we will now show that the vector space $H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)})$ is trivial if and only if $d \neq i$, for $l = 1, \ldots, n_j$. For $l = 1$, we have, since $\mathcal{O}_A \cong \mathcal{U}_j^{(1)}/\mathcal{U}_j^{(0)} \cong \mathcal{U}_j^{(1)}$,

$$H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(1)}) \cong H^d(A, T_{x_j}^*\mathcal{L}_0) \quad ,$$

and this is trivial if and only if $d \neq i$. Now assume that the hypothesis

$$H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l')}) = 0 \qquad \Longleftrightarrow \qquad d \neq i$$

holds for $l' < l$. The short exact sequence

$$0 \longrightarrow T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l-1)} \longrightarrow T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)} \longrightarrow T_{x_j}^*\mathcal{L}_0 \longrightarrow 0$$

gives rise to the long exact cohomology sequence

$$\cdots \longrightarrow H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l-1)}) \longrightarrow H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)}) \longrightarrow H^d(A, T_{x_j}^*\mathcal{L}_0) \longrightarrow \cdots \quad .$$

Let $d \neq i(\mathcal{L}_0)$. Then, as above, $H^d(A, T_{x_j}^*\mathcal{L}_0) = 0$. Furthermore, by the induction hypothesis, $H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l-1)}) = 0$, and so $H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)}) = 0$. If, on the other side, $d = i(\mathcal{L}_0)$, we get the sequence

$$0 \longrightarrow H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l-1)}) \longrightarrow H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)}) \longrightarrow H^d(A, T_{x_j}^*\mathcal{L}_0) \longrightarrow 0 \quad .$$

Here we have $H^d(A, T_{x_j}^*\mathcal{L}_0) \neq 0$; from the short exact sequence, it follows immediately that $H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)}) \neq 0$.

This concludes the proof, since $H^d(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0)$ is the direct sum over j of the cohomology groups $H^d(A, T_{x_j}^*\mathcal{L}_0 \otimes \mathcal{U}_j)$. ■

Remark 3.1.1. The use of the filtration of $\mathcal{U}_i$ in the proof is equivalent to the use of the filtration of $p_*G_L(\mathcal{L})$ in [37], Thm. 6.1.

If one assumes that $H^0(A, \mathcal{L}_0) \neq 0$, ampleness and nondegeneracy are equivalent properties (see for instance [19], Prop. 7.1, pg. 33). This is the case, if $\mathcal{L}_0 = \mathcal{L}_0(D)$ for an effective divisor D on the abelian variety (as in [21], Application 1, pg. 60). A direct consequence is the

Corollary 3.1.3. *Let $\mathcal{L}_0$ be an ample invertible sheaf on A, with $H^0(A, \mathcal{L}_0) \neq 0$. Let $\mathcal{L}$ and X be as in the Theorem. Then, $H^0(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) \neq 0$ and furthermore $H^d(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) = 0$ for all $d > 0$.*

We now compute the Euler characteristic

$$\chi(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) = \sum_i (-1)^i \dim \left(H^i(G_L(X), G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0)\right)$$

of the sheaf $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$ as a function of $\chi(\mathcal{L}_0)$.

Theorem 3.1.4. *Let $\mathcal{L}_0$ be an invertible sheaf on the abelian variety A, and $G_L(\mathcal{L})$ the line bundle on $G_L(X)$ constructed above. Then,*

$$\chi(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) = \dim \left(H^0(X, \mathcal{L})\right) \cdot \chi(\mathcal{L}_0) \quad .$$

Proof: As in the proof of Theorem 3.1.2, a combined application of Mukai's Theorem and the projection formula yields

$$\chi(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) = \chi\left(\mathcal{L}_0 \otimes \bigoplus_{j=1}^{k} \mathcal{P}_j \otimes \mathcal{U}_j\right) = \sum_{j=1}^{k} \chi(\mathcal{L}_0 \otimes \mathcal{P}_j \otimes \mathcal{U}_j)$$

with $\mathcal{P}_j \in \mathrm{Pic}^0(A)$ and $\mathcal{U}_j \in \mathrm{U}(A)$, and we get $x_1, \ldots, x_k \in A$ with $\mathcal{P}_j \cong T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{L}_0^{\otimes -1}$, so that

$$\chi(\mathcal{L}_0 \otimes \mathcal{P}_j \otimes \mathcal{U}_j) = \chi(T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{U}_j) \quad .$$

We compute this Euler characteristic as follows: from the filtration

$$0 = \mathcal{U}_j^{(0)} \subset \mathcal{U}_j^{(1)} \subset \ldots \subset \mathcal{U}_j^{(n_j-1)} \subset \mathcal{U}_j^{(n_j)} = \mathcal{U}$$

with trivial quotients, we get again the exact sequences

$$0 \longrightarrow T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{U}_j^{(l-1)} \longrightarrow T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)} \longrightarrow T^*_{x_j}\mathcal{L}_0 \longrightarrow 0$$

for $l = 0, \ldots, n_j$, so that

$$\chi(T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{U}_j^{(l)}) = \chi(T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{U}_j^{(l-1)}) + \chi(T^*_{x_j}\mathcal{L}_0)$$

(by [12], Ex. 5.1, pg. 230). An iterated application of this relation gives

$$\chi(T^*_{x_j}\mathcal{L}_0 \otimes \mathcal{U}_j) = n_j \cdot \chi(T^*_{x_j}\mathcal{L}_0) = n_j \cdot \chi(\mathcal{L}_0) \quad ,$$

and so

$$\chi(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) = \sum_{j=1}^{k} n_j \cdot \chi(\mathcal{L}_0) = \mathrm{rank}(p_*G_L(\mathcal{L})) \cdot \chi(\mathcal{L}_0) \quad .$$

This concludes the proof since, as showed in Prop. 2.2.4, the rank of $p_*G_L(\mathcal{L})$ is equal to the dimension of $H^0(X, \mathcal{L})$. ■

We recall the Riemann-Roch Theorem for an invertible sheaf on an abelian variety:

Theorem 3.1.5. *For a line bundle $\mathcal{L}_0$ on an n-dimensional abelian variety A, we have*

$$\chi(\mathcal{L}_0) = \frac{\left(\mathcal{L}_0^n\right)}{n!} \quad ,$$

where $\left(\mathcal{L}_0^n\right)$ denotes the n-fold self intersection number of $\mathcal{L}_0 \in \mathrm{Pic}(A)$.

Proof: See [21], pg. 150. ■

From this, together with Theorem 3.1.4, we get the following result for $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$:

Corollary 3.1.6. *Let G be a connected algebraic group, L its largest linear and connected subgroup and A the abelian variety G/L. Let X be a projective L-variety, $\mathcal{L}$ an L-linearized invertible sheaf on X and $\mathcal{L}_0$ a nondegenerate, invertible sheaf on A. Let $G_L(\mathcal{L})$ be the line bundle constructed above on the variety $G_L(X)$, and $p : G_L(X) \to A$ the natural projection. Then,*

$$\chi(G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0) = \dim\left(H^0(X, \mathcal{L})\right) \cdot \frac{\left(\mathcal{L}_0^{\dim(A)}\right)}{\dim(A)!} \quad .$$

If $\mathcal{L}$ and $\mathcal{L}_0$ are both very ample, we know from Theorem 2.3.8 that $G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$ is also very ample. This implies, with Theorem 3.1.2, that its Euler characteristic coincides with the dimension of the space of its global sections. Hence, the corollary provides an upper bound for the dimension of a projective embedding of $G_L(X)$.

3.2 Normal generation and normal presentation

We describe two further applications of the vanishing theorem 3.1.2. The first one regards the projective normality of the embedding of the variety $G_L(X)$ in projective space, while the second one gives conditions under which the embedded variety is defined by homogeneous polynomials of degree two.

Let X be a projective variety, and $\mathcal{L}$ an ample line bundle on X. Following [22], we call $\mathcal{L}$ *normally generated*, if the natural map

$$H^0(X, \mathcal{L})^{\otimes k} \longrightarrow H^0(X, \mathcal{L}^{\otimes k})$$

is surjective for all $k \geq 1$. By [22], pp. 38-39, if $\mathcal{L}$ is normally generated, then it is very ample, and X is projectively normal with respect to the embedding $X \hookrightarrow \mathbb{P}^M = \mathbb{P}(H^0(X, \mathcal{L}))$.
The following criterion for normal generation is also taken from [22]: for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, define two groups $\mathcal{R}(\mathcal{F}, \mathcal{G})$ and $\mathcal{S}(\mathcal{F}, \mathcal{G})$ by means of the exact sequence

$$0 \to \mathcal{R}(\mathcal{F}, \mathcal{G}) \to H^0(X, \mathcal{F}) \otimes_k H^0(X, \mathcal{G}) \xrightarrow{\psi} H^0(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \to \mathcal{S}(\mathcal{F}, \mathcal{G}) \to 0 \quad ,$$

i.e. let $\mathcal{R}(\mathcal{F},\mathcal{G})$ and $\mathcal{S}(\mathcal{F},\mathcal{G})$ be respectively the kernel and the cokernel of the natural map ψ. Then, the ample sheaf $\mathcal{L}$ on X is normally generated, if and only if $\mathcal{S}(\mathcal{L}^{\otimes i},\mathcal{L})=0$ for all $i\geq 1$.
The following theorem, the so-called Generalized Lemma of Castelnuovo, is very useful in this context, since it relates projective normality to cohomology:

Theorem 3.2.1 (Mumford). *Let $\mathcal{L}$ be an ample, invertible sheaf on a projective variety X, generated by its global sections. Suppose $\mathcal{F}$ is a coherent sheaf on X, such that*

$$H^i(X,\mathcal{F}\otimes\mathcal{L}^{\otimes -i})=0 \quad , \quad i\geq 1 \quad .$$

Then

1. $H^i(X,\mathcal{F}\otimes\mathcal{L}^{\otimes j})=0$, *if* $i+j\geq 0, i\geq 1$
2. $\mathcal{S}(\mathcal{F}\otimes\mathcal{L}^{\otimes i},\mathcal{L})=0$, *if* $i\geq 0$.

Proof: See [22], pp. 41 and ff. ■

Consider again the sheaf $\mathcal{M}:=G_L(\mathcal{L})\otimes p^*\mathcal{L}_0$ on $G_L(X)$, constructed out of an L-linearized sheaf $\mathcal{L}$ on X and a nondegenerate invertible sheaf $\mathcal{L}_0$ on A. As in [22], Theorem 3, pg. 45, an application of the Generalized Lemma of Castelnuovo, together with the Vanishing Theorem 3.1.2, yields the following result:

Lemma 3.2.2. *Let $\mathcal{L}$ and $\mathcal{L}_0$ be ample and generated by their global sections, and $\mathcal{M}$ as above. Then*

$$\mathcal{S}(\mathcal{M}^{\otimes k},\mathcal{M}^{\otimes \ell})=0$$

if $k\geq \dim G_L(X)+1$ *and* $\ell\geq 1$.

