Doctoral Thesis

Flow dynamics of Vatnajökull ice cap, Iceland

Author(s):
Aðalgeirs dóttir, Guðfinna

Publication Date:
2002

Permanent Link:
https://doi.org/10.3929/ethz-a-004489563

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FLOW DYNAMICS OF VATNAJÖKULL ICE CAP, ICELAND

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZURICH
for the degree of
Doctor of Natural Science

presented by
Guðfinna Aðalgeirsdóttir
M.S. University of Alaska Fairbanks
B.S. University of Iceland
born 20. March 1972
citizen of Iceland

accepted on the recommendation of
Prof. Dr. H.-E. Minor, examiner
Dr. G. Gudmundsson, co-examiner
Prof. Dr. H. Björnsson, co-examiner

2002
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<td>AAR</td>
<td>Accumulation Area Ratio</td>
</tr>
<tr>
<td>ADS-I</td>
<td>Alternating Direction Semi-Implicit method</td>
</tr>
<tr>
<td>ELA</td>
<td>Equilibrium Line Altitude</td>
</tr>
<tr>
<td>ΔELA</td>
<td>Shift in the ELA relative to reference ELA</td>
</tr>
<tr>
<td>SIA</td>
<td>Shallow Ice Approximation</td>
</tr>
<tr>
<td>VAW</td>
<td>Laboratory of Hydraulics, Hydrology and Glaciology</td>
</tr>
<tr>
<td>w.e.</td>
<td>water equivalent</td>
</tr>
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# List of Symbols

## Greek letters

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<th>Description</th>
<th>Units</th>
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<tr>
<td>( \alpha^a )</td>
<td>mass balance gradient above ELA</td>
<td>( a^{-1} )</td>
</tr>
<tr>
<td>( \alpha^b )</td>
<td>mass balance gradient below ELA</td>
<td>( a^{-1} )</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>surface slope</td>
<td></td>
</tr>
<tr>
<td>( \dot{\epsilon} )</td>
<td>strain rate tensor with components ( \dot{\epsilon}_{ij} )</td>
<td>( a^{-1} )</td>
</tr>
<tr>
<td>( \rho )</td>
<td>density</td>
<td>( \text{kg m}^{-3} )</td>
</tr>
<tr>
<td>( \rho_{\text{ice}} )</td>
<td>density of ice = 917 kg m(^{-3})</td>
<td>( \text{kg m}^{-3} )</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>stress tensor with components ( \sigma_{ij} )</td>
<td>bar</td>
</tr>
<tr>
<td>( \sigma' )</td>
<td>stress deviator tensor ( (\sigma'<em>{ij} := \sigma</em>{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} ) )</td>
<td>bar</td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>error in each mass balance measurement</td>
<td>m ( a^{-1} )</td>
</tr>
<tr>
<td>( \tau )</td>
<td>second invariant of ( \sigma'_{i,j} )</td>
<td>bar</td>
</tr>
<tr>
<td>( \tau_b )</td>
<td>basal shear stress</td>
<td>bar</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>balance flow field</td>
<td>m(^3) ( a^{-1} )</td>
</tr>
<tr>
<td>( \Psi_{i,j}^{\text{in}}, \Psi_{i,j}^{\text{out}} )</td>
<td>scalar in and out flow</td>
<td>m(^3) ( a^{-1} )</td>
</tr>
</tbody>
</table>
## Latin letters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
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<tr>
<td>$A$</td>
<td>rate factor or flow parameter</td>
<td>$s^{-1} \text{kPa}^{-3}$</td>
</tr>
<tr>
<td>$\tilde{A}$</td>
<td>constant in the ice sheet equation ($\tilde{A} = \frac{2A}{n+2}(\rho g)^n$)</td>
<td></td>
</tr>
<tr>
<td>$a_0, a_1, a_2$</td>
<td>constants of the ELA plane</td>
<td>$\text{m, l, l}$</td>
</tr>
<tr>
<td>$a, c$</td>
<td>average accumulation and ablation rates</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$\dot{b}$</td>
<td>mass balance rate</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$b_t$</td>
<td>ablation at terminus</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$B$</td>
<td>glacier annual balance</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$\bar{B}$</td>
<td>mean specific annual balance</td>
<td>$\text{m a}^{-1} \text{m}^{-2}$</td>
</tr>
<tr>
<td>$C$</td>
<td>sliding parameter</td>
<td>$\text{m a}^{-1} \text{Pa}^{-3}$</td>
</tr>
<tr>
<td>$C$</td>
<td>boundary of area $S$</td>
<td></td>
</tr>
<tr>
<td>$D_{i,j}$</td>
<td>coefficient in the ice sheet equation</td>
<td>$\text{m}^3 \text{a}^{-1}$</td>
</tr>
<tr>
<td>$d$</td>
<td>distance from the summit</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration due to gravity</td>
<td>$\text{m s}^{-2}$</td>
</tr>
<tr>
<td>$h, z_s$</td>
<td>surface of the ice sheet</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$H$</td>
<td>glacier or ice cap thickness</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$H_{ELA}$</td>
<td>height of the ELA</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$L$</td>
<td>glacier or ice cap length</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$M$</td>
<td>accumulation rate (Eismint experiments)</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$m$</td>
<td>exponent in the sliding law</td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>exponent in Glen’s flow law</td>
<td></td>
</tr>
<tr>
<td>$\tilde{n}$</td>
<td>normal vector</td>
<td></td>
</tr>
<tr>
<td>$q$</td>
<td>ice flux</td>
<td>$\text{m}^3 \text{a}^{-1}$</td>
</tr>
<tr>
<td>$R, R_{el}$</td>
<td>distance from summit to ELA</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$s$</td>
<td>slope of the mass balance function</td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>area within the boundary $C$</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
<td>$\text{s, a}$</td>
</tr>
<tr>
<td>$\tilde{u}$</td>
<td>average velocity over depth</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>components of the velocity vector $v$</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity vector, $v = (u, v, w)$</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$V_{def}$</td>
<td>deformation velocity</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$V_{slid}$</td>
<td>sliding velocity</td>
<td>$\text{m a}^{-1}$</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>space coordinates</td>
<td>$\text{m}$</td>
</tr>
<tr>
<td>$x$</td>
<td>position vector, $x = (x, y, z)$</td>
<td>$\text{m}$</td>
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Abstract

In this thesis the flow dynamics and the stability of Vatnajökull ice cap in Iceland are analysed. Two types of models are used to model the flow and the evolution of the surface. A full system model is used on a small portion in the center of the ice cap to model how the ice flows into a surface depression created during glacier outburst flood that occurred in 1996. The full equation system is solved with a commercial finite element program. The full system model can be simplified with the Shallow Ice Approximation (SIA) which is valid if the thickness of the ice is much smaller than its length. It is shown with a model-model comparison that the SIA model gives essentially the same result as the full system model when the spatial length scale of the computation is larger than about 10 times the ice thickness. It is, thus, appropriate to use the SIA model to compute the surface evolution of the whole ice cap. The SIA equations are solved with a finite difference method on a square grid.

The basis for the numerical modeling are the radio-echo sounding data of the bedrock topography of Vatnajökull, along with surface elevation and velocity of the ice, and mass balance measurements provided by Helgi Björnsson at the Science Institute, University of Iceland. The data are used to select the model parameters to describe the flow of Vatnajökull.

Several numerical schemes to solve the SIA equations are tested. It is shown that a commonly used method (alternating direction semi-implicit) is not mass conserving due to ice free points within the ice cap. It is necessary to use an upstream scheme to ensure mass conservation. All computations on Vatnajökull presented in the thesis are done with the upstream scheme.

A non-linear regression model that describes the mass-balance distribution of Vatnajökull during the years 1992-2000 is developed. The regression model uses six adjustable parameters, the slope, direction and the equilibrium line altitude (ELA), two altitude mass-balance gradients, and a maximum value of the surface mass balance. It is found that the temporal variation of the observed mass-balance distribution can be accurately described through annual shifts of the ELA.

Model computations with the SIA model forced with the regression model for the mass-balance distribution, including mass balance-elevation feedback, show that Vatnajökull ice cap can not be modeled with a constant ELA. Depending on the altitude of the ELA the ice cap either grows without bound or it settles to a steady state that is considerably smaller than the present ice cap. The ice cap is sensitive to the mass-balance forcing. Changes in the ELA that are within the range measured 1992-2000 cause large volume changes.

With time varying ELA the ice cap fluctuates around a size that is considerably smaller than computed with a constant ELA. Model experiments with periodic changes in basal sliding velocity, that imitate surges, show similar decrease in size. The time dependent changes, which also are observed in nature, are possible mechanism that stops the unlimited growth and allows Vatnajökull to maintain its present size.
It is concluded that a critical size of an ice cap can occur where it is very sensitive to changes in mass balance input and can respond in an unstable manner by unlimited growth. Vatnajökull is presently close to this critical size. Ice caps that are smaller and glaciers resting on steep bedrock slopes are not as sensitive to mass balance changes.
Zusammenfassung


Die Datengrundlage für die numerische Modellierung besteht aus Radio-Echolotdaten der Gletscherbetttopographie, der Oberflächentopographie, dem Geschwindigkeitsfeld des Gletschers und Massenbilanzmessungen, die alle von Helgi Björnsson (Science Institute, University of Iceland) zur Verfügung gestellt wurden. Diese Datensätze werden benutzt, um die Modellparameter zu bestimmen, die das Fliesen des Vatnajökull beschreiben.


Es wird gefolgt, dass eine Eiskappe eine kritische Ausdehnung besitzen kann, wo sie dann sehr empfindlich ist gegenüber Massenbilanzänderungen und bereits auf kleine Störungen unstabil mit unlimitiertem Wachstum reagiert. Vatnajökull ist heute nahe an diesem kritischen Zustand. Kleinere Eiskappen oder Gletscher, die auf einem steilen Felsbett aufliegen, sind dagegen weniger empfindlich gegenüber Massenbilanzänderungen.
Chapter 1

Introduction

1.1 Motivation

Vatnajökull ice cap is located at the south east coast of Iceland. Situated in a maritime climate, with relatively low summer temperatures and heavy winter precipitation, Vatnajökull has high rates of surface mass exchange. The highest annual precipitation has been measured at the summit plateau of Öræfajökull which in the period 1993-1998 was in the range 7-8 m water equivalent units (w.e.) (Gudmundsson, 2000) whereas values for ablation can be as high as 11-12 m w.e. (Ahlmann and Thorarinsson, 1938). Ice surface velocities of up to 60-80 m a$^{-1}$ are typical and during the frequent surges velocities can exceed 2-3 m day$^{-1}$ (Björnsson et al., 2002b). Not only is the ice cap very dynamical in nature but, it partly covers active volcanic areas, which allows fire and ice to interact when volcanic eruptions melt their way through the ice (Gudmundsson et al., 1997). The constant melting in the Grímsvötn area (Björnsson, 1988; Gudmundsson and Björnsson, 1991) and Skaftárkatlars feed the outburst floods, termed jökulhlaups, which occur regularly (Björnsson, 1998). All these activities on Vatnajökull make it a fascinating place to study and provide perpetual challenging projects for research.

The aim of this thesis is to develop an ice flow model of Vatnajökull. The purpose is not to tackle all the complexities and the peculiarities observed on Vatnajökull but to develop a model that is capable of describing the main properties of the ice cap, ice flow dynamics, shape, thickness, extent, flow lines, velocity distribution and location of ice divides. This work follows the tenet that models should start with the simplest case and gradually increase in complexity.

The dynamics of the ice cap is of high interest to the local energy authorities. The information on the location of the ice and water divides, mass transport, melt water production and distribution, and how it may change in the future, is valuable for the present and planned power plants which employ water originating in the northern and western parts of the ice cap.

1.2 Previous work on Vatnajökull

Glaciological investigations in Iceland were launched in 1930 when the Icelandic meteorologist Jón Eyþórsson and Helgi Hermann Eiríksson began recording glacier snout variations. The joint Swedish-Icelandic expedition to Vatnajökull 1936-38, led by the Swedish glaciologist
Hans Ahlmann and Jón Eyþórsson, investigated the mass balance regime of eastern Vatnajökull. Ahlmann (1936) concluded that Vatnajökull belongs to one of the most intensive glaciated areas under investigation. The French-Icelandic Expedition took place in 1951 and 1955 which carried out seismic sounding of ice thickness. Björnsson (1988) gives a historical outline of glaciological studies in Iceland with an exhaustive list of references. He also gives a thorough account of the work on the mapping of glacier surface and bedrock topography.

The work presented in this thesis is based on data that has been gathered in numerous expeditions on Vatnajökull and knowledge that has been collected by many people during the last decades. Since 1977 the Science Institute of the University of Iceland has performed radio-echo soundings of the ice thicknesses on Vatnajökull and several other glaciers and ice caps in Iceland (Björnsson, 1986, 1988; Björnsson and Pålsson, 1991; Björnsson et al., 1992), which is the basis for the model study presented here. The mass balance and surface velocity have been measured on a large portion of Vatnajökull since 1991 (Björnsson et al., 1998b, Björnsson personal communication). These measurements provide means to calibrate the flow model towards the conditions on the ice cap and determine the parameterization for the mass balance distribution which is necessary for presented the study.

1.3 Model

In this thesis two types of models are applied to model the flow regime of Vatnajökull, a shallow ice approximation model (SIA) and a full system model. Model inter comparison reveals that at spatial scales larger than about 10 times the ice thickness the two models give essentially the same result. It is thus appropriate to apply the SIA model (Hutter, 1983) for the overall flow regime of Vatnajökull. This approximation is solved with a finite difference method and is computationally efficient, which is necessary when model simulations are done for long time periods. The full system model, which uses finite element formulation to solve the equations, is applied on the surface depression east of Grímsvötn in the center of the ice cap. The flow and sliding parameters are determined by comparing the model results with surface velocity measurements. The mass-balance distribution is parameterized with statistical methods by comparing the model results with mass-balance measurements.

1.4 Contents of thesis

This thesis presents a flow model for Vatnajökull. The aim of this work is to improve the understanding of the dynamic of the ice cap and its response to climatic changes. To ease the reading, for the readers who are not familiar with Iceland or Vatnajökull, two figures are drawn. Figure 1.1 shows the outline of Iceland and the largest glaciers with the location of Hofsjökull and Vatnajökull indicated. Figure 1.2 shows Vatnajökull along with topographical names of all the outlet glaciers and mountains within the ice cap, or other features that are mentioned in the text.

The data which are the basis for this modeling work are briefly described in Chapter 2. These data belong to the database of the Science Institute, University of Iceland and are more thoroughly described and discussed in other publications (Björnsson et al., 1995b,a, 1997, 1998b, 2002a).
Figure 1.1: Map of Iceland with the largest glaciers outlined. Vatnajökull and Hofsjökull are indicated.

Figure 1.2: Outline of Vatnajökull along with the names of the outlet glaciers and mountains or other features mentioned in the text.
In Chapter 3 the SIA model is presented and tested. Different numerical schemes are tested on a simple ice sheet geometry and compared with analytical solution. The parameters of the model are selected to fit the conditions at Vatnajökull by comparing the model results with surface velocity measurements and melting at the bed in Grímsvötn area is included.

Some effort was put in selecting a numerical method to solve the equation for the surface evolution. A mass inconsistency is observed when the numerical codes are applied on Vatnajökull which brought the attention to the boundary treatment in the model. In Chapter 4 the mass inconsistency is analysed and tests are done on simple ice sheets. It is concluded that an upstream scheme is necessary to maintain mass preservation at glacier boundaries within the ice cap, the so called nunataks. Sensitivity tests with the upstream scheme and the model parameters are also presented.

In Chapter 5 mass-balance parameterization to use together with the flow model to study Vatnajökull is developed. The mass-balance distribution is described with a linear function of the altitude, with two different gradients. The equilibrium line altitude (ELA), where mass balance is zero and the gradient changes, is linearly dependent on the distance from the coast. Annual variation in mass balance is presented with a shift in the ELA. This parameterization is tested statistically and found to simulate closely the measured mass balance for the period 1992 – 2000.

Steady state computations of Hofsjökull the neighboring ice cap of Vatnajökull are presented in Chapter 6.

The state of balance is analysed in Chapter 7. The mass-balance distribution developed in Chapter 6 is used to analyse each of the outlet glaciers. Two methods were used. First method uses the simple integral of the mass balance over each outlet glacier to predict whether the glacier can be expected to grow or contract. The other method compares two models of the flow to assess the state of balance. It is assumed that discrepancies between the two models reflect imbalance in the flow regime of the ice cap.

The SIA model is applied on Vatnajökull in Chapter 8. It is shown that Vatnajökull is presently not stable with respect to perturbations in ELA. The ice cap either grows rapidly to a large size apparently without bound or it contracts to small detached ice caps on the mountains.

In chapter Chapter 9 model experiments with time dependent forcing are presented. The time dependent changes in the height of the ELA and the amount of sliding provide a mechanism that can decrease, or even stop the unlimited growth observed in the model experiments with constant model parameters.

The thesis is concluded in Chapter 10 and some suggestions are made for further research and field investigations.

The volcanic eruption in Gjálp 1996, beneath the center of the ice cap, and the subsequent jökulhlaup provided an opportunity to investigate small scale disturbances on glacier surface. Data from the Science Institute together with data collected with instruments from the Section of Glaciology, VAW, ETH Zürich, were used to calibrate a full-system model to study the response of the glacier to surface disturbance. The flow law parameter, A, for Vatnajökull is determined. This model study is presented in Appendix A.

In Appendices B-G, data from 3 expeditions on Vatnajökull are presented. Two field trips were done to measure surface velocities around the surface depression which formed during the catastrophic jökulhlaup in 1996. The third field trip was on Tungnaárjökull in an area where
the ice is flowing over bedrock undulations. Surface velocity and elevation were surveyed. These data have not yet been used to their full extent but they are listed in Appendices H-I. All field work was organized in cooperation with the Science Institute, University of Iceland, with support from the Iceland Glaciology Society (JÖRFÍ) and Landsvirkjun.

From this thesis two papers are already published or in press. The published paper is presented in almost the exact form, in which it is published in Appendix A. Chapter 6 in this thesis is a somewhat longer version of the paper in press.


Chapter 2

Available data

Vatnajökull ice cap is located close to the southeastern coast of Iceland, extending approx. 150 km from west to east and 100 km form south to north, 8100 km$^2$ in area (Figure 2.1). It is the largest ice cap outside the polar areas. The ice cap covers active volcanic areas where eruptions are frequent. Many of the outlet glaciers surge regularly. During the measurement period three outlet glaciers have surged and impeded measurements because of very rough surface due to the surges.

The basis for the modeling work is the data of the surface and bed topography, annual mass balance and surface velocity measurements that have been collected during the last two decades at the Science Institute, University of Iceland. This data has been compiled and tailored for use in the flow model (Björnsson et al., 1998a). This chapter discusses briefly the data which are used for the model geometry, the flow and sliding parameter selection and the mass-balance parameterization. These data are more thoroughly described and discussed in other publications (Björnsson et al., 1995b,a, 1997, 1998b, 2002a, and annual reports, Landsvirkjun (National Power Company, Iceland)).

2.1 Surface and bed geometry

The measured surface and bedrock topography are the results of radio-echo sounding surveys which were done by Science Institute, University of Iceland in the years 1980-1996 in cooperation with Landsvirkjun (Björnsson, 1988; Björnsson and Pálsson, 1991; Björnsson et al., 1992).

The bed topography is however not completely known, as some parts of the eastern ice cap as well as Öræfajökull have not been surveyed. The bed topography in these areas was estimated by interpolating the bedrock outside the glacier. The bed topography outside the glacier was taken from a grid from Landmælingar Íslands (Icelandic Geodetic Survey). All the model computations were done on a grid with a 1 km spacing in the horizontal.

2.2 Surface velocity measurements

Surface velocity has been measured on various locations on Vatnajökull since 1992. Figure 2.1 shows the locations of all available measurements. These measurements from the basis for the
determination of the values of the flow parameters of the model.

The location of poles or wires drilled into the ice is measured two times a year, in the spring and again in the fall. This allows for summer and winter velocities to be computed. There are several measurements in the Grímsvötn area and around the depressions created by the subglacial eruption in Gjálp and the following jökulhaup. Some of those measurements have been used to calibrate a model, which is based on a finite element method and uses all the terms of the momentum equations (Aðalgeirsdóttir et al., 2000). The SIA model used for the whole ice cap is not capable of describing the detailed flow field around the depressions or in the neighborhood of Grímsvötn, due to the neglected longitudinal strain rates that are important for flow on this spatial scale. These velocity measurements were thus not used in the parameter determination of the SIA model.

There are measurements available on the flow lines of 5 outlet glaciers of Vatnajökull; on Tungnaárjökull, Köldukvíslarjökull, Dyngujökull, Brúarjökull and Eyjabakkajökull. These are indicated with lines on Figure 2.1 and each flow line is shown in Figures 2.2 - 2.6. The difference in elevation between the surface profile and the location of the stake, visible in the figures, reveals elevation changes between the time of the surface survey and the time of the velocity measurements, or a slight offset in location from the actual profile. In general, there are several measurements available on each location and temporal variations in the velocity are observed. In the following sections the velocity measured on each of the flow lines is discussed. The measurements outside the flow lines are not shown but they are also compared to the modeled velocity to determine the flow parameters.

2.2.1 Tungnaárjökull

The profile along the flow line on Tungnaárjökull is shown in Figure 2.2. It is approximately 40 km long glacier on the western side of Vatnajökull. The width is not clearly defined as
the position of the ice divides between Tungnaárjökull and neighboring Sylgju-, Skaftár- and Síðujökull is not fixed and often during surges ice is shifted from its usual path.

There exists a series of velocity measurements done during the surge of Tungnaárjökull in 1992. These measurements give the opportunity to quantify how much the velocity increases during a surge. Figure 2.2 shows all the measurements, including the surge velocities.

There is no regular seasonal variation in the data. The lowest velocity was often measured during the summer and the measured winter velocities were higher than many of the summer velocities.

2.2.2 Köldukvíslarjökull

The profile along the flow line on Köldukvíslarjökull is shown in Figure 2.3. It is a 25 km long and approximately 20 km wide glacier north of Tungnaárjökull. The profile reaches all the way up to Bárðarbunga (see Figure 2.1). 4-7 stakes have been repeatedly measured since 1994, 6 stakes are included in the profile. When summer and winter velocities are compared no regular seasonal variation in the velocity is observed.

2.2.3 Dyngjujökull

The profile of the flow line on Dyngjujökull is shown in Figure 2.4. It is a 43 km long and 20-30 km wide glacier on the north-western side of Vatnajökull. Measurements have been done on 15 stakes on this glacier since 1992 and 10 stakes are included in the profile. Measurements do not show significant seasonal variation in the velocity, but there are, however, not many winter velocity measurements available.

2.2.4 Brúarjökull

The profile of the flow line on Brúarjökull is shown in figure 2.5. It is the largest outlet glacier, about 60 km long and 20-40 km wide glacier east of Kverkfjöll with relatively flat bed topography. Measurements were done on 19 stakes 1993 and since then on 6-13 stakes, 9 stakes are included in the profile. The winter velocity, only measured on two stakes though, is considerably smaller than the summer velocity.

2.2.5 Eyjabakkajökull

The profile of the flow line on Eyjabakkajökull is shown in figure 2.6. This glacier is a relatively small, 15 km long and 5-7 km wide, east of Brúarjökull. Velocity has been repeatedly measured on 4 stakes since 1995. There is no significant seasonal variation observed in the measurements.
Figure 2.2: Tungnaárjökull. Upper panel shows the profile down the glacier approximately along a flow line. Crosses indicate the location of velocity measurements. Lower panel shows the surface velocity measurements. Red crosses are the summer velocity, blue pluses the winter velocity and the green stars the mean annual velocity. The accuracy in each measurement is indicated with an error bar. Velocities during the surge in 1992 are included in this figure.
Figure 2.3: Köldukvíslarjökull. Upper panel shows the profile down the glacier approximately along a flow line. Crosses indicate the location of velocity measurements. Lower panel shows the surface velocity measurements. Red crosses are the summer velocity, blue pluses the winter velocity and the green stars the mean annual velocity. The accuracy in each measurement is indicated with an error bar.
Figure 2.4: Dyngjujökull. Upper panel shows the profile down the glacier approximately along a flow line. Crosses indicate the location of velocity measurements. Lower panel shows the surface velocity measurements. Red crosses are the summer velocity, blue pluses the winter velocity and the green stars the mean annual velocity. The accuracy in each measurement is indicated with an error bar.
Figure 2.5: Brúarjökull. Upper panel shows the profile down the glacier approximately along a flow line. Crosses indicate the location of velocity measurements. Lower panel shows the surface velocity measurements. Red crosses are the summer velocity, blue pluses the winter velocity and the green stars the mean annual velocity. The accuracy in each measurement is indicated with an error bar.
Figure 2.6: Eyjabakkajökull. Upper panel shows the profile down the glacier approximately along a flow line. Crosses indicate the location of velocity measurements. Lower panel shows the surface velocity measurements. Red crosses are the summer velocity, blue pluses the winter velocity and the green stars the mean annual velocity. The accuracy in each measurement is indicated with an error bar.
2.3 Mass balance measurements

Mass-balance monitoring has been carried out on the western and northern outlets of the ice cap since 1991 and on Breiðamerkurjökull in the southeast since 1996 (Björnsson et al., 1998b, 1995b,a, 1997). The mass-balance measurements have been done at similar locations as the velocity measurements (see Figure 2.1). In addition, one point on Öræfajökull has been measured since 1993 which gives indication of the mean specific annual mass balance where the maximum accumulation is expected (Gudmundsson, 2000). The coverage of measurements is good so that most of the larger outlet glaciers are monitored. The smaller glaciers flowing towards the east, Skeiðarárjökull in the south and the outlet glaciers of Öræfajökull are, however, not well covered. In Figure 2.7 the height dependence of the net mass balance is shown for the measured outlet glaciers. On this figure it can be observed that the equilibrium line altitude (ELA), where the mass balance is zero, is varying across the ice cap. Breiðamerkurjökull has ELA at about 1100 m a.s.l. but the other glaciers, on the northern and western side of the ice cap, have ELA at 1200-1400 m a.s.l. This is expected as most of the precipitation comes from the south arriving at the coast and the northern and western outlet glaciers lie in a precipitation shadow of the coastal mountains. Another attribute of the mass-balance data can be observed in Figure 2.7. The mass balance has a strong piecewise linear dependence on altitude. The increase in mass balance with altitude is slower above the ELA than below it. This well known feature can clearly be seen in the data from Tungnaárájökull (Figure 2.7 F).

These characteristics of the mass-balance data, the dependence of the ELA on the distance from the coast and the dependence on the altitude, will be used to parameterize the mass balance distribution of Vatnajökull.

The mean specific net balance has been estimated by extrapolating the measurements and integrate over the whole glacier surface. It was positive, but decreasing, the first three years (1992-1994). After that the mean specific net balance became negative and decreased year by year until 1997. The mass balance then increased again but remained negative for the years 1998 - 2000. (Björnsson et al., 1998b, Björnsson, personal communication).
Figure 2.7: Measured mass balance as a function of elevation for various outlet glaciers. (A) Breiðamerkurjökull. (B) Brúarjökull. (C) Dyngjujökull. (D) Eyjabakkajökull. (E) Köldukvíslarjökull. (F) Tungnaárjökull. (Björnsson et al., 1998b, 2002, personal communication).
Chapter 3

Model description

This chapter describes the flow model used to analyse Vatnajökull, the approximations made and the numerical methods. The codes are tested on simple geometry by repeating the Eismint experiment (Huybrechts and Payne, 1996) and comparing with analytical 2D solution. Flow law parameters are selected to tune the model for use on Vatnajökull by comparing the model velocities with all available measured velocities. Melting at the base in the Grímsvötn area is added to include the geothermal activity there, which cannot be ignored in the model computations.

3.1 Selection of model

Models can be useful to reduce complex real situations to a simple closed system that describes the fundamental aspects and to which the laws of physics can be applied. Models of the flow of large ice masses have been widely used to gain better understanding on how they are created, their stability, and how they react to and influence large scale climate changes. Nye (1952a,b) examined the flow of valley glaciers, the mechanical properties of ice and set forth a method to calculate the thicknesses of the ice sheets. Bodvarsson (1955) used perturbation method to show that ice sheets are unstable, where small perturbations from the steady state will lead to either growth or decrease of the ice sheet, depending on the perturbation. Laboratory experiments lead to establishment of a constitutive equation for the ice, that describes the relation of stresses and strain in ice (Glen, 1955), which was generalized by Nye (1957). Nye (1959a) analyzed the motion and steady states of ice sheets and Weertman (1961) showed that ice sheet dimensions are very sensitive to the distribution of mass balance in space. Nye and Weertman published numerous papers on different theoretical aspects of ice sheet modeling and investigated among other things the responses to seasonal and climate changes with kinematic wave theory (Nye, 1960) and non-equilibrium ice-sheets (Weertman, 1964). Robin (1967) pointed out the importance of including the longitudinal stresses in ice sheet computations. The time evolution of ice sheets was investigated by developing analytical similarity solutions in two and three dimensions (Halfar, 1981, 1983; Hindmarsh, 1990) and the bistability of ice sheets was confirmed. Hindmarsh (1990) accentuated the importance of the dependence of ice sheet response upon the slope of the equilibrium line altitude (ELA) and the ablation/accumulation ratio, and showed weak dependence upon the rate factor (temperature) and the magnitude of the accumulation and ablation, which had been pointed out by Weertman (1961). The stable, unstable and multiple steady states and the importance of the mass balance-elevation feedback were
investigated further with relatively simple numerical models (Oerlemans, 1981, 1983; Van der Veen and Oerlemans, 1984; Abe-Ouchi, 1993).

The development of computers and increased computer power gave way to increased sophistication in ice-sheet modeling work. The first numerical models were developed and applied on the large ice sheets in the seventies (Budd and Jenssen, 1975; Mahaffy, 1976; Jenssen, 1977; Bindschadler, 1982).

