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[parameter] $|x| - > |x|[parameter] \|x\| - > ||x||$

Lattices and ℓ_2-Betti numbers

Beno Eckmann

0. Introduction

0.1. It is well-known that the Lie group $L = PSL_2(\mathbb{R})$ contains lattices Γ which are non-Abelian free, and lattices which are isomorphic to the fundamental group of closed surfaces of genus ≥ 2 (we will call them in short "surface groups"). The former are non-cocompact and the latter cocompact (uniform). Actually these are the only possibilities for a torsion-free lattice in $PSL_2(\mathbb{R})$: Indeed the symmetric space $M = L/K$ where K is a maximal compact subgroup of L is the universal covering of a $K(\Gamma,1)$-manifold of dimension 2, open in the non-cocompact case and closed in the cocompact case; thus the cohomology dimension is 1 in the former and 2 in the latter case, so that we get the two types of torsion-free lattices above. Their first ℓ_2-Betti number $\beta_1(\Gamma)$ is $\neq 0$.

If we admit torsion then the lattice Γ is virtually torsion-free, i.e. contains a subgroup of finite index which is torsion-free. Thus the above remarks apply "virtually": The virtual cohomology dimension vcd(Γ) is 1 or 2 respectively; Γ is virtually non-Abelian free or a virtual surface group, and $\beta_1(\Gamma) \neq 0$.

0.2. In this note we show, in the general case of arbitrary connected Lie groups L, that the non-vanishing of β_1 of a lattice is exceptional, in the following sense. *If for a lattice Γ in a connected Lie group the first ℓ_2-Betti number is non-zero then Γ is commensurable with a torsion-free lattice Δ in $PSL_2(\mathbb{R})$* (this means that there is a subgroup Γ_0 of Γ of finite index and an exact sequence

$$1 \longrightarrow N \longrightarrow \Gamma_0 \longrightarrow \Delta \longrightarrow 1$$

where N is finite and Δ a torsion-free lattice in $PSL_2(\mathbb{R})$, i.e. a surface group or a non-Abelian free group). Thus in general the first Betti number of a lattice is zero. This implies properties of the deficiency of a lattice, and of the signature of a 4-manifold with fundamental group isomorphic to a lattice in a connected Lie group.

0.3. We first consider (Section 2) the case of a connected semisimple linear Lie group L without compact factors. Here we discuss the non-vanishing of β_k of a lattice Γ for arbitrary $k \geq 0$. Using deep results (see Section 2) it turns out that $\beta_k(\Gamma) = 0$ for all k except possibly for the middle dimension of the symmetric space $M = L/K$. This has some applications. We then turn in Section 3 to the non-vanishing of β_1. Arbitrary Lie groups are considered in Section 4; the general case is reduced in several steps to the case of a semisimple linear Lie group without compact factors.

For more general G-spaces we use the singular ℓ_2-theory of [Ch-G] which coincides with the former in the cocompact case.

1. Preliminaries

We recall for later use at different places some facts concerning $\beta_i(G)$ of a finitely presented infinite group G.

Proposition 1.1. If G_1 is a subgroup of finite index in G then, for all $i \geq 0$, $\beta_i(G) \neq 0$ if and only if $\beta_i(G_1) \neq 0$.

Proposition 1.2. If N is a finite normal subgroup of G then $\beta_1(G) \neq 0$ if and only if $\beta_1(G/N) \neq 0$.

Proposition 1.1 follows from the fact that for a subgroup of finite index the β_i are multiplied by the index.

To prove Propostion 1.2 consider the Lyndon-Hochschild-Serre spectral sequence leading from G/N to G. Since $\beta_0(G/N) = \beta_1(N) = 0$, $\beta_0(N) \neq 0$ the only term contributing to $\beta_1(G)$ is $H_1(G/N; H_0(N))$ (reduced ℓ_2-homology). Thus $H_1(G) \neq 0$ if and only if $H_1(G/N) \neq 0$.

Proposition 1.3. (The Cheeger-Gromov Theorem). If the finitely presented group G admits an infinite amenable normal subgroup N then $\beta_1(G) = 0$; if G is of type F_m, i.e. if there is a $K(G,1)$ with finite m-skeleton, then all $\beta_i(G)$, $i < m$ are $= 0$.

Proposion 1.4. If $\beta_1(G) = 0$ then the deficiency $\mathrm{def}(G)$ is ≤ 1, and $\mathrm{def}(G) = 1$ implies that $\mathrm{cd}(G) \leq 2$.

To prove this let Z be the finite 2-dimensional cell complex constructed from the presentation, $\tilde{Z}$ its universal covering. We recall from [E] that $\mathrm{def}(G) = 1 + \beta_1(G) - \beta_2(\tilde{Z})$. If $\beta_1(G) = 0$ then $\mathrm{def}(G) \leq 1$, and if $\mathrm{def}(G) = 1$ then $\beta_2(\tilde{Z}) = 0$ whence $H_2(\tilde{Z}) = 0$ (reduced ℓ_2-homology). Since integral finite cycles are contained in $H_2(\tilde{Z})$ it follows that ordinary integral homology in dimension 2 of $\tilde{Z}$ is 0; thus $\tilde{Z}$ is contractible.