Proof:(see also [22], Thm. 3, pg. 45) Let first $\ell=1$, and set in the hypotheses of Theorem 3.2.1 $\mathcal{F}:=\mathcal{M}^{\otimes k}$ and $\mathcal{L}:=\mathcal{M}$. The invertible sheaf $\mathcal{M}$ is ample (Cor. 2.3.9) and generated by its global sections (since both $p^*\mathcal{L}_0$ and $G_L(\mathcal{L})$ are, for the latter see pg. 45). Furthermore,

$$H^i(G_L(X),\mathcal{F}\otimes\mathcal{L}^{\otimes -i})=H^i(G_L(X),\mathcal{M}^{\otimes k-i}).$$

If $i\geq \dim G_L(X)+1$, cohomology vanishes by the Grothendieck Vanishing Theorem (see [12], pg. 208). If $1\leq i\leq \dim G_L(X)$, we have $k-i>0$, and

$$H^i(G_L(X),\mathcal{M}^{\otimes k-i})=H^i(G_L(X),G_L(\mathcal{L}^{\otimes k-i})\otimes p^*\mathcal{L}_0^{\otimes k-i})=0$$

by ”our” vanishing theorem, since $G_L(\mathcal{L}^{\otimes k-i})\otimes p^*\mathcal{L}_0^{\otimes k-i}=(G_L(\mathcal{L})\otimes p^*\mathcal{L}_0)^{\otimes k-i}$ is again very ample. This proves that $\mathcal{S}(\mathcal{M}^{\otimes k},\mathcal{M})=0$ for $k\geq \dim G_L(X)+1$. Explicitely, that means

$$H^0(G_L(X),\mathcal{M}^{\otimes k})\otimes H^0(G_L(X),\mathcal{M})\longrightarrow H^0(G_L(X),\mathcal{M}^{\otimes k+1})$$

is surjective, if $k \geq \dim G_L(X) + 1$. By an inductive argument, one gets surjectivity for

$$H^0(G_L(X), \mathcal{M}^{\otimes k}) \otimes H^0(G_L(X), \mathcal{M})^{\otimes \ell} \xrightarrow{\psi} H^0(G_L(X), \mathcal{M}^{\otimes k+\ell}) \quad ,$$

if $\ell \geq 1$. Now, since ψ factorizes through the maps

$$\begin{aligned} H^0(G_L(X), \mathcal{M}^{\otimes k}) \otimes H^0(G_L(X), \mathcal{M})^{\otimes \ell} &\longrightarrow H^0(G_L(X), \mathcal{M}^{\otimes k}) \otimes H^0(G_L(X), \mathcal{M}^{\otimes \ell}) \\ &\longrightarrow H^0(G_L(X), \mathcal{M}^{\otimes k+\ell}) \quad , \end{aligned}$$

it follows that the latter is also surjective for $\ell \geq 1$, if $k \geq \dim G_L(X) + 1$. That means by definition

$$\mathcal{S}(\mathcal{M}^{\otimes k}, \mathcal{M}^{\otimes \ell}) = 0 \quad .$$

This concludes the proof. ■

In particular, for $\widetilde{\mathcal{M}} := \mathcal{M}^{\otimes \dim G_L(X)+1}$, we get $\mathcal{S}(\widetilde{\mathcal{M}}^{\otimes i}, \widetilde{\mathcal{M}}) = 0$ for all $i \geq 1$. By the criterion for normal generation, together with Theorem 2.3.8, we get the following

Corollary 3.2.3. *Let*

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$$

be an extension of an abelian variety A with a linear algebraic group L. Let $\mathcal{L}_0$ be an ample line bundle on A, generated by its global sections, and let $G_L(\mathcal{L})$ be constructed out of an L-linearized, ample line bundle $\mathcal{L}$ on X, generated by its global sections. Let $k \geq \dim G_L(X)+1$. Then, the line bundle

$$G_L(\mathcal{L}^{\otimes k}) \otimes p^* \mathcal{L}_0^{\otimes k}$$

is very ample on $G_L(X)$, and its global sections define a projectively normal embedding of the variety $G_L(X)$ (where p denotes again the canonical projection $G_L(X) \to A$).

Let us now sketch the second promised application of the vanishing theorem. Let $\mathcal{L}$ be a normally generated line bundle on a projective variety X. Let

$$R_X = \bigoplus_{k \geq o} H^0(X, \mathcal{L}^{\otimes k})$$

be the projective coordinate ring of X with respect to its projective embedding in $\mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{L}))$. Denote by $I_X^{(k)}$ the kernel of the natural map

$$S^k H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^{\otimes k}) \quad ,$$

so that

$$I_X = \bigoplus_{k \geq 0} I_X^{(k)}$$

is the graded ideal of X in $\mathbb{P}^N$:

$$R_X = R/I_X \cong \bigoplus_{k\geq 0} \left(S^k H^0(X,\mathcal{L})/I_X^{(k)} \right) \quad ,$$

where $R = \bigoplus_{k\geq 0} S^k H^0(X,\mathcal{L}) \cong k[X_0,\ldots,X_N]$. Again following [22], we call $\mathcal{L}$ *normally presented*, if for all $k \geq 2$ the natural map

$$I_X^{(2)} \otimes S^{k-2} H^0(X,\mathcal{L}) \longrightarrow I_X^{(k)}$$

is surjective. In that case, the ideal I_X of X in $\mathbb{P}^N$ is generated by homogeneous polynomials of degree 2. One says in this case that X is *cut out by quadrics*.
Recall that we defined, for two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, the group $\mathcal{R}(\mathcal{F},\mathcal{G})$ as the kernel of the natural map $H^0(X,\mathcal{F}) \otimes H^0(X,\mathcal{G}) \to H^0(X,\mathcal{F}\otimes\mathcal{G})$. In [22], the following criterion for normal presentation is given :

Lemma 3.2.4. *Let $\mathcal{L}$ be a normally generated invertible sheaf on a projective variety X. Then, $\mathcal{L}$ is normally presented if and only if the natural map*

$$\mathcal{R}(\mathcal{L}^{\otimes i},\mathcal{L}^{\otimes j}) \otimes H^0(X,\mathcal{L}^{\otimes k}) \longrightarrow \mathcal{R}(\mathcal{L}^{\otimes i+k},\mathcal{L}^{\otimes j})$$

is surjective for all $i, j, k \geq 1$.

Proof: See [22], pp. 39-40. Note that we exchanged the roles of $\mathcal{L}^{\otimes i}$ and $\mathcal{L}^{\otimes j}$, but this is no problem, since the proof is symmetric in i and j. ■

The following Theorem, which is proven in [22], relates normal presentation to the vanishing of the higher cohomology groups:

Theorem 3.2.5 (Mumford). *Let $\mathcal{L}$ be an ample invertible sheaf on a projective variety X. Assume that $\mathcal{L}$ is generated by its global sections, and that $H^i(X,\mathcal{L}^{\otimes j}) = 0$ for $i, j \geq 1$. Then, the natural map*

$$\mathcal{R}(\mathcal{L}^{\otimes i},\mathcal{L}^{\otimes j}) \otimes H^0(X,\mathcal{L}^{\otimes k}) \longrightarrow \mathcal{R}(\mathcal{L}^{\otimes i+k},\mathcal{L}^{\otimes j})$$

is surjective, if $i \geq \dim X + 2$ and $j, k \geq 1$.

Proof: See [22], pp. 50-51. ■

Recall the definitions of the variety $G_L(X)$ and the sheaf $\mathcal{M} = G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$ on it. By Theorem 3.2.5, if we set $\widetilde{\mathcal{M}} := \mathcal{M}^{\otimes \dim G_L(X)+2}$, the natural map

$$\mathcal{R}(\widetilde{\mathcal{M}}^{\otimes i},\widetilde{\mathcal{M}}^{\otimes j}) \otimes H^0(G_L(X),\widetilde{\mathcal{M}}^{\otimes k}) \longrightarrow \mathcal{R}(\widetilde{\mathcal{M}}^{\otimes i+k},\widetilde{\mathcal{M}}^{\otimes j})$$

is surjective for all $i, j, k \geq 1$, if both $\mathcal{L}$ and $\mathcal{L}_0$ are ample and generated by their global sections.
Together with Lemma 3.2.4, we get the following sharpening of Corollary 3.2.3:

Corollary 3.2.6. *Let*

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0$$

be an extension of an abelian variety A with a linear algebraic group L. Let $\mathcal{L}_0$ be an ample line bundle on A, generated by its global sections, and let $G_L(\mathcal{L})$ be constructed out of an L-linearized, ample line bundle $\mathcal{L}$ on X, generated by its global sections. Let $k \geq \dim G_L(X)+2$. Then, the line bundle

$$G_L(\mathcal{L}^{\otimes k}) \otimes p^*\mathcal{L}_0^{\otimes k}$$

is very ample on $G_L(X)$, and its global sections define a projectively normal embedding of the variety $G_L(X)$ in some $\mathbb{P}^M$, such that the image of $G_L(X)$ in $\mathbb{P}^M$ is cut out by quadrics.

3.3 Families of translations

In the previous sections, we showed how it is possible to embed an algebraic group in projective space. For number-theoretical purposes, it is often convenient to have bounds for the degree of homogeneous polynomials describing the translation on the group. In his work [17], H. Lange shows that, for commutative algebraic groups, translation can be described locally by quadratic forms. The aim of this section will be to extend these results to algebraic groups without the restriction of commutativity. As a matter of fact, we shall show that, once the results of the previous sections are known, Lange's ideas can be immediately applied also to the noncommutative case.

We begin by recalling the notations of [17], and adapting them to our purposes. Let, as before,

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

be an extension of an abelian variety with a linear algebraic group. Let X be a projective variety with an action of L. As before, we construct the projective variety $G_L(X)$ and, if $\mathcal{L}$ is an L-linearized line bundle on X, the line bundle $G_L(\mathcal{L})$ on $G_L(X)$. If $\mathcal{L}$ is very ample, and $\mathcal{L}_0$ is a very ample line bundle on the abelian variety A, we showed that

$$\mathcal{M} := G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$$

(where we denote again by $p : G_L(X) \to A$ the natural map) is very ample. Furthermore, by a result in the previous section, we can assume that the projective embedding defined by $\mathcal{M}$ is projectively normal (since we can always replace $\mathcal{L}$ and $\mathcal{L}_0$ by suitable powers).
As we have already seen, $G_L(X)$ admits in a natural way an action of G which we shall denote by $\phi : G \times G_L(X) \to G_L(X)$. Let $U = \mathrm{Spec}(R)$ be a nonempty affine open set in G; following [17], we call the map

$$\begin{array}{rcl}(p_1, \phi) : U \times G_L(X) & \longrightarrow & U \times G_L(X) \\ (u, g) & \longmapsto & (u, ug)\end{array}$$

(p_1 denoting the first projection) the *family of translations on $G_L(X)$ parametrized by U*. Now assume that $G_L(X)$ is projectively embedded as above. Let $V \subseteq U \times G_L(X)$ be a nonempty open subset and $n \geq 1$ an integer. The family $(p_1, \phi) : U \times G_L(X) \to U \times G_L(X)$ is said to be *described on V by forms of degree n*, if there exist $f_0, \ldots, f_N \in R[X_0, \ldots, X_N]$ homogeneous of degree n such that

$$(p_1, \phi)(u, g) = (u,\ f_0(u; \underline{X}(g)), \ldots, f_N(u; \underline{X}(g))) \quad \forall (u, g) \in V \quad .$$

By the notation $f_i(u; \underline{X}(g))$ we mean that coordinates of u in a local chart are inserted in the coefficients of the polynomial $f_i \in R[X_0, \ldots, X_N]$, and $\underline{X}(g) = (X_0(g), \ldots, X_N(g))$ denote projective coordinates for $g \in G_L(X) \subseteq \mathbb{P}^N$. If $U \times G_L(X)$ admits an open cover $\mathcal{V} = \{V_i\}_{i \in I}$ such that (p_1, ϕ) is described on V_i by forms of degree n for every $i \in I$, one says that (p_1, ϕ) can be described *completely* on $U \times G_L(X)$ by forms of degree n.
Following [17], we shall now show that in our case n can always be chosen to be equal to 2 (i.e. (p_1, ϕ) is described by *quadratic* forms on $U \times G_L(X)$).
Assume again that $G_L(X)$ is embedded in $\mathbb{P}^N$ by means of the invertible sheaf $\mathcal{M}$ in a projectively normal way, and let $U = \mathrm{Spec}(R)$ as above. Denote by $p_2 : U \times G_L(X) \to G_L(X)$ the second projection. Then, the sheaf

$$\overline{\mathcal{L}} := p_2^* \mathcal{M} = p_2^* \left(G_L(\mathcal{L}) \otimes p^* \mathcal{L}_0 \right)$$

induces an embedding

$$U \times G_L(X) \hookrightarrow U \times \mathbb{P}^N = \mathbb{P}^N_U \quad .$$

The following proposition expresses the fact that a family of translations is described by forms of degree n as above in terms of the sheaf $\overline{\mathcal{L}}$:

Proposition 3.3.1. *With the notations above, assume that the invertible sheaf*

$$(p_1, \phi)^* \overline{\mathcal{L}}^{\otimes -1} \otimes \overline{\mathcal{L}}^{\otimes n}$$

on $U \times G_L(X)$ is generated by its global sections; then, the family of translations (p_1, ϕ) can be described completely by forms of degree n on $U \times G_L(X)$.

