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Abstract

We analyze mixed *hp*-discontinuous Galerkin finite element methods (DGFEM) for Stokes flow in polygonal domains. In conjunction with geometrically refined meshes and linearly increasing approximation orders, we prove that the *hp*-DGFEM leads to exponential rates of convergence for piecewise analytic solutions exhibiting singularities near corners

Keywords: Mixed FEM, *hp*-DGFEM, Exponential Convergence

Subject Classification: 65N30

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1. Introduction

In the last years, several mixed discontinuous Galerkin finite element methods (DGFEM) have been proposed for the discretization of incompressible fluid flow problems. We mention here only the piecewise solenoidal discontinuous Galerkin methods introduced in [5, 24], the local discontinuous Galerkin methods of [12, 11], and the interior penalty methods studied in [23, 32, 17]. Some of the main motivations that lead to the above methods are the following: First of all, the discontinuous nature of the finite element spaces allows one to easily treat convective terms by suitable upwind fluxes, similarly to the original discontinuous Galerkin discretizations of (non-linear) hyperbolic equations (see [13, 10, 14] and the references therein). Thus, mixed DG methods provide robust and high-order accurate approximations particularly in transport-dominated regimes; see, e.g., [24, 11, 17] for mixed DGFEM for the Navier-Stokes and Oseen equations. Moreover, discontinuous Galerkin approaches are extremely flexible in the mesh-design; meshes with hanging nodes, elements of various types and shapes, and local spaces of different orders can be easily dealt with. Finally, mixed DG methods are considerably flexible in the choice of velocity-pressure combinations, without extensive stabilization techniques. In the discontinuous Galerkin context, for example, no extra stabilization is needed to use optimal mixed-order elements where the approximation degree for the pressure is of one order lower than that of the velocity; see [23, 32] for details.

The recent work in [28] presented a unifying framework for the analysis of mixed hp-DGFEM for pure Stokes flow. For $\mathcal{Q}^k - \mathcal{Q}^{k-1}$ elements, the dependence of the discrete inf-sup constant on the polynomial degree k was shown to be of the order $\mathcal{O}(1/k)$, for two- and three-dimensional domains. In three dimensions, this is exactly the same bound as that of [31] for conforming mixed hp-FEM, but with an optimal gap of one order in the finite element spaces for the velocity and the pressure. The results in [28] then ensure (slightly suboptimal) error bounds for the p-version of the DGFEM where convergence is obtained by increasing the polynomial approximation order on a fixed (quasi-uniform) mesh. However, these bounds give algebraic rates of convergence and are restricted to piecewise smooth solutions; an assumption that is unrealistic in domains with corners, due to the presence of corner singularities, see, e.g., [25, 22]. For conforming mixed methods, similar p-version results can be found in, e.g., [6, 31, 8, 30, 7] and the references therein.

In this paper, we extend the hp-approaches of [28] to mixed hp-DGFEM for Stokes flow in non-smooth polygonal domains where the exact solutions are piecewise analytic, but exhibit singularities at the corners. To describe the regularity of the exact solutions, we use a recent result from [22] that measures analytic regularity in terms of the countably normed, weighted spaces that were introduced by Babuška and Guo for closely related potential and elasticity problems; see [19, 20, 18, 3, 2, 4, 21, 29] and the references therein. The reduced regularity near corners imposes several technical difficulties and requires a careful treatment of the elements and the numerical fluxes near vertices of the domain. By the use of new trace theorems for functions in weighted Sobolev spaces, we first show that the mixed hp-DGFEM is in fact well-defined. Then, we employ standard hp-version mesh design principles to resolve corner singularities: namely, we use meshes that are geometrically refined towards corners and approximation degrees that increase linearly away from corners. We show that this combination of h- and p-refinement leads to exponential rates of convergence. For hp-DGFEM discretizations of scalar diffusion problems an analogous result was recently obtained in [34].

To prove exponential convergence for our mixed methods, we use several ingredients from the analysis of conforming mixed hp-FEM for Stokes flow on geometric meshes; see, e.g., [30, 29, 27], combined with the techniques that were developed in [34, 33] to treat diffusion and elasticity problems in polygons. Furthermore, we use the setting [28] to derive the exponential convergence result. Exemplarily, we consider only the interior penalty DGFEM, but point out that our results hold true verbatim for all the DG methods studied in [28]. We also note that our analysis can be straightforwardly extended to mixed formulations of linear elasticity problems with nearly incompressible materials; see, e.g., [9, 15].

The outline of the paper is as follows: In Section 1.1 we begin by introducing some notational conventions that we use throughout the paper. Section 2 reviews the analytic regularity of the Stokes problem in polygonal domains. In Section 3, we introduce meshes and establish several properties of functions on the elements near the corners of the domain. The *hp*-DGFEM discretization of the Stokes problem is introduced in Section 4. In Section 5, we derive abstract error estimates for piecewise analytic solutions. Section 6 is devoted to the main result of this paper: we prove that the *hp*-DGFEM is exponentially convergent. We end our presentation with concluding remarks in Section 7.

1.1. **Notation.** For a bounded Lipschitz domain D in $\mathbb{R}^d$, $d \geq 1$, we denote by $L^p(D)$, $1 \leq p \leq \infty$, the Lebesgue space of p-integrable functions, endowed with the norm $\|\cdot\|_{L^p(D)}$. We set $L^2_0(D) := \{q \in L^2(D) : \int_D q\, d\mathbf{x} = 0\}$. The space of p-times continuously differentiable functions on D is $C^p(\overline{D})$, $0 \leq p \leq \infty$, equipped with the usual norm $\|\cdot\|_{C^p(\overline{D})}$. The standard Sobolev space of functions with integer or fractional regularity exponent $s \geq 0$ is denoted by $H^s(D)$. We write $\|\cdot\|_{H^s(D)}$ and $|\cdot|_{H^s(D)}$ for its norm and seminorm, respectively, and set $H^0(D) = L^2(D)$. The trace space of $H^1(D)$ is denoted by $H^{\frac{1}{2}}(\partial D)$ and, as usual, we define $H^1_0(D)$ as the subspace of functions in $H^1(D)$ with zero trace on ∂D. The dual space of $H^1_0(D)$ is denoted by $H^{-1}(D)$. For a function space $X(D)$ we write $X(D)^d$ and $X(D)^{d\times d}$ to denote vector and tensor fields whose components belong to $X(D)$, respectively. Without further specification, these spaces are equipped with the usual product norms (which we simply denote by $\|\cdot\|_{X(D)}$). For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, and matrices $\underline{\sigma}, \underline{\tau} \in \mathbb{R}^{d\times d}$, we use the standard notation $(\nabla\mathbf{v})_{ij} = \partial_j v_i$, $(\nabla\cdot\underline{\sigma})_i = \sum_{j=1}^d \partial_j \sigma_{ij}$, and $\underline{\sigma} : \underline{\tau} = \sum_{i,j=1}^d \sigma_{ij}\tau_{ij}$. Furthermore, we denote by $\mathbf{v}\otimes\mathbf{w}$ the matrix whose ij-th component is $v_i w_j$, and use the identity $\mathbf{v}\cdot\underline{\sigma}\cdot\mathbf{w} = \sum_{i,j=1}^d v_i\sigma_{ij}w_j = \underline{\sigma} : (\mathbf{v}\otimes\mathbf{w})$.

2. The Stokes Problem with Piecewise Analytic Data

2.1. **The Stokes Equations.** Let $\Omega \subset \mathbb{R}^2$ be a polygonal and bounded domain. The Stokes problem is to find a velocity field $\mathbf{u}$ and a pressure p such that

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here, the right-hand side $\mathbf{f} \in H^{-1}(\Omega)^2$ is an exterior body force, and $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^2$ a prescribed Dirichlet datum satisfying the compatibility condition $\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\, ds = 0$, with $\mathbf{n}$ denoting the outward unit normal vector to $\partial\Omega$. Due to the continuous inf-sup condition, the Stokes system (2.1) has a unique solution $(\mathbf{u}, p)$ in $H^1_0(\Omega)^2 \times L^2_0(\Omega)$; see, e.g., [9, 16] for details.

