A sharp Trudinger - Moser type inequality for unbounded domains in $\mathbb{R}^2$

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A sharp Trudinger - Moser type inequality
for unbounded domains in \( \mathbb{R}^2 \)

Bernhard Ruf

Abstract

The classical Trudinger-Moser inequality says that for functions with Dirichlet norm
smaller or equal to 1 in the Sobolev space \( H^1_0(\Omega) \) (with \( \Omega \subset \mathbb{R}^2 \) a bounded domain), the
integral \( \int_{\Omega} e^{4\pi u^2} dx \) is uniformly bounded by a constant depending only on \( \Omega \). If the volume
\( |\Omega| \) becomes unbounded then this bound tends to infinity, and hence the Trudinger-Moser
inequality is not available for such domains (and in particular for \( \mathbb{R}^2 \)).

In this paper we show that if the Dirichlet norm is replaced by the standard Sobolev norm,
then the supremum of \( \int_{\Omega} e^{4\pi u^2} dx \) over all such functions is uniformly bounded,
independently of the domain \( \Omega \). Furthermore, a sharp upper bound for the limits of Sobolev normalized
concentrating sequences is proved for \( \Omega = B_R \), the ball of radius \( R \), and for \( \Omega = \mathbb{R}^2 \). Finally,
the explicit construction of optimal concentrating sequences allows to prove that the above
supremum is attained on balls \( B_R \subset \mathbb{R}^2 \) and on \( \mathbb{R}^2 \).

1 Introduction

Let \( \Omega \subset \mathbb{R}^N \) denote a bounded domain. The Sobolev imbedding theorem states that \( H^1_0(\Omega) \subset L^p(\Omega) \), for \( 1 \leq p \leq 2^* = \frac{2N}{N-2} \), or equivalently, using the Dirichlet norm \( \|u\|_D = (\int_{\Omega} |\nabla u|^2 dx)^{1/2} \)
on \( H^1_0(\Omega) \),
\[ \sup_{\|u\|_D \leq 1} \int_{\Omega} |u|^p dx < +\infty, \quad \text{for} \quad 1 \leq p \leq 2^*, \]
while this supremum is infinite for \( p > 2^* \). The maximal growth \( |u|^{2^*} \) is called “critical” Sobolev
growth. In the case \( N = 2 \), every polynomial growth is admitted, but one knows by easy
examples that \( H^1_0(\Omega) \subset L^\infty(\Omega) \). Hence, one is led to look for a function \( g(s) : \mathbb{R} \to \mathbb{R}^+ \)
with maximal growth such that
\[ \sup_{\|u\|_D \leq 1} \int_{\Omega} g(u) dx < +\infty. \]

It was shown by Pohozhaev [12], Trudinger [14] and Moser [11] that the maximal growth is
of exponential type. More precisely, the Trudinger-Moser inequality states that for \( \Omega \subset \mathbb{R}^2 \)
bounded
\begin{equation}
\sup_{\|u\|_D \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) dx = c(\Omega) < +\infty \quad \text{for} \quad \alpha \leq 4\pi, \tag{1.1}
\end{equation}
The inequality is optimal: for any growth \( e^{\alpha u^2} \) with \( \alpha > 4\pi \) the corresponding supremum is
\( +\infty \).

The supremum (1.1) becomes infinite for domains \( \Omega \) with \( |\Omega| = \infty \), and therefore the
Trudinger-Moser inequality is not available for unbounded domains. Related inequalities for
unbounded domains have been proposed by Cao [5] and Tanaka [2], however they assume a growth \( e^{\alpha u^2} \) with \( \alpha < 4\pi \), i.e. with subcritical growth.

In this paper we show that replacing the Dirichlet norm \( \|u\|_D = \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2} \) by the standard Sobolev norm on \( H^1_0(\Omega) \), namely

\[
\|u\|_S = \left( \|u\|^2_D + \|u\|^2_{L^2} \right)^{1/2} = \left( \int_{\Omega} (|\nabla u|^2 + |u|^2) \, dx \right)^{1/2}
\]

yields a bound independent of \( \Omega \). More precisely, we prove

**Theorem 1.1** There exists a constant \( d > 0 \) such that for any domain \( \Omega \subset \mathbb{R}^2 \)

\[
\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) \, dx \leq d
\]

The inequality is sharp: for any growth \( e^{\alpha u^2} \) with \( \alpha > 4\pi \) the supremum is \( +\infty \).

In an interesting paper, L. Carleson and A. Chang [6] proved that the supremum in (1.1) is attained if \( \Omega = B_1(0) \), the unit ball in \( \mathbb{R}^2 \). This result was extended to arbitrary bounded domains in \( \mathbb{R}^2 \) by M. Flucher [9]. In their proof, Carleson and Chang used a “concentration-compactness” argument. They consider “normalized concentrating sequences”, i.e. normalized (in the Dirichlet norm) sequences which converge weakly to 0 and (being radial) blow up at the origin. They showed that for any such sequence \( \{u_n\} \) one has

\[
\lim_{n \to \infty} \int_{B_1(0)} (e^{4\pi u_n^2} - 1) \, dx \leq e \, |B_1|
\]

Hence, one may say that \( e \, |B_1| \) is the highest possible ”concentration” or ”non-compactness” level (see also P.L. Lions [10], and H. Brezis - L. Nirenberg [3] for the related situation for Sobolev embeddings). Carleson and Chang went on to show that

\[
\sup_{\|u\|_D \leq 1} \int_{B_1} (e^{4\pi u^2} - 1) \, dx > e \, |B_1|
\]

and hence, since no concentration can happen at a level above \( e \, |B_1| \), they concluded that the supremum in (1.1) is attained.

Let us call the maximal limit in (1.4) the *Carleson-Chang limit*, in symbol: \( cc-lim \). In [7] an explicit normalized concentrating sequence \( \{y_n\} \) with

\[
\lim_{n \to -\infty} \int_{B_1} (e^{4\pi y_n^2} - 1) \, dx = \lim_{\|u_n\|_D \leq 1} \int_{B_1} (e^{4\pi u_n^2} - 1) \, dx = e \, |B_1|
\]

was constructed.

In this paper we analyze the corresponding Carleson-Chang limit for concentrating sequences which are *normalized in the Sobolev norm*. We will show
Theorem 1.2
1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let $R > 0$ such that $|\Omega| = |B_R|$. Then

\begin{equation}
\limsup_{\|u_n\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \leq \pi e^{1-D(R)},
\end{equation}

where

\[ D(R) = 2K_0(R)[2RK_1(R) - 1/I_0(R)] > 0, \quad \text{with} \quad \lim_{R \to +\infty} D(R) = 0. \]

Here, $I_k(x)$ and $K_k(x)$ denote the $k$-th modified Bessel functions of the first and second kind, i.e. the solutions of the equation

\[ -x^2 u''(x) - xu'(x) + (x^2 + k^2)u(x) = 0, \quad k = 0, 1, 2, ... \]

2. Let $\Omega \subseteq \mathbb{R}^2$ be an arbitrary domain. Then

\begin{equation}
\limsup_{\|u_n\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \leq \pi e.
\end{equation}

3. The bound in (1.7) is sharp for $\Omega = B_R(0)$, and the bound in (1.8) is sharp for $\Omega = \mathbb{R}^2$.

It is remarkable that for $\Omega = B_1(0)$ with Dirichlet normalization and for $\Omega = \mathbb{R}^2$ with Sobolev normalization the corresponding Carleson-Chang limits coincide, that is

\[ \limsup_{\|u_n\|_{D} \leq 1} \int_{B_1} (e^{4\pi u_n^2} - 1) dx = \limsup_{\|u_n\|_{S} \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u_n^2} - 1) dx = e \pi. \]

In the final result of the paper we prove

Theorem 1.3 For any ball $\Omega = B_R(0)$ and for $\Omega = \mathbb{R}^2$ holds

\begin{equation}
\sup_{\|u\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx > e^{1-D(R)} \pi.
\end{equation}

This implies in particular that the supremum (1.9) is attained in the cases of $\Omega = B_R(0)$ and $\Omega = \mathbb{R}^2$.

