A sharp Trudinger - Moser type inequality for unbounded domains in $\mathbb{R}^2$
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Abstract

The classical Trudinger-Moser inequality says that for functions with Dirichlet norm smaller or equal to 1 in the Sobolev space $H^1_0(\Omega)$ (with $\Omega \subset \mathbb{R}^2$ a bounded domain), the integral $\int_{\Omega} e^{4\pi u^2} \, dx$ is uniformly bounded by a constant depending only on $\Omega$. If the volume $|\Omega|$ becomes unbounded then this bound tends to infinity, and hence the Trudinger-Moser inequality is not available for such domains (and in particular for $\mathbb{R}^2$).

In this paper we show that if the Dirichlet norm is replaced by the standard Sobolev norm, then the supremum of $\int_{\Omega} e^{4\pi u^2} \, dx$ over all such functions is uniformly bounded, independently of the domain $\Omega$. Furthermore, a sharp upper bound for the limits of Sobolev normalized concentrating sequences is proved for $\Omega = B_R$, the ball of radius $R$, and for $\Omega = \mathbb{R}^2$. Finally, the explicit construction of optimal concentrating sequences allows to prove that the above supremum is attained on balls $B_R \subset \mathbb{R}^2$ and on $\mathbb{R}^2$.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ denote a bounded domain. The Sobolev imbedding theorem states that $H^1_0(\Omega) \subset L^p(\Omega)$, for $1 \leq p \leq 2^* = \frac{2N}{N-2}$, or equivalently, using the Dirichlet norm $\|u\|_D = (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2}$ on $H^1_0(\Omega)$,

$$\sup_{\|u\|_D \leq 1} \int_{\Omega} |u|^p \, dx < +\infty, \text{ for } 1 \leq p \leq 2^*,$$

while this supremum is infinite for $p > 2^*$. The maximal growth $|u|^{2^*}$ is called “critical” Sobolev growth. In the case $N = 2$, every polynomial growth is admitted, but one knows by easy examples that $H^1_0(\Omega) \not\subset L^\infty(\Omega)$. Hence, one is led to look for a function $g(s) : \mathbb{R} \to \mathbb{R}^+$ with maximal growth such that

$$\sup_{\|u\|_D \leq 1} \int_{\Omega} g(u) \, dx < +\infty.$$ 

It was shown by Pohozhaev [12], Trudinger [14] and Moser [11] that the maximal growth is of exponential type. More precisely, the Trudinger-Moser inequality states that for $\Omega \subset \mathbb{R}^2$ bounded

$$\sup_{\|u\|_D \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) \, dx = c(\Omega) < +\infty \text{ for } \alpha \leq 4\pi,$$

The inequality is optimal: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the corresponding supremum is $+\infty$.

The supremum (1.1) becomes infinite for domains $\Omega$ with $|\Omega| = \infty$, and therefore the Trudinger-Moser inequality is not available for unbounded domains. Related inequalities for
unbounded domains have been proposed by Cao [5] and Tanaka [2], however they assume a growth $e^{\alpha u^2}$ with $\alpha < 4\pi$, i.e. with subcritical growth.

In this paper we show that replacing the Dirichlet norm $\|u\|_D = \left(\int_{\Omega} |\nabla u|^2 dx\right)^{1/2}$ by the standard Sobolev norm (1.2)

$$\|u\|_S = \left(\|u\|_D^2 + \|u\|_{L^2}^2\right)^{1/2} = \left(\int_{\Omega} (|\nabla u|^2 + |u|^2)dx\right)^{1/2},$$

yields a bound independent of $\Omega$. More precisely, we prove

**Theorem 1.1** There exists a constant $d > 0$ such that for any domain $\Omega \subset \mathbb{R}^2$

$$\sup_{\|u\|_S \leq 1} \int_{\Omega} \left(e^{4\pi u^2} - 1\right) dx \leq d \quad (1.3)$$

The inequality is sharp: for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum is $+\infty$.

In an interesting paper, L. Carleson and A. Chang [6] proved that the supremum in (1.1) is attained if $\Omega = B_1(0)$, the unit ball in $\mathbb{R}^2$. This result was extended to arbitrary bounded domains in $\mathbb{R}^2$ by M. Flucher [9]. In their proof, Carleson and Chang used a "concentration-compactness" argument. They consider "normalized concentrating sequences", i.e. normalized (in the Dirichlet norm) sequences which converge weakly to 0 and (being radial) blow up at the origin. They showed that for any such sequence $\{u_n\}$ one has

$$(1.4) \quad \lim_{n \to \infty} \int_{B_1(0)} \left(e^{4\pi u_n^2} - 1\right) dx \leq |B_1|$$

Hence, one may say that $e |B_1|$ is the highest possible "concentration" or "non-compactness" level (see also P.L. Lions [10], and H. Brezis - L. Nirenberg [3] for the related situation for Sobolev embeddings). Carleson and Chang went on to show that

$$(1.5) \quad \sup_{\|u\|_D \leq 1} \int_{B_1} \left(e^{4\pi u^2} - 1\right) dx > e |B_1|$$

and hence, since no concentration can happen at a level above $e |B_1|$, they concluded that the supremum in (1.1) is attained.

Let us call the maximal limit in (1.4) the **Carleson-Chang limit**, in symbol: $cc$-$\lim$. In [7] an explicit normalized concentrating sequence $\{y_n\}$ with

$$(1.6) \quad \lim_{n \to -\infty} \int_{B_1} \left(e^{4\pi y_n^2} - 1\right) dx = \text{cc-}\lim_{\|u_n\|_D \leq 1} \int_{B_1} \left(e^{4\pi u_n^2} - 1\right) dx = e |B_1|$$

was constructed.

In this paper we analyze the corresponding Carleson-Chang limit for concentrating sequences which are normalized in the Sobolev norm. We will show
**Theorem 1.2**

1. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain, and let \( R > 0 \) such that \( |\Omega| = |B_R| \). Then

\[
\text{cc-lim}_{\|u_n\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \leq \pi e^{1 - D(R)},
\]

where

\[
D(R) = 2K_0(R)[2RK_1(R) - 1/I_0(R)] > 0, \quad \text{with} \quad \lim_{R \to +\infty} D(R) = 0.
\]

Here, \( I_k(x) \) and \( K_k(x) \) denote the \( k \)-th modified Bessel functions of the first and second kind, i.e. the solutions of the equation

\[
-x^2 u''(x) - xu'(x) + (x^2 + k^2)u(x) = 0, \quad k = 0, 1, 2, ...
\]

2. Let \( \Omega \subseteq \mathbb{R}^2 \) be an arbitrary domain. Then

\[
\text{cc-lim}_{\|u_n\|_{S} \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \leq \pi e.
\]

3. The bound in (1.7) is sharp for \( \Omega = B_R(0) \), and the bound in (1.8) is sharp for \( \Omega = \mathbb{R}^2 \).

It is remarkable that for \( \Omega = B_1(0) \) with Dirichlet normalization and for \( \Omega = \mathbb{R}^2 \) with Sobolev normalization the corresponding Carleson-Chang limits coincide, that is

\[
\text{cc-lim}_{\|u_n\|_{D} \leq 1} \int_{B_1} (e^{4\pi u_n^2} - 1) dx = \text{cc-lim}_{\|u_n\|_{S} \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u_n^2} - 1) dx = e\pi.
\]

In the final result of the paper we prove

**Theorem 1.3** For any ball \( \Omega = B_R(0) \) and for \( \Omega = \mathbb{R}^2 \) holds

\[
\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx > e^{1 - D(R)} \pi.
\]

This implies in particular that the supremum (1.9) is attained in the cases of \( \Omega = B_R(0) \) and \( \Omega = \mathbb{R}^2 \).

