Report

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Asymptotics of the admissible growth of the coefficient of quasiconformality at infinity and injectivity of immersions of Riemannian manifolds

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1. Introduction. Locally invertible quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible, provided $n > 2$. This statement was formulated by M.A. Lavrentiev [1] for $n = 3$ and later was proved in [2].

Then it was developed in several directions (see survey paper [3]). In particular, the asymptotics of admissible growth of the coefficient of quasiconformality at infinity (sharp in a sense) which still guarantees the global invertibility of arbitrary immersion $f : \mathbb{R}^n \to \mathbb{R}^n$ was indicated in [4].

The passage from immersions of Euclidean space to the general case of immersions of Riemannian manifolds [5], [6] and the analysis of the proof in [2] stimulated the following conformally invariant form of the initial theorem related to asymptotic geometry and conformal classification of Riemannian manifolds [7].

For the global invertibility of a quasiconformal immersion $f : M^n \to N^n$ of a Riemannian manifold $M^n$ to a simply connected Riemannian manifold $N^n$ of the same dimension $n > 2$ it suffice the manifold $M^n$ be of conformally parabolic type.

In the present paper we show how do the asymptotics of admissible growth of coefficient of quasiconformality looks like in a general theorem on the global invertibility for immersions of Riemannian manifolds of conformally parabolic

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type. (For the notion of the conformal type of a manifold see, for example, [7], [8]).

Recall (see [8]) that, for instance, the divergence of the integral \( \int_0^\infty S^{1-n} \) may serve as a sufficient condition of conformal parabolicity of a noncompact complete Riemannian manifold \( M^n \). Here \( S = S(r) \) is a surface \((n - 1)\)-measure of the sphere of radius \( r \) with a fixed center \( o \in M^n \).

This assertion follows immediately from the following useful (and used below) general estimate of the conformal capacity of the condenser \( R_{r_0}^{r_1} \) bounded by geodesic spheres of radii \( r_0 \) and \( r_1 \) (see [9], [10], [8]):

\[
\text{cap } R_{r_0}^{r_1} \leq \left( \int_{r_0}^{r_1} S^{\frac{1}{1-n}} \right)^{1-n}.
\]

2. **Formulation of the main theorem.** By the coefficient of quasiconformality of a mapping defined in a domain we mean the least upper bound of linear distortions over all points of this domain. The linear distortion in a point is measured by the ratio of the maximal dilatation to the minimal one. (We skip a detailed discussion of this definition which is actually applicable to nonsmooth mappings as well.)

Let \( k(r) \) be the coefficient of quasiconformality of immersion \( f : M^n \to N^n \) within the ball of radius \( r \) with a fixed center \( o \in M^n \).

If the mapping \( f \) is quasiconformal, the function \( k(r) \) is bounded. In the general case (not a quasiconformal immersion) the magnitude \( k(r) \) can grows infinitely with \( r \). The following theorem gives the asymptotics of admissible rate of growth of function \( k(r) \) for which the immersion \( f : M^n \to N^n \) is still globally injective.

**Theorem.** Suppose that \( M^n \) and \( N^n \) are Riemannian manifolds of dimension \( n > 2 \); \( N^n \) is simply connected, and \( M^n \) is a complete noncompact manifold of conformally parabolic type.

If the immersion \( f : M^n \to N^n \) satisfies

\[
\int_0^\infty \frac{dr}{\left(kS^{\frac{1}{1-n}}(r)\right)} = \infty,
\]

then \( f \) is injective and the Hausdorff dimension of the set \( N^n \setminus f(M^n) \) is equal to zero.

In particular, if \( M^n = \mathbb{R}^n \) then \( S(r) \propto r^{n-1} \), and we obtain the result of paper [4].
3. Sketch of the proof. The following hint is actually sufficient. Morally, the proof follows the one in [4] where we replace the estimates of conformal capacities of rings in $\mathbb{R}^n$ by the above mentioned general estimate of the value $\text{cap} R_{r_0}^{R_{r_0}}$ on a Riemannian manifold $M^n$.