Rigorous asymptotic scaling analysis where the flow of large natural ice masses under gravity was described by thermomechanically coupled system of mass, momentum and energy balance of an incompressible, homogeneous, heat conducting, non-linearly viscous fluid resulted in a lead order reduced model which at present is known as the Shallow Ice Approximation (SIA) (Fowler and Larson, 1978; Hutter, 1982a,b, 1983; Morland and Johnson, 1980; Morland, 1984). The SIA is valid when the thickness of the ice is much smaller than the length. Then the variations in longitudinal directions are small, as compared to the vertical, so that stresses and velocities are functions of local thickness and slope. The advantage of these reduced equations is that they are easily and efficiently programmable and the relative error is known to be the square of the aspect ratio. The SIA has now been applied to three dimensional, unsteady, thermomechanically coupled systems for grounded ice sheets as well as ice shelves to investigate the behavior during glacial-interglacial cycles and responses to future climate changes (for example: Herterich, 1988; Huybrechts, 1986, 1990a,b; Huybrechts and Oerlemans, 1990; Huybrechts et al., 1991; Letreguilly et al., 1991; Letreguilly and Ritz, 1993; Fabre et al., 1995; Calov and Hutter, 1996; Weis et al., 1996; Greve, 1997). For a comprehensive reference list the paper of Calov and Hutter (1997) is recommended. A thorough review of most aspects of modeling ice sheets dynamics is given by Hindmarsh (1993). Several authors are applying higher order models (Blatter, 1995) and models which solve the full Stokes equations on glacial systems (for example: Gudmundsson, 1999; Lüthi and Funk, 2000; Pattyn, 2000; Vieli et al., 2000).

By comparing the results of both analytical zeroth- and first-order theories, as well as the SIA and a full system model, Ædalgeirsdóttir et al. (2000) showed that the spatial scale of the problem under consideration is important. For spatial scales larger than about 10 times the ice thickness the different models give essentially the same result for the surface evolution, but at smaller scales the horizontal stress gradients become important and SIA models are not applicable. Leysinger-Vieli and Gudmundsson (2002) show, with a similar model inter-comparison, that the rate-of-advance of the snout is similar with both SIA model and a full system model, even though at the same time the surface velocity is different. This can be understood by the fact that advance or retreat of a glacier is a response to integrated changes in mass balance and not particular flow dynamics of the snout. They conclude that long-term surface evolution can be predicted accurately with a SIA model, without accounting for horizontal stress gradients in the calculations.

The purpose of the modeling work presented here is to analyze the present state of Vatnajökull and assess whether it is possible, with the present knowledge of the mass balance, to predict future responses to probable changes in the mass balance. A SIA model is thus suitable. The length scale of Vatnajökull is much larger than its thickness. The total length of the ice cap is about 100 km, the outlet glaciers are 20-60 km long and the maximum thickness is 900 m. The focus of this study is on the advance and retreat of the outlet glaciers and therefore it is not necessary to include longitudinal stresses. Paterson (1994) has pointed out that uncertainties in specifying the mass balance, rather than deficiencies in the glacier model, are probably the main source of inaccuracy in calculations of this type.
3.2 Shallow Ice Approximation Model

The flow of ice sheets can be mathematically described by balance equations based on general continuum-mechanical balance statements:

\[
\begin{align*}
  v_{i,i} &= 0 \quad \text{(mass)}, \\
  \sigma_{ij,j} + \rho g_i &= 0 \quad \text{(momentum)}, \\
  \sigma_{ij} - \sigma_{ji} &= 0 \quad \text{(moment of momentum)}.
\end{align*}
\]

Here \( v_i \) is a component of the velocity vector, \( \sigma_{ij} \) the stress tensor, \( \rho \) the density of the material, here assumed to be a constant and \( g \) is the acceleration due to gravity. These equations are simplified by scaling them with the aspect ratio \( \epsilon = \frac{H}{L} \ll 1 \), in which \( H \) and \( L \) are a typical depth and horizontal distance. The zeroth-order equations, using this scaling, are the so-called shallow ice approximation (SIA) (Hutter, 1983). The solution for the stresses is

\[
\begin{align*}
  \sigma_{zz}^{(0)} &= \rho g (z_s - z), \\
  \sigma_{xz}^{(0)} &= -\rho g (z_s - z) \frac{\partial h}{\partial x}, \\
  \sigma_{yz}^{(0)} &= -\rho g (z_s - z) \frac{\partial h}{\partial y}.
\end{align*}
\]

In order to calculate the velocity in the ice sheet the material properties have to be defined. The generally used constitutive equation for ice is the Glen’s flow law (Glen, 1955; Paterson, 1994), which describes the relation between the strain rate \( \dot{\epsilon}_{ij} = \frac{1}{2} (v_{ij} + v_{ji}) \) and the deviatoric stress \( \sigma'_{ij} \) which is defined as \( \sigma'_{ij} = \sigma_{ij} - \frac{1}{2} \delta_{ij} \sigma_{kk} \), \( (\delta_{ij} \) is the Kronecker delta). It has the form:

\[
\dot{\epsilon}_{ij} = A \tau^{n-1} \sigma'_{ij},
\]

where \( \tau \) is the second invariant of \( \sigma'_{ij} \) defined as \( \tau = \sqrt{\frac{2}{3} \sigma'_{ij} \sigma'_{ij}} \). Here \( n \) is a constant, usually taken to be equal to three, but \( A \) depends on the ice temperature, crystal size and orientation, impurity and water content, density and perhaps other factors. As Vatnajökull is a temperate ice cap (Rist, 1961) the temperature is everywhere at the melting point. Thus, the value of \( A \) is assumed to be a constant in the following calculations. Due to all the uncertainties in the value of this parameter it is used to tune the model towards the measured surface velocity.

The zeroth order SIA leads to the following expression for the deformation velocity (Mahaffy, 1976; Hutter, 1983):

\[
V_i^{\text{def}}(x, y, z, t) = \frac{2A}{n+1} (\rho g)^n \alpha^{n-1} \frac{\partial h}{\partial x_i} \left( (h - z)^{n+1} - (h - z_0)^{n+1} \right), \quad (3.1)
\]

where \( z_0 \) is the bedrock elevation, \( h \) the surface elevation, and \( \alpha \) is the slope of the surface.
The evolution of the surface can be computed with the continuity equation, which often is called the ice sheet equation (Hindmarsh and Payne, 1996):

\[ \frac{\partial h}{\partial t} = \dot{b} - \nabla \cdot q \]  

(3.2)

Here \( \dot{b} \) is the mass balance rate, also called accumulation – ablation function, at the surface and the bed, and \( q \) is the ice flow or volume flux, which in the model is vertically integrated over the whole ice depth:

\[ q_i(x, y, t) = \int_{z_b}^{z_s} \left[ V_i^{\text{def}}(x, y, z, t) + V_i^{\text{slid}}(x, y, t) \right] dz. \]  

(3.3)

\( V_i^{\text{slid}}(x, y, t) \) is the velocity at which the glacier is sliding over the bedrock. The sliding at the bed is approximated by using a Weertman type of sliding law (Paterson, 1994). The sliding velocity is assumed to be dependent on the basal shear stress, \( \tau_b \), according to:

\[ V_i^{\text{slid}}(x, y, t) = C m \frac{\tau_b}{m} = C (\rho g H)^m \alpha^{m-1} \frac{\partial h}{\partial x_i}. \]  

(3.4)

In the following calculations we take the constant \( m \) to be equal to the flow law component \( n \). With these deformation and sliding velocities, the ice sheet equation becomes a nonlinear partial differential equation:

\[ \frac{\partial h}{\partial t} = \dot{b} - \nabla \left( \tilde{A} H(x, y, t) \alpha^{n+2} - \alpha^{n-1} \nabla h + C (\rho g)^m H(x, y, t) \alpha^{m+1} \alpha^{m-1} \nabla h \right), \]  

(3.5)

where

\[ \tilde{A} = \frac{2 A}{n + 2} (\rho g)^n. \]

This equation for the thickness, \( h \), of the ice sheet can be numerically solved on a horizontal grid where each point is defined as:

\[ x = i \Delta x, i \in (0, N_x) \quad \text{and} \quad y = j \Delta y, j \in (0, N_y) \]

where \( N_x \) and \( N_y \) are the number of grid points in the \( x \) and \( y \) direction, respectively.

There are several ways possible to solve this equation numerically and carry out the horizontal space discretization. An explicit upstream scheme and three alternating direction semi-implicit
schemes (ADS-I) were tested. The space discretization is similar to methods 1 and 2 of Hindmarsh and Payne (1996) and the ADS-I method is adapted from Huybrechts (1986). All the schemes were tested on the Eismint experiment setup (Huybrechts and Payne, 1996). The reason that all these different schemes were tested is that the first code used to compute Vatnajökull (ADS-I method) showed inconsistency in the treatment of ice free points within the ice sheet (nunataks) so an alternative solution method was sought. The idea with using an explicit up-stream scheme was that the first order scheme can be corrected with a synchronous and iterative flux-correction formalism for coupled transport equations (Schär and Smolarkiewicz, 1996). The code for this correction was kindly provided by C. Schär (personal communication, 2002), but after a lengthy battle, attempts to apply it on Vatnajökull were abandoned. It would, however, be very interesting to pursue this further and compare the results with the higher order ADS-I schemes.

3.3 Finite-difference discretization methods

3.3.1 Explicit upstream scheme

An upstream explicit scheme was applied on Equation 3.2. The advective velocity is computed on a staggered grid (see Figure 3.1)

\[
 u_{i+\frac{1}{2},j} = -\frac{2 A}{n+1} (\rho g)^n h_{i+\frac{1}{2},j}^{n+1} \alpha^{n-1} \left( \frac{\Delta h}{\Delta x} \right)_{i+\frac{1}{2},j} - C (\rho g)^m h_{i+\frac{1}{2},j}^m \alpha^{m-1} \left( \frac{\Delta h}{\Delta x} \right)_{i+\frac{1}{2},j}
\]

\[
 v_{i,j+\frac{1}{2}} = -\frac{2 A}{n+1} (\rho g)^n h_{i,j+\frac{1}{2}}^{n+1} \alpha^{n-1} \left( \frac{\Delta h}{\Delta y} \right)_{i,j+\frac{1}{2}} - C (\rho g)^m h_{i,j+\frac{1}{2}}^m \alpha^{m-1} \left( \frac{\Delta h}{\Delta y} \right)_{i,j+\frac{1}{2}}
\]

where

\[
 \alpha = \sqrt{\left( \frac{\Delta h}{\Delta x_{i+\frac{1}{2},j}} \right)^2 + \left( \frac{\Delta h}{\Delta y_{i,j+\frac{1}{2}}} \right)^2},
\]

\[
 h_{i+\frac{1}{2},j} = \frac{1}{2} (h_{i,j} + h_{i+1,j})
\]

\[
 h_{i,j+\frac{1}{2}} = \frac{1}{2} (h_{i,j} + h_{i,j+1}),
\]

and

\[
 \frac{\Delta h}{\Delta x_{i+\frac{1}{2},j}} = \frac{1}{\Delta x} (h_{i+1,j} - h_{i,j})
\]

\[
 \frac{\Delta h}{\Delta y_{i,j+\frac{1}{2}}} = \frac{1}{\Delta y} (h_{i,j+1} - h_{i,j}).
\]

the upstream scheme computes the flux according to
Figure 3.1: Schematic figure that illustrates the different discretization methods to compute the coefficient $D$ and the discharge $q$. For each method the left hand figure shows which points are used in the computation of the thickness and slope used to compute the velocity (for the explicit scheme) or $D$ (for the ADS-I Scheme). The right hand figure shows the location of $u/q$ and the upstream thickness (for explicit scheme) or $D$ and the points used to compute $\frac{\partial h}{\partial x}$ required to compute the discharge (for ADS-I Scheme).

\[
q_x^{i+\frac{1}{2},j} = \begin{cases} 
u_{i+\frac{1}{2},j}h_{i,j}, & \text{for } u > 0 \\ u_{i+\frac{1}{2},j}h_{i+1,j}, & \text{for } u < 0 \end{cases}
\]

and

\[
q_y^{i,j+\frac{1}{2}} = \begin{cases} v_{i,j+\frac{1}{2}}h_{i,j}, & \text{for } v > 0 \\ v_{i,j+\frac{1}{2}}h_{i,j+1}, & \text{for } v < 0 \end{cases}
\]

the forward time stepping is then computed explicitly with

\[
\frac{h^{k+1} - h^k}{\Delta t} = b^k - \left( \frac{q_x^{i+\frac{1}{2},j} - q_x^{i-\frac{1}{2},j}}{\Delta_x} + \frac{q_y^{i,j+\frac{1}{2}} - q_y^{i,j-\frac{1}{2}}}{\Delta_y} \right)
\]
3.3.2 Alternating Direction Semi-Implicit method

In this scheme a coefficient $D$ is defined

\[ D = \tilde{A} h(x, y, t)^{n+2} \alpha^{n-1} + C (\rho g)^m h(x, y, t)^{m+1} \alpha^{m-1}, \]

so that Equation (3.5) becomes:

\[ \frac{\partial h}{\partial t} = b - \nabla (D \nabla h). \]

$D$ is computed between the gridpoints similar to Method 1 of Hindmarsh and Payne (1996) (see Figure 3.1)

\[ D_{i+\frac{1}{2}, j+\frac{1}{2}} = \tilde{A} h_{i+\frac{1}{2}, j+\frac{1}{2}}^{n+2} \alpha_{i+\frac{1}{2}, j+\frac{1}{2}}^{n-1} + C (\rho g)^m h_{i+\frac{1}{2}, j+\frac{1}{2}}^{m+1} \alpha_{i+\frac{1}{2}, j+\frac{1}{2}}^{m-1} \]

where

\[ h_{i+\frac{1}{2}, j+\frac{1}{2}} = \frac{1}{4} (h_{i,j} + h_{i+1,j} + h_{i,j+1} + h_{i+1,j+1}) \]

and

\[ \alpha_{i+\frac{1}{2}, j+\frac{1}{2}} = \frac{1}{2 \Delta x} (h_{i+1,j} + h_{i+1,j+1} - h_{i,j} - h_{i,j+1}) + \frac{1}{2 \Delta y} (h_{i,j+1} + h_{i+1,j+1} - h_{i,j} - h_{i+1,j}). \]

The flux in the x-direction, $q^x$, is computed as

\[ q^x_{i+\frac{1}{2}, j} = -\frac{1}{2} (D_{i+\frac{1}{2}, j+\frac{1}{2}} + D_{i, j+\frac{1}{2}}) \frac{h_{i+1,j} - h_{i,j}}{\Delta x} \]

and similarly for $q^y_{i+\frac{1}{2}, j}$, $q^y_{i, j+\frac{1}{2}}$ and $q^y_{i, j-\frac{1}{2}}$

The time evolution of the surface elevation is computed using constant time steps, $t = k \Delta t$ and semi-implicit method

\[ \frac{h^{k+1} - h^k}{\Delta t} = b^k - \nabla (D^k \nabla h^{k+\theta}) \]  \hspace{1cm} (3.6)

the superscripts refer to the time-step. This equation is solved by using ADI method (Alternating Direction Implicit method) (Hirsch, 1988). The $\theta$ is alternating equal 0 or 1 in x and y direction so in each time step one direction is computed implicit while the other is explicit. This is similar to the method Huybrechts (1986) used in his model. Due to the nonlinearity of the coefficient D the solution method is not fully implicit, but semi-implicit, as D is evaluated at the previous time step, but $\nabla h$ is implicitly determined.
3.3.3 Alternating Direction Semi-Implicit upstream A

In order to avoid the problem arising at the nunatak points the scheme is made upstream. This ensures that no flow will be computed out of bedrock points sticking out of the ice sheet, as happened in the previous method, causing considerable amount of mass being created as a result. This method is based on the discretization of Mahaffy (1976) (method 2 in Hindmarsh and Payne (1996)). This method evaluates $D$ at $x$-midway and $y$-midway points that are staggered in the direction of flow only (see Figure 3.1 for the location of the points used in the computation)

\[
D_{i+\frac{1}{2}, j} = \bar{A} h_{i+\frac{1}{2}, j}^{n+2} \alpha_{i+\frac{1}{2}, j}^{n-1} + C (\rho g)^m h_{i+\frac{1}{2}, j}^{m+1} \alpha_{i+\frac{1}{2}, j}^{m-1}
\]

\[
D_{i, j+\frac{1}{2}} = \bar{A} h_{i, j+\frac{1}{2}}^{n+2} \alpha_{i, j+\frac{1}{2}}^{n-1} + C (\rho g)^m h_{i, j+\frac{1}{2}}^{m+1} \alpha_{i, j+\frac{1}{2}}^{m-1}
\]

where

\[
h_{i+\frac{1}{2}, j} = \begin{cases} h_{i,j} & \text{for } h_{i,j} > h_{i+1,j} \\ h_{i+1,j} & \text{for } h_{i,j} < h_{i+1,j} \end{cases}
\]

\[
h_{i, j+\frac{1}{2}} = \begin{cases} h_{i,j} & \text{for } h_{i,j} > h_{i,j+1} \\ h_{i,j+1} & \text{for } h_{i,j} < h_{i,j+1} \end{cases}
\]

and

\[
\alpha_{i+\frac{1}{2}, j} = \frac{1}{\Delta x} (h_{i+1,j} - h_{i,j}) + \frac{1}{4\Delta y} (h_{i,j+1} + h_{i+1,j+1} - h_{i,j-1} - h_{i+1,j-1})
\]

\[
\alpha_{i, j+\frac{1}{2}} = \frac{1}{4\Delta x} (h_{i+1,j} + h_{i+1,j+1} - h_{i-1,j} - h_{i-1,j+1}) + \frac{1}{\Delta y} (h_{i,j+1} - h_{i,j}).
\]

The flux in the x-direction is computed at the same x-midway points according to

\[
q_{i+\frac{1}{2}, j}^{\pi} = -D_{i+\frac{1}{2}, j} \frac{h_{i+1,j} - h_{i,j}}{\Delta x}
\]

and similarly for $q_{i-\frac{1}{2}, j}^{\pi}$. The y-direction flux $q_{i+\frac{1}{2}, j}^{\eta}$ and $q_{i-\frac{1}{2}, j}^{\eta}$ are evaluated at the y-midway points according to

\[
q_{i, j+\frac{1}{2}}^{\eta} = -D_{i, j+\frac{1}{2}} \frac{h_{i,j+1} - h_{i,j}}{\Delta y}.
\]

The time evolution is computed with Equation 3.6 with $\theta$ alternating between 0 and 1 in x and y directions, respectively, as before.
3.3.4 Alternating Direction Semi-Implicit upstream B

This scheme is similar to upstream A, except that only one power of the thickness is an upstream value. \( D \) is now computed as

\[
D_{i+\frac{1}{2},j} = \frac{1}{2} h_{i+\frac{1}{2},j}^{n+1} \alpha_{i+\frac{1}{2},j}^{n+1} + C (p g)^{m} h_{i+\frac{1}{2},j}^{m} \alpha_{i+\frac{1}{2},j}^{m} 
\]

where

\[
\tilde{h} = \frac{1}{2} (h_{i,j} + h_{i+1,j})
\]

and similar for \( D_{i,j+\frac{1}{2}} \)

3.4 Comparison with the Eismint experiment and an analytical solution

The numerical code that solves the ice sheet equation can be tested by repeating the Eismint benchmark experiments (Huybrechts and Payne, 1996) and compare the results with the published results. These are a series of experiments designed to compare ice sheet models and assess how accurately these models compute basic variables such as ice thickness, velocity or flow, and to identify the most accurate and efficient numerical techniques. The benchmark experiments, that consider vertically integrated isothermal ice flow, were performed with all the above presented numerical schemes. The results were compared with the published ice sheet models. This comparison includes four different experiments, two steady state and two time dependent, with and without a fixed boundary. The numerical domain is the same in both experiments. It consists of a square grid of sides 1500 \( \times \) 1500 km with a flat bed at zero elevation, shown in Figure 3.2. The grid spacing is chosen to be 50 km, leading to 31 \( \times \) 31 = 961 regularly spaced points. This experiment does not include basal sliding. The schemes presented here are comparable with the 3D mass-conserving model of type I (Huybrechts and Payne, 1996). Time step was chosen to be 10 years. A comparison run was done with a time step of 5 years and the result was identical.

During the computations for the comparison it was noticed that the grid size chosen by the Eismint group is not small enough for the result to have converged to a constant value, it is rather a compromise between the computing time and the accuracy of the result. With a smaller grid size, the height of the steady state ice sheet changes more than the published standard deviation which is representative for the variation between the participating models. To assess the accuracy of the numerical codes it was thus considered better to compare the results with an analytical solution. The analytical solution for a 2D ice sheet is presented. The numerical codes compute an ice sheet which is 100 times longer in one direction than the other and the center height is compared with the analytical solution.

3.4.1 The fixed margin experiment

In the first experiment the ice sheet fills the entire grid area, leading to a square shape. Model runs were done to test both the steady-state and the time-dependent model behavior. All simulations are over 200 000 years.
Figure 3.2: Numerical grid used for the Eismint calculations. The left panel shows the steady state ice sheet computed in the fixed margin experiment and the right panel the ice sheet computed in the moving margin experiment. The orientation of the axes, the length scale and the numbering conventions are shown. The cross indicates point (24,16), the midpoint, where the flux is examined.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Ice thickness at divide (m)</th>
<th>Mass flux at midpoint (\left(10^2 m^2 a^{-1}\right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eismint 3d type I</td>
<td>3419.9 ± 1.7</td>
<td>789.95 ± 1.83</td>
</tr>
<tr>
<td>AD Semi-Implicit</td>
<td>3421.80</td>
<td>791.07</td>
</tr>
<tr>
<td>ADS-I upstream A</td>
<td>3241.20</td>
<td>808.88</td>
</tr>
<tr>
<td>ADS-I upstream B</td>
<td>3380.27</td>
<td>791.93</td>
</tr>
<tr>
<td>Explicit</td>
<td>3273.86</td>
<td>778.42</td>
</tr>
</tbody>
</table>

Table 3.1: Results of the fixed-margin experiment in steady state.

**Steady state**  Climatic boundary condition is a fixed accumulation rate over the entire ice covered area

\[
M = 0.3 \text{ m a}^{-1}
\]

where \(M\) is expressed in m a\(^{-1}\) of ice equivalent. Initial condition is no ice. The resulting ice thickness and mass flux per unit width are shown in Figure 3.3. The comparison with the Eismint values for the ice thickness at the divide, in the center of the grid, and the mass flux at gridpoint (24,16), shown with a cross on the left panel of Figure 10, is given in Table 3.1. The resulting shape and the values for the ADS-I method is very similar to the published values. The values published for the 3D models of type I are mean values and their standard deviations taken from the results of the 6 models participating in the experiment. The ice thickness at the divide is 50-150 m lower for the upstream methods, which is only a few percent of the total thickness. It is, however, outside the published standard deviation and is due to the large grid size. With a smaller grid size the ice sheet thickens, as shown in a later section. The ADS-I upstream method B has 1.2 % thinner ice. The computed mass flux is similar for ADS-I and ADS-I upstream method B but the results of the other methods are outside the published values.
Figure 3.3: Surface geometry to the left and mass flux to the right resulting from the fixed-margin experiment in steady state. The profile starts at the center of the steady state ice cap and extends to the margin.

Figure 3.4: Ice thickness change for the fixed-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka.

**Time dependent mass balance forcing**  The time dependent behavior of the model is tested with a sinusoidally varying climatic boundary condition

\[ M = 0.3 + 0.2 \sin\left(\frac{2\pi t}{T}\right) \text{ (ma}^{-1}) \text{).} \]

The driving periods used in these experiments are of Milankovitch time-scales, which are 20 000 and 40 000 years. The steady-state solution from the first experiment is specified as the initial condition. The resulting ice thickness and mass flux are shown in Figures 3.4 and 3.5. The comparison with the Eismint values is given in Table 3.2.

Again, the ADS-I method gives very similar results as the published results, all values are within the standard deviation, except the ice thickness range for the T=20 ka experiment, which is just barely outside it. The ADS-I method B gives results which also are within the standard
Figure 3.5: Mass flux for the fixed-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Ice thickness at divide (m)</th>
<th>Mass flux at midpoint ($10^2$ m, a$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>At t = 200 ka</td>
<td>Range</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Range</td>
</tr>
<tr>
<td>$T = 20000$ a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eismint 3d type I</td>
<td>3264.8 ± 5.6</td>
<td>968.28 ± 4.85</td>
</tr>
<tr>
<td>AD Semi-Implicit</td>
<td>3266.10</td>
<td>965.44</td>
</tr>
<tr>
<td>ADS-I upstream A</td>
<td>3099.54</td>
<td>995.51</td>
</tr>
<tr>
<td>ADS-I upstream B</td>
<td>3227.76</td>
<td>968.38</td>
</tr>
<tr>
<td>Explicit</td>
<td>3130.15</td>
<td>955.02</td>
</tr>
<tr>
<td>$T = 40000$ a</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eismint 3d type I</td>
<td>3341.7 ± 3.9</td>
<td>1021.49 ± 6.04</td>
</tr>
<tr>
<td>AD Semi-Implicit</td>
<td>3344.39</td>
<td>1020.33</td>
</tr>
<tr>
<td>ADS-I upstream A</td>
<td>3173.27</td>
<td>1046.73</td>
</tr>
<tr>
<td>ADS-I upstream B</td>
<td>3305.10</td>
<td>1022.24</td>
</tr>
<tr>
<td>Explicit</td>
<td>3204.10</td>
<td>1006.10</td>
</tr>
</tbody>
</table>

Table 3.2: Results of the fixed-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka.
deviation, except the ice thickness itself is a little lower, in both experiments the thickness is about 1% smaller than the published values. The results for the other methods are not as similar to the published values.

The difference between the ADS-I and ADS-I upstream B methods and the published values is so small that it is taken as not significant and it is concluded that both codes perform similar to the models of the Eismint group for the fixed margin experiment.

### 3.4.2 The moving margin experiment

The experiment with moving margin includes ice ablation and aims at simulating the position of a free margin. The model setup resembles that for the fixed-margin experiments, except for different climatic boundary conditions.

**Steady state** The steady state experiment prescribes the mass balance, $M$, as a function of the radial distance, $d$, from the center of the grid

$$M = \min\{0.5, s(R_{el} - d)\},$$

where

$$d = \sqrt{(x - x_{\text{summit}})^2 + (y - y_{\text{summit}})^2}.$$  

\[\]

![Image](image.png)

Figure 3.6: Surface and mass flux for the moving-margin experiment in steady-state.

$R_{el}$ is the distance at which the mass balance changes from positive to negative values and $s$ is the slope of the mass-balance function. The values of those constants were 450 km and $10^{-2}$ m a$^{-1}$ km$^{-1}$, respectively, according to the Eismint prescription. This results in a steady state ice sheet configuration which is axi-symmetric. The resulting ice thickness and mass flux are shown in Figure 3.6. The comparison with the Eismint values is given in Table 3.3. The evolution to a steady state from zero ice thickness occurs over a period of 20 000 years. The ADS-I and ADS-I upstream B method result in a steady state ice sheet which is very similar to the Eismint ice sheet for summit elevation and extent. The summit elevation is about 0.3% and 0.8% lower than the published value and slightly outside the standard deviation range. The computed mass flux is within the standard deviation of the Eismint values. The other methods give results that are outside the standard deviation, except the mass flux of the Explicit method, it is very close to the Eismint value but the ice thickness at the divide is about 140 m lower.
Experiment | Ice thickness at divide (m) | Mass flux at midpoint ($10^2 m^2 a^{-1}$)  
--- | --- | ---  
Eismint 3d type I | 2997.5 ± 7.4 | 999.24 ± 17.91  
AD Semi-Implicit | 2986.60 | 989.01  
ADS-I upstream A | 2862.48 | 973.19  
ADS-I upstream B | 2973.95 | 988.00  
Explicit | 2860.51 | 997.07  

Table 3.3: Results of the moving-margin experiment in steady state.

**Time dependent mass balance forcing** The time dependent behavior for the moving boundary experiment is tested by varying the distance at which the mass balance changes from positive to negative sinusoidally.

$$R_{el} = 450 + 100 \sin\left(\frac{2\pi t}{T}\right) \text{ (km)}.$$  

Again the periods used are 20,000 and 40,000 years. The resulting ice thickness and mass flux are shown in Figures 3.7 and 3.8. The extent of the glacier evaluated as the grid point of the margin is shown in Figure 3.9. The comparison with the Eismint values is given in Table 3.4.

![Figure 3.7: Ice thickness change for the moving-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka.](image)

The results of this experiment do not compare as well with the Eismint values as before. The ADS-I method has slightly lower ice thickness for the $T=20$ ka experiment, but it is within the standard deviation range for the $T=40$ ka. The ice thickness range is, however, within the given standard deviation in both experiments. The mass flux range is slightly smaller for the $T=20$ ka experiment, but larger in the $T=40$ ka experiment. The ADS-I upstream B method has only the ice thickness range in the $T=20$ ka experiment similar to the Eismint values. The difference
Figure 3.8: Mass flux for the moving-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka. The sharp peaks within the cycle correspond to times when the margin position is jumping from one grid point to the next.

Figure 3.9: Glacier extent for the moving-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka.
<table>
<thead>
<tr>
<th>Experiment</th>
<th>Ice thickness at divide at t = 200 ka (m)</th>
<th>Range</th>
<th>Mass flux at midpoint (10^2 m^2 a^{-1})</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eismint 3d type I</td>
<td>2813.5 ± 2.0</td>
<td>528.6 ± 11.3</td>
<td>578.17 ± 3.29</td>
<td></td>
</tr>
<tr>
<td>AD Semi-Implicit</td>
<td>2804.94</td>
<td>534.50</td>
<td>568.66</td>
<td></td>
</tr>
<tr>
<td>ADS-I upstream A</td>
<td>2666.05</td>
<td>546.85</td>
<td>607.12</td>
<td></td>
</tr>
<tr>
<td>ADS-I upstream B</td>
<td>2785.24</td>
<td>533.48</td>
<td>598.23</td>
<td></td>
</tr>
<tr>
<td>Explicit</td>
<td>2677.44</td>
<td>526.71</td>
<td>599.21</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>T = 40000 a</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eismint 3d type I</td>
<td>2872.5 ± 6.8</td>
<td>591.4 ± 4.6</td>
<td>534.94 ± 7.28</td>
<td></td>
</tr>
<tr>
<td>AD Semi-Implicit</td>
<td>2871.49</td>
<td>595.60</td>
<td>560.76</td>
<td></td>
</tr>
<tr>
<td>ADS-I upstream A</td>
<td>2737.00</td>
<td>614.30</td>
<td>557.85</td>
<td></td>
</tr>
<tr>
<td>ADS-I upstream B</td>
<td>2847.96</td>
<td>599.59</td>
<td>554.38</td>
<td></td>
</tr>
<tr>
<td>Explicit</td>
<td>2742.07</td>
<td>587.13</td>
<td>589.40</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4: Summary of the results for the moving-margin experiment forced by varying the mass balance input with a sinusoidal function of periods 20 and 40 ka.
in the other values is small, the ice thickness is about 1% smaller and the mass flux range is slightly larger than the published values. The other methods have results that are outside the standard deviation given by the Eismint group, except the Explicit method has similar ice thickness range in both experiments. The flux (Figure 3.8) shows very similar behavior of several local maxima and minima over each cycle. The figure in the paper shows flux values after averaging and, thus, is Figure 3.8 with somewhat sharper peaks. Inspection reveals that these points correspond to times when the margin position is jumping from one grid point to the next. The extent of the ice sheet, shown in Figure 3.9, is identical to the Eismint ice sheet. The margin migrated over a distance of 200 km (four grid points) at roughly the same times.