2 A uniform bound

In this section we prove Theorem 1.1. We begin with

**Proposition 2.1** Let \( \Omega \subset \mathbb{R}^2 \) denote a domain in \( \mathbb{R}^2 \), and let \( H_0^1(\Omega) \) denote the standard Sobolev space equipped with the norm

\[
\|u\|_S = \left( \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \right)^{1/2}.
\]

Then there exists a constant \( d \) (independent of \( \Omega \)) such that

\[
\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) dx \leq d.
\]
Proof. It is clear that

\[(2.2) \quad \sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1)dx \leq \sup_{\|u\|_S \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)dx\]

since any function \(u \in H^1_0(\Omega)\) can be extended by zero outside of \(\Omega\), obtaining a function in \((H^1(\mathbb{R}^2), \| \cdot \|_S)\). Hence, it is sufficient to show that

\[(2.3) \quad \sup_{\|u\|_S \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)dx \leq d\]

We use symmetrization (see e.g. J. Moser [11]) by defining the radially symmetric function \(u^*\) as follows:

for every \(\rho > 0\) let 
\[
m((\{x \in \mathbb{R}^2; u^*(x) > \rho\}) = m((\{x \in \mathbb{R}^2; u(x) > \rho\}).
\]

Then \(u^*\) is a non-increasing function in \(|x|\). By construction

\[
\int_{\mathbb{R}^2} (e^{4\pi |u^*|^2} - 1)dx = \int_{|x| \leq r_0} (e^{4\pi u^2} - 1)dx + \int_{|x| \geq r_0} (e^{4\pi u^2} - 1)
\]

and it is known that

\[
\int_{\mathbb{R}^2} |\nabla u^*|^2 \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx.
\]

It is therefore sufficient to prove (2.3) for radially symmetric functions \(u(x) = u(|x|)\).

Thus, we may assume that \(u\) in (2.3) is radially symmetric and non-increasing. We divide the integral (2.3) into two parts, with \(r_0 > 0\) to be chosen:

\[(2.4) \quad \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1) = \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) + \int_{|x| \geq r_0} (e^{4\pi u^2} - 1)
\]

We write the second integral as

\[(2.5) \quad \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) = \sum_{k=1}^{\infty} \int_{|x| \geq r_0} \frac{(4\pi)^k |u|^{2k}}{k!}
\]

We estimate the single terms by the following ”radial lemma” (see Berestycki - Lions, [4], Lemma A.IV):

\[(2.6) \quad |u(r)| \leq \frac{1}{\sqrt{\pi}} \|u\|_{L^2} \frac{1}{r}, \text{ for all } r > 0,
\]

Hence we obtain for \(k \geq 2:\)

\[(2.7) \quad \int_{|x| \geq r_0} |u|^{2k} \leq \|u\|_{L^2}^{2k} \frac{2}{\pi^{k-1}} \int_{r_0}^{\infty} \frac{1}{r^{2k}} r dr = \frac{1}{k - 1} \|u\|_{L^2}^{2} \left( \frac{\|u\|_{L^2}^{2}}{r_0^{2}} \right)^{k-1}.
\]

This yields

\[(2.8) \quad \int_{|x| \geq r_0} (e^{4\pi u^2} - 1) \leq 4\pi \|u\|_{L^2}^{2} + 4\pi \|u\|_{L^2}^{2} \sum_{k=2}^{\infty} \frac{1}{k!} \left( \frac{4\pi \|u\|_{L^2}^{2}}{r_0^{2}} \right)^{k-1}
\]

\[
\leq c(r_0),
\]

since \(\|u\|_{L^2} \leq 1\).
To estimate the first integral in (2.4), let

\[ v(r) = \begin{cases} 
  u(r) - u(r_0) & 0 \leq r \leq r_0 \\
  0 & r \geq r_0 
\end{cases} \]

Then, by (2.6)

\[
    u^2(r) = v^2(r) + 2u(r)u(r_0) + u^2(r_0) \\
    \leq v^2(r) + v^2(r) \frac{1}{\pi r_0} \|u\|_{L^2}^2 + 1 + \frac{1}{\pi r_0} \|u\|_{L^2}^2 \\
    \leq v^2(r) \left[ 1 + \frac{1}{\pi r_0} \|u\|_{L^2}^2 \right] + d(r_0)
\]

hence

\[
    u(r) \leq v(r) \left( 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 \right)^{1/2} + d^{1/2}(r_0) =: w(r) + d^{1/2}(r_0)
\]

By assumption

\[
    \int_{B_{r_0}} |\nabla v|^2 dx = \int_{B_{r_0}} |\nabla u|^2 dx \leq 1 - \|u\|_{L^2}^2
\]

and hence

\[
    \int_{B_{r_0}} |\nabla w|^2 dx = \int_{B_{r_0}} |\nabla v(1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2)^{1/2}|^2 dx \\
    = \left( 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 \right) \int_{B_{r_0}} |\nabla u|^2 dx \\
    \leq \left( 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 \right) (1 - \|u\|_{L^2}^2) \\
    = 1 + \frac{1}{\pi r_0^2} \|u\|_{L^2}^2 - \|u\|_{L^2}^2 - \frac{1}{\pi r_0^2} \|u\|_{L^2}^4 \leq 1
\]

provided that \( r_0^2 \geq \frac{1}{\pi} \). Since by (2.9) \( u^2(r) \leq w^2(r) + d \) we get

\[
    \int_{|x| \leq r_0} (e^{4\pi u^2} - 1) dx \leq e^{4\pi d} \int_{B_{r_0}} e^{4\pi u^2} dx
\]

The result follows by the Trudinger-Moser inequality, since \( w \in H_0^1(B_{r_0}) \) with \( \|w\|_{D}^2 = \int_{B_{r_0}} |\nabla w|^2 dx \leq 1 \).

\[ \blacksquare \]

In the next proposition we show that the result is optimal (as in the Dirichlet-norm case), namely that the supremum in (2.1) becomes infinite if the exponent \( 4\pi \) is replaced by a number \( \alpha > 4\pi \).

**Proposition 2.2** Suppose that \( \alpha > 4\pi \). Then, for any domain \( \Omega \subseteq \mathbb{R}^2 \)

\[
    \sup_{\|u\|_{s} \leq 1} \int_{\Omega} (e^{\alpha u^2} - 1) dx = +\infty.
\]

**Proof.**
We may suppose that $0 \in \Omega$, and that for some $\rho > 0$ the ball $B_\rho(0) \subset \Omega$. We use a modified "Moser-sequence", see [11], defined in $B_\rho(0)$ and continued by zero in $\Omega \setminus B_\rho(0)$, and with Sobolev-norm $\leq 1$:

$$m_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} 
\frac{\log(\rho/|x|)}{(\log n)^{1/2}}(1 - \frac{\rho^2}{4\log n})^{1/2}, & \frac{\rho}{n} \leq |x| \leq \rho \\
(\log n)^{1/2}(1 - \frac{\rho^2}{4\log n})^{1/2}, & 0 \leq |x| \leq \rho/n
\end{cases}$$

One checks that $\|m_n\|_{H^1_0(\Omega)} \leq 1$, for $n$ large. Hence one has

$$\sup_{\|u\|_S \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx \geq \lim_{n \to \infty} \int_{B_\rho} (e^{4\pi m_n^2} - 1) dx$$

$$\geq 2\pi \int_0^{\rho/n} \left( e^{\frac{\alpha}{2\pi} \log n \frac{1}{1-\rho^2/(4\log n)} - 1} \right) r dr$$

$$= 2\pi \left( n^{\frac{\alpha}{2\pi}} e^{-\frac{\alpha^2}{2\pi} - 1} \right) \frac{\rho/n}{2} \to +\infty, \text{ as } n \to \infty$$

\(\blacksquare\)

3 Critical growth and concentration

Numerous studies in recent years have shown the close connection of critical growth with concentration phenomena, see e.g. the pioneering work of H. Brezis - L. Nirenberg [3].