Proof: See [17], Lemma 2, pg. 263. ■

We are ready to prove the

Theorem 3.3.2. *Let $G_L(X) \hookrightarrow \mathbb{P}^N$ be a projectively normal embedding as above. Let $g_0 \in G$ be a given point. Then, there is an open neighborhood U of g_0 such that the family of translations (p_1, ϕ) can be described completely by quadratic forms on $U \times G_L(X)$.*

According to Proposition 3.3.1, the Theorem follows from the

Lemma 3.3.3. *Let the notations be as above. For every $g_o \in G$ there is an open neighborhood $U \subseteq G$ containing g_0 such that $(p_1, \phi)^* \overline{\mathcal{L}}^{\otimes -1} \otimes \overline{\mathcal{L}}^{\otimes 2}$ is generated by its global sections on $U \times G_L(X)$.*

Proof: As we already noticed in the proof of Proposition 2.3.5, $G_L(\mathcal{L})$ is invariant under the action of G on $G_L(X)$:

$$\left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L}) \cong G_L(\mathcal{L}) \quad \forall g \in G \quad .$$

This has the consequence that

$$\begin{aligned} \phi^* G_L(\mathcal{L}) \otimes p_2^* G_L(\mathcal{L})^{\otimes -1}|_{\{g\} \times G_L(X)} &\cong \left(\tau_g^{G_L(X)}\right)^* G_L(\mathcal{L}) \otimes G_L(\mathcal{L})^{\otimes -1} \cong \\ &\cong \mathcal{O}_{G_L(X)} \quad \forall g \in G \end{aligned}$$

(under the natural identification $\{g\} \times G_L(X) \xrightarrow{\sim} G_L(X)$). Since $G_L(X)$ is a complete variety, the Seesaw Theorem ([21], Cor. 6, pg. 54) implies the existence of a line bundle $\mathcal{P}$ on G such that

$$\phi^* G_L(\mathcal{L}) \otimes p_2^* G_L(\mathcal{L})^{\otimes -1} \cong p_1^* \mathcal{P} \quad ,$$

i.e.

$$\phi^* G_L(\mathcal{L}) \cong p_1^* \mathcal{P} \otimes p_2^* G_L(\mathcal{L}) \quad .$$

Choose an open neighborhood U_1 of g_0 such that $\mathcal{P}|_{U_1} \cong \mathcal{O}_{U_1}$, and so

$$\phi^* G_L(\mathcal{L}) \cong p_2^* G_L(\mathcal{L}) \quad \text{on } U_1 \times G_L(X) \quad .$$

Recall that $\overline{\mathcal{L}} = p_2^* \left(G_L(\mathcal{L}) \otimes p^* \mathcal{L}_0\right)$; on $U_1 \times G_L(X)$, we have

$$\begin{aligned} (p_1, \phi)^* \overline{\mathcal{L}}^{\otimes -1} \otimes \overline{\mathcal{L}}^{\otimes 2} &= (p_1, \phi)^* p_2^* \left(G_L(\mathcal{L}) \otimes p^* \mathcal{L}_0\right)^{\otimes -1} \otimes p_2^* \left(G_L(\mathcal{L}) \otimes p^* \mathcal{L}_0\right)^{\otimes 2} = \\ &= \phi^* G_L(\mathcal{L})^{\otimes -1} \otimes \phi^* p^* \mathcal{L}_0^{\otimes -1} \otimes p_2^* G_L(\mathcal{L})^{\otimes 2} \otimes p_2^* p^* \mathcal{L}_0^{\otimes 2} = \\ &\cong \phi^* p^* \mathcal{L}_0^{\otimes -1} \otimes p_2^* p^* \mathcal{L}_0^{\otimes 2} \otimes p_2^* G_L(\mathcal{L}) \quad . \end{aligned}$$

Together with the relations

$$\begin{aligned} p \circ \phi &= m \circ (\pi \times p) : U_1 \times G_L(X) \longrightarrow A \\ \text{and} \quad p \circ p_2 &= \overline{p_2} \circ (\pi \times p) : U_1 \times G_L(X) \longrightarrow A \end{aligned}$$

where $m : A \times A \to A$ is the product map on A, $\pi : G \to A$ is the quotient map and $\overline{p_2}$ is the second projection on $A \times A$, we get

$$(p_1, \phi)^* \overline{\mathcal{L}}^{\otimes -1} \otimes \overline{\mathcal{L}}^{\otimes 2} \cong (\pi \times p)^* \left(m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2}\right) \otimes p_2^* G_L(\mathcal{L}) \quad .$$

The line bundle $\mathcal{L}$ is very ample, and so generated by its global sections on X; by Corollary 2.2.3 and the remark just thereafter the same holds for $G_L(\mathcal{L})$, and so for $p_2^* G_L(\mathcal{L})$, too. Therefore, it will be sufficient to show that there is a neighborhood $U \subseteq U_1$ of x such that $(\pi \times p)^* \left(m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2} \right)$ is effective on $U \times G_L(X)$. This is implied by the existence of a neighborhood W of $\pi(g_0)$ in $\pi(U_1)$ such that $m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2}$ is effective on $W \times A$, since then $(\pi \times p)^* \left(m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2} \right)$ is effective on $(\pi \times p)^{-1}(W \times A) = \pi^{-1}(W) \times G_L(X) \subseteq U_1 \times G_L(X)$.
By [18], Prop. 2.3, pg. 609, the line bundle

$$m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_1}^* \mathcal{L}_0^{\otimes 3} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2}$$

is ample and effective, if $\mathcal{L}_0$ is[1], and in our case this holds since $\mathcal{L}_0$ is very ample. We choose the open set $W \subseteq A$ in such a way that $\mathcal{L}_0$ is trivial on W. It follows that

$$m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_1}^* \mathcal{L}_0^{\otimes 3} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2}|_{W \times A} \cong m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2}|_{W \times A}$$

and so that $m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 2}$ is effective on $W \times A$.
We set $U := \pi^{-1}(W)$; by the discussion above, $(p_1, \phi)^* \overline{\mathcal{L}}^{\otimes -1} \otimes \overline{\mathcal{L}}^{\otimes 2}$ is generated by its global sections on $U \times G_L(X)$, and so the proof is concluded. ■

Assume now that $X = \overline{L}$, an L-equivariant, projective completion of the linear algebraic group L, and so that $G_L(X) = G_L(\overline{L}) = \overline{G}$ is a G-equivariant, projectively normal completion of G. Then, $G \times \overline{G}$ can be embedded in $\mathbb{P}^N \times \mathbb{P}^N$, and the coefficients of the polynomials $f_0, \dots, f_N \in R[X_0, \dots, X_N]$ describing a family of translations on $V \subseteq U \times \overline{G}$ can be seen as rational functions in the homogeneous coordinates $T_0, \dots, T_N$ of the first $\mathbb{P}^N$. After eventually passing to smaller open sets V, we can assume that the coefficients of $f_0, \dots, f_N$ are homogeneous in $T_0, \dots, T_N$, all of the same degree. We get the following

Corollary 3.3.4. *Let $\overline{G} \hookrightarrow \mathbb{P}^N$ be a projectively normal embedding, such as the one constructed above. Then, there is an affine open covering $\mathcal{V} = \{V_i\}_{i \in I}$ of $G \times \overline{G}$, and for every $i \in I$ there are bihomogeneous polynomials $f_0^i, \dots, f_N^i \in k[T_0, \dots, T_N, X_0, \dots, X_N]$ of degree 2 in $X_0, \dots, X_N$ such that*

$$gg' = \phi(g, g') = \left(f_0^i \left(\underline{T}(g), \underline{X}(g') \right), \dots, f_N^i \left(\underline{T}(g), \underline{X}(g') \right) \right)$$

for every $(g, g') \in V_i$.

[1] Actually, the Proposition in [18] asserts that $m^* \mathcal{L}_0^{\otimes -1} \otimes \overline{p_1}^* \mathcal{L}_0^{\otimes 2} \otimes \overline{p_2}^* \mathcal{L}_0^{\otimes 3}$ is ample and effective, but a rapid check shows that the exponents of $\overline{p_1}^* \mathcal{L}_0$ and $\overline{p_2}^* \mathcal{L}_0$ are interchangeable (this is implicitely meant in the main Theorem of [18]).

3.4 Serre's compactification

In this section, we discuss some explicit compactifications for the linear part of an algebraic group, and we draw some consequences from the theory exposed up to this point.

As we already noticed, in order to get a compactification of a connected algebraic group by Serre's method, one has to start by compactifying the fibre L in the fibration

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

given by Chevalley's theorem, and to do it in an equivariant way. A general strategy to follow in order to obtain such a compactification is to consider an action of L on a suitable space $\mathbb{P}^N$ such that the stabilizer L_x of some point $x \in \mathbb{P}^N$ is trivial. In this case, we shall be able to identify the group $L \cong L/L_x$ with the orbit $L \cdot x$. By [13], 8.3, pg. 60 the closure $\overline{L \cdot x}$ of the orbit is itself invariant under the action of L, and $L \cdot x$ is open in its closure. Therefore, the projective variety $\overline{L} := \overline{L \cdot x}$ can be considered as an equivariant compactification of L, containing L as an open, invariant subset.
The *existence* of such a compactification is a consequence of the following

Theorem 3.4.1 (Sumihiro). *Let L be a connected linear algebraic group and let Y be a normal quasi-projective variety on which L acts morphically. Then there is a projective embedding $\psi : Y \to \mathbb{P}^N$ and a group representation $\rho : L \to \mathrm{PGL}_N$ such that $\rho(g)\psi(y) = \psi(gy)$ for every $g \in L$ and $y \in Y$.*

Proof: See [31], Thm. 1, pg. 5. ■

Indeed, if we set $Y := L$ and $y := e_L$, the neutral element of L, the relation $\rho(g)\psi(e_L) = \psi(g)$ shows that L can be equivariantly embedded in $\mathbb{P}^N$ as the orbit of $y := \psi(e_L) \in \mathbb{P}^N$ under the action of L on $\mathbb{P}^N$ induced by the representation ρ.
In many cases, as shown by the following examples, the orbit $L \cdot y$ is dense in $\mathbb{P}^N$, so that one obtains the equivariant completion $\overline{L} = \mathbb{P}^{\dim(L)}$.

