Diploma Project

Tuning of Musical Glasses

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Chapter 1

Introduction

Apart from cheering (or annoying) table mates at the end of a good lunch, playing on – empty or partially filled – wine glasses with a moistened finger is also a rather unknown way of making music. Through the last three centuries, a certain tradition of glass instrument building has been developed.

As soon as a sound emitting item is meant to be used musically, it faces the need to be tuned. Musical glasses can be tuned by filling them with different amount of liquid, but this method is not satisfactory, since a careful tuning needs a certain amount of time prior to the performance and is very easily disturbed while the liquids evaporate in the ambient atmosphere. To avoid this problem, glass instruments are tuned by mechanically altering the structure of the glass.

The tradition of the instrument makers tells us that by grinding down the glass just over the stem, the glass pitch can be lowered to about a half tone, whereas by reducing the glass’ height or by dimming its walls at the top, the pitch can be raised to up to an octave and a half.

Although very interesting research has previously been done on the physics of musical glasses, there is still no satisfying explanatory theory on the physical mechanism that hide underneath these tuning techniques.

Resting upon the knowledge that has already been acquired, this paper aims to investigate the tuning problem by modeling the material removal using simple and physically explainable concepts. A very simple finite-elements model of the glass rim has been developed for this purpose.

The paper begins with a short review of the state-of-the-art in musical glasses physics and some experiments. Then, an analytical and numerical model of the glass and of the grinding technique are established. The obtained results will then be compared against experimental results and related work.
2.1 Excitation Source

The rubbing of a wet finger on the glass rim – obviously the source of the glass’ vibration – is an example of the *stick-slip* phenomenon, which is also responsible for the excitation of strings by a violin bow. This effect relies on the fact that frictional forces depend on velocity and results in a saw-tooth like motion, inducing a vibration [11].

In the case of musical glasses, the water rises by capilarity into the interstices of the skin and lubricates the finger, which *slips* smoothly on the glass rim. Since the velocity of a liquid is zero on the surface of a solid, water is disposed on the rim during the slip phase (like a paint brush). After a certain distance, the finger dries, looses its lubrication and *sticks* on the surface. Then, the water present on the glass rim wets the finger again by capilarity and the cycle repeats itself [12].

It may be interesting to notice than every glass music player insists on the "quality" of the water he uses to wet his fingers. Some scholars from the 17th century wrote that different tones can be obtained when the liquid used is pure water, salted water, wine, alcohol or oil [9, 8]. In our experiments, we discovered that better results were obtained when we added alcohol to the water, which after all "simulates" wine, which is basically a mixture of water and alcohol.

A more precise examination of this phenomenon would have been clearly outside the range of this paper, but we could speculate that the viscosity of the liquid affects capilarity and coefficient of friction, thus modifying the ideal composition of pressure and velocity to make the glasses ring.

2.2 Vibrational Mode

The vibrational modes of a wineglass very closely resemble the flexural modes of a large church bell or a small handbell. The principal modes of vibration result from the propagation of bending waves around the glass rim, resulting in $2n$ nodes around the circumference. For a wine glass, the lowest mode is $n = 2$ (see Figure 2.1), which means that the upper rim of the glass changes from circular to elliptical twice (perpendicularly) per
The fact that the exciting finger rests on the glass while it is vibrating suggests that the finger is imposing a node (where normal vibration is absent). Since the finger rotates around the glass rim, a node must be following the finger in its rotation, which means that rubbing a glass creates rotating modes.

2.2.1 Analysis using speckle interferometry

We used speckle interferometry to illustrate the vibrational modes concerned by the rubbed glass. Studying the glass excited with a rotating finger confirmed the fact that the $n = 2$ mode is the fundamental mode in this case and that the nodes rotate along with the finger. No higher modes were seen while rubbing the glass, except in extreme cases (the finger applying very little pressure on the glass rim and moving very slowly) and for very brief moments.

To illustrate the higher modes of vibration, we excited the glasses with a small loudspeaker emitting a sinusoidal tone. Figure 2.2 shows the first three modes of a cognac glass, that is similar in shape to the glasses of the Glasharfe.

With this method, we were able to obtain better results than by recording the rotating modes induced by the finger. Since the modes are moving, the integration time of the speckle analysis must be short and the quality of the pictures stays low. A second difference is that by using a loudspeaker, we were able to adjust the amplitude of the glass vibration, whereas in the case of the rubbing finger, the amplitude can not be reduced at will because the stick-slip activity only starts with a certain pressure and velocity combination. The images obtained by speckle analysis tend to be better when the amplitude of the observed vibration is moderate.

We noted that the glasses ring with the same tone when they are rubbed with a wet finger, struck with a mallet or excited on the edge with a violin bow. Since the
fundamental mode can be excited by applying either a normal or a tangential force, it suggests that the glass’ motion is both normal and tangential.

The reason for the predominance of the $n = 2$ mode is probably that this mode supposes the circumferential length of the glass rim to stay approximately constant. Indeed, glass is highly resistant to extension or compression, so vibrations of the system tend to occur more easily in situations where such deformations are minimized, which is the case if the rim deforms from a circle into an ellipse, as seen above.

### 2.2.2 Apparatus

For the measurements of a glass excited by a wet finger, we needed an apparatus to ensure that the finger’s pressure and rotational velocity stay constant over time. An "artificial finger" was therefore developed to replace human hands. The "finger" is a piece of rubber set into rotation by a small electrical motor over a fixed glass. To isolate the glass from the motor’s vibration – which would have disturbed the vibration analysis – the glass is glued into a metal plate firmly fixed on the base (Figure 2.3).

The measurements were done on glasses of different shapes and size, including a cylindrical one which we believed would be easier to modelize, if needed. We found that the big glasses vibrated with an amplitude too important to be measured, the best results were obtained with the "cognac" glass, which resembles the actual glasses used in the Glasharfe. The glasses are painted in white to provide a smooth reflecting surface for the laser beam (using plain acrylic aerosol white paint).

### 2.3 Low Frequency Amplitude Modulation

When hearing the sound of a rubbed glass, a low-frequency modulation is immediately noticeable. And indeed, this modulation can be made visible when plotting a recording, as shown on figure 2.4(a). One sees that even after the player removes her hand (at $t \approx 1.6$ s), this modulation is still present, although not with the same frequency, which hints that two distinct phenomena are acting.

During the excitation with a finger, the vibrational mode rotates around the glass’
rim. To a fixed observation point (i.e. a microphone or an auditor), this results in a mobile source effect. The maximum sound amplitude occurs on the antinodes, whereas on the nodes there is no vibration and therefore no sound is emitted. As the antinodes pass in front of the observation point, the amplitude of the perceived sound is modulated.

Since there are four antinodes on the circumference, we expect to hear four ”peaks” during one finger rotation. This is confirmed in the recording on Figure 2.4(a): the player rotates her finger at about one revolution per second and there actually are four modulations per second.

When the player releases her hand \( t \approx 1.6 \text{ s} \), the vibration is no longer excited but free. The standing waves observed in Section 2.2 result from the addition of flexural waves traveling along the glass rim in both directions. When the glass is left free, these waves are still in motion. Depending on their relative position, they will add or subtract themselves, resulting in a modulation.

![Figure 2.4: Sound sample of a glass excited and then left free after about 1.6 seconds](image)

\( \text{(a) Sound wave} \quad \text{(b) Spectrogramm} \)

2.4 Spectral Analysis

From the perceived purity of the sound of musical glasses, one would expect a spectral composition with only a few overtones, if any. In fact, we found that the glass tones present harmonics that are near to exact multiples of the fundamental frequency \( f_q = qf_1 \), \( q \in \mathbb{N} \). Figure 2.5 shows the spectral analysis of four glasses, representing the note \( F \) in the four octaves that span the instrument.

One one hand, Meyer and Allen [13] are surprised because they do not see a reason for the harmonics, on the other hand, in the model proposed by French [1] and used thereafter, the frequency spectrum should be approximately given by

\[
\nu_{m,n} = K(a) \sqrt{\frac{(n^2 - 1)^2 + (\beta_m b)^4}{1 + \frac{1}{n^2}}} 
\]  

(2.1)

where \( K(a) \) is a constant function of the glass wall thickness \( a \) and \( b \) is a constant (in the order of unity). \( n \) is the circumferential mode number and \( \beta_m \) is a quantity that rises with
the vertical mode number $m$. Changes in $n$ would imply great frequency hops (indeed, $n$ is of power 4), therefore we understand why exciting higher circumferential modes (see Figure 2.2) is difficult with a rotating finger, so vertical modes $m$ gain importance. However, with a constant $n$, the trend would be hyperbolic and not linear as observed.

This may unveil the fact that the varying thickness of the glass may indeed compensate this hyperbolic growth. In the model we are considering, the glass walls have a constant thickness, whereas in reality the shape of a glass is much more complicated. It will be seen in Chapter 3 that this very thickness variation is acting when a glass is tuned, so a direct relation exists between thickness variation and resonant frequencies.

While the analysis of this surprising spectral composition could be of great interest, our study limits itself to the analysis of the fundamental frequency. We are interested in tuning the glasses, which basically consists in changing this fundamental frequency.

An interesting phenomenon is observed when the player removes her hand and let the glass ring: when the glass is set free, the perceived note raises. If we compare spectrograms of a glass before and after the finger is removed, one can see that the spectral composition of the sound changes in that most of the higher frequency components disappear over time.