2 A uniform bound

In this section we prove Theorem 1.1. We begin with

Proposition 2.1 Let $\Omega \subset \mathbb{R}^2$ denote a domain in $\mathbb{R}^2$, and let $H_0^1(\Omega)$ denote the standard Sobolev space equipped with the norm

\[ \|u\|_{S} = \left( \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right)^{1/2} \]

Then there exists a constant $d$ (independent of $\Omega$) such that

\begin{equation}
\sup_{\|u\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx \leq d.
\end{equation}
Proof. It is clear that

\[(2.2) \quad \sup_{\|u\| \leq 1} \int_\Omega (e^{4\pi u^2} - 1) dx \leq \sup_{\|u\| \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) dx \]

since any function \( u \in H^1_0(\Omega) \) can be extended by zero outside of \( \Omega \), obtaining a function in \((H^1(\mathbb{R}^2), \| \cdot \|_S)\). Hence, it is sufficient to show that

\[(2.3) \quad \sup_{\|u\| \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) dx \leq d \]

We use symmetrization (see e.g. J. Moser [11]) by defining the radially symmetric function \( u^* \) as follows:

for every \( \rho > 0 \) let \( m(\{ x \in \mathbb{R}^2 ; u^*(x) > \rho \}) = m(\{ x \in \mathbb{R}^2 ; u(x) > \rho \}) \).

Then \( u^* \) is a non-increasing function in \(|x|\). By construction

\[ \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) dx = \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) dx + \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) \]

and it is known that

\[ \int_{\mathbb{R}^2} |\nabla u^*|^2 \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx . \]

It is therefore sufficient to prove (2.3) for radially symmetric functions \( u(x) = u(|x|) \).

Thus, we may assume that \( u \) in (2.3) is radially symmetric and non-increasing. We divide the integral (2.3) into two parts, with \( r_0 > 0 \) to be chosen:

\[(2.4) \quad \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) = \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) + \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) \]

We write the second integral as

\[(2.5) \quad \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) = \sum_{k=1}^\infty \int_{|x| \geq r_0} \frac{(4\pi)^k |u|^{2k}}{k!} \]

We estimate the single terms by the following ’radial lemma’ (see Berestycki - Lions, [4], Lemma A.IV):

\[(2.6) \quad |u(r)| \leq \frac{1}{\sqrt{\pi}} \|u\|_{L^2} \frac{1}{r} , \quad \text{for all } r > 0 , \]

Hence we obtain for \( k \geq 2 \):

\[(2.7) \quad \int_{|x| \geq r_0} |u|^{2k} \leq \|u\|_{L^2}^{2k} \frac{2}{\pi^{k-1}} \int_{r_0}^\infty \frac{1}{r^{2k}} r dr = \frac{1}{k-1} \|u\|_{L^2}^{2k} \left( \frac{\|u\|_{L^2}^{2}}{\pi r_0^2} \right)^{k-1} . \]

This yields

\[(2.8) \quad \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) \leq 4\pi \|u\|_{L^2}^{2} + 4\pi \|u\|_{L^2}^{2} \sum_{k=2}^\infty \frac{1}{k!} \left( \frac{4\|u\|_{L^2}^{2}}{r_0^2} \right)^{k-1} \leq c(r_0) , \]

since \( \|u\|_{L^2} \leq 1 \).
To estimate the first integral in (2.4), let

\[ v(r) = \begin{cases} 
  u(r) - u(r_0), & 0 \leq r \leq r_0 \\
  0, & r \geq r_0 
\end{cases} \]

Then, by (2.6)

\[ u^2(r) = v^2(r) + 2v(r)u(r_0) + u^2(r_0) \leq v^2(r) + u^2(r_0) \frac{1}{\pi r_0^2} \|u\|^2_{L^2} + 1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2} \]

\[ \leq v^2(r) \left[ 1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2} \right] + d(r_0) \]

hence

\[ u(r) \leq v(r) \left( 1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2} \right)^{1/2} + d^{1/2}(r_0) =: w(r) + d^{1/2}(r_0) \]

By assumption

\[ \int_{B_{r_0}} |\nabla v|^2 \, dx = \int_{B_{r_0}} |\nabla u|^2 \, dx \leq 1 - \|u\|^2_{L^2} \]

and hence

\[ \int_{B_{r_0}} |\nabla w|^2 \, dx = \int_{B_{r_0}} |\nabla v(1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2})^{1/2}|^2 \]

\[ = (1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2}) \int_{B_{r_0}} |\nabla u|^2 \, dx \]

\[ \leq (1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2})(1 - \|u\|^2_{L^2}) \]

\[ = 1 + \frac{1}{\pi r_0^2} \|u\|^2_{L^2} - \|u\|^2_{L^2} - \frac{1}{\pi r_0^2} \|u\|^4_{L^2} \leq 1 \]

provided that \( r_0^2 \geq \frac{1}{\pi} \). Since by (2.9) \( u^2(r) \leq w^2(r) + d \) we get

\[ \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) \, dx \leq e^{4\pi d} \int_{B_{r_0}} e^{4\pi w^2} \, dx \]

The result follows by the Trudinger-Moser inequality, since \( w \in H^1_0(B_{r_0}) \) with \( \|w\|^2_D = \int_{B_{r_0}} |\nabla w|^2 \, dx \leq 1 \).

In the next proposition we show that the result is optimal (as in the Dirichlet-norm case), namely that the supremum in (2.1) becomes infinite if the exponent \( 4\pi \) is replaced by a number \( \alpha > 4\pi \).

**Proposition 2.2** Suppose that \( \alpha > 4\pi \). Then, for any domain \( \Omega \subseteq \mathbb{R}^2 \)

\[ \sup_{\|u\|_{S} \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) \, dx = +\infty. \]  

**Proof.**
We may suppose that $0 \in \Omega$, and that for some $\rho > 0$ the ball $B_\rho(0) \subset \Omega$. We use a modified "Moser-sequence", see [11], defined in $B_\rho(0)$ and continued by zero in $\Omega \setminus B_\rho(0)$, and with Sobolev-norm $\leq 1$:

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \frac{\log(\rho/|x|)}{(\log n)^{1/2}} (1 - \frac{\rho^2}{4\log n})^{1/2}, & \frac{\rho}{n} \leq |x| \leq \rho \\ (\log n)^{1/2} (1 - \frac{\rho^2}{4\log n})^{1/2}, & 0 \leq |x| \leq \rho/n \end{cases}$$

One checks that $\|m_n\|_{H^1_0(\Omega)}^2 \leq 1$, for $n$ large. Hence one has

$$\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) dx \geq \lim_{n \to \infty} \int_{B_\rho} (e^{\alpha m_n^2} - 1) dx$$

$$= 2\pi \left( \frac{\alpha}{e^{\alpha^2/4}} - 1 \right) \frac{\rho/n}{2} \to +\infty, \text{ as } n \to \infty$$

\[\square\]

3 Critical growth and concentration

Numerous studies in recent years have shown the close connection of critical growth with concentration phenomena, see e.g. the pioneering work of H. Brezis - L. Nirenberg [3].