Example 3.4.1. (*Commutative algebraic groups*) Let L be a commutative and connected linear algebraic group (as usual, defined over an algebraically closed field k of characteristic zero). Then, the Jordan decomposition

$$L \cong L_s \times L_u$$

(where L_s and L_u denote the semisimple resp. unipotent part of L, see [13], pp. 98-100) allows us to identify L with a direct product of multiplicative and additive groups. Indeed, the closed, connected subgroup L_s consists of commuting semisimple elements, and so it is isomorphic to a torus $\mathbb{G}_m(k)^{\ell_m}$, while the unipotent part L_u is a successive extension of additive groups $\mathbb{G}_a(k)$, which can only be trivial (see [28], §2.7, pg. 172). Hence, we can write

$$L \cong \mathbb{G}_m^{\ell_m} \times \mathbb{G}_a^{\ell_a} \quad , \quad \ell_m + \ell_a = \dim(L)$$

(this result was first proven by I. Barsotti, see [1], Thm. 3.3, pg. 104). Each factor $\mathbb{G}_m$ resp. $\mathbb{G}_a$ can be naturally embedded in $\mathbb{P}^N$ in an equivariant way (see for instance [29] or [37]); this gives immediately the projective, equivariant completion $\overline{L} = \left(\mathbb{P}^1\right)^{\dim(L)}$.
Another method by which one can construct $\overline{L}$ (see [14], pp. 569-570) is the following: we consider the representation of $L \cong \mathbb{G}_m^{\ell_m} \times \mathbb{G}_a^{\ell_a}$ in $\mathrm{PGL}_n(k)$, $n = \dim(L)$ given by

$$\begin{array}{rcl} \sigma : \mathbb{G}_m^{\ell_m} \times \mathbb{G}_a^{\ell_a} & \longrightarrow & \mathrm{GL}_{n+1}(k) \\ (\alpha_1, \dots, \alpha_{\ell_m}, \alpha_{\ell_m+1}, \dots, \alpha_n) & \longmapsto & \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \alpha_1 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \dots & 0 & \alpha_{\ell_m} & 0 & & \vdots \\ \alpha_{\ell_m+1} & 0 & \dots & 0 & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ \alpha_n & 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix} \end{array} .$$

This gives us an action of L on $\mathbb{P}^n$ given explicitely by

$$\begin{array}{rcl} \eta : L \times \mathbb{P}^n & \to & \mathbb{P}^n \\ (\alpha, x) & \mapsto & \alpha \cdot x := [x_0 : \alpha_1 x_1 : \ldots : \alpha_{\ell_m} x_{\ell_m} : \alpha_{\ell_m+1} x_0 + x_{\ell_m+1} : \ldots : \alpha_n x_0 + x_n] \end{array}$$

(where the square brackets stand for homogeneous coordinates), and the orbit $L \cdot x$ of the point

$$x = [1 : \underbrace{1 : \ldots : 1}_{\ell_m} : \underbrace{0 : \ldots : 0}_{\ell_a}] \in \mathbb{P}^n$$

can be identified with L itself, since the isotropy group L_x is trivial. Furthermore, $L \cdot x$ is dense in $\mathbb{P}^n$ (note that $\dim(L) = \dim(\mathbb{P}^n) = n$). This means that $\overline{L} = \mathbb{P}^n$ is an equivariant compactification of L.
As special cases of this compactification, we recover for $(\ell_m, \ell_a) = (1, 0)$ resp. $(\ell_m, \ell_a) = (0, 1)$ the completions for $\mathbb{G}_m$ resp. $\mathbb{G}_a$ given in [37], §1/2.

Example 3.4.2. (*The general linear group*) Let $L = \mathrm{GL}_n(k)$, and denote by 1_L resp. ρ_L the trivial representation of L on k and the identity representation of L on k^n. Define a new representation

$$\rho := 1_L \oplus \underbrace{\rho_L \oplus \ldots \oplus \rho_L}_{n}$$

of L on $k \oplus k^n \oplus \ldots \oplus k^n = k^{n^2+1}$. This representation can also be interpreted as a left action of L on $k \oplus \mathrm{M}_n(k)$, where $\mathrm{M}_n(k)$ denotes the k-vector space of $n \times n$-matrices with entries in k; let namely $\rho_L(g) = (g_{ij})_{i,j=1}^n \in L$, $\alpha \in k$ and $x = (x_{ij})_{i,j=1}^n \in \mathrm{M}_n(k)$: since GL_n acts on M_n columnwise by ρ_L, we can write

$$\rho(g)(\alpha, x) = (\alpha, ((gx)_{ij})) \quad ,$$

where by $((gx)_{ij})$ we denote matrix multiplication (so that $(gx)_{ij} = \sum_{m=1}^{n} g_{im}x_{mj}$). This action of L on k^{n^2+1} gives rise to an action on $\mathbb{P}^{n^2}$, given by

$$\begin{array}{rcl} \eta : L \times \mathbb{P}^{n^2} & \longrightarrow & \mathbb{P}^{n^2} \\ (g, x) & \longmapsto & g \cdot x := [x_0 : ((gx)_{ij})] \quad , \end{array}$$

(and so again to a representation of L in PGL_n). Since the point $x = [1 : (\delta_{ij})]$ (where δ_{ij} denotes the Kronecker delta) has trivial isotropy group, its orbit $L \cdot x$ can be identified with L: this gives rise to the embedding

$$\begin{array}{rcl} L & \hookrightarrow & \mathbb{P}^{n^2} \\ (g_{ij}) & \longmapsto & [1 : (g_{ij})] \quad . \end{array}$$

The orbit closure $\overline{L \cdot x}$ coincides with $\mathbb{P}^{n^2}$; this means that also in this case we can choose $\mathbb{P}^{\dim(L)}$ as an equivariant compactification.

Example 3.4.3. (*Solvable groups*) Let L be a connected and solvable group, given as a subgroup of some $\mathrm{GL}(V)$. The action of L on V fixes a full flag

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n = V \quad , \quad n = \dim(V)$$

of subspaces of V: this follows from a repeated application of the Lie-Kolchin Theorem (see [13], pg. 113), which claims that a solvable subgroup of $\mathrm{GL}(V)$ has a common eigenvector in V.
Choose the representation of L on

$$W := V_1 \oplus V_2 \oplus \ldots \oplus V_n \quad :$$

an appropriate choice of a basis of W identifies L with the full diagonal group

$$\mathrm{D}_n = \{(a_{ij}) \in \mathrm{GL}_n(k) \mid a_{ij} = 0 \quad \text{if} \quad i > j\} \quad ;$$

in particular, we can again let it act on a vector space of matrices, the subspace of upper triangular matrices in M_n (i.e. we "forget" the entries which lie under the the diagonal in the previous example). Proceeding as with GL_n, we find an open, equivariant embedding of L in $\mathbb{P}^N$, with $N = \frac{n(n-1)}{2}$. Since this is also the dimension of D_n, we notice that also in this case we can choose $\overline{L} = \mathbb{P}^{\dim(L)}$.

Besides an equivariant completion $\overline{L}$ of the linear part L of an algebraic group G, Serre's recipe for a compactification of G requires a very ample, L-linearized invertible sheaf on $\overline{L}$. Since $\overline{L}$ is equivariantly embedded in a projective space $\mathbb{P}^N$, its existence follows from the existence of an L-linearized invertible sheaf on $\mathbb{P}^N$ (recall Lemma 2.2.2, pg. 43). This is provided by the following

Proposition 3.4.2. *Let a connected linear algebraic group L act on an algebraic variety X, proper over k. Let $\mathcal{L}$ be an invertible sheaf on X. Then, if X is a normal variety, some power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ is always linearizable.*

Proof: See [23], Prop. 1.5 and Cor. 1.6, pp. 34-35. ■

If $X = \mathbb{P}^N$, it follows from the proposition that $\mathcal{O}(n)$ is linearizable for some n. Of particular interest is the case $n = 1$, as we shall see later. This holds whenever the Picard group of L is trivial, as shown by the next lemma. We assume again that L acts on $\mathbb{P}^N$ by means of a faithful representation in PGL_N:

Lemma 3.4.3. *Let L be a connected linear algebraic group, acting on a projective space $\mathbb{P}^N$. Assume that* $\mathrm{Pic}(L)$ *is trivial. Then, the sheaf* $\mathcal{O}_{\mathbb{P}^N}(1)$ *admits an L-linearization.*

Proof: (see [37], pg. 284) Denote by

$$\mu : \mathrm{PGL}_N \times \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

the projective action, and by $\rho : L \hookrightarrow \mathrm{PGL}_N$ the representation of L in PGL_N. Then the action of L on $\mathbb{P}^N$ is given by

$$\eta = \mu \circ (\rho \times \mathrm{Id}_{\mathbb{P}^N}) : L \times \mathbb{P}^N \longrightarrow \mathbb{P}^N \quad .$$

Let us denote by $q_1 : L \times \mathbb{P}^N \to L$ and $q_2 : L \times \mathbb{P}^N \to \mathbb{P}^N$ the natural projections. A linearization of $\mathcal{O}_{\mathbb{P}^N}(1)$ is an isomorphism

$$\eta^* \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\sim} q_2^* \mathcal{O}_{\mathbb{P}^N}(1)$$

of sheaves on $L \times \mathbb{P}^N$ satisfying the cocycle condition.
We shall denote by p_1 and p_2 the natural projections of $\mathrm{PGL}_N \times \mathbb{P}^N$ to PGL_N and $\mathbb{P}^N$ respectively. Being a morphism to $\mathbb{P}^N(k) = \mathrm{Proj}\, k[X_0, \ldots, X_N]$, μ can be defined by specifying the $\mu^*(X_i)$, $i = 1, \ldots, n$ (see [12], Thm. 7.1, pg. 150). In our case we have

$$\mu^*(X_i) = \sum_{k=0}^{n} p_1^*(a_{ik}) \otimes p_2^*(X_k) \quad ; \tag{3.1}$$

this gives an isomorphism

$$\mu^* \left(\mathcal{O}_{\mathbb{P}^N}(1) \right) \cong p_1^* \left(\mathcal{O}_{\mathrm{PGL}_N}(1) \right) \otimes p_2^* \left(\mathcal{O}_{\mathbb{P}^N}(1) \right) \tag{3.2}$$

(see [23], pg. 33), where we denote by $\mathcal{O}_{\mathrm{PGL}_N}(1)$ the restriction of $\mathcal{O}_{\mathbb{P}^{N^2+2N}}(1)$ to PGL_N (since PGL_N can be considered as the pricipal open subset of $\mathbb{P}^{N^2+2N}$ defined by the nonvanishing of

the determinant).
The "restriction" of (3.2) to $L \times \mathbb{P}^N$ (i.e. its inverse image with respect to $\rho \times \mathrm{Id}_{\mathbb{P}^N}$) gives

$$\eta^* \mathcal{O}_{\mathbb{P}^N}(1) \cong (\rho \times \mathrm{Id}_{\mathbb{P}^N})^* p_1^* \mathcal{O}_G(1) \otimes (\rho \times \mathrm{Id}_{\mathbb{P}^N})^* p_2^* \mathcal{O}_{\mathbb{P}^N}(1) \quad .$$

The evident relations

$$p_1 \circ (\rho \times \mathrm{Id}_{\mathbb{P}^N}) = \rho \circ q_1 \quad \text{and} \quad p_2 \circ (\rho \times \mathrm{Id}_{\mathbb{P}^N}) = q_2$$

show that

$$(\rho \times \mathrm{Id}_{\mathbb{P}^N})^* p_1^* \mathcal{O}_G(1) \cong q_1^* \rho^* \mathcal{O}_G(1)$$

and

$$(\rho \times \mathrm{Id}_{\mathbb{P}^N})^* p_2^* \mathcal{O}_{\mathbb{P}^N}(1) \cong q_2^* \mathcal{O}_{\mathbb{P}^N}(1)$$

hold; since $\mathrm{Pic}(L)$ is trivial, it follows that $\rho^* \mathcal{O}_G(1)$ is trivial on L and so that $q_1^* \rho^* \mathcal{O}_G(1) \cong \mathcal{O}_{L \times \mathbb{P}^N}$. Followingly,

$$\eta^* \left(\mathcal{O}_{\mathbb{P}^N}(1) \right) \cong q_2^* \left(\mathcal{O}_{\mathbb{P}^N}(1) \right) \quad ,$$

and this shows that (3.1) defines an L-linearization of $\mathcal{O}_{\mathbb{P}^N}(1)$. ■

In general, the Picard group of a linear algebraic group cannot be expected to be trivial: for instance, this is not the case for $L = \mathrm{PGL}_N$ (see [23], pg. 35). But a nice characterization of a family of groups for which Pic is trivial is given by R. Fossum and B. Iversen in [7]: we meet again the "special" groups, of which we already made mention in Remark 1.2.5, pg. 23 ("special" in the terminology of [27]).