2.2. **Analytic Regularity in Polygonal Domains.** For piecewise analytic data, the regularity of the exact solution $(\mathbf{u}, p)$ of (2.1) was recently described by Guo and Schwab [22] in terms of the weighted Sobolev spaces that were originally introduced by Babuška and Guo for closely related elasticity and potential problems; see [19, 20, 18, 3, 2, 4, 21, 29] and the references therein. To define these weighted spaces, let $\{A_i\}_{i=1}^M$ denote the vertices of the domain Ω. To each vertex A_i we assign a weight $\beta_i \geq 0$ and store these numbers in the M-tuple $\underline{\beta} = (\beta_1, \ldots, \beta_M)$. We define $\underline{\beta} \pm j := (\beta_1 \pm j, \ldots, \beta_M \pm j)$ and use the shorthand notation $C_1 > \underline{\beta} > C_2$ to mean

$C_1 > \beta_i > C_2$ for $i = 1, \dots, M$. For $r_i^*(\mathbf{x}) = \min\{1, |\mathbf{x} - A_i|\}$ we define the weight function $\Phi_{\underline{\beta}}(\mathbf{x}) := \prod_{i=1}^M r_i^*(\mathbf{x})^{\beta_i}$, and introduce the seminorms

$$|u|^2_{H^{k,l}_{\underline{\beta}}(\Omega)} := \sum_{|\alpha| \geq l}^{k} \|\Phi_{\underline{\beta}+|\alpha|-l} D^\alpha u\|^2_{L^2(\Omega)}, \qquad k \geq l \geq 0.$$

We denote by $H^{k,l}_{\underline{\beta}}(\Omega)$ the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\begin{aligned} \|u\|^2_{H^{k,l}_{\underline{\beta}}(\Omega)} &:= \|u\|^2_{H^{l-1}(\Omega)} + |u|^2_{H^{k,l}_{\underline{\beta}}(\Omega)}, \qquad l \geq 1, \\ \|u\|^2_{H^{k,0}_{\underline{\beta}}(\Omega)} &:= \sum_{|\alpha| \geq 0}^{k} \|\Phi_{\underline{\beta}+|\alpha|} D^\alpha u\|^2_{L^2(\Omega)}. \end{aligned}$$

Definition 2.1. For an M-tuple $\underline{\beta} = (\beta_1, \dots, \beta_M)$ and $l \geq 0$, the countably normed space $B^l_{\underline{\beta}}(\Omega)$ consists of all functions u for which $u \in H^{k,l}_{\underline{\beta}}(\Omega)$ for $k \geq l$ and

$$\|\Phi_{\underline{\beta}+k-l} D^\alpha u\|_{L^2(\Omega)} \leq C d^{(k-l)} (k-l)!, \qquad |\alpha| = k \geq l,$$

for some constants $C > 0$, $d \geq 1$ independent of k.

We remark that, in general, $B^2_{\underline{\beta}}(\Omega) \not\subset H^2(\Omega)$ and $B^1_{\underline{\beta}}(\Omega) \not\subset H^1(\Omega)$. However, $B^2_{\underline{\beta}}(\Omega) \subset C^0(\overline{\Omega})$ and $B^1_{\underline{\beta}}(\Omega_{\text{int}}) \subset C^0(\overline{\Omega}_{\text{int}})$, for all interior domains Ω_{int} with $\overline{\Omega}_{\text{int}} \subset \overline{\Omega} \setminus \{A_i\}_{i=1}^M$.

For a noninteger exponent k, the space $H^{k,l}_{\underline{\beta}}(\Omega)$ is defined by interpolation. Finally, we define $H^{k-\frac{1}{2},l-\frac{1}{2}}_{\underline{\beta}}(\partial\Omega)$ and $B^{l-\frac{1}{2}}_{\underline{\beta}}(\partial\Omega)$ as spaces of traces of functions in $H^{k,l}_{\underline{\beta}}(\Omega)$ and $B^l_{\underline{\beta}}(\Omega)$, respectively. The space $H^{k-\frac{1}{2},l-\frac{1}{2}}_{\underline{\beta}}(\partial\Omega)$ is endowed with the norm

$$\|g\|_{H^{k-\frac{1}{2},l-\frac{1}{2}}_{\underline{\beta}}(\partial\Omega)} = \inf\{\, \|u\|_{H^{k,l}_{\underline{\beta}}(\Omega)} : u|_{\partial\Omega} = g\,\}.$$

The following regularity result will be the basis of our analysis; its proof can be found in [22].

Theorem 2.2. *There exist a weight vector* $0 \leq \underline{\beta}_{\min} < 1$ *depending on the opening angles of* Ω *at the vertices* $\{A_i\}_{i=1}^M$ *such that for weight vectors* $\underline{\beta}$ *with* $\underline{\beta}_{\min} < \underline{\beta} < 1$ *and piecewise analytic data*

$$\mathbf{f} \in B^0_{\underline{\beta}}(\Omega)^2, \qquad \mathbf{g} \in B^{\frac{3}{2}}_{\underline{\beta}}(\partial\Omega)^2, \tag{2.2}$$

the solution $(\mathbf{u}, p)$ *of the Stokes system (2.1) satisfies*

$$\mathbf{u} \in B^2_{\underline{\beta}}(\Omega)^2, \qquad p \in B^1_{\underline{\beta}}(\Omega). \tag{2.3}$$

We point out that, in particular, Theorem 2.2 implies that

$$\mathbf{u} \in H^{2,2}_{\underline{\beta}}(\Omega)^2, \qquad p \in H^{1,1}_{\underline{\beta}}(\Omega), \qquad \nabla\mathbf{u} \in H^{1,1}_{\underline{\beta}}(\Omega)^{2\times 2}, \tag{2.4}$$

and

$$-\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } H^{0,0}_{\underline{\beta}}(\Omega)^2. \tag{2.5}$$

Throughout the paper, the smoothness property in Theorem 2.2 is assumed to hold for a weight vector $\underline{\beta}$ with $\underline{\beta}_{\min} < \underline{\beta} < 1$.

3. Meshes and Trace Operators

In this section, we introduce the trace operators that are needed to define the interelemental terms in our discontinuous Galerkin methods. Furthermore, we prove a series of technical results that allow us to properly treat the elements at the vertices of the domains. Similar results were used recently in [34, 33] to analyze *hp*-DGFEM for diffusion and elasticity problems.

3.1. **Meshes.** Throughout, let $\mathcal{T}_h = \{K\}$ be a shape-regular affine mesh on Ω consisting of parallelograms. For each $K \in \mathcal{T}_h$, we denote by $\mathbf{n}_K$ the outward unit normal vector to the boundary ∂K, and by h_K the elemental diameter. Furthermore, we assign to each element $K \in \mathcal{T}_h$ an approximation order $k_K \geq 1$. The local quantities h_K and k_K are stored in the vectors $\underline{h} = \{h_K\}_{K\in\mathcal{T}_h}$ and $\underline{k} = \{k_K\}_{K\in\mathcal{T}_h}$, respectively. We set $h = \max_{K\in\mathcal{T}_h} h_K$ and $|\underline{k}| = \max_{K\in\mathcal{T}_h} k_K$.

An interior edge of $\mathcal{T}_h$ is the (non-empty) one-dimensional interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}_h$. Similarly, a boundary edge of $\mathcal{T}_h$ is the (non-empty) one-dimensional interior of $\partial K \cap \partial\Omega$ which consists of entire edges of ∂K. We denote by $\mathcal{E}_\mathcal{I}$ the union of all interior edges of $\mathcal{T}_h$, by $\mathcal{E}_\mathcal{D}$ the union of all boundary edges, and set $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{D}$. Generally, we allow for irregular meshes, i.e., meshes with hanging nodes (see [29, Sect. 4.4.1]), but suppose that the intersection between neighboring elements is either a common vertex or a common edge of one of the two elements. We also assume the local mesh sizes and approximation degrees to be of bounded variation: that is, there is a constant $\kappa > 0$ such that

$$\kappa h_K \leq h_{K'} \leq \kappa^{-1} h_K, \qquad \kappa k_K \leq k_{K'} \leq \kappa^{-1} k_K, \tag{3.1}$$

whenever K and K' share a common edge.