As pointed out in the introduction, it is of particular interest to study the “highest level of noncompactness” for the functional $\int_{\Omega} (e^{4\pi u_n^2} - 1) dx$, under the restriction $\|u\|_S \leq 1$. In view of this, we make the following definition:

**Definition 3.1** A sequence $\{u_n\} \subset H^1_0(\Omega)$ is a Sobolev-normalized concentrating sequence (for short, SNC-sequence), if

a) $\|u_n\|_S = 1$

b) $u_n \rightharpoonup 0$, weakly in $H^1_0(\Omega)$

c) $\exists x_0 \in \Omega$ such that $\forall \rho > 0 : \int_{\Omega \setminus B_\rho(x_0)} (|\nabla u_n|^2 + |u_n|^2) dx \to 0$

Next, we define the Carleson-Chang limit as the maximal limit of SNS-sequences:

**Definition 3.2** Let

$$\Sigma := \{ \{u_n\} \subset H^1_0(\Omega) \mid \{u_n\} \text{ is a SNC-sequence} \}$$

and define the Carleson-Chang limit as

$$\text{cc–lim} \sup_{\|u_n\|_S \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx := \sup_{\Sigma} \lim_{n \to \infty} \int_{\Omega} (e^{4\pi u^2} - 1) dx$$

The following “concentration-compactness alternative” by P.L. Lions (restated in our notation) is relevant for our purposes:

**Proposition** (P.L. Lions, [10], Theorem I.6). Let $\{u_n\} \subset H^1_0(\Omega)$ satisfy $\|u_n\|_S \leq 1$; we may assume that $u_n \rightharpoonup u$. Then either
\{u_n\} is a SNC-sequence or
\[ \int_\Omega (e^{4\pi u_n^2} - 1)dx \rightarrow \int_\Omega (e^{4\pi u^2} - 1)dx; \] this holds in particular if \( u \neq 0 \).

Then one has

**Proposition 3.3** Suppose that
\[ S := \sup_{|u|_S \leq 1} \int_\Omega (e^{4\pi u^2} - 1)dx > \text{cc-} \lim_{\|u_n\|_S \leq 1} \int_\Omega (e^{4\pi u_n^2} - 1)dx. \]

Then the supremum \( S \) is attained.

**Proof.** Let \( \{y_n\} \) denote a maximizing sequence for \( S \), and assume that \( S \) is not attained. We may assume that \( y_n \rightharpoonup y \). By the alternative of P.L. Lions we get \( y = 0 \), and \( \{y_n\} \) is a SNC-sequence. Hence
\[ S = \lim_{n \to \infty} \int_\Omega (e^{4\pi y_n^2} - 1)dx \leq \text{cc-} \lim_{\|u_n\|_S \leq 1} \int_\Omega (e^{4\pi u_n^2} - 1)dx < S \]
Contradiction!

**4 Upper bound for the Carleson-Chang limit**

In this section we prove an explicit upper bound for the Carleson-Chang limit. In particular, we prove the estimates (1.7) and (1.8) of Theorem 1.2. In section 7 we will show that the bound in (1.7) is sharp for \( \Omega = B_R \), with any radius \( R > 0 \), and the bound in (1.8) is sharp for \( \Omega = \mathbb{R}^2 \).

**Proof.**

1. Using symmetrization as in section 2, we see that it is sufficient to prove (1.7) for radial functions in \( B_R(0) \). Following J. Moser [11] we perform the change of variables
\[ r = e^{-t/2}, \text{ and setting } w_n(t) = (4\pi)^{1/2}y_n(r), \]
we transform the radial integrals on \([0, R]\) into integrals on the half-line \([-2 \log R, +\infty)\). We will write throughout the paper: \( \alpha_R = -2 \log R \), with \( \alpha_R = -\infty \) if \( R = +\infty \). One checks that
\[ \int_{B_R} |\nabla y_n(x)|^2dx = 2\pi \int_0^R \left| \frac{d}{dr}y_n(r) \right|^2rdr = \int_{\alpha_R}^\infty |w_n'(t)|^2dt \]
and
\[ \int_{B_R} (e^{4\pi y_n^2(x)} - 1)dx = 2\pi \int_0^R (e^{4\pi y_n^2(r)} - 1)rdr = \pi \int_{\alpha_R}^\infty (e^{w_n^2(t)} - 1)e^{-t}dt \]
and similarly
\[ \int_{B_R} |y_n(x)|^2dx = 2\pi \int_0^R |y_n(r)|^2rdr = \frac{1}{4} \int_{\alpha_R}^\infty |w_n(t)|^2e^{-t}dt. \]

The SNC-sequences in this new setting are characterized by:

a) \( \|w_n\|_S^2 := \int_{\alpha_R}^\infty (|w_n'|^2 + \frac{1}{4}|w_n|^2e^{-t})dt = 1, \quad w_n(\alpha_R) = 0 \)

b) \( w_n \rightharpoonup 0 \), weakly in \( H^1([\alpha_R, +\infty)) \)
\( c) \int_{\alpha R}^{A} \left( |w_n'|^2 + \frac{1}{4} |w_n|^2 e^{-t} \right) dt \to 0 \) for any fixed \( A > 0 \),
and the estimate (1.7) (which we seek to prove) becomes
\[
\text{cc-}\lim_{\|w_n\|_{S} \leq 1} \pi \int_{\alpha R}^{\infty} (e^{w_n^2(t)} - 1)e^{-t} dt \leq \pi e^{1-D(R)}
\]
for SNC-sequences \( \{w_n\} \subset H^1([\alpha R, +\infty)) \).

Let now denote \( \{w_n\} \) a maximizing SNC-sequence for the Carleson-Chang limit (1.7). We may assume that the sequence \( \{w_n\} \) satisfies
\[
\lim_{n \to \infty} \pi \int_{\alpha R}^{\infty} (e^{w_n^2(t)} - 1)e^{-t} dt > 2 \pi e^{-D(R)},
\]
(4.5)
since otherwise the theorem is proved. Note that we may assume that \( w_n(t) \) is an increasing function on \([\alpha R, +\infty)\). Fix \( A_R \geq 1 \) such that
\[
t - 2 \log t - D(R) > 1, \ \forall \ t \geq A_R.
\]
(4.6)

**Claim 1:** There exists a number \( n_1 \) such that
\[
w_n(t) < 1, \ \forall \ t \leq A_R, \ \forall \ n \geq n_1
\]
Indeed, for \( 0 < R < +\infty \) we can estimate
\[
w_n(t) \leq (A_R + 2 \log R)^{1/2} \left( \int_{\alpha R}^{A_R} |w_n'|^2 dt \right)^{1/2}
\]
(4.7)
with \( \delta_n \to 0 \) as \( n \to 0 \), by c).