As pointed out in the introduction, it is of particular interest to study the “highest level of noncompactness” for the functional $\int_\Omega (e^{4\pi u_n^2} - 1) dx$, under the restriction $\|u\|_S \leq 1$. In view of this, we make the following definition:

**Definition 3.1** A sequence $\{u_n\} \subset H^1_0(\Omega)$ is a Sobolev-normalized concentrating sequence (for short, SNC-sequence), if

a) $\|u_n\|_S = 1$

b) $u_n \rightharpoonup 0$, weakly in $H^1_0(\Omega)$

c) \(\exists x_0 \in \Omega\) such that $\forall \rho > 0 : \int_{\Omega \setminus B_\rho(x_0)} (|\nabla u_n|^2 + |u_n|^2) dx \to 0$

Next, we define the **Carleson-Chang limit** as the maximal limit of SNS-sequences:

**Definition 3.2** Let

$$\Sigma := \{ \{u_n\} \subset H^1_0(\Omega) \mid \{u_n\} \text{ is a SNC-sequence} \},$$

and define the Carleson-Chang limit as

$$\text{cc–lim} \sup_{\|u_n\|_S \leq 1} \int_{\Omega} (e^{4\pi u_n^2} - 1) dx := \sup_{\Sigma} \lim_{n \to \infty} \sup_{\Omega} \int (e^{4\pi u_n^2} - 1) dx.$$

The following “concentration-compactness alternative” by P.L. Lions (restated in our notation) is relevant for our purposes:

**Proposition** (P.L. Lions, [10], Theorem I.6). Let $\{u_n\} \subset H^1_0(\Omega)$ satisfy $\|u_n\|_S \leq 1$; we may assume that $u_n \rightharpoonup u$. Then either
\{u_n\} is a SNC-sequence

or

\[ \int_Ω (e^{4\pi u_n^2} - 1)dx \to \int_Ω (e^{4\pi u^2} - 1)dx; \text{ this holds in particular if } u \neq 0. \]

Then one has

**Proposition 3.3** Suppose that

\[ S := \sup_{\|u\|_S \leq 1} \int_Ω (e^{4\pi u^2} - 1)dx > \lim_{\|u_n\|_S \to 1} \int_Ω (e^{4\pi u_n^2} - 1)dx. \]

Then the supremum \( S \) is attained.

**Proof.** Let \( \{y_n\} \) denote a maximizing sequence for \( S \), and assume that \( S \) is not attained. We may assume that \( y_n \rightharpoonup y \). By the alternative of P.L. Lions we get \( y = 0 \), and \( \{y_n\} \) is a SNC-sequence. Hence

\[ S = \lim_{n \to \infty} \int_Ω (e^{4\pi y_n^2} - 1)dx \leq \lim_{\|u_n\|_S \leq 1} \int_Ω (e^{4\pi u_n^2} - 1)dx < S \]

Contradiction!

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4 Upper bound for the Carleson-Chang limit

In this section we prove an explicit upper bound for the Carleson-Chang limit. In particular, we prove the estimates (1.7) and (1.8) of Theorem 1.2. In section 7 we will show that the bound in (1.7) is sharp for \( \Omega = B_R \), with any radius \( R > 0 \), and the bound in (1.8) is sharp for \( \Omega = \mathbb{R}^2 \).

**Proof.**

1. Using symmetrization as in section 2, we see that it is sufficient to prove (1.7) for radial functions in \( B_R(0) \). Following J. Moser [11] we perform the change of variables

\[ (4.1) \quad r = e^{-t/2}, \text{ and setting } w_n(t) = (4\pi)^{1/2} y_n(r), \]

we transform the radial integrals on \([0, R]\) into integrals on the half-line \([-2 \log R, +\infty)\). We will write throughout the paper: \( \alpha_R = -2 \log R \), with \( \alpha_R = -\infty \) if \( R = +\infty \). One checks that

\[ \int_{B_R} |\nabla y_n(x)|^2dx = 2\pi \int_0^R \frac{d}{dr}y_n(r)^2rdr = \int_{\alpha_R}^{\infty} |w'_n(t)|^2dt \]

and

\[ (4.2) \quad \int_{B_R} (e^{4\pi y_n^2(x)} - 1)dx = 2\pi \int_0^R (e^{4\pi y_n^2(r)} - 1)rdr = \pi \int_{\alpha_R}^{\infty} (e^{w_n^2(t)} - 1)e^{-t}dt \]

and similarly

\[ (4.3) \quad \int_{B_R} |y_n(x)|^2dx = 2\pi \int_0^R |y_n(r)|^2rdr = \frac{1}{4} \int_{\alpha_R}^{\infty} |w_n(t)|^2e^{-t}dt. \]

The SNC-sequences in this new setting are characterized by:

a) \( \|w_n\|_S^2 := \int_{\alpha_R}^{\infty} (|w'_n|^2 + \frac{1}{4}|w_n|^2e^{-t})dt = 1 \), \( w_n(\alpha_R) = 0 \)

b) \( w_n \rightharpoonup 0 \), weakly in \( H^1([\alpha_R, +\infty)) \)
c) \( \int_{\alpha R}^A (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t}) dt \to 0 \) for any fixed \( A > 0 \),

and the estimate (1.7) (which we seek to prove) becomes

\[
\text{cc-} \lim_{\|w_n\|_{S} \leq 1} \pi \int_{\alpha R}^\infty (e^{w_n^2(t)} - 1)e^{-t} dt \leq \pi e^{1-D(R)}
\]

for \( SNC \)-sequences \( \{w_n\} \subset H^1([\alpha R, +\infty)) \).

Let now denote \( \{w_n\} \) a maximizing \( SNC \)-sequence for the Carleson-Chang limit (1.7). We may assume that the sequence \( \{w_n\} \) satisfies

\[
\lim_{n \to \infty} \pi \int_{\alpha R}^\infty (e^{w_n^2(t)} - 1)e^{-t} dt > 2 \pi e^{-D(R)},
\]

since otherwise the theorem is proved. Note that we may assume that \( w_n(t) \) is an increasing function on \([\alpha R, +\infty)\). Fix \( A_R \geq 1 \) such that

\[
t - 2 \log t - D(R) > 1, \ \forall \ t \geq A_R.
\]

**Claim 1:** There exists a number \( n_1 \) such that

\[
w_n(t) < 1, \ \forall \ t \leq A_R, \ \forall \ n \geq n_1
\]

Indeed, for \( 0 < R < +\infty \) we can estimate

\[
w_n(t) \leq (A_R + 2 \log R)^{1/2} \left( \int_{\alpha R}^A |w_n'|^2 dt \right)^{1/2}
\]

\[
=: (A_R + 2 \log R)^{1/2} \delta_n, \ \text{for} \ t \leq A_R,
\]

with \( \delta_n \to 0 \) as \( n \to 0 \), by c).

