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Groups of Tree-Automorphisms and their Unitary Representations

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Abstract

English

In this work, we indroduce a property for automorphism groups of locally finite trees, which we call the Independence Property and which, for closed groups, is equivalent to Property (P) introduced in [Tits]. Using this, we proceed to the classification of all continuous unitary irreducible representations of closed non compact locally 2-transitive (cf. [B; M]) automorphism groups of locally finite homogeneous or semi-homogeneous trees which have the Independence Property. Further we treat the case of the locally finite homogeneous tree where we show that all closed edge transitive automorphism groups which have the Independence Property are locally transitive Universal Groups (cf. [B; M]). At last we give two necessary and sufficient conditions for a locally transive Universal Group to be topologically finitely generated (cf. [Mozes]).

Deutsch

In dieser Arbeit führen wir die "unabhängigkeits Eigenschaft" (Independence Property) für Automorphismengruppen von lokal endlichen Bäumen ein, die, falls die Automorphismengruppe geschlossen ist, äquivalent zur Eigenschaft (P) in [Tits] ist. Ferner geben wir eine Klassifizierung sämtlicher stetiger unitärer irreduktibler Darstellungen von geschlossenen nicht kompakten lokal 2-transitiven (s. [B; M]) Automorphismengruppen mit unabhängigkeits Eigenschaft von lokal endlichen homogenen oder semi-homogenen Bäumen. Den Fall der lokal endlichen homogenen Bäumen behandelnd, zeigen wir, dass alle geschlossenen Automorphismengruppen mit unabhängigkeits Eigenschaft und transitiver Aktion auf den Kanten des Baumes lokal transitive Universal-Gruppen (s. [B; M]) sind. Wir geben auch für lokal transitive Universal-Gruppen zwei äquivalente Bedingungen um topologisch endlich erzeugt zu sein (s. [Mozes]).

Contents

Introduction

The group aut (T) of all automorphisms of a locally finite tree is locally compact. In this work we study a large class of closed automorphism groups G with praticular focus on their continuous unitary representations.

In Chapter one we introduce the independence property of an automorphism group of some tree. For closed groups, this property is indeed equivalent to the Property (P) introduced by J. Tits in [Tits] (4.2, p. 197).

Considering a continuous unitary representation π of a closed automorphism group G of a locally finite tree T, it turns out that there exists a non trivial vector which is invariant under the action of the subgroup of automorphisms fixing some finite subtree. Among all the finite subtrees for which there exists such invariant non trivial vector, one takes the set M_π of complete subtrees which are minimal for the inclusion. If the set M_π contains a subtree which has more than one edge, then we call the representation π super cuspidal. The representation π is called special, if M_π contains a subtree which has only one edge, and is called spherical if M_π contains a subtree which has only one vertex.

In the first part of the second chapter we discuss the existence of super cuspidal representations in the case where the group has the independence property. Generalizing Ol'shanski's basic idea (c.f. [Ol'sh]) to any closed tree-automorphism group G with the independence property, we find that G has a super cuspidal representation if G does not fix any point at the boundary of the tree T and admits no G-invariant non trivial proper subtree.

In the second part of the second chapter we proceed to the complete classification of the continuous unitary representations of a closed non compact automorphism group G with the independence property and acting transitively on the boundary of the tree T, which in this case is automatically homogeneous or semi-homogeneous. In particular we see that an irreducible continuous unitary representations of such a group is of exactly one of the three types: super cuspidal, special or spherical. This is a generalization of the classifications given in [Ol'sh] and [F-T; N], as it covers the case of a great variety of closed automorphism groups of the homogeneous or semi-homogeneous tree, in particular of the so called Universal Group $U(F)$ of the homogeneous tree of degree $d \geq 3$ in [B; M] with F a transitive permutation group on $\{1, \ldots, d\}$.

The third chapter treats of the Universal Group $U(F)$. We shall see that all closed edge transitive groups with the Independence Property are (locally transitive) Universal Groups.

In [Mozes], Shahar Mozes, discussing irreducible uniform lattices Γ in aut $(T_1) \times$ aut (T_2), with T_1, T_2 homogeneous trees, is interested in those subgroups $U(F)$ which arise as closure of projections of such lattices Γ. As these must be topologically finitely generated, he stated without proof that, if F is a transitive permutation group on $\{1, \ldots, d\}$ and $F_1 = \mathrm{Stab}_F(1)$ is non-trivial, then $U(F)(x)$ is topologically finitely generated if and only if the stabilizer group F_1 is perfect and equal to its normalizer. In the second part of Chapter 3, we shall prove that, if F is a transitive permutation group on $\{1, \ldots, d\}$ and $F_1 = \mathrm{Stab}_F(1)$ is non-trivial, the following conditions are equivalent:

1. The subgroup $U(F)(x)$ is topologically finitely generated.
2. The stabilizer group F_1 is perfect and equal to its normalizer.
3. For every real positive number M, the group $U(F)$ has finitely many equivalence classes of super cuspidal representations with formal degree less than M.

Chapter 1

Trees and their Automorphisms

For the beginning, we want to give some notations and definitions concerning trees and their automorphisms. In particular we shall introduce what we call the independence property of a group of automorphisms of a tree. For closed groups, this property is equivalent to the Property (P) introduced by J. Tits in [Tits] (4.2, p. 197).