On Figure 2.4(b), an exponential drop starting at $t \simeq 1.6$ can clearly be seen. Indeed, when the flexural waves are not excited by the rubbing finger any more, they encounter a damping process. The waves responsible for higher frequencies are traveling with the higher speed and thus are the most damped. This is why they disappear before the lower frequency components. Figure 2.6 shows the spectral composition of the tone displayed in Figure 2.4, (a) averaged from $t_0 = 1s$ to $t_1 = 1.5s$ and (b) averaged from $t_0 = 2s$ to $t_1 = 2.5s$. 
Figure 2.5: Spectral analysis of glasses of four different sizes

(a) $F_2$ glass, fundamental frequency 177 Hz
(b) $F_3$ glass, fundamental frequency 355 Hz
(c) $F_4$ glass, fundamental frequency 705 Hz
(d) $F_5$ glass, fundamental frequency 1405 Hz

Figure 2.6: Higher frequencies quickly disappear when the player removes her hand (b)

(a) Excited glass
(b) Glass free
Chapter 3

The Tuning Problem

Apart from the desire to understand the physics of the vibrating glass, what piqued our curiosity was the method used by the instrument builders for tuning their glasses: mechanical grinding of the glass walls. At first sight, removing material seems a natural way to lower the vibrating frequency since reducing the thickness of the wall should make it more flexible. However, a surprising fact is that when material is removed on the other end of the wall – that is, near the rim – the frequency rises. This suggests that the phenomenon acting there is more complex than a simple variation of the mean thickness or mass.

To understand more precisely the role of the wall’s shape, a model of the wineglass could enable to vary the thickness of the walls.

3.1 An approach with the energy method: brief résumé of [1]

This work is based on an article written by French [1], in which the energy method is used to compute the natural frequencies of a vibrating glass. The problem is separated in an azimuthal and a vertical component. The vertical problem is reduced to the classical case of a beam of uniform thickness, clamped at one end and free at the other. This article mentions tuning by adding water into the glass and proposes a solution of the problem with a partially filled wine glass.

The work described by French appears to be a very reliable comparison for this work and provides us with a stable base for our investigations. However, we should be warned that his theory on partially filled glasses must not be completely correct since he neglects the effect of the liquid on the vertical vibrational form. It seems very unlikely that adding such mass and inertia to the clamped-free beam system does not affect the vibrational form.

By using the results for an empty glass, this paper aims to bring a contribution to the proposed model with walls of nonuniform thickness to simulate the tuning process.
A first approximation of the shape of a wine glass is a thin-walled cylinder on a rigid circular base, as shown on Figure 3.1. One can see that the complete glass can be generated by rotating this section about the vertical axis of symmetry.

Let the displacement of an antinodal point on the rim at the top of the glass (where the amplitude is maximal) be

$$\Delta(t) = \Delta_0 \cos(\omega t) \quad (3.1)$$

The displacement of any other point of the glass is related to this by a time-dependent geometrical factor that also depends on the azimuth $\theta$ and the vertical coordinate $z$. Assuming linear elastic restoring forces, the total energy of the system can be written as:

$$E_{tot} = A \left( \frac{\partial \Delta}{\partial t} \right)^2 + B \Delta^2(t) \quad (3.2)$$

$$= A \Delta_0^2 \omega^2 \sin^2(\omega t) + B \Delta_0^2 \cos^2(\omega t) \quad (3.3)$$

where $A$ and $B$ are constants. Ignoring damping, $E_{tot}$ must be constant, independently of the time $t$, and

$$\omega^2 = \frac{B}{A} \quad (3.4)$$

Section 2.2 hints that the vibrational mode of a rubbed glass is mode \(^1(1,2)\), as shown on Figure 3.1. With the co-ordinates used on this figure, the nodes (no vibration) of the oscillation occur at $\theta = \pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$. The antinodes (maximum amplitude) are on $\theta = 0$, $\pi/2$, $\pi$ and $3\pi/2$. With this scheme, the horizontal radial displacement $x$ of any point of the glass wall can be written

$$x(z, \theta, t) = C \zeta(z) \cos(2\theta) \cos(\omega t) \quad (3.5)$$

where $\zeta(z)$ describes the vertical form of the oscillation and $C$ is a constant. A particular form is $\zeta_m(y)$ that corresponds to the vertical mode $m$.

Suppose that a wineglass is modeled as in Figure 3.1 as a vertical cylinder of radius $R$ and height $H$, with walls of uniform thickness $a$ and the density of glass $\rho_g$. If at the antinodes, the displacement is only radial, it is not the case between nodes and antinodes. A tangential displacement is added to the radial one, and so the total kinetic energy $K$ of the vibrating glass is:

\(^1\)In this paper, the mode number will be coded as $(m,n)$ with $m$ and $n$ the mode number of the vertical and circumferential modes respectively.
\[ K = \frac{1}{2} \rho g a R \omega^2 \Delta_0^2 \sin^2(\omega t) \int_{z=0}^{H} \int_{\theta=0}^{2\pi} [\zeta_m(z)]^2 \left[ \cos^2(2\theta) + \frac{1}{4} \sin^2(2\theta) \right] d\theta dz \quad (3.6) \]

\[ = \frac{5\pi}{8} \rho g a R \omega^2 \Delta_0^2 \sin^2(\omega t) \int_{0}^{H} [\zeta_m(z)]^2 dz \quad (3.7) \]

The potential energy \( U \) represents the flexural energy of the glass in both horizontal and vertical planes. For small variations, it appears that this total potential energy is:

\[ U = \frac{\pi E a^3}{24 R^3} \Delta_0^2 \cos^2(\omega t) \left( 9 \int_{0}^{H} [\zeta_m(z)]^2 dz + R^4 \int_{0}^{H} \left[ \frac{\partial^2 \zeta_m}{\partial z^2} \right]^2 dz \right) \quad (3.8) \]

where \( E \) is the Young’s modulus in glass. In section 3.2.1, the equation of motion of a vibrating bar (that approximates a section of the glass wall) will be developed. A general solution of this equation is

\[ \zeta_m(z) = a_1 \cos(\gamma_m z) + a_2 \sin(\gamma_m z) + a_3 \cosh(\gamma_m z) + a_4 \sinh(\gamma_m z) \quad (3.9) \]

where \( a_i \) and \( \gamma_m \) are constants. If the glass wall is modeled as a bar clamped at the bottom and free at the top with the boundary conditions corresponding to this situation and in a variational form where the test functions are \( \zeta_m \), then the following equation is true

\[ \int_{0}^{H} \left[ \frac{\partial^2 \zeta_m}{\partial z^2} \right]^2 dz = \gamma_m^4 \int_{0}^{H} [\zeta_m(z)]^2 \quad (3.10) \]

Equation 3.8 now becomes

\[ U = \frac{\pi E a^3}{24 R^3} \Delta_0^2 \cos^2(\omega t)(9 + \gamma_m^4 R^4) \int_{0}^{H} [\zeta_m(z)]^2 dz \quad (3.11) \]

Adding Equations 3.7 and 3.11 and reorganizing provide the values of the coefficients \( A \) and \( B \) in Equation 3.3:

\[ A = \frac{5\pi}{8} \rho g a R \int_{0}^{H} [\zeta_m(z)]^2 dz \quad (3.12) \]

\[ B = \frac{\pi E a^3}{24 R^3} (9 + \gamma_m^4 R^4) \int_{0}^{H} [\zeta_m(z)]^2 dz \quad (3.13) \]

These coefficients inserted into Equation 3.4 provide the natural frequency of oscillation in mode \( (m,2) \):

\[ \omega_{m2}^2 = \frac{a^2 E(9 + \gamma_m^4 R^4)}{15 R^4 \rho g} \quad (3.14) \]
3.2 Model of a vibrating beam with variable section

Equation 3.14 provides an approximation of the expected resonant frequency of a glass. The coefficient $\gamma_m^4$ is different for each mode $m$. However, a first limitation is that the glass is modeled as a regular, thin-walled cylinder, which only approximates very roughly the shape of a real wineglass. French proposes a solution with a "corrected height" $H^*$, obtained by interpolating the real (measured) frequency with the corresponding height according to his model.

The second limitation is the model of the walls as clamped-free vibrating beams of fixed thickness. The walls of a wine glass are somehow thicker at the base and thinner at the rim, and if we want to investigate the tuning of glasses by material removal, we must be able to adjust the thickness variation.

In this section, the equation of the fixed-thickness beam used by French will first be established, the model is then extended with variable thickness. The approach followed in this paper is based on Prof. Martin’s lecture notes [4].

3.2.1 Elementary establishment of the equation of motion for a vibrating beam

The form of $\zeta(z)$ used in Equation 3.5 is needed. A vertical section of the glass can be modeled as a beam, clamped at the bottom and free at the top end. Note that to comply with the traditional notation, we use a different coordinate system than in Section 3.1. Here, the beam is parallel to $y$ and its thickness is expressed on $z$, the displacement of the bar is thus noted $\zeta(y)$.

![Figure 3.3: Forces in a bending bar](image)

First, the static equation of a bar in bending will be posed. In Figure 3.3, a slice $\Delta y$ of the bar (size before bending) is considered and the forces acting on it can be listed:

1. Forces causing the flexion: transverse force $T + \Delta T$ and momentum $C + \Delta C$
2. Reaction forces holding the slice back: $T$ and momentum $C$
3. External forces:
   (a) Weight $mg = \rho_y S \Delta y = Z \Delta y$
      With $\rho_y$ the mass density of glass and $S$ the cross-section. We pose $Z$ to be the linear mass along $y$.
   (b) Pressure along $z$: $pl_x \Delta y = F_y \Delta y$
4. External momentum $\Gamma_x \Delta y$

The static equations are therefore:
• Equation of the forces

\[ T + F_z \Delta y - Z \Delta y - (T + \Delta T) = 0 \]
\[ \frac{\Delta T}{\Delta y} = F_z - Z \]  \hspace{1cm} (3.15)

• Equation of the momentums

\[ C + \Gamma_x \Delta y - (C + \Delta C) - T \Delta y = 0 \]
\[ \frac{\Delta C}{\Delta y} = \Gamma_x - T \]  \hspace{1cm} (3.16)

Substituting 3.16 into 3.15 leads to the static equation of the bar in bending, expressed with the momentums:

\[ \frac{\Delta^2 C}{\Delta y^2} = Z - F_z + \frac{\Delta \Gamma_x}{\Delta y} \]  \hspace{1cm} (3.17)

written later

\[ \frac{\partial^2 C}{\partial y^2} = Z - F_z + \frac{\partial \Gamma_x}{\partial y} \]  \hspace{1cm} (3.18)