For \( R = +\infty \) and \( 0 < t \leq A_R \) we estimate

\[
w_n(t) = w_n(0) + \int_0^t w'(t)dt \leq w_n(0) + t^{1/2} \left( \int_0^t |w_n'|^2 \right)^{1/2} dt
\]

The second term goes to zero, as above. For the estimate of \( w_n(0) \) we use the following Radial Lemma (see W. Strauss, [13]), valid for radial functions \( v(r) \) in \( H^1(\mathbb{R}^2) \) and for \( r \geq 1 \):

\[
(r + \frac{1}{2})v^2(r) \leq \frac{5}{4} \int_r^\infty (|v'|^2 + |v|^2)pdp
\]

We transform this inequality (as before) by the change of variables \( r = e^{-t/2} \) and \( w(t) = (4\pi)^{1/2} v(r) \) and get, for \( t \leq 0 \):

\[
(e^{-t/2} + \frac{1}{2})w^2(t) \leq \frac{5}{4} \int_{-\infty}^t (|w'(t)|^2 + \frac{1}{4}|w(t)|^2 e^{-t}) dt.
\]

Hence, we get for \( w_n(0) \), using the concentration property of \( w_n \)

\[
u_n^2(0) \leq \frac{5}{3} \int_{-\infty}^0 (|w'(t)|^2 + \frac{1}{4}|w(t)|^2 e^{-t}) dt =: \sigma_n^2 \to 0, \ \text{as} \ n \to \infty.
\]
Thus the claim is proved.

By claim 1 we conclude that for \( n \) sufficiently large (0 \( < R \leq +\infty \))
\[
w_n^2(t) < 1 < A_R - 2 \log A_R - D(R), \quad \alpha_R \leq t \leq A_R.
\]

Let now \( a_n > A_R \) denote the first \( t > A_R \) with
\[
w_n^2(a_n) = a_n - 2 \log a_n - D(R).
\]
Such an \( a_n \) exists (for \( n \) sufficiently large), since otherwise
\[
w_n^2(t) < t - 2 \log t - D(R), \quad \forall t \geq A_R \geq 1, \quad \text{as} \quad n \to \infty,
\]
and thus
\[
\pi \int_{\alpha_R}^{\infty} (e^{w_n^2} - 1)e^{-t} dt \leq \pi \int_{\alpha_R}^{A_R} (e^{w_n^2} - 1)e^{-t} + \pi \int_{A_R}^{\infty} e^{-2 \log t - D(R) - t}
\]
The second term on the right is bounded by \( \pi e^{-D(R)} \), and in the following claim 2 we prove that the first term goes to 0, for \( n \to \infty \), and thus we have a contradiction to assumption (4.5).

**Claim 2:**
\[
\pi \int_{\alpha_R}^{A_R} (e^{w_n^2} - 1)e^{-t} \to 0 \quad \text{as} \quad n \to \infty.
\]
This is immediate for \( 0 < R < +\infty \), since then this term can be estimated, using (4.7), by
\[
\pi(R^2 - e^{-A_R})(e^{2\alpha}(A_R + \alpha_R) - 1) \to 0 \quad \text{as} \quad n \to \infty.
\]
If \( R = +\infty \) we write
\[
\int_{-\infty}^{0} (e^{w_n^2} - 1)e^{-t} dt + \int_{0}^{A_R} (e^{w_n^2} - 1)e^{-t} dt
\]
The second term is now estimated as before, while for the first term we use a series expansion:
\[
\int_{-\infty}^{0} (e^{w_n^2} - 1)e^{-t} dt = \int_{-\infty}^{0} \sum_{k=1}^{\infty} \frac{|w_n(t)|^{2k}}{k!} e^{-t} dt
\]
\[
= \int_{-\infty}^{0} |w_n(t)|^2 e^{-t} dt + \int_{-\infty}^{0} \frac{1}{2} |w_n(t)|^4 e^{-t} dt + \sum_{k=3}^{\infty} \int_{-\infty}^{0} \frac{|w_n(t)|^{2k}}{k!} e^{-t} dt
\]
The first term goes to zero by concentration, the second term can be estimated by Sobolev (by returning to the variable \( r \) and back to \( t \))
\[
\int_{-\infty}^{0} w_n^4 e^{-t} dt \leq c_0 \left( \int_{-\infty}^{0} (|w_n|^2 + \frac{1}{4} |w_n|^2 e^{-t}) dt \right)^2
\]
and hence also goes to zero by concentration. For the third term, observe that by (4.8) we get for \( t \leq 0 \)
\[
w_n^2(t) \leq \frac{5}{4} \frac{1}{e^{-t/2} + 1/2} \sigma_n^2 \leq c e^{t/2} \sigma_n^2
\]
Hence we can estimate the series as
\[
\sum_{k=3}^{\infty} \frac{c^k \sigma_n^2 k^{t/2} e^{-t} dt}{k!} \leq \sum_{k=3}^{\infty} c^k \sigma_n^{2k} \int_{-\infty}^{0} e^{t/2} dt \leq c_1 \sigma_n^6 2,
\]
and thus claim 2 is proved.

Thus we have proved the existence of a number \( a_n > A_R \) as claimed in (4.9).

We now prove, for \( 0 < R \leq +\infty \)

i) \( \pi \int_{a_R}^{a_n} (e^{w_n^2} - 1)e^{-t}dt \to 0, \text{ as } n \to \infty. \)

ii) \( \lim_{n \to \infty} \pi \int_{a_n}^{\infty} (e^{w_n^2} - 1)e^{-t}dt \leq \pi e^{1-D(R)} \)

Proof of i): Note that the argument above shows that \( a_n \to +\infty \) as \( n \to \infty \), since for an arbitrarily large number \( A_R \) there exists \( n_0(\alpha) \) such that \( a_n > A_R \) for \( n \geq n_0 \). By (4.9) we have

\[
\pi \int_{a_R}^{a_n} (e^{w_n^2} - 1)e^{-t}dt \leq \int_{a_R}^{A} (e^{w_n^2} - 1)e^{-t}dt + \pi \int_{A}^{a_n} e^{-2\log t-D(R)}dt
\]

Let \( \epsilon > 0 \): for the second term we get \( \pi e^{-D(R)}(\frac{1}{A} - \frac{1}{a_n}) < \epsilon/2, \) for \( A \) sufficiently large, and then the first term becomes \( \leq \epsilon/2, \) for \( n \geq n_0(A, \epsilon) \), proceeding as in Claim 2.

Proof of ii): We apply the following basic estimate which was proved in [6] (we cite it here in the form given in [7], Proposition 2.2):

**Lemma** (Carleson-Chang): For \( a > 0 \) and \( \delta > 0 \) given, suppose that

\[
\int_{a}^{\infty} |w'(t)|^2 dt \leq \delta
\]

Then

\[
\int_{a}^{\infty} e^{w^2-t}dt \leq e \frac{1}{\frac{1}{a} - \delta} e^K, \quad \text{with } K = w^2(a)(1 + \frac{\delta}{\frac{1}{a} - \delta}) - a .
\]

We apply this Lemma to our sequence \( \{w_n\} \), with \( a = a_n \) given in (4.9), and \( \delta = \delta_n = \int_{a_n}^{\infty} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt \). Furthermore, in the following section 5, (5.1) and section 6, Proposition 6.4, it is shown that:

For \( a > 0 \) and \( b > 0 \) given, let

\[
S_{a,b} = \{ u \in H^1(\alpha_R, a), \ u(\alpha_R) = 0, \ \int_{\alpha_R}^{a} (|u'|^2 + \frac{1}{4}|u|^2 e^{-t})dt = b \}
\]

Then the supremum

\[
\sup\{\|u\|_{\infty}^2 : u \in S_{a,b}\}
\]

is attained by a function \( y \), with

\[
\|y\|_{\infty}^2 = y^2(a) = b(a - D(R)) + O(\frac{1}{a}).
\]