3.2. **Averages and Jumps.** Next, we define average and jump operators. To that end, let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$; let $\mathbf{x}$ be an arbitrary point of the interior edge $e = \partial K^+ \cap \partial K^- \subset \mathcal{E}_\mathcal{I}$. Let q, $\mathbf{v}$, and $\underline{\tau}$ be scalar-, vector-, and matrix-valued functions, respectively, that are smooth inside each element $K^\pm$, and let us denote by $(q^\pm, \mathbf{v}^\pm, \underline{\tau}^\pm)$ the traces of $(q, \mathbf{v}, \underline{\tau})$ on e taken from within the interior of $K^\pm$. Then, we define the following averages at $\mathbf{x} \in e$

$$\{\!\{q\}\!\} = (q^+ + q^-)/2, \qquad \{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2, \qquad \{\!\{\underline{\tau}\}\!\} = (\underline{\tau}^+ + \underline{\tau}^-)/2.$$

Similarly, the jumps at $\mathbf{x} \in e$ are given by

$$[\![q]\!] = q^+\,\mathbf{n}_{K^+} + q^-\,\mathbf{n}_{K^-}, \qquad [\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}_{K^+} + \mathbf{v}^- \cdot \mathbf{n}_{K^-},$$
$$\underline{[\![\mathbf{v}]\!]} = \mathbf{v}^+ \otimes \mathbf{n}_{K^+} + \mathbf{v}^- \otimes \mathbf{n}_{K^-}, \qquad [\![\underline{\tau}]\!] = \underline{\tau}^+\mathbf{n}_{K^+} + \underline{\tau}^-\mathbf{n}_{K^-}.$$

On boundary edges $e \subset \mathcal{E}_\mathcal{D}$, we set $\{\!\{q\}\!\} = q$, $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$, $\{\!\{\underline{\tau}\}\!\} = \underline{\tau}$, as well as $[\![q]\!] = q\mathbf{n}$, $[\![\mathbf{v}]\!] = \mathbf{v}\cdot\mathbf{n}$, $\underline{[\![\mathbf{v}]\!]} = \mathbf{v}\otimes\mathbf{n}$, and $[\![\underline{\tau}]\!] = \underline{\tau}\mathbf{n}$.

3.3. **Elements Near Vertices.** To account for the singular behavior of solutions near the vertices $\{A_i\}_{i=1}^M$ of the domain, we define the sets

$$\mathcal{K}_{\mathrm{vert}} = \{\, K \in \mathcal{T}_h : \overline{K} \cap A_i \neq \emptyset \text{ for some } 1 \leq i \leq M \,\},$$
$$\mathcal{K}_{\mathrm{int}} = \{\, K \in \mathcal{T}_h : \overline{K} \cap A_i = \emptyset \text{ for all } 1 \leq i \leq M \,\}.$$

Let $K \in \mathcal{K}_{\mathrm{vert}}$. We always assume that the partitions $\mathcal{T}_h$ are fine enough so that exactly one vertex belongs to K. We denote this vertex by A_K and the corresponding weight exponent by $\beta_K \in (0,1)$. The spaces $H^{k,\ell}_{\beta_K}(K)$ are defined as in Section 2, but equipped with the weight function $\Phi_{\beta_K}(\mathbf{x}) = r^{\beta_K}$, with r denoting the distance to the corner A_K. We have the following auxiliary results.

Lemma 3.1. *Let $K \in \mathcal{K}_{\mathrm{vert}}$. Then:*

(1) *We have $H^{0,0}_{\beta_K}(K) \subset L^1(K)$ and*

$$\|\varphi\|_{L^1(K)} \leq C h_K^{1-\beta_K} \|\varphi\|_{H^{0,0}_{\beta_K}(K)}, \qquad \forall \varphi \in H^{0,0}_{\beta_K}(K).$$

(2) *Let $\varphi \in H^{0,0}_{\beta_K}(K)$ and $v \in L^\infty(K)$. Then the integral $\int_K \varphi v \, d\mathbf{x}$ is well-defined and $|\int_K \varphi v \, d\mathbf{x}| \leq C h_K^{1-\beta_K} \|v\|_{L^\infty(K)} \|\varphi\|_{H^{0,0}_{\beta_K}(K)}$.*

(3) *Let* $\varphi \in H^{1,1}_{\beta_K}(K)$*. Then the trace* $\varphi|_{\partial K}$ *belongs to* $L^1(\partial K)$ *and satisfies*

$$\|\varphi\|_{L^1(\partial K)} \le C\big(\|\varphi\|_{L^2(K)} + h_K^{1-\beta_K}|\varphi|_{H^{1,1}_{\beta_K}(K)}\big).$$

All the constants $C > 0$ *are independent of* $\underline{h}$ *and* $\underline{k}$*.*

Proof. For $\varphi \in H^{0,0}_{\beta_K}(K)$, we have

$$\int_K |\varphi|\,d\mathbf{x} \le \|r^{-\beta_K}\|_{L^2(K)}\|r^{\beta_K}\varphi\|_{L^2(K)} = \|r^{-\beta_K}\|_{L^2(K)}\|\varphi\|_{H^{0,0}_{\beta_K}(K)}.$$

Since $\|r^{-\beta_K}\|_{L^2(K)} \le Ch_K^{1-\beta_K}$, the first assertion follows. The second assertion follows then straightforwardly from Hölder's inequality. To prove the third assertion, let $\varphi \in H^{1,1}_{\beta_K}(K)$. From the standard trace theorem and a scaling argument, we have

$$\|\varphi\|_{L^1(\partial K)} \le C\big(h_K^{-1}\|\varphi\|_{L^1(K)} + \|\nabla\varphi\|_{L^1(K)}\big).$$

First, we note that $h_K^{-1}\|\varphi\|_{L^1(K)} \le C\|\varphi\|_{L^2(K)}$. Next, since $\nabla\varphi \in H^{0,0}_{\beta_K}(K)^2$, we have $\|\nabla\varphi\|_{L^1(K)} \le Ch_K^{1-\beta_K}|\varphi|_{H^{1,1}_{\beta_K}(K)}$, which is a consequence of the first assertion and the definition of the seminorm $|\cdot|_{H^{1,1}_{\beta_K}(K)}$. This completes the proof. □

Lemma 3.2. *Let* $K \in \mathcal{K}_{\mathrm{vert}}$*,* $\underline{\tau} \in H^{1,1}_{\beta_K}(K)^{2\times 2}$ *and* $\mathbf{v} \in C^1(\overline{K})^2$*. Then the following integration by parts formula holds*

$$\int_K \nabla\cdot\underline{\tau}\cdot\mathbf{v}\,d\mathbf{x} = -\int_K \underline{\tau} : \nabla\mathbf{v}\,d\mathbf{x} + \int_{\partial K} \underline{\tau} : (\mathbf{v}\otimes\mathbf{n}_K)\,ds,$$

where the term on the left and the boundary term are understood as $L^1 \times L^\infty$ *pairings.*

Proof. We start by noting that all the integrals above are well-defined due to Lemma 3.1 and the fact that $\nabla\cdot\underline{\tau} \in H^{0,0}_{\beta_K}(K)^2$. Furthermore, since $C^\infty(\overline{K})$ is dense in $H^{1,1}_{\beta_K}(K)$, there exists a sequence $\{\underline{\tau}_n\} \subset C^\infty(\overline{K})^{2\times 2}$ with $\underline{\tau}_n \to \underline{\tau}$ in $H^{1,1}_{\beta_K}(K)^{2\times 2}$. Clearly,

$$\int_K \nabla\cdot\underline{\tau}_n\cdot\mathbf{v}\,d\mathbf{x} = -\int_K \underline{\tau}_n : \nabla\mathbf{v}\,d\mathbf{x} + \int_{\partial K} \underline{\tau}_n : (\mathbf{v}\otimes\mathbf{n}_K)\,ds.$$

The trace estimate from Lemma 3.1 yields

$$\Big|\int_{\partial K} (\underline{\tau}-\underline{\tau}_n) : (\mathbf{v}\otimes\mathbf{n}_K)\,ds\Big| \le C\|\mathbf{v}\|_{L^\infty(\partial K)}\|\underline{\tau}-\underline{\tau}_n\|_{H^{1,1}_{\beta_K}(K)}.$$

Furthermore, again with Lemma 3.1,

$$\begin{aligned}\Big|\int_K \nabla\cdot(\underline{\tau}-\underline{\tau}_n)\cdot\mathbf{v}\,d\mathbf{x}\Big| &\le \|\mathbf{v}\|_{L^\infty(K)}\|\nabla\cdot(\underline{\tau}-\underline{\tau}_n)\|_{L^1(K)}\\ &\le Ch_K^{1-\beta_K}\|\mathbf{v}\|_{L^\infty(K)}\|\underline{\tau}-\underline{\tau}_n\|_{H^{1,1}_{\beta_K}(K)},\end{aligned}$$

and

$$\begin{aligned}\Big|\int_K (\underline{\tau}-\underline{\tau}_n) : \nabla\mathbf{v}\,d\mathbf{x}\Big| &\le \|\nabla\mathbf{v}\|_{L^2(K)}\|\underline{\tau}-\underline{\tau}_n\|_{L^2(K)}\\ &\le \|\nabla\mathbf{v}\|_{L^2(K)}\|\underline{\tau}-\underline{\tau}_n\|_{H^{1,1}_{\beta_K}(K)}.\end{aligned}$$

Passing to the limit finishes the proof. □

Lemma 3.3. *Let the exact solution* $(\mathbf{u}, p)$ *of the Stokes system satisfy (2.3). For an interior edge* $e \subset \mathcal{E}_{\mathcal{I}}$*, we have that* $[\![\nabla\mathbf{u} - p\underline{I}]\!] = \mathbf{0}$ *on* e*.*