The next step is to write the bending momentum \( C \) as a function of the displacement \( \zeta(y) \) along \( z \). Considering a "fiber" of cross section \( dS \) inside the beam, the force needed to compress (or extend) it by an amount \( dy \) is

\[ dF = EdS \frac{\partial v}{\partial y} \]  \hspace{1cm} (3.19)

where \( E \) is the Young’s coefficient and \( \frac{\partial v}{\partial y} \) is the relative elongation (Hooke’s law). The bending momentum \( C \) results from these forces, which must be integrated over the whole cross-section:

\[ C(y) = -\int_{S(y)} zdF = -E \int_{S(y)} z \frac{\partial v(y, z)}{\partial y} \, ds \]  \hspace{1cm} (3.20)

With a particular hypothesis using the geometry of the cross-section during the deformation, it can be shown that in the case of a bending beam, we have

\[ \frac{z}{R(y)} = \frac{\partial v(y, z)}{\partial y} \]  \hspace{1cm} (3.21)

where \( R(y) \) is the bending radius. The geometry of differential curves provides

\[ R(y) = \left[ 1 + \frac{\partial \zeta}{\partial y} \right]^{3/2} \approx \frac{1}{\frac{\partial^2 \zeta}{\partial y^2}} \]  \hspace{1cm} (3.22)

Equation 3.20 now becomes
\[ C(y) = -\frac{E}{R(y)} \int_{S(y)} z^2 ds = -E \frac{\partial^2 \zeta}{\partial y^2} I_x \] (3.23)

with \( I_x = \int_{S(y)} z^2 ds \) the moment of inertia along \( x \). Assuming a constant section (\( I_x \) is not a function of \( y \)), the static equation 3.18 can now be written as follows:

\[
\frac{\partial^2 C}{\partial y^2} = -E I_x \frac{\partial^4 \zeta}{\partial y^4} = \frac{\partial \Gamma_x}{\partial y} + Z - F_z
\]
\[
\Leftrightarrow E I_x \frac{\partial^4 \zeta}{\partial y^4} + \frac{\partial \Gamma_x}{\partial y} + Z - F_z = 0
\] (3.24)

The dynamic equation is obtained by adding the acceleration into 3.24:

\[
E I_x \frac{\partial^4 \zeta}{\partial y^4} + \frac{\partial \Gamma_x}{\partial y} + Z - F_z = m a_z = \rho g S \frac{\partial^2 \zeta}{\partial t^2}
\] (3.25)

We pose that there is no external momentum \( \Gamma_x \) (applies to a free vibrating bar) and that the weight \( Z \) can be negligible compared to the inertial forces. Equation 3.25 finally simplifies to the classical form

\[
\frac{\partial^4 \zeta}{\partial y^4} + \frac{\rho g S \partial^2 \zeta}{E I_x \partial t^2} = \frac{F_z}{E I_x}
\] (3.26)

For the case of a clamped-free bar, the boundary conditions are

\[
\zeta(y = 0) = \frac{\partial \zeta}{\partial y}(y = 0) = 0
\]
\[
\frac{\partial^2 \zeta}{\partial y^2}(y = H) = \frac{\partial^3 \zeta}{\partial y^3}(y = H) = 0
\] (3.27)

### 3.2.2 General solution

The theory of differential equations suggests a general solution of Equation 3.26 in the form

\[
\zeta(z) = a_1 \cos(\gamma z) + a_2 \sin(\gamma z) + a_3 \cosh(\gamma z) + a_4 \sinh(\gamma z)
\] (3.28)

with

\[
\gamma := \frac{\rho g S}{E I_x}
\] (3.29)

Using numerical methods, we can find the values of \( \gamma \) for which there exists a solution, that is which are the possible bending modes on the bar. Introducing the dimensionless variable \( \beta = \frac{\gamma H}{\rho g} \), the first modes are
\[ \beta_1 = 0.597 \quad \beta_2 = 1.494 \quad \beta_3 = 2.500 \ldots \] (3.30)

which are confirmed in [2].

### 3.2.3 Extension to the case of a non-uniform beam

If the cross-section is variable, then \( I_x \) becomes \( I_x(y) \) and Equation 3.23 now reads

\[
C(y) = -\frac{E}{R(y)} I_x(y) \simeq -E I_x(y) \frac{\partial^2 \zeta}{\partial y^2} = -E \left( \frac{\partial^2 I \partial^2 \zeta}{\partial y^2 \partial y^2} + 2 \frac{\partial I \partial^3 \zeta}{\partial y \partial y^3} + I \frac{\partial^4 \zeta}{\partial y^4} \right) \] (3.31)

(3.32)

Again neglecting the weight \( Z \) and for no external momentum \( \Gamma_x \), the dynamic equation 3.25 becomes

\[
\frac{\partial^2 C}{\partial y^2} + mg = F_z - ma_z \] (3.33)

\[
\iff -E \left( \frac{\partial^2 I \partial^2 \zeta}{\partial y^2 \partial y^2} + 2 \frac{\partial I \partial^3 \zeta}{\partial y \partial y^3} + I \frac{\partial^4 \zeta}{\partial y^4} \right) = F_z - \rho S(y) \frac{\partial^2 \zeta}{\partial t^2} \] (3.34)

Considering a general harmonic solution \( \zeta(y,t) = \zeta(y) e^{i\omega t} \), this equation can be written as:

\[
\frac{\partial^4 \zeta}{\partial y^4} - \rho S(y) \omega^2 \zeta(y) + 2 \frac{\partial I}{\partial y} \frac{\partial^2 \zeta}{\partial y^3} + \frac{\partial^2 I}{\partial y^2} \frac{\partial^2 \zeta}{\partial y^2} = 0 \] (3.35)

### 3.3 Analytical solution

To begin, a general case of a non-uniform beam in bending with small variations is considered, following the approach taken in [2]. The beam’s density \( \rho(y) \) and its cross-sectional area \( S(y) \) encounter very small variations versus \( y \):

\[
\rho(y) = \rho_0[1 + g(y)] \quad S(y) = S_0[1 + \alpha(y)] \quad a^2(y) = a_0^2[1 + \sigma(y)] \] (3.36)

(3.37)
a^2(y) is the radius of gyration and is defined as

\[
I(y) = a^2(y) S(y) \] (3.37)

By neglecting the products of small quantities
\[
\begin{align*}
\frac{\partial I}{\partial y} &= \frac{\partial}{\partial y} \left[ \alpha(y) + \sigma(y) \right] \approx \frac{\partial}{\partial y} \left[ (1 + \alpha + \sigma)(y) \right] \approx \frac{\partial}{\partial y} \left[ (1 + \sigma)(y) \right] \quad (3.38) \\
\frac{\partial^2 I}{\partial y^2} &= \frac{\partial^2}{\partial y^2} \left[ \alpha(y) + \sigma(y) \right] \approx \frac{\partial^2}{\partial y^2} \left[ (1 + \sigma)(y) \right] \approx \frac{\partial^2}{\partial y^2} \left[ (1 + \sigma)(y) \right] \quad (3.39) \\
\gamma(y) &= \frac{\rho_0(y)S(y)\omega^2}{EI(y)} = \frac{\rho_0(y)\omega^2}{Ea_0^2[1 + \sigma(y)]} \approx \frac{\rho_0\omega^2}{Ea_0^2}[1 + g(y) - \sigma(y)] \quad (3.40)
\end{align*}
\]

By posing
\[
E \frac{a_0^2}{\rho_0} = c_{L0}^2 a_0^2 \quad (3.41)
\]

Equation 3.35 now becomes
\[
\frac{\partial^4 \zeta}{\partial y^4} - \frac{\omega^2}{c_{L0}^2 a_0^2} [1 + g(y) - \sigma(y)] \zeta(y) + 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \frac{\partial^3 \zeta}{\partial y^3} + \frac{\partial^2}{\partial y^2} [\alpha(y) + \sigma(y)] \frac{\partial^2 \zeta}{\partial y^2} = 0 \quad (3.42)
\]

Assuming now that each mode \( \zeta_{\text{m},n}(y) \) of a non-uniform beam can be expressed as a combination of all modes \( \zeta_{k,0} \) of the uniform beam:
\[
\zeta_n(y) = \sum_{k=1}^{\infty} A_{nk} \zeta_{k,0}(y) \quad (3.43)
\]

with \( A_{mn} \gg A_{nk} \) for \( k \neq n \).

With the assumption that the frequency that corresponds to one mode of the non-uniform case is near to the frequency of the equivalent mode of the uniform case, it can be expressed with a small variation \( \theta_n \):
\[
f_n = f_{n0}(1 + \theta_n) \quad (3.44)
\]

Inserted into Equation 3.42 with \( \gamma_{n0}^4 := \frac{4\pi^2 f_{n0}^2}{c_{L0}^2 a_0^2} \):
\[
\frac{\partial^4 \zeta_n}{\partial y^4} - \gamma_{n0}^4 \zeta_n + 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \frac{\partial^3 \zeta_n}{\partial y^3} + \frac{\partial^2}{\partial y^2} [\alpha(y) + \sigma(y)] \frac{\partial^2 \zeta_n}{\partial y^2} - \gamma_{n0}^4 [2\theta_n + g(y) - \sigma(y)] \zeta_n = 0 \quad (3.45)
\]

Equation 3.43 can be graphically represented as on Figure 3.4. What is in fact considered is the projection of \( \zeta_n \) on a "plane" spun by all modes \( \zeta_{nk} \) of the homogenous problem. Since the thickness variation is so small, the projection is very near to the actual value of \( \zeta_n \). For the same reason, the contribution of \( A_{mn} \) will be far greater than that of the other modes, and it will be possible to neglect all the terms in \( A_{nk} \) with \( k \neq n \) of orders higher than 1.