Thus, choosing \( a = a_n \) and \( b = b_n = 1 - \delta_n \) we get for \( w_n \in S_{a_n,b_n} \)

\[
w_n^2(a_n) \leq a_n - a_n \delta_n - D(R) + O(\delta_n) + O(\frac{1}{a_n}),
\]

which implies together with (4.9)

(4.10) \[
\delta_n \leq \frac{2 \log a_n}{a_n} + O(\frac{\log a_n}{a_n^2})
\]
Thus we have for $K = K_n$ in the Lemma of Carleson and Chang

\[
K_n = w_n^2(a_n)(1 + \frac{\delta_n}{1 - \delta_n}) - a_n
\leq \left( a_n - a_n\delta_n - D(R) + O\left(\frac{\log a_n}{a_n}\right) \right) \left( 1 + \delta_n + O(\delta_n^2) \right) - a_n
\]

(4.11)

\[
= -D(R) - \delta_n D(R) + O\left(\frac{\log a_n}{a_n}\right) + a_n O(\delta_n^2)
\]

\[
= -D(R) + O\left(\frac{(\log a_n)^2}{a_n}\right)
\]

Hence we obtain by the Lemma of Carleson and Chang for any maximizing SNC-sequence $\{w_n\}$

\[
\lim_{n \to \infty} \pi \int_{a_n}^{\infty} (e^{w_n^2} - 1) e^{-t} dt \leq \lim_{n \to \infty} \pi e^{1} \frac{1}{1 - \delta_n} e^{K_n} \leq \pi e^{1 - D(R)} ;
\]

thus ii) is proved.

With i) and ii) we now easily complete the proof of the first statement of Theorem 1.2

2. It is clear that for $\Omega_0 \subset \Omega_1$ the corresponding cc-limits are increasing. Thus, it is sufficient to prove 2) for $\Omega = \mathbb{R}^2$; this corresponds to setting $R = +\infty$, which was included in the proof of 1).

5 An auxiliary variational problem

In this section we consider the following variational problem: Determine

\[
\sup \{ \|u\|_{\infty}^2 | u \in S_{a,b} \} ,
\]

where

\[
S_{a,b} = \left\{ u \in H^1(\alpha R, a) | u(\alpha R) = 0, \int_{\alpha R}^a \left( |u'|^2 + \frac{R^2}{4} |u|^2 e^{-t} \right) dt = b > 0 \right\}
\]

Note that $S_{a,b} \subset L^\infty(\alpha R, a)$, with compact embedding, and hence it is easily seen that the supremum in (5.1) is attained: let $y_a \in S_{a,b}$ such that

\[
\|y_a\|_{\infty}^2 = \sup \{ \|u\|_{\infty}^2 | u \in S_{a,b} \} .
\]

In order to determine the value of (5.2) we need to identify the maximizing function $y_a \in S_{a,b}$. The natural way to do this consists in deriving the Euler-Lagrange equation associated to (5.1), but we encounter the difficulty that the functional $y \mapsto \|y\|_{\infty}^2$ is not differentiable. However, this functional is convex, and hence its subdifferential exists. We briefly recall this notion, and then derive the Euler-Lagrange equation for (5.1). For the proofs of some of the results we refer to [8].

**Definition 5.1** Let $E$ be a Banach space, and $\psi : E \to \mathbb{R}$ continuous and convex. Then we denote by $\partial \psi(u) \subset E'$ the subdifferential of $\psi$ in $u \in E$, given by

\[
\mu_u \in \partial \psi(u) \iff \psi(u + v) - \psi(u) \geq \langle \mu_u, v \rangle , \forall v \in E ;
\]

here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $E$ and $E'$. An element $\mu_u \in \partial \psi(u)$ is called a subgradient of $\psi$ at $u$. 

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In [8], Lemma 2.2, it is proved that

Lemma: If \( \psi \) satisfies in addition

\[
\psi(x) \geq 0 \quad \forall x \in E, \text{ and } \psi(tx) = t^2\psi(x) \quad \forall t \geq 0,
\]

then

\[
\mu \in \partial \psi(u) \iff \left\{ \begin{array}{c}
\langle \mu, u \rangle = 2\psi(u) \\
\langle \mu, x \rangle \leq \langle \mu, u \rangle, \quad \forall x \in \psi^u = \{ x \in E; \psi(x) \leq \psi(u) \}
\end{array} \right.
\]

Furthermore, by an easy variation of [8], Lemma 2.3 and Corollary 2.4, one has:

Lemma 5.2 Suppose that \( \psi : E \to \mathbb{R} \) satisfies (5.3), and \( \phi \in C^1(E, \mathbb{R}) \) satisfies \( \langle \phi'(x), x \rangle = 2\phi(x) \), \( \forall x \in E \). If \( y \in E \) is such that

\[
\psi(y) = \sup_{\{u \in E, \phi(u) = b\}} \psi(u),
\]

then

\[
\phi'(u) \in \frac{b}{\psi(u)} \partial \psi(u)
\]

Proof. The Euler-Lagrange equation

\[
\phi'(u) \in \lambda \partial \psi(u) \quad \text{for some } \lambda > 0
\]

is obtained as in [8], Lemma 2.3 and Corollary 2.4. The value

\[
\lambda = \frac{b}{\psi(u)}
\]

is found by testing (5.4) with \( u \):

\[
2b = 2\phi(u) = \langle \phi'(u), u \rangle = \lambda \langle \mu_u, u \rangle = \lambda 2\psi(u).
\]

We now apply Lemma 5.2 to our situation, and obtain

Theorem 5.3 Let \( E = \{ v \in H^1(\alpha_R, a); v(\alpha_R) = 0 \} \), and consider

\[
\psi(u) = \|u\|^2_{\infty} : E \to \mathbb{R}
\]

and

\[
\phi(u) = \int_{\alpha_R}^a (|u'(x)|^2 + \frac{1}{4}|u(x)|^2 e^{-x})dx.
\]

Suppose that \( y \in E \) satisfies

\[
\psi(y) = \sup\{ \psi(u) \mid u \in E, \phi(u) = b \};
\]

then \( y \) satisfies (weakly) the equation

\[
y''(x) + \frac{1}{4} y(x) e^{-x} = \frac{b}{\|y\|^2_{\infty}} \mu_y, \quad \text{where } \mu_y \in \partial \psi(y) \subset E'.
\]
6 The auxiliary Euler-Lagrange equation

It remains to determine the subgradient $\mu_y$ in equation (5.5). Again following [8], Lemma 2.6, 2.7 and 2.8 we find:

**Proposition 6.1** Let $K_y = \{x \in [\alpha_R, a]; |y(x)| = \|y\|_\infty\}$. Then

i) $\text{supp } \mu_y \subset K_y$

ii) $K_y = \{a\}$

iii) $\mu_y = \|y\|_\infty \delta_a$, the Dirac delta-function concentrated in the point $a$.

Thus, equation (5.5) becomes

\[
\begin{cases}
-y'' + \frac{1}{4} ye^{-t} = \frac{b}{\|y\|_\infty} \delta_a, & \alpha_R \leq t \leq a \\
y(\alpha_R) = 0
\end{cases}
\]

From this one now concludes easily that equation (5.5) is equivalent to solving the equation

\[
\begin{cases}
-w'' + \frac{1}{4} we^{-t} = 0, & \alpha_R \leq t < a \\
w(\alpha_R) = 0
\end{cases}
\]

with the condition that

\[
\int_{\alpha_R}^{a} (|w'(t)|^2 + \frac{1}{4}|w(t)|^2 e^{-t}) dt = b;
\]

the last condition is obtained by multiplying equation (6.1) by $y$ and integrating.