2. ℓ_2-Betti numbers of lattices in semisimple linear groups.

2.1. Let Γ be a torsion-free lattice in the connected semisimple linear Lie group L without compact factors.

The Riemannian symmetric space $M = L/K$, where K is a maximal compact subgroup, is a contractible free Γ-manifold of dimension d (cocompact or not) whence the universal covering of a $K(\Gamma, 1)$-manifold X, closed or open respectively.

We are using in M the singular reduced ℓ_2-cohomology, defined in [Ch-G]; and $\beta_k(M)$ is its von Neumann dimension relative to Γ. For $k = 1$ it can be computed from a $K(\Gamma, 1)$-cell complex with finite 2-skeleton (one adds cells of dimensions $3, 4, ...$ to Z in the proof of proposition 1.4). In the cocompact case all $\beta_k(\Gamma)$ can be computed as usual from a finite cell decomposition of the closed manifold X.

At this point we use the fact that Γ is a lattice, i.e. that the volume of M/Γ is finite. Then according to [Ch-G, (0.28)] the reduced singular ℓ_2-cohomology space $H^k(M)$ of M is Γ-isomorphic to the space $\mathcal{H}^k(M)$ of harmonic L_2-forms of degree k.

2.2. If we now assume that $\beta_k(\Gamma) \neq 0$ then $\mathcal{H}^k(M) \neq 0$ and therefore M admits non-zero harmonic L_2-forms of degree k. According to Borel-Wallach [B-W, Section 5 in Chap.II] and Connes-Moscovici [C-M, Theorem 6.1 ff] there is only one dimension q where such harmonic q-forms $\neq 0$ can exist, namely $q = 1/2 \dim M = 1/2\ d$. Thus the dimension d of M must be $2k$.

Theorem 2.1. Let Γ be a torsion-free lattice in the connected semisimple linear Lie group L without compact factors. If the dimension d of the symmetric space $M = L/K$ is odd then all $\beta_i(\Gamma)$ are zero. If d is even then all $\beta_i(\Gamma)$ are zero except possibly for the middle dimension $i = 1/2\ d$.

If the lattice Γ is not asssumed torsion-free then it contains a subgroup Γ_1 of finite index which is torsion-free. Γ_1 is again a lattice in L. The non-vanishing of $\beta_k(\Gamma)$ implies the non-vanishing of $\beta_k(\Gamma_1)$ and thus d must be $= 2k$:

Theorem 2.2. The conclusions of Theorem 1.1 hold for arbitrary lattices in L.

2.3. As a corollary of Theorem 2.1 one obtains the following known result, which is a special case of the "Singer Conjecture".

Theorem 2.3. Let X be a closed Riemannian manifold. If the universal covering $\tilde{X}$ is a Riemannian symmetric space of non-compact type then all $\beta_i(X) = \beta_i(\tilde{X} \mathrm{rel}\ \pi_1(X))$ are zero except possibly for the middle dimension.

Indeed $\Gamma = \pi_1(X)$ considered as covering transformation group is a cocompact lattice in the isometry group of $\tilde{X}$. Thus $\beta_i(\Gamma) = \beta_i(\tilde{X}) = 0$ except possibly for the middle dimension.

Example 2.4. If X is a closed hyperbolic manifold of dimension d then $\tilde{X} = \mathbb{H}_d$ is the hyperbolic d-space. The ℓ_2-Betti numbers are zero except for even $d = 2k$ where $\beta_k \neq 0$, see [D].

Example 2.5. We consider the non-cocompact lattice $\Gamma = SL_n(\mathbb{Z})$ in $SL_n(\mathbb{R})$ for $n \geq 3$, and let Γ_1 be a torsion-free subgroup of finite index in Γ. Although here $X = M/\Gamma_1$ is not compact it was shown by Borel-Serre that X is homotopy equivalent to a finite cell complex. The ℓ_2-Betti numbers of Γ_1, whence of Γ, are all zero if d is odd ($n = 4m$ and $n = 4m + 3$). They are all zero except possibly for the middle dimension if d is even ($n = 4m + 1$ or $n = 4m + 2$); but since it is known that for $n \geq 3$ the Euler characteristic of Γ_1 is $= 0$ they are also all zero.

3. Non-vanishing of β_1.

3.1. Let again L be a connected semisimple linear Lie group without compact factors, and Γ a lattice in L. If $\beta_1(\Gamma) \neq 0$ then according to Theorem 2.1 the dimension d of the symmetric space $M = L/K$ must be 2. Looking at the list of all L one notices that $d = 2$ for $Sl_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$, and there are no other possibilities.

Since the virtual cohomology dimension of Γ is 2 in the cocompact case and 1 in the non-cocompact case, Γ is either a virtual surface group (of genus ≥ 2) or virtually a free non-Abelian group.