This comparison with the results of the Eismint group shows that the space discretization and the solution methods can give somewhat different results even for a very simple geometry. The explicit method, which is a first order method and less accurate than the ADI methods, gives results that are considerably different from the Eismint result. The attempt to flux-correct the explicit method, to make it second order accurate, was not successful, and thus not possible to test if the result improves. The comparison above shows that the ADS-I scheme and the ADS-I upstream B scheme give very similar results to the published Eismint values. To further test the schemes computations were done on a very long ice sheet and the resulting ice thickness in the center compared with a 2D analytical solution. This comparison follows.

3.4.3 Analytical solution

Following Nye (1959b) the analytical solution for a two dimensional ice sheet can be found by assuming steady state and incompressibility, then

\[ Mx = h\bar{u}, \]  

(3.7)

where \( M \) is the accumulation function and \( \bar{u} \) is the average velocity over the depth. By using Glen’s flow law this velocity is (Paterson, 1994, page 252)

\[ \bar{u} = u_b + \frac{2A}{n+2} (\rho g \frac{dh}{dx})^n h^{n+1}. \]

There is no basal velocity so \( u_b \) is zero. With this average velocity Equation (3.7) can be integrated by taking the origin at the center of the bed and consider the right-hand half of the ice sheet of length \( L \),

\[ M^{\frac{1}{2}} \int_x^L x^{\frac{1}{2}} \, dx = \left( \frac{2A}{n+2} \right)^{\frac{1}{2}} \rho g \int_0^h h^{1+\frac{n}{2}} \, dh. \]

The height of the ice sheet is thus

\[ h^{2+\frac{1}{n}} = \frac{2}{\rho g} M^{\frac{n}{2}} \left( \frac{n+2}{2A} \right)^{\frac{1}{2}} \left[ L^{1+\frac{1}{n}} - x^{1+\frac{1}{n}} \right] \]

with same parameters as used before the height at the center is \( h_0^{2d} = 3575.06 \text{ m} \)

To make the numerical solution similar to a 2D solution the numerical ice sheet was extended 100 times in y-direction and a steady state computed with all the numerical schemes and different grid sizes. The resulting surface elevations at the center of the ice sheet for the numerical
computations and the analytical height are shown in Figure 3.10. With decreasing grid size the results of all the methods converge to the analytical solution, the explicit method has the slowest convergence. The time step is decreased when the grid size is decreased to ensure numerical stability. With grid size 25 km the time step is 1 year, and with grid size 10 km the time step is 0.5 year. This resulting center heights shown in Figure 3.10 indicate that the two schemes that compare best with the Eismint results also simulate the analytical solutions best. The difference between the results computed with 50 km and 10 km grid size is less than 1% which justifies the selection of a 50 km grid size in the Eismint computations, because of increased computing time with the smaller grid size. It is concluded that these schemes, the ADS-I and ADS-I upstream B, are the best to use and are used in all the following computations.

![Figure 3.10: Analytical 2D solution and numerical heights at the center of a very long ice sheet. The dot is for the analytical solution and the other symbols for the different numerical methods as noted.](image)

3.5 Determination of the flow and sliding parameters

The two equations (Equations 3.1 and 3.4) that describe the deformation and the sliding velocities were derived using the shallow ice approximation. Each equation has a parameter which can be tuned to fit the surface velocity measurements on Vatnajökull, the rate factor $A$ and the sliding parameter $C$.

The value $A$ in Glen’s flow law is the rate factor which relates the strain rate and the deviatoric stress. It depends on temperature and other properties of the ice itself, for example crystal size and orientation, impurity, water content and density. The value of this parameter has been measured in laboratory and a strong temperature dependence observed. The recommended value for $A$ at the melting point of ice is $6.8 \times 10^{-15} \text{s}^{-1} \text{kPa}^{-3}$ (Paterson, 1994). This value is about three times larger than a value that was obtained in a study on Vatnajökull (Chapter 5). That value for the flow law parameter was obtained by comparing measured surface velocity with a model that solves the full system of the Stokes equations. Similar values for the flow parameter of temperate glaciers were obtained by analogous comparison of surface velocity
measurements with calculations based on numerical flow models on two different glacier in the Swiss Alps (Hubbard et al., 1998; Gudmundsson, 1999) and one in Sweden (Albrecht et al., 2000).

The flow law parameter which is suitable for Vatnajökull is determined by comparing the computed deformation velocity (Equation 3.1) to the available surface velocity measurements. The ratio between deformation and sliding velocity in the measured values is, however, not known. Therefore, the upper bound for the deformation velocity is the lowest measured surface velocity at each location. The value for the parameter was adjusted until best fit with only the lowest measured values was obtained. This was done on each outlet glacier separately and the results are presented in the following section. It is not likely that the flow parameter is variable within the ice sheet and therefore the lowest value obtained on all the glacier is selected to represent the whole ice sheet.

After the flow law parameter has been determined an estimate for the sliding parameter $C$ (Equation 3.4) can be made by comparing the computed velocity to all other surface velocity measurements and adjusting the sliding parameter until best fit is obtained. It is evident from the measurements that sliding is not only spatially heterogeneous but also temporally variable. The measurements provide, thus, only and upper bound for the sliding law parameter. Due to the large variation in the measured velocities two estimates were made for the sliding parameter, one for the maximum velocity and another which represents intermediate velocities. There are many reasons for the sliding parameter to be spatially variable. For example, the bedrock geology, the bed topography and the amount of melt water at the interface of ice and bedrock influence basal motion. Since sliding is both spatially and temporally variable the suggested values for the sliding parameter, which are based on the velocity measurements and the sliding law used in the model, are perhaps speculative. It, however, gives a better representation of the overall flow field of the ice cap to introduce sliding than when sliding is omitted in the model.

### 3.5.1 The flow law parameter

The deformation velocity for the ice cap was computed by using Equation (3.1) and the available surface and bedrock data. This velocity field is shown in Figure 3.11. It shows a flow field that is rather uneven and has many relatively large jumps. This is due to high frequency noise in the surface data. It can be eliminated by using a low-pass filter on the surface data or simply by letting the flow model run for a short period of time leaving the surface free to evolve. The latter method was chosen and the flow model was run over a period of one year. The resulting surface geometry was used to calculate the velocity again and this new flow field is presented in Figure 3.12. It is clear from this figure that the surface irregularities have diffused, resulting in a smoother velocity field. The northernmost part of the ice cap has relatively low velocities as compared to the southern and eastern part.

The surface geometry, that resulted from the one year flow computation, was used to calculate the velocities which are compared with the velocity measurements to determine the flow law parameter $A$. The lowest measured velocity on each stake was compared with the modeled deformational velocity and the parameter tuned until the best fit was obtained. For each outlet glacier a flow parameter that minimizes the difference between the modeled and measured velocities was determined. The comparison is shown with crosses on Figure 3.13. The values for the flow parameter and the root-mean-square difference between the measured and modeled velocity along with the number of stakes on each glacier are given in Table 3.5. The values
Figure 3.11: Deformation velocity computed from the measured geometry of the ice cap.

Figure 3.12: Deformation velocity computed from the surface that has been adjusted by running the fbw model for a period of one year.
<table>
<thead>
<tr>
<th>Glacier</th>
<th>Value for $A$ $\times 10^{-15} \text{s}^{-1} \text{(kPa)}^{-3}$</th>
<th>rms $\text{m a}^{-1}$</th>
<th>Number of stakes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tungnaárjökull</td>
<td>4.36</td>
<td>7.83</td>
<td>21</td>
</tr>
<tr>
<td>Köldukvíslarjökull</td>
<td>1.83</td>
<td>2.65</td>
<td>6</td>
</tr>
<tr>
<td>Dyngjujökull</td>
<td>3.32</td>
<td>6.98</td>
<td>15</td>
</tr>
<tr>
<td>Brúarjökull</td>
<td>4.28</td>
<td>8.37</td>
<td>19</td>
</tr>
<tr>
<td>Eyjabakkajökull</td>
<td>2.60</td>
<td>2.49</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 3.5: The determined flow law parameter for the various glaciers. The rms is the root-mean-square difference between the modeled and measured velocities on each flow line.

are slightly different between the glaciers, but the order of magnitude is everywhere the same, they are all considerably lower than the recommended value for temperate ice. This indicates a somewhat higher viscosity or stiffer ice than previously attributed to temperate ice.

The best fit and the lowest value for the flow parameter were obtained on Eyjabakkajökull and Köldukvíslarjökull. These glaciers are rather small with relatively uneven bedrock topography (see Figures 2.3 and 2.6). The best fit with the measurements on the larger glaciers was found with a higher value for the flow parameter. This indicates that the measured surface velocity on these larger glaciers may contain a contribution from sliding at the bed. The magnitude of this contribution is not known, therefore, the lowest flow parameter obtained from all the glaciers is selected and it is assumed that the velocity exceeding the modeled velocity with this parameter is due to sliding at the bed. Comparison of the measured velocity and the modeled velocity computed with the selected flow parameter, $A = 2 \times 10^{-15} \text{s}^{-1} \text{kPa}^{-3}$, is shown with diamonds in Figure 3.13. This comparison shows that with this parameter the measured velocities are generally above the modeled velocities. This value for $A$ is similar to the value of $A = 2.3 \times 10^{-15} \text{s}^{-1} \text{kPa}^{-3}$ which was obtained in the study around the depression east of Grímsfjall using a full system model (Chapter 5).

### 3.5.2 The sliding parameter

The sliding parameter, $C$, in Equation 3.4, was determined by comparing all available velocity measurements on each location with the modeled velocity and adjusting the value of $C$ until a good fit was obtained. A lower and a higher estimate for the sliding parameter was established and they are presented in Figure 3.14. Before using the sliding parameters with the flow model the values were smoothed with a boxcar average with length 9 km so that no sharp changes appear in the value of the sliding parameter.

The total model velocity, computed with the selected flow parameter and these values for the sliding parameter, is compared with the measured surface on each of the outlet glaciers in Figures 3.15 - 3.19. These figures show that the flow model, with the selected flow and sliding parameters, presents the measured flow field relatively well.
Figure 3.13: Measured velocity plotted against the modeled velocity for the five outlet glaciers. Crosses show the model results for the fbw parameter that minimizes the difference between the modeled and measured velocity. Diamonds show model results with the selected fbw parameter, $A = 2 \times 10^{-15}$ s$^{-1}$ kPa$^{-3}$. (A) Tungnaárjökull. (B) Köldukvislarjökull. (C) Dyngjujökull. (D) Brúarjökull. (E) Eyjabakkajökull.
Figure 3.14: The determined sliding parameter for the various glaciers $(m\,a^{-1}\,Pa^{-3})$.

Figure 3.15: Computed velocities compared with the measure ones for Tungnaáriðjökull. The deformation velocity is drawn with a solid line. The combined deformation and sliding velocity, computed with the lower estimate for the sliding parameter is drawn with a dashed line, and with the higher estimate for the sliding parameter with a dotted line.
Figure 3.16: Computed velocities compared with the measured ones for Köl-dukvíslarjökull. The deformation velocity is drawn with a solid line. The combined deformation and sliding velocity, computed with the lower estimate for the sliding parameter is drawn with a dashed line, and with the higher estimate for the sliding parameter with a dotted line.

Figure 3.17: Computed velocities compared with the measured ones for Dyn-gjujökull. The deformation velocity is drawn with a solid line. The combined deformation and sliding velocity, computed with the lower estimate for the sliding parameter is drawn with a dashed line, and with the higher estimate for the sliding parameter with a dotted line.
Figure 3.18: Computed velocities compared with the measured ones for Brúarjökull. The deformation velocity is drawn with a solid line. The combined deformation and sliding velocity, computed with the lower estimate for the sliding parameter is drawn with a dashed line, and with the higher estimate for the sliding parameter with a dotted line.

Figure 3.19: Computed velocities compared with the measured ones for Eyjabakkajökull. The deformation velocity is drawn with a solid line. The combined deformation and sliding velocity, computed with the lower estimate for the sliding parameter is drawn with a dashed line, and with the higher estimate for the sliding parameter with a dotted line.
3.6 Melting at the bed in Grímsvötn

Vatnajökull covers the most active volcanic areas in Iceland. During last century several volcanic eruptions have occurred underneath the ice cap causing vast outburst flood (jökulhlaups) to emerge from the glacier. A detailed account of the activity and development of the geothermal areas is given by Björnsson (1988). The most active geothermal area is Grímsvötn which is located in the center of the ice cap. The heat from Grímsvötn sustains a subglacial lake from which jökulhlaups originate approximately every 5-10 years. The area contributing ice flow to the subglacial lake is approximately 160 km$^2$ and the area affected by geothermal activity is about 60 km$^2$, which is roughly the same size as the visible Grímsvötn depression. Björnsson (1988) concludes that the long term average for the rate of accumulation of ice over the total surface of the ice drainage basin equals the long term average for the mass released in jökulhlaups. He estimates the amount of ice melt, due to the geothermal activity, to be $4 \times 10^{11}$ kg a$^{-1}$. That amounts to about 7 m a$^{-1}$ ice melt over the 60 km$^2$ area. This is a substantial melt that can not be ignored in the flow model. Björnsson and Guðmundsson (1993) give account of the evolution of the heat output from the geothermal area over a 70 year period. They show that there is a considerable fluctuation in the heat flux which is closely related to volcanic activity. With the model computations it is not intended to simulate the Grímsvötn area, but the whole ice field, and thus a constant heat flux is assumed.

Ideally, the flow model should be used to obtain an independent estimate for the energy needed to melt the ice at the bed in the Grímsvötn area by adjusting the applied melt to a value that maintains the observed surface depression. However, as the flow model does not settle to a steady state which is similar to the present day ice cap, this is not possible.

Model runs were done with the estimated 7 m a$^{-1}$ over an area of 60 km$^2$ but it appeared that this is an overestimate, as the area becomes ice free. This could possibly be due to an underestimate in the accumulation in this area. The area was thus reduced to 35 km$^2$, which is approximately
the area of the subglacial lake, but the melt kept at 7 m a$^{-1}$. The grid points where basal melt is included in the computations are shown with crosses in Figure 3.20. This melt estimate in the flow model produces a more realistic ice cover in this area. The sensitivity of the model results to this estimate are analysed in Chapter 4.2.4.

In addition to Grímsvötn, there are two ice cauldrons located some 10-15 km northwest of Grímsvötn, Skaftárkatlar. The eastern one drains approximately every 2-3 years and the western one every 3-4 years in the so called Skaftárhlaups. These are not included in the model as their influence is only in several grid-points causing a surface depression with dimensions that cannot be resolved with the SIA model applied on the ice cap. Influences of extraordinary events like the eruption in Gjálp volcanic fissure north of Vatnajökull (Gudmundsson et al., 1997) which caused a large depression in the ice surface due to the eruption itself as well as large ice canyon in the flood path of the subsequent jökulhlaup are excluded as well in flow model for the whole ice cap. The evolution of the canyon has been modeled with a full system model (Appendix A) due to the small spatial dimension which cannot be resolved with the SIA model.
Chapter 4

Selection of numerical scheme and sensitivity tests

The first computations with the tuned model reveal a mass inconsistency when the ADS-I method is used. During volume increase the net specific mass balance is negative. It appears that this problem arises at the boundary within the ice sheet, on the nunatak points. In this chapter this mass inconsistency is analysed and it is shown that by using the upstream numerical methods it vanishes. The results of the two ADS-I numerical schemes, the first, mass inconsistent one, and the upstream B, are compared on a simple ice sheet geometry with one nunatak point penetrating the ice sheet and shown that by using the ADS-I upstream B method mass is conserved.

The sensitivity of the result to the model parameters determined in the previous chapter is then analysed. The grid size, the rate factor, the amount of sliding and the melt in Grímsvötn is altered and the model result compared with reference model runs. It is shown that the grid spacing, of 1 km, is sufficiently small to capture the characteristics of Vatnajökull and large enough to keep the computing time within reasonable limit. The flow law and sliding parameters have influence on the thickness of the ice cap through the amount of flux, and due to the mass-balance elevation feedback the volume evolution is sensitive to changes in the value of these parameters. The amount of melt in Grímsvötn has also significant influence on the volume evolution of Vatnajökull. It is not possible to determine the amount from the model computations because the ice cap does not settle to a steady state similar to the present ice cap. The estimate from Chapter 3 is thus used for the melt in Grímsvötn.

4.1 Selection of numerical scheme

4.1.1 Volume and net mass balance evolution

The four numerical schemes, with the determined flow and sliding parameters and melting at the bed in Grímsvötn, along with the mass balance parameterization presented in Chapter 6, were applied on the geometry of Vatnajökull. The volume evolution for two values of $\Delta$ELA (see Chapter 6 for explanation of the mass balance distribution and $\Delta$ELA) and the mean specific net balance during these model runs are shown in Figures 4.1 and 4.2. The volume of the ice
Figure 4.1: Volume evolution computed with two values of $\Delta ELA$ and the four different numerical methods.

Figure 4.2: Mass balance evolution computed with two values of $\Delta ELA$ and the four different numerical methods.
cap computed with the ADS-I scheme is growing. With $\Delta \text{ELA}=79 \text{ m}$ the volume increase accelerates after about 6000 years and the ice cap grows seemingly without bounds. With $\Delta \text{ELA}=80 \text{ m}$ the volume of the ice cap is slowly increasing. The volume evolution with the two upstream methods is different. Instead of increasing in volume, the ice cap retreats and diminishes to several small ice caps and settles to a steady state. The mean specific net balance shows differences between the ADS-I method and the upstream methods. The net balance of the decreasing ice caps computed with the upstream methods is initially negative but as the steady state is reached the net balance becomes slightly positive. During the growth of the ice cap computed with the ADS-I method the net balance is negative. This negative mass balance provokes the suspicion that the ADS-I method is not mass conserving and a further investigation of the volume change and net balance during the computation is carried out. The attention is drawn towards the ice boundary where the cause for this mass inconsistency is found.

4.1.2 Flow at the ice margin

In the numerical codes no special attention is given to the ice boundary. Whenever melting is more than the amount of ice at a particular point and the calculated ice surface is below the bedrock the value is simply raised back to the bedrock elevation and the point becomes ice free. In the code this is done by checking the elevation of all points after each time step and raising the ones that are below bedrock back to the bedrock level. This seems reasonable to do, as it provides a natural constraint on the glacier boundary; where melting exceeds the inflow at the boundary point it becomes ice free and the glacier retreats. Neither the volume change nor the mass balance of this particular point is counted as it is now an ice free point. This will lead to a somewhat positive mean specific mass balance when a glacier is in a steady state. The negative mass balance at the first ice free point outside the glacier is not counted in the net balance sum, but it does contribute to the total mass balance by melting away the ice flowing to this point out of the boundary of the glacier.

There is another boundary which is not as simple for the numerical code to treat. This is at the ice free points within the ice cap, the so called nunatak points. In these points the mass balance can be positive but the computed ice flow out of this point can be larger than the available volume. The mass inconsistency can also appear in ice covered points where the slope is steep so the SIA flows are large. The computed outflow could then exceed the sum of the inflow, the accumulated mass over the area, and the volume that already is in the point. To avoid negative thickness to appear as a solution for this situation the following inequality must hold in each point

$$q_{\text{out}} \Delta x \leq q_{\text{t}} \Delta x + q_{\text{in}} \Delta x + \Delta V. \quad (4.1)$$

Each term of this equation is shown in Figure 4.3. To ensure that the inequality holds the flow must be explicitly computed in each time step. This is not done in the Semi-Implicit schemes used to compute the ice sheet evolution and thus not possible to apply this inequality in the computations. However, the analyses below and the results shown in Table 4.3 indicate that the error that might be due to failure of inequality 4.1 is not large in ice covered points and thus ignored. The Flux Correction formalism suggested by Schär (Schär and Smolarkiewicz, 1996) should be compared to the computations with the ADS-I upstream method as well as method called Source Linearization for Always- Positive Variables (Patankar, 1980, page 145). This is a future project. The more demanding mass inconsistency appears in the nunatak points and is analysed below.
As shown in Figure 3.1 for the ADS-I method, the constant D, slopes and thickness are computed from the surrounding points. This means that when a single or a few points are ice free within the ice cap, it is possible to compute ice flow out of this point. Ice is transported downslope from this point and the calculated ice surface in the ice free point becomes below the bedrock elevation. Before the next time step is taken this point is raised back up to the level of the bed. In doing so a volume of ice, which is not available, has been created and transported to the adjacent point. This can create a considerable amount of volume during the computation. The upstream method, on the other hand, uses only the upstream thickness to compute the flow, which ensures that out of an ice free point no ice flow is possible, or where the ice is thin the ice flow is less than when the average between the adjacent thicknesses is used to compute the ice flow. The importance of this difference in the numerical schemes is shown below by comparing each term of the ice sheet equation using both methods.

Figure 4.4 shows the ice free points where mass flux is computed (both in and out flow) using the ADS-I method. These are all the points around the glacier boundary and the nunatak points. Points marked with an asterisk are points where the solution of the numerical code is considerably below the bedrock level, the other points are only very slightly below it, or there is mass flowing into the point. There are in fact two surfaces created after each time step, the numerical solution of the ice sheet equation (Eq. 3.5) and another surface which is the former corrected.
Table 4.1: The sum of each term in the balance equation over different areas for the initial surface geometry. The unchanged surface elevation (solution of the equation system) is used. First column lists values summed over only ice covered points, second column over all the area where flux is computed and the last column over only ice free points.

* These points are shown as dots on Figure 4.4

Comparison of the values in the two tables reveals the mass inconsistency. In Table 4.1 the terms balance each other, as expected, since this is the solution of the equation system. The sum of all the terms on the right hand side (second last line) matches the sum of the elevation changes (second line). This is not the case in Table 4.2. The surface used to compute the fluxes in this table is the one that has been altered so that points that were calculated below bedrock have been raised. For the ice covered points the difference is -19.5 m which averaged over the whole surface equals 0.0024 m. The sum of the mass flux in the ice free points is positive which leads to a surface lowering, there is ice flowing out of the ice free points, which is unrealistic and is caused by the numerical scheme used in the calculation. The upstream schemes ensure that no flow can be computed out of points that have no thickness. The analysis was repeated with the Explicit upstream method and the result computed with the changed surface is shown in Table 4.3. In this case there are much fewer ice free points where flux is computed and the value of the flux in these points is everywhere negative, which means that ice is flowing into the ice free points. No ice flows out of ice free points, as was the case with the ADS-I method. The ice free points have a large negative mass balance which removes all the ice that is transported into these
Table 4.2: The sum of each term in the balance equation over different areas for the initial geometry. The surface that has been raised to the bedrock elevation is used. First column lists values summed over only ice covered points, second column over all the area where flux is computed and the last column over only ice free points.

<table>
<thead>
<tr>
<th>Term</th>
<th>Ice covered points</th>
<th>All points</th>
<th>Ice free points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of points</td>
<td>8239</td>
<td>9229</td>
<td>990</td>
</tr>
<tr>
<td>$\sum \frac{\Delta h}{\Delta t}_{i,j}$</td>
<td>3930.0877</td>
<td>3961.8501</td>
<td>31.7624</td>
</tr>
<tr>
<td>$\sum \nabla q_{i,j}$</td>
<td>-4189.5323</td>
<td>$-6.925 \times 10^{-12}$</td>
<td>4189.5323</td>
</tr>
<tr>
<td>$\Sigma B_{i,j}$</td>
<td>-1.9304593</td>
<td>-2836.6985</td>
<td>-2834.7681</td>
</tr>
<tr>
<td>$\Sigma M_{i,j}$</td>
<td>238.000</td>
<td>245.000</td>
<td>7.000</td>
</tr>
<tr>
<td>$\Sigma - \nabla q_{i,j} + B_{i,j} - M_{i,j}$</td>
<td>3949.6018</td>
<td>-3081.6985</td>
<td>-7031.3004</td>
</tr>
<tr>
<td>Difference</td>
<td>-19.5141</td>
<td>7043.5486</td>
<td>7063.0627</td>
</tr>
</tbody>
</table>

* These points are shown as dots on Figure 4.4

Table 4.3: The sum of each term in the balance equation over different areas for the initial surface geometry using the explicit upstream scheme. First column lists values summed over only ice covered points, second column over all the area where flux is computed and the last column over only ice free points.

<table>
<thead>
<tr>
<th>Term</th>
<th>Ice covered points</th>
<th>All points</th>
<th>Ice free points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of points</td>
<td>8239</td>
<td>8735</td>
<td>496</td>
</tr>
<tr>
<td>$\sum \frac{\Delta h}{\Delta t}_{i,j}$</td>
<td>-402.4804</td>
<td>-247.0341</td>
<td>155.4463</td>
</tr>
<tr>
<td>$\sum \nabla q_{i,j}$</td>
<td>164.0807</td>
<td>$-1.480 \times 10^{-12}$</td>
<td>-164.0807</td>
</tr>
<tr>
<td>$\Sigma B_{i,j}$</td>
<td>-1.9305</td>
<td>-1386.8405</td>
<td>-1384.9100</td>
</tr>
<tr>
<td>$\Sigma M_{i,j}$</td>
<td>238.000</td>
<td>238.000</td>
<td>7.000</td>
</tr>
<tr>
<td>$\Sigma \nabla q_{i,j} + B_{i,j} + M_{i,j}$</td>
<td>-404.0111</td>
<td>-1624.8405</td>
<td>-1220.8293</td>
</tr>
<tr>
<td>Difference</td>
<td>1.5308</td>
<td>1377.8064</td>
<td>1376.2756</td>
</tr>
</tbody>
</table>

points and causes the large difference between the left- and right hand side of the equation for the entire grid (middle column) when the points are raised back to the level of the bedrock. This can not be avoided without making the time step so small that the mass balance matches the smallest amount of flow into the boundary points. This is not required, but it must be noted that for a glacier in a steady state the mean specific net balance over the ice covered points can be positive due to this effect. The difference between the left- and right hand sides of the equation in the ice covered points is now very small, only 1.5 m over 8239 points which is equivalent to 0.00019 m in each point. The use of an upstream method removes the problem of flow being computed out of ice free points. It is concluded that it is necessary to use an upstream numerical code to prevent the mass inconsistency which appeared when the the ADS-I code was applied.
4.1.3 Flat bed with a steep mountain in the center

The above analyses reveals that the mass inconsistency that appeared when the ADS-I method is used is caused by nunatak points, sticking out of the ice surface, and the inconsistency vanishes when an upstream method is used. To analyse this further the two methods that converge fastest towards the analytical solution (ADS-I and ADS-I upstream B, see Figure 3.10) are used to compute a simple ice sheet with one nunatak point. A steep mountain is created at the center of the flat bed geometry used to perform the Eismint experiments. The base of the mountain is 200 km in all experiments, and the height is varying between 3200 and 6000 m. The fixed boundary experiment was performed and a constant mass balance of 0.3 m a\(^{-1}\) was applied in all experiments. Computations were done with the ADS-I method and the ADS-I upstream method B, with grid sizes of 50 km (same as in the Eismint experiments) and 10 km. The resulting surface geometry from the ADS-I upstream B method computed with both grid sizes is shown in Figure 4.5.

![Figure 4.5: Surface geometry of the experiment with a mountain in the center computed with the ADS-I upstream B method. The left panel shows the results of the computation with dx = 50 km and the right panel with dx = 10 km, surface is shown every 500 years. The height of the mountain is 4000 m.](image)

A closer look of the center region, computed with both methods with a 4000 m high mountain, is given in Figures 4.6 and 4.7. The evolution of the center point height for all experiments is shown in Figures 4.8 and 4.9, and the values of the center height and the thickness is given in Table 4.4. The size of the time step is varying. In the computations with 50 km grid size results are the same with dt = 10 and 5 years. With a 10 km grid size the time step is 0.5 years.

The computations with the ADS-I method and grid size 50 km show that the height of the mountain controls whether it becomes ice free or not. The evolution of the center point elevation computed with a 4000 m high mountain (Figure 4.6 and second dashed line from above in Figure 4.8) is thickening at the beginning, then ice starts to flow down the slopes and the ice thins for a short period, then it thickens again and finally ice flows away and the top becomes ice free. The steady state solution for the ice surface is below the bedrock, the computed ice flow out of the point is larger than the mass balance. For dx = 50 km and dt = 5 years the solution is 25 m below the bedrock, after each time step the ice surface is raised back to the bedrock elevation and mass, which is not available, is transported to the adjacent points. This model is, thus, not mass conserving. With a grid size of 10 km and dt = 0.5 years the flow is not larger than the mass balance and the steady state ice thickness is 161.55 m (see Table 4.4). For a lower mountain, with height closer to the steady state thickness without a nunatak, the evolution of
Figure 4.6: Surface geometry of the center region computed with the ADS-I method. The left panel shows the results of the computation with $dx = 50$ km and the right panel with $dx = 10$ km, surface is shown every 500 years. The height of the mountain is 4000 m. The surface evolution of the center point is shown in Figure 4.8

Figure 4.7: Surface geometry of the center region computed with the ADS-I upstream B method. The left panel shows the results of the computation with $dx = 50$ km and the right panel with $dx = 10$ km, surface is shown every 500 years. The height of the mountain is 4000 m. The surface evolution of the center point is shown in Figure 4.9
Figure 4.8: Surface elevation of the center point computed with the ADS-I method. Bed is flat with a mountain in the center of 3200, 3400, 4000 and 4500 m height. Solid lines are computed with 10 km grid size and the dashed lines with 50 km grid size.