We now determine the explicit solution of equation (6.2).

**Theorem 6.2** The solution of equation (6.2) is given by

- for $0 < R < +\infty$:

\[
w(t) = \gamma \left( K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-t/2}) \right) =: \gamma z(t)
\]

- for $R = +\infty$:

\[
w(t) = \gamma K_0(e^{-t/2}),
\]

with unique coefficients $\gamma = \gamma(R, a, b) \in \mathbb{R}^+$.

Here $I_k(x)$ and $K_k(x)$ are the $k$-th modified Bessel functions of first and second kind, i.e. the solutions of the equation

\[-x^2 u''(x) - xu'(x) + (x^2 + k^2) u(x) = 0, \quad k = 1, 2, ...
\]

**Proof.** By inspection.

It is crucial to determine with precision the value of the coefficient $\gamma = \gamma(R, a, b)$ of $w(t)$. This requires some lengthy calculations.

We begin by recalling the following relations for the modified Bessel functions (see e.g. [1], 9.6.27,28):

\[
\frac{d}{dx} I_0(x) = I_1(x), \quad \frac{d}{dx} K_0(x) = -K_1(x), \quad \frac{d}{dx} (xK_1(x)) = -x K_0(x),
\]

\[
\int_{\alpha_R}^{a} (|w'(t)|^2 + \frac{1}{4}|w(t)|^2 e^{-t}) dt = b;
\]
and the following integral relations

\[ \begin{align*}
\int_a^b |K_0(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (K_0^2(r) - K_1^2(r)) \right]_a^b \\
\int_a^b |K_1(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (K_1^2(r) - K_0(r)K_2(r)) \right]_a^b \\
\int_a^b |I_0(r)|^2 r dr &= \left[ \frac{1}{2} r^2 (I_0^2(r) - I_1^2(r)) \right]_a^b \\
\int_a^b |I_1(r)K_1(r) - I_0(r)K_0(r)| r dr &= \left[ I_0(r)K_1(r)r \right]_a^b
\end{align*} \]

(6.7)

see [1]; for the last relation use integration by parts and (6.6).

Using these relations we will prove:

**Theorem 6.3**

1) Condition (6.3) yields for the coefficient \( \gamma = \gamma(R, a, b) \) in (6.4)

\[ \gamma^2 = 4 \frac{b}{a} \left[ 1 - \frac{4}{a} C(R) \right] + O\left( \frac{1}{a^3} \right), \]

for a large, with

\[ C(R) = \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - I_1(R)) \right) + 2RK_0(R)K_1(R) - \frac{2K_0(R)}{I_0(R)} \]

(6.8)

and \( C(+\infty) = 0 \).

2) The solution \( w(t), \alpha R \leq t \leq a \), of equation (6.2) is given by

- for \( 0 < R < +\infty \):

\[ w(t) = 2 \sqrt{\frac{b}{a}} \left( 1 - \frac{4}{a} C(R) + O\left( \frac{1}{a^2} \right) \right)^{1/2} \left( K_0(e^{-t/2}) - \frac{K_0(R)}{I_0(R)}I_0(e^{-t/2}) \right) \]

(6.9)

- for \( R = +\infty \):

\[ w(t) = 2\sqrt{\frac{b}{a}} \left( 1 + O\left( \frac{1}{a^2} \right) \right)^{1/2} K_0(e^{-t/2}) \]

(6.10)

**Proof.** Recall the definition of \( w(t) \) given in (6.4). We begin by evaluating the expression

\[ W^2(a) := \int_{\alpha R}^a \left( |w'(x)|^2 + \frac{1}{4} |w^2(x)|^2 e^{-x} \right) dx \]

Using the explicit form of \( w(t) \) in (6.4), the change of variable \( r = e^{-x/2} \), and the relations (6.6), we get
Using the relations (6.7) we get

\[ W^2(a) = \frac{1}{4} \int_{\alpha R}^a \left\{ \left| K_0'(e^{-x/2}) - \frac{K_0(R)}{I_0(R)} I_1'(e^{-x/2}) \right|^2 + \left| K_0(e^{-x/2}) - \frac{K_0(R)}{I_0(R)} I_0(e^{-x/2}) \right|^2 \right\} e^{-x} dx \]

\[ = \frac{1}{2} \int_{e^{-a/2}}^R \left\{ -K_1(r) - \frac{K_0(R)}{I_0(R)} I_1(r) \right\}^2 + \left| K_0(r) - \frac{K_0(R)}{I_0(R)} I_0(r) \right|^2 \right\} rdr \]

\[ = \frac{1}{2} \int_{e^{-a/2}}^R \left\{ (K_1(r))^2 + \frac{K^2_0(R)}{I^2_0(R)} |I_1(r)|^2 + |K_0(r)|^2 + \frac{K^2_0(R)}{I^2_0(R)} |I_0(r)|^2 \right\} rdr + 2 \frac{K_0(R)}{I_0(R)} (K_1(r) I_1(r) - K_0(r) I_0(r)) \right\} rdr \]

(6.11)

Using the relations (6.7) we get

\[
\frac{1}{2} \left\{ \left[ \frac{1}{2} r^2 (K_0^2(r) - K_0(r) K_2(r)) \right]_{e^{-a/2}}^R + \frac{K_0^2(R)}{I_0^2(R)} \left[ \frac{1}{2} r^2 (I_1^2(r) - I_0(r) I_2(r)) \right]_{e^{-a/2}}^R \\
+ \left[ \frac{1}{2} r^2 (K_0^2(r) - K_2^2(r)) \right]_{e^{-a/2}}^R + \frac{K_0^2(R)}{I_0^2(R)} \left[ \frac{1}{2} r^2 (I_0^2(r) - I_1^2(r)) \right]_{e^{-a/2}}^R \right\}
\]

(6.12)

\[
= \frac{1}{2} \left\{ \left[ \frac{1}{2} r^2 (K_0^2(r) - K_0(r) K_2(r) + K_0^2(R) \frac{I_0^2(R)}{I_0^2(R)} (I_0^2(r) - I_0(r) I_2(r)) \right]_{e^{-a/2}}^R \\
+ 2 \frac{K_0(R)}{I_0(R)} \left[ I_0(r) K_1(r) \right]_{e^{-a/2}}^R \right\}
\]

Evaluating at the boundaries we obtain

\[
\frac{1}{4} R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 2 R K_0(R) K_1(R)
\]

\[ - \frac{1}{4} e^{-a} \{ K_0^2(e^{-a/2}) - K_0(e^{-a/2}) K_2(e^{-a/2}) \}
\]

(6.13)

\[
+ \frac{K_0^2(R)}{I_0^2(R)} \left[ I_0^2(e^{-a/2}) - I_0(e^{-a/2}) I_2(e^{-a/2}) \right] \}
\]

\[-2 e^{-a/2} \frac{K_0(R)}{I_0(R)} I_0(e^{-a/2}) K_1(e^{-a/2}) \}
\]

For the terms with argument $e^{-a/2}$, a large, we now use the following behavior of the Bessel functions for $x > 0$ small, see [1].9.6.7-9: :

\[
K_0(x) \sim - \log x \quad K_1(x) \sim \frac{1}{x} \quad K_2(x) \sim \frac{2}{x^2}
\]

(6.14)

\[
I_0(x) \sim 1 \quad I_1(x) \sim \frac{1}{2} x \quad I_2(x) \sim \frac{1}{12} x^2
\]