Proposition 3.4.4 (Fossum-Iversen). *Let L be a linear and connected algebraic group with the property that all locally isotrivial principal L-bundles are locally trivial. Then,* $\mathrm{Pic}(L) = 0$.

Proof: See [7], Cor. 3.2, pg. 276. ■

The Proposition allows us to give a list of groups for which Lemma 3.4.3 holds: it comprises

- the general linear group GL_n, and all its linear subgroups L for which the fibration $\mathrm{GL}_n \to \mathrm{GL}_n/L$ is locally trivial ([27], Théorème 2, pg. 1-24);

- all connected and solvable linear algebraic groups ([27], Prop. 14., pg. 1-25), and so in particular all connected and commutative linear algebraic groups;

- the groups SL_n and $\mathrm{Sp_n}$ (these groups, and their direct products, are the only semisimple groups which are "special", see [11], Théorème 3, pg. 5-22).

In any case, the results of this section yield a projective completion $\overline{L}$ of a connected linear algebraic group L and a very ample, L-linearized invertible sheaf $\mathcal{L}$ on $\overline{L}$. Being projective, the variety $\overline{L}$ satisfies the finiteness condition **(F)** of page 29, and so if L is the largest linear and connected algebraic subgroup of a connected algebraic group G it follows that $\overline{G} = G_L(\overline{L})$ is an equivariant completion of G. Together with the results of the previous sections, this implies the following (qualitative) result:

Theorem 3.4.5. *Let G be a connected algebraic group, and L its largest linear and connected algebraic subgroup. Then, the complete variety $\overline{G} = G_L(\overline{L})$ is projective, and the projective embedding of $\overline{G}$ can be chosen in such a way that $\overline{G}$ is projectively normal and cut out by quadrics, and such that the translation on $\overline{G}$ is defined locally by quadratic forms.*

The strength of the results of the previous sections lies in their effectivity; if we apply them more carefully, we obtain a quantitative version of Theorem 3.4.5. We recall once again the notations: the connected algebraic group G is given as an extension

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

of an abelian variety A with a linear algebraic group L; $\overline{L}$ is an equivariant compactification of L, and $\mathcal{L} = \mathcal{O}_{\overline{L}}(1)$ is a very ample, L-linearized invertible sheaf on $\overline{L}$ satisfying $H^i(\overline{L}, \mathcal{L}) = 0$ for $i > 0$. We denote by $\overline{G} = G_L(\overline{L})$ the completion of G and by $p : \overline{G} \to A$ the natural projection. For a very ample invertible sheaf $\mathcal{L}_0$ on A, we denote by $\mathcal{M}(\mathcal{L}, \mathcal{L}_0)$ the sheaf

$$\mathcal{M}(\mathcal{L}, \mathcal{L}_0) := G_L(\mathcal{L}) \otimes p^*\mathcal{L}_0$$

on $\overline{G}$.

Theorem 3.4.6.

1. *The invertible sheaf $\mathcal{M}(\mathcal{L}, \mathcal{L}_0)$ is very ample on $\overline{G}$; its global sections define a projective embedding of $\overline{G}$ in $\mathbb{P}^N$, with $N = \dim H^0(\overline{L}, \mathcal{L}) \cdot \dim H^0(A, \mathcal{L}_0) - 1$.*

2. *Let $k \geq \dim(G) + 1$. Then, the global sections of the very ample invertible sheaf $\mathcal{M}(\mathcal{L}, \mathcal{L}_0)^{\otimes k}$ define a projectively normal embedding of $\overline{G}$ such that there is an affine open covering $\mathcal{V} = \{V_i\}_{i \in I}$ of $G \times \overline{G}$ and for every $i \in I$ there are bihomogeneous polynomials $f_0^i, \dots, f_N^i \in k[T_0, \dots, T_N, X_0, \dots, X_N]$ of degree 2 in $X_0, \dots, X_N$ with*

$$g \cdot x = \phi(g, x) = \left(f_0^i \left(\underline{T}(g), \underline{X}(x) \right), \dots, f_N^i \left(\underline{T}(g), \underline{X}(x) \right) \right)$$

   for every $(g, x) \in V_i \subseteq G \times \overline{G}$.

3. *Let $k \geq \dim(G) + 2$. Then, the global sections of the very ample invertible sheaf $\mathcal{M}(\mathcal{L}, \mathcal{L}_0)^{\otimes k}$ define a projectively normal embedding of $\overline{G}$ such that the homogeneous ideal of the image of $\overline{G}$ in projective space is generated by polynomials of degree 2.*

Proof:

1. Follows from Theorem 2.3.8 and Theorem 3.1.4, together with the fact that $\chi(\mathcal{L}_0) = \dim H^0(A, \mathcal{L}_0)$.

2. See Corollaries 3.2.3 and 3.3.4.

3. See Corollary 3.2.6. ■

Remark 3.4.1. If, as in the examples above, $\overline{L} = \mathbb{P}^{\dim(L)}$ is a projective space and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{\dim(L)}}(1)$ is L-linearized, one has $\dim H^0(\overline{L}, \mathcal{L}) = \dim(L) + 1$. In this case, we get the effective bound

$$N = (\dim(L) + 1) \cdot \dim H^0(A, \mathcal{L}_0) - 1$$

for the embedding dimension of $\overline{G}$.

Chapter 4

An affine analogon

The final chapter is an appendix which can be read independently from the rest of this work. It is based on V.L. Popov's work [24], which was recently brought to our attention by Professor Wüstholz. Although the original idea of using this material in order to improve the results from Section 3.4 did not bear the hoped fruits, we have chosen to include Popov's results in our work since they can be seen as the affine counterpart of what we showed for the quasiprojective groups (especially Theorems 3.4.5 and 3.4.6).

4.1 Popov's results on semisimple groups

In [6], D. E. Flath and J. Towber outline a method for the description of the affine coordinate ring $k[G]$ of a connected and reductive linear algebraic group, defined over an algebraically closed field k of characteristic 0, by means of generators of relations. They formulate a conjecture about the structure of $k[G]$, which they prove for the classical groups by rather explicit methods. In [24] V. L. Popov proves the conjecture for the semisimple algebraic groups, and he goes a step further: incorporating results from [4] and [15], he is able to give a method for describing the ring $k[G]$ out of the fundamental representations of G. The aim of this section is to briefly resume Popov's methods and results.

Let also G be a reductive and connected algebraic group; let B be a Borel subgroup of G, T a maximal torus of B and U the unipotent radical of B. Let furthermore B^- be the Borel subgroup of G *opposite* to B, i.e. the uniquely determined maximal solvable subgroup of G such that $B \cap B^- = T$, and let U^- be its unipotent radical. We consider $k[G]$ as a G-module with respect to the action given by the left translation, and we define two G-submodules S and S^- of $k[G]$ as follows:

$$\begin{aligned} S &:= \{f \in k[G] \mid f(gu) = f(g),\ g \in G,\ u \in U\} \quad , \\ S^- &:= \{f \in k[G] \mid f(gu) = f(g),\ g \in G,\ u \in U^-\} \quad . \end{aligned}$$

The G-algebra S is sometimes referred to as the *flag algebra* of G, and denoted by $\Lambda^+(G)$ (for instance in [15]). It admits a direct sum decomposition as follows: denote by P_{++} the monoid of the highest weights of the simple G-modules (i.e. the *dominant weights*). We shall denote by $R(\lambda)$ a simple G-module with the highest weight $\lambda \in P_{++}$, and by λ^* the highest weight of $R(\lambda)^*$. For any $\lambda \in P_{++}$, define a G-submodule

$$S_\lambda := \{ f \in S \mid f(gb) = \lambda(b) f(g),\ g \in G\,,\, b \in B\}$$

of S. It is well-known (see for instance [25], pg. 173) that S_λ is a simple G-submodule of $k[G]$ with the highest weight λ^*, and furthermore that S is graded by the S_λ's:

$$S = \bigoplus_{\lambda \in P_{++}} S_\lambda \qquad , \qquad S_\lambda \cdot S_{\lambda'} = S_{\lambda+\lambda'} \quad . \tag{4.1}$$

In a similar way, we get a decomposition of S^-: let $w_0 \in N_G(T)$ be an element of the normalizer of T such that $w_0 U w_0^{-1} = U^-$; then, S^- is obtained from S by right translation by w_0, and if we define S_λ^- as the right translation of S_λ by w_0 we get

$$S^- = \bigoplus_{\lambda \in P_{++}} S_\lambda^- \qquad , \qquad S_\lambda^- \cdot S_{\lambda'}^- = S_{\lambda+\lambda'}^- \quad . \tag{4.2}$$

The G-submodule S_λ^- is simple, of highest weight λ^* and it admits the explicit description

$$S_\lambda^- = \{ f \in S^- \mid f(gt) = w_0\lambda(t) f(g),\ g \in G\,,\, t \in T\} \quad ,$$

where the action of $N_G(T)$ on the roots is given by $w_0\lambda(t) = \lambda(w_0^{-1} t w_0)$.

Let us now consider the homomorphism of G-algebras

$$\mu : S \otimes_k S^- \longrightarrow k[G]$$

given by the rule $\mu(f \otimes h) := fh$. Flath and Towber's conjecture is formulated in terms of the morphism μ:

Conjecture 4.1.1 (Flath, Towbers).

> **(Sur)** μ *is surjective ,*
>
> **(Ker)** $\ker(\mu)$ *is generated by the G-invariant elements.*

The conjecture yields the presentation

$$k[G] \cong S \otimes_k S^- / \left(\ker(\mu)^G\right) \quad ,$$

where $\left(\ker(\mu)^G\right)$ is the ideal in $S \otimes_k S^-$ generated by the G-invariant elements in $\ker(\mu)$.
As we already mentioned, the conjecture is proven for the classical groups in [6]:

Theorem 4.1.2 (Flath, Towbers). **(Sur)** *and* **(Ker)** *hold if G is any of the groups*

$$\mathrm{SL}_n(k),\ \mathrm{GL}_n(k),\ \mathrm{SO}_n(k), \mathrm{Sp}_n(k) \quad ,$$

with $n \geq 1$.