For \( R = +\infty \) and \( 0 < t \leq A_R \) we estimate
\[
w_n(t) = w_n(0) + \int_{0}^{t} w'(t) dt \leq w_n(0) + t^{1/2} \left( \int_{0}^{t} |w_n'|^2 \right)^{1/2} dt
\]
The second term goes to zero, as above. For the estimate of \( w_n(0) \) we use the following Radial Lemma (see W. Strauss, [13]), valid for radial functions \( v(r) \) in \( H^1(\mathbb{R}^2) \) and for \( r \geq 1 \):
\[
(r + \frac{1}{2})v^2(r) \leq \frac{5}{4} \int_{r}^{\infty} (|v'|^2 + |v|^2) \rho d\rho
\]
We transform this inequality (as before) by the change of variables \( r = e^{-t/2} \) and \( w(t) = (4\pi)^{1/2} v(r) \) and get, for \( t \leq 0 \):
\[
(e^{-t/2} + \frac{1}{2})w^2(t) \leq \frac{5}{2} \int_{-\infty}^{-t/2} (|w'(t)|^2 + \frac{1}{4} |w(t)|^2 e^{-t}) dt.
\]
(4.8)
Hence, we get for \( w_n(0) \), using the concentration property of \( w_n \)
\[
w_n^2(0) \leq \frac{5}{3} \int_{-\infty}^{0} (|w'(t)|^2 + \frac{1}{4} |w(t)|^2 e^{-t}) dt =: \sigma_n^2 \to 0, \ \text{as} \ n \to \infty.
\]
Thus the claim is proved.

By claim 1 we conclude that for \( n \) sufficiently large \((0 < R \leq +\infty)\)
\[
w_n^2(t) < 1 < A_R - 2\log A_R - D(R)\quad,\quad \alpha_R \leq t \leq A_R.
\]
Let now \( a_n > A_R \) denote the first \( t > A_R \) with
\[
(4.9)\quad w_n^2(a_n) = a_n - 2\log a_n - D(R).
\]
Such an \( a_n \) exists (for \( n \) sufficiently large), since otherwise
\[
w_n^2(t) < t - 2\log t - D(R)\quad,\quad \forall \quad t \geq A_R \geq 1\quad,\quad \text{as} \quad n \to \infty\,
\]
and thus
\[
\pi \int_{\alpha_R}^\infty (e^{w_n^2} - 1)e^{-t} dt \leq \pi \alpha_R (e^{w_n^2} - 1)e^{-t} + \pi \int_{A_R}^\infty e^{-2\log t-D(R)-t}
\]
The second term on the right is bounded by \( \pi e^{-D(R)} \), and in the following claim 2 we prove that the first term goes to 0, for \( n \to \infty \), and thus we have a contradiction to assumption (4.5).

**Claim 2:**
\[
\pi \int_{\alpha_R}^A (e^{w_n^2} - 1)e^{-t} dt \to 0 \quad \text{as} \quad n \to \infty.
\]
This is immediate for \( 0 < R < +\infty \), since then this term can be estimated, using (4.7), by
\[
\pi (R^2 - e^{-A_R})(e^{w_n^2(A_R+\sigma_R)} - 1) \to 0 \quad \text{as} \quad n \to \infty.
\]
If \( R = +\infty \) we write
\[
\int_{-\infty}^0 (e^{w_n^2} - 1)e^{-t} dt + \int_0^{A_R} (e^{w_n^2} - 1)e^{-t} dt
\]
The second term is now estimated as before, while for the first term we use a series expansion:
\[
\int_{-\infty}^0 (e^{w_n^2} - 1)e^{-t} dt = \int_{-\infty}^0 \sum_{k=1}^\infty \frac{|w_n(t)|^{2k}}{k!} e^{-t} dt
\]
\[
= \int_{-\infty}^0 |w_n(t)|^2 e^{-t} dt + \int_{-\infty}^0 \frac{1}{2} |w_n(t)|^4 e^{-t} dt + \sum_{k=3}^\infty \int_{-\infty}^0 \frac{|w_n(t)|^{2k}}{k!} e^{-t} dt
\]
The first term goes to zero by concentration, the second term can be estimated by Sobolev (by returning to the variable \( r \) and back to \( t \))
\[
\int_{-\infty}^0 w_n^4 e^{-t} dt \leq c_0 \left( \int_{-\infty}^0 (|w_n|^2 + \frac{1}{4} |w_n|^2 e^{-t}) dt \right)^2
\]
and hence also goes to zero by concentration. For the third term, observe that by (4.8) we get for \( t \leq 0 \)
\[
w_n^2(t) \leq \frac{5}{4} \frac{1}{e^{-t/2} + 1/2} \sigma_n^2 \leq c e^{t/2} \sigma_n^2
\]
Hence we can estimate the series as
\[
\sum_{k=3}^\infty \int_{-\infty}^0 \frac{c^k}{k!} \sigma_n^{2k} e^{k/2} e^{-t} dt \leq \sum_{k=3}^\infty c^k \sigma_n^{2k} \int_{-\infty}^0 e^{k/2} dt \leq c_1 \sigma_n^6 2
\]
and thus claim 2 is proved.

Thus we have proved the existence of a number \( a_n > A_R \) as claimed in (4.9).

We now prove, for \( 0 < R \leq +\infty \)
\[
i) \quad \pi \int_{\alpha_R}^{a_n} (e^{w_n^2} - 1)e^{-t}dt \to 0, \text{ as } n \to \infty. \\
ii) \quad \lim_{n \to \infty} \pi \int_{a_n}^{\infty} (e^{w_n^2} - 1)e^{-t}dt \leq \pi e^{1-D(R)}
\]

Proof of i): Note that the argument above shows that \( a_n \to +\infty \) as \( n \to \infty \), since for an arbitrarily large number \( A_R \) there exists \( n_0(A_R) \) such that \( a_n > A_R \) for \( n \geq n_0 \). By (4.9) we have
\[
\int_{\alpha R}^{a_n} \alpha \left( e^{w_n^2} - 1 \right) e^{-tdt} \leq \pi \int_{\alpha R}^{A} \left( e^{w_n^2} - 1 \right) e^{-tdt} + \pi \int_{A}^{a_n} e^{-2\log t - D(R)}dt
\]
Let \( \epsilon > 0 \): for the second term we get
\[
\pi e^{1-D(R)}(1 - \frac{1}{a_n}) < \epsilon/2, \text{ for } A \text{ sufficiently large, and then the first term becomes } \leq \epsilon/2, \text{ for } n \geq n_0(A, \epsilon), \text{ proceeding as in Claim 2.}
\]

Proof of ii): We apply the following basic estimate which was proved in [6] (we cite it here in the form given in [7], Proposition 2.2):

**Lemma (Carleson-Chang):** For \( a > 0 \) and \( \delta > 0 \) given, suppose that
\[
\int_{a}^{\infty} |w'(t)|^2 dt \leq \delta. \text{ Then } \\
\int_{a}^{\infty} e^{w^2 - t} dt \leq \epsilon \frac{1}{1 - \delta} e^K, \quad \text{with } K = w^2(a)(1 + \frac{\delta}{1 - \delta}) - a.
\]

We apply this Lemma to our sequence \( \{w_n\} \), with \( a = a_n \) given in (4.9), and \( \delta = \delta_n = \int_{a_n}^{\infty} (\frac{1}{4}|w_n|^2 + |w_n|^2 e^{-t}) dt \). Furthermore, in the following section 5, (5.1) and section 6, Proposition 6.4, it is shown that:

For \( a > 0 \) and \( b > 0 \) given, let
\[
S_{a,b} = \{ u \in H^1(\alpha_R, a), \ u(\alpha_R) = 0, \ \int_{\alpha_R}^{a} (|u'|^2 + \frac{1}{4}|u|^2 e^{-t}) dt = b \}.
\]

Then the supremum
\[
\sup\{\|u\|^2_\infty : u \in S_{a,b}\}
\]
is attained by a function \( y \), with
\[
\|y\|^2_\infty = y^2(a) = b(a - D(R)) + O\left(\frac{1}{a}\right).
\]