Inserting Equation 3.43 into Equation 3.45 leads to
Figure 3.4: The flexural displacement $\zeta_n$ represented as a projection on the space spun by the homogenous modes $\zeta_{nk}$

$$\sum_{k=1}^{\infty} A_{nk} \left( \frac{\partial^4 \zeta_{k0}}{\partial y^4} - \gamma_{n0}^4 \zeta_{k0} \right) + \sum_{k \neq n} A_{nk} (\ldots) =$$

$$-A_{nn} \left[ 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \frac{\partial^2 \zeta_{n0}}{\partial y^2} + \frac{\partial^2}{\partial y^2} \left[ \alpha(y) + \sigma(y) \right] \frac{\partial^2 \zeta_{n0}}{\partial y^2} - \gamma_{n0}^4 [2\theta_n + g(y) - \sigma(y)] \zeta_{n0} \right]$$

(3.46)

where (\ldots) represent the first-order terms in $A_{nk}$ with $n \neq k$ that can also be neglected compared to $A_{nn}$. The first term of Equation 3.46 is therefore

$$\sum_{k=1}^{\infty} A_{nk} \left( \frac{\partial^4 \zeta_{k0}}{\partial y^4} - \gamma_{n0}^4 \zeta_{k0} \right) = \sum_{k=1}^{\infty} A_{nk} \left( \gamma_{k0}^4 - \gamma_{n0}^4 \right) \zeta_{k0}$$

(3.47)

To obtain the values of $A_{nk}$, the vectors $\zeta_n$ are projected on the vectors $\zeta_{m0}$:

$$\sum_{k=1}^{\infty} A_{nk} \left( \gamma_{k0}^4 - \gamma_{n0}^4 \right) \int_0^H \zeta_{m0}(y) \zeta_{k0}(y) dy =$$

$$-A_{nn} \int_0^H \zeta_{m0} \left\{ -\gamma_{n0}^4 [2\theta_n + g(y) - \sigma(y)] + 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \frac{\partial^2 \zeta_{n0}}{\partial y^2} + \frac{\partial^2}{\partial y^2} \left[ \alpha(y) + \sigma(y) \right] \frac{\partial^2 \zeta_{n0}}{\partial y^2} \right\} dy$$

(3.48)

When $k = n$, the left term of Equation 3.48 disappears because $\gamma_{k0}^4 = \gamma_{n0}^4$. When $k \neq n$, however, two distinct cases can arise:

**when $k = m$:**

The left term of Equation 3.48 is zero because the vectors $\zeta_{k0}$ and $\zeta_{m0}$ are supposed orthogonal, so their product is zero and the equation simplifies to:

$$2\theta_n = \frac{2}{H} \left\{ \int_0^H [\sigma(y) - g(y)] \zeta_{n0}^2 dy \right\}$$
\[ + \frac{1}{\gamma_{n0}} \int_0^H \left[ 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \zeta_{n0}(y) \frac{\partial^3 \zeta_{n0}}{\partial y^3} + \frac{\partial^2}{\partial y^2} [\alpha(y) + \sigma(y)] \zeta_{n0}(y) \frac{\partial^2 \zeta_{n0}}{\partial y^2} \right] dy \]  

(3.49)

and provides the value of \( \theta_n \).

**when** \( m = k \):

The left term of Equation 3.48 is no longer zero and

\[ A_{nk} \left( \gamma_{k0}^4 - \gamma_{n0}^4 \right) \int_0^H \zeta_{k0}^2(y) dy = -A_{nn} \gamma_{n0}^4 \left\{ -2\theta_n \int_0^H \zeta_{k0} \zeta_{n0} dy + \int_0^H \left[ \sigma(y) - g(y) \right] \zeta_{k0} \zeta_{n0} dy + \frac{1}{\gamma_{n0}^4} \int_0^H \zeta_{k0} \left[ 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \zeta_{n0}(y) \frac{\partial^3 \zeta_{n0}}{\partial y^3} + \frac{\partial^2}{\partial y^2} [\alpha(y) + \sigma(y)] \zeta_{n0}(y) \frac{\partial^2 \zeta_{n0}}{\partial y^2} \right] dy \} \]

\( \Leftrightarrow \)

\[ A_{nk} = -2 H \frac{\gamma_{n0}^2}{\gamma_{k0}^4 - \gamma_{n0}^4} \left\{ \int_0^H \left[ \sigma(y) - g(y) \right] \zeta_{k0} \zeta_{n0} dy + \frac{1}{\gamma_{n0}^4} \int_0^H \zeta_{k0} \left[ 2 \frac{\partial}{\partial y} [\alpha(y) + \sigma(y)] \zeta_{n0}(y) \frac{\partial^3 \zeta_{n0}}{\partial y^3} + \frac{\partial^2}{\partial y^2} [\alpha(y) + \sigma(y)] \zeta_{n0}(y) \frac{\partial^2 \zeta_{n0}}{\partial y^2} \right] dy \} \]

(3.51)

providing the value of the \( A_{nk} \).

### 3.3.1 Linear thickness variation

Equations 3.49 and 3.50 provide an analytical approximation of the frequency variation for an arbitrary thickness variation. However, the particular case of a linearly varying cross section \( h(y) \) will now be considered (Figure 3.5):

\[ h(y) = h_0 [1 + \epsilon y] \]  

(3.52)

then \( S(y) \) and \( a_2(y) \) become

\[ S(y) = l_s h_0 [1 + \epsilon y] = S_0 [1 + \epsilon y] \]  

(3.53)

\[ a_2(y) = \frac{h_0^2}{12} \approx \frac{h_0^2 [1 + \epsilon y]^2}{12} \approx \frac{a_0^2 [1 + 2\epsilon y]}{12} \]  

(3.54)
Or, to use the notation introduced in 3.36

\[
\begin{align*}
\alpha(y) &= \epsilon y \\
\frac{\partial \alpha}{\partial y} &= \epsilon \\
\frac{\partial^2 \alpha}{\partial y^2} &= 0 \\
\sigma(y) &= 2\epsilon y \\
\frac{\partial \sigma}{\partial y} &= 2\epsilon \\
\frac{\partial^2 \sigma}{\partial y^2} &= 0 \\
g(y) &= 0
\end{align*}
\]

(3.55)  (3.56)  (3.57)

The equation of motion 3.42 simplifies to

\[
\frac{\partial^4 \zeta}{\partial y^4} - \gamma^4 \zeta(y) + 2\epsilon y \gamma_0^4 \zeta(y) + 6\epsilon \frac{\partial^3 \zeta}{\partial y^3} = 0
\]

(3.58)

The frequency variation in Equation 3.49 becomes

\[
2\theta_n = \frac{2}{H} \left\{ 2\epsilon \int_0^H y \zeta_{n0}^2(y) dy + \frac{1}{\gamma_{n0}^4} 6\epsilon \int_0^H \zeta_{n0}(y) \frac{\partial^3 \zeta_{n0}}{\partial y^3} dy \right\}
\]

(3.59)

Integrating by parts the second integral leads to

\[
\zeta_{n0}(y) \frac{\partial^2 \zeta_{n0}}{\partial y^2} \bigg|_y^H - \int_0^H \frac{\partial \zeta_{n0}}{\partial y} \frac{\partial^2 \zeta_{n0}}{\partial y^2} dy
\]

(3.60)

The first term is 0 due to the boundary conditions of a clamped-free bar, c.f. 3.27. The second term develops to

\[
-\frac{1}{2} \left[ (\frac{\partial \zeta_{n0}}{\partial y})^2 \bigg|_y=H - (\frac{\partial \zeta_{n0}}{\partial y})^2 \bigg|_y=0 \right]
\]

(3.61)

The first term approximates to \( \gamma_{n0}^2 \) (by property of the characteristic function \( \zeta_{n0}(y) \), c.f. [2]), the second one disappears due to the boundary conditions in \( y = 0 \).

Equation 3.59 now reads:

\[
\theta_n = \frac{1}{H} \left\{ 2\epsilon \int_0^H y \zeta_{n0}^2(y) dy - \frac{1}{\gamma_{n0}^4} 6\epsilon \gamma_{n0}^2 \right\}
\]

(3.62)

3.3.2 Results

The primary interest is knowing wether this approach is going in the right direction. The form of \( \zeta_{n0}(y) \) is in principle the one arising from Equation 3.28, taking into account the boundary conditions. However, a good approximation [1] of this function for the lowest flexural mode is

\[
\zeta_{n0}(y) \approx \sqrt{2} \left( \frac{y}{H} \right)^{3/2}
\]

(3.63)

If we set this approximation into Equation 3.62, we get
\[ \theta_n \simeq \frac{2\epsilon}{H} \int_0^H 2y \frac{y^3}{H^3} dy - \frac{6\epsilon}{\gamma_{n0}^2 H} = \frac{4}{5} \epsilon H - \frac{6\epsilon}{\gamma_{n0}^2 H} = \left( \frac{4}{5} - \frac{6}{\beta_{n0}^2 \pi^2} \right) \epsilon H = K \epsilon H \] (3.64)

with \( \gamma_{n0}^2 = \frac{\beta_{n0}^2 \pi^2}{H^2} \). For the fundamental frequency, \( \beta_1 \simeq 0.6 \) (see 3.30) so \( K \) is negative.

- By removing material near the top of the glass (near \( y = H \)), \( \epsilon \) is negative (the thickness in \( y = 0 \) remains constant) so the variation \( \theta_1 \) of the fundamental frequency is positive. The frequency raises when material is removed at the top of the glass, which is coherent with the tradition of instrument making.

- Adding material near \( y = H \) leads to a positive \( \epsilon \) and so the variation \( \theta_n \) is negative.

- To study more precisely the case when material is removed near \( y = 0 \), Equation 3.44 must be rewritten in function of the frequency \( f_{nH} \) that correspond to a bar with a constant thickness at the top (\( y = H \)):

\[ f_n = f_{n0}(1 + K\epsilon H) \] (3.65)
\[ = f_{nH} \frac{h_0}{h_H} (1 + K\epsilon H) \] (3.66)
\[ \simeq f_{nH}(1 - \epsilon H)(1 + K\epsilon H) \] (3.67)
\[ \simeq f_{nH}[1 + (K - 1)\epsilon H] \] (3.68)

In this situation, material removal near \( y = 0 \) leads to a positive \( \epsilon \). With the thickness in \( H \) constant, the frequency drops with the removal, which meets our expectations.