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We rewrite (6.16) as
\[\frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_x(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R)\]
with
\[\frac{1}{4} e^{-a}\{ (-\log(e^{-a/2}))^2 - (-\log(e^{-a/2})) \frac{2}{e^{-a}} \}
+ \frac{K_0^2(R)}{I_0(R)} \left[ 1 - \frac{1}{8} e^{-a} \right] \} - 2e^{-a/2}K_0(R) \frac{1}{I_0(R)} \]
\[= \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_x(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R)\]
\[-\frac{1}{4} e^{-a}\{ (\frac{a}{2})^2 - \frac{a}{2} 2e^{-a} + \frac{K_0^2(R)}{I_0(R)} \left[ 1 - \frac{1}{8} e^{-a} \right] \} - 2K_0(R) \frac{1}{I_0(R)}\]
\[= \frac{1}{4} R^2 \left( K_0^2(R) - K_0(R)K_2(R) + K_0^2(R)(1 - \frac{I_x(R)}{I_0(R)}) \right) + 2RK_0(R)K_1(R)\]
\[+ \frac{1}{4} a - 2 \frac{K_0(R)}{I_0(R)} + O(a^2 e^{-a})\]
\[= \frac{1}{4} a + C(R) + O(a^2 e^{-a}),\]

with $C(R)$ as in (6.8). Conditions (6.3) and (6.4) yield now
\[b = \gamma^2W^2(a) = \gamma^2 \left( \frac{1}{4} a + C(R) + O(a^2 e^{-a}) \right)\]

We rewrite (6.16) as
\[\gamma \frac{a}{4} \left( 1 + \frac{4}{a} C(R) + O(\gamma e^{-a}) \right) = b\]
which yields for $\gamma = \gamma(a, b)$
\[\gamma^2 = \frac{4}{a} b \left[ 1 - \frac{4}{a} C(R) \right] + O(\frac{1}{a^3})\]

This proves 1). Assertion 2) follows now from (6.4). Formula (6.10) follows from (6.9), noting that $C(\infty) = 0$ and $K_0(\infty)/I_0(\infty) = 0$.

With this information we can now calculate the value $\|w\|^2_\infty = w^2(a)$:

**Proposition 6.4** Let $w(t)$ denote the solution of (6.2), (6.3) and hence of (5.1). Then
\[\|w\|^2_\infty = w^2(a) = b \left[ a - D(R) \right] + O(\frac{1}{a}) .\]

**Proof.** By (6.4) we have, using (6.14)
\[w^2(a) = \gamma^2 \left( K_0(e^{-a/2}) - \frac{K_0(R)}{I_0(R)}I_0(e^{-a/2}) \right)^2\]
\[= 4 \frac{b}{a} \left[ (1 - \frac{4}{a} C(R)) + O(\frac{1}{a^3}) \right] \left( K_0(e^{-a/2}) - \frac{K_0(R)}{I_0(R)}I_0(e^{-a/2}) \right)^2\]
\[= 4 \frac{b}{a} \left[ (1 - \frac{4}{a} C(R)) \left( \frac{a}{2} - \frac{K_0(R)}{I_0(R)} \right)^2 + O(\frac{\log a}{a^3}) \right]\]
\[= b \left[ a - 4C(R) - \frac{4}{a} K_0(R) \right] + O(\frac{1}{a})\]

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Set
\[ D(R) = 4C(R) + 4 \frac{K_0(R)}{I_0(R)} ; \]
then (6.19) becomes
\[ w^2(a) = b \left(a - D(R)\right) + O\left(\frac{1}{a}\right) \]
(6.21)

### 7 Construction of optimal concentrating sequences

In this section we show that the upper bounds for the Carleson-Chang limit
\[ \text{cc–lim} \int_{\|u_n\|_\infty \leq 1} (e^{4\pi u^2} - 1) dx \leq \pi e^{1-D(R)} , \]
given in Theorem 1.2 are sharp for \( \Omega = B_R \) and \( \Omega = \mathbb{R}^2 \). We do this by constructing explicit optimal SNC-sequences \( \{w_n\} \) for (7.1) for which the Carleson-Chang limit is equal to the bound on the right.

The construction of this sequence follows closely the proof of the upper bound for the Carleson-Chang limit, section 4, in combination with information on the optimal sequence for the corresponding Dirichlet-norm problem, see [7].

We begin by defining the sequence \( \{w_n(t)\} \) on \([\alpha R, n]\): in Theorem 6.3, set \( a = n \) and \( b = 1 - \frac{2 \log n}{n} \). Then, for \( 0 < R \leq +\infty \), let \( w_n(t) \) be given by (6.9) or (6.10), respectively. Thus, \( w_n(t) \) satisfies equation (6.2) with \( a = n \), and condition (6.3) with \( b = 1 - \frac{2 \log n}{n} \). Furthermore, we have by Proposition 6.4
\[ w^2_n(n) = \sup\{\|w_n\|_\infty^2 \mid w_n \in S_n\} = n - 2 \log n - D(R) + O\left(\frac{1}{n}\right) , \]
(7.2)
where \( S_n = \{u \in H^1(\alpha R, n) \mid u(\alpha R) = 0, \int_{\alpha R}^n (|u'|^2 + \frac{1}{4} |u|^2 e^{-t}) dt = 1 - \frac{2 \log n}{n}\} \). We remark that formula (7.2) constitutes a (late) motivation for the choice of \( a_n \) in (4.9).

It remains to define \( \{w_n(t)\} \) in \([n, +\infty)\). Here we can follow [7] where an optimal Dirichlet normalized concentrating sequence was constructed by analyzing carefully the proof of Carleson-Chang [6].

The complete definition of the optimal SNC-sequence \( \{w_n(t)\} \) is:

**Definition 7.1** Let \( w_n(t) \) be given by:
\[ w_n(t) = \begin{cases} w_n(t) , & \text{given by (6.9) or (6.10), respectively,} \\ \text{with } a = n \text{ and } b = 1 - \frac{2 \log n}{n} \\ w_n(n) + \frac{1}{w_n(n)} \log \frac{1 + A_n}{A_n + e^{-(t-n)}} & t \geq n \end{cases} \]
(7.3)
where \( A_n \in \mathbb{R}^+ \) is such that
\[ \int_{\alpha R}^{\infty} (|w_n'(t)|^2 + \frac{1}{4} |w_n(t)|^2 e^{-t}) dt = 1 . \]
(7.4)
We show that $A_n \in \mathbb{R}^+$ can be chosen as in Definition 7.1, i.e. satisfying (7.4), with the estimate

**Lemma 7.2**

(7.5) \[ A_n = \frac{1}{n^2 e} + O\left(\frac{1}{n^4}\right) \]

**Proof.** First note that by condition (6.3)

(7.6) \[ \int_n^{\infty} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt = 1 - \frac{2 \log n}{n} \]

Thus, we look for a constant $A_n$ such that

(7.7) \[ \int_n^{\infty} (|w_n'|^2 + \frac{1}{4}|w_n|^2 e^{-t})dt = \frac{2 \log n}{n} \]

Assume that $A_n \geq \frac{1}{3n^2}$, then one has

\[ \log\left(\frac{1 + A_n}{A_n + e^{-(t-n)}}\right) \leq \log(1 + \frac{1}{A_n}) \leq \log(1 + 3n^2) \]

and then by (7.3) and using that $w_n(n) = n + O(\log n)$ (by Proposition 6.4)

\[ w_n(t) \leq w_n(n) + \frac{1}{w_n(n)} \log(1 + 3n^2) \leq 2n, \quad \text{for } t \geq n, \ n \ \text{large}, \]

and hence

\[ \int_n^{\infty} |w_n|^2 e^{-t} dt \leq 4n^2 e^{-n} \]

Therefore, condition (7.7) becomes

(7.8) \[ \int_n^{\infty} |w_n'|^2 = \frac{2 \log n}{n} + O(n^2 e^{-n}) \]