Some more one can find below.

**Lemma.** Let $M^n$ be complete Riemannian noncompact manifold, $D$ — unbounded domain in $M^n$, and $\Gamma = \{ \gamma \}$ — a family of curves (paths) in $D$, ending at infinity (i.e., they finally leave any compact set). Suppose that $\varphi : D \to N^n$ is a homeomorphic mapping of the domain $D$ to a Riemannian manifold $N^n$ and $\Gamma := \varphi(\Gamma)$.

Under this conditions if $\mod \tilde{\Gamma} > 0$, then

$$\int_{0}^{\infty} \frac{dr}{(kS^{\frac{1}{n-1}}(r))} < \infty.$$  

Here, as before, $S = S(r)$ is a surface of the sphere of radius $r$ with a fixed center $o \in M^n$, but now $k = k_\varphi(r)$ is a coefficient of quasiconformality of mapping $\varphi : D \to N^n$ in the bounded part of the domain $D$ cuted off by this sphere.

**Proof.** Notice that the function $k = k_\varphi(r)$ is nondecreasing, therefore one may assume that it is everywhere finite (otherwise the lemma is trivial).

Let $d$ be the metric in $M^n$ generated by the Riemannian structure of the manifold.

Take some monotone unbounded sequence $0 < r_0 < r_1 < ... < r_m < ...$ of values of radius $r$.

Consider the sequence of $M^n$-rings $R_m = R_{r_m}^{r_{m+1}} = \{ x \in M^n \mid r_m < d(o, x) < r_{m+1} \}$, and the corresponding sequence of $D$-rings $D_m := D \cap R_m$.

First suppose that all curves of the family $\Gamma$ intersect the sphere of radius $r_0$. Then on each curve $\gamma \in \Gamma$ one can choose the following sequence of simple arcs $\gamma_m$: each arc is situated in the corresponding domain $D_m$ and connects boundary spheres of the ring $R_m$. In this way we obtain a sequence $\Gamma_m := \cup_{\gamma \in \Gamma} \gamma_m$ of families of curves.

The curves of the family $\Gamma_m$ form a part of all possible curves in the ring $R_m$ connecting its boundary spheres. Therefore $\mod \Gamma_m$ is not greater than the conformal module $\text{cap} R_{r_m}^{r_{m+1}}$ of this general family. Thus, by the estimate
of conformal capacity of a geodesic ring mentioned in section 1, we have

$$\text{mod } \Gamma_m \leq \left( \int_{r_m}^{r_{m+1}} S_{\frac{1}{1-n}} \right)^{1-n}.$$  

The families $\Gamma_m$ are situated in disjoint domains $D_m$ and each of them
minorises the initial family $\Gamma$ (i.e., for each curve $\gamma \in \Gamma$ there exists a curve $\gamma_m \in \Gamma_m$ such that $\gamma_m \subset \gamma$).

The same holds for their images $\tilde{\Gamma}_m$ and $\tilde{\Gamma}$.

Thus, by virtue of classical principles of Grötzsch

$$\text{mod } \frac{1}{1-n} \tilde{\Gamma} \geq \sum_{m=0}^{\infty} \text{mod } \frac{1}{1-n} \Gamma_m.$$  

The mapping under consideration is $(k(r_{m+1}) := k_\phi(r_{m+1}))$-quasiconformal within the domain $D_m$. Therefore

$$\text{mod } \tilde{\Gamma}_m \leq k^{n-1}(r_{m+1}) \text{ mod } \Gamma_m.$$  

From the last three inequalities we obtain

$$\text{mod } \frac{1}{1-n} \tilde{\Gamma} \geq \sum_{m=0}^{\infty} k^{-1}(r_{m+1}) \left( \int_{r_m}^{r_{m+1}} S_{\frac{1}{1-n}} \right).$$  