Figure 3.6 shows the frequency change in function of material removal near the clamped end, that is near the glass’ stem. We know from the instrument makers that a diminution down to half a tone (6 %) is possible. Our results show that in order to obtain this diminution, one should remove material in the order of 1.5 % of the total vessel mass, which seems a reasonable value. This justifies the first order approximations that led to these results.

### 3.4 Finite Elements Method

As an alternate method to study the effect on variable section on the vibration frequencies of the observed beam, we now propose a numerical solution, using the Finite Elements Method (FEM).
3.4.1 Weak form

To solve this problem, a Finite Elements model is constructed, using the Galerkine method. Equation 3.58, expressed in the weak form, leads to

\[
\int_0^H z(y) \left[ \frac{\partial^4 \zeta}{\partial y^4} - \gamma_0^4 \zeta(y) \right] dy + 6\epsilon \int_0^H z(y) \frac{\partial^2 \zeta}{\partial y^2} dy + \gamma_0^4 2\epsilon \int_0^H z(y) \zeta(y) dy = 0 \tag{3.69}
\]

The first two terms are the classical terms for a vibrating beam with a constant section, the last two express the section variation (we can see that they disappear as soon as the section becomes constant, i.e. \( \epsilon = 0 \)). The test function \( z(y) \) can be shown to be in fact \( \delta \zeta(y) \).

3.4.2 Variational form

By integrating by parts Equation 3.69 and using the boundary conditions 3.27, we get the variational form:

\[
\int_0^H \frac{\partial^2 z}{\partial y^2} \frac{\partial^2 \zeta}{\partial y^2} dy - \gamma_0^4 \int_0^H z(y) \zeta(y) dy - 6\epsilon \int_0^H \frac{\partial z}{\partial y} \frac{\partial^2 \zeta}{\partial y^2} dy + \gamma_0^4 2\epsilon \int_0^H z(y) \zeta(y) dy = 0 \tag{3.70}
\]

It should be kept in mind that by doing this, the essential boundary conditions \( z(y = 0) = \frac{\partial z}{\partial y}(y = 0) = 0 \) have been "lost" and should be taken into account while constructing the matrix form.

3.4.3 Discretization

The beam in bending is discretized in finite elements as shown on Figure 3.7. All elements have the same length \( L = \frac{H}{N-1} \). The variational form 3.70 must be verified for the sum.
of all elements. In this particular case, it is equivalent if the variational form is verified on each element:

\[
\int_{y_j}^{y_{j+1}} \frac{\partial^2 z}{\partial y^2} \frac{\partial^2 \zeta}{\partial y^2} dy - \gamma_0 \int_{y_j}^{y_{j+1}} z(y) \zeta(y) dy - 6\epsilon \int_{y_j}^{y_{j+1}} \frac{\partial z}{\partial y} \frac{\partial^2 \zeta}{\partial y^2} dy + \gamma_0^4 2\epsilon \int_{y_j}^{y_{j+1}} z(y) y \zeta(y) dy
\]

(3.71)

Or, using change of variables

\[
y = y_j + u \quad y \in [y_j, y_{j+1}] \quad y_{j+1} = y_j + L
\]

Equation 3.71 becomes for every element

\[
\int_0^L \frac{\partial^2 z}{\partial u^2} \frac{\partial^2 \zeta}{\partial u^2} du - \gamma_0^4 \int_0^L z(u) \zeta(u) du - 6\epsilon \int_0^L \frac{\partial z}{\partial u} \frac{\partial^2 \zeta}{\partial u^2} du +
\gamma_0^4 2\epsilon \int_0^L z(u) u \zeta(u) du + \gamma_0^4 2\epsilon y_j \int_0^L z(u) \zeta(u) du
\]

(3.73)

This equation is of 2nd order, so a solution must be at least of order 2. This would mean 3 coefficients for which 3 independent equations would be needed. However, it appears that using the displacement and its variation on the joints Equation 3.71 becomes for every element leads to 4 equations. The general solution \( \zeta(u) \) on element \( j \) is thus chosen of order 3:

\[
\zeta(u) = a_j u^3 + b_j u^2 + c_j u + d_j
\]

(3.74)

This leads to the 4 independent equations corresponding to the 4 degrees of freedom:

\[
\begin{bmatrix}
    u_j^3 & u_j^2 & u_j & 1 \\
    3u_j^2 & 2u_j & 1 & 0 \\
    u_{j+1}^3 & u_{j+1}^2 & u_{j+1} & 1 \\
    3u_{j+1}^2 & 2u_{j+1} & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    a_j \\
    b_j \\
    c_j \\
    d_j
\end{bmatrix}
= 
\begin{bmatrix}
    \zeta_j \\
    \theta_j \\
    \zeta_{j+1} \\
    \theta_{j+1}
\end{bmatrix}
\]

(3.75)

This system of equations is solved by the MATLAB programm listed in Appendix B.1. On each element \( j \), the transversal displacement \( \zeta(u) \) can thus be expressed as
\[
\zeta(u) = N_{1j}\zeta_j + N_{2j}\theta_j + N_{3j}\zeta_{j+1} + N_{4j}\theta_{j+1} = (N_{1j}, N_{2j}, N_{3j}, N_{4j}) \left\{ \begin{array}{c}
\zeta_j \\
\theta_j \\
\zeta_{j+1} \\
\theta_{j+1}
\end{array} \right\} \tag{3.76}
\]

with

\[
N_1 = \frac{2}{L^3}u^3 - \frac{3}{L^2}u^2 + 1 \tag{3.77}
\]
\[
N_2 = \frac{1}{L^2}u^3 + \frac{2}{L}u^2 + u \tag{3.78}
\]
\[
N_3 = \frac{-2}{L^3}u^3 + \frac{3}{L^2}u^2 \tag{3.79}
\]
\[
N_4 = \frac{1}{L^2}u^3 - \frac{1}{L}u^2 \tag{3.80}
\]

### 3.4.4 Matrix form

The test functions \(z_i\) are constructed on the same model as the displacement \(\zeta_i\), their variation (the counterpoint of \(\theta_i\)) are noted \(\mu_i\). With this discretization scheme, Equation 3.73 (on element \(j\)) finally leads to

\[
\langle z_j, \mu_j, z_{j+1}, \mu_{j+1} \rangle \left( [K] + \frac{\gamma_0^4}{2} (2\epsilon y_j - 1) [M] - 6\epsilon [X] + \frac{\gamma_0^4}{2} 2\epsilon [Y]) \right) \left\{ \begin{array}{c}
\zeta_j \\
\theta_j \\
\zeta_{j+1} \\
\theta_{j+1}
\end{array} \right\} = \langle z_j, \mu_j, z_{j+1}, \mu_{j+1} \rangle \{f\} \tag{3.81}
\]

The matrices are obtained with the MATLAB programm in Appendix B.1:

\[
[K] = \frac{1}{L^3} \begin{bmatrix}
12 & 6L & -12 & 6L \\
6L & 4L^2 & -6L & 2L^2 \\
-12 & -6L & 12 & -6L \\
6L & 2L^2 & -6L & 4L^2
\end{bmatrix} \tag{3.82}
\]
\[
[M] = \frac{L}{420} \begin{bmatrix}
156 & 22L & 54 & -13L \\
22L & 4L^2 & 13L & -3L^2 \\
54 & 13L & 156 & -22L \\
-13L & -3L^2 & -22L & 4L^2
\end{bmatrix} \tag{3.83}
\]
\[
[X] = \frac{1}{2L} \begin{bmatrix}
0 & 2 & 0 & -2 \\
-2 & -L & 2 & -L \\
0 & -2 & 0 & 2 \\
2 & L & -2 & L
\end{bmatrix} \tag{3.84}
\]
\[
[Y] = \frac{L^2}{580} \begin{bmatrix}
72 & 14L & 54 & -12L \\
14L & 3L^2 & 14L & -3L^2 \\
54 & 14L & 240 & -30L \\
-12L & -3L^2 & -30L & 5L^2
\end{bmatrix} \tag{3.85}
\]
It is apparent that the first two matrices are respectively the stiffness and inertia matrix of the classical bending beam problem [3] and that when the thickness is constant, $\epsilon = 0$ and the problem simplifies to this classical case.

### 3.4.5 Multiple-elements assembly and sollicitation

Let $[W_e]$ be the (4x4)-matrix that results from the equation of motion 3.81. Extended to $N - 1$ elements ($N$ joints), this equation becomes

\[
\begin{bmatrix}
\zeta_1 \\
\theta_1 \\
\vdots \\
\zeta_N \\
\theta_N
\end{bmatrix} = \begin{bmatrix} \zeta_1 & \zeta_2 & \ldots & \zeta_N \\
\theta_1 & \theta_2 & \ldots & \theta_N
\end{bmatrix} \begin{bmatrix} z_1 \\
z_2 \\
\vdots \\
z_N
\end{bmatrix} = \begin{bmatrix} f_1 \\
f_2 \\
\vdots \\
f_N
\end{bmatrix}
\]

\[
\Leftrightarrow [W] \begin{bmatrix} z_1 \\
z_2 \\
\vdots \\
z_N
\end{bmatrix} = \{ f \}
\]

where $[W]$ is the (2Nx2N)-matrix constructed from $[W_e]$ as shown in Figure 3.8.

![Figure 3.8: Assembly of matrix $[W]$](image)

The excitation vector $\{ f \}$ represents forced values of $\zeta$ and/or $\theta$ at any joint. We arbitrarily set it to an initial displacement at node $N$ (free end):

\[
\{ f \} = \begin{bmatrix} 0 \\
0 \\
\vdots \\
1 \\
0
\end{bmatrix}
\]

The simulation is run on these equations in the MATLAB program in Appendix B.2 by letting $\gamma_0$ (and thus $\omega$) vary and detecting the resonant frequencies. The program can compute the resonant frequencies for any number of elements, for a constant thickness or for every value of the thickness variation $\epsilon$. 