Proof. We note that $\nabla \mathbf{u} - p\underline{I}$ belongs to $C^0(\overline{\Omega}_{\text{int}})$ for all interior domains Ω_{int} with $\overline{\Omega}_{\text{int}} \subset \overline{\Omega} \setminus \{A_i\}_{i=1}^M$. Hence, if $\overline{e} \cap \{A_i\}_{i=1}^M = \emptyset$, we immediately have that $[\![\nabla \mathbf{u} - p\underline{I}]\!] = \mathbf{0}$ on e. Let us then consider the case where $\overline{e} \cap \{A_i\}_{i=1}^M = A_j$ for a vertex A_j. We may assume that the edge is parameterized by $\overline{e} = \varphi(t)$, $t \in [0,1]$, with $\varphi(0) = A_j$. Then,

$$\int_{\varepsilon}^{1} |[\![\nabla \mathbf{u} - p\underline{I}]\!]|\, |\varphi'(t)|\, dt = 0,$$

for all $\varepsilon > 0$. Thanks to (2.4), we have $\nabla \mathbf{u} - p\underline{I} \in H^{1,1}_{\underline{\beta}}(\Omega)^{2\times 2}$. Thus, $[\![\nabla \mathbf{u} - p\underline{I}]\!] \in L^1(e)^2$, according to Lemma 3.1. We conclude with Lebesgue's dominated convergence theorem that

$$\int_{0}^{1} |[\![\nabla \mathbf{u} - p\underline{I}]\!]|\, |\varphi'(t)|\, dt = 0,$$

and thus $[\![\nabla \mathbf{u} - p\underline{I}]\!] = \mathbf{0}$ on e. □

4. Discontinuous Galerkin Discretization

In this section, we introduce discontinuous Galerkin methods for the Stokes problem and review their well-posedness, using the recent results in [28].

4.1. **Mixed DGFEM.** Given a mesh $\mathcal{T}_h$ and a degree vector $\underline{k} = \{k_K\}$, $k_K \geq 1$, we approximate the Stokes problem by finite element functions $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ where

$$\begin{aligned} \mathbf{V}_h &= \{\, \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in \mathcal{Q}^{k_K}(K)^2,\ K \in \mathcal{T}_h \,\}, \\ Q_h &= \{\, q \in L^2_0(\Omega) : q|_K \in \mathcal{Q}^{k_K-1}(K),\ K \in \mathcal{T}_h \,\}. \end{aligned}$$

Here, $\mathcal{Q}^k(K)$ denotes the space of polynomials of degree at most $k \geq 0$ in each variable on K. For further reference, we also define the space

$$\widetilde{Q}_h = \{\, q \in L^2(\Omega) : q|_K \in \mathcal{Q}^{k_K-1}(K),\ K \in \mathcal{T}_h \,\}.$$

We consider the mixed method: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\left\{ \begin{array}{rcccl} A_h(\mathbf{u}_h, \mathbf{v}) & + & B_h(\mathbf{v}, p_h) & = & F_h(\mathbf{v}) \\ -B_h(\mathbf{u}_h, q) & & & = & G_h(q) \end{array} \right. \tag{4.1}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$. The forms A_h and B_h are discontinuous Galerkin forms that discretize the Laplacian and the incompressibility constraint, respectively, with corresponding right-hand sides F_h and G_h. These forms are given by

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla_h \mathbf{u} : \nabla_h \mathbf{v}\, d\mathbf{x} - \int_{\mathcal{E}} \left(\{\!\{\nabla_h \mathbf{v}\}\!\} : \underline{[\![\mathbf{u}]\!]} + \{\!\{\nabla_h \mathbf{u}\}\!\} : \underline{[\![\mathbf{v}]\!]} \right) ds \\ &\quad + \int_{\mathcal{E}} \mathsf{c} \underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\, ds, \\ B_h(\mathbf{v}, q) &= -\int_\Omega q\, \nabla_h \cdot \mathbf{v}\, d\mathbf{x} + \int_{\mathcal{E}} \{\!\{q\}\!\} [\![\mathbf{v}]\!]\, ds, \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} - \int_\Omega (\mathbf{g} \otimes \mathbf{n}) : \nabla_h \mathbf{v}\, ds + \int_{\mathcal{E}_{\mathcal{D}}} \mathsf{c}\mathbf{g} \cdot \mathbf{v}\, ds, \\ G_h(q) &= -\int_{\mathcal{E}_{\mathcal{D}}} q\, \mathbf{g} \cdot \mathbf{n}\, ds. \end{aligned} \tag{4.2}$$

Here, ∇_h and $\nabla_h \cdot$ denote the discrete gradient and divergence operator, taken elementwise. The function $\mathsf{c} \in L^\infty(\mathcal{E})$ is the so-called discontinuity stabilization function that is chosen as follows. Define the functions $\mathtt{h} \in L^\infty(\mathcal{E})$ and $\mathtt{k} \in L^\infty(\mathcal{E})$ by

$$\mathtt{h}(\mathbf{x}) := \begin{cases} \min\{h_K, h_{K'}\}, & \mathbf{x} \in e = \partial K \cap \partial K' \subset \mathcal{E}_\mathcal{I}, \\ h_K, & \mathbf{x} \in e = \partial K \cap \partial \Omega \subset \mathcal{E}_\mathcal{D}, \end{cases}$$

$$\mathtt{k}(\mathbf{x}) := \begin{cases} \max\{k_K, k_{K'}\}, & \mathbf{x} \in e = \partial K \cap \partial K' \subset \mathcal{E}_\mathcal{I}, \\ k_K, & \mathbf{x} \in e = \partial K \cap \partial \Omega \subset \mathcal{E}_\mathcal{D}. \end{cases}$$

Then we set

$$\mathtt{c} = \gamma \mathtt{h}^{-1} \mathtt{k}^2, \tag{4.3}$$

with a parameter $\gamma > 0$ that is independent of $\mathtt{h}$ and $\mathtt{k}$.

Remark 4.1. It can be seen from (2.4) and the trace properties in Lemma 3.1 that the forms A_h and B_h are well-defined when inserting the exact solution $(\mathbf{u}, p)$ satisfying (2.3). Similarly, F_h and G_h are well-defined due to (2.2) and Lemma 3.1.

Remark 4.2. The form A_h corresponds to the so-called symmetric interior penalty discretization of the Laplace operator; see [1] and [28] where the presentation and analysis of several different DG methods were unified for diffusion problems and the Stokes system, respectively. We emphasize that all the results presented in this paper hold true verbatim for all the mixed discontinuous Galerkin methods investigated in [28].

4.2. **Well-posedness and Basic Error Estimates.** Well-posedness of the discrete system (4.1) was established in [28]. Indeed, by introducing the space $\mathbf{V}(h) = \mathbf{V}_h + H^1(\Omega)^2$, endowed with the broken norm

$$\|\mathbf{v}\|_h^2 = \|\nabla_h \mathbf{v}\|_{L^2(\Omega)}^2 + \int_\mathcal{E} \mathtt{h}^{-1} \mathtt{k}^2 |[\![\mathbf{v}]\!]|^2 \, ds, \qquad \mathbf{v} \in \mathbf{V}(h),$$

we first note that the forms A_h and B_h are continuous on $\mathbf{V}_h$ and Q_h, that is

$$\begin{aligned} |A_h(\mathbf{v}, \mathbf{w})| &\le C \|\mathbf{v}\|_h \|\mathbf{w}\|_h, & \mathbf{v},\ \mathbf{w} \in \mathbf{V}_h \\ |B_h(\mathbf{v}, q)| &\le C \|\mathbf{v}\|_h \|q\|_{L^2(\Omega)}, & \mathbf{v} \in \mathbf{V}_h,\ q \in Q_h, \end{aligned}$$

with continuity constants $C > 0$ independent of $\underline{h}$ and $\underline{k}$. Furthermore, there exists a parameter $\gamma_{\min} > 0$ independent of $\underline{h}$ and $\underline{k}$ such that for any $\gamma \ge \gamma_{\min}$ there exists a coercivity constant $C > 0$ independent of $\underline{h}$ and $\underline{k}$ with

$$A_h(\mathbf{v}, \mathbf{v}) \ge C \|\mathbf{v}\|_h^2, \qquad \mathbf{v} \in \mathbf{V}_h.$$

Throughout, we assume that $\gamma \ge \gamma_{\min}$. Finally, for $k_K \ge 2$, the following discrete inf-sup condition for the finite element spaces $\mathbf{V}_h$ and Q_h holds true:

$$\inf_{0 \ne q \in Q_h} \sup_{0 \ne \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_{L^2(\Omega)}} \ge C |\underline{k}|^{-1} > 0,$$

with a constant $C > 0$ that is independent of $\underline{h}$ and $\underline{k}$.