Because of the condition $\text{mod } \tilde{\Gamma} > 0$, the last series must converge, and the
convergence does not depend on the choice of the sequence $r_1 < ... < r_m < ...$

of values of radius $r$. This implies the convergence of the integral indicated
in the lemma. In fact, this becomes clear, e.g., after the following estimate
concerning any interval $[a, b]$ on $r$ (with keeping in mind that we are free to
choose any partition of the interval):

$$0 \leq \sum \left( k^{-1}(r_m) - k^{-1}(r_{m+1}) \right) \left( \int_{r_m}^{r_{m+1}} S_{\frac{1}{1-n}} \right) \leq$$  

$$\leq \left( k^{-1}(r_{m_a}) - k^{-1}(r_{m_b}) \right) \max_{m \leq m_a \leq m_b} \left( \int_{r_m}^{r_{m+1}} S_{\frac{1}{1-n}} \right).$$  

To complete the proof we should eliminate our additional assumption
that all curves of the family $\Gamma$ intersect a sphere of fixed radius $r_0$.  

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The family $\Gamma$ can be presented as a countable union of the following families $L_m$. Curves of the family $L_m$ do not intersect the sphere of radius $m$ but intersect the sphere of radius $m+1$. Since the family $\Gamma$ is not exceptional, one of the families $L_m$ must be nonexceptional too. It can be admit for $\Gamma$.

Now we can prove the theorem.

Notice that the construction used in [7] for the investigation of quasiconformal immersion $f : M^n \to N^n$, besides the condition $n > 2$, used only two things: conformal module of any family $\Gamma$ of curves on $M^n$ going to infinity within the domain $D \subset M^n$ where the mapping $f$ (or, to say more correctly, the restriction $f|_D = \varphi$) is injective, is equal to zero; and the image of such a family also has the conformal module equal to zero.

In our case the condition mod $\Gamma = 0$ is ensured by the conformal parabolicity of the manifold $M^n$, and the condition mod $\Gamma = 0$ holds by virtue of the Lemma if the integral indicated in the theorem diverges.

**4. Comments.** Already Hadamard [11] observed that a locally invertible mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is globally invertible if and only if $f(x) \to \infty$ when $x \to \infty$.

Obviously, this is equivalent to the condition that the mapping $f$ is a proper one (i.e., the pre-image of a compact set is a compact set); thus $f$ is a covering of a simply connected manifold, and is a homeomorphism.

Certainly, this reasoning and the condition of Hadamard remain valid for immersion $f : M^n \to N^n$ of any manifold to a simply connected one of the same dimension.

Proofs of the global invertibility property for mappings of special classes, e.g., quasiconformal in the Lavrentiev’s problem or polynomial in the (still unsolved) Jacobian problem, reduce in a sense (although, certainly, not literally) to checking the condition of Hadamard.

In the form convenient for application to quasiconformal mappings the condition of Hadamard can be reformulated as follows.

Let $\Gamma_\infty$ be the set of all paths on $M^n$ ending at infinity, and let mod $f(\Gamma_\infty)$ be the conformal module of its image under an immersion $f : M^n \to N^n$. If $M^n$ is a manifold of conformally parabolic type, then, certainly, mod $\Gamma_\infty = 0$.

In this setting the condition of Hadamard is equivalent to the relation mod $f(\Gamma_\infty) = 0$ provided $n > 2$.

If the mapping $f$ is quasiconformal, then the image of any exceptional family of curves (e.g., the family $\Gamma_\infty$) is exceptional. The condition mentioned
in the above formulated theorem related the admissible rate of growth of the function \( k(r) \) do guaranties the necessary equality \( \text{mod} f(\Gamma_\infty) = 0 \) if \( \text{mod} \Gamma_\infty = 0 \).

The sharpness of the asymptotics indicated in the theorem can be confirmed by the example constructed in the paper [4].

Notice finally that we considered here immersions of manifolds of conformally parabolic type. However there are serious reasons to expect the global invertibility property for quasiconformal immersions in a much more general settings. As a model example in this context one can recall the problem formulated in [3] relating the removability of a segment of singular points.

References