Figure 4.9: Surface elevation of the center point computed with the ADS-I upstream B method. Bed is flat with a mountain in the center of 3200, 4000, 4500, 5000 and 6000 m height. Solid lines are computed with 10 km grid size and the dashed lines with 50 km grid size.
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<th>Ice (m)</th>
<th>Ice thickness at center</th>
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</table>

Table 4.4: Surface heights and thicknesses of ice at the center point for all mountain experiments computed with both methods. The evolution of the center point is shown in Figures 4.8 and 4.9.

The center point is similar at the beginning, the point becomes ice free, but as the ice around the center becomes thicker the top is covered with ice again (the two lower dashed lines in Figure 4.8), these model runs are mass conserving. Computations with a smaller grid size result in an ice covered center and are mass conserving. The mass accumulation at the center point in the beginning is not as large and the ice sheet settles faster to a steady state which covers the bedrock point.

The computations with the upstream method have similar initial evolution for the center height, with growing ice thickness in the beginning and then thinning, but the bedrock point is covered with ice in all computations, even though the bedrock point is raised very high. The reason is that this method computes flow in proportion to the upstream thickness, not the average between the upstream and downstream, and thus not as large flow is computed out of the thin ice region. This numerical method is always mass conserving.

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4.1.4 Results

The computations with the ADS-I method are not mass conserving when nunatak points are present. The model runs done on Vatnajökull (Figure 4.1) show that the error made by using this method can be considerable. By using the ADS-I upstream B method this mass inconsistency is no longer present. The upstream method ensures that no flow is computed out of ice free points and in areas where there is thin ice but steep slopes the flow is proportional to the upstream thickness. Comparison of the results from these two methods without nunatak points (Figure 3.10) shows that there is only small height difference (44 m with dx= 50 km which is 1.2 % of the total thickness) between the solutions and they converge towards the analytical solution. The difference is caused by the fact that not exactly the same grid points are used to compute the slopes and thicknesses. The difference increases with steeper slopes.

In all the following computations the ADS-I upstream B method is used. Due to the boundary treatment at the terminus it can be expected that for a glacier in a steady state the mean specific net balance is positive.

4.2 Sensitivity tests

In this section the model sensitivity to grid spacing, the rate factor, A, the sliding parameters as well as the amount of melt in the Grímsvötn area is examined. Several model runs with altered parameters are compared with the model runs computed with the selected values.

4.2.1 Grid spacing

The data on the bedrock and surface geometry has been compiled on a 500 m grid. This grid spacing depicts the geometry in some detail. The grid spacing influences the maximum time step in model computation. The von Neumann stability analysis gives that \(|v \frac{\Delta t}{\Delta x}| < 1\), which indicates that a particle should not travel more than on spatial step-size \(\Delta x\) in one time step \(\Delta t\) (Courant-Friedrichs-Lewy condition (Fletcher, 1988)). The 500 m grid is detailed and requires a long computing time. For the purpose of modeling the overall behavior of Vatnajökull with the SIA model such detailed map is not necessary. It was thus decided to work with a coarser grid, which still represents the surface and bed geometry well, and a 1 km grid was generated from the first grid. 7000 years long model run with the coarser grid, with time step of 0.02 years, takes about 37 hours on sun blade work station and 60 hours on sun ultra 2 station. To examine whether the 1 km grid spacing causes loss of accuracy in the model results, a coarser grid, with grid size 2 km, was generated and several model runs done. With the coarser grid, the bedrock is not as precisely depicted, slopes are less steep, and the are of melt in Grímsvötn has slightly different extent. Instead of melt in 35 gridpoints with a size of 1 km\(^2\) there is melt in 9 points of 4 km\(^2\) each. Computations were done with \(\Delta ELA=55\) m, which is close to the critical value discussed in Chapter 8, with and without sliding, and with and without melting in the Grímsvötn area.

Figure 4.10 shows the volume evolution computed with two grid spacings. The evolution is almost identical during the first few hundred years of computation, but the difference between the results increases with time. In each time step the ice cap computed with the 2 km grid
Figure 4.10: Volume evolution computed with two grid spacings. The lines with the steepest slope are computed with no sliding and no melting in Grímsvötn, the lines in the middle are computed with maximum estimate for sliding and no melting and the lowest lines include melting in Grímsvötn.

Figure 4.11: Volume evolution computed with the 2 km grid and with three different values of ΔELA.
spacing grows less due to the less steep surface slopes and thus the difference between the solutions accumulates. The discrepancy between the results is small in the beginning and only reflects the difference in the resolution of the grids. To test the general response of the ice cap computed with the 2 km grid spacing, several model runs were done with different values for the \( \Delta \text{ELA} \). The volume evolution of these model runs is shown in Figure 4.11. It shows similar volume evolution as observed in the computations with the 1 km grid. The ice cap either grows or contracts.

It is concluded that the 1 km grid spacing is sufficiently small for calculating the volume evolution of Vatnajökull.

### 4.2.2 Flow parameter

Several model runs were done with different values of the flow law parameter \( A \). Computations were done using the maximum estimate of the sliding and with two different values for \( \Delta \text{ELA} \), which are close to the critical value of \( \Delta \text{ELA} \). The resulting volume evolution is shown in Figures 4.12 and 4.13. The lowest value of the flow parameter \( A \) is the one determined for the whole ice cap (see Chapter 3.5.1). This value is considerably lower than the value recommended for temperate ice (Paterson, 1994), which is the highest value tested. The middle value corresponds to the value that fits the velocity measurements on Brúarjökull, that probably include some portion of sliding at the bed. The velocity is linearly dependent on the flow law parameter. The mass flux thus increases with a higher value of \( A \). Increased mass flux lowers the height of the steady state ice sheet and due to the mass-balance elevation feedback a higher value of \( A \) results in a smaller steady state volume. Both figures show similar decrease in volume with increased value of the flow parameter. The steady states computed with the same flow parameter but different value of \( \Delta \text{ELA} \) have different volumes. The value of the \( \Delta \text{ELA} \) also influences the size of the steady state ice cap. With a lower ELA the steady state is larger, as expected since the accumulation area increases when the ELA is lowered.

These model runs show that the value of the flow law parameter has a significant influence on the steady state volume of the ice cap. The lowest value which is used for Vatnajökull was determined by comparing model results with measurements and it is considered to be the most correct one for the whole ice cap.

### 4.2.3 Amount of sliding

The value for the parameter in the Weertman type sliding law was determined by comparing the model results with the measured velocity (see Chapter 3.5.2). On each measurement location there are several velocity measurements available that show temporal variation in the velocity. To account for this variation two estimates for the value of the sliding parameter were established. The volume evolutions shown in Figure 4.14 are computed with the flow law parameter determined for the whole ice cap, \( \Delta \text{ELA}=60 \text{ m} \), and no sliding, the lower estimate for the sliding parameter and the higher estimate for the sliding. This figure shows that the amount of sliding in the model affects the volume evolution significantly. The sliding affects the flux of the ice and the resulting steady state volume in a similar way as the flow parameter. Decreased, or no sliding in the model decreases the flux and the steady state thickness of the ice increases. The mass-balance elevation feedback in the mass balance input causes the ice cap to grow rapidly when it thickens.
Figure 4.12: Volume evolution computed with $\Delta ELA = 55$ m and three different values for the flow law parameter $A$.

Figure 4.13: Volume evolution computed with $\Delta ELA = 60$ m and three different values for the flow law parameter $A$. 
It must be noted that outside the area distribution of the sliding parameter shown in Figure 3.14 no sliding is included. This means that when the ice cap has extended outside this margin the modeled sliding is not realistic. The volume evolution is thus not meaningful for the lower estimate of the sliding after the ice cap has grown beyond this area.

These results emphasize the fact that the amount of sliding included in the model significantly alters the volume evolution. Further investigation of sliding processes are required to enable sliding and its seasonal, or long term, changes to be incorporated in a more physically correct way in the model. For the purpose of modeling the Vatnajökull ice cap, and its stability, this preliminary estimate for the sliding amount describes the sliding velocity sufficiently well, as it reproduces the measured surface velocities.

![Figure 4.14: Volume evolution computed with ΔELA = 60 m and the three estimates for the sliding amount.](image)

### 4.2.4 Melting in Grímsvötn

The basal melt rate in the Grímsvötn area was determined in Chapter 3.6. The estimate for the melt rate is based on the estimate for the heat output from the subglacial geothermal area (Björnsson, 1988). For simplicity the heat flux is assumed constant in these computations even though the variation is considerable (Björnsson and Guðmundsson, 1993). The estimated basal melt in the model, due to the geothermal heat, is about 7% of the total surface ablation. It can have a significant influence of the volume evolution of the ice cap.

To assess the effect of basal melt in Grímsvötn a model run without melting was done and compared to the one that includes melting. The volume evolution for both runs is shown in Figure 4.15 and the surface geometries, after 6500 years, are shown in Figure 4.16.

These figures show that the basal melt in Grímsvötn area has an important effect on the volume evolution of the ice cap. Without the basal melt the initial retreat is smaller and the following growth of the ice cap is significantly faster. Comparison of the ice cap geometries in Figure 4.16, that show the geometry of the ice cap after 6500 years, reveals that not only the area around the depression, but the whole ice cap is affected. The outlet glaciers to the west and
south of Grímsvötn are primarily affected, which could have been expected as the Grímsvötn accumulation area is closely related to the accumulation area of these glaciers. If the basal melt in Grímsvötn is turned off all the major outlet glaciers grow faster than they do when the basal melt is included. This demonstrates the importance of the geothermal area within Grímsvötn for the overall balance of the ice cap.

In Chapter 8 it will be shown that the modeled ice cap does not settle to a steady state that is similar in size as the present ice cap. It is thus not possible to adjust the basal melt to an amount which would maintain the visible surface depression.

Figure 4.15: Volume evolution computed with $\Delta ELA = 55$ m, maximum sliding and with and without basal melt in the Grímsvötn area.
Figure 4.16: The surface geometry after 6500 years with (top figure) and without (bottom figure) basal melt in Grímsvötn area. The Grímsvötn area is shown with a circle. Both ice caps are growing. The volume evolution of the model runs is shown in Figure 4.15
Chapter 5

Mass balance input

Dynamical models of the flow regime of glaciers are frequently employed to analyze their present state and estimate their response to climate changes. The paramount controlling factor for these models is the boundary condition at the ice-sheet surface, the surface mass balance. Hindmarsh (1990) pointed out that the ice sheet is primarily sensitive to the ELA slope and the ratio of ablation to accumulation and has only a weak dependence upon the rate factor, which presents deformation and temperature, and the magnitudes of the accumulation and ablation. Moreover, model studies have shown that the mass-balance-elevation feedback has a very decisive influence on the stability and the size of ice sheets (Weertman, 1961; Oerlemans, 1981). This chapter describes a parameterization of the observed mass-balance distribution, which contains the basic features of climate control, and is applied with the ice-dynamic model in the following chapters.

Several methods have been developed to quantify the mass-balance distribution on the ice sheet surface and the relationship to climatic variability. The most commonly applied model is the so called degree-day model which determines melting of snow and ice as a function of positive air temperature at a meteorological station (for example: Reeh, 1991; Braithwaite, 1995; Johannesson et al., 1995; Hock, 1999). Another method computes the melt that occurs at the surface of a glacier by considering the energy flux to the surface. Energy can be transferred to the surface by radiation in the form of short- or long wave radiation fluxes, or via turbulent exchange of latent and sensible heat (for example: Braithwaite and Olesen, 1990; van de Wal and Oerlemans, 1994). Both these methods have been widely used on many different glaciated areas. The research project TEMBA and the succeeding project ICEMASS have been working towards establishing such an energy balance model for Vatnajökull, Iceland (de Ruyter de Wildt et al., in press). The data required for the model calculations are not readily available in all areas. It is also difficult to determine the variation of input data and the model parameters away from the measurement location over the area covered by the ice cap.

The model presented here is not an attempt to relate temperature or other global variables to mass balance but, rather, set forth a model which is capable of describing the current mass-balance distribution of Vatnajökull, which will be used to force a dynamical flow model. This is a parameterization of the mass balance based on measurements that have been collected over the ice cap during the period 1992 – 2000 (Björnsson et al., 1998b, 2002a) The dependence of ELA on location is presented with a linear function. Two mass balance gradients are determined, one above the ELA and the other below it. Temporal variation in mass balance is described with a shift in the height of the equilibrium line. The parameters in this model are determined with an optimization procedure.
5.1 Parameterization of the mass balance

Two characteristics of the mass balance data are observed. The first is the dependence of the ELA on the distance from the coast. An approximation of this attribute is to describe the ELA with a linear plane:

\[
\text{ELA}(x, y) = a_0 + a_1 x + a_2 y,
\]

where \(x\) and \(y\) are the coordinates of the rectangular transformation for Vatnajökull. How well this describes the actual data is tested by analysing the distribution of the residuals.

The second is the linear altitude dependence of the mass balance with a larger gradient below the ELA than above it. The relationship between annual mass balance and the ELA makes it possible to translate a shift of the ELA into change in mass balance (Ohmura et al., 1992). Here it is assumed that the temporal variation of the mass-balance distribution can be presented with a shift of the ELA plane by a constant value; \(\Delta \text{ELA}(t)\). The mass balance can then be computed as:

\[
\hat{b}(x, y, z, t) = (z - \text{ELA}(x, y) + \Delta \text{ELA}(t)) \times \left\{ \begin{array}{ll}
\alpha^a, & \text{above ELA} \\
\alpha^b, & \text{below ELA} \end{array} \right.
\]

This model for the mass-balance distribution has six parameters, three in the equation for the ELA plane, which includes the time dependent shift, two mass balance gradients and a maximum value for the surface mass balance, which is set to the value 3.5 m a\(^{-1}\), is incorporated. These parameters are determined with optimizing procedure by finding the values which minimize the sum of squares of the residuals, i.e. the difference between the model output and the measured mass balance

\[
\chi^2 = \sum_{i=1}^{N} \left[ \frac{b_{i\text{meas}} - \hat{b}(x_i, y_i, z_i, t_i)}{\sigma_i} \right]^2,
\]

where \(\sigma_i\) is the error in each measured value. This is a nonlinear least-squares problem which was solved with the Levenberg-Marquardt method (Press et al., 1996). This was done iteratively, first the linear parameters for the ELA plane \((a_0, a_1, a_1)\) and the mass balance gradients \((\alpha^a, \alpha^b)\) were determined for each year separately, then the shift in the ELA plane \(\Delta \text{ELA}\) was determined for each year, also by minimizing the residues. With these values of \(\Delta \text{ELA}\) new parameters were determined for all the data simultaneously and with the new parameters the shift was determined again, this was done until convergence was reached. The resulting parameters for the mass balance model are given in Table 5.1. The ELA planes for each of the years are shown in Figure 5.1. The ELA planes for each of the years are shown in Figure 5.1. The best fit to the data yields the ELA plane tilted towards the ocean, parallel to the coast. The direction and the tilt of the plane is given in the two last columns of Table 5.1. The tilt is less than one degree and the direction is towards the coast so that the lowest ELA is at the coast. With these parameters the mass balance distribution for each year is computed. The measurements of the mass balance are plotted against the modeled values in Figure 5.2. This figure shows that the best fit is for the values close to zero mass balance. The difference between the model and the measurements is shown as circles in Figure 5.3. The modeled values are subtracted from the mass balance measurements, red circles indicate locations where
the model yields higher values than the measured and blue circles indicate locations where the model yields lower values. No systematic error can be detected such that some areas of the ice cap have large errors, the error is randomly distributed over the ice cap. Figure 5.4 shows the distribution of the errors for each year. The distribution of the measurements is, however, not uniform over the whole ice cap, as can be seen in Figure 5.3. This could limit the possibility of the model to present the actual mass balance in the areas where no measurements are available.

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</table>

Table 5.1: The resulting parameters for the mass balance model, the values minimize the sum for each year separately. The last two columns, direction and slope refer to the orientation (to the right from north) and the tilt of the ELA plane, shown in Figure 5.1. The last line in the table lists the values for the model that fits all the data simultaneously, with a shift in the ELA, shown in Figure 5.8.

As described above the model with the shift in the ELA, \(\Delta\text{ELAs}\), is fitted with all the data simultaneously resulting in the parameters shown in the last line of Table 5.1. A reference mass balance distribution is defined so that the mean specific net balance for the present ice cap is zero. This reference mass balance distribution is shown in Figure 5.5. The accumulation area ratio (AAR) for this distribution is 0.65. The values of the \(\Delta\text{ELAs}\) are determined relative to this reference. To assess how well this model presents the measured mass balance, a mass balance is computed corresponding to each measurement. The measurements of the mass balance are plotted against the modeled values in Figure 5.6. This figure shows that the model reproduces the measurements relatively well, with a correlation between the modeled and the measured values of \(r = 0.96\). The model is not capable of reproducing the most extreme values of measured accumulation or ablation. The distribution of the residuals between the model and the measurements is shown in Figure 5.7. It is a Gaussian shaped distribution which is taken as an indication that there are no significant systematic errors in the model. It is concluded that this parameterization for the mass balance which describes the mass balance with a piece wise linear dependence on the elevation and the ELA with a simple linear approximation is successful in computing the observed mass balance distribution.
Figure 5.1: Resulting ELA planes for each year separately. The straight lines are level lines of the plane with equidistant levels of 100 m.
Figure 5.2: The measured mass balance plotted against the modeled values for each year separately.
Figure 5.3: The distribution, drawn as circles, of the difference between the modeled mass balance values and the measured for each year separately. Red circles indicate locations where the model yields higher values than the measured, and blue circles where the model is lower.
Figure 5.4: Distribution of the residuals between the model and the measurements for each year separately. The difference is grouped into 0.25 m bins.
Figure 5.5: Mass balance distribution for Vatnajökull computed with an ELA which gives zero average net balance. The accumulation area ratio is 0.65.

Figure 5.6: All the measured mass balance values plotted against the corresponding modeled values.
Figure 5.7: Distribution of the residuals between the model and the measurements. The difference is grouped into 0.1 m bins.

5.2 Discussion

The mass balance gradients determined with the regression model are $0.0040$ m water equivalent $a^{-1}m^{-1}$ (m w.e. $a^{-1}m^{-1}$) above the ELA and $0.0082$ m w.e. $a^{-1}m^{-1}$ below the ELA (see Table 5.1). These values are comparable with values measured on two glaciers in the Swiss Alps 1993 - 1999. During this time Griesgletscher had mass balance gradients in the range $0.0010 - 0.0015$ m w.e. $a^{-1}m^{-1}$ and $0.0057 - 0.0080$ m w.e. $a^{-1}m^{-1}$ and Silvretta glacier had the values $0.0022 - 0.0026$ m w.e. $a^{-1}m^{-1}$ and $0.0071 - 0.0092$ m w.e. $a^{-1}m^{-1}$ above and below the ELA, respectively (Herren et al., 1999a,b, 2001). Jóhannesson (1991) finds a similar value for Hofsjökull, a neighboring glacier of Vatnajökull, of $0.0075$ m w.e. $a^{-1}m^{-1}$ based on measurements in the ablation area. Also, measurements in the ablation area of Unteraargletscher in the Swiss Alps indicate gradients of $0.011$, $0.009$ and $0.016$ m w.e. $a^{-1}m^{-1}$ for the years 1996/97, 1997/8 and 1998/9, respectively (Bauder, 2001). It is thus apparent that the mass balance gradients have a fairly constant value on temperate glaciers.

Another determined model parameter is the $\Delta$ELA. For the period of measurements this shift is plotted in Figure 5.8. The variation in the ELA during this period shows the same pattern as the estimated total balance of Vatnajökull (Björnsson et al., 1998b) with a high ELA corresponding to a negative mass balance and vice versa. This gives confidence in the assumption that the variation in mass balance can be presented with a shift in the ELA.

Both energy balance models and degree day models have been used to relate climate changes, presented as temperature or temperature and precipitation changes, to changes in the ELA. Oerlemans (1992) calculated changes in the ELA for a 1 K temperature increase with an energy balance model for three glaciers in Norway. The resulting changes are $110$, $108$ and $135$ m for Nigardsbreen, Hellstugubrean and Alfortbreen, respectively. Johannesson et al. (1995) use a
degree day model to compute changes in the ELA due to warmer climate. They predict that a warming of 2 K will lead to a 220 m or 180 m rise in the ELA of Sátujökull, central Iceland, for no precipitation change or 10 % precipitation increase, respectively. These values are similar in magnitude as the observed yearly variation in Figure 5.8.

During the time of observation a slight trend towards more negative mass balance is observed. A positive mass balance was observed the first three years and since 1995 the net balance has been negative. The year-to-year variation in the ELA is large, on the order of magnitude similar to the computed changes in the ELA due to expected climatic warming during the next decades.

![Figure 5.8: Temporal change in the ELA for the measured years.](image)

The mass-balance distributions for the highest and the lowest value of the ELA are shown in Figures 5.9 and 5.10, respectively. These figures show how large influence these yearly fluctuations have on the net balance. The mean specific net balance of the ice cap for the high value of the ELA is -1.17 m with an AAR of 0.44 and for the low value it is 0.75 m with AAR of 0.77. Table 5.2 shows the changes of AAR and the $b_{net}$ given variations in the parameters of the model. The large sensitivity of the net balance to changes in parameters of the ELA plane ($a_0$, $a_1$ and $a_2$) is due to the nature of the mass balance, it is very dependent on the height of the equilibrium line. The turning of the plane, by changing the $a_1$ and $a_2$, produces similar changes in the net balance as shifting the plane up and down. It is, however, resolved that changes in the net balance will be determined with a shift in the height ($a_0$) and the other parameters kept constant at presentative mean values. The net balance is not sensitive to changes in the mass balance gradients, $a^a$ and $a^b$, a 5% change in the values of the gradients changes the net balance by 3-4 cm.

The comparison between the results from the mass balance model and the measurements can be presented on each outlet glacier separately. Figures 5.11-5.15 show the mass balance computed with the model as lines and the mass balance measurements on the pole locations as crosses. The profile starts at the tongue of the glacier. The paths of these profiles are shown as lines in Figure 2.1. The model result, computed with the ELA which gives a zero net balance for the Vatnajökull, is the middle line in the figure and the results computed with the highest and the lowest values of the ELA are the lines below and above this line, respectively. The measurements have a large temporal variation, which is reflected in the variation in the ELA. It can be
Figure 5.9: Mass balance distribution for Vatnajökull computed with the highest obtained ELA value that fits the data from 1997 (ΔELA = 198.9 m). The average net balance is -1.17 m and the accumulation area ratio is 0.44.

Figure 5.10: Mass balance distribution for Vatnajökull computed with the lowest obtained ELA value that fits the data from 1993 (ΔELA = -143.9 m). The average net balance is 0.75 m and the accumulation area ratio is 0.77.
Table 5.2: Sensitivity of the AAR and the $b_{\text{net}}$ to changes, $\delta$, in the parameters of the mass balance model. The reference values for the AAR and $b_{\text{net}}$ are 0.648 and 0.006 m, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\delta$ (5 %)</th>
<th>AAR (%)</th>
<th>$b_{\text{net}}$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0 (-2138.29 \text{ m})$</td>
<td>+50.0</td>
<td>0.600</td>
<td>-0.273</td>
</tr>
<tr>
<td>-50.0</td>
<td>0.693</td>
<td>0.276</td>
<td></td>
</tr>
<tr>
<td>$a_1 (-0.0025)$</td>
<td>+0.000125</td>
<td>0.697</td>
<td>0.304</td>
</tr>
<tr>
<td>-0.000125</td>
<td>0.596</td>
<td>-0.303</td>
<td></td>
</tr>
<tr>
<td>$a_2 (0.0049)$</td>
<td>+0.00024</td>
<td>0.543</td>
<td>-0.591</td>
</tr>
<tr>
<td>-0.00024</td>
<td>0.742</td>
<td>0.560</td>
<td></td>
</tr>
<tr>
<td>$a^a (0.0040 \text{ a}^{-1})$</td>
<td>+0.00020</td>
<td>0.648</td>
<td>0.043</td>
</tr>
<tr>
<td>-0.00020</td>
<td>0.648</td>
<td>-0.030</td>
<td></td>
</tr>
<tr>
<td>$a^b (0.0082 \text{ a}^{-1})$</td>
<td>+0.00041</td>
<td>0.648</td>
<td>-0.030</td>
</tr>
<tr>
<td>-0.00041</td>
<td>0.648</td>
<td>0.043</td>
<td></td>
</tr>
</tbody>
</table>

seen in these figures that this variation is captured by the model, most of the measured values lie within the range of the model results. There are only a few high mass balance values on Dyngjujökull, Eyjabakkajökull and Tungnaárjökull that are above the model result computed with the lowest obtained ELA value. This indicates that the variance in the ELA can even be larger than the optimization procedure indicates.

This parameterization for the mass balance has the advantage of being very simple and transparent in the way that climatic variation is related directly with changes in the ELA. The mass balance at each location is a function of only the location and elevation. It does not depend on meteorological data from stations outside the glacier, which often are not available, and are difficult to extrapolate over the area covered by the ice cap. It must be emphasized that this parameterization is only valid for an ice cap similar in size to the present ice cap. How the mass balance distribution and the micro climate on the ice cap is coupled with the size of the ice mass itself is not known. In scenarios where the ice cap will change its size considerably the more traditional mass balance models have to be applied.

### 5.3 Conclusion

The parameterization of the mass balance distribution which is based on the following assumptions; (a) the mass balance is piecewise linearly dependent on the elevation, (b) the ELA is linearly dependent on the distance from the coast and (c) the yearly variation in mass balance can be presented with a shift in the ELA closely simulates the measured mass balance for the period 1992 – 2000. It provides a necessary surface boundary condition for dynamic model studies of Vatnajökull.

The shift in the ELA determined with the parameter selection shows a large annual variation, which is of the same magnitude as expected climate variation during the next decades. Decrease in precipitation is observed during this short period of available measurements, which causes the mean specific net balance to become negative and raises the height of the ELA.
Figure 5.11: Mass balance measurements along with the results of the mass balance model computed with the highest, lowest and the ELA that gives zero total mass balance for Vatnajökull on a profile along Brúarjökull.

Figure 5.12: Mass balance measurements along with the results of the mass balance model computed with the highest, lowest and the ELA that gives zero total mass balance for Vatnajökull on a profile along Dyngjujökull.
Figure 5.13: Mass balance measurements along with the results of the mass balance model computed with the highest, lowest and the ELA that gives zero total mass balance for Vatnajökull on a profile along Eyjabakkajökull.

Figure 5.14: Mass balance measurements along with the results of the mass balance model computed with the highest, lowest and the ELA that gives zero total mass balance for Vatnajökull on a profile along Köldukvíslarjökull.
Figure 5.15: Mass balance measurements along with the results of the mass balance model computed with the highest, lowest and the ELA that gives zero total mass balance for Vatnajökull on a profile along Tungnaárjökull.
Chapter 6

Numerical computations of Hofsjökull

In this chapter the flow model, along with the mass balance parameterization determined in Chapter 5, are applied on Hofsjökull, a neighboring ice cap of Vatnajökull. Both the ADS-I code and the ADS-I upstream B code are used and the results compared. The codes give almost identical results. The bedrock and the surface topography of Hofsjökull have been measured with radio-echo sounding methods (Björnsson, 1988) and were kindly provided by Helgi Björnsson. The steady-state surface geometry and the mass balance from which it resulted are used in the flow analysis in Chapter 7.

6.1 Steady-state computations

Hofsjökull is a neighboring ice cap of Vatnajökull, to the northwest, in the center of Iceland. It covers an inactive volcanic mountain and the bedrock topography is in general somewhat steeper than on Vatnajökull. The ice fills a caldera and flows outwards and forms almost a circular ice sheet (Figure 6.1). This ice cap has been modeled with a cylindrically symmetric model to investigate the time-scale of possible glacier variations (Jóhannesson, 1991). Here, the measured bed and surface topography are used as initial condition. The mass balance parameterization that was developed for Vatnajökull in Chapter 5 is used to force the computations for Hofsjökull. The height of the ELA plane was shifted by 200-400 m relative to the reference plane used for Vatnajökull in order to get a realistic values for the ELA on Hofsjökull. This shift indicates that the ELA does not increase monotonously away from the coast, but reaches a value in the center of Iceland that is similar to the values at the northwestern side of Vatnajökull. The height of the ELA plane that gives an ice cap similar in size to the present is shown in the right panel of Figure 6.1. The heights are in good agreement with measurements on three outlet glaciers of Hofsjökull 1987-1997 (Sigurðsson, 2000). No sliding was included in the computations.

The evolution of the volume and the net mass balance during several model runs computed with the ADS-I method are shown in Figure 6.2. For the different heights of the ELA plane the ice cap evolves to different sizes, the size that is closest to the present ice cap is the one computed with the ELA plane 340 m lower than the reference plane of Vatnajökull. For all the model runs the final mean specific net balance is positive. The mass balance inconsistency that appeared on Vatnajökull does not appear on Hofsjökull, as there are only two points that are ice free within the ice cap. Number of runs with other ELA heights, as well as another flow parameter A, were
Figure 6.1: Surface geometry of Hofsjökull. The straight lines indicate the location of the cross sections shown in Figure 6.5. The right panel shows the height of the ELA plane, with a shift of -340 m, which produces an ice cap that is similar to the present ice cap in size and extent. The straight lines are level lines of the plane with equidistant levels of 50 m.

computed. The resulting steady-state volume as a function of the ELA shift is shown in Figure 6.3. There is a steady state for each of the model runs. By lowering the height of the ELA plane the volume increases and by applying a higher flow parameter $A$ the velocity increases and, thus, volume decreases, as expected.

The ice cap was also computed with the ADS-I upstream B method by using the height of the ELA plane that supports an ice cap similar to the present ice cap in size. The resulting ice caps computed with the two numerical methods are shown in Figure 6.4 and two cross sections through the ice caps in Figure 6.5. The location of the cross sections is shown in Figure 6.1. The two methods give almost identical steady states, there is only a slight difference in the extent and the shape at the edges, where the slopes are steep. This is not surprising because the two methods compute the slopes and the coefficient $D$ with different points (see Figure 3.1) and the difference between the methods increases with steeper slopes. The maximum height computed with the ADS-I method is 1783.63 m and with the ADS-I upstream B method the maximum height is 1781.62 m, or only 0.7 % of the total thickness in this point.