The above properties of the forms A_h and B_h show the well-posedness of the system (4.1). The following abstract error bounds were obtained in [28, Sect. 3 and 4]: let $(\mathbf{u}, p)$ be the exact solution of the Stokes system and $(\mathbf{u}_h, p_h)$ the discontinuous Galerkin approximation (4.1). Then we have

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le C |\underline{k}| \Big[\inf_{\mathbf{w} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{w}\|_h + \inf_{q \in Q_h} \|p - q\|_{L^2(\Omega)} + \sup_{\mathbf{v} \in \mathbf{V}_h} \frac{|R_h(\mathbf{u}, p; \mathbf{v})|}{\|\mathbf{v}\|_h} \Big], \tag{4.4}$$

as well as

$$\|p - p_h\|_{L^2(\Omega)} \le C |\underline{k}|^2 \Big[\inf_{q \in Q_h} \|p - q\|_{L^2(\Omega)} + \inf_{\mathbf{w} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{w}\|_h + \sup_{\mathbf{v} \in \mathbf{V}_h} \frac{|R_h(\mathbf{u}, p; \mathbf{v})|}{\|\mathbf{v}\|_h} \Big], \tag{4.5}$$

where the constants $C > 0$ are independent of $\underline{h}$ and $\underline{k}$. In the above estimates (4.4) and (4.5), the term $R_h(\mathbf{u}, p; \mathbf{v})$ is a residual term that stems from the nonconformity of the method and is defined and investigated next.

To define the term $R_h(\mathbf{u}, p; \mathbf{v})$, we introduce the auxiliary space

$$\underline{\Sigma}_h := \{\, \underline{\tau} \in L^2(\Omega)^{2\times 2} : \underline{\tau} \in \mathcal{Q}^{k_K}(K)^{2\times 2},\ K \in \mathcal{T}_h \,\}.$$

Moreover, we introduce the lifting operators $\underline{\mathcal{L}} : \mathbf{V}(h) \to \underline{\Sigma}_h$, as well as $\mathcal{M} : \mathbf{V}(h) \to Q_h$ given by

$$\int_\Omega \underline{\mathcal{L}}(\mathbf{v}) : \underline{\tau}\, d\mathbf{x} = \int_\mathcal{E} \underline{[\![\mathbf{v}]\!]} : \{\!\{\underline{\tau}\}\!\}\, ds, \qquad \forall \underline{\tau} \in \underline{\Sigma}_h,$$
$$\int_\Omega \mathcal{M}(\mathbf{v}) q\, d\mathbf{x} = \int_\mathcal{E} [\![\mathbf{v}]\!] \{\!\{q\}\!\}\, ds, \qquad \forall q \in Q_h.$$

The residual can be expressed as follows; see [28] for details.

Lemma 4.3. *Let $\mathbf{f} \in B^0_{\underline{\beta}}(\Omega)^2$. For test functions $\mathbf{v} \in \mathbf{V}_h$, we have*

$$R_h(\mathbf{u}, p; \mathbf{v}) = \int_\Omega [\nabla\mathbf{u} - p\underline{I}] : \nabla_h \mathbf{v}\, d\mathbf{x} - \int_\Omega \nabla\mathbf{u} : \underline{\mathcal{L}}(\mathbf{v})\, d\mathbf{x} + \int_\Omega p\mathcal{M}(\mathbf{v})\, d\mathbf{x} - \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}.$$

Remark 4.4. We point out that the regularity assumption (2.3) is not needed to obtain the abstract error estimates (4.4) and (4.5) and the expression for the residual in Lemma 4.3. The reason for this is that A_h and B_h can be extended in a non-consistent way to continuous forms on $\mathbf{V}(h)\times\mathbf{V}(h)$ and $\mathbf{V}(h)\times L^2(\Omega)$, respectively; see [28] for details. The only assumption that is needed in Lemma 4.3 is that $\mathbf{f} \in B^0_{\underline{\beta}}(\Omega)^2$ so as to make the integral $\int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}$ well-defined for test function $\mathbf{v} \in \mathbf{V}_h$. We will invoke the regularity assumption (2.3) in the next section in order to show that R_h is convergent.

5. Error Analysis

In this section, we present an error analysis valid for piecewise analytic solutions. Special care is needed for the elements $K \in \mathcal{K}_{\text{vert}}$ near the vertices.

5.1. The Residual.

For smooth solutions, the residual expression in Lemma 4.3 has been shown to be optimally convergent in [28]. For solutions satisfying the regularity assumption (2.3) a more careful investigation is needed.

Lemma 5.1. *Assume (2.2) and (2.3). Let $\underline{P} : L^2(\Omega)^{2\times 2} \to \underline{\Sigma}_h$ and $P : L^2_0(\Omega) \to Q_h$ denote the L^2-projections onto $\underline{\Sigma}_h$ and Q_h, respectively. Then we have*

$$R_h(\mathbf{u}, p; \mathbf{v}) = \int_\mathcal{E} \{\!\{\nabla\mathbf{u} - \underline{P}(\nabla\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]}\, ds - \int_\mathcal{E} \{\!\{p - P(p)\}\!\}[\![\mathbf{v}]\!]\, ds$$

for all $\mathbf{v} \in \mathbf{V}_h$.

Proof. We first note that, by definition of the lifting operators,

$$\int_\Omega \nabla\mathbf{u} : \underline{\mathcal{L}}(\mathbf{v})\, d\mathbf{x} = \int_\Omega \underline{P}(\nabla\mathbf{u}) : \underline{\mathcal{L}}(\mathbf{v})\, d\mathbf{x} = \int_\mathcal{E} \{\!\{\underline{P}(\nabla\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]}\, ds$$

and

$$\int_\Omega p\mathcal{M}(\mathbf{v})\, d\mathbf{x} = \int_\Omega P(p)\mathcal{M}(\mathbf{v})\, d\mathbf{x} = \int_\mathcal{E} \{\!\{P(p)\}\!\}[\![\mathbf{v}]\!]\, ds.$$

Furthermore, integrating by parts the expression in Lemma 4.3 over each element $K \in \mathcal{T}_h$ gives

$$\begin{aligned} R_h(\mathbf{u}, p; \mathbf{v}) &= \int_\Omega [-\Delta\mathbf{u} + \nabla p - \mathbf{f}]\cdot\mathbf{v}\, d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nabla\mathbf{u} - p\underline{I}) : (\mathbf{v}\otimes\mathbf{n}_K)\, ds \\ &\quad - \int_\mathcal{E} \{\!\{\underline{P}(\nabla\mathbf{u})\}\!\} : \underline{[\![\mathbf{v}]\!]}\, ds + \int_\mathcal{E} \{\!\{P(p)\}\!\}[\![\mathbf{v}]\!]\, ds. \end{aligned}$$

Note that all the integrals are well-defined thanks to Lemma 3.1, Lemma 3.2, (2.4) and (2.5). Elementary manipulations then show that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\nabla\mathbf{u}-p\underline{I}):(\mathbf{v}\otimes\mathbf{n}_K)\,ds=\int_{\mathcal{E}_\mathcal{I}}[\![\nabla\mathbf{u}-p\underline{I}]\!]\cdot\{\!\{\mathbf{v}\}\!\}\,ds+\int_{\mathcal{E}}\{\!\{\nabla\mathbf{u}-p\underline{I}\}\!\}:\underline{[\![\mathbf{v}]\!]}\,ds.$$

Application of Lemma 3.3 yields

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\nabla\mathbf{u}-p\underline{I}):(\mathbf{v}\otimes\mathbf{n}_K)\,ds=\int_{\mathcal{E}}\{\!\{\nabla\mathbf{u}\}\!\}:\underline{[\![\mathbf{v}]\!]}\,ds-\int_{\mathcal{E}}\{\!\{p\}\!\}[\![\mathbf{v}]\!]\,ds.$$

Combining the above results and observing that $-\Delta\mathbf{u}+\nabla p=\mathbf{f}$ in $H^{0,0}_{\underline{\beta}}(\Omega)^2$, from (2.5), yields the assertion. $\square$

We have the following estimate of R_h.

Lemma 5.2. *Assume (2.2) and (2.3). For $\mathbf{v}\in\mathbf{V}_h$, we have*

$$\begin{aligned}|R_h(\mathbf{u},p;\mathbf{v})|\leq&C\|\mathbf{v}\|_h\big[\|\mathbf{u}-\mathbf{w}\|_h+\|p-q\|_{L^2(\Omega)}\big]\\&+\Big|\int_{\mathcal{E}}\{\!\{\nabla\mathbf{u}-\nabla\mathbf{w}\}\!\}:\underline{[\![\mathbf{v}]\!]}\,ds-\int_{\mathcal{E}}\{\!\{p-q\}\!\}[\![\mathbf{v}]\!]\,ds\Big|\end{aligned}$$

for any $(\mathbf{w},q)\in\mathbf{V}_h\times Q_h$.