Proof: See [6], §3 through §6. Note that, although they are given for $k = \mathbb{C}$, the proofs work for any algebraically closed field of characteristic zero. ■

As we already mentioned, thanks to Popov the conjecture is known to hold in the semisimple case:

Theorem 4.1.3 (Popov). **(Sur)** *and* **(Ker)** *hold for any connected semisimple algebraic group.*

Proof: See [24], Thm. 3 and 4. We do not repeat Popov's entire proof here: this would take us too far away, since it requires a considerable amount of invariant theory from [25]. We just sketch the main geometric constructions behind it, which are quite enlightening, skipping most of the technical details.
The core of the proof consists in a reformulation of the problem in the language of algebraic geometry: the rings S and S^- are realized as the coordinate rings of affine G-varieties X and X^-, and G is equivariantly embedded in the product $X \times X^-$ as a closed orbit $G \cdot z$ of a suitable point z, so that the ideal $\ker(\mu)$ can be identified with the ideal of $G \cdot z$ in $k[X \times X^-] \cong S \otimes S^-$. Since S is defined as the subring of $k[G]$ consisting of the U-invariant elements, a natural candidate for X could seem to be the homogeneous space G/U; unfortunately, this space is only quasi-affine (see [25], pg. 172), and so its coordinate ring does not coincide with its ring of regular functions. The variety X is constructed as follows: let $\{\lambda_1, \ldots, \lambda_s\}$ be a system of generators for P_{++} (i.e. a basis of *fundamental dominant weights*), and define the G-module

$$V := R(\lambda_1) \oplus \ldots \oplus R(\lambda_s)$$

(the direct sum of the *fundamental representations* associated to $\lambda_1, \ldots, \lambda_s$). For $v_i \in R(\lambda_i)$ a highest weight vector, $i = 1, \ldots, s$ and $v := (v_1, \ldots, v_s)$ define the morphism

$$\begin{array}{rcl} \delta : G & \longrightarrow & V \\ g & \longmapsto & g \cdot v \quad ; \end{array}$$

then, $\delta(G) = G \cdot v$, the orbit of v under the action of G on V. The restriction of functions from V to the orbit closure $\overline{G \cdot v}$ gives rise to an isomorphism

$$\delta^* : k[\overline{G \cdot v}] \longrightarrow S$$

(for details see [24], pg. 6). Hence we see that we can choose $X = \overline{G \cdot v}$. The affine variety X^- can be constructed in an analoguous way: one takes

$$V^* = R(\lambda_1^*) \oplus \ldots \oplus R(\lambda_s^*)$$

(where λ_i^* is the highest weight of the simple G-module $R(\lambda_i)^*$) and, for highest weight vectors u_i of $R(\lambda_i^*)$, one sets $w := (w_0 \cdot u_1, \ldots, w_0 \cdot u_s)$ and $X^- := \overline{G \cdot w}$.
Let now $z := (v, w) \in X \times X^-$; the isotropy group G_z for the action of G on $X \times X^-$ is $U \cap U^- = \{1_G\}$; hence, we can identify G with the orbit $G \cdot z$, and this gives a (left) equivariant embedding

$$i : G \hookrightarrow X \times X^- \quad .$$

This embedding is closed (i.e. $G \cdot z$ is closed in $X \times X^-$) if and only if the comorphism

$$i^* : k[X \times X^-] \longrightarrow k[G]$$

(which amounts to the restriction of functions from $X \times X^-$ to $G \cdot z$) is surjective; but the isomorphism $k[X \times X^-] \cong S \otimes S^-$ identifies i^* with $\mu : S \otimes S^- \to k[G]$ and $\ker(\mu)$ with the kernel of the restriction of functions from $X \times X^-$ to $G \cdot z$. Therefore, **(Sur)** is reduced to the verification that $G \cdot z$ is a closed subvariety of $X \times X^-$.
We shall be very sketchy at this point: z is the sum of the weight vectors v_i, $i = 1, \ldots, s$ of weight λ_i and $w_0 \cdot u_i$, $i = 1, \ldots, s$ of weight $w_0 \lambda_i^* = -\lambda_i$. Hence, 0 is an interior point of the convex hull of the weights of $v_1, \ldots, v_s, w_0 \cdot u_1, \ldots, w_0 \cdot u_s$. By [24], Thm. 2 this is equivalent to the closedness of $G \cdot z$ in $X \times X^-$, and this proves **(Sur)**.
A proof of **(Ker)** can also be derived from the geometric situation: Thm. 5 of [24] shows that, since $G \cdot z$ is closed in $X \times X^-$ and G_z is trivial, the kernel of the morphism

$$\begin{array}{rcl} k[X \times X^-] & \longrightarrow & k[G \cdot z] \\ f & \longmapsto & f|_{G \cdot z} \end{array}$$

is generated by its intersection with $k[X \times X^-]^G$. Under our identifications, this means that $\ker(\mu) = \big(\ker(\mu) \cap (S \otimes S^-)^G\big) = \big(\ker(\mu)^G\big)$. This proves **(Ker)**. ■

We now begin with Popov's explicit description of $k[G]$ for a semisimple algebraic group. The first aim is the determination of the kernel of μ. By (4.1) and (4.2), this algebra admits a decomposition

$$\big(S \otimes_k S^-\big)^G \cong \bigoplus_{\lambda, \lambda' \in P_{++}} \big(S_\lambda \otimes_k S^-_{\lambda'}\big)^G \quad ;$$

as in [24], (8) an application of Schur's Lemma implies that this decomposition reduces to

$$\big(S \otimes_k S^-\big)^G \cong \bigoplus_{\lambda \in P_{++}} \big(S_\lambda \otimes_k S^-_{\lambda^*}\big)^G \quad .$$

Proposition 4.1.4. *The restriction of μ to $\big(S \otimes_k S^-\big)^G$ gives isomorphisms*

$$\mu : \big(S_\lambda \otimes_k S^-_{\lambda^*}\big)^G \xrightarrow{\sim} k[G]^G \cong k \quad .$$

Proof: See [24], Thm. 2. ■

It follows that for each dominant weight λ there is a uniquely determined element h_λ of $\left(S_\lambda \otimes_k S^-_{\lambda^*}\right)^G$ such that $\mu(h_\lambda) = h_\lambda(e,e) = 1$ (where e denotes the neutral element of G). The h_λ's are exactly what is needed for the construction of $\ker(\mu)^G$:

Theorem 4.1.5 (Popov). *Let $\lambda_1, \ldots, \lambda_s$ be a set of generators for P_{++}. Then, ker$(\mu)^G$ coincides with the ideal generated by the set*

$$\left\{h_{\lambda_1} - 1, h_{\lambda_2} - 1, \ldots, h_{\lambda_s} - 1\right\} \quad .$$

Proof: See [24], Thm. 7. ■

The element h_λ for a fixed λ can be constructed as follows: let

$$\langle\,\cdot\,,\,\cdot\,\rangle : S_\lambda \times S^-_{\lambda^*} \longrightarrow k$$

be a nonzero G-invariant pairing, and let $\{p_i\}$ and $\{q_j\}$ be a pair of dual bases of S_λ and $S^-_{\lambda^*}$ respectively; then, the sum $\sum_i p_i \otimes q_i$ is a nonzero element of $\left(S_\lambda \otimes_k S^-_{\lambda^*}\right)^G$, independent on the choice of the bases (see [24], Lemma 1, (ii)); therefore, we can set

$$h_\lambda := \frac{\sum_i p_i \otimes q_i}{\sum_i p_i(e) q_i(e)} \quad .$$

Up to now, we have described the ring $k[G]$ as a quotient of $S \otimes_k S^-$, which is not a priori given in an explicit way. This algebra can be described by means of the irreducible representations of G, as showed in [15] and [4]. Popov chooses to follow the more geometric approach from [4]. Assume that P_{++} admits a free system of generators, i.e. that there exist $\{\lambda_1, \ldots, \lambda_s\} \subseteq P_{++}$, $s = \dim(T)$ such that each element of P_{++} can be written in a unique way as a positive linear combination of $\lambda_1, \ldots, \lambda_s$ (this holds, for instance, if G is simply connected). Consider again the G-module

$$V := R(\lambda_1) \oplus \ldots \oplus R(\lambda_s) \quad ,$$

the direct sum of the corresponding simple G-modules. The decomposition of V induces in a natural way an $\mathbb{N}^s$-grading of the ring $k[V]$ of regular function on V:

$$k[V] \cong S(V^*) = \bigoplus_{(a_1,\ldots,a_s)\in\mathbb{N}^s} \left(S^{a_1} R(\lambda_1)^* \oplus \ldots \oplus S^{a_s} R(\lambda_s)^*\right) \quad .$$

In particular, if $e_i \in \mathbb{N}^s$ is the s-tuple $(\delta_{1i}, \ldots, \delta_{si})$ (where δ_{ij} denotes the Kronecker delta), we have $k[V]_{e_i} \cong R(\lambda_i)^*$, and

$$k[V]_{e_i+e_j} \cong \begin{cases} R(\lambda_i)^* \otimes_k R(\lambda_j)^* & \text{if } i \neq j, \\ S^2 R(\lambda_i)^* & \text{if } i = j. \end{cases}$$

The G-module $k[V]_{e_i+e_j}$ $(1 \leq i \leq j \leq s)$ contains a unique simple submodule with the highest weight $\lambda_i^* + \lambda_j^*$, the *Cartan product* of the representations $R(\lambda_i)^*$ and $R(\lambda_j)^*$. Let Q_{ij} be its G-invariant direct complement. It can be determined with the help of Kostant's Theorem 1.1 of [15]. Let J be the ideal generated by all Q_{ij}, $1 \leq i \leq j \leq s$.

Theorem 4.1.6 (Popov). *Assume that P_{++} admits a free system of generators $\{\lambda_1, \ldots, \lambda_s\}$. Then, the G-algebras S and $k[V]/J$ are isomorphic.*

Proof: See [24], Thm. 8 for Popov's geometric version of the proof and [15], Thm. 1 for Kostant's original, more algebraic-oriented proof.
The idea behind Popov's proof goes as follows: in [4], Thm. 4.1, it is shown that J is the ideal of the closed subvariety $G \cdot V^U$ of V in $k[V]$ (where V^U denotes the subset of the invariant elements); since the proof of Theorem 4.1.3 shows that S is the affine coordinate ring of $\overline{G \cdot v}$ in V, in order to prove the claim it is sufficient to show that $\overline{G \cdot v} = G \cdot V^U$. ■

Since the ideal J is graded, it follows as expected that $S = k[V]/J$ is a graded G-algebra. In particular, we recover S_{λ_i} as the image of $R(\lambda_i)^*$ under the natural projection $k[V] \to S$, for $i = 1, \ldots, s$. Once S is known, translation by $w_0 \in N_G(T)$ allows one to construct $S^- = S^{w_0}$.

Remark 4.1.1.

1. The standard way of linearizing an affine algebraic group (see for instance [13]) is to consider its action on a suitable G-module, for instance a finite-dimensional G-submodule of its affine coordinate ring which contains a set of generators for the algebra $k[G]$. In Popov's presentation such module is given by twice all irreducible representations of G (once for S and once for S^-).

2. Since $\ker(\mu)^G$ and the ideal J are generated by (inhomogeneous) polynomials of degree 2, it follows that all semisimple connected algebraic groups whose monoid P_{++} of the dominant weights is freely generated can be cut out by quadrics in an affine space.

4.2 An application

As an illustration of the usefulness of Popov's construction, we shall now apply it to the special linear group $G := \mathrm{SL}_n(k)$.