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3.4.6 Results

Figure 3.9 shows the frequency variation in function of material removal near the clamped end, obtained through FEM simulation. Since $\epsilon$ is defined as a variation in function of the thickness $h_0$ in $y = 0$, attention must be paid to it when this thickness $h_0$ is varying. In these simulations, a fixed $h_H$ was given and then $\epsilon$ was redefined in each step to reflect the variation of $h_0$.

Whereas these results meet our expectations in showing a diminution of the fundamental frequency by grinding down the stem, the drop is flatter than in the analytical results and for a diminution of a half-tone (6 %), the corresponding removal of material is 3 %, that is twice the amount obtained by analytical means. However, this value remains within an acceptable range.

3.5 Discussion

The first conclusion that can be drawn from these results is that if the natural frequency of the glasses essentially depends on its vertical comportment, this method certainly goes in the right direction. Figure 3.10 shows the variation in frequency in function of the material removal and compares the results obtained with the analytical model used in Section 3.3, the results of the FEM analysis used in Section 3.4 and experimental results issued from the article [5].

The three experimental values correspond to three glasses of different sizes but similar shapes. Table 3.1 gives their dimensions in mm. $H$ is the height of the vessel, $H_{stem}$ the height of the glass’ stem, $R_{top}$ the radius of the vessel at the top and $R_{max}$ the maximal radius.

Although there is not a very close agreement between the analytical, numerical and experimental results, they remain in the same order of magnitude. The exact geometry of the glass clearly plays a very important role and must be better modeled in order to obtain more coherent results. This is why we cannot clearly state that the FEM method is better than the analytical one.
<table>
<thead>
<tr>
<th>Glass type</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>91</td>
<td>76</td>
<td>62</td>
</tr>
<tr>
<td>$H_{stem}$</td>
<td>99</td>
<td>94</td>
<td>78</td>
</tr>
<tr>
<td>$R_{top}$</td>
<td>33</td>
<td>30</td>
<td>25</td>
</tr>
<tr>
<td>$R_{max}$</td>
<td>40</td>
<td>35</td>
<td>29</td>
</tr>
</tbody>
</table>

Table 3.1: Dimensions in $mm$ of the experimental glasses used in [5]

Figure 3.10: Comparison between the analytical and numerical models and experimental values
Figure 3.11: Relationship between glass mass shaved and fundamental frequency [5]

The experimental results are extrapolated from a paper published by Nakanishi et al. [5]. The goal of this team was to provide a tuning "method" for musical glasses, using commercially available glasses. Their approach is to use a combination of mechanical grinding and filling with water. To investigate this, they built a very precise numerical model of a glass, reflecting the exact shape and size of a wine glass. The results were all obtained with simulations on this model and were confirmed with experimental results. No analytical method was envisaged.

The team developed a method that makes the mechanical shaving and the analysis easier by removing material circumferentially (see Figure 3.11(a)). Given the high precision of the numerical model used, the predicted results were very close to the experimental values, as seen on Figure 3.11(b).

Another interesting fact unveiled by Nakanishi is that the harmonics follow the same trend as the fundamental frequency when the glass is shaved, that is the timber of the glass is not altered by the shaving. The sole limitation to the shaving method is an evident mechanical one: the glass stability. Depending on the glass' size, changes up to a semi-tone are possible.
Chapter 4

Conclusion and Outlook

After a first set of elementary experiments to obtain knowledge about some fundamental phenomenons acting in a vibrating glass, an attempt to simplify and model the system has been developed. By focusing on the vibrating beam problem, a great number of questions concerning the exact modeling of thickness variation have been unveiled.

Both the analytical and numerical model discussed here lead to mainly correct results, but lack greatly in precision. We have shown that the variation of thickness certainly plays a great role in the tuning process and that it cannot only be resumed to a mass diminution and finally it turns out that the acoustics of glass instruments are very complex and only partly understood.

Should this work be continued, the following points could be considered:

- Before making a more complex model, experiments should be run on simplified systems to validate the existing models:
  - measurements on a vibrating bar with linearly variable thickness
  - differences if the clamped or the free end is thinned
  - measurements on a cylindrical glass that resembles the model of Section 3.1
  - experiment the role of the position of the grinding (smooth slope, grinding at one spot)

- Go further to take the circumferential modes into account (see below)

- Enhance the model to better reflect the real geometry of the glass

- Study the role of the boundary conditions at the bottom of the vessel (see below)

About the importance of the azimuthal modes, the interferometry images show clearly that the azimuthal modes primarily act on the top of the glass, so intuitively, a variation in the bottom should have less importance on these modes. On the other hand, the tradition of instrument making and experimental results show that removing material at the bottom does indeed affect the frequency, therefore it must be the vertical modes that are the most affected.

The frequency spectrum obtained by French [1] is of the form
\[ \nu_{m,n} = K \hat{h} \sqrt{\frac{(n^2 - 1)^2 + (\beta_m b)^4}{1 + \frac{1}{n^2}}} \] (4.1)

that, within the present scope of the work could perhaps be understood as

\[ \nu_{m,n} \simeq K \hat{h} \sqrt{(n^2 - 1)^2 + (\beta_m b)^4[1 - 1/n^2]} \]
\[ = K \hat{h}[A(n) + B(m, n) + C(m)] \] (4.2)

Not taking the azimuthal modes into account \((n \text{ constant})\), the effect of variable section on the vertical modes leads to a new frequency through term \(C\) in Equation 4.2. To a given variation \(\epsilon\), one can make correspond a ”corrected” thickness \(\hat{h}\) to homogenize the equation. Numerical results show that, contrarily to intuition, \(\hat{h}\) drops when material is removed at the bottom but rises when material is removed at the top.

Given the apparent feeble importance of circumferential modes, a reasonable hypothesis is to state that the thickness variation primarily acts on the vertical modes and that its effect on the circumferential mode (term \(A\)) is limited. It could be that a second order coupling term \(B(m, n)\) can be neglected.

Another phenomenon could eventually occur: the boundary conditions. By removing material near the clamped end of a beam narrows the surface on which the boundary conditions are applied. Intuitively, this should also mean going from a clamped end to a mere supported end, thus varying the boundary conditions themselves.

To investigate this field, an attempt to model the vibrating beam problem with ”mixed” boundary conditions has been done. Recall that the clamped conditions requires that

\[ \zeta(y = 0) = 0, \quad \frac{\partial \zeta}{\partial y}(y = 0) = 0 \] (4.3)

whereas an end that is merely supported requires

\[ \zeta(y = 0) = 0, \quad \frac{\partial^2 \zeta}{\partial y^2}(y = 0) = 0 \] (4.4)

A model of ”mixed” boundary conditions that are between these two cases could be imagined. Consider that the following equation must be true

\[ \frac{\partial \zeta}{\partial y}(y = 0) + \alpha \frac{\partial^2 \zeta}{\partial y^2}(y = 0) = 0 \] (4.5)

with the coefficient \(\alpha\). On first sight, if \(\alpha\) is zero, then one falls back into the supported case whereas when \(\alpha\) becomes bigger, it approximates the clamped case.

The general solution of the vibrating beam is Equation 3.28, which is repeated here

\[ \zeta(z) = a_1 \cos(\gamma z) + a_2 \sin(\gamma z) + a_3 \cosh(\gamma z) + a_4 \sinh(\gamma z) \] (4.6)
Taking the boundary conditions exposed in 4.5 at one end \((y = 0)\) and the free case at the other end \((y = H)\):

\[
\frac{\partial^2 \zeta}{\partial y^2}(y = H) = \frac{\partial^3 \zeta}{\partial y^3}(y = H) = 0
\]  

(4.7)

leads to the following system:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
-\gamma^2 & 0 & \alpha \gamma & \gamma^2 & 0 & \alpha \gamma \\
-\gamma^2 \cos(\gamma H) & -\gamma^2 \sin(\gamma H) & \gamma^2 \cosh(\gamma H) & \gamma^2 \sinh(\gamma H) \\
\gamma^3 \sin(\gamma H) & -\gamma^3 \cos(\gamma H) & \gamma^3 \sinh(\gamma H) & \gamma^3 \cosh(\gamma H)
\end{bmatrix}
\begin{Bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{Bmatrix}
= \{0\}
\]  

(4.8)

By solving this problem and trying to let \(\alpha\) vary, it appears that the situation is not so clear. Indeed, the first mode obtained when \(\alpha = 0\) is a mode with very low (if not zero) frequency. When \(\alpha > 0\), the first mode corresponds to a non-zero frequency. The interface between the two cases is therefore delicate to establish, essentially because each eigenfrequency obtained stem from an iterative process, with an initial value that greatly influences the final result. This way remains to be explored.

Some interesting hints on an influence of the boundary conditions can be found in a paper by Gautherin and Mollet [6]. By varying the boundary conditions at the bottom of the vessel from fixed in all directions (clamped) to the totally free case, the fundamental frequency is seen to drop by 7\%. The higher frequencies are less affected by this variation.
Bibliography

Fundamental scientific resources


Musicological sources


Informational sources


Appendix A

Glass Music

A.1 Brief History of the Glass Instruments

Almost since its invention, glass seems to have been used as a mean for producing sounds. At least in the Persian and Chinese antiquity, where traces of different glass (or ceramic) instruments were found. These were used by tapping them with sticks or mallets, until it was discovered that a tone could be produced by rubbing a wet finger around the rim of a glass.

It was also discovered quite early that by filling the ringing recipients with different quantities of water, several different sounds could be produced, thus enabling real music to be played on these glass instruments.

Like many other discoveries, glass music slowly propagated out of its asian cradle and eventually reached Europe in the 15th century. Later, Father Athanasius Kircher (1602-1680) conduced several experiments on glass instruments and studied the effects of the different amounts of water in the recipients. He also noted that the tone quality was altered by the liquid used to tune the glasses: water, wine, alcohol or oil.