Proof. Let $(\mathbf{w},q)\in\mathbf{V}_h\times Q_h$ be arbitrary. From the result in Lemma 5.1 and since the L^2-projections reproduce polynomials in $\underline{\Sigma}_h$ and Q_h, respectively, we obtain

$$R_h(\mathbf{u},p;\mathbf{v})=\int_{\mathcal{E}}\{\!\{\nabla\mathbf{u}-\nabla_h\mathbf{w}-\underline{P}(\nabla\mathbf{u}-\nabla_h\mathbf{w})\}\!\}:\underline{[\![\mathbf{v}]\!]}\,ds-\int_{\mathcal{E}}\{\!\{p-q-P(p-q)\}\!\}[\![\mathbf{v}]\!]\,ds.$$

The term T with the L^2-projections can be bounded by

$$\begin{aligned}|T|&=\Big|\int_{\mathcal{E}}\{\!\{\underline{P}(\nabla\mathbf{u}-\nabla_h\mathbf{w})\}\!\}:\underline{[\![\mathbf{v}]\!]}\,ds-\int_{\mathcal{E}}\{\!\{P(p-q)\}\!\}[\![\mathbf{v}]\!]\,ds\Big|\\&\leq C\|\mathbf{v}\|_h\sum_{K\in\mathcal{T}_h}\Big[\frac{h_K}{k_K^2}\|\underline{P}(\nabla\mathbf{u}-\nabla_h\mathbf{w})\|^2_{L^2(\partial K)}+\frac{h_K}{k_K^2}\|P(p-q)\|^2_{L^2(\partial K)}\Big]^{1/2}\\&\leq C\|\mathbf{v}\|_h\big[\|\underline{P}(\nabla\mathbf{u}-\nabla_h\mathbf{w})\|_{L^2(\Omega)}+\|P(p-q)\|_{L^2(\Omega)}\big]\\&\leq C\|\mathbf{v}\|_h\big[\|\mathbf{u}-\mathbf{w}\|_h+\|p-q\|_{L^2(\Omega)}\big].\end{aligned}$$

Here, we used the Cauchy-Schwarz inequality, the definition of `h` and `k`, the fact that $|[\![\mathbf{v}]\!]|^2\leq|\underline{[\![\mathbf{v}]\!]}|^2$, the discrete trace inequality

$$\|\varphi\|^2_{L^2(\partial K)}\leq Ck_K^2h_K^{-1}\|\varphi\|^2_{L^2(K)},$$

valid for polynomials $\varphi\in\mathcal{Q}^{k_K}(K)$, and the stability of the L^2-projections. The triangle inequality completes the proof. $\square$

5.2. **Error Estimates.** In this section, we combine the bounds (4.4) and (4.5) with the ones in Lemma 5.2 to obtain the following result.

Theorem 5.3. *Let the exact solution $(\mathbf{u},p)$ of the Stokes system satisfy (2.3), and let $(\mathbf{u}_h,p_h)$ be the discontinuous Galerkin approximation (4.1) with $k_K\geq 2$, for all $K\in\mathcal{T}_h$. Then, for any $(\mathbf{w},\widetilde{q})\in\mathbf{V}_h\times\widetilde{Q}_h$, we have the error bound*

$$\|\mathbf{u}-\mathbf{u}_h\|_h+\|p-p_h\|_{L^2(\Omega)}\leq C\,|\underline{k}|^3\,\big[E_1+E_2+E_3\big],$$

where

$$\begin{aligned}E_1^2&=\sum_{K\in\mathcal{T}_h}\big[\,|\mathbf{u}-\mathbf{w}|^2_{H^1(K)}+h_K^{-2}\|\mathbf{u}-\mathbf{w}\|^2_{L^2(K)}+\|p-\widetilde{q}\|^2_{L^2(K)}\,\big],\\E_2^2&=\sum_{K\in\mathcal{K}_{\mathrm{int}}}h_K^2\big[\,|\mathbf{u}-\mathbf{w}|^2_{H^2(K)}+|p-\widetilde{q}|^2_{H^1(K)}\,\big],\\E_3^2&=\sum_{K\in\mathcal{K}_{\mathrm{vert}}}h_K^{2(1-\beta_K)}\big[\,|\mathbf{u}-\mathbf{w}|^2_{H^{2,2}_{\beta_K}(K)}+|p-\widetilde{q}|^2_{H^{1,1}_{\beta_K}(K)}\,\big].\end{aligned}$$

The constant $C > 0$ is independent of $\underline{h}$ and $\underline{k}$.

Proof. Let $\mathbf{w} \in \mathbf{V}_h$, $\widetilde{q} \in \widetilde{Q}_h$ be arbitrary. Set $q := \widetilde{q} - \frac{1}{|\Omega|}\int_\Omega \widetilde{q}\, d\mathbf{x} \in Q_h$. Then, the bounds from (4.4), (4.5) and Lemma 5.2 yield

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-p_h\|_{L^2(\Omega)} \\
&\qquad \le C|\underline{k}|^2\Big[\|\mathbf{u}-\mathbf{w}\|_h + \|p-q\|_{L^2(\Omega)} + \sup_{\mathbf{v}\in\mathbf{V}_h}\frac{|E_h(\mathbf{u}-\mathbf{w},p-q;\mathbf{v})|}{\|\mathbf{v}\|_h}\Big],
\end{aligned} \tag{5.1}
$$

with E_h given by

$$
E_h(\mathbf{u}-\mathbf{w},p-q;\mathbf{v}) = \int_{\mathcal{E}} \{\!\{\nabla\mathbf{u}-\nabla_h\mathbf{w}\}\!\} : \underline{[\![\mathbf{v}]\!]}\, ds - \int_{\mathcal{E}} \{\!\{p-q\}\!\}[\![\mathbf{v}]\!]\, ds.
$$

In the following, we estimate the right-hand side of (5.1) in terms of $\{E_i\}_{i=1}^3$.

First, using the shape-regularity of the mesh, property (3.1), and the trace inequality

$$
\|\varphi\|^2_{L^2(\partial K)} \le C\big[h_K^{-1}\|\varphi\|^2_{L^2(K)} + h_K|\varphi|^2_{H^1(K)}\big], \qquad \forall \varphi\in H^1(K),
$$

valid with a constant $C > 0$ independent of $\underline{h}$ and $\underline{k}$, yields

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{w}\|_h^2 &= \sum_{K\in\mathcal{T}_h} |\mathbf{u}-\mathbf{w}|^2_{H^1(K)} + \int_{\mathcal{E}} \mathtt{h}^{-1}\mathtt{k}^2|\underline{[\![\mathbf{u}-\mathbf{w}]\!]}|^2\, ds \\
&\le \sum_{K\in\mathcal{T}_h} |\mathbf{u}-\mathbf{w}|^2_{H^1(K)} + C\sum_{K\in\mathcal{T}_h} h_K^{-1}k_K^2\|\mathbf{u}-\mathbf{w}\|^2_{L^2(\partial K)} \\
&\le C|\underline{k}|^2 \sum_{K\in\mathcal{T}_h}\big[h_K^{-2}\|\mathbf{u}-\mathbf{w}\|^2_{L^2(K)} + |\mathbf{u}-\mathbf{w}|^2_{H^1(K)}\big] \\
&\le C|\underline{k}|^2 E_1^2.
\end{aligned} \tag{5.2}
$$

Next, since $\int_\Omega p\, d\mathbf{x} = \int_\Omega q\, d\mathbf{x} = 0$, we have

$$
\begin{aligned}
\|p-q\|_{L^2(\Omega)} &= \|p-\widetilde{q}-|\Omega|^{-1}\int_\Omega (p-\widetilde{q})\, d\mathbf{x}\|_{L^2(\Omega)} \\
&\le \|p-\widetilde{q}\|_{L^2(\Omega)} + |\Omega|^{-1/2}\int_\Omega |p-\widetilde{q}|\, d\mathbf{x} \\
&\le 2\|p-\widetilde{q}\|_{L^2(\Omega)} \\
&\le 2E_1.
\end{aligned} \tag{5.3}
$$