One proves as in [7] that this yields

\[ A_n = \frac{1}{n^2 e} + O\left(\frac{1}{n^4}\right) \]

We now give an asymptotic lower bound for $\pi \int_{\alpha_R} (e^{w_n^2} - 1)e^{-t}dt$, as $n \to \infty$:

**Theorem 7.3** Let $\{w_n\}$ denote the sequence (7.3), and let $D(R)$ be given by (6.20). Then

\[ \pi \int_{\alpha_R} (e^{w_n^2} - 1)e^{-t} \geq e \pi e^{-D(R)} (1 + 2D(R) \frac{\log n}{n}) + O\left(\frac{1}{n}\right). \]

**Proof.**

a) First note that

(7.9) \[ \pi \int_{\alpha_R} (e^{w_n^2} - 1)e^{-t} dt \geq 0, \quad \text{for all } n \]

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b) Consider now
\[ \pi \int_{n}^{\infty} (e^{w^2_n} - 1)e^{-t} = \pi \int_{n}^{\infty} e^{w^2_n - t} + O(e^{-n}) . \]
Performing the change of variables \( s = t - n \), setting
\[ v_n(s) = \frac{1}{w_n(n)} \log \frac{A_n + 1}{A_n + e^{-s}} \]
and using that by Proposition 6.4
\[ w^2_n(n) = (1 - \frac{2 \log n}{n})[n - D(R)] + O(\frac{1}{n}) \]
we obtain
\[ \pi \int_{\alpha R}^{\infty} \exp \left( \left[ w_n(n) + v_n(s) \right]^2 - s - n \right) ds \]
\[ \geq \pi \int_{\alpha R}^{\infty} \exp \left( w^2_n(n) + 2w_n(n)v_n(s) - s - n \right) ds \]
\[ \geq \pi \int_{\alpha R}^{\infty} \exp \left( n - 2 \log n - D(R) + 2D(R) \frac{\log n}{n} + O(\frac{1}{n}) + 2 \log \frac{A_n + 1}{A_n + e^{-s}} - s - n \right) \]
\[ = \pi \int_{\alpha R}^{\infty} \exp(-2 \log n - D(R) + 2 \log \frac{A_n + 1}{A_n + e^{-s}} - s + 2D(R) \frac{\log n}{n} + O(\frac{1}{n})) \]
\[ = e^{\pi e^{-D(R) - 2 \log R + 2D(R) \frac{\log n}{n} + O(\frac{1}{n})}} , \quad \text{as} \quad n \to \infty . \]

Joining (7.9) and (7.10) we get
\[ \pi \int_{\alpha R}^{\infty} (e^{w^2_n} - 1)e^{-t} dt \geq e \pi e^{-D(R) - 2 \log R + 2D(R) \frac{\log n}{n}} + O(\frac{1}{n}) , \]
and hence the theorem is proved.

We conclude this section by proving some properties of the function \( D(R) \):

**Lemma 7.4** Let \( D(R) \) given by (6.20). Then

\[ D(R) = 4R K_0(R)K_1(R) - 2 \frac{K_0(R)}{I_0(R)} . \]

Furthermore, \( D(R) > 0 \), for all \( R \in \mathbb{R}^+ \), and
\[ D(R) \sim -2 \log R , \quad \text{as} \quad R \to 0 \]
and
\[ D(R) \sim \frac{\pi}{R} e^{-2R} , \quad \text{as} \quad R \to +\infty . \]
Proof. The explicit form of $D(R)$ is

$$D(R) = 4C(R) + 4K_0(R) I_0(R)$$

$$= R^2 \left( K_0^2(R) - K_0(R) K_2(R) + K_0^2(R) (1 - \frac{I_2(R)}{I_0(R)}) \right) + 8RK_0(R)K_1(R) - 4\frac{K_0(R)}{I_0(R)}$$

Using the relations (see [1], 9.6.26)

$$K_2(x) - K_0(x) = \frac{2}{x} K_1(x) \quad \text{and} \quad I_0(x) - I_2(x) = \frac{2}{x} I_1(x)$$

we get

$$D(R) = 6RK_0(R)K_1(R) + (2RK_0(R)I_1(R) - 4) \frac{K_0(R)}{I_0(R)} .$$

which simplifies, using (see [1], 9.6.15)

$$K_1(x)I_0(x) + K_0(x)I_1(x) = \frac{1}{x}$$

(7.13)

to (7.11).

We prove that $D(R) > 0$, for all $R > 0$: by (7.11) we get, using again (7.13)

$$D(R) = 2\frac{K_0(R)}{I_0(R)} [RK_1(R)I_0(R) - 1 + RK_1(R)I_0(R)]$$

$$= 2\frac{K_0(R)}{I_0(R)} [RK_1(R)I_0(R) - 1 + 1 - RK_0(R)I_1(R)] > 0 ,$$

since $K_1(x) > K_0(x)$ and $I_0(x) > I_1(x)$, for all $x > 0$.

Next, using the behavior of the Bessel functions (6.14), for $R > 0$ small, we have

$$D(R) \sim -4\log R - 2(-\log R) = -2 \log R , \quad \text{for} \quad R > 0 \quad \text{small} .$$

For the behavior of $D(R)$ at $+\infty$ we use the asymptotic behavior of the Bessel functions at $+\infty$, see [1], 9.7.1-2:

$$I_i(x) \sim \frac{1}{\sqrt{2\pi x}} e^x (1 - \frac{4^{2i-1}}{8x})$$

(7.14)

$$K_i(x) \sim \frac{\pi}{\sqrt{2\pi x}} e^{-x} (1 + \frac{4^{2i-1}}{8x})$$

Hence, we obtain by (7.11)

$$D(R) \sim 4R \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 - \frac{1}{8R} \right) \frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 + \frac{3}{8R} \right)$$

$$- 2\frac{\pi}{\sqrt{2\pi R}} e^{-R} \left( 1 + \frac{1}{8R} \right) \sqrt{2\pi R} e^{-R} \left( 1 - \frac{1}{8R} + O(\frac{1}{R^2}) \right)$$

$$\sim 2\pi e^{-2R} \left( 1 + \frac{1}{4R} \right) - 2\pi e^{-2R} \left( 1 - \frac{1}{4R} \right) = \frac{\pi}{R} e^{-2R} .$$

\[\blacksquare\]
The Supremum is attained

In this section we show that the supremum
\[ \sup_{\|u\| \leq 1} \int_{\Omega} (e^{4\pi u^2} - 1) \, dx \]
is attained for any ball $\Omega = B_R(0)$, as well as for $\Omega = \mathbb{R}^2$.

By Proposition 3.3 it suffices to prove

**Theorem 8.1** Let $0 < R \leq +\infty$. Then
\[ \sup_{\|u\| \leq 1} \pi \int_{\alpha R}^{\infty} (e^{u^2} - 1)e^{-t} \, dt > \lim_{\|u_n\| \leq 1} \pi \int_{\alpha R}^{\infty} (e^{u^2_n} - 1)e^{-t} \, dt \]

**Proof.** This follows immediately by Theorem 7.3: Choose an element of the maximizing sequence \( \{w_n\} \), with $n$ sufficiently large. Then
\[ \sup_{\|u\| = 1} \pi \int_{\alpha R}^{\infty} (e^{u^2} - 1)e^{-t} \, dt \geq \lim_{\|u_n\| \leq 1} \pi \int_{\alpha R}^{\infty} (e^{u^2_n} - 1)e^{-t} \, dt < \pi e^{1-D(R)} \]

This completes the proof of Theorem 1.3.

References


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