Thus, choosing \( a = a_n \) and \( b = b_n = 1 - \delta_n \) we get for \( w_n \in S_{a_n,b_n} \)
\[
w_n^2(a_n) \leq a_n - a_n\delta_n - D(R) + O(\delta_n) + O\left(\frac{1}{a_n}\right),
\]
which implies together with (4.9)
\[
\delta_n \leq \frac{2\log a_n}{a_n} + O\left(\frac{\log a_n}{a_n^2}\right)
\]
Thus we have for $K = K_n$ in the Lemma of Carleson and Chang

\begin{align}
K_n &= w_n^2(a_n)(1 + \frac{\delta_n}{1 - \delta_n}) - a_n \\
&\leq \left(a_n - a_n\delta_n - D(R) + O\left(\frac{\log a_n}{a_n}\right)\right)(1 + \delta_n + O(\delta_n^2)) - a_n \\
&= -D(R) - \delta_n D(R) + O\left(\frac{\log a_n}{a_n}\right) + a_n O(\delta_n^2) \\
&= -D(R) + O\left(\frac{(\log a_n)^2}{a_n}\right)
\end{align}

(4.11)

Hence we obtain by the Lemma of Carleson and Chang for any maximizing SNC-sequence $\{w_n\}$

\[\lim_{n \to \infty} \pi \int_{a_n}^{\alpha R} (e^{w_n^2} - 1)e^{-t}dt \leq \lim_{n \to \infty} \pi e^{1 - \delta_n} e^{K_n} \leq \pi e^{1 - D(R)};\]

thus ii) is proved.

With i) and ii) we now easily complete the proof of the first statement of Theorem 1.2

2. It is clear that for $\Omega_0 \subset \Omega_1$ the corresponding cc-limits are increasing. Thus, it is sufficient to prove 2) for $\Omega = \mathbb{R}^2$; this corresponds to setting $R = +\infty$, which was included in the proof of 1).

\section{An auxiliary variational problem}

In this section we consider the following variational problem: Determine

\[\sup \{\|u\|_{\infty}^2 \mid u \in S_{a,b}\},\]

where

\[S_{a,b} = \left\{u \in H^1(\alpha R, a) \mid u(\alpha R) = 0, \int_{\alpha R}^{a} \left(|u'|^2 + \frac{R^2}{4}|u|^2 e^{-t}\right)dt = b > 0\right\}\]

Note that $S_{a,b} \subset L^\infty(\alpha R, a)$, with compact embedding, and hence it is easily seen that the supremum in (5.1) is attained: let $y_a \in S_{a,b}$ such that

\[\|y_a\|_{\infty}^2 = \sup \{\|u\|_{\infty}^2 \mid u \in S_{a,b}\}.\]

In order to determine the value of (5.2) we need to identify the maximizing function $y_a \in S_{a,b}$. The natural way to do this consists in deriving the Euler-Lagrange equation associated to (5.1), but we encounter the difficulty that the functional $y \mapsto \|y\|_{\infty}^2$ is not differentiable. However, this functional is convex, and hence its subdifferential exists. We briefly recall this notion, and then derive the Euler-Lagrange equation for (5.1). For the proofs of some of the results we refer to [8].

\textbf{Definition 5.1} Let $E$ be a Banach space, and $\psi : E \to \mathbb{R}$ continuous and convex. Then we denote by $\partial \psi(u) \subset E'$ the subdifferential of $\psi$ in $u \in E$, given by

\[\mu_u \in \partial \psi(u) \iff \psi(u + v) - \psi(u) \geq \langle \mu_u, v \rangle, \forall v \in E;\]

here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $E$ and $E'$. An element $\mu_u \in \partial \psi(u)$ is called a subgradient of $\psi$ at $u$. 11
In [8], Lemma 2.2, it is proved that

**Lemma:** If ψ satisfies in addition

(5.3) \[ \psi(x) \geq 0, \forall x \in E, \text{ and } \psi(tx) = t^2 \psi(x), \forall t \geq 0, \]

then

\[ \mu \in \partial \psi(u) \iff \begin{cases} \langle \mu, u \rangle = 2\psi(u) \\ \langle \mu, x \rangle \leq \langle \mu, u \rangle, \forall x \in \psi^u = \{x \in E; \psi(x) \leq \psi(u)\}. \end{cases} \]

Furthermore, by an easy variation of [8], Lemma 2.3 and Corollary 2.4, one has:

**Lemma 5.2** Suppose that ψ : E → R satisfies (5.3), and φ ∈ C^1(E, R) satisfies \[ \langle \phi'(x), x \rangle = 2\phi(x), \forall x \in E. \] If y ∈ E is such that

\[ \psi(y) = \sup_{\{u \in E, \phi(u) = b\}} \psi(u), \]

then

\[ \phi'(u) \in \frac{b}{\psi(u)} \partial \psi(u) \]

**Proof.** The Euler-Lagrange equation

(5.4) \[ \phi'(u) \in \lambda \partial \psi(u) \text{ for some } \lambda > 0 \]

is obtained as in [8], Lemma 2.3 and Corollary 2.4. The value

\[ \lambda = \frac{b}{\psi(u)} \]

is found by testing (5.4) with u:

\[ 2b = 2\phi(u) = \langle \phi'(u), u \rangle = \lambda \langle \mu_u, u \rangle = \lambda 2\psi(u). \]

We now apply Lemma 5.2 to our situation, and obtain

**Theorem 5.3** Let \( E = \{v \in H^1(\alpha_R, a); v(\alpha_R) = 0\}, \) and consider

\[ \psi(u) = \|u\|_\infty^2 : E \to \mathbb{R} \]

and

\[ \phi(u) = \int_{\alpha_R}^a (|u'(x)|^2 + \frac{1}{4}|u(x)|^2 e^{-x})dx. \]

Suppose that y ∈ E satisfies

\[ \psi(y) = \sup\{\psi(u) \mid u \in E, \phi(u) = b\}; \]

then y satisfies (weakly) the equation

(5.5) \[ -y''(x) + \frac{1}{4} y(x)e^{-x} = \frac{b}{\|y\|_\infty^2} \mu_y, \text{ where } \mu_y \in \partial \psi(y) \subset E'. \]
# The auxiliary Euler-Lagrange equation

It remains to determine the subgradient \( \mu_y \) in equation (5.5). Again following [8], Lemma 2.6, 2.7 and 2.8 we find:

**Proposition 6.1** Let \( K_y = \{ x \in [\alpha_R, a]; |y(x)| = \|y\|_\infty \} \). Then

i) \( \text{supp } \mu_y \subset K_y \)

ii) \( K_y = \{ a \} \)

iii) \( \mu_y = \|y\|_\infty \delta_a \), the Dirac delta-function concentrated in the point \( a \).

Thus, equation (5.5) becomes

\[
\begin{cases}
-y'' + \frac{1}{4}y e^{-t} = \frac{b}{\|y\|_\infty} \delta_a , & \alpha_R \leq t \leq a \\
y(\alpha_R) = 0
\end{cases}
\]

From this one now concludes easily that equation (5.5) is equivalent to solving the equation

\[
\begin{cases}
-w'' + \frac{1}{4}w e^{-t} = 0 , & \alpha_R \leq t < a , \\
w(\alpha_R) = 0
\end{cases}
\]

with the condition that

\[
\int_{\alpha_R}^{a} (|w'(t)|^2 + \frac{1}{4}|w(t)|^2 e^{-t}) dt = b ;
\]

the last condition is obtained by multiplying equation (6.1) by \( y \) and integrating.

We now determine the explicit solution of equation (6.2).