These computations show that in the absence of nunatak points ADS-I and ADS-I upstream B methods give almost indentical results. The computations for Hofsjökull are used in the steady state analyses of Chapter 7. Hofsjökull is an example of an ice cap that is not sensitive to the mass balance input, as opposed to Vatnajökull. This will be discussed further in Chapter 8.
Figure 6.2: Evolution of the volume (upper panel) and the net mass balance (lower panel) of Hofsjökull for four different heights of the ELA plane.
Figure 6.3: Steady-state volume of Hofsjökull as a function of the shift of the ELA plane, relative to the Vatnajökull reference height, for two values of the flow law parameter.

Figure 6.4: The steady state ice caps computed with the ADS-I method in the left panel and the ADS-I upstream B in the right panel. The solid line indicates the extent of the present, measured ice cap.
Figure 6.5: Profiles through the ice cap. The location of the profiles is shown in Figure 6.1. The dotted line indicates the initial surface elevation and the two solid lines the numerical solutions with the ADS-I method and the ADS-I upstream B method.
Chapter 7

Analysis of the state of balance

The information on surface and bed geometry, mass-balance distribution, surface velocity and ice divides provide an opportunity to analyse the state of balance for an ice cap. In this chapter two methods to assess the state of balance are applied on Vatnajökull. The first is to integrate the mass-balance distribution over the entire glacier surface in order to determine the mean specific annual balance. The mean specific annual balance can also be estimated for each outlet glacier separately. If the mean specific annual balance is positive, one would expect the glacier to grow and to contract if the net balance is negative. This method gives no information on the flow regime of the ice cap and variations in mass balance can make the assessment of the state of balance of the ice cap more complicated.

The second method, to assess the state of balance, is to compare two models of the flow regime that are based on different assumptions. The first model is the flow model presented in Chapter 3. It is based on the Shallow Ice Approximation (SIA) and uses Glen’s flow law as a constitutive equation. This model was adjusted to the flow regime measured on Vatnajökull by comparing the model results with the surface velocity measurements and select the rate factor and the sliding parameter so that the modeled velocity field resembles the measured one. The flux is computed by using Equation (3.3) and will be termed dynamical flux. The dynamical flux is assumed to represent the measured flow of the ice cap. The other model is based on steady state assumption and uses the mass conservation to compute the flux which maintains the surface geometry for a given mass-balance distribution, this flux will be termed balance flux. The mass-balance distribution that is used to compute the balance flux is the parameterization of the mass-balance measurements presented in Chapter 5. The difference between the two modeled fluxes indicates the areas where the modeled flow is not in balance with the modeled mass balance distribution. The difference of the models is interpreted to have relevance to the actual state of balance of the ice cap and it is used to identify outlet glaciers that are likely to retreat, advance or even surge. Variations in mass balance and surface velocity could be used to compute time series of dynamical and balance flux and monitoring of the difference can refine the balance assessment.

The two models for the flux give similar results in the two test cases, on the simple spherical ice sheet, and on the steady state of Hofsjökull which is computed in Chapter 6. The dynamical flux and the balance flux are, on the other hand, different on large areas of Vatnajökull. The difference is analysed and it is concluded that the larger outlet glaciers of Vatnajökull are not presently in a steady state.
7.1 Integration of mass-balance distribution

The most direct way to assess the state of balance for an ice sheet, given a specific mass-balance distribution, is to integrate the mass balance over the glacier surface and evaluate the mean specific annual balance. The mean specific annual balance gives information about whether the ice cap can be expected to grow or shrink, for positive or negative balance, respectively. This method takes no account of the prevailing flow regime of the ice cap, only the mass-balance distribution is considered. The same can also be done for each outlet glacier separately to assess its state of balance. To do that the ice divides must be located. There seems to be no straightforward automated method available to draw the ice divides. They were, thus, drawn by hand from a tightly spaced contour map of the surface. The resulting ice divides define each drainage basin of Vatnajökull and these are in accordance with named outlet glaciers. The divides are, however, not fixed in time as the frequent surges on most of the outlet glacier cause the ice divides to migrate. The annual balance, \( b(x, y) \), provided by the mass-balance parameterization described in Chapter 5 is integrated over the area, \( S \), of each outlet glacier to give the corresponding glacier annual balance

\[
B = \int_S b(x, y) \, dS,
\]

and the mean specific annual balance

\[
\bar{B} = \frac{B}{S}.
\]

The position of the equilibrium line is given by the mass-balance parameterization so the accumulation-area ratio (AAR) can be calculated. It is the area of the accumulation area divided by the area of the whole glacier. For comparison the maximum and the minimum height of the ELA determined from the mass-balance measurements were also used to compute the mean specific net balance and the AAR (Table 7.1). The variation in the mass balance during

<table>
<thead>
<tr>
<th>Glacier</th>
<th>Area (( \text{km}^2 ))</th>
<th>AAR (%)</th>
<th>( B ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 max</td>
<td>min</td>
</tr>
<tr>
<td>The whole ice cap</td>
<td>8294</td>
<td>0.65 0.44 0.77</td>
<td>0.01 -1.17 0.75</td>
</tr>
<tr>
<td>Breiðamerkurjökull</td>
<td>826</td>
<td>0.60 0.44 0.70</td>
<td>-0.37 -1.56 0.42</td>
</tr>
<tr>
<td>Brúarjökull</td>
<td>1728</td>
<td>0.65 0.41 0.77</td>
<td>-0.02 -1.20 0.73</td>
</tr>
<tr>
<td>Dyngjújökull</td>
<td>1066</td>
<td>0.68 0.51 0.79</td>
<td>0.23 -0.89 0.97</td>
</tr>
<tr>
<td>Köldukvíslarjökull</td>
<td>353</td>
<td>0.64 0.48 0.78</td>
<td>0.33 -0.83 1.08</td>
</tr>
<tr>
<td>Skeiðarárjökull</td>
<td>1428</td>
<td>0.68 0.52 0.78</td>
<td>0.04 -1.09 0.78</td>
</tr>
<tr>
<td>Sy- Tun-Siðujökull</td>
<td>1070</td>
<td>0.52 0.29 0.70</td>
<td>-0.39 -1.69 0.42</td>
</tr>
<tr>
<td>Eyjabakkajökull</td>
<td>136</td>
<td>0.55 0.13 0.75</td>
<td>-0.39 -1.72 0.39</td>
</tr>
</tbody>
</table>

Table 7.1: The total area, accumulation-area ratio (AAR) and the mean specific annual balance for the whole ice cap and each of the outlet glaciers. The three columns for the AAR and the mean specific annual balance correspond to the height of the ELA chosen to compute the mass-balance distribution, the reference height, the maximum height of the ELA indicated by the measurements (+198.9 m, 1997) and the minimum height (-142.9 m, 1993).
the period of measurements, presented by the minimum and the maximum height of the ELA, shows that each glacier has had years of both negative and positive mean specific annual balance. Due to the large variation in the mass balance it is thus difficult to draw any conclusions about the state of balance of the ice cap with this analysis.

Figure 7.1: Mass- balance distribution of the initial surface. The ice divides are drawn on the surface and the mean specific annual balance of each of the outlet glacier is shown.

Figure 7.1 shows the reference mass-balance distribution along with the location of the ice divides and the mean specific annual balance of each outlet glacier. Despite the fact that the mean specific annual balance for the whole ice cap is zero this is not the case for the separate outlet glaciers. Brúarjökull in north and Skeiðarárjökull in the south have almost zero mean specific annual balance but the other outlet glaciers have 20 - 40 cm positive or negative values. Positive values imply that the glacier should grow when this mass-balance distribution is applied in the dynamical computations and negative that it should contract. This is indeed the case. The initial response of the ice cap is in accordance with the mean specific annual balance, Dyngjujökull grows, Skeiðarárjökull does not change so much, Breiðamerkurjökull retreats etc. However, the thickness in the center of the ice cap increases and due to the elevation feed back in the mass-balance computation the ice cap start to grow after the initial adjustments.

The elevation distribution or the hypsometry can have a pronounced influence on the stability and response of glaciers subjected to similar climatic conditions (see for example: Furbish and Andrews, 1984; Mercer, 1961). To analyse this influence on Vatnajökull the elevation distribution for each of the outlet glaciers is drawn (Figure 7.2). It appears that the elevation distribution is similar on Breiðamerkur-, Dyngju- and Skeiðarárjökull with the ELA below the peak in the distribution. The other glaciers have somewhat different distribution shapes and the location of ELA relative to this shape is different. The relation between the AAR and the mean specific net balance is drawn in Figure 7.3. The slope of the lines for Breiðamerkurjökull, Dyngjujökull and Skeiðarárjökull is similar but different for the other glaciers, which demonstrate the influence of the elevation distribution. Particularly the slope is different for Eyjabakkajökull, which is the smallest glacier and the ELA is close to the peak in its elevation distribution. The raising of the
Figure 7.2: Elevation distribution for each of the outlet glaciers. The approximate height of the ELA is shown with a straight line. (A) Breiðamerkurjökull. (B) Brúarjökull. (C) Dyngujökull. (D) Köldukvíslarjökull. (E) Skeiðarársandurjökull. (F) Sylgju- Tungnaár- og Síðujökull. (G) Eyjabakkajökull.
7.2 Comparison of dynamical flux and balance flux

The second method, to assess the state of balance for Vatnajökull, is to compare two models that compute the flux of the ice cap. The first model is the dynamical flow model presented in Chapter 3. The flux is computed with Equation (3.3) by using the SIA and using the Glen’s flow law for the ice rheology, this flux will hereafter be called dynamical flux. The dynamical flow model uses the continuity equation to compute the elevation evolution by subtracting the divergence of the flux from the mass-balance distribution

$$\frac{\partial h}{\partial t} = b(x, y, t) - \nabla (q).$$

If steady state is assumed and the divergence theorem applied on the area integral for the flux we get

$$\int \int_S b(x, y) \, dx \, dy = \int \int_S \nabla (q) \, dx \, dy = \int_C q \cdot \vec{n} \, dl,$$

which shows that in a steady state the total flux through a defined boundary, C, equals the total mass balance over the area, S, enclosed by C. This flux, that maintains the geometry of the ice cap, the balance flux, can be computed without assuming anything about the ice rheology by making use of mass conservation. The comparison of the two fluxes can identify areas where the dynamical flux is slower or faster than the balance flux, which are then expected to retreat or advance, respectively. This method makes use of available data on both surface velocity, to determine the dynamical flux, and mass-balance distribution, to determine the balance flux.

Several methods to compute the balance flux have been proposed. A flowline or flux-gate approach for spatial discretization has been used (for example Budd et al., 1971; Joughin et al.,...
1997), but recently many authors have adopted a two dimensional finite difference scheme (Budd and Warner, 1996) to compute the balance flux and balance velocities (Bamber et al., 2000a,b,c; Fricker et al., 2000; Huybrechts et al., 2000). The computer code for this method is freely available for general use from the authors and was kindly provided by Roland Warner at the Cooperative Research Centre for Antarctic and Southern Ocean Environment (Antarctic CRC), University of Tasmania, Hobart Australia. The computing scheme was tested on several ice sheets to assess the applicability of the method. First, the dynamical flux and balance flux were compared on a simple circular ice sheet similar to the geometry of the Eismint ice sheet with three different grid sizes. Second, the flux comparison was done on a steady state configuration of the neighboring ice cap Hofsjökull from Chapter 7. Lastly, the dynamical and balance fluxes were compared on the initial geometry of Vatnajökull.

7.2.1 Method to compute the balance flux

The code to compute the balance flux and the optional balance velocity implements a finite difference scheme. It is version 2, called BalanceV2. It computes the balance flux and (optionally) the balance velocities for an ice sheet given the accumulation, surface elevation and (optionally) ice thickness over a regular grid. The method is outlined by Budd and Warner (1996) and here it will only be briefly described. The algorithm used to compute the balance flow distribution was altered somewhat in the newer version of the code and the subsequent description follows Fricker et al. (2000). The basic assumption is that for an ice sheet in a steady state there exists a time-invariant flow which maintains the surface geometry. The method assumes moreover that the direction of ice flow is orthogonal to the elevation contours and that the flux in the direction of flow can be resolved into rectangular \((x,y)\) components which can be suitably represented by finite-difference scheme to the resolution of interest. The former assumption has been found to apply reasonably well for horizontal scales which are large (20 times) relative to the ice thickness and Budd and Warner (1996) suggest that smoothing the surface over a horizontal scale of 10-20 ice thicknesses before the balance flow is computed will ensure the applicability of this assumption. The latter assumption entails that the grid-point density chosen is sufficient to represent the surface topography in enough detail to capture the essential flow characteristics. No interior surface hollows (or exactly flat regions) are allowed (in the absence of net surface ablation). Before the balance flux is computed the code scans through the surface elevation matrix, fills up local troughs, and removes flat spots. The modified surface topography is output for comparison purposes.

The continuity in discrete form is:

\[
\Psi_{i,j}^{\text{out}} = \Psi_{i,j}^{\text{in}} + b_{i,j} \Delta x \Delta y,
\]

where \(\Psi_{i,j}^{\text{in}}\) and \(\Psi_{i,j}^{\text{out}}\) are the scalar in- and out flows of the grid cell \(i,j\) with the accumulation distribution \(b(x,y)\) and \(\Delta x\) and \(\Delta y\) are the grid spacings. The cell outflow rate or the scalar flux field \(\Psi_{i,j}^{\text{out}}\) is found sequentially by sorting the cells by elevation, starting at the highest point and proceeding downslope. The flow direction is incorporated by apportioning the outflow from a gridcell between those neighbors that lie downslope, in proportion to the elevation differences. For example, the flux from \(\text{cell}_{i,j}\) to \(\text{cell}_{i,j+1}\) is:

\[
\frac{h_{i,j} - h_{i,j+1}}{N} \Psi_{i,j}^{\text{out}}
\]
Figure 7.4: The flux of the circular ice-sheet. The left hand side shows the dynamical flux and the right hand side the balance flux. The top figures have grid size 50 km, the same as in the Eismint experiment. The middle figures have grid size 25 km and the bottom figures have grid size 10 km.
Figure 7.5: Surface velocity of the circular ice-sheet. The left hand side shows the dynamical surface velocity and the right hand side is the balance velocity multiplied with \((n + 2)/(n + 1)\). The top figures have grid size 50 km, the same as in the Eismint experiment. The middle figures have grid size 25 km and the bottom figures have grid size 10 km.
Figure 7.6: Flux of a steady-state ice cap Hofsjökull. The top figure shows the dynamical flux and the bottom figure the balance flux. The lines indicate the outline of the present ice cap, which is slightly different in extent from the steady state ice cap.
Figure 7.7: The flux of the initial geometry of Vatnajökull ice sheet. The top figure shows the dynamical flux and the bottom figure the balance flux.
where $h_{i,j}$ is the surface elevation in cell $i,j$ and $N$ is the sum of all downward slopes from the same cell. The total scalar inflow of ice is the sum of the out-flowing contributions from neighboring upslope cells. This procedure yields the balance flux on the same staggered grid points as the dynamical flux and can be directly compared with it. The optional column-averaged balance velocity can be computed by dividing the balance flux with the ice thickness at each point.

### 7.2.2 Test on simple circular ice-sheet

The first test was to compute the dynamical flux and balance flux on the same, simple, circular ice-sheet as the flow model was tested on. The moving boundary set up of the Eismint experiment was chosen. The grid size of the Eismint ice-sheet is 50 km which represents the surface topography reasonably well. The dynamical flux is shown on the left hand side of Figure 7.4 and the balance flux on the right hand side. For this large grid size both computed fluxes are not symmetric. There are local maxima in the flux which are clearly controlled by the grid orientation (see top panel of Figure 7.4). The grid size was halved and the flux computed again. With a grid size of 25 km the dynamical flux is symmetric but the balance flux is still influenced by the grid orientation. However, to a lesser extent than before. With a yet smaller grid size of 10 km both flux distributions are symmetric and similar to what one intuitively expects (bottom panel of Figure 7.4). The surface velocity was also computed with the dynamical model and by dividing the balance flux with the ice thickness and multiply with $(n+2)/(n+1)$ the column-average velocity is changed to surface velocity (see Paterson (1994), page 252 for this ratio).

Both models show a similar, very simple, flow pattern for all grid sizes (Figure 7.5). There is, however, a difficulty in computing the velocity at the very edge of the ice sheet resulting in large peaks in both models of the outermost grid cell. This is due to the fact that at the edge the slope is the greatest and the thickness the smallest leading to a break down in the assumptions of the flow model.

### 7.2.3 Flux comparison on a steady-state ice cap, Hofsjökull

The above flux computations were done on a simple geometry resulting in an identical flux for both models, when the grid size is small enough. To test the comparison method further the dynamical and balance fluxes were computed for the steady state geometry of Hofsjökull. The topography and the mass-balance distribution used for the flux computations are the results from the flow model computations in Chapter 6. The flux that maintains this topography was computed with both models. The comparison is shown in Figure 7.6. The figure shows that the two methods produce similar flux pattern, the location of the ice divides is similar and the shape of the outlet glaciers is roughly the same. The magnitude of the flux of some of the outlet glaciers differs somewhat between the two methods. This is, however, not a large difference and the overall flux field is similar. This results shows that for an ice cap in a steady state both methods compute the same flux field.

### 7.2.4 Flux comparison on the initial geometry of Vatnajökull

The method was subsequently applied on the initial condition of Vatnajökull (see section 3.5.1). From the dynamical model computations it is known that the initial geometry is not in a steady
state with the applied mass-balance distribution (see Chapter 9). The basic assumption of the balance flux computation is, thus, not met in this case. The purpose of the dynamical-balance flux comparison is to identify the regions of the ice cap that are or are not in balance with the applied mass-balance distribution.

The dynamical- and the balance flux are shown in Figure 7.7. As mentioned above the balance flux method does not allow any hollows in the accumulation area. The code, therefore, changes the surface before the balance flux is computed. There is a significant change in the surface geometry in Grímsvötn, which is a local depression caused by subglacial geothermal heat that melts the ice at the bed and maintains a subglacial lake. The consequence of this artificial filling can be seen in Figure 7.7. There is a narrow ice stream flowing out of the Grímsvötn area, that has now been filled, which has no analogy in reality. It should, therefore, be ignored as it is an artifact of the method. The dynamical flux was computed using the altered geometry, which is an output of the balance flux computation, to make the resulting fluxes comparable. The top figure of Figure 7.7 shows the flux computed with the dynamical flow model with the selected flow law parameter and the maximum estimate for sliding. As described in Chapter 3 the flow law parameter and the sliding estimates are adjusted to simulate the measured surface velocity and produce a flow pattern which resembles the measurements. There is smaller flux towards north, on the outlet glacier Brúarjökull and Dynjújökull, than towards south where Skeiðarárjökull and Breiðamerkurjökull flow relatively fast. The computed balance flux shows a somewhat different flow regime. The difference between the modeled balance flux and the dynamical flux can be analyzed to evaluate the state of balance of each of the outlet glaciers.

Following is an analysis of each outlet glacier starting in the south on Skeiðarárjökull and going clockwise around the ice cap. It must be kept in mind that the mass-balance distribution used to compute the balance flux is the reference distribution which gives mean specific annual balance equal to zero for this ice geometry. The measurements that this mass-balance distribution is based on show a large variation during the nine years period (Figure 5.8). In the beginning of the period the mass balance was positive and after 1995 it was negative. The reference mass-balance distribution used to compute the balance flux represents an average over the period of measurements. Moreover, the dynamical flux is a temporarily averaged presentation of the flow field which is known, from measurements, to be varying constantly in time (Figures 3.15-3.19). Both fluxes are, thus, a presentation of the average flow and mass balance regime prevailing on Vatnajökull during the nine years measurements period. The coverage of the mass-balance and velocity data is not everywhere good (Figure 2.1) so the reliability of both modeled fluxes is not everywhere the same. In addition, to make the analysis more complicated, there have been several surges during this period, which have altered the surface considerably. The date of the surface elevation survey relative to the occurrence of a surge is, thus, important in the balance assessment to know whether the glacier was preparing a surge or just had had one. The dates of the survey and the surges are listed in Table 7.2 and Figure 7.8 shows all known surges (Björnsson et al., 2002b).

**Skeiðarárjökull.** The surface was surveyed in 1993-1994, shortly after the surge in 1991, the geometry is, thus, relatively flat because the glacier recently transported large amount of ice down to the ablation area. There are only a few surface velocity measurements available for this glacier, on one location at the head and another close to the terminus. The dynamical flux is, thus, not well constrained with measurements. The shape of the flow regime is similar in the accumulation area for both models (except for the artificial flow out of Grímsvötn). In the center of the glacier the dynamical and balance fluxes are similar, above the ELA the dynamical flux
Figure 7.8: Known surges of the outlet glaciers of Vatnajökull (Björnsson et al., 2002b).

is slightly higher and below it the balance flux is higher. The calculated balance flux suggest that enough mass is available to maintain the western part of the terminus but not the eastern part of the terminus. This and the facts that the dynamical flux is only slightly higher in the accumulation area and balance flux is higher below the ELA indicate that the glacier is close to be in balance. This is in agreement with the slightly positive mean specific annual balance of this outlet glacier which is 0.04 m.

Síðujökull. The surface was surveyed in 1990 and 1991, shortly before the surge in 1992-1994. In the accumulation area the glacier was probably thicker than usually and the surface relatively steep. These changes in the surface geometry of the glacier result in larger predicted dynamical flux than when the glacier is in a quiescent phase. The calculated balance flux is smaller than the calculated dynamical flux and there appears to be more ice melted away than is

<table>
<thead>
<tr>
<th>Glacier</th>
<th>Time of surface measurement</th>
<th>Time of Surge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Köldukvíslárjökull</td>
<td>1982</td>
<td>1992</td>
</tr>
<tr>
<td>Brúarjökull</td>
<td>1988</td>
<td>1963-1964</td>
</tr>
<tr>
<td>Eyjabakkajökull</td>
<td>1980</td>
<td>1972</td>
</tr>
<tr>
<td>Breiðamerkurjökull</td>
<td>1990</td>
<td>1978</td>
</tr>
</tbody>
</table>

Table 7.2: The time of the radio-echo sounding measurements for each of the outlet glaciers along with the time of the most recent surge (Björnsson et al., 2002b, personal communication F. Pálsson, 2000).
accumulated. Due to the time of the surface survey, relative to the surge, it is thus not possible to conclude about the state of balance with only the comparison of the predicted dynamical and balance fluxes. Here the data on the observed surface velocity should be used directly. The surface velocity did not increase with the steeper surface until the onset of the surge. The excess of the calculated dynamical flux relative to the balance flux is in this case caused by the thick and steep pre-surge surface geometry.

**Tungnaárjökull.** Surface measurements were done in 1981 and the glacier surged 1992-1995, at the same time as Síðujökull. The previous surge was in 1945 so at the time of the survey the glacier was in the latter part of the quiescent period. There is not as large difference between the calculated dynamical and balance fluxes as for Síðujökull. This is presumably because the survey was done about 10 years ahead of the surge. The predicted dynamical flux is larger than the balance flux, which is caused by change in surface geometry due to the surge activity and not a possible retreat.

**Sylgjujökull.** This is the small glacier between Tungnaárjökull and Köldukvíslarjökull. It surged after Tungnaárjökull in 1995-1998 and the surface was surveyed in 1981, 15 years before the surge. There was a surge 1945 as in Tungnaárjökull so the survey was in the latter part of the quiescent period. The balance flux is stronger than the dynamical flux which indicates that excess mass is accumulating and the glacier will advance or is preparing for a surge.

These three glaciers, Síðu-, Tungnaár- and Sylgjujökull (there are actually four, the fourth one between Síðujökull and Tungnaárjökull, Skaffárjökull, is difficult to distinguish from the others) on the western side of the ice cap share the accumulation area. The divides migrate as the glaciers surge. In the surge that started 1992 it seems that enough mass had accumulated to let all the glaciers in this area surge but there are also occasions when only one glacier surged at a time (Figure 7.8). In this area the mass-balance distribution used to compute the balance flux probably influences the prediction of the balance flux. All the surges start before the mass balance became negative in 1995. Before the surges the total mass balance was positive and thus more mass accumulated in that period than the years after the surges. Also, the surface survey is made relatively shortly before the surge so the surface is relatively steep which increases the predicted dynamical flux. In this area it is better to use the data on the observed surface velocity to compare with the balance velocity to assess the state of balance. This method of comparing the dynamical and balance flux must clearly be used with all the knowledge of the previous surge history of the glaciers, the average surface geometry and the surface velocity measurements. The net average mass balance for the region of the four glaciers is negative, -0.39 m, which is the average during the nine years period of measurements.

**Köldukvíslarjökull.** The surface was surveyed 1982 between the surges that occurred 1972 and 1992. The surveyed surface is thus in the middle of a quiescent phase. The computed dynamical and balance fluxes have very similar magnitude. Also here it must be taken into account that the mass balance used to compute the balance flux is lower than just before the surge started and thus in the time before the surge the mass balance was higher and the glacier could accumulate mass to prepare the surge. The calculated balance suggests that similar amount of ice is melted as is accumulated. Therefore, all additional mass will be accumulated. The mean specific annual mass balance for Köldukvíslarjökull is 0.33 m.
**Dyngjujökull.** Surface was measured in 1989 and surges occurred in 1977 and 1998–2000. The surface is measured in between two surges. The dynamical flux is adjusted to surface velocity measurements made in the period 1991–1998 (during the surge it was not possible to do measurements on the surface). The mass balance used for the balance flux computation is approximately the mean of the measurements in the same period, before 1995 the mass balance was higher and after it was lower. The computed balance flux is considerably larger for the center of the glacier than the predicted dynamical flux. This indicates that the glacier is not capable of transporting the accumulated mass down to the ablation area and the glacier builds up and prepares for the surge. The mean specific annual mass balance for Dyngjujökull is 0.23 m, indicating an excess mass. In the area around the volcano Bárðarbunga and on the western part of Dyngjujökull the balance flux and dynamical flux are very similar, this region did not surge in 1998.

**Brúarjökull.** Surface was surveyed in 1988 and the last surge occurred in 1963–1964. The surges before that were in 1810 and 1890, so there is much longer time between surges of Brúarjökull than on the other glaciers. Measured surface is thus in the earlier part of the quiescent period. This glacier is the largest outlet glacier of Vatnajökull, it comprises about 21 % of the total area. It is relatively broad and flat glacier and the underlying bedrock is also without large undulations. The flow regimes of the two methods are very different. The predicted dynamical flux indicates strong flow from southeast and southwest towards the center and then it slows down and towards the terminus the flow is very slow. The computed balance flux on the other hand is concentrated in the center where flow is channeled towards the terminus. There is also large amount of mass transport suggested from the southwestern part, the center of the ice cap, but very little from the eastern side. The dynamical flux is larger than the balance flux in the accumulation area but the balance flux is larger in the ablation area. This indicates that the glacier can not transport mass fast enough in the lower part of the ablation area and it is accumulating excess mass. However, the mean specific annual balance is slightly negative, -0.02 m, which indicates that there is close balance between the accumulation and ablation for this mass-balance distribution. The two methods to assess the state of balance for Brúarjökull lead to contradicting conclusions.

**Eyjabakkajökull.** Surface was measured in 1980 and there was a surge in 1972. The computed dynamical and balance fluxes are both small for this glacier. The balance flux indicates that there is more ablation than accumulation. The mean specific annual mass balance is -0.39 m, which is consistent with the balance flux that does not reach the terminus. A retreat of this glacier can be expected.

**Eastern part.** There are several glaciers flowing off the plateau down steep slopes towards the coast at the eastern side of Vatnajökull. These glaciers and their surroundings have not yet been surveyed. The grids are based on topographic maps made from aerial photographs taken 1945–1946 (AMS maps). There are neither surface velocity measurements, to constrain the dynamical flux, nor mass-balance measurements for the mass-balance distribution. Both modeled fluxes are, thus, not well adjusted to data in this area. The flux of both models is of similar magnitude and increases towards each of the small outlet glaciers. The balance flux has slightly more concentrated flow towards the center of each glacier but it does not reach the terminus, which indicates that the glaciers could be expected to retreat in the near future as the
modeled mass balance is not enough to maintain the glacier. This is however difficult to verify due to lack of data.

**Breiðamerkurjökull.** The surface was surveyed in 1990 and the last surge was 1978. This glacier is presently terminating into a lake and calving into it. This lake is created as the glacier retreats from its post-glacial maximum position which it reached in the 1890s. Björnsson (1996) describes the advance and retreat from the settlement of Iceland. The glacier advanced by 9 km between 1730 and 1890 with several rapid advances which are definitely surges. The inhabitants reported rapid advances occurring approximately every fifth year followed by recession. The last advance of this nature was reported early in the 20th century and since then the glacier has been retreating and increasing the size of the proglacial lake at the same time. During the advance a sediment volume of \(5 \times 10^9 m^3\) was excavated and a 20 km long, 2–5 km wide trench, which extends to 300 m below sea level is believed to have been created (Björnsson, 1996). The ice fluxes and calving rates are described and simulated with a model by Björnsson et al. (2001). Their calculations suggest similar retreat rate of the calving front position for the next approximately 70 years.

The computed balance flux is much smaller than the predicted dynamical flux which indicates that there is not enough mass accumulated in order to maintain the present size of the glacier. The dynamical flux regime shows well defined ice streams flowing between the nunataks at Mavabyggoir and Eslujjoll. There are only a few surface velocity measurements available to constrain the flow- and sliding parameters. Along the center of the glacier are 4 poles measured only in the year 1997 and at the terminus there were 3 poles measured in 1998. Close to the head of the glacier in the central part one pole has been measured 1998–1999. In order to confine the dynamical flow, to be able to estimate the retreat rate, more surface velocity measurements are needed.

**Öræfajökull.** The glaciers flowing outwards from the volcano Öræfajökull are, like the eastern glaciers, not yet surveyed. Neither surface velocity nor mass-balance measurements are available to constrain the dynamical flux and the mass-balance distribution. The glaciers are small, approximately the same magnitude as the grid size of the model. It is thus difficult to use the comparison between the dynamical flux and balance flux to infer anything about that state of balance of these small outlet glaciers. The balance flux does not indicate any excess flow and there is no history of surges in these glaciers.