Theorem 3.1. Let Γ be a lattice in the connected semisimple linear Lie group L. If the first ℓ_2-Betti number $\beta(\Gamma)$ is non-zero then Γ is a lattice in $SL_2(\mathbb{R})$ or $PSl_2\mathbb{R}$. Thus Γ is a virtual surface group or virtually a non-Abelian free group.

3.2. We add two immediate corollaries of Theorem 3.1. L and Γ are as in that Theorem.

Corollary 3.2. If Γ is not isomorphic to a lattice in $PSL_2(\mathbb{R})$ then $\beta_1(\Gamma) = 0$. This is the case in particular if $\mathrm{vcd}(\Gamma)$ is > 2 for cocompact Γ and > 1 for non-cocompact Γ.

According to Proposition 4 the vanishing of β_1 implies strong properties of the deficiency of the finitely presented group Γ: namely $\mathrm{def}(\Gamma) \leq 1$ and if $\mathrm{def}(\Gamma) = 1$, then the cohomology dimension of Γ is ≤ 2.

Corollary 3.3. If Γ is not isomorphic to a lattice in $PSL_2(\mathbb{R})$ then its deficiency $\mathrm{def}(\Gamma)$ is ≤ 1 and if $\mathrm{def}(\Gamma) = 1$ then $\mathrm{cd}(\Gamma) \leq 2$ and Γ is torsion-free.

4. Lattices in connected Lie groups

4.1. Let L be a connected Lie group and Γ an (infinite) lattice in L with $\beta_1(\Gamma) \neq 0$. Since the intersection $\Gamma \cap \mathrm{Rad}(L)$ of Γ with the radical of L is solvable, whence amenable, it must be finite by Proposition 1.3. Note that if $L/\mathrm{Rad}(L)$ has no compact factor then $\Gamma/\Gamma \cap \mathrm{Rad}(L) = \Gamma\mathrm{Rad}(L)/\mathrm{Rad}(L)$ is discrete in $L/\mathrm{Rad}(L)$ whence a lattice.

The proof that $\Gamma/\Gamma \cap \mathrm{Rad}(L)$ is discrete (see [A, Proposition 2] or [R, Corollary 8.27]) is based on various results. We recall that one first shows that its closure in $L/\mathrm{Rad}(L)$ is solvable and then that the 1-component of the closure is normal in $L/\mathrm{Rad}(L)$, whence $= 1$ (this in turn because a lattice has property (S), see [R, 5.1 ff]).

We write R for the product of $\mathrm{Rad}(L)$ with a suitable compact normal subgroup of L so that $L_1 = L/R$ is semisimple without compact factors. Then $\Gamma_1 = \Gamma/\Gamma \cap R$ is an infinite lattice in L_1 with $\beta_1(\Gamma_1) \neq 0$.

4.2. The intersection of Γ_1 with the discrete center $Z(L_1)$ must be finite, again by Proposition 1.3. The adjoint representation of L_1 yields a linear Lie group $L_1/Z(L_1)$. It contains the lattice $\Gamma_1/\Gamma_1 \cap Z(L_1)$, i.e. the factor group Γ/N of the original lattice Γ by a finite normal subgroup N, and $\beta_1(\Gamma/N) \neq 0$.

4.3. Let Γ_0/N be a torsion-free subgroup of finite index in Γ/N. It is a lattice in a connected semisimple linear Lie group without compact factors; by Theorem 3.1 it is isomorphic to a torsion-free lattice Δ in $PSL_2(\mathbb{R})$. Since Γ_0 has finite index in Γ and N is finite we get the exact sequence

$$1 \longrightarrow N \longrightarrow \Gamma_0 \longrightarrow \Delta \longrightarrow 1$$

as requested.

Theorem 4.1. Let Γ be a lattice in a connected Lie group. If the first ℓ_2-Betti number of Γ is non-zero then Γ is commensurable with a torsion-free lattice Δ in $PSL_2(\mathbb{R})$, i.e. with a surface group of genus ≥ 2 or a non-Abelian free group.

Thus "in general" the first ℓ_2-Betti number of a lattice in a connected Lie group is zero. As in **3.3** this yields information about the deficiency.

Theorem 4.2. If the infinite lattice Γ in the connected Lie group L is not commensurable with a torsion-free lattice in $PSL_2(\mathbb{R})$ then $\beta_1(\Gamma) = 0$. The deficiency of Γ is then ≤ 1; and if $\mathrm{def}(\Gamma) = 1$ then $\mathrm{cd}(\Gamma) \leq 2$.

4.4. Two applications.

1)Again we can say that "in general" the deficiency of a lattice in a connected Lie group is ≤ 0. We return to the question of deficiency in the exceptional cases in a separate note.

2) Let Y be a 4-manifold with fundamental group isomorphic to a lattice Γ in a connected Lie group, Γ not commensurable with a lattice in $PSL_2(\mathbb{R})$, and let χ be its Euler characteristic and σ its signature. Then (cf [E2]) one has $|\sigma| \leq \chi$.

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