The subgroup B of all lower triangular matrices is a Borel subgroup of G; its maximal torus T consists of the diagonal matrices in G, and the unipotent radical U of G is the subgroup of all lower triangular matrices with 1's on the main diagonal. Let σ_G be the identity representation of G on k^n; then the fundamental representations are given by the irreducible representations

$$\wedge^r \sigma_G : G \to \mathrm{GL}\left(\wedge^r k^n\right)$$

for $1 \leq r \leq n-1$ (see [8], pg. 234). Let $\{e_1, \dots, e_n\}$ be the standard base in k^n. With respect to the above choice of B, the vector

$$e_{n-r+1} \wedge \dots \wedge e_n \in \wedge^r k^n$$

is a maximal vector of the highest weight

$$\lambda_r(t) = \prod_{i=n-r+1}^{n} t_i = \prod_{i=1}^{n-r} t_i^{-1} \quad ,$$

where t is a diagonal matrix with entries $t_1, \dots, t_n$ satisfying $t_i \dots t_n = 1$. The verifications of this facts are straightforward: $e_{n-r+1} \wedge \dots \wedge e_n$ is invariant under the action of U since, for $u \in U$, ue_j is obtained by adding to e_j a linear combination of $e_{j+1}, \dots, e_n$ and in particular $ue_n = e_n$, and furthermore it is clear that

$$\wedge^r \sigma_G(t)\,(e_{n-r+1} \wedge \dots \wedge e_n) = t_{n-r+1}e_{n-r+1} \wedge \dots \wedge t_n e_n = \lambda_r(t)e_{n-r+1} \wedge \dots \wedge e_n$$

by linearity. The vector $e_{n-r+1} \wedge \dots \wedge e_n$ generates the whole G-module $R(\lambda_r) := \wedge^r k^n$, since the action of SL_n permutes the one-dimensional subspaces of k^n.
Let us denote by W_r the family of all subsets of $\{1, \dots, n\}$ with r elements: an element of W_r is a set $A = \{i_1, \dots, i_r\}$, where we shall always assume that $1 \leq i_r < \dots < i_r \leq n$. For such an A, we denote by e_A the vector $e_{i_1} \wedge \dots \wedge e_{i_r}$; a base for $R(\lambda_r)$ is then given by $\{e_A | A \in W_r\}$, and $e_{\{n-r+1,\dots,n\}}$ is the maximal vector described above.
Let $\{e_A^* | A \in W_r\} \subseteq (\wedge^r k^n)^*$ be the dual base to $\{e_A | A \in W_r\}$. The relation

$$e_A^*\left(\wedge^r \sigma_G(g)e_{\{n-r+1,\dots,n\}}\right) = e_A^*\left(\sigma_G(g)e_{n-r+1} \wedge \dots \wedge \sigma_G(g)e_n\right) = \det\left(g_{ij}\right)_{i \in A,\, j=n-r+1,\dots,n}$$

implies that we can identify the elements of the dual base with the *right minors* of an $(n \times n)$-matrix. Kostant's theorem in [15] can then be used in order to determine the relations between the minors, i.e. the ideal J. The result is given by Towber's Theorem 3.1 in [33].
Let us denote by Z_A the image of e_A^* in $S = k[V]/J$, where as above V is the direct sum of all $R(\lambda_i)$'s. We write also

$$S = k[Z_A \mid A \in W_r,\ r = i, \dots, n-1] \quad ,$$

and furthermore we have that the image of $R(\lambda_r)^*$ in S is equal to

$$Q_{G,r} := S_{\lambda_r} = \operatorname{span}\{Z_A \mid A \in W_r\} \quad .$$

In order to get the G-algebra S^-, we need to look for the element $w_0 \in N_G(T)$ such that $w_0 U w_0^{-1} = U^-$, where U^- is the unipotent radical of the Borel subgroup of G opposite to B.

From the condition $B \cap B^- = T$ it follows that B^- is the subgroup of all upper triangular matrices in G, and U^- is the group of all upper triangular matrices with 1's on the main diagonal. Therefore, w_0 can be given by the matrix

$$w_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots} \\ 0 & 1 & \dots & 0 \\ \pm 1 & 0 & \cdots & 0 \end{pmatrix}$$

where the lower left entry has to be chosen in order to ensure that $w_0 \in \mathrm{SL}_n$ (also -1 for $n = 2, 3, 6, 7, \dots$ and $+1$ for $n = 4, 5, 8, 9 \dots$). The right translation of $Q_{G,r}$ by w_0 gives

$$P_{G,r} := S^-_{\lambda_r} = \left(Q_{G,r}\right)^{w_0} = \mathrm{span}\,\{Y_A \mid A \in W_r\}$$

(for $r = 1, \dots, n-1$), where we denote by Y_A the *left* r-minor given by

$$Y_A(g) = Z^{w_0}_A(g) = \det\left(g_{ij}\right)_{i \in A,\, j=1,\dots,r} \quad .$$

It follows that

$$S^- = S^{w_0} = k[Y_A \mid A \in W_r,\ r = i, \dots, n-1] \quad .$$

Since **(Sur)** is proven for the semisimple groups, we know that

$$\mu : S \otimes_k S^- \longrightarrow k[\mathrm{SL}_n]$$

is surjective. It remains to look for the kernel of μ. We do it by Popov's method. First of all, we have to determine the G-module $S^-_{\lambda_r^*}$. From [13], pp. 193-194, we know that

$$R(\lambda_r)^* = R(-w_0(\lambda_r)) \quad ,$$

where w_0 is seen as an element of the Weyl group, acting on the (abstract) weights. Furthermore,

$$(-w_0(\lambda_r))(t) = \prod_{i=1}^{r} t_i^{-1} = \lambda_{n-r}(t) \quad ,$$

and so $R(\lambda_r)^* = R(\lambda_{n-r})$. From this, it follows that $S^-_{\lambda_r^*} = S_{\lambda_{n-r}} = Q_{G,n-r}$, and so

$$S^-_{\lambda_r^*} = \left(Q_{G,n-r}\right)^{w_0} = P_{G,n-r} = \mathrm{span}\,\{Y_A \mid A \in W_{n-r}\} \quad .$$

In order to describe the G-invariant pairing $\langle\cdot,\cdot\rangle : S_{\lambda_r} \times S^-_{\lambda_r^*} \to k$ and the dual bases, we introduce some new notations. For $A \in W_r$, we let $A^\vee \in W_{n-r}$ be its complement in $\{1, \dots, n\}$. For $A \in W_r$ and $B \in W_{r'}$, $r, r' \in \{1, \dots, n\}$, we define

$$\mathrm{sign}(A, B) := (-1)^{|\{(a,b) \in A \times B | a > b\}|} \quad .$$

We denote by $\mathcal{S}_n$ the symmetric group on n elements; for a permutation $\pi \in \mathcal{S}_n$, we shall denote by π also the corresponding permutation matrix in $\mathrm{SL}_n(k)$ acting by left matrix multiplication.

Lemma 4.2.1. *Let* $1 \leq r \leq n-1$*. Then, the mapping*

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle : P_{G,n-r} \times Q_{G,r} &\longrightarrow k \\ (p,q) &\longmapsto \sum_{\pi\in\mathcal{S}_n} \operatorname{sign}(\pi)\, p(\pi e) q(\pi e) \end{aligned}$$

(where e denotes the neutral element of G) is a G-invariant pairing, and the bases

$$\{Y_A \mid A \in W_{n-r}\} \subseteq P_{G,n-r} \quad \textit{and} \quad \{\operatorname{sign}(B^\vee, B) Z_B \mid B \in W_r\} \subseteq Q_{G,r}$$

are dual with respect to it.

Proof: Let $A \in W_{n-r}$, $B \in W_r$. We begin by calculating

$$\sum_{\pi\in\mathcal{S}_n} \operatorname{sign}(\pi) Y_A(\pi e) Z_B(\pi e) \quad ;$$

if A and B are not disjoint, this expression is zero, since either $Y_A(\pi g)$ or $Z_B(\pi g)$ vanishes for all π's (the left and right minors have a line in common, and on each line only one entry is nonzero). This shows that, if $B \neq A^\vee$,

$$\langle Y_A, \operatorname{sign}(B^\vee, B) Z_B\rangle = 0 \quad .$$

Let also $B = A^\vee$; write $A = \{a_1, \ldots, a_{n-r}\}$ and $B = \{b_1, \ldots, b_r\}$ and define the permutation

$$\pi_A := \begin{pmatrix} 1 & \ldots & n-r & n-r+1 & \ldots & n \\ a_1 & \ldots & a_{n-r} & b_1 & \ldots & b_r \end{pmatrix} \quad .$$

The map

$$\begin{aligned} \mathcal{S}_n &\longrightarrow \mathcal{S}_n \\ \pi &\longmapsto \pi_A \circ \pi \end{aligned}$$

is a bijection on $\mathcal{S}_n$. Let $g \in \mathrm{SL}_n(k)$. It follows that, for all $g \in G$,

$$\begin{aligned} \sum_{\pi\in\mathcal{S}_n} \operatorname{sign}(\pi) Y_A(\pi g) Z_{A^\vee}(\pi g) &= \sum_{\pi\in\mathcal{S}_n} \operatorname{sign}(\pi_{\mathrm{A}}\pi) Y_A(\pi_A \pi g) Z_{A^\vee}(\pi_A \pi g) = \\ &= \operatorname{sign}(\pi_{\mathrm{A}}) \sum_{\pi\in\mathcal{S}_n} \operatorname{sign}(\pi) Y_{\{1,\ldots,n-r\}}(\pi g) Z_{\{n-r+1,\ldots,n\}}(\pi g) \quad . \end{aligned}$$

Noting that $\operatorname{sign}(\pi_A) = \operatorname{sign}(A, A^\vee)$, and that the rest of the expression is a Laplace expansion for the determinant of g, we get finally

$$\sum_{\pi\in\mathcal{S}_n} \operatorname{sign}(\pi) Y_A(\pi g) Z_{A^\vee}(\pi g) = \operatorname{sign}(A, A^\vee)\det(g) = \operatorname{sign}(A, A^\vee) \quad .$$

This proves that the pairing is G-invariant (the expression is independent on g), and if we set $g = e$ that

$$\operatorname{sign}(A, A^\vee)\langle Y_A, Z_{A^\vee}\rangle = \langle Y_A, \operatorname{sign}(A, A^\vee) Z_{A^\vee}\rangle = 1 \quad ,$$

and with that the proof is concluded. ■

As a corollary to this lemma, we recover Flath and Towber's presentation of the affine coordinate ring of $\mathrm{SL}_n(k)$:

Corollary 4.2.2. *The map*

$$\begin{array}{rcl} \mu : S \otimes_k S^- & \longrightarrow & k[\mathrm{SL}_n] \\ (f, g) & \longmapsto & fg \end{array}$$

is surjective, and its kernel is generated by

$$\ker(\mu)^G = (F_1 - 1, \ldots, F_{n-1} - 1) \quad ,$$

where

$$F_r = \sum_{A \in W_r} \operatorname{sign}(A, A^\vee) Y_A \otimes Z_{A^\vee} \quad .$$

Proof: The surjectivity of μ is clear by Popov's Theorem 4.1.3. Furthermore, by Thm. 4.1.5, the kernel of μ is generated by the elements $h_{\lambda_1} - 1, \ldots, h_{\lambda_{n-1}} - 1$ with

$$h_{\lambda_r} = \frac{\sum_{A \in W_r} Y_A \otimes \operatorname{sign}(A, A^\vee) Z_{A^\vee}}{\sum_{A \in W_r} Y_A(e) \cdot \operatorname{sign}(A, A^\vee) Z_{A^\vee}(e)} = F_r \quad ,$$

since the denominator is just the determinant of the neutral element e of G, also 1. ■

Remark 4.2.1. The standard definition as the zero set of the determinant gives a closed embedding of SL_n in $\mathbb{A}^{n^2}$ as the zero set of a polynomial of degree n; here, for $n \geq 3$, we get a different presentation, as the zero set of an ideal generated by polynomials of degree 2 in an the affine space of dimension $2(2^n - 2) = 2^{n+1} - 4$ (one coordinate for each left and one for each right minor).