### 7.2.5 Discussion

The two models to compute the flux pattern of an ice sheet are based on different assumptions. The dynamical flux is computed by using the SIA and the rheology of the ice, independent of the mass-balance distribution. The balance flux, on the other hand, uses mass conservation and assumes steady state. The mass balance is apportioned to the flux in proportion to the surface slope. These models compute similar fluxes on the simple circular ice sheet and the steady state configuration of Hofsjökull, which indicates that the numerical methods used are correctly implemented. The difference between the dynamical and balance flux of Vatnajökull, on the other hand, is pronounced and reflects the state of imbalance of Vatnajökull. The difference is eminent on the largest outlet glaciers, Dyngjujökull, Brúarjökull and Breiðamerkurjökull. Dyn- gujökull and Brúarjökull have balance flux larger than the dynamical flux which indicates that
mass is accumulated and, indeed, Dyngjujökull surged at the end of the measurement period. On Breiðamerkurjökull the dynamical flux is larger so a retreat of this outlet glacier can be expected. Skeiðarárjökull is close to balance and Eyjabakkajökull can be expected to retreat. The assessment of balance for the glaciers on the western side of Vatnajökull is difficult because the surface was surveyed shortly before surge started and thus the geometry of the glaciers influenced by pre-surge thickness and steepness in the accumulation area, causing the dynamical flux to be larger than the balance flux. Because the surface is not surveyed as regularly as the mass balance and surface velocity measurements are done it is presently difficult to employ this method to assess the state of balance further than the above analysis. As more data become available time series of dynamical and balance flux can be made and by comparing those accurate monitoring of the excess or deficit of mass can be made. Such monitoring is made possible with increased availability of satellite images and interferograms derived from them. The technique and methods to separate simultaneously changing velocity field and surface topography are available and well tested. Interferograms retrieved from images with long baselines provide information on the topography and from interferograms with short baselines motion can be distracted (Andrea Fischer, personal communication, 2002). Due to the short period of time of which the satellite data have been available (1995-2000) it is presently difficult to use them in this method, but as more becomes available the applicability of the method presented here escalates. Then it will become possible to monitor the evolving surface, how the ice divides migrate and how the steepness of the surface changes in addition to the constantly collected data on surface velocity, mass flux and mass balance. With this information the dynamical flux can be accurately modeled and adjusted to measured surface velocity at a particular time. With the concurrent mass-balance distribution the balance flux can be computed and the difference between the two indicates the state of balance of the ice cap. To compare the balance flux directly with the measured surface velocity it must be converted to balance velocity. That requires the ratio between the column-averaged balance velocity and surface velocity. This ratio is between 0.8 - 1.0, the lower limit is when the surface velocity is due to deformation only and the upper limit when the surface velocity is due to sliding only. The flow model can be used to compute this ratio (Bamber et al., 2000a) or the whole range can be used and compared with the measured surface velocity. In the comparison on western Vatnajökull this alternative method would probably be more suitable because the surface was surveyed shortly before a surge and thus relatively steep, causing the predicted dynamical flow to be larger than the actual movement of the ice. A combination of both methods, using all available data, should be applied when the flow history is complicated and surges occur regularly.

The method used here to compute the balance flux starts at the highest elevation of the ice sheet and proceeds downslope. A monotonously lowering surface is necessary and no local depressions allowed. This requires all surface hollows in the accumulation area to be filled before the computation starts. This is generally not a large alteration in the surface but can be significant as in the case of Grímsvötn where a surface depression is maintained by geothermal heat. This hollow filling would not be necessary if the vector flow field would be solved directly with a sparse matrix system. This was suggested by Budd and Warner (1996) but thought to be problematic for Antarctica as the flow differs spatially by several orders of magnitude. This is not a problem on the glaciers that were analysed here and might be worthwhile to solve the balance flow field $\Phi$, which is equivalent to the dynamical flow $q$, directly from the equation

$$ b_{i,j} = \frac{1}{2\Delta x} (\Phi_{i+1,j} \cos \theta_{i+1,j} - \Phi_{i-1,j} \cos \theta_{i-1,j}) $$
\[ + \frac{1}{2\Delta y} (\Phi_{i,j+1} sin\theta_{i,j+1} - \Phi_{i,j-1} sin\theta_{i,j-1}) \].

This method, to compute the balance flux, would give a supplementary estimate for the amount of melt at the bed of Grímsvötn required to maintain the surface depression.

### 7.3 Conclusion

Presented are two methods to analyze the state of balance of an ice cap. The first method gives no information on the flow regime and can only indicate whether growth or retreat of the glacier can be expected. The other method is a comparison of two models for the flow regime. One model uses the rheology of ice to compute the flux and the other is based on a steady state assumption. Discrepancies in the modeled fluxes indicate areas where the modeled dynamical flow, which is assumed to represent the actual flow of the glacier, is in balance with the modeled mass balance distribution and where it is not in balance. The difference in the models is assumed to reflect the imbalance in the actual flow and mass-balance regime prevailing on the ice cap and conclusions about which regions are presently not in balance are drawn. This kind of conclusions are not possible to make with a simple summation of the mass balance over drainage basins as was demonstrated on Vatnajökull.

The method to compute the balance flux of an ice sheet may be improved by solving the vector flow field with a sparse matrix method. This is not done here but would be worthwhile to do as it would provide an additional estimate on how much melting is required to maintain the surface depression in Grímsvötn.

In order to compare the balance flux directly with surface velocity measurements it is necessary to estimate the average to surface-velocity ratio. This ratio determines the dynamic of the ice sheet, including ice rheology and the amount of sliding. To estimate this ratio another method, for example the dynamical model, is needed. Then the two methods are no longer independent of each other but the combined models can be compared to measurements in order to assess the state of balance.
Chapter 8

Stability of ice sheets

In this chapter the stability of ice sheets is analysed. The stability of the solution of the continuity equation is analysed with simplifying assumptions. The role of the slope of the bedrock and the ELA for the stability and steady state volume is evaluated with a 2D numerical model. These considerations show that an ice sheet on a flat bed with mass-balance elevation feedback is unstable.

Numerical modeling experiments on Vatnajökull, with the SIA model that has been developed, tested and adjusted to the conditions on Vatnajökull in Chapter 3, along with the mass balance parameterization developed in Chapter 5, are then presented. The climate forcing is kept constant in each model experiment but the height of the ELA is varied between each model run. It is shown that the modeled Vatnajökull does not settle to a steady state similar to the present ice cap. Vatnajökull reacts in an unstable way to perturbations in the ELA. Depending on the height of the ELA, the ice cap either grows to a large size or contracts to steady states that are considerably smaller than the present ice cap. It is not possible to model Vatnajökull as it appears today with constant mass balance forcing.

8.1 Analysis of the stability of ice sheets

In ice age theories the stability of continental ice sheets plays a major role in explaining how ice ages are initiated and what causes them to recede. Weertman (1961) showed, with a simple ice sheet model, that a small ice cap can become unstable and grow into a large ice age ice sheet, if its size exceeds a critical size. Further, he showed that the large ice sheet can also become unstable and shrink to nothing, as a result of relatively small changes in accumulation, or ablation.

Bodvarsson (1955) predicted a similar unstable response with a perturbation method. He showed that there is only one solution for a thin linear ice sheet moving on a horizontal plane and small perturbations from this solution will either lead to a retreat ending in complete disappearance, or growth without limit. Nye (1960) considered an ice sheet in a steady state with the mass balance as a function of the location and not the altitude. He did not expect to find in nature a system in an unstable steady state, and claimed that kinematic waves on the surface will restore the stability of the initial unstable response. He found, however, that including mass balance-elevation feedback in the model will introduce the instability of Bodvarsson (1955).
Weertman (1961) and Oerlemans (1981) found by solving the continuity equation for steady state ice sheet and by using simple numerical ice sheet models, respectively, that the mass-balance elevation feedback introduces a non-linear response which causes the ice sheet to either grow to a continental size or contract, with a possible unstable steady state in between.

There are other factors that can introduce instability in ice sheets. These are the bedrock geometry into which the ice sheet extends, steep outward slopes stabilize the glacier, but flat bed or inverse slopes have reverse effect, the isostatic rebound of the crust, the sinking of a growing ice sheet lowers the surface and decreases the accumulation, and variation in the bed conditions and sliding velocity. Small thickness and large accumulation rates are favorable for an ice sheet to be frozen to its bed. Large thickness and small accumulation rates, on the other hand, increase the probability that the ice at the bed is at melting point and the ice can slide over its bed. Increased or initiated sliding can have similar influence on the ice sheet as increase in the ablation rate and cause the ice sheet to disappear.

Model experiments with 3D models and realistic bedrock geometry have confirmed these predictions. Mahaffy (1976) modeled Barnes Ice Cap in the Northwest Territories of Canada including mass balance-elevation feed back and did not obtain any steady state. Similar model experiments for the Laurentide Ice Sheet (Andrews and Mahaffy, 1976) also failed to find a steady state after 10 000 years of computation. Payne and Sugden (1990) observe similar ice sheet evolution with a numerical ice sheet model applied to the uplands of Scotland. The ice sheet either settles to a small steady state on the mountain tops or grows to a large size. They relate this instability to the topography of the underlying bedrock and emphasize the importance of the vertical amplitude and the spatial distribution of bedrock basins and ridges in determining the pattern, rate and extent of ice sheet growth. When smaller ice caps that are formed on mountain tops grow together, a considerable amount of the previous ablation area can be raised above the ELA and the rate of growth increases.

There are three factors that primarily influence the stability of the ice sheet, the mass-balance regime which is presented with the ELA slope and the ratio of ablation to accumulation, the mass balance-elevation feed back, and the bedrock topography. In the following section the stability of the ice sheet relative to the ablation-accumulation ratio and the ELA is analysed with continuity consideration following Weertman (1961).

8.1.1 Stability of a steady state geometry with respect to shifts in ELA

The stability of the steady state solution can be investigated with similar considerations as Weertman (1961) applied on ice age ice sheets. The model for Vatnajökull uses different assumptions for the sliding and deformation velocity so the analysis is repeated here.

Consider a 2D steady state ice sheet resting on a flat bed with total length L and the center of the ice sheet at the origin of the x-coordinate. The total volume of ice passing through a vertical cross section through the ice cap per unit time must balance the integrated accumulation or ablation upstream from this cross section. If $\bar{U}$ is the average velocity in the vertical cross section and $B$ is the accumulation or ablation function, the following equation for the horizontal mass flux is valid

$$\bar{U} h = \int_0^x B \, dx.$$
The average velocity can be computed with Equation (3.1) and (3.4). The accumulation or ablation is simplified according to Weertman (1961), who assumed that the accumulation rate at every point in the accumulation area is equal to the average accumulation rate, \( a \), and that everywhere in the ablation area the ablation rate is the average ablation rate, \( c \). This assumption is reasonable, since it is known that accumulation and ablation rates are usually slowly varying functions of the distance \( x \) from the center of the ice sheet. Assume that the ELA intersect the glacier surface at distance \( R \) from the center and the length of the ice cap from the center is \( L \).

The average accumulation and the average ablation rates are defined by

\[
a = \frac{1}{R} \int_0^R B \, dx
\]

\[
c = \left| \frac{1}{L - R} \int_R^L B \, dx \right|.
\]

The requirement for the ice sheet to be in a steady state is satisfied if the accumulation and ablation are equal

\[a \, R = c \, (L - R).
\]

The average velocity can be computed with Equation (3.1). The steady state profile can then be obtained by solving

\[
\tilde{A} \, h^5 \left( \frac{\partial h}{\partial x} \right)^3 = a \, x
\]

in the accumulation area \((x < R)\). The solution is

\[
h^{8/3} = H^{8/3} - \frac{2}{A^{1/3} \, a} (a \, x)^{4/3},
\]

where \( H \) is the thickness of the ice sheet at the center. In the ablation area \((x > R)\) the equation for the steady state profile is

\[
\tilde{A} \, h^5 \left( \frac{\partial h}{\partial x} \right)^3 = a \, R - c(x - R),
\]

with \( H(R) = H_{ELA} \) the solution is

\[
h^{8/3} = H_{ELA}^{8/3} - \frac{2}{A^{1/3} \, c} \left( [a \, R - c(x - R)]^{4/3} - [a \, R]^{4/3} \right).
\]

The distance \( R \) can be obtained from this equation by setting \( \dot{h} = 0 \) at \( x = L \).
The steady state determined with these equations is unstable. The equations for H and L show that the steady state size of ice caps increases with increased value for ELA and decreased value for the accumulation. On the other hand, if the ELA is raised or the accumulation decreased, on an ice cap in a steady state, the volume of the ice cap will decrease, because the accumulation area decreases for the raised ELA and when the accumulation on a steady state ice cap is decreased the volume decreases. The new steady state corresponding to the changed value can not be reached and the ice cap will shrink until it disappears. Similarly, if the accumulation rate is increased or the ELA lowered, which both cause an ice cap to grow, the ice cap will grow to infinite. The steady state profile corresponding to the new value of ELA, or accumulation, is smaller than the original steady state and the growing ice cap cannot reach this steady state.

Weertman (1961) extended the analysis to include a linear increase in the height of the ELA towards the equator. The analysis is the same but the result is that H and L increase with increasing accumulation rate and decreasing ablation rate and decrease with decreasing accumulation and increasing ablation. Hence, the profile can be a stable one. If the rates of accumulation and ablation are changed, the ice sheet will be able to approach a new equilibrium profile. There are values of \( a \) and \( c \) for which no possible equilibrium ice sheet can exist, this is when \( a \) is small, or \( c \) or the slope of the ELA is large. A steady state ice sheet already in existence will retreat and disappear if the rates will changes to such values. Small changes in the rate of accumulation can change the size of an ice sheet in stable equilibrium by relatively large amounts and can even make it impossible for such an ice sheet to exist. Weertman (1961, page 3790) shows that the value of the slope of the ELA influences the sensitive of the stable ice sheet. The smaller the slope is the more sensitive is the size of the ice sheet to changes in accumulation, or ablation. The analysis above shows that when the slope is zero only unstable ice sheets can exist.

### 8.2 Influence of topography and the slope of the ELA on stability

The bedrock topography and the slope of the ELA relative to the bedrock can have a decisive effect on the evolution of ice caps. When the volume evolution of Vatnajökull and Hofsjökull are computed with similar model and mass balance distributions, it can be observed that while
Hofsjökull always find a steady state Vatnajökull does not. The bedrock topography on which Hofsjökull rests is basically a circular caldera with relatively steep slopes in all directions (Figure 6.5). The bedrock topography on which Vatnajökull rests has different slopes. Figure 8.2 shows cross sections through the ice cap along with the height of the reference ELA, the location of each cross section is shown in Figure 8.1. An important characteristic of the bedrock is observed. Towards the coast there are steep slopes that allow the ice to advance towards lower elevations of ablation, but on the inland side the bed is relatively flat at a high elevation. As the ice advances inland, it is not significantly lowered and thus is not ablated as much as on the coastal side. This causes the rapid growth towards north which is observed in the model computations (Figure 8.8 (A)).

8.2.1 2D model experiments

To assess the influence of the bedrock slope and the slope of the ELA on the volume evolution and steady state size of the ice cap a 2D model is used. The model bedrock is a simplified version of the actual bedrock of Vatnajökull. Figure 8.3 shows an example of the model geometry used and a resulting steady state ice cap. The mass balance gradients determined in Chapter 5 are used and the reference slope of the ELA line is 0.0055 m/m. Two experiments were done. In the first one the right side bedrock slope is changed while the ELA slope is kept at a constant value. In the second experiment the bedrock slope is kept constant and the ELA slope is varied. The steep bedrock slope on the left side is kept constant in all experiments.

The results of the model experiments are summarized in Figures 8.4 and 8.5. For a constant ELA slope the steady state volume increases as the bedrock slope is decreased. Steady states for a flat bedrock exist but the rate of growth increases with decreasing ELA slope. If the bedrock slope is kept constant and the ELA slope is decreased the steady state volume increases, with
Figure 8.2: Profiles 1–4 through Vatnajökull. The bedrock and the overlying ice is shown and the ELA is drawn with a solid straight line.
Figure 8.3: The 2D model geometry. The bedrock and the growth to a steady state, starting from no ice is shown. ELA is shown with a thin straight line. In the experiments the bedrock slope on the right side and the slope of the ELA are varied. The steady state shown is computed with bedrock slope 0.005 and ELA slope 0.0035.

Figure 8.4: Steady state volume as a function of the bedrock slope for two ELA slopes.
the exception of a symmetric bedrock (solid line in Figure 8.5). For the symmetric bedrock the steady state volume decreases when the ELA slope is decreased. Only very small ice caps exist for the symmetric bedrock. The decrease is due to the asymmetric accumulation and ablation areas and rates.

When the bedrock is asymmetric with a flatter bedrock on the side of higher ELA the steady state volume increases with decreased ELA slope (Figure 8.5). As the bedrock slope is decreased the rate of growth with decreased ELA slope increases. With a relatively flat bedrock the steady state volume grows beyond the computational domain. This can be understood from the analysis of Weertman (1961). The smaller the ELA slope is the more sensitive is the steady state volume to changes in accumulation and ablation. Tests were done by changing the accumulation or ablation rate after a steady state was reached. These changes cause the modeled ice cap to shift to another steady state, or disappear.

8.3 Model experiments on Vatnajökull with constant model parameters

The stability of Vatnajökull is analysed with the SIA model which is forced with constant mass balance input. The initial condition in all the model experiments is the present day surface geometry of the ice cap. The mass balance input is the parameterization of the mass balance data developed in Chapter 5. A reference mass balance distribution and the corresponding ELA height has been defined such that the mean specific annual balance for the present ice cap is equal to zero. Variation in the mass balance is assumed to be caused by shifts in the height of the ELA. These shifts are called $\Delta$ELA and refer to the height of the ELA plane relative to the ELA that gives the reference mass-balance distribution. In the model experiments the height of the ELA is kept constant, but as the ice cap adjusts to the mass-balance input, the mass-balance distribution changes due to changes in the surface height of the ice cap. Several model runs with different values for the $\Delta$ELA were used to analyse the adjustment of the ice cap to the mass
balance distribution. A steady state is defined such that the ice cap is in equilibrium with the applied mass balance distribution and the volume has reached a constant value. In the following sections the volume evolution, and the geometry of the adjusting ice cap is presented.

8.3.1 Volume evolution

The volume evolution of the ice cap is shown in Figure 8.6. The top panel shows all model runs with different values of ΔELA. Depending on the value for ΔELA the ice cap either grows or it retreats. When ΔELA is set to zero, the ice cap grows rapidly. Despite the initial mean specific annual balance of zero the ice cap thickens in the center and reaches higher elevation, which causes the accumulation to increase, due to the mass-balance elevation feedback, and the ice cap grows.

As the ΔELA value is increased the growth rate decreases. The initial response of the ice cap is to retreat at the largest outlet glaciers, but thicken in the accumulation area which allows the ice cap to start grow again. With values for the ΔELA up to 55 m the ice cap grows after the initial retreat. The lower panel of Figure 8.6 shows the first 500 years of the model runs. On this figure it seems that the ice cap is reaching a steady state but the top three lines, that are computed with ΔELA at and below 55 m, start to grow after 500 years as the top panel shows. With ΔELA = 55.5 m the ice cap does not grow after the initial retreat. It is slowly decreasing in volume for about 15000 computational years and then the retreat rate accelerates during about 3000 years and the ice cap settles to a steady state that is considerable smaller than the present ice cap. The increased value for the ΔELA causes the ice cap to reach a gradually smaller steady state earlier. The largest steady state obtained with this model is considerably smaller than the present size of the ice cap. When the ice cap is close to the size of the largest possible steady state, small changes in the height of the ELA cause large changes in the volume. Raising the ELA by less than 0.5 m causes the ice cap to grow rapidly. On the other hand, if the ice cap is small, changes in the ELA have smaller effect. The steady state computed with ΔELA = 100 m does not change much in volume as the ELA is raised by 20 m.

There is a critical value of the ΔELA for which a steady state is possible. If ΔELA is lower than this critical value, the ice cap does not find a steady state and grows. Figure 8.7 shows the volume of the ice cap plotted against the ΔELA at several different times (these are time slices of Figure 8.6). Steady state is only reached in model runs for which ΔELA is 55.5 m or higher.

8.3.2 Evolution of the ice cap geometry

In this section the ice cap is shown in several states of advance, retreat and steady state. The volume corresponding to each figure is shown with black dots in Figure 8.6, except the volume of the ice cap shown in Figure 8.8 (A) which is outside the margin of Figure 8.6.

Figure 8.8 (A) shows the ice cap geometry at 4000 years computed with ΔELA = 40 m. This ice cap is in a growing state. It is extending towards north and south, Dyngjujökull and Brúarjökull have grown together and are rapidly advancing. Skeiðarárjökull and Breiðamerkurjökull have also advanced considerably, and Breiðamerkurjökull has reached the ocean. Tungnaárjökull and Síðujökull, on the other hand, are not advancing as fast. The initial response of the western side of Vatnajökull is to retreat and at 4000 years it is just starting to grow again so the glaciers on this side of the ice cap are smaller relative to the present size. The outlet glaciers of Öræfajökull
Figure 8.6: Volume evolution computed with different values for the ΔELA. (A) All model runs. Dots indicate the size of the ice cap shown in Figures 8.8 - 8.10. (B) A closeup of the first 500 years of computation.
in the south and the smaller glaciers at the eastern margin have not changed significantly, they are similar to today's glaciers.

The slowly shrinking ice cap computed with $\Delta ELA = 55.5$ m is shown in Figure 8.8 (B). This is an unstable state, the ice cap continues to retreat and eventually reaches a steady state similar to the one shown in Figure 8.9 (A). This almost steady state ice cap is somewhat different in shape from the present ice cap. Many of the larger outlet glaciers have retreated but the northwestern side remains alike. Köldukvíslarjökull is very similar to the present size and Dyngjujökull is somewhat larger. The ice on the northeastern side has diminished, Brúarjökull has retreated considerably and there is a small detached ice cap on Goðahnjúkar. The western side of Vatnajökull is almost ice free and there are only small ice tongues flowing down to the former outlets of Skeiðarárjökull and Breiðarmerkurjökull. Öræfajökull and its outlet glaciers remain, however, similar in size.

The steady states computed with gradually higher value for the $\Delta ELA$ are shown in Figures 8.9 and 8.10. As the ELA is raised the resulting steady state size decreases. Parts of the ice cap segregate as the ice cap gets smaller and thins in the center. For the highest ELA, the resulting steady state consists of small separate ice caps located at the highest bedrock elevations.

### 8.3.3 Glacier profiles

Profiles along Dyngjujökull and Skeiðarárjökull and two cross sections through the glaciers were selected to elucidate the evolution of the glacier surface. These are shown in Figures 8.12-8.15 and the location of the profiles and the cross sections is shown in Figure 8.11.

The evolution of the profile along Dyngjujökull, computed with $\Delta ELA = 55.5$ and 40 m, is shown in Figure 8.12. The volume evolution for these model experiments is shown in Figure 8.6 and the surface geometry of the ice cap is shown in Figure 8.8. Figure 8.12 (A) shows how the
Figure 8.8: Ice cap geometry. (A) At 4000 years computed with $\Delta ELA = 40$ m. The ice cap is rapidly growing. (B) At 10000 years computed with $\Delta ELA = 55.5$ m. This is not a steady state, the ice cap is slowly decreasing in size and retreats to a steady state similar to Figure 8.9 (A). The line indicates the extent of the initial state.
Figure 8.9: Steady state surface geometry. (A) Computed with $\Delta ELA = 60$ m. (B) Computed with $\Delta ELA = 80$ m. The line indicates the extent of the initial state.
Figure 8.10: Steady state surface geometry. (A) Computed with ∆ELA = 100 m. (B) Computed with ∆ELA = 120 m. The line indicates the extent of the initial state.
terminus position fluctuates during the computation, at the beginning the glacier retreats three gridpoints (3 km) at the same time as the glacier thickens in the upper part, then it advances six gridpoints and finally retreats two points again where a steady state is reached. Figure 8.12 (B) shows similar initial response of retreat and thickening but the advance does not stop but rather continues rapidly. A shift in the position of the ice divide can be observed in the figure.

Figure 8.13 shows the same model runs for Skeiðarárjökull. A continuous retreat is observed in Figure 8.13 (A) and a concomitant migration of the ice divide. Figure 8.13 (B) shows how the initial retreat is followed by an advance.

The east-west (EW) and north-south (NS) cross sections in Figures 8.14 and 8.15 show how the entire ice cap evolves. The location of these profiles is shown in Figure 8.11. These figures show the importance of the Grímsvötn area on the volume evolution of the ice cap (at x = -470 000 m and y = 430 000 m on Figure 8.14 and 8.15, respectively). With $\Delta\text{ELA} = 55.5$ m the determined melt is too large, but it is too small to maintain the measured surface depression, when the $\Delta\text{ELA}$ is lowered to 40 m. Due to the fact, that there is no steady state found with this model, resembling the present ice cap, it is not possible to determine the amount of melt that would maintain the current depression.

### 8.4 Summary and Conclusions

The analysis of the continuity equation shows that the solution is unstable. There are, however, several factors that can stabilize the solution. These are the bedrock geometry, the slope of the ELA and the ratio of accumulation and ablation. The 2D model experiments confirm the result of Weertman (1961) which showed that the sensitivity of the ice cap increases with decreased slopes of the ELA and the bedrock. The spreading of the terminus of glaciers is also stabilizing. As the tongue of the glacier advances it spreads out and the ablation area increases with length.
All these factors contribute to the stability of ice caps and glaciers in nature. Hofsjökull, the neighboring ice cap of Vatnajökull, is computed with the SIA model in Chapter 6 and it is shown that this ice cap always finds a steady state. The modeled Vatnajökull, on the other hand, is not stabilized by these factors. The ice cap either grows without bounds, or it contracts to several small ice caps on the highest bedrock. The model experiments on Vatnajökull show that there is a critical size for which the ice cap is very sensitive to small changes in the climate input. Lowering the ELA by less than 0.5 m causes the ice cap to grow. This sensitivity to the height of the ELA confirms the ice age ice sheet analysis that small changes in the climate can cause unstable response of ice sheets, if the ice cap has reached a critical size. The model computations show further that the smaller glaciers and ice caps are not as sensitive to changes in the mass balance input. The glaciers on the eastern side of Vatnajökull and the outlet glaciers of Öræfajökull do not change significantly in size as the ELA is changed, even though the total volume of the ice cap changes considerably. The smallest computed steady states are not sensitive to changes in the mass balance input, 20 m change in the ELA does not change the size significantly (Figure 8.10).

Vatnajökull is presently close to the critical size, it is very sensitive to small changes in the mass-balance input. There is a critical value of ELA for which a steady state ice cap can exist. For Vatnajökull this value is ΔELA between 55 and 55.5 m. Lower values lead to unstable growth and higher values to steady states of smaller size. The critical value for Vatnajökull is approximately in the middle of the range of measured ELA heights during the last 10 years (Figure 5.8). This means that under present climatic condition volume changes of Vatnajökull could vary between rapid growth or retreat to smaller size.

The largest possible steady state computed with the constant forcing model is considerably smaller than the present ice cap. Figure 8.8 (B) and Figure 8.9 (A) show the geometry of the unstable steady state and the largest steady state, respectively. Brúarjökull in the northeast; Tungnaárjökull, Síðujökull and Skeiðarárjökull in the southwest and Breiðarmerkurjökull in southeast have retreated, indicating that these glaciers are sensitive to the climate input.

The unstable growth state is shown in Figure 8.8 (A). The ice cap is rapidly growing towards north and south but the glaciers in the east and west and the outlet glaciers of Öræfajökull are not growing as fast. This can be understood by considering the bedrock geometry (Figure 8.2). On the eastern and southern margin the bedrock is steep towards the coast which brings the ice from the accumulation area to the lower, ablation area. The northern side, on the contrary, has relatively flat bed, where the highland plateau extends towards the center of the island. The ice on the north side can, thus, not be transported to as low elevation as on the south side, where it would melt, but is accumulated and moved forward resulting in a rapidly growing ice cap. As a result the center part of the ice cap thickens and the glaciers in the south increase in size as well.

The model experiments in this chapter show that the Vatnajökull ice cap, as it is measured today, cannot be modeled with constant climate forcing. There are other factors in the nature that contribute to its present size. These can be time dependent variations in the meteorological forcing, which cause fluctuations in the mass balance distribution, seasonal or longer variations in the sliding velocity, and the frequent surges observed on many of the larger outlet glaciers of Vatnajökull. The influence of the above mentioned factors on the model computations is examined in the next chapter which discusses time dependent model experiments.
Figure 8.12: Evolution of the profile along Dyngjújökull computed with (A) \( \Delta \text{ELA} = 55.5 \, \text{m} \), and (B) \( \Delta \text{ELA} = 40 \, \text{m} \). The broken line indicates the present surface. The color indicates the time, it starts with black, then changes to red, green, and finally blue color. The steady state geometry in the model runs computed with \( \Delta \text{ELA} = 55.5 \, \text{m} \) is drawn with light green color.
Figure 8.13: Evolution of the profile along Skeiðarárjökull computed with (A) $\Delta ELA = 55.5$ m, and (B) $\Delta ELA = 40$ m. The broken line indicates the present surface. The color indicates the time, it starts with black, then changes to red, green, and finally blue color. The steady state geometry in the model runs computed with $\Delta ELA = 55.5$ m is drawn with light green color.
Figure 8.14: Evolution of the east-west profile computed with (A) $\Delta ELA = 55.5$ m, and (B) $\Delta ELA = 40$ m. The broken line indicates the present surface. The color indicates the time, it starts with black, then changes to red, green, and finally blue color. The steady state geometry in the model runs computed with $\Delta ELA = 55.5$ m is drawn with light green color.
Figure 8.15: Evolution of the north-south profile computed with (A) $\Delta ELA = 55.5$ m, and (B) $\Delta ELA = 40$ m. The broken line indicates the present surface. The color indicates the time, it starts with black, then changes to red, green, and finally blue color. The steady state geometry in the model runs computed with $\Delta ELA = 55.5$ m is drawn with light green color.
Chapter 9

Model experiments with time dependent forcing

The model computations in Chapter 8 show that present Vatnajökull can not be modeled with constant climate forcing. Either the ice cap grows without bound or it retreats to steady states that are considerably smaller than the present ice cap. Current geometry of Vatnajökull is a product of past mass-balance variations and other time dependent factors. For example varying basal motion, which in extreme cases leads to surges. During years without surges the seasonally varying basal motion causes significant changes in the surface velocity. In this chapter the influence of time dependent factors on the volume evolution of the ice cap is explored. The purpose is not to reconstruct the historical extent, which in itself is an interesting project, but rather elucidate to what extent the evolution of the ice sheet is influenced by these temporal factors.