Moreover,

$$
\begin{aligned}
|E_h(\mathbf{u}-\mathbf{w},p-q;\mathbf{v})| &\le \sum_{e\subset\mathcal{E}}\int_e \big[|\{\!\{\nabla\mathbf{u}-\nabla_h\mathbf{w}\}\!\} : \underline{[\![\mathbf{v}]\!]}| + |\{\!\{p-q\}\!\}[\![\mathbf{v}]\!]|\big]\, ds \\
&\le \sum_{e\subset\mathcal{E}}\int_e \big[|\{\!\{\nabla\mathbf{u}-\nabla_h\mathbf{w}\}\!\}| + |\{\!\{p-q\}\!\}|\big]|\underline{[\![v]\!]}|\, ds \\
&\le \sum_{e\subset\mathcal{E}} \|\underline{[\![v]\!]}\|_{L^\infty(e)}\int_e \big[|\{\!\{\nabla\mathbf{u}-\nabla_h\mathbf{w}\}\!\}| + |\{\!\{p-q\}\!\}|\big]\, ds.
\end{aligned}
$$

Note that $\underline{[\![\mathbf{v}]\!]}$ is a polynomial on each edge $e\subset\mathcal{E}$. Applying a standard inverse inequality for polynomials (see, e.g., [26]) and using property (3.1) yields

$$
\|\underline{[\![v]\!]}\|_{L^\infty(e)} = \||\underline{[\![v]\!]}|^2\|^{1/2}_{L^\infty(e)} \le C\frac{\mathtt{k}|_e}{\sqrt{\mathtt{h}|_e}}\||\underline{[\![v]\!]}|^2\|^{1/2}_{L^1(e)} \le C\frac{\mathtt{k}|_e}{\sqrt{\mathtt{h}|_e}}\|\underline{[\![v]\!]}\|_{L^2(e)}.
$$

Therefore, using the shape-regularity of the mesh it follows that

$$\begin{aligned}
&|E_h(\mathbf{u}-\mathbf{w},p-q;\mathbf{v})| \\
&\quad\leq\ C\sum_{e\subset\mathcal{E}}\Big\|\frac{\mathtt{k}}{\sqrt{\mathtt{h}}}\underline{[\![v]\!]}\Big\|_{L^2(e)}\int_e\big[|\{\!\{\nabla\mathbf{u}-\nabla_h\mathbf{w}\}\!\}|+|\{\!\{p-q\}\!\}|\,\big]\,ds \\
&\quad\leq\ C\Big[\int_{\mathcal{E}}\mathtt{h}^{-1}\mathtt{k}^2|\underline{[\![v]\!]}|^2\,ds\Big]^{1/2}\Big[\sum_{K\in\mathcal{T}_h}\|\nabla\mathbf{u}-\nabla_h\mathbf{w}\|^2_{L^1(\partial K)}+\|p-q\|^2_{L^1(\partial K)}\Big]^{1/2} \\
&\quad\leq\ C\,\|\mathbf{v}\|_h\Big[\sum_{K\in\mathcal{T}_h}\|\nabla\mathbf{u}-\nabla_h\mathbf{w}\|^2_{L^1(\partial K)}+\|p-q\|^2_{L^1(\partial K)}\Big]^{1/2}.
\end{aligned}$$

In addition, the third assertion in Lemma 3.1 implies that

$$\begin{aligned}
\frac{|E_h(\mathbf{u}-\mathbf{w},p-q;\mathbf{v})|}{\|\mathbf{v}\|_h}\ \leq\ & C\Big[\sum_{K\in\mathcal{T}_h}\big[|\mathbf{u}-\mathbf{w}|^2_{H^1(K)}+\|p-q\|^2_{L^2(K)}\,\big] \\
&+\sum_{K\in\mathcal{K}_{\mathrm{int}}}h_K^2\big[|\mathbf{u}-\mathbf{w}|^2_{H^2(K)}+|p-q|^2_{H^1(K)}\,\big] \\
&+\sum_{K\in\mathcal{K}_{\mathrm{vert}}}h_K^{2-2\beta_K}\big[|\mathbf{u}-\mathbf{w}|^2_{H^{2,2}_{\beta_K}(K)}+|p-q|^2_{H^{1,1}_{\beta_K}(K)}\,\big]\Big]^{1/2}.
\end{aligned}$$

Finally, applying (5.3) and using the fact $\nabla(q-\widetilde{q})\equiv 0$ results in

(5.4)
$$\begin{aligned}
\frac{|E_h(\mathbf{u}-\mathbf{w},p-q;\mathbf{v})|}{\|\mathbf{v}\|_h}&\leq C\Big[E_1^2+\sum_{K\in\mathcal{K}_{\mathrm{int}}}h_K^2\big[|\mathbf{u}-\mathbf{w}|^2_{H^2(K)}+|p-\widetilde{q}|^2_{H^1(K)}\,\big] \\
&\qquad+\sum_{K\in\mathcal{K}_{\mathrm{vert}}}h_K^{2-2\beta_K}\big[|\mathbf{u}-\mathbf{w}|^2_{H^{2,2}_{\beta_K}(K)}+|p-\widetilde{q}|^2_{H^{1,1}_{\beta_K}(K)}\,\big]\Big]^{1/2} \\
&\leq C(E_1+E_2+E_3),
\end{aligned}$$

for all $\mathbf{v}\in\mathbf{V}_h$. Combining (5.2)–(5.4) with (5.1) completes the proof. □

6. Exponential Rates of Convergence

The aim of this section is to show that the error estimates in Theorem 5.3 are exponentially convergent on geometric meshes.

6.1. **Geometric Meshes.** In order to resolve singular solution behavior near corners we introduce meshes that are geometrically refined towards the vertices of Ω. First, we define the basic geometric meshes on $\hat{Q}=(0,1)^2$.

Definition 6.1. Fix $n\in\mathbb{N}_0$ and $\sigma\in(0,1)$. On $\hat{Q}$, the geometric mesh $\Delta_{n,\sigma}$ with $n+1$ layers and grading factor σ is created recursively as follows: If $n=0$, $\Delta_{0,\sigma}=\{\hat{Q}\}$. Given $\Delta_{n,\sigma}$ for $n\geq 0$, $\Delta_{n+1,\sigma}$ is generated by subdividing the square K with $0\in\overline{K}$ into four smaller rectangles by dividing its sides in a $\sigma:(1-\sigma)$ ratio.

An example of a basic geometric mesh is shown in Figure 1. We denote the elements in the basic geometric mesh by $\{K_{ij}\}$ as indicated there. We say that the elements K_{1j}, K_{2j} and K_{3j} constitute layer j for $j\geq 2$ while K_{11} is the element at the origin.

Definition 6.2. A geometric mesh $\mathcal{T}_{n,\sigma}$ in the polygon $\Omega\subset\mathbb{R}^2$ is obtained by mapping the basic geometric meshes $\Delta_{n,\sigma}$ from $\hat{Q}$ affinely to a vicinity of each convex corner of Ω. At reentrant corners three suitably scaled copies of $\Delta_{n,\sigma}$ are used (as shown in Figure 2). The remainder of Ω is subdivided with a fixed affine and quasi-uniform partition.

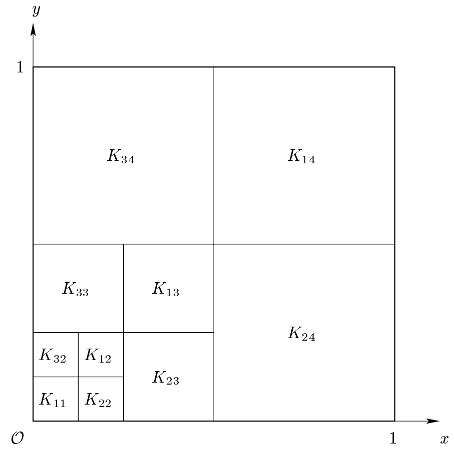

FIGURE 1. The geometric mesh $\Delta_{n,\sigma}$ with $n = 3$ and $\sigma = 0.5$. The elements are numbered as indicated.

In Figure 2 this local geometric refinement is illustrated. For ease of exposition, we consider only mesh patches that are identically refined with the same parameters σ and n, although different grading factors and numbers of layers may be used for the partition of each corner patch.

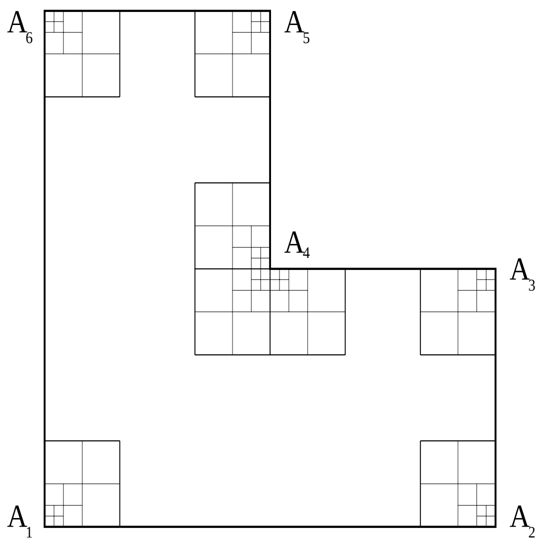

FIGURE 2. Local geometric refinement near vertices $\{A_i\}$ of Ω. At the reentrant corner A_4 three suitably scaled copies of $\Delta_{n,\sigma}$ are used. In all corners, $n = 3$ and $\sigma = 0.5$.