The first European to perceive the musical potential of the glass instruments is said to be the Irishman Richard Pockrich (1690-1759). He built an instrument composed of several glasses, tuned with water, which he called the ”angelick organ”. It is difficult to know exactly wether it was a friction or a percussion instrument. Pockrich obtained great success in 18th century England, giving concerts and teaching his art to an increasing number of scholars, in particular to Ann Thickenesse-Ford (1738-1824), who developed the playing technique and even wrote a treaty on musical glasses, one of the few that actually survived until today [10].

The extraordinary sounds of glass instruments became quite en vogue in the 18th century. According to a great number of texts, paintings and musical scores, glass music was a very fashionable recreation and was even used to accompany worship in British churches [8].

1 Other forms of his name are also in use in the literature: Puckeridge, Pockridge, Pockeridge, etc.
A.1.1 The Glasharmonica

Around 1760, the well-known scientist and politician Benjamin Franklin combined the antique tradition of glass music with the modern occidental technology in inventing the Armonica, an instrument made of concentric glass bowls that were brought into rotation by a pedal mechanism similar to the one that was used in sewing machines. The musician played this instrument by delicately touching the rotating bowls with its wet fingers.

For the next 40 years, the Glasharmonica enjoyed a rising popularity in Europe and America. Soon, certain virtuoso performers like Marianne Davies (1744-1792) or Marianne Kirchgessner (1769-1808) were sought after for their uncommon skills and great composers like Mozart and Beethoven wrote works for the instrument. The uncommon and beautiful music was praised by Schiller and Goethe, and to meet the increasing demand, a Glasharmonica factory was founded in Bohemia, a place of great tradition in glass and ceramic skills.

Firmly convinced of the effects of glass music on the human soul and body, the Viennese physician and hypnotiseur Franz Anton Mesmer began using glass instruments in his studies on hypnosis. The public quickly associated glass music with the increasingly bad reputation of Mesmer as a ”mad scientist”. The sudden and mysterious death of the great virtuoso Marianne Kirchgessner, who was a young woman of great beauty, spurred all romantic spirits of the time and it was soon declared that the player’s death was caused by her instrument [8]. The sequel to this was a total diabolization of the Glasharmonica and other glass instruments. The public quickly lost interest in this kind of music, which became entirely forgotten.

A.1.2 The Grand Harmonicon

While the era of glass music seemed over in Europe, an american novelist named Francis Hopkinson Smith (1797-1872) patented his "Grand Harmonicon", a somewhat refined version of Puckridge’s ”angelick organ” and a direct precursor of the Glasharfe. His instruments were made out of 25 glasses mounted in a square case. The glasses were carefully fine-tuned by slight structural modifications, so that they could be played without the cumbersome need to tune them with water before each performance. It is quite clear that Smith discovered where to remove matter from the glasses: in order to lower the pitch, material was removed by shaving the bottom place of the glass (near the stem), whereas to raise the pitch, material was removed around the lip and sides of the glass (thinning the walls) [13] [14].

Around a hundred Grand Harmonicons were manufactured by Smith, although few of them were really used to produce music. It appears that the instrument were appreciated for their beauty and were used more as decorative pieces than for the sake of making music. The real breakthrough came a few decades later from Bruno Hoffmann, that rediscovered the same instrument, improved it and more significantly actually played it.

Apart from Bruno Hoffmann, one should also mention Gerhard Finkenbeiner, a master glassblower from Massachusets, that recreated a Glasharmonica based on Franklin’s initial design and achieved to manufacture and sell a certain number of instruments. He used carefully blown and turned bowls, which were fine-tuned by a method similar to Smith’s: grinding down the glass lowers the pitch while thinning the walls in a hydrofluoric acid etching bath raises it [13].
A.2 Bruno Hoffmann’s *Glasharfe*

A.2.1 A Life for the *Glasharfe*

By devoting his whole life to the *Glasharfe*, Bruno Hoffmann certainly contributed more than anyone to the renaissance of the glass instruments in the 20th century. Born in Stuttgart, Germany in 1913, he built his first instrument at age 16 in December 1929, less than a week after hearing a glass instrument for the first time.

At 17, he was broadcasted live during a Christmas play on the *Stuttgarter Rundfunk*. After his *Abitur* in early 1931, he began his architecture studies at the *Technische Hochschule Stuttgart*. Together with his studies, he played in parties, concerts and more radio broadcasts and began composing his first music.

In 1934, he built a second instrument (in A), improving the technique by which the glasses were fixed on the table (the glass feet were clamped on the table by rubber and copper stripes). In 1935, he was asked to play in Berlin for the *Reichssender* and other radio companies. After a first round of concerts and Revues in Berlin in 1936 and 1937, he travels to London, where he gives a series of concerts and is recorded on film.

Abandoning the use of common stem glasses, clamped at the feet, Bruno Hoffmann developed a new kind of instrument, letting the glasses be blown exactly for his requirements, and without feet. The glasses were blown by Sußmuth in Kassel. He ordered glasses with a simple cylindrical stem, which he fitted directly into a wooden case, using a little rubber ring to fasten the glass. The first instrument of this kind was finished in 1948 and had 46 glasses.

During and after the Second World War, he resumed concerts in Germany, then in Zurich, Paris, and London. In 1957, he works with Carl Orff and begins thinking of improving his instrument again, adding 4 lower glasses (D₂ to F♯₂). Two instruments were built in the 1960s on this model.

Bruno Hoffmann never stopped his efforts to develop his instrument and built several exemplars, each slightly different during his active life. His last two instruments were finished in 1976. Setting aside the original prototypes, built with common glasses, Hoffmann has built 5 instruments, all of which are still preserved today, although only one is still being played with.

Until his death in 1991, Bruno Hoffmann almost never stopped playing his wonderful instrument, giving concerts, being broadcasted on radio and television and recorded hours of music in all continents. [7]

The *Glasharfe* is an uncommon instrument in many senses, and due to its very originality, the instrument is not one that is being taught in music academies. The only teachers are the rare musicians (who often learned by themselves). It was clear to Bruno
Hoffmann that, living such an active life as his, he would not be able to properly teach his art to anyone.

In 1965, Hoffmann met a German interpreter, Mrs. Ingeborg Emge, in Madrid. Being fluent in French, Mrs. Emge was of great help for Bruno Hoffmann, who began working on a project in French-speaking Switzerland. Fascinated by the beauty of the Glasharfe, Mrs. Emge learned to play on this instrument and has thus let Bruno Hoffmann’s extraordinary work still be known today.

A.2.2 His instrument

From the beginning, Hoffmann renounced to use Franklin’s mechanized Harmonica technique. Since it was difficult to blow perfectly round glasses, the rotating cups of an Harmonica were constantly shifting and made it difficult to obtain sounds of constant intensity. Moreover, the glasses are not rubbed on their rim, but on the uppermost part of their sides, creating more unwanted harmonic sounds. It was technically difficult to maintain the whole rubbed surface constantly wet and since all glasses where of different diameter, they all were rubbed at a different speed, making a proper intensity control impossible (to make glasses sound louder, one has to augment finger pressure and rubbing speed).

Although his first instrument was made out of glasses Hoffmann picked at a local store, he quickly realized that plain drinking glasses were not of an optimal shape for music production. He then let the glasses of his instruments be blown in custom shape and size. He also renounced to use lead crystal since it ”swings too slowly and rings too lengthily”.

He also abandoned tuning the glasses with different level of water. Beside the evident problem of constantly having to re-tune the instrument, glass after glass, because of the evaporation of the water, liquid-filled glasses also have a poorer timbre [5]. Since Hoffmann already let his glass be custom blown, he directed the blowers to produce glasses of different size and with walls of different thickness. The bigger the glass and the thinner its wall, the lower it rings. Fine tuning was achieved by shaving the bottom part of the glass. Such glasses ring at defined tones and need no more be tuned with water.

The subject of this study is one of the last two instruments built by Bruno Hoffmann in 1976. It is formed by 50 glasses, blown by Eisch in Frauenau (former Bohemia).
The glasses have no "feet" like common wine glasses but nearly cylindrical stems (see Figure A.3) that are fitted with rubber into the instrument’s body. There are four different sets of glasses on the instrument with diameters of approximately 9-9.5 cm, 8.5 cm, 7.5 cm and 7 cm (bass to treble). The bass glasses have very thin walls (∼0.4 mm) and vibrate with such great amplitude that they can be seen vibrating.

![Glass foot](image1) ![Hole](image2)

(a) Glass foot. The glass has been tuned by grinding the bottom of the vessel (white patch)  
(b) Hole

Figure A.3: Fitting of the glasses

The instrument body consists of a casing in indian palisander (122x54x16cm) set on four removable feet. The bottom is built stepwise so as to bring the rim of all glasses to the same height, a necessary condition for playing. Beneath the 3rd-sized glasses, a tin basin contains the water used by the performer to wet his fingers. Figure A.4 shows a schematic top and side view of the instrument.

The Glasharfe is a tempered chromatic instrument (like a piano) and spans nearly 4 octaves, from $D_2$ to $C_6$. Since Bruno Hoffmann used to play with orchestras, the instrument is tuned a little bit higher than a piano, with the $A_3$ at about 446 Hz (probably because the tuning of the Glasharfe is very stable whereas other instruments notably shift in tone when the atmospheric conditions vary, as it is often the case during performances). Figure A.4 shows the notes corresponding to each glass. On this schema, the octaves are coded with colors, from rosa for octave 2 to green for octave 6.