4.3 Bounds for the affine embedding

Let us step back to the general case of a connected semisimple group G. By Popov's method (see the proof of Theorem 4.1.3), we can construct a closed, left-equivariant embedding of G in $V \oplus V^*$, where $V = \bigoplus_{i=1}^s R(\lambda_i)$ and $V^* = \bigoplus_{i=1}^s R(\lambda_i^*)$ is its dual; identifying both V and V^* with $\mathbb{A}^N$, $N = \sum_{i=1}^s \dim R(\lambda_i)$ we see that this is the same as a closed embedding of G in

$\mathbb{A}^N \times \mathbb{A}^N \cong \mathbb{A}^{2N}$.
For $k = \mathbb{C}$, the dimension of the irreducible representations can be explicitely computed by means of a formula due to H. Weyl (see for instance [32], pg. 9): hence, with Popov's construction we are able to exhibit explicit upper bounds for the dimension of an affine embedding of a connected semisimple algebraic group which depends only on the type of the group.
Let us denote by $N(\mathsf{T})$ the number N (see above) for a group of type T. Denote furthermore by $n_i = n_i(\mathsf{T})$ the degree of the i-th fundamental representation $R(\lambda_i)$ ($i = 1, \ldots, \ell$, where ℓ is the rank of the root system), so that $N(\mathsf{T}) = \sum_{i=1}^{\ell} n_i(\mathsf{T})$. With the help of Tits' tables from [32], we obtain the following list:

- $\boxed{\text{Type } \mathsf{A}_\ell}$ $(\ell \geq 1)$
  here we have $n_i = \binom{\ell+1}{i}$, $i = 1, \ldots, \ell$ and so

$$N(\mathsf{A}_\ell) = \sum_{i=1}^{\ell} \binom{\ell+1}{i} = 2^{\ell+1} - 2 \quad ;$$

- $\boxed{\text{Type } \mathsf{B}_\ell}$ $(\ell \geq 2)$
  here we have $n_i = \binom{2\ell+1}{i}$ for $i \leq \ell - 1$ and $n_\ell = 2^\ell$, and so

$$N(\mathsf{B}_\ell) = \sum_{i=1}^{\ell-1} \binom{2\ell+1}{i} + 2^\ell \quad ;$$

- $\boxed{\text{Type } \mathsf{C}_\ell}$ $(\ell \geq 3)$
  here we have $n_1 = 2\ell$ and $n_i = \binom{2\ell}{i} - \binom{2\ell}{i-2}$ for $i = 2, \ldots, \ell$, and so

$$N(\mathsf{C}_\ell) = 2\ell + \sum_{i=2}^{\ell} \left(\binom{2\ell}{i} - \binom{2\ell}{i-2} \right) = \binom{2\ell+1}{\ell} - 1 \quad ;$$

- $\boxed{\text{Type } \mathsf{D}_\ell}$ $(\ell \geq 4)$
  here we have $n_i = \binom{2\ell}{i}$ for $i \leq \ell - 2$ and $n_{\ell-1} = n_\ell = 2^{\ell-1}$, and so

$$N(\mathsf{D}_\ell) = \sum_{i=1}^{\ell-2} \binom{2\ell}{i} + 2 \cdot 2^{\ell-1} = \sum_{i=1}^{\ell-2} \binom{2\ell}{i} + 2^\ell \quad ;$$

- $\boxed{\text{Type } \mathsf{E}_6}$
  here we have $n_1 = n_5 = 27$, $n_2 = n_4 = 351$, $n_3 = 2\,925$, $n_6 = 78$ and so
  $$N(\mathsf{E}_6) = n_1 + \ldots + n_6 = 3759 \quad ;$$
- $\boxed{\text{Type } \mathsf{E}_7}$
  here we have $n_1 = 56$, $n_2 = 1\,539$, $n_3 = 27\,664$, $n_4 = 365\,750$, $n_5 = 8\,645$, $n_6 = 133$, $n_7 = 912$ and so
  $$N(\mathsf{E}_7) = n_1 + \ldots + n_7 = 404\,699 \quad ;$$
- $\boxed{\text{Type } \mathsf{E}_8}$
  here we have $n_1 = 248$, $n_2 = 30\,380$, $n_3 = 2\,450\,240$, $n_4 = 146\,325\,270$, $n_5 = 6\,899\,079\,264$, $n_6 = 6\,696\,000$, $n_7 = 3\,875$, $n_8 = 147\,250$ and so
  $$N(\mathsf{E}_8) = n_1 + \ldots + n_8 = 7\,054\,732\,527 \quad ;$$
- $\boxed{\text{Type } \mathsf{F}_4}$
  here we have $n_1 = 26$, $n_2 = 273$, $n_3 = 1\,274$, $n_4 = 52$ and so
  $$N(\mathsf{F}_4) = n_1 + \ldots + n_4 = 1\,625 \quad ;$$
- $\boxed{\text{Type } \mathsf{G}_2}$
  here we have $n_1 = 7$, $n_2 = 14$ and so
  $$N(\mathsf{G}_2) = n_1 + n_2 = 21 \quad .$$

If the monoid P_{++} of the dominant weights is freely generated, Theorem 4.1.5 and Theorem 4.1.6 show that the ideal of G in $\mathbb{A}^{2N}$ is generated by polynomials of degree 2. We get the

Theorem 4.3.1. *Let G be a connected, semisimple linear algebraic group of type T, defined over $\mathbb{C}$. Assume that the monoid of the dominant weights is freely generated. Then, the group G admits a closed, equivariant embedding as an intersection of quadrics in an affine space of dimension $2N(\mathsf{T})$, where $N(\mathsf{T})$ is determined by the list above.*

At this point, one could ask oneself whether the results of this section can be combined with those of section 3.4 in order to gain some more informations on an extension

$$0 \longrightarrow L \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

of an abelian variety A with a linear algebraic group L, and in particular if they can be used to compute the dimension of $H^0(\overline{L}, \mathcal{L})$ in Theorem 3.4.6. Unfortunately, two problems arise: the first one is that by Popov's method we get a closed embedding of L in an affine space, and not an open embedding in a $\mathbb{P}^N$. The second problem is that the only semisimple groups for which the Picard group is trivial are the products of SL_n and Sp_n, and so even if we could construct an open embedding of L in a projective space Lemma 3.4.3 would not apply in general. This obstacle could be overcome by considering suitable powers of an ample line bundle (since the Picard group of a connected linear algebraic group is finite, see [7], Cor. 4.4, pg. 278), but this would still require an explicit compactification of L. For the moment, we leave this question open.

Bibliography

[1] Iacopo Barsotti: *Structure theorems for group varieties*, Annali di mat. **38**, 77-119 (1955).

[2] Armand Borel: *Linear algebraic groups. Second Enlarged Edition*, Graduate Texts in Mathematics **126**, Springer, New York 1991.

[3] Siegfried Bosch, Werner Lütkebohmert, Michel Raynaud: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, Bd. **21**, Springer, Berlin 1990.

[4] Michel Brion: *Représentation exceptionelles des groupe de Lie semi-simples*, Ann. scient. Éc. Norm. Sup., 4e série **18**, 345-387 (1985).

[5] Gerd Faltings and Gisbert Wüstholz: *Einbettungen kommutativer algebraischer Gruppen und einige ihrer Eigenschaften*, J. reine angew. Math. **333**, 144-161 (1982).

[6] Daniel Flath and Jacob Towber: *Generators and relations for the affine rings of the classical groups*, Commun. in algebra **20(10)**, 2877-2902 (1992).

[7] Robert Fossum and Birger Iversen: *On Picard groups of algebraic fibre spaces*, J. of Pure and Applied Algebra **3**, 269-280 (1973).

[8] Roe Goodman and Nolan R. Wallach: *Representations and invariants of the classical groups*, Cambridge University Press, Cambridge 1998.

[9] Alexandre Grothendieck: *Eléments de géométrie algébrique*, Publ. math. IHES **4, 8, 11, 17, 20, 24, 28, 32**, Paris 1960-1967.

[10] Alexandre Grothendieck: *Technique de descente et théorèmes d'existence en géometrie algébrique I: Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki 1959.

[11] Alexandre Grothendieck: *Torsion homologique et sections rationelles*, Séminaire Chevalley 1958.

[12] Robin Hartshorne: *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York 1981.

[13] James E. Humphreys: *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer, New York 1981.

[14] Friedrich Knop and Herbert Lange: *Commutative algebraic groups and intersections of quadrics*, Math. Ann. **267**, 555-571 (1984).

[15] Glenn Lancaster and Jacob Towber: *Representation-functors and flag-algebras for the classical groups I*, J. of algebra **59**, 16-38 (1979).

[16] Serge Lang: *Abelian varieties*, Interscience publishers, New York 1959.

[17] Herbert Lange: *Families of translations of commutative algebraic groups*, J. of algebra **109**, 260-265 (1987).

[18] Herbert Lange and Wolfgang Ruppert: *Complete systems of addition laws on an abelian variety*, Inv. math. **79**, 603-610 (1985).

[19] James S. Milne: *Abelian varieties (course notes)*, `www.jmilne.org/math` (1998).

[20] Shigeru Mukai: *Semi-homogeneous vector bundles on an abelian variety*, J. math. Kyoto univ. **18-2**, 239-272 (1978).

[21] David Mumford: *Abelian varieties*, Oxford university press, 1974.

[22] David Mumford: *Varieties defined by quadratic equations*. In: *Questions on algebraic varieties*, CIME, 29-100 (1970).

[23] David Mumford and John Fogarty: *Geometric invariant theory, second enlarged edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete **34**, Springer, Berlin 1982.

[24] Vladimir L. Popov: *Generators and relations of the affine coordinate rings of connected semisimple algebraic groups*, Preprint ESI **972**, The Erwin Schrödinger international institute for mathematical physics, `www.esi.ac.at/ESI-Preprints.html`, Vienna (2000).

[25] Vladimir L. Popov and Ernest B. Vinberg: *Invariant theory*, In: A.N. Parshin, I.R. Shafarevich (Eds.): *Algebraic geometry IV*, Encyclopaedia of mathematical sciences Vol. **55**, Springer, New York 1994.

[26] Maxwell Rosenlicht: *Some basic theorems on algebraic groups*, Amer. j. math. **78**, 401-443 (1956).

[27] Jean-Pierre Serre: *Espaces fibrés algébriques*, Séminaire Chevalley 1958.

[28] Jean-Pierre Serre: *Groupes algébriques et corps de classes*, Hermann et cie. , Paris 1959.

[29] Jean-Pierre Serre: *Quelques proprietés des groupes algébriques commutatifs*, in: M. Waldschmidt, *Nombres Transcendants et groupes algébriques*, Astérisque **69-70**, 191-202 (1978).

[30] Francesco Severi: *Funzioni quasi abeliane*, Pont. Acad. Scient., Rome 1947.

[31] Hideyasu Sumihiro: *Equivariant completion*, J. Math. Kyoto Univ **14-1**, 1-28 (1974).

[32] Jacques Tits: *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Lecture Notes in Mathematics **40**, Springer, Berlin 1967.

[33] Jacob Towber: *Young Symmetry, the flag manifold, and representations of* $\mathrm{GL}(n)$, J. of algebra **61**, 414-462 (1979).

[34] André Weil: *Varietés abéliennes*, Colloque d' Algèbre et Théorie des Nombres (1950), C.N.R.S., 125-127. (Reprinted in: A. Weil, *Collected Papers*, Vol. **I**, Springer, New York 1979.)

[35] Gisbert Wüstholz: *Multiplicity estimates on group varieties*, Ann. of math. **129**, 471-500 (1989).

[36] Gisbert Wüstholz: *Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen*, Ann. of math. **129**, 501-517 (1989).

[37] Gisbert Wüstholz: *Computations on commutative group varieties.* In: *Arithmetic geometry, Cortona 1994, Ed. by F. Catanese*, Symposia mathematica Vol. **XXXVII**, Cambridge university press, Cambridge 1997.

Curriculum Vitae

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