**Theorem 6.2** The solution of equation (6.2) is given by

- for \( 0 < R < +\infty \):

\[
w(t) = \gamma \left( K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-t/2}) \right) =: \gamma z(t)
\]

- for \( R = +\infty \):

\[
w(t) = \gamma K_0(e^{-t/2})
\]

with unique coefficients \( \gamma = \gamma(R,a,b) \in \mathbb{R}^+ \).

Here \( I_k(x) \) and \( K_k(x) \) are the \( k \)-th modified Bessel functions of first and second kind, i.e. the solutions of the equation

\[-x^2 u''(x) - xu'(x) + (x^2 + k^2)u(x) = 0 , \ k = 1,2,...\]

**Proof.** By inspection.

It is crucial to determine with precision the value of the coefficient \( \gamma = \gamma(R,a,b) \) of \( w(t) \). This requires some lengthy calculations.

We begin by recalling the following relations for the modified Bessel functions (see e.g. [1], 9.6.27,28):

\[
\frac{d}{dx} I_0(x) = I_1(x) , \quad \frac{d}{dx} K_0(x) = -K_1(x) , \quad \frac{d}{dx} (x K_1(x)) = -x K_0(x) ,
\]

\[
\frac{d}{dx} (x K_0(x)) = -x K_1(x) .
\]
and the following integral relations
\[
\begin{align*}
\int_a^b |K_0(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (K_0^2(r) - K_1^2(r)) \right]_a^b \\
\int_a^b |K_1(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (K_1^2(r) - K_0(r)K_2(r)) \right]_a^b \\
\int_a^b |I_0(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (I_0^2(r) - I_1^2(r)) \right]_a^b \\
\int_a^b |I_1(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (I_1^2(r) - I_0(r)I_2(r)) \right]_a^b \\
\int_a^b [I_1(r)K_1(r) - I_0(r)K_0(r)]r dr &= [I_0(r)K_1(r)r]_a^b
\end{align*}
\]
\[(6.7)\]

see [1]; for the last relation use integration by parts and (6.6).

Using these relations we will prove:

**Theorem 6.3**

1) Condition (6.3) yields for the coefficient \( \gamma = \gamma(R, a, b) \) in (6.4)

\[
\gamma^2 = 4 \frac{b}{a} \left[ 1 - \frac{4}{a} C(R) \right] + O\left( \frac{1}{a^3} \right), \]

for a large, with

\[(6.8)\]

\[
C(R) = \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_2(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R) - 2K_0(R)K_1(R) \frac{K_0(R)}{I_0(R)}
\]

and \( C(\infty) = 0 \).

2) The solution \( w(t), \alpha_R \leq t \leq a \), of equation (6.2) is given by

- for \( 0 < R < \infty \):

\[
(6.9) \quad w(t) = 2 \sqrt{\frac{b}{a}} \left( 1 - \frac{4}{a} C(R) + O\left( \frac{1}{a^2} \right) \right)^{1/2} \left( K_0(\frac{e^{-t/2}}{R^2}) - \frac{K_0(R)}{I_0(R)} \frac{K_0(e^{-t/2})}{I_0(R)} \right)
\]

- for \( R = \infty \):

\[
(6.10) \quad w(t) = 2 \sqrt{\frac{b}{a}} \left( 1 + O\left( \frac{1}{a^2} \right) \right)^{1/2} K_0(e^{-t/2})
\]

**Proof.** Recall the definition of \( w(t) \) given in (6.4). We begin by evaluating the expression

\[
W^2(a) := \int_{\alpha R}^a (|w'(x)|^2 + \frac{1}{4} |w^2(x)|^2 e^{-x}) dx
\]

Using the explicit form of \( w(t) \) in (6.4), the change of variable \( r = e^{-x/2} \), and the relations (6.6), we get
Using the relations (6.7) we get

\[ W^2(a) = \frac{1}{4} \int_{\alpha R}^{a} \left\{ K_0'(e^{-x/2}) - \frac{K_0(R)}{I_0(R)} I_0'(e^{-x/2}) \right\}^2 + \left| K_0(e^{-x/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-x/2}) \right|^2 e^{-x} \, dx \]

\[ = \frac{1}{2} \int_{e^{-a/2}}^{R} \left\{ \left| -K_1(r) - \frac{K_0(R)}{I_0(R)} I_1(r) \right|^2 + \left| K_0(r) - \frac{K_0(R)}{I_0(R)} I_0(r) \right|^2 \right\} r \, dr \]

\[ = \frac{1}{2} \int_{e^{-a/2}}^{R} \left\{ |K_1(r)|^2 + \frac{K_0^2(R)}{I_0^2(R)} |I_1(r)|^2 + |K_0(r)|^2 + \frac{K_0^2(R)}{I_0^2(R)} |I_0(r)|^2 \right\} r \, dr \]

\[ + 2 \frac{K_0(R)}{I_0(R)} (K_1(r)I_1(r) - K_0(r)I_0(r)) \right\} r \, dr \]

(6.11)

Using the relations (6.7) we get

\[ \frac{1}{2} \left\{ \left[ \frac{1}{2} r^2 (K_1^2(r) - K_0(r)K_2(r)) \right] \right\}^{R}_{e^{-a/2}} + \frac{K_0^2(R)}{I_0^2(R)} \left[ \frac{1}{2} r^2 (I_1^2(r) - I_0(r)I_2(r)) \right]^{R}_{e^{-a/2}} \]

\[ + \left[ \frac{1}{2} r^2 (K_0^2(r) - K_2^2(r)) \right]^{R}_{e^{-a/2}} + \frac{K_0^2(R)}{I_0^2(R)} \left[ \frac{1}{2} r^2 (I_0^2(r) - I_1^2(r)) \right]^{R}_{e^{-a/2}} \]

\[ + 2 \frac{K_0(R)}{I_0(R)} [I_0(r)K_1(r) I_1(r)]^{R}_{e^{-a/2}} \]

(6.12)

Evaluating at the boundaries we obtain

\[ \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_2(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R) \]

\[ - \frac{1}{4} e^{-a} \left\{ K_0^3(e^{-a/2}) - K_0(e^{-a/2})K_2(e^{-a/2}) \right\} \]

\[ + \frac{K_0^2(R)}{I_0^2(R)} \left[ I_0^2(e^{-a/2}) - I_0(e^{-a/2})I_2(e^{-a/2}) \right] \]

\[ - 2e^{-a/2} \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2})K_1(e^{-a/2}) \]

(6.13)

For the terms with argument \( e^{-a/2} \), a large, we now use the following behavior of the Bessel functions for \( x > 0 \) small, see [1],9.6.7-9: :

\[ K_0(x) \sim - \log x \quad K_1(x) \sim \frac{1}{x} \quad K_2(x) \sim \frac{2}{x^2} \]

\[ I_0(x) \sim 1 \quad I_1(x) \sim \frac{1}{x} \quad I_2(x) \sim \frac{2}{x^2} \]

(6.14)

\[ x \]
We rewrite (6.16) as

\[ \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_2(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R) \]

\[-\frac{1}{4} e^{-a} \left\{ \left(-\log(e^{-a/2})\right)^2 - \left(-\log(e^{-a/2})\right) \frac{2}{e^a} \right\} + \frac{K_0^2(R)}{I_0(R)} \left[ 1 - \frac{1}{8} e^{-a} \right] \}

\[ + 2e^{-a/2}K_0(R) \frac{1}{I_0(R)} e^{-a/2} \]

\[ = \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_2(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R) \]

\[-\frac{1}{4} e^{-a} \left\{ \left( \frac{a}{2} \right)^2 - \frac{a}{2} e^a + \frac{K_0^2(R)}{I_0(R)} \left[ 1 - \frac{1}{8} e^{-a} \right] \} - 2K_0(R) \frac{1}{I_0(R)} e^{-a} \]

\[ = \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_2(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R) \]

\[ + \frac{1}{4} a - 2K_0(R) \frac{1}{I_0(R)} + O(a^2 e^{-a}) \]

\[ = \frac{1}{4} a + C(R) + O(a^2 e^{-a}), \]

with C(R) as in (6.8). Conditions (6.3) and (6.4) yield now

\[ (6.16) \quad b = \gamma^2 W^2(a) = \gamma^2 \left( \frac{1}{4} a + C(R) + O(a^2 e^{-a}) \right) \]