Definition 6.3. A polynomial degree distribution $\underline{k}$ on a geometric mesh $\mathcal{T}_{n,\sigma}$ is called linear with slope $\mu > 0$ if the elemental polynomial degrees are layerwise constant in the geometric patches and given by $k_j := \max(2, \lfloor \mu j \rfloor)$ in layer j, $j = 1, \dots, n+1$. In the interior of the domain the elemental polynomial degree is set constant to $\max(2, \lfloor \mu(n+1) \rfloor)$.

6.2. **Exponential Convergence.** Our main result establishes exponential convergence of the mixed hp-DGFEM.

Theorem 6.4. *Assume that the exact solution* $(\mathbf{u}, p)$ *of the Stokes equations satisfies (2.3) with* $\underline{\beta}_{\min} < \underline{\beta} < 1$. *Let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the DGFEM approximation (4.1) on geometric meshes* $\mathcal{T}_{n,\sigma}$. *Then there exists* $\mu_0 = \mu_0(\sigma, \underline{\beta}) > 0$ *such that for linear degree vectors* $\underline{k}$ *with slope* $\mu \geq \mu_0$ *there holds the error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_{L^2(\Omega)} \leq C \exp(-bN^{1/3})$$

with constants $C, b > 0$ *independent of* $N = \dim(\mathbf{V}_h) \approx \dim(Q_h)$.

Remark 6.5. If the polynomial degree is chosen to be constant throughout the mesh, i.e., $k_K = k$ for all $K \in \mathcal{T}_h$, exponential convergence is still obtained by choosing k proportionally to the number n of layers. This is due to the fact that the interpolant constructed for the proof of Theorem 6.4 still can be used for $k = \max(2, \lfloor \mu(n+1) \rfloor)$.

Proof. We proceed in two steps.

Step 1: We consider first the case where $\Omega = \hat{Q}$ and $\mathcal{T}_{n,\sigma} = \Delta_{n,\sigma}$ is the basic geometric mesh from Definition 6.1. From [27, Proposition 27] and [19] or [29, Lemma 4.25], there exist $\widetilde{q}_{11} \in \mathcal{Q}^0(K_{11})$ and $\mathbf{w}_{11} \in \mathcal{Q}^1(K_{11})^2$ such that

$$\|p - \widetilde{q}_{11}\|^2_{L^2(K_{11})} + h_{K_{11}}^{2-2\beta_{K_{11}}} |p - \widetilde{q}_{11}|^2_{H^{1,1}_{\beta_{K_{11}}}(K_{11})} \leq C\sigma^{2n(1-\beta_{K_{11}})} |p|^2_{H^{1,1}_{\beta_{K_{11}}}(K_{11})}$$

and

$$h_{K_{11}}^{-2} \|\mathbf{u} - \mathbf{w}_{11}\|^2_{L^2(K_{11})} + |\mathbf{u} - \mathbf{w}_{11}|^2_{H^1(K_{11})} + h_{K_{11}}^{2-2\beta_{K_{11}}} |\mathbf{u} - \mathbf{w}_{11}|^2_{H^{2,2}_{\beta_{K_{11}}}(K_{11})}$$
$$\leq C\sigma^{2n(1-\beta_{K_{11}})} |\mathbf{u}|^2_{H^{2,2}_{\beta_{K_{11}}}(K_{11})}.$$

Moreover, for $K_{ij} \in \mathcal{K}_{\text{int}}$ there are $\widetilde{q}_{ij} \in \mathcal{Q}^{k_{K_{ij}}-1}(K_{ij})$ and $\mathbf{w}_{ij} \in \mathcal{Q}^{k_{K_{ij}}}(K_{ij})^2$ such that

$$\|p - \widetilde{q}_{ij}\|^2_{L^2(K_{ij})} + h^2_{K_{ij}} |p - \widetilde{q}_{ij}|^2_{H^1(K_{ij})}$$
$$\leq C\sigma^{2(n+2-j)(1-\beta_{K_{ij}})} \frac{\Gamma(k_{K_{ij}} - s_{ij} + 1)}{\Gamma(k_{K_{ij}} + s_{ij} - 1)} \left(\frac{\varrho}{2}\right)^{2s_{ij}} \|p\|^2_{H^{s_{ij}+3,1}_{\beta_{k_{K_{ij}}}}(K_{ij})}$$

and

$$h^{-2}_{K_{ij}} \|\mathbf{u} - \mathbf{w}_{ij}\|^2_{L^2(K_{ij})} + |\mathbf{u} - \mathbf{w}_{ij}|^2_{H^1(K_{ij})} + h^2_{K_{ij}} |\mathbf{u} - \mathbf{w}_{ij}|^2_{H^2(K_{ij})}$$
$$\leq C\sigma^{2(n+2-j)(1-\beta_{K_{ij}})} \frac{\Gamma(k_{K_{ij}} - s_{ij} + 1)}{\Gamma(k_{K_{ij}} + s_{ij} - 1)} \left(\frac{\varrho}{2}\right)^{2s_{ij}} \|\mathbf{u}\|^2_{H^{s_{ij}+3,2}_{\beta_{k_{K_{ij}}}}(K_{ij})}$$

for any $1 \leq i \leq 3, 2 \leq j \leq n+1$ and $s_{ij} \in [1, k_{K_{ij}}]$. Here, $\varrho = \max(1, \sigma^{-1}(1-\sigma))$. This was proved, e.g., in [27, Sect. 5.2] in all details. Referring to Theorem 5.3 implies that

$$\|\mathbf{u} - \mathbf{u}_h\|^2_h + \|p - p_h\|^2_{L^2(\Omega)} \leq C\sigma^{2n(1-\beta_{K_{11}})} \Big[\Psi^{2,1}_{\beta_{K_{11}}}(\mathbf{u}, p)$$
$$+ \sum_{i=1}^{3} \sum_{j=2}^{n+1} \sigma^{2(2-j)(1-\beta_{K_{ij}})} \frac{\Gamma(k_{K_{ij}} - s_{ij} + 1)}{\Gamma(k_{K_{ij}} + s_{ij} - 1)} \left(\frac{\varrho}{2}\right)^{2s_{ij}} \Psi^{s_{ij}+3, s_{ij}+3}_{\beta_{K_{ij}}}(\mathbf{u}, p) \Big], \tag{6.1}$$

where

$$\Psi^{m,l}_{\beta_K}(\mathbf{u}, p) := \|\mathbf{u}\|^2_{H^{m,2}_{\beta_K}(K)} + \|p\|^2_{H^{l,1}_{\beta_K}(K)}.$$

In [3, 18] or [29, Sect. 4.5.3] it was shown that there exist s_{ij}, $1 \leq i \leq 3, 2 \leq j \leq n+1$ and $\mu_0 > 0$ such that, for linear polynomial degree distributions as in Definition 6.3 with slope $\mu \geq \mu_0$, the right-hand side of (6.1) is exponentially small with respect to N. More precisely, there holds:

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_{L^2(\hat{Q})} \leq C \exp(-bN^{1/3}).$$

Step 2: Let now $\mathcal{T}_{n,\sigma}$ be a geometric mesh on the polygon Ω, as in Definition 6.3. We recall that $\mathcal{T}_{n,\sigma}$ is obtained by mapping affinely up to three geometric mesh patches $\Delta_{n,\sigma}$ to a neighborhood of each corner. On each of these patches, we can construct an interpolant $(\mathbf{w}, q)$ as in Step 1, remarking that a generalization of the result there to affinely mapped meshes can be established straightforwardly; see, e.g., [19, 18, 29] and the references therein.

This completes the proof. □

7. Conclusions

In this paper, we have presented the first proof of exponential convergence for mixed *hp*-DGFEM for Stokes flow on geometric meshes with linearly increasing approximation orders. The proof relies on a combination of new trace theorems for functions in weighted Sobolev spaces and standard *hp*-approximation techniques. We point out that the exponential convergence result proved in this work can be straightforwardly extended to mixed formulations of linear elasticity problems with nearly incompressible materials. The numerical validation of the *hp*-scheme proposed in this paper is the subject of ongoing work and will be presented elsewhere.

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