The mass balance distribution used for the following computations is the one developed in Chapter 5. It approximates the present mass balance distribution. In this chapter a mass balance distribution for an ice cap considerably different in size from todays ice cap is produced by shifting the height of the ELA and keeping the mass balance gradients constant. By using the model in this way no account is taken for changes that the ice mass itself has on meteorological factors or general changes in the climate that caused the changes in the ice cap, but rather it is assumed that the climate is similar to present climate and variations are caused by shifts in the ELA, as it is observed today (Chapter 5). For studies of historical extent or for predictions of future changes it is more appropriate to use mass balance input based on more physics that take into account temperature and precipitation variations, such as a degree day model or an energy balance model.

Four model experiments are presented. First, the height of the ELA plane is varied by a sinusoidal function and the evolution of the ice cap observed. Second, the time it takes for the ice cap to grow, or retreat, from one steady state to another is analysed. Third, the height of the ELA plane is changed by same pattern as estimated temperature in Iceland during the last 1000 years. As an initial state the small steady states from previous chapter are used and the two estimates for the amount of sliding established from the surface velocity measurements are used. Finally, the influence of periodic surges on ice cap geometry is evaluated.
9.1 Periodic changes in the height of the ELA plane

A few model runs were done with a periodic change in the height of the ELA plane. A sinusoidal function with an amplitude of 200 m, similar to the observed variation over the last 10 years (Figure 5.8), and various periods were chosen and the height of the ELA plane shifted according to it. As a reference the model run with $\Delta \text{ELA} = 55.5$ m was used and that is the height about which the ELA plane varies. The resulting volume evolutions are shown in Figure 9.1. For comparison the volume evolution without changes in the height is also shown. The dotted line is computed with 10 years period, but because an output was only written every 10 years the periodic changes are not visible in this line. The other lines that are shown are computed with periods 50 years and 400 years. Computations with 400 years period were started both by raising and by lowering the plane. The retreat of the ice cap is faster and larger than when the ELA plane is not changing. The volume around which the ice cap fluctuates is significantly smaller than the resulting steady state volume without periodic changes in the height of the ELA plane.

![Figure 9.1: Volume evolution computed with ELA shift = 55.5 m and periodic changes in the height of the ELA plane. The solid line is computed without changes in the ELA plane, the dotted line with period 10 years, and the other lines with period 50 years and 400 years, one run starting by raising the plane, the other by lowering it.](image)

9.2 Shift from one steady state to another

The steady states computed with the constant mass balance forcing model runs in Chapter 8 are considerably smaller than present ice cap (Figures 8.9 and 8.10). To estimate the volume time-scale, the time it takes to reach a new steady state, model runs with a shift in the $\Delta \text{ELA}$ were done. Initial condition was each of the four steady states computed with $\Delta \text{ELA} = 60, 80, 100$ and 120 m. At $t = 0$ the value of the $\Delta \text{ELA}$ was shifted by $\pm 20$ m. Figure 9.2 shows the volume evolution of these model runs.
Figure 9.2: Volume evolution computed with step changes in ΔELA. Initial geometries are the steady states computed with the constant mass balance forcing (Figures 8.9 and 8.10). The volume of these steady states are shown with the straight dotted lines. At $t = 0$ the value of ΔELA is shifted by ± 20 m.

The time it takes the ice cap to reach the new steady state is dependent on the size of the ice cap, the larger ice cap, the longer it takes to reach a steady state. It takes about 10,000 years to change between the two largest steady states and about 1000 years between the two smallest steady states. The model run shown with a dash-dotted line in Figure 9.2) indicates that there are two possible steady states with the ΔELA = 80 m, one has volume of about 1000 km$^3$ and is reached from larger ice cap, the other has volume of 700 km$^3$ and is reached from a smaller ice cap.

<table>
<thead>
<tr>
<th>Ice cap geometry</th>
<th>Number of points</th>
<th>$H_{\text{max}}$ (m)</th>
<th>$b_{\text{min}}$ (m a$^{-1}$)</th>
<th>$\tau_M$ (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The initial ice cap</td>
<td>8247</td>
<td>916</td>
<td>7.2</td>
<td>127</td>
</tr>
<tr>
<td>ΔELA = 60 m</td>
<td>5437</td>
<td>951</td>
<td>7.7</td>
<td>123</td>
</tr>
<tr>
<td>ΔELA = 80 m</td>
<td>4134</td>
<td>812</td>
<td>6.9</td>
<td>117</td>
</tr>
<tr>
<td>ΔELA = 100 m</td>
<td>3022</td>
<td>797</td>
<td>6.5</td>
<td>123</td>
</tr>
<tr>
<td>ΔELA = 120 m</td>
<td>2712</td>
<td>793</td>
<td>6.9</td>
<td>134</td>
</tr>
</tbody>
</table>

Table 9.1: The number of ice covered points (each with area of 1 km$^2$), the maximum thickness, the lowest mass balance and the resulting volume timescale for the different ice cap geometries.

A method for determining the response time, or the volume time-scale, of glacier was introduced by Jóhannesson et al. (1989) who used glaciers in a steady state to find the time lag between a climate change and the glacier response. According to this method the response time of a glacier $\tau_M$, may be estimated by the equation

$$\tau_M = \frac{H}{-b_t},$$

where $H$ is a thickness scale of the glacier and $(-b_t)$ is a scale of the ablation along its terminus.
Values of the maximum thickness and the lowest ablation on the ice cap along with the value of $\tau_M$ for the initial geometry and the different steady states of Vatnajökull are given in Table 9.1. The values for $\tau_M$ are similar for all the ice cap geometries, approximately 120-130 years. This value is considerably smaller than the time it takes to change between the steady states. The theoretical estimates do not take the mass balance-elevation feed back into account. As discussed in Jóhannesson et al. (1989) this feed back leads to somewhat longer response time than is expected from Equation (9.1).

### 9.3 Estimate of temperature in Iceland in the last 1000 Years

An estimate of temperature in Iceland for most of the era of human settlement in Iceland has been made (Bergthórsson, 1969). Three sources of information were used; temperature measurements dating back to 1846, annual number of days affected by ice on the coast back to 1591, and number of severe years in every decade from historical information. The definition of such severe years is that there has been starvation, people dying from lack of food, or that the drift ice has reached the SW-coast of Iceland. Correlation between annual temperature and incidence of drift ice is used to estimate temperature for 1591-1846. Then temperature and ice incidence are correlated with frequency of severe years 1591-1969 which makes it possible to estimate the temperature and ice incidence in 930-1591.

![Figure 9.3: Estimated temperature in Iceland in 1000 years. Both running 30-year means of temperature and decadal running means (from 1591) are shown. The solid lines are temperature measurements (from 1846) and the dotted lines are based on estimates from ice incidence and severe years. Reproduced from Figures 5 and 6 in Bergthórsson (1969).](image)

Figure 9.3 shows the estimated temperature, reproduced from Figures 5 and 6 in Bergthórsson (1969). 30 years running mean temperature, one value plotted for every decade, is shown, except for a period in the 15th century when information is considered to be too meager. For comparison the decadal running mean after 1591 is also shown. The solid lines are the temperature measurements available since 1846. This figure shows that at the time of settlement
Figure 9.4: Volume evolution computed from the two steady states (computed with $\Delta ELA = 55.5$ and 80 m), the corresponding heights of the ELA, and both lower and higher estimate for the sliding parameter. The dots indicate the volumes of the ice geometry shown in Figure 9.6.

Figure 9.5: Mass balance evolution computed with two different initial conditions, two heights of the ELA, and both lower and higher estimate for the sliding parameter.
Figure 9.6: Evolution of Vatnajökull computed with the climate history for 1000 years. The left figures show results computed with initial ELA=55.5 m and the right figures with ELA=80m. The volume evolution is shown in Figure 9.4.
the temperature was similar to present temperature followed by several shifts of approximately one degree during the next 300 years. From 13th century temperature stayed low with smaller fluctuation and not until beginning of 20th century did it rise again more than one degree. The gap in the temperature data was interpolated by using a cubic spline function. This estimate can be used to force the flow model of Vatnajökull to investigate the influence of varying mass balance on the size of the ice cap.

Temperature and precipitation changes influence the height of the ELA, which in turns affects the mass balance of the ice cap (Ohmura et al., 1992). Johannesson et al. (1995) used a degree day model to predict that a warming of 2 K will lead to a 220 m or 180 m rise in the ELA of Sátujökull, which is an outlet glacier of Hofsjökull, for no precipitation change or 10 % precipitation increase, respectively. Here, a 100 m change in the height of the ELA for each temperature degree is assumed. The height of the ELA is varied with the same pattern as the 30-year running mean temperature line shown in Figure 9.3. Two initial conditions were tested, the steady states resulting from computations with $\Delta \text{ELA} = 55.5 \text{ m}$ and $\Delta \text{ELA} = 80 \text{ m}$ (Figure 8.9). The initial value of the $\Delta \text{ELA}$ was either 55.5 m or 80 m, depending on the initial condition. The temperature record was shifted down 4 degrees to make the reference close to either the initial or present temperatures. Colder temperatures lower the ELA height. Both the lower and the higher estimate for the sliding parameter were applied.

The resulting volume evolution is shown in Figure 9.4 and the mass balance evolution is shown in Figure 9.5. These figures show a range of volume increase which depends on the initial conditions, the initial height of the ELA, and the amount of sliding. The values of these factors are chosen within a range obtained from the available measurements. All model runs indicate that the ice cap constantly grows during the 1000 years. The present volume of Vatnajökull is within the range of the resulting volumes. The geometries of the ice cap corresponding to the dots in Figure 9.4 are shown in Figure 9.6.

9.4 Periodic surges

All the above described model computations are done without incorporating an important characteristic of the flow regime of Vatnajökull, the surges. As discussed in Chapter 7 there are regular surges occurring on almost all of the larger outlet glaciers of Vatnajökull (Figure 7.8). A detailed account of known surges in Iceland is given by Björnsson et al. (2002b). Surges can have substantial influence on the size and evolution of the ice cap. It is probable that a considerable portion of the ice is transported to ablation area during surges. The surges of Tungnaárjökull 1945-1946 and Brúarjökull 1963-1964 have been investigated in detail (Guðmundsson and Björnsson, 1992a,b; Guðmundsson et al., 1996) and there is much information available from the recent surges on Tungnaárjökull in 1992-1995, Síðujökull in 1992-1994 and Dyngjujökull 1998-2000. This data, along with the dynamical flow model could be used to analyse the influence of surges on the flow regime, ice divides, water paths and the overall stability of the ice cap.

Recent model experiments with a simple ice sheet geometry which include periodic surges (Guðmundsson et al., 2002) show that surging ice caps are smaller than non-surging ones. The change in size is dependent on the duration of the surge cycle relative to the time it takes the ice cap to reach a new steady state in the absence of surges. Similar 2D model experiments done with the bed geometry which is similar to the bedrock geometry displayed in Figure 8.2, in
Figure 9.7: Surface geometry of surging ice cap. The steady state geometry without surges is shown along with the longest and shortest extent of the surging ice cap. The slope of the ELA is the same as the reference slope determined for Vatnajökull.

Figure 9.8: The length of the surging ice cap.
order to imitate the situation on Vatnajökull, give essentially the same results as Gudmundsson et al. (2002) obtained. First a steady state ice cap was computed with a constant value of sliding on the right half of the steady state ice cap, no sliding was assumed on the left half of the ice cap. Then the sliding factor, C, in Equation (3.4), was increased thousandfold during one year, every 80 years. The steady state ice cap and the maximum and minimum extents of the surging ice cap are shown in Figure 9.7. The length of the ice cap during the computations is shown in Figure 9.8. After the onset of periodic surges the mean size of the ice cap decreases and after about 3000 years the surging ice cap oscillates around a mean size that is considerably smaller than the steady state size without surges. The left side of the ice cap also retreats, due to the relocation of the ice divide and overall smaller volume of the ice cap. Several model runs with varying model input reveal that the reduction in size and the length increase during surge is dependent on various factors; the mass balance regime (the ELA slope and the ratio of accumulation and ablation), the slope of the bedrock, the duration of the surge cycle, the response time of the glacier, the increase in velocity during surge and probably others factors as well. The kink in the surface of the largest extent of the surging glacier is caused by the spatial change in the value for the sliding factor. There is only sliding on the right side from the ice divide of the initial steady state ice cap and no sliding on the left side. During the surge the surface lowers above the ELA line but it is raised below the ELA line.

Several model experiments were done with the 3D SIA model of Vatnajökull to explore the influence that surges have on the volume evolution of Vatnajökull. The complicated interplay between the varying periods of surges observed on each of the outlet glaciers, the amount of increase in the sliding velocity during surges and the mass-balance elevation feedback make it a challenging problem. Another factor that makes the model experiments troublesome is that due to the increased velocity during surges, the time step of the computations must be reduced. This increases the computing time of each model experiments manifold and reduces the possibility to perform such experiments on the available computer sources. For practical reasons the model experiments with the geometry of Vatnajökull including surges were abandoned.

9.5 Discussion

The response time of Vatnajökull obtained by the model computations from one steady state to another is considerably longer than predicted with theoretical methods, assuming that the response time is equal to the volume time scale. It takes 1000 years to grow or retreat between the smallest steady states and 10 000 years between the largest steady states, which are considerably smaller than the present ice cap. The theoretical estimate gives volume time scale of about 100 years.

The model computations with periodically changing ELA show that there is a time lag between the varying ELA height and the volume response. A close look at Figure 9.1 reveals that this time lag is about one quarter of the period, for both the 50 and the 400 years period. The growth of the ice cap continues after the ELA starts to rise again and the maximum volume is reached at the same time as the ELA is half way to its highest value. This period dependent time lag indicates that the volume time scale is larger than both periods and as Figure 9.2 indicates is the volume time scale on the order of 1000 years or longer. The discrepancy between the observed and the theoretical value can partly be explained by the fact that the mass-balance elevation feedback is not taken into account in the theoretical value, which would lead to a somewhat longer response time (Jóhannesson et al., 1989).
Similar model computations for the neighboring ice cap Hofsjökull indicate that the response time of Hofsjökull is between 50 and 100 years (Jóhannesson, 1991). This response time is in good agreement with the theoretical predictions. The reason for the discrepancy between the response time and the volume time scale on Vatnajökull is not clear.

By observing the response of Vatnajökull during the time dependent computations another estimate for the response time can be obtained. The applied temperature history has several fluctuations which cause the mean specific net balance to change from positive to negative values for a brief period. The response to this negative mass balance can be seen in Figure 9.4 as a slow down in the volume increase. The last 100 years are interesting for the time scale considerations, the temperature increases relatively fast and then lowers briefly at the very end of the record. The resulting volume response shows that the increased temperature has influenced the volume by turning the growth to contraction, but the brief cooling has not yet influenced the volume. This indicates that the volume response time of Vatnajökull is larger than several decades because the response in Figure 9.4 has not reacted to the very last abrupt temperature decrease, but smaller than 100 years because the temperature increase during the 20th century has caused retreat of the growth of the ice cap. This is a similar estimate for the time scale as Jóhannesson (1986) obtained from observations of terminus positions and the same temperature record that was used in this experiment, he concludes that the response time of the Icelandic glaciers is considerably longer than 30 years (the length of the increase in temperature 1930-1960) and considerably shorter than 700 years (the duration of the colder period).

The model runs which used the pattern of the temperature change during the last 1000 years in Iceland to shift the ELA evoke more questions than answers. They give good indication to where both the model parameters and the input data must be amended. These points are briefly discussed in the following.

The extent of the glaciers at the time of settlement 1000 years ago is not well known. There are, however, evidence that the glaciers were smaller than at present (for example: Thorarinsson, 1956) and, thus, initial states for the model runs were chosen the steady states obtained with the raised ELA planes. Either state is plausible as an initial condition and here it is assumed that they present a realistic range for the condition at this time.

The speed of growth is clearly dependent on the initial height of the ELA plane, around which the temperature induced changes fluctuate. Not only the initial height of the ELA plane influences the speed of growth, but also the amount of sliding. Figure 9.4 shows clearly that with the initial $\Delta ELA = 55.5$ m and the lower estimate of sliding the ice cap growth is the largest. The available surface velocity measurement show that the amount of sliding is continuously changing. The two estimates represent the range for sliding determined from present day measurements, how it may have evolved during these 1000 years is hard to estimate.

The amount of melting in Grímsvötn can have a considerable influence on the evolution of the ice cap, as shown in Figure 4.15. In these model runs the melt was kept constant at the present estimate. How the volcanic activity has evolved during this 1000 years is almost impossible to simulate, but it has certainly not been constant (Björnsson and Guðmundsson, 1993).

The model runs do not include any isostatic rebound or attempt to estimate the excavating power of the glaciers. As can be observed in Figure 9.6 the present bedrock depressions, that were created by the growing glaciers, are simply sitting there waiting for the glaciers to come and fill them again. This is a very crude way to model the evolution of the ice cap and can have a considerable influence on the result. There are existing estimates for the glacio-isostatic crustal
movements caused by historical volume changes of Vatnajökull (Sigmundsson and Einarsson, 1992; Einarsson et al., 1996) which should be incorporated in the model to make the ice cap evolution more realistic.

All the above addressed points have an influence on the resulting volume evolution. The model parameters and input could partly be confined by using measured extent of the glaciers during the last 250 years (Thorarinsson, 1943) and using mass balance input based on a degree day model along with the shorter, more detailed, temperature record. There will, however, many uncertainties regarding the evolution of these parameters remain. Despite the many ambiguities, an important question can partly be answered with the results of this model experiment.

The question is whether the ice cap can grow from a small size to its present size during the 1000 years of human settlement in Iceland. The answer is yes, as the model computations indicate that applying the suggested temperature history with the flow model a growth from small size to present size is possible. Figure 9.6 shows how the detached small ice caps grow together and form an ice cap similar to the present ice cap. The faster growth, computed with the larger initial condition and the initial \( \Delta \text{ELA} = 55.5 \text{ m} \), shows that during the 1000 years most of the glaciers reach their present extent. Tungnaárdjökull and Skeiðarárjökull in the south become similar to present state, most of the smaller glacier remain similar in size, and Brúarjökull and Breiðamerkurjökull have almost grown back to their present size, while Dyngjujökull and the northwestern part has grown well beyond its present margin. The slower growth presented on the right side of Figure 9.6 is the lower limit and shows that while most of the glaciers do not reach its present extent the northwestern side reaches a state similar to today's state. As noted in the computations in Chapter 9 it can be seen that Öræfajökull and its outlet glaciers are not sensitive to the mass balance forcing and do not change much during the model runs.

9.6 Conclusion

The model experiments with periodic changes in the ELA and the simple ice sheet model reveal that both the periodic changes in the height of the ELA plane and in the sliding velocity, which imitate surges, cause the ice cap to retreat and fluctuate about a state considerably smaller than the steady state without the periodic changes. This result suggests that these temporal factors could possibly prevent the apparent unlimited growth of Vatnajökull computed with the steady state model and stabilize the model to a size which is similar to the present ice cap.

There is a discrepancy in the estimate of the response time of Vatnajökull. On one hand, the theoretical estimate of the volume time scale and the observation of the response to the time varying climate forcing indicate that the volume time scale of Vatnajökull is about 100 years. On the other hand, the model computations show that the time it takes to grow or retreat from one steady state to another is much longer. It takes about 1000 years between the smallest steady states and 10 000 years to grow or retreat between the largest steady states. The response time depends strongly on mass-balance elevation feedback and the size of the climate change.

Preliminary model experiments with time dependent mass balance forcing show that the 1000 year period starting at the time of settlement in Iceland, when glaciers were believed to be much smaller than today, is enough time to build up an ice sheet similar in size as the present ice sheet.
Chapter 10

Summary and Outlook

Each chapter in the thesis has its own discussion and conclusions, here the main results of the modeling work are summarized, and some suggestions for further work given.

10.1 Summary

Two types of models have been used to model Vatnajökull, a full system model that solves all terms of the model equations and Shallow Ice Approximation (SIA) model which is a zeroth order approximation that is valid when the thickness of the ice is much smaller than the length. Model inter comparison showed that when the horizontal scale is larger than about 10 times the ice thickness the SIA gives essentially the same results as full system model. For the overall dynamical response of Vatnajökull the SIA model is appropriate. It is shown that a grid size of 1 km is sufficient to compute the volume evolution of Vatnajökull. In the case of the surface depression east of Grímsvötn the inclusion of horizontal stress gradients is necessary to obtain a reasonably good approximation to the solution of the full system. In Appendix A the full system model is used to model the surface depression and the flow parameter for Vatnajökull is determined.

Data collected by Science Institute, University of Iceland on the bedrock and surface geometry, surface velocity and mass balance are used to adjust the SIA ice flow model to describe Vatnajökull. The modeled velocity is compared with the surface velocity measurements to determine the flow law and sliding parameters.

The mass-balance distribution on Vatnajökull is described with a parameterization of the mass-balance measurements 1992–2000, using statistical methods. The observed changes in the mass balance can be presented with a shift in the ELA. During the period of measurements a trend towards more negative mass balance is observed.

Several numerical schemes were tested to solve the ice sheet equation. It is shown that in areas where there are ice free, nunatak points within the ice sheet, care must be taken in treating the boundary to avoid creating mass. An upstream scheme is necessary to prevent mass flow out of the ice free points. In applications with simple geometry or ice cap with no nunatak points the ADS-I and the ADS-I upstream methods give essentially the same result, but in the presence of nunatak points the ADS-I methods is not mass conserving. The upstream method is used in the model experiments on Vatnajökull.
A method to assess the state of balance is presented in Chapter 7. This method is a model inter comparison, one model based on the modeled mass balance distribution and the other on the flow model. Discrepancies in the two models are assumed to reflect imbalance in the flow regime of Vatnajökull. With the method it is possible to identify surge type glaciers.

The model experiments with the SIA model, using the upstream scheme to solve the ice sheet equation show that Vatnajökull is presently not in a stable state. With constant mass balance forcing it is not possible to model an ice cap similar in size to the present size. Depending on the height of the ELA the ice cap either grows without bound or it contracts to steady states that are considerably smaller than the present size. The ice cap is very sensitive to the mass balance forcing, small changes in the ELA that are within the range measured 1992 – 2000, cause large volume changes. The steady states, obtained with the constant mass balance forcing, are less sensitive to changes in the ELA than the present ice cap. Óræfajökull, in the south of the ice cap, and the small outlet glaciers in the east of the ice cap are not sensitive, variations in the ELA cause small changes in the volume of these glaciers. These model results on Vatnajökull confirm theoretical predictions of Weertman (1961) that small ice caps or glaciers on steep bedrock are stable. If the ice cap grows it becomes very sensitive to small changes and when it has reached a critical size it can grow to a continental size. Vatnajökull is presently close to this critical size.

Similar model experiments on Hofsjökull, the neighboring ice cap, show that Hofsjökull is stable. With constant mass balance forcing in the flow model this ice cap always settles to a steady state.

The model experiments presented in Chapter 9 indicate that time dependent forcing in the model computation can have a considerable influence on the volume evolution. Periodic changes in both the ELA and in sliding velocity cause the ice cap to retreat to smaller size than when the ELA and the sliding are kept constant. Similar factors in the nature, varying climate, seasonal or longer changes in sliding velocity and surges, are probably the reason that Vatnajökull is presently larger than the largest possible steady state but does not grow to cover the whole island.

10.2 Outlook

The dynamical flow model presented in this thesis is a tool to investigate the dynamical response of Vatnajökull. The mass balance forcing, used here to force the flow model, describes the present mass-balance distribution and serves well to analyse the present state and stability of the ice cap. In order to analyse past response or possible future changes it is more appropriate to couple a degree-day mass-balance model (Johannesson et al., 1995) or energy mass balance model (de Ruyter de Wildt et al., ress) with the dynamical flow model.

The numerical schemes tested in this thesis are efficient for the presented model computations. When including surges in the model the computing time increases manifold and partly due to the long computing time the computations including surges were abandoned. It might prove more efficient to use a flux corrected explicit scheme (Schär and Smolarkiewicz, 1996). This is not tested in this thesis. A more efficient solution method would make it easier to include surges in the computations.

Evidence of historical glacier extent can be used to constrain the model. The preliminary model computations with time dependent mass balance show that it is possible to estimated how the
The flow model can be forced with estimated future climate change scenarios to compute possible changes in the volume. The modeled changes provide possibility to predict changes in water supply from Vatnajökull and the future location of ice and water divides. This is valuable information for the power plants that employ water originating from Vatnajökull.

Better spatial coverage of mass balance and surface velocity measurements, especially at Skeiðarárjökull, Breiðarmerkurjökull and the southeastern part, is needed to set better constraints on the model parameters in these areas.

The model computations presented in this thesis indicate that Óræfajökull and the small glaciers at the eastern margin of Vatnajökull are not as sensitive to changes in the climate as the larger outlet glaciers that rest on a relatively flat bedrock. There are no data available to verify this model results. It would be valuable to measure surface mass balance and velocity at a few locations in these areas to test the model prediction.

With longer time series of seasonal changes in the surface velocity the estimate of sliding amount can be refined. In this thesis an estimate for the amount of basal motion was made by comparing the measured surface velocity with the model result and adjust the parameter in the Weertman type sliding law to match the measurements. No attempt was made to include basal hydrology or meltwater supply in the computations. Tests on Vatnajökull with coupled dynamic and basal hydrology models have shown that the ice-volume evolution is sensitive to various sliding parameterizations (Flowers et al., 2002). It is, thus, an important physical process that has been neglected in the modeling study here. Incorporating this process in the model might improve the results.

There are available data which could be used to refine the estimated basal melt in Grímsvötn. Because the model does not settle to a steady state similar to todays ice cap not much effort was put into adjusting this estimate. It was shown that the amount of melt in the geothermal area can have a significant influence on the volume evolution of the ice.

Vatnajökull influences the isostatic crustal movements with fluctuations in its size and at the same time the location of the coast. Including the isostatic movements in the model computations is an important next step in the model development. With knowledge of the past volume changes of Vatnajökull along with the rate of the crustal rising it would be possible to estimate the viscosity of the mantle (Sigmundsson and Einarsson, 1992).

This work was focused on developing and testing numerical model to describe Vatnajökull ice cap in Iceland. It was shown that the ice cap is very sensitive to changes in the climate. Changes that are within the measured year-to-year variation during the last 10 years can cause the ice cap to respond in an unstable manner. There are, however, time dependent factors, such as varying mass balance and basal sliding, that can stabilize the response. Future work should aim at improving the understanding of the dynamical behavior of Vatnajökull by including the time dependent changes observed in nature and employ the steadily increasing amount of data on these changes.
Appendix A

The response of a glacier to a surface disturbance, a case study on Vatnajökull ice cap

A.1 Abstract

In the course of a tremendous outburst flood (jökulhlaup) following the subglacial eruption in Vatnajökull in October 1996, a depression in the surface of the ice cap was created as a result of melting of ice from the walls of a subglacial tunnel. The surface depression was initially approximately 6 km long, 1 km wide and about 100 m deep. This “canyon” represents a significant perturbation in the geometry of the ice cap in this area where the total ice thickness ranges from about 200 to 400 m.

We present results of repeated measurements of flow velocities and elevation changes in the vicinity of the canyon made over a period of about two years. The measurements show a reduction in the depth of the canyon and a concomitant decrease in surface flow towards it over time. By calculating the transient evolution of idealized surface depressions using both analytical zeroth and first-order theories, as well as the shallow-ice approximation and a finite-element model incorporating all the terms of the momentum equations, we demonstrate the importance of horizontal stress gradients at the spatial scale of this canyon.

The transient evolution of the canyon is calculated with a two-dimensional time-dependent finite-element model with flow parameters (the parameters $A$ and $n$ of Glen’s flow law) that are tuned towards an optimal agreement with measured flow velocities. Although differences between measured and calculated velocities are comparable to measurement errors, the differences are not randomly distributed. The model is therefore not verified in detail. Nevertheless the model reproduces observed changes in the geometry over a 15 month time period reasonably well. The model also reproduces both changes in velocities and geometry considerably better than an alternative model based on the shallow-ice approximation.
A.2 Introduction

The propagation and diffusion of surface undulations on glaciers and ice caps have traditionally been a subject of considerable interest to glaciology (e.g. Hutter, 1983; Paterson, 1994; Hooke, 1998). The transient evolution of surface undulations is described by two time scales: the diffusion time scale ($t_d$) and the propagation time scale ($t_p$). These time scales are fundamental to the dynamics of large ice masses. A number of different theoretical estimates for $t_d$ and $t_p$ have been put forward (e.g. Finsterwalder, 1907; Nye, 1960; Jóhannesson et al., 1989). On short to intermediate spatial scales (about $< 10h$, where $h$ is the mean ice thickness) these time scales are strongly affected by horizontal stress gradients (Jóhannesson, 1992; Gudmundsson et al., 1998). These theoretical concepts about the transient evolution of surface undulations have seldom been validated by a direct comparison with field measurements. This may be due to the fact that surface undulations seen on glaciers are often the result of ongoing processes such as flow over bedrock bumps or differential ablation. For this reason it is often difficult to estimate accurately the changes in geometry that are solely due to viscous relaxation of the ice.

Figure A.1: Map of the northwestern part of Vatnajökull showing the location of the 1996 eruption fissure, Gjálp, and the subglacial lake Grímsvötn. The contour labels give the ice surface topography in meters above sea level. Inset shows the Vatnajökull ice cap.

An unusual event following a subglacial eruption of the volcano Gjálp (Figure A.1) in Vatnajökull, Iceland, in October 1996 unexpectedly opened the possibility of observing directly the transient response of an ice cap to sudden changes in geometry. The water that was created when large volumes of ice were melted during the eruption, did not immediately drain away from the glacier but accumulated in a subglacial lake called Grímsvötn about five km south of the eruption site (Figure A.1). Only after the water level in Grímsvötn had reached a critical level, and about 3 km$^3$ of water had accumulated in the lake, did water escape out of the lake towards the glacier margin resulting in a tremendous outburst flood (jökulhlaup) on the 5th and...
the 6th of November 1996. The water drained out of the subglacial lake Grímsvötn forming an ice tunnel beneath the ice dam along the eastern flank of the nunatak Grímsfjall (Figures A.1 and A.2). Following the jökulhlaup, the roof of the ice tunnel collapsed leaving behind a canyon about 100 m deep, 6 km long and 800 m wide.

Jökulhlaup originate from Grímsvötn approximately every five years (Björnsson, 1992; Gudmundsson et al., 1995). No visible surface depressions have, however, resulted from previous floods although they all follow a similar subglacial route around Grímsfjall as did the jökulhlaup in November 1996. This can be understood in the light of the fact that the meltwater released in the jökulhlaup in November 1996 was about 6 degrees warm whereas usually the temperature of the water is at the melting point (Gudmundsson et al., 1997).