We rewrite (6.16) as

\[ (6.17) \quad \gamma \frac{a}{4} \left( 1 + \frac{4}{a} C(R) + O(e^{-a}) \right) = b \]

which yields for \( \gamma = \gamma(a, b) \)

\[ (6.18) \quad \gamma^2 = 4 \frac{b}{a} \left[ 1 - \frac{4}{a} C(R) \right] + O\left( \frac{1}{a^3} \right) \]

This proves 1). Assertion 2) follows now from (6.4). Formula (6.10) follows from (6.9), noting that \( C(\infty) = 0 \) and \( K_0(\infty)/I_0(\infty) = 0. \]

With this information we can now calculate the value \( \|w\|_{\infty}^2 = w^2(a) \):

**Proposition 6.4** Let \( w(t) \) denote the solution of (6.2), (6.3) and hence of (5.1). Then

\[ \|w\|_{\infty}^2 = w^2(a) = b \left[ a - D(R) \right] + O\left( \frac{1}{a} \right). \]

**Proof.** By (6.4) we have, using (6.14)

\[ w^2(a) = \gamma^2 \left( K_0(e^{-a/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2}) \right)^2 \]

\[ = 4 \frac{b}{a} \left[ (1 - \frac{4}{a} C(R)) + O\left( \frac{1}{a^2} \right) \right] \left( K_0(e^{-a/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2}) \right)^2 \]

\[ = 4 \frac{b}{a} \left[ (1 - \frac{4}{a} C(R)) \left( \frac{a}{2} - \frac{K_0(R)}{I_0(R)} \right)^2 + O\left( \frac{\log a}{a^4} \right) \right] \]

\[ = b \left[ a - 4C(R) - 4 \frac{K_0(R)}{I_0(R)} \right] + O\left( \frac{1}{a} \right) \]

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Set
\begin{equation}
D(R) = 4C(R) + 4\frac{K_0(R)}{I_0(R)} ;
\end{equation}
then (6.19) becomes
\begin{equation}
w^2(a) = b \left[ a - D(R) \right] + O\left(\frac{1}{a} \right)
\end{equation}

\section{Construction of optimal concentrating sequences}

In this section we show that the upper bounds for the Carleson-Chang limit
\begin{equation}
\text{cc–lim } \| u_n \|_S \leq \int_\Omega \left( e^{4\pi u^2} - 1 \right) dx \leq \pi e^{1-D(R)} ,
\end{equation}
given in Theorem 1.2 are sharp for \( \Omega = B_R \) and \( \Omega = \mathbb{R}^2 \). We do this by constructing explicit optimal SNC-sequences \( \{ w_n \} \) for (7.1) for which the Carleson-Chang limit is equal to the bound on the right.

The construction of this sequence follows closely the proof of the upper bound for the Carleson-Chang limit, section 4, in combination with information on the optimal sequence for the corresponding Dirichlet-norm problem, see [7].

We begin by defining the sequence \( \{ w_n(t) \} \) on \([\alpha_R, n] \): in Theorem 6.3, set \( a = n \) and \( b = 1 - \frac{2\log n}{n} \). Then, for \( 0 < R \leq +\infty \), let \( w_n(t) \) be given by (6.9) or (6.10), respectively. Thus, \( w_n(t) \) satisfies equation (6.2) with \( a = n \), and condition (6.3) with \( b = 1 - \frac{2\log n}{n} \). Furthermore, we have by Proposition 6.4
\begin{equation}
w_n^2(n) = \sup\{ \| w_n \|^2_{L_{\infty}} \mid w_n \in S_n \} = n - 2\log n - D(R) + O\left(\frac{1}{n} \right) ,
\end{equation}
where \( S_n = \{ u \in H^1(\alpha_R, n) \mid u(\alpha_R) = 0, \int_{\alpha_R}^{n} (|u'|^2 + \frac{1}{4}|u|^2 e^{-t}) dt = 1 - \frac{2\log n}{n} \} \). We remark that formula (7.2) constitutes a (late) motivation for the choice of \( a_n \) in (4.9).

It remains to define \( \{ w_n(t) \} \) in \([n, +\infty) \). Here we can follow [7] where an optimal Dirichlet normalized concentrating sequence was constructed by analyzing carefully the proof of Carleson-Chang [6].

The complete definition of the optimal SNC-sequence \( \{ w_n(t) \} \) is:

\begin{definition}
Let \( w_n(t) \) be given by:
\begin{equation}
w_n(t) = \begin{cases} 
  w_n(t) , & \text{given by (6.9) or (6.10), respectively, } \alpha_R \leq t \leq n \\
  w_n(n) + \frac{1}{w_n(n)} \log \frac{1 + A_n}{A_n + e^{-(t-n)}} & t \geq n
\end{cases}
\end{equation}

where \( A_n \in \mathbb{R}^+ \) is such that
\begin{equation}
\int_{\alpha_R}^{\infty} (|w_n'(t)|^2 + \frac{1}{4}|w_n(t)|^2 e^{-t}) dt = 1 .
\end{equation}
\end{definition}
We show that $A_n \in \mathbb{R}^+$ can be chosen as in Definition 7.1, i.e. satisfying (7.4), with the estimate

**Lemma 7.2**

(7.5) \[ A_n = \frac{1}{n^2 e} + O\left(\frac{1}{n^4}\right) \]

**Proof.** First note that by condition (6.3)

(7.6) \[ \int_{\alpha R}^n \left(\rho_n^2 + \frac{1}{4} |w_n|^2 e^{-t}\right) dt = 1 - \frac{2\log n}{n} \]

Thus, we look for a constant $A_n$ such that

(7.7) \[ \int_{n}^\infty \left(\rho_n^2 + \frac{1}{4} |w_n|^2 e^{-t}\right) dt = \frac{2\log n}{n} \]

Assume that $A_n \geq \frac{1}{3n^2}$, then one has

\[
\log\left(\frac{1 + A_n}{A_n + e^{-t-n}}\right) \leq \log(1 + \frac{1}{A_n}) \leq \log(1 + 3n^2)
\]

and then by (7.3) and using that $w_n(n) = n + O\left(\log n\right)$ (by Proposition 6.4)

\[
w_n(t) \leq w_n(n) + \frac{1}{w_n(n)} \log(1 + 3n^2) \leq 2n, \text{ for } t \geq n, \text{ n large},
\]

and hence

\[
\int_{n}^\infty |w_n|^2 e^{-t} dt \leq 4n^2 e^{-n}
\]

Therefore, condition (7.7) becomes

(7.8) \[ \int_{n}^\infty |w_n'|^2 dt = 2\frac{\log n}{n} + O\left(n^2 e^{-n}\right) \]

One proves as in [7] that this yields

\[
A_n = \frac{1}{n^2 e} + O\left(\frac{1}{n^4}\right)
\]

We now give an asymptotic lower bound for $\pi \int_{\alpha R}^{\infty} (e^{w_n^2} - 1)e^{-t} dt$, as $n \to \infty$:

**Theorem 7.3** Let \( \{w_n\} \) denote the sequence (7.3), and let $D(R)$ be given by (6.20). Then

\[
\pi \int_{\alpha R}^{\infty} (e^{w_n^2} - 1)e^{-t} \geq e \pi e^{-D(R)} \left(1 + 2D(R) \frac{\log n}{n}\right) + O\left(\frac{1}{n}\right).
\]

**Proof.**

a) First note that

(7.9) \[ \pi \int_{\alpha R}^{n} (e^{w_n^2} - 1)e^{-t} dt \geq 0, \text{ for all } n \]
b) Consider now
\[ \pi \int_{n}^{\infty} (e^{w^2_n} - 1)e^{-t} = \pi \int_{n}^{\infty} e^{w^2_n - t} + O(e^{-n}). \]
Performing the change of variables \( s = t - n \), setting
\[ v_n(s) = \frac{1}{w_n(n)} \log \frac{A_n + 1}{A_n + e^{-s}} \]
and using that by Proposition 6.4
\[ w^2_n(n) = (1 - \frac{2 \log n}{n})[n - D(R)] + O(\frac{1}{n}) \]
we obtain
\[ \pi \int_{n}^{\infty} \exp \left( [w_n(n) + v_n(s)]^2 - s - n \right) ds \]
\[ \geq \pi \int_{n}^{\infty} \exp \left( w^2_n(n) + 2w_n(n)v_n(s) - s - n \right) ds \]
\[ \geq \pi \int_{n}^{\infty} \exp \left( n - 2 \log n - D(R) + 2D(R) \frac{\log n}{n} + O(\frac{1}{n}) + 2 \frac{A_n + 1}{A_n + e^{-s}} - s - n \right) \]
\[ = \pi \int_{0}^{\infty} \exp(-2 \log n - D(R) + 2 \frac{A_n + 1}{A_n + e^{-s}} - s + 2D(R) \frac{\log n}{n} + O(\frac{1}{n})) \]
\[ = e^{-D(R)} \frac{1}{n^2} \int_{0}^{\infty} \left( 1 + \frac{A_n}{A_n + e^{-s}} \right)^2 e^{-s} ds (1 + 2D(R) \frac{\log n}{n} + O(\frac{1}{n})) \]
\[ = e^{-D(R)} \frac{1}{n^2} \frac{1 + A_n}{A_n} (1 + 2D(R) \frac{\log n}{n} + O(\frac{1}{n})) \]
\[ = e^{-D(R)} \left( 1 + 2D(R) \frac{\log n}{n} \right) + O(\frac{1}{n}), \quad \text{as} \ n \to \infty. \]
Joining (7.9) and (7.10) we get
\[ \pi \int_{\alpha R}^{\infty} (e^{w^2_n} - 1)e^{-t} dt \geq e \pi \frac{e^{-D(R)}}{n} \left( 1 + 2D(R) \frac{\log n}{n} \right) + O(\frac{1}{n}), \]
and hence the theorem is proved.

We conclude this section by proving some properties of the function \( D(R) \):

**Lemma 7.4** Let \( D(R) \) given by (6.20). Then
\[ D(R) = 4R K_0(R)K_1(R) - 2 \frac{K_0(R)}{I_0(R)}. \]
Furthermore, \( D(R) > 0, \) for all \( R \in \mathbb{R}^+ \), and
\[ D(R) \sim -2 \log R, \quad \text{as} \ R \to 0 \]
and
\[ D(R) \sim \frac{\pi}{R} e^{-2R}, \quad \text{as} \ R \to +\infty. \]
**Proof.** The explicit form of $D(R)$ is

$$
D(R) = 4C(R) + 4 \frac{K_0(R)}{I_0(R)}
$$

$$
= R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_0(R)}{I_0(R)}) \right) + 8R K_0(R) K_1(R) - 4 \frac{K_0(R)}{I_0(R)}
$$

Using the relations (see [1], 9.6.26)

$$
K_2(x) - K_0(x) = \frac{2}{x} K_1(x) \quad \text{and} \quad I_0(x) - I_2(x) = \frac{2}{x} I_1(x)
$$

we get

$$
D(R) = 6RK_0(R) K_1(R) + (2RK_0(R) I_1(R) - 4) \frac{K_0(R)}{I_0(R)} .
$$

which simplifies, using (see [1], 9.6.15)

$$
K_1(x) I_0(x) + K_0(x) I_1(x) = \frac{1}{x}
$$

to (7.11).

We prove that $D(R) > 0$, for all $R > 0$: by (7.11) we get, using again (7.13)

$$
D(R) = 2 \frac{K_0(R)}{I_0(R)} [RK_1(R) I_0(R) - 1 + RK_1(R) I_0(R)]
$$

$$
= 2 \frac{K_0(R)}{I_0(R)} [RK_1(R) I_0(R) - 1 + 1 - RK_0(R) I_1(R)] > 0 ,
$$

since $K_1(x) > K_0(x)$ and $I_0(x) > I_1(x)$, for all $x > 0$.

Next, using the behavior of the Bessel functions (6.14), for $R > 0$ small, we have

$$
D(R) \sim -4 \log R - 2(-\log R) = -2 \log R , \quad \text{for} \quad R > 0 \quad \text{small} .
$$

For the behavior of $D(R)$ at $+\infty$ we use the asymptotic behavior of the Bessel functions at $+\infty$, see [1], 9.7.1-2:

$$
I_1(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \left( 1 - \frac{4^2 - 1}{8x} \right)
$$

$$
K_1(x) \sim \frac{\pi}{\sqrt{2\pi x}} e^{-x} \left( 1 + \frac{4^2 - 1}{8x} \right)
$$

Hence, we obtain by (7.11)

$$
D(R) \sim 4R \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 - \frac{1}{8R} \right) \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 + \frac{3}{8R} \right)
$$

$$
- 2 \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 + \frac{1}{8R} \right) \sqrt{2\pi R} e^{-R} \left( 1 - \frac{1}{8R} + O\left( \frac{1}{R^2} \right) \right)
$$

$$
\sim 2\pi e^{-2R} \left( 1 + \frac{1}{4R} \right) - 2\pi e^{-2R} \left( 1 - \frac{1}{4R} \right) = \frac{\pi}{R} e^{-2R} .
$$

\[\blacksquare\]
8 The Supremum is attained

In this section we show that the supremum
\[
\sup_{\|u\|_S \leq 1} \int_\Omega (e^{4\pi u^2} - 1)dx
\]
is attained for any ball \( \Omega = B_R(0) \), as well as for \( \Omega = \mathbb{R}^2 \).

By Proposition 3.3 it suffices to prove

**Theorem 8.1** Let \( 0 < R \leq +\infty \). Then
\[
\sup_{\|u\|_S \leq 1} \pi \int_\alpha_R (e^{u^2} - 1)e^{-t}dt > \cc \lim_{\|u_n\|_S \leq 1} \pi \int_\alpha_R (e^{u^2_n} - 1)e^{-t}dt
\]

**Proof.** This follows immediately by Theorem 7.3: Choose an element of the maximizing sequence \( \{w_n\} \), with \( n \) sufficiently large. Then
\[
\sup_{\|u\|_S = 1} \pi \int_\alpha_R (e^{u^2} - 1)e^{-t}dt \geq \pi \int_\alpha_R (e^{w_n^2} - 1)e^{-t}dt > \pi e^{1-D(R)} = \cc \lim_{\|u_n\|_S \leq 1} \int_\alpha_R (e^{u^2_n} - 1)dx .
\]

This completes the proof of Theorem 1.3.

References


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