Some notes are duplicated ($F_{#3}$, $F_4$, $C_5$ and $E_5$) for convenience. The notes are arranged in such a way that the performer can make certain common chords with one hand, using three or four fingers. For example, the notes $D_4-F_{#4}-A_4$ just above the water basin are arranged into a triangle that allows the performer to ring all three tones with one hand, producing a $D$-major chord.
Figure A.4: Plan of the *Glasharfe* with glass tones

![Plan of the Glasharfe with glass tones](image)

Figure A.5: Bruno Hoffmann’s signature on his instrument

![Bruno Hoffmann’s signature on his instrument](image)
Appendix B

MATLAB-Files

B.1 Computation of the FEM matrices

% matrices.m
%
% This program computes the matrices for finite-element
% analysis of a clamped-free bar (approx for glass wall).
%
% EPFL/LEMA Thomas Guignard - 2003-01-15

%yj = sym('yj');
yj = 0;
%yjj = sym('yjj');
L = sym('L');
yjj = L;
y = sym('y');
zetaj = sym('zetaj');
thetaj = sym('thetaj');
zetajj = sym('zetajj');
thetajj = sym('thetajj');
N1 = sym('N1');
N2 = sym('N2');
N3 = sym('N3');
N4 = sym('N4');

A = [[yj^3 yj^2 yj 1];
     [3*yj^2 2*yj 1 0];
     [yjj^3 yjj^2 yjj 1];
     [3*yjj^2 2*yjj 1 0]];

B = [zetaj; thetaj; zetajj; thetajj];

Y = A\B;

% Find out the coefficients, reorganize
coeff_zj = collect([y^3 y^2 y 1] * Y, zetaj);
\[
\text{coeff}_tj = \text{collect}\left(\left[ y^3 \ y^2 \ y \ 1 \right] \ast Y, \ \text{thetaj} \right);
\text{coeff}_zjj = \text{collect}\left(\left[ y^3 \ y^2 \ y \ 1 \right] \ast Y, \ \text{zetajj} \right);
\text{coeff}_tjj = \text{collect}\left(\left[ y^3 \ y^2 \ y \ 1 \right] \ast Y, \ \text{thetajj} \right);
\]

% Outputs of previous operations
\[
\text{N}1j = \left( 2 \ast y^3 / L^3 - 3 \ast y^2 / L^2 + 1 \right);
\text{N}2j = \left( -2 \ast y^2 / L + y \ast y^3 / L^2 \right);
\text{N}3j = \left( 3 \ast y^2 / L^2 - 2 \ast y^3 / L^3 \right);
\text{N}4j = \left( -y^2 / L + y^3 / L^2 \right);
\]

% control
\[
\text{y}1 = \left[ y^3 \ y^2 \ y \ 1 \right] \ast Y;
\text{y}2 = \left[ \text{zetaj} \ \text{thetaj} \ \text{zetajj} \ \text{thetajj} \right] \ast \left[ \text{N}1j; \ \text{N}2j; \ \text{N}3j; \ \text{N}4j \right];
\]

\[
\text{N} = \left[ \text{N}1j; \ \text{N}2j; \ \text{N}3j; \ \text{N}4j \right];
\text{Nt} = \left[ \text{N}1j \ \text{N}2j \ \text{N}3j \ \text{N}4j \right];
\]

% Classical terms

% Stiffness matrix
\[
k = \int (\text{diff}(\text{N},y,2) \ast \text{diff}(\text{Nt},y,2)), \ y, 0, L)
kk = \int (\text{diff}(\text{N},y,2) \ast \text{diff}(\text{Nt},y,2)), \ y, 0, L) \ast L^3
\]

% Mass matrix
\[
m = \int (\text{N} \ast \text{Nt}), \ y, 0, L)
\text{mm} = \int (\text{N} \ast \text{Nt}), \ y, 0, L) \ast 420 / L
\]

% Non-constant section, non-classical terms
\[
x = \int (\text{diff}(\text{N},y,1) \ast \text{diff}(\text{Nt},y,2)), \ y, 0, L)
xx = \int (\text{diff}(\text{N},y,1) \ast \text{diff}(\text{Nt},y,2)), \ y, 0, L) \ast 2 \ast L
\]
\[
z = \int (\text{N} \ast y \ast \text{Nt}), \ y, 0, L)
zz = \int (\text{N} \ast y \ast \text{Nt}), \ y, 0, L) \ast 840 / L^2
\]
B.2 FEM simulation

% FEM_variable.m
%
% This program assembles the matrix holding the equation
% of motion of a vibrating beam, then records the resonances
% by sweeping the frequencies (in the form of gamma0).
%
% EPFL/LEMA - Thomas Guignard - 2003-03-06

function [omega_1] = FEM_variable(nrofelements, maxiterations, varargin)

if size(varargin) == 1
    % 1 argument was passed -> value of material removal at top
    ht = 0.0015 - varargin{1}; % glass thickness, top, m
    hb = 0.0015; % glass thickness, bottom, m
else
    % 2 arguments were passed -> values of bottom(1) and top(2) thicknesses
    ht = varargin{2}; % glass thickness, top, m
    hb = varargin{1}; % glass thickness, bottom, m
end

R = 0.03; % Glass radius, m
H = 0.05; % Glass height, m
L = H/nrofelements; % element length

epsilon = (ht-hb)/H;
h = 0.0016; % (virtual) bar thickness, m
rho = 2880; % mass per unit volume, kg/m3
E = 6.15e10; % Young's Modulus, N/m3

% Inertia Matrix

% Stiffness Matrix
K = [12 6*L -12 6*L; 6*L 4*L^2 -6*L 2*L^2; -12 -6*L 12 -6*L; 6*L 2*L^2 -6*L 4*L^2] .* (1/L^3);

% Non-classical terms
X = [[ 0, 2, 0, -2];
\[
\begin{bmatrix}
-2, -L, 2, -L; \\
0, -2, 0, 2; \\
2, L, -2, L
\end{bmatrix} \cdot \frac{1}{2L};
\]

\[
Y = \begin{bmatrix}
72, 14L, 54, -12L; \\
14L, 3L^2, 14L, -3L^2; \\
54, 14L, 240, -30L; \\
-12L, -3L^2, -30L, 5L^2
\end{bmatrix} \cdot \frac{L^2}{580};
\]

result = [1:maxiterations];

for ite = 1:maxiterations

    gamma0 = 0.1 \times ite;

    % Matrix assembly

    matrix = zeros(2*nrofelements + 2);

    for i = 1:nrofelements

        % Position

        yj = (i-1)*L;

        % Equation Matrix

        elementmat = K - gamma0^4 \times M + (2*epsilon/hb) \times \ldots
         \left(-3 \times X + gamma0^4 \times Y + gamma0^4*yj \times M\right);

        % Matrix construction

        matrix((2*i-1):(2*i+2),(2*i-1):(2*i+2)) = ...
         matrix((2*i-1):(2*i+2),(2*i-1):(2*i+2)) + elementmat;

    end

    matrix = matrix(3:(2*nrofelements + 2),3:(2*nrofelements + 2));

    sollic = zeros(2*nrofelements,1);

    sollic(2*nrofelements-1) = 1;

    x = matrix\backslash sollic;

    result(ite) = x(2*nrofelements-1);
end

beta = 0.1 \times [1:maxiterations]\times H/\pi;

gammam = 1.8751/H;

[maxvalue maxgamma] = max(abs(result(1:640)));
maxgamma = 0.1 \times maxgamma;
omega_1 = \sqrt{E/hb^2/(12*rho)} \times maxgamma^2;
hmiddle = (ht+hb)/2;
htilde = \sqrt{(12*rho/E)/gammam^2} \times omega_1;
beta_1 = maxgamma \times H/\pi;
B.3 Plotting of the results

function mixed_results(hbstart, hbstop, precision)

ht = 0.0015; \%htop = 1.5mm
H = 0.05; \% Glass height, m
betan = 0.6;
K = 4/5 - 6/(betan^2*pi^2);

nrofiterations = floor(abs(hbstart-hbstop)/precision)
omega_1 = [1:nrofiterations];
varfreqFEM = [1:nrofiterations];
percent = [1:nrofiterations];

\% Compute FEM results for different thickness variations
\% (hb is varying)
for ite = 1:nrofiterations
hb = hbstart - (ite-1)*precision;
[omega_1(ite)] = FEM_variable(5,5000,hb,ht);
varfreqFEM(ite) = (omega_1(ite)/omega_1(1) -1) * 100;
epsilon(ite) = (ht-hb)/(hb * H);
percent(ite) = (ht-hb)/(2*ht)*100;
end

\% This is the analytical results, in percent
varfreqAn = (K-1)*H .* epsilon*100;

\% Extrapolation of experimental results
japC = - 6/2.6 .* percent;
japB = - 6/4.1 .* percent;
japA = - 6/5.5 .* percent;

figure(1)
plot(percent,varfreqAn, percent, varfreqFEM)
title('Frequency variation vs thickness variation')
xlabel('Material removal [in \%]')
ylabel('Variation of natural frequency [in %]')
legend('Analytical', 'FEM')
figure(2)
plot(percent, varfreqAn, '-', percent, varfreqFEM, '+-', percent, japA, '-', ...
    percent, japB, '-', percent, japC, '-')
title('Frequency variation vs thickness variation')
xlabel('Material removal [in %]')
ylabel('Variation of natural frequency [in %]')
legend('Analytical', 'FEM', 'Experimental (A-glass)', 'Experimental (B-glass)', ...
    'Experimental (C-glass)')
B.4 Frequency analysis

\%
\%
\% This program asks the user for a sound file to open and plots the
\% frequency analysis of the sound file.
\%
\% EPFL/LEMA - Thomas Guignard - 2003-02-27
\%
\clear all, close all, clc
\cd 'E:\glasharfe\sounds'
[filename, pathname] = uigetfile('*.wav');
\cd(pathname);
\filewav = wavread(filename);
\figure;
[Pxx, F] = psd(filewav, 8192, 44100, 8192);
plot(F, 10*log10(abs(Pxx))), grid on
\axis([0 2000 -100 0]);
xlabel('Frequency'), ylabel('Power Spectrum Magnitude (dB)');
freqfond_index = find(max(Pxx) == Pxx);
freqfond = F(freqfond_index)
title(['File: ' filename ' / Fundamental frequency: ' num2str(freqfond) 'Hz'])
\cd '\Lemapc4\commun LEMA\dipl_Th_Guignard\matlab';