![Map of the study area showing the location of measurement stakes where surface velocities in summer 1997 and 1998 (crosses) and autumn velocities in 1998 (diamonds) were measured. The thin dashed and dash-dotted lines show various transverse profiles which are referred to in the text. The thicker lines mark the path of surface-elevation surveys.](image)

The purpose of this paper is threefold. First we present measurements of changes in surface geometry and surface velocities along three transverse profiles across the canyon. Second, we discuss what kind of flow models are needed in order to estimate the closure rate of a surface depression of this type. We do so by calculating the transient surface evolution of idealized surface depressions having similar geometrical aspect ratios as the canyon using two different analytical and two different numerical flow models. Essentially this discussion revolves around the question of how strongly the closure rate is affected by the presence of horizontal stress gradients. Finally, we compare the results of a numerical simulation with field measurements.
A.3 Study area

Vatnajökull is a temperate ice cap with an area of about 8 300 km$^2$ (Björnsson, 1978). Its western part covers an active volcanic zone. The Grímsvötn caldera (Figure A.1), approximately in the middle of the ice cap, is among the most active geothermal areas in Iceland with frequent eruptions and associated jökulhlaups (Björnsson, 1988, 1992; Gudmundsson et al., 1995). The subglacial lake Grímsvötn is sustained by geothermal activity within the Grímsvötn caldera. The fissure eruption in October 1996 of the volcano Gjálp took place about 5 km to the north of the Grímsvötn caldera (Gudmundsson et al., 1997).

The canyon created as a result of the jökulhlaup in 1996 has an approximately north-south orientation (Figure A.2). To the west of it lies the mountain Grímsfjall. Grímsfjall is mostly covered by ice with a thickness of a few tens of meters, apart from the top which is ice free. East of the canyon lies the glacier Skeiðarárjökull. There the ice is much thicker or around 300 to 400 m.

A.4 Measurements

The surface and bed topography of northwestern Vatnajökull were surveyed prior to the eruption with radio-echo sounding methods (Björnsson, 1988; Björnsson et al., 1992). The bed geometry of the area affected by the jökulhlaup in 1996 is, therefore, known. In June 1997 the surface elevation of the canyon and the surrounding area was surveyed using kinematic GPS techniques. The measured profile is shown in Figure A.2 as a heavy dash-dotted line. Three lines of stakes were drilled into the ice and surveyed in June and August of that year with GPS equipment. Similar stake measurements were made again in June and September of the following year (profiles 1, 2a and 3a in Figure A.2). In addition, velocity over a period of 29 days was measured on another set of 3 lines in August and September 1998 (profiles 1, 2b and 3b in Figure A.2). The surface elevation along the centre of the canyon and to the east of it was measured again in September 1998 (heavy dotted line in Figure A.2). In total there are two sets of elevation measurements within a time interval of 15 months, and several sets of velocity measurements along a total of six transverse profiles.

The results of the surface-elevation measurements were used together with existing topographic maps to create surface maps of the canyon and the surrounding area (Data base of the Science Institute, Univ. of Iceland. M.T. Gudmundsson, F. Pálsson and Þ. Högnadóttir pers. comm., 1998). Unfortunately, no continuous transverse profiles of surface elevation could be measured in any of the field campaigns. Apart from the measured position of the velocity stakes, the surface elevation along the transverse profiles is, thus, not known in detail. Transverse profiles of ice thickness along the measured velocity profiles were therefore generated by interpolating the surface and bedrock maps.

The measurements in 1997 and June 1998 were made with GPS equipment that has an accuracy of about 0.5 m in the horizontal direction and 1-2 m in the vertical direction. The August and September measurements in 1998 were, on the other hand, made with instruments that have an accuracy of about 2 cm in both horizontal and about 5 cm in the vertical direction.

Figure A.3 shows these three sets of velocity measurements along the transverse profiles 1, 2, and 3. One of the objectives of the measurements was to determine if the presence of the canyon
leaves a clear and unambiguous signature in the flow field along these transverse profiles. From an inspection of Figure A.3 it becomes clear that this is only the case for profile 1 and not for profiles 2 and 3, and in fact only for the 1997 measurements of profile 1. Evidently, the velocity field, especially along profiles 2 and 3, is being determined primarily by factors other than the presence of the canyon. Data for profiles 2 and 3 are, thus, not suitable for modelling purposes. The general flow direction along profiles 2 and 3 is towards the south, or the same as the overall slope direction. Since the surface slope increases somewhat in the down-flow direction going from profile 1 towards profile 3, and because the canyon is shallower along profiles 2 and 3 than along profile 1, it is not surprising that the effects of the canyon are observed most strongly along profile 1.

Figure A.3: Measured horizontal velocities along profiles 1, 2a, 2b, 3a and 3b (Figure A.2). Only the components of the horizontal velocity vector in the direction of the corresponding profiles are shown. Average velocities from June to August 1997 (summer 1997) are shown as plus symbols, average velocities from June to August 1998 (summer 1998) as crosses, and average velocities from August to September 1997 (autumn 1997) as diamonds.
It must be stressed that Figure A.3 shows the components of the measured horizontal velocity vectors in the direction of the corresponding profiles. Directions of measured velocities are never exactly along the direction of the profiles. The difference between measured orientation of the velocity vector and the orientation of the profiles varies from stake to stake and with time. For profile 1 this difference is, for 13 of the 15 available velocity measurements, less than 20°. The two stakes were the difference was larger than 20° were situated more than five km from the canyon.

There is no indication of a seasonal variation in measured surface velocities. It is, conceivable, that such changes are masked by ongoing changes in velocities due to the evolving geometry of the canyon. That would, however, be a rather unlikely fortuitous cancellation of two effects.

A.5 Model selection

We consider the closure of the canyon to be an example of a viscous relaxation process of an incompressible creeping medium. In principle, such a relaxation process can be modelled by solving the incompressibility equation

\[ v_{i,i} = 0, \]

the linear and angular momentum equations

\[ \sigma_{ij,j} + f_i = 0, \quad \text{and} \quad \sigma_{ij} - \sigma_{ji} = 0, \]

together with a constitutive law such as Glen’s flow law and a suitable set of boundary conditions. In the above equations \( v_i \) (for \( i = 1, 2, 3 \)) are the components of the velocity vector, \( \sigma_{ij} \) are the components of the stress tensor, and \( f_i \) are the components of the volume force.

In most (two dimensional) modelling work on glaciers the starting point is not the system of equations listed above, but the continuity equation

\[ \partial_t h + \partial_x q = b, \]

where \( h \) is the surface elevation, \( q \) the ice flux, and \( b \) the mass-balance function. The flux \( q \) is then assumed to be a function of local slope and ice thickness, such that \( q \propto -h^{n+2}\partial_x h|^{n-1}\partial_x h, \)

where \( n \) is a parameter in Glen’s flow law. In this approach all terms of the linear momentum equations involving horizontal stress gradients are ignored. Examples of such models are the shallow-ice approximation (SIA) (e.g. Hutter, 1983), and the kinematic wave equation (e.g Nye, 1960). These models are sometimes referred to as zeroth-order models.

The kinematic wave equation can be written as

\[ \partial_t \Delta h + \partial_x (c_0 \Delta h) - \partial_x (D_0 (\partial_x \Delta h)) = \Delta b \]

where \( c_0 := \partial_x q(h(x), \alpha(x)) \) and \( D_0 := \partial_\alpha q(h(x), \alpha(x)) \). Here \( \alpha \) is the surface slope, \( \Delta h \) is the surface perturbation with respect to some datum geometry, and \( \Delta b \) is the accumulation function. If the datum geometry does not depend on \( x \), this equation can be solved analytically. This leads to analytical expressions for the time scales \( t_p \) and \( t_d \). We refer to the resulting zeroth-order theory of transient evolution of surface disturbances as the traditional kinematic wave theory (TKWT).
The canyon has a width of about 800 m, which is about twice the mean ice thickness of the surrounding region. This rather small width-to-thickness ratio suggests that only models which rigorously account for horizontal stress gradients will give accurate approximations to solutions of the full system of the field equations (Nye, 1969; Landon and Raymond, 1978; Kamb and Echelmeyer, 1986; Jóhannesson, 1992). The question, thus, arises how accurately the transient evolution of the canyon can be calculated with flow models such as the commonly used shallow-ice approximation (SIA). In order to answer this question the transient evolution of a Gaussian-shaped surface depression, with similar geometrical aspects ratios as the canyon, was calculated using a number of different models for both a linear medium \((n = 1)\) and a non-linear medium with \(n = 3\).

Figure A.4 shows the geometry of a Gaussian-shaped surface depression for \(t = 0.5\) as calculated using four different models for a linear medium. The predictions of the zeroth-order theories SIA and TKWT are almost identical, which is not surprising as the underlying set of assumptions of both these theories are the same. The only essential difference is that TKWT is a perturbation theory were the surface perturbations are assumed to be small compared to the mean ice thickness. For an initial depth at the centre of the depression of 100 m and a mean ice thickness of 400 m, the depth-to-thickness ratio is clearly not small. Reducing the depth of the depression by a factor of 10 results in an almost perfect agreement between the prediction of the SIA and the TKWT.

The dash-dotted lines in Figure A.4 are calculated using a transient three-dimensional first-order perturbation theory developed by Gudmundsson (unpublished). The relevant analytical solutions can be seen in Gudmundsson et al. (1998) where the solution procedure is outlined. Previously, Jóhannesson (1992) worked out the transient solutions in two dimensions using different methods. The perturbation theory takes horizontal stress gradients into account.

The transient evolution of the surface depression is also calculated with a commercial finite-element model which solves the full system of the field equations in one horizontal dimension. The finite element program was customized for ice flow by Gudmundsson (1994), and further developed by Leysinger (1998) for calculations of transient flow.

The shapes at \(t = 0.5\) predicted by the first-order perturbation theory (dash-dotted line) and the full-system finite-element calculation (long dashes) are almost identical, but differ strongly from the predictions of SIA and the TKWT (Figure A.4). This demonstrates the importance of horizontal stress gradients for the transient evolution of surface depressions having geometrical aspect ratios similar to those of the canyon. Calculations using other Gaussian-shaped surface depressions having width-to-ice-thickness ratios larger than 20 and depth-to-ice-thickness ratios smaller than 0.05 resulted, on the other hand, in identical results for all four models. The exercise was repeated for \(n = 3\) using the SIA and the (full system) numerical model, with essentially identical results with regards to the importance of stress gradients at different spatial scales. It can, thus, be concluded that a model including stress gradients must be used for a realistic simulation of the transient evolution of the canyon.
Figure A.4: Transient evolution of a Gaussian-shaped surface depression. The initial shape of the depression at $t = 0$ is shown as a solid line. All other lines depict the shape at $t = 0.5 \text{ a}$ as predicted by four different flow models. The two dash-dotted lines are based on an analytical first-order perturbation theory, for an infinitely long depression (2D), and a depression having a length in $y$ direction comparable to that of the canyon (3D). The long-dashed line follows from a numerical calculation with a transient finite-element model which solves all terms of the field equations. The dotted line is the prediction of the shallow-ice approximation, and the short-dashed line is based on the traditional kinematic wave theory. The flow parameters used in the calculations were $n = 1$ and $A = 1.0 \times 10^{-6} \text{ Pa}^{-1} \text{ a}^{-1}$. The mean ice thickness was 400 m, the half width ($\sigma_x$) was set equal to 200 m, the half length ($\sigma_y$) to 3 km (only relevant for the dash-dotted curve (3D)), and the initial depth of the surface depression was 100 m.
A.6 Model calculations

A.6.1 Numerical aspects

The numerical model used for simulating the transient evolution of the canyon is a two-dimensional finite-element model which solves equations (1) and (2), and uses Glen’s flow law as a constitutive relation. The preceding discussion shows that commonly used approximations such as the SIA cannot be used. Basal sliding is ignored.

Quadrilateral strictly-incompressible plain-strain elements are used in the model calculations. The size of the elements varies within the mesh with widths in the range of 80 to 90 m and heights in the range of 3 to 60 m. The surface evolution is calculated by explicit forward integration in time of the kinematic boundary condition

\[
\frac{\partial h}{\partial t} = -v_x\frac{\partial h}{\partial x} + v_z.
\]

(ignoring accumulation) using a two-step Lax-Wendroff scheme. The numerical model has been extensively tested by comparing it with analytical solutions based on perturbation analysis (Leysinger, 1998). The size of the time steps must fulfill the Courant stability condition \((v\frac{\Delta t}{\Delta x})^2 \leq 1\), where \(v\) is ice speed, \(\Delta x\) is a typical elements size, and \(\Delta t\) the size of the time step. For most of the calculations time steps of 3 months were used.

A.6.2 Model tuning and verification

The use of Glen’s flow law introduces two parameters \(A\) and \(n\). We treat the rate factor \(A\) as a freely tunable parameter which can be set at any value that will minimize differences between measurements and model calculations. The sensitivity of the model with respect to variations in \(n\) was also tested.

The model was run forward in time for a time period of 15 months, with the measured surface geometry along profile 1 in June 1997 as an initial upper boundary. Figure A.5a shows the velocity along four selected vertical sections for the initial model geometry. Note the extrusion flow (increase in horizontal velocity with depth) in the east flank (to the left in the figure) of the canyon, which would not be predicted by the SIA.

Figure A.5b shows calculated and measured velocities components along profile 1. There are three sets of velocity measurements available along this profile: summer 1997, summer 1998, and autumn 1998. The smallest root-mean-square error between calculated and measured surface velocities from these three measurement periods resulted in \(n = 3\) and \(A = 23 \times 10^{-16} \text{ s}^{-1} \text{ kPa}^{-3}\) (Figure A.6). This optimal value of \(A\) is about three times smaller than the value recommended for temperate ice (Paterson, 1994, page 96). It is, however, identical to the value of \(A = 23.7 \times 10^{-16} \text{ s}^{-1} \text{ kPa}^{-3}\) obtained by Gudmundsson (1999), and similar to the estimate of \(A = 20 \times 10^{-16} \text{ s}^{-1} \text{ kPa}^{-3}\) by Hubbard et al. (1998). Both Gudmundsson (1999) and Hubbard et al. (1998) arrived at their estimates for the rate factor using a calibration process of the same type as the one used here, that is, by comparing measurements of surface velocities with calculations based on numerical flow models which incorporate horizontal stress gradients.

The root-mean-square error (rms-error) between measured and modelled surface velocities is shown in Figure A.6 as a function of \(A\) for a number of different values of \(n\). The rms-error is
Figure A.5:  a) Calculated velocities along four vertical sections on profile 1 for the initial model geometry. The flags indicate the location of velocity measurements.  b) Calculated velocities along profile 1 for the initial model geometry (dotted lines), after one year (dashed line) and after 15 months (dash-dotted line). Measured velocities are plotted for comparison. Plus symbols correspond to the initial velocity, crosses to the velocity after one year and diamonds to the velocity after 15 months.

Figure A.6: The root-mean-square error between the measured and the modelled velocities for various $n$ in the fbw law, as a function of the fbw parameter $A$. 
Figure A.7: The measured velocity plotted against the modelled velocity computed with $n = 3$ and $A = 2.34 \times 10^{-16} \text{ kPa}^{-3} \text{s}^{-1}$.

Figure A.7 shows modelled velocities plotted against measured stake velocities. The data points do not all fall on a straight line with slope equal to one (shown as long dashes) indicating that there is not a perfect agreement between measured on modelled velocities. The figure shows clearly that the deviations from this line are not randomly distributed. Modelled velocities are, for example, after 12 and 15 months of forward integration considerably smaller than those measured. The model is, thus, not free from systematic errors. It is difficult to pinpoint exactly the cause for these systematic differences between measured and modelled velocities. Some deviations are to be expected because the modelled cross section is not exactly along a flow line.

Figure A.8 shows the initial surface geometry (solid line), surface geometry after 15 months of forward integration (dotted line), and the resulting surface geometry after 15 years (dashed line). The measurements of the surface geometry (crosses) were not used in the tuning procedure. Therefore, the performance of the model with respect to transient changes in geometry can be judged by comparing the dotted curve with the cross symbols. Although not all the data points autumn on the calculated curve, the depth at the centre of the canyon and the width of the canyon are reproduced with a reasonable accuracy. The model performance is, thus, acceptable with respect to changes in geometry, but poor with respect to changes in surface velocities. This can be understood given the high sensitivity of modelled surface velocities to small variations in slope and ice thicknesses.

The initial surface geometry used in the model is not well constrained by measurements. The sensitivity of the modelled velocities and surface elevation to errors in initial surface elevation was estimated through a number of model runs. The deviation between observed and modelled quantities can be reduced significantly by adjusting the initial geometry of the model within the constraints given by measurements. It is possible that systematic errors may be largely, or even fully, eliminated in this way. Although it might improve the model performance, repeated trial-and-error adjustment of the initial surface geometry is hardly a worthwhile exercise and was therefore not carried out. Further elevation measurements are needed in order to determine
whether differences in measured and modelled surface velocities are caused solely by errors in surface geometry, and not by failures of some of the modelling assumptions.

Figure A.8: Modelled and measured transient evolution of the surface along profile 1 for \( n = 3 \) and \( A = 2.34 \times 10^{-16} \text{ kPa}^{-3} \text{s}^{-1} \). Crosses and plus symbols are measured velocities towards the canyon at the beginning and 15 months after the start of the numerical simulation, respectively.

A.6.3 Discussion of model results

The optimal value of the rate factor \( A \) is three times smaller than the recommended value by Paterson (1994), but nevertheless in good agreement with results from two recently published studies (Hubbard et al., 1998; Gudmundsson, 1999). The estimated value of \( A \) would be even smaller if basal sliding, contrary to the model assumptions, significantly contributes to measured surface velocities. Changing the surface geometry of the model gives different estimates for the rate factor. Based on oblique photographs we judge the surface slopes of the model to have been underestimated rather than overestimated. Increasing the surface slopes of the model would also lead to a smaller estimate for \( A \).

The model predicts that the depth at the centre of the canyon will have reduced to 10% of its initial depth after about 15 years. This result mainly depends on the overall magnitude of the velocities, and not on the finer details of the velocity profile along the surface. The changes in geometry are caused by diffusion of the initial thickness anomaly. Small surface slopes of the surrounding area (less than one degree), and the flow direction of the undisturbed flow along the axis of the canyon rather than perpendicular to it, both make the initiation of a kinematic wave impossible. This situation is very different from the one on the steeply sloped Drift Glacier, Alaska, where the 1966-68 eruptions of Mount Redoubt lead to the triggering of a kinematic wave (Sturm et al., 1986).

The transient evolution of surface depressions on Vatnjökull having similar geometrical aspects ratios as the canyon have previously been modelled using the SIA (Jonsson et al., 1998). This modelling approach ignores the effects of horizontal stress gradients. We find that horizontal
stress gradients have a very significant effect on the time evolution of the canyon. The calculated width of the canyon using the SIA after 15 months of forward integration is more than twice as large as the measured width. Although the finite-element model did not predict surface velocities accurately, the agreement between observations and calculations is much better than that for the SIA model. Surface velocities calculated with the SIA are at least one order of magnitude too large. A rough agreement with observed surface velocities can, nevertheless, be obtained by reducing the value of the rate parameter accordingly. The order-of-magnitude lower value for $A$ than typically estimated for temperate ice obtained by Jonsson et al. (1998) may result from the shortcomings of their SIA model.

A.7 Summary

Surface velocities and surface elevation along three transverse profiles with respect to the canyon were measured repeatedly over a period of about two years. The effect of the canyon on the flow velocities could only be clearly and unambiguously detected in one of these three profiles (profile 1). Most likely the canyon affects the flow along the other two profiles as well, but the effect is to a large extent masked by large-scale flow not related to the presence of the canyon.

The transient evolution of a Gaussian-shaped surface depression having similar geometrical ratios as the canyon was calculated using a series of different flow models. The transient evolution predicted by both the shallow-ice approximation and the kinematic wave theory are essentially identical. Including horizontal stress gradients leads to significantly different predictions. The shallow-ice approximation can, thus, not be used for modelling the diffusion of a surface disturbance with a spatial scale similar to that of the canyon. The inclusion of horizontal stress gradients is essential for obtaining a reasonably good approximation to solution of the full system.

Temporal changes in surface velocities and surface geometry along profile 1 were modelled using a two-dimensional finite-element model which solves for all the terms of the momentum equations. The model has two rheological parameters $n$ and $A$ which are constant in space and time. The values for these parameters were determined from a comparison between measured and modelled surface velocities. Available measurements of elevation changes over time were on the other hand not used for model calibration. For no pair of $n$ and $A$ could an agreement between measured and observed surface velocities be obtained which was free from systematic errors. Differences between measured and modelled surface velocities are, nevertheless, on the whole similar to estimated errors in the velocity measurements. The predicted changes in surface geometry over a 15 month period were also in reasonably good agreement with measurements. This agreement was obtained without any tuning. Estimating the sensitivity of the model with respect to changes in initial geometry revealed that further measurements of elevation changes over time are desirable for future model validation.
### Appendix B

Velocity measurements 1997 and 1998

#### Mean velocity summer 1997 (estimated error 3.9 m a\(^{-1}\))

| Pole | v  | dir. | \(v_{||}\) | v  | dir. | \(v_{||}\) | v  | dir. | \(v_{||}\) |
|------|----|------|-----------|----|------|-----------|----|------|-----------|
| 1    | 36.8 | 201  | 36.5      | 14.2 | 202  | 13.1      |    |      |            |
| 2    | 21.4 | 197  | 21.0      | 53.1 | 169  | 30.0      | 31.3 | 181  | 12.1      |
| 3    | 14.9 | 180  | 13.1      | 13.4 | 195  | 11.6      | 37.5 | 187  | 18.0      |
| 4    | 10.8 | 173  | 8.9       | 10.2 | 184  | 7.7       | 31.3 | 193  | 17.9      |
| 5    | 11.4 | 153  | 6.5       |    |      |           |    |      |            |

#### Mean velocity summer 1998 (estimated error 3.6 m a\(^{-1}\))

| Pole | v  | dir. | \(v_{||}\) | v  | dir. | \(v_{||}\) | v  | dir. | \(v_{||}\) |
|------|----|------|-----------|----|------|-----------|----|------|-----------|
| 1    | 10.9 | 190  | 10.3      | 15.2 | 184  | 11.5      | 23.7 | 181  | 9.2       |
| 2    | 13.1 | 189  | 12.4      | 18.3 | 181  | 13.3      | 24.2 | 182  | 9.7       |
| 3    | 13.1 | 189  | 12.3      | 16.8 | 183  | 12.6      | 27.9 | 188  | 13.8      |
| 4    | 12.8 | 192  | 12.3      | 14.0 | 190  | 11.5      | 26.9 | 194  | 15.7      |
| 5    | 11.3 | 186  | 10.5      |    |      |           |    |      |            |

#### Mean velocity fall 1998 (estimated error 0.35 m a\(^{-1}\))

| Pole | v  | dir. | \(v_{||}\) | v  | dir. | \(v_{||}\) | v  | dir. | \(v_{||}\) |
|------|----|------|-----------|----|------|-----------|----|------|-----------|
| 1    | 13.5 | 210  | 13.5      | 19.7 | 190  | 14.6      | 26.8 | 205  | 18.3      |
| 2    | 23.1 | 188  | 16.6      | 32.1 | 191  | 15.5      |    |      |            |
| 3    | 11.1 | 195  | 10.5      | 23.4 | 189  | 17.1      | 36.3 | 193  | 18.7      |
| 4    | 10.6 | 210  | 10.5      | 20.6 | 205  | 18.4      | 30.6 | 206  | 21.2      |
| 5    | 9.6  | 195  | 9.2       | 20.0 | 191  | 15.1      | 30.2 | 211  | 22.8      |
Figure B.1: Map of the area east of Grímsfjall. Lines 1-3 are indicated with thin dash dot lines and the location of the poles with crosses and diamonds. The thick dotted line indicates the surface elevation measurements.
Appendix C

Location of velocity measurements summer 1999

H is the height above ellipsoide (WGS-84), the height of the pole is subtracted. N is the estimated difference between ellipsoide and geoide.

Reference: Fix on Hut; 64° 24’ 24.57126” N  17° 15’ 57.39289” W  1785.625 m

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Appendix D


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Appendix E

Location of velocity measurements fall 1999

H is the height above ellipsoid (WGS-84), the height of the pole is subtracted. N is the estimated difference between ellipsoid and geoid.

Reference: Fix on Hut; 64° 24’ 24.57126” N 17° 15’ 57.39289” W 1785.625 m

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<td>17°12’59.88292”</td>
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<td>-60</td>
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<td>1491.2196</td>
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</table>
Figure E.1: Location of the poles and the velocity. Mean velocity during summer 1999 was measured on poles shown with crosses, the fall velocity 1999 on poles that are shown with diamonds. Mass balance measurements were done on locations shown with stars.
## Appendix F

**Velocity measurements 22. September - 25. September 1999**

<table>
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<th>Pole</th>
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<th>Time</th>
<th>Displacement (m)</th>
<th>Velocity (cm d&lt;sup&gt;-1&lt;/sup&gt;)</th>
<th>Velocity (m a&lt;sup&gt;-1&lt;/sup&gt;)</th>
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</thead>
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Appendix G

Surface measurements

The surface evolution was surveyed around the depression and the new dent that was formed after the volcanic activity in 1998. The purpose of the surface survey is to map the surface to verify the model computations. It is, however, improbable that this will be possible because the surface has changed considerably due to the more recent volcanic activity and thus not possible to distinguish the surface evolution around the old depression from the new alterations in the surface. The path of the surface measurements is shown in Figure G.1, the location of both the old depression and the new dent is visible as it was not possible to measure there due to steep slopes and crevasses.

Figure G.1: Path of the surface elevation measurements. The location of the new depression is clearly visible as it was not possible to measure there due to steep slopes and crevasses.
## Appendix H

### Location of velocity measurements April 2000 A

H is the height above ellipsoide (WGS-84)

<table>
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<th>Time</th>
<th>Latitude</th>
<th>Longitude</th>
<th>H (m.a.e.)</th>
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</thead>
<tbody>
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<td>64°18'30.38814&quot;</td>
<td>17°59'01.25662&quot;</td>
<td>1147.1310</td>
</tr>
<tr>
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<td>17°57'29.65012&quot;</td>
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## Appendix I

**Location of velocity measurements April 2000 B**

H is the height above ellipsoid (WGS-84)

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## Appendix J

### Velocity measurements 9/13. April - 16/17. April 2000

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<th>Velocity (cm d⁻¹)</th>
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Figure J.1: Velocity measured in April 2000.

Figure J.2: Velocity measured in April 2000, a closer look.
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Acknowledgments

I am grateful for the opportunity I have had to work on this interesting and challenging ice cap which has generated continuous inspiring problems to deal with. Vatnajökull will continue to provide endless projects and puzzling problems to tackle in the future. I feel very privileged that I could work on this project in Switzerland, which not only provided pleasant and stimulating academic environment, but also fantastic opportunities to be carried away towards the mountains, and the remaining glaciers, both for pleasure and for fieldwork which I had the pleasure to participate in.

My sincere thanks go to Prof. Helgi Björnsson who gave me the opportunity to work on this thesis. He initiated and supported the project, and critically reviewed the manuscript. Without his generous support and the access to all the data that has been collected on Vatnajökull by the Science Institute, University of Iceland this thesis would not have been realised.

Valuable support and guidance was given by Dr. Hilmar Guðmundsson. He initiated the project and made it possible to do the work at VAW. He provided continuous inspiration, but also gave me the chance to dig the ditch myself. His critical review of the manuscript of this thesis improved it and is appreciated.

Prof. H.-E. Minor and Dr. Martin Funk provided all facilities and the necessary equipment at VAW, I appreciate this support. Prof. H.-E. Minor critically reviewed the manuscript and gave valuable comments.

I am grateful for the opportunity I had to receive the lectures on theoretical glaciology given by Prof. Kolumban Hutter. His patience during times when I was slow was remarkable. I am grateful for continuous and generous support as well as stimulating discussions during the progress of the thesis.

Data collection on Vatnajökull during the last 10 years has been supported by the National Power Company (Landsvirkjun), the Icelandic Road Authority, the University of Iceland Research Fund and the Iceland Glaciological Society (JÖRFÍ). I would like to thank all the people that have participated in the fieldwork. Important support was given by Finnur Pálsson, Þórdís Högnadóttir, Magnús Tumi Guðmundsson and Þorsteinn Jónsson at the Science Institute, Hannes Haraldsson at Landsvirkjun and Halldór Gíslason from JÖRFÍ.

I thank Kolumban Hutter, Martin Lüthi, Martin Funk and Heinz Blatter for spending time on parts of earlier versions of the manuscript and giving critical and valuable comments for improvements. Thomas Schuler translated the abstract to German.

All fellow students and co-workers at VAW contributed to a very pleasant and stimulating atmosphere. I am grateful for all the support I have received with computer problems and all kind of other types of problems. Matthias Wegmann warmly welcomed me in his office. He and Martin Lüthi, Andreas Bauder, Andreas Vieli and Gwendolyn Leysinger-Vieli generously gave me
access to their computer programs and supported me in my first steps in the programming and Unix worlds. Hermann Bösch was helpful concerning the use of the GPS equipment and pleasant to work with. I thank Urs Fischer, Bruno Nedela, Thomas Schuler, Stephan Suter, Maura von Moos, Jakob Helbing, Marie Rousselot, Aurel Schwerzmann, Antoine Pralong, Melanie Raymond, Karin Mellini, Elsbeth Kuriger and Monika Weber for the time spent together in front of the computer, at coffee breaks, on glaciers and in the mountains, this has been a wonderful time.

Having said this, I remember that there is more to life than science and I gratefully acknowledge Ingrid Wolf, Guðný Guðlaugsdóttir and Andri Stefánsson who spiced up life and provided important support during my time in Zürich. My parents always guided, supported and cheered me up. Thank you for always being there. Lili, Harry, Ryan, Guðrún, Disa and Dóri, thank you for all the correspondence, which helped to keep the focus, but at the same time keep thinking and extending the limits.

The work on this thesis was financially supported by the National Research Council of Iceland (Grant No. 10501), the National Power Company of Iceland (Landsvirkjun), through collaboration with the Science Institute, University of Iceland, and VAW.
ekkert mal ad setja inn bladsidu!
**Curriculum vitae**

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