1.1 Definitions and Notations

For the definitions of graphs, trees and automorphisms, we refer to the book [Serre] or to its English translation [Serre 2] by J.-P. Serre. We just recall here the essential and introduce some notation.

A *graph* $\Gamma = (X, Y)$ consists a set set of *vertices* X and a set of *edges* Y and the maps $(o, t) : Y \to X \times X, e \mapsto \big(o(e), t(e)\big)$ and $Y \to Y, e \mapsto \overline{e}$ satisfying $\overline{\overline{e}} = e, \overline{e} \neq e$ and $o(e) = t(\overline{e})$. The vertex $o(e)$ is called the *origine* and the vertex $t(e)$ is called the *terminus* of the edge e. A *path* is a sequence of edges $(e_i)_{i \in I}$, *finite of length* n (*ie* $I = \{1, \dots, n\}$ for some positive natural number n), *infinite* (*ie* I is the set $\mathbb{N}$ of natural numbers) or *doubly infinite* (*ie* I is the set $\mathbb{Z}$ of relative numbers), such that $t(e_i) = o(e_{i+1})$ for each $i \in I$ with $i + 1 \in I$. We call *chain* a path $(e_i)_{i \in I}$ without backtracking, i.e. such that $e_i \neq \overline{e}_{i+1}$ for each $i \in I$ with $i+1 \in I$. A graph is *connected*, if any two vertices can be joined by a finite path, *ie* such that x is the original vertex of the first edge and y the terminal vertex of the last edge of the path. On the set of vertices of a connected graph exists a natural *distance* $d(x, y)$ between two vertices x and y, which is the length of the shortest path (chain) linking x to y. A *cycle* is a path

$e_1, \dots, e_k$ such that for each i, $e_i \neq \overline{e}_{i+1}$ and $o(e_1) = t(e_k)$. A *tree* is a connected, non empty graph without cycle. We call *degree* or *index* of a vertex x and write $d(x)$ the cardinality of the set $o^{-1}(x) = t^{-1}(x)$. A graph is *locally finite* if any of its vertices is of finite index.

A *morphism* g of the graph $\Gamma_1 = (X_1, Y_1)$ to $\Gamma_2 = (X_2, Y_2)$ is a map $g = g_X \times g_Y : X_1 \times Y_1 \to X_2 \times Y_2$ such that for every edge e of Γ_1, $\big(g_X(o(e)), g_X(t(e))\big) = \big(o(g_Y(e)), t(g_Y(e))\big)$ and $\overline{g_Y(e)} = g_Y(\overline{e})$. By abuse we shall write $g_X = g$ and $g_Y = g$.

The graph $\Gamma' = (X', Y')$ is a *subgraph* of the graph $\Gamma = (X, Y)$ if the inclusion maps $X' \subseteq X$ and $Y' \subseteq Y$ form a morphism of graphs and if $\overline{Y'} = Y'$, where $\overline{Y'} = \{\bar{e} \mid e \in Y'\}$. We write $\Gamma' \subseteq \Gamma$.

A connected graph $\Gamma = (X, Y)$ has a *bipartite structure* if there exist two subsets $X_1, X_2 \subseteq X$ such that $X = X_1 \sqcup X_2$ and $o(Y) \times t(Y) \subseteq (X_1 \times X_2) \sqcup (X_2 \times X_1)$.

An *automorphism* of a graph $\Gamma = (X, Y)$ is a morphism of Γ to Γ which is bijective on the sets X and Y. We write $\mathrm{aut}\,(\Gamma)$ for the group of all automorphisms of Γ and we take as convention that $\mathrm{aut}\,(\Gamma)$ acts from the left on the sets X and Y.

We consider now a tree $T = (X, Y)$. A *subtree* of T is a connected subgraph of the tree T.

A vertex of a (sub-)tree S is called a *leaf* or *end* of S if it is the terminal vertex of at most one edge in S; this edge is then also called a *terminal edge* of S. The subtree S of T is *complete* if all its vertices which are not leaves of S have no adjacent vertices outside of S. We call *interior* of the complete subtree S the subtree S° obtained by removing all leaves and their adjacent edges of S.

For an edge e of T we let T_e be the unique maximal subtree of T having e as terminal edge.

We say that two infinite chains $(e_i)_{i \in \mathbb{N}}$ and $(f_i)_{i \in \mathbb{N}}$ on T are *equivalent*, if they have infinite intersection, this means, that there exist $n, K \in \mathbb{N}$ such that $e_k = f_{k+n}$ for all $k \geq K$. It is easy to see that this defines indeed an equivalence relation on the set of infinite chains on T. We call *boundary* of the tree T and write $T(\infty)$, the set of equivalence classes of this equivalence relation. The elements of the boundary are sometimes called *leaves* or *ends* of T and can be seen as an extension of the above definition of leaves of a (sub-) tree. For a fixed vertex x of T, there corresponds to each end ϵ of T a unique infinite chain $(e_i)_{i \in \mathbb{N}} \in \epsilon$ starting at x, i.e. with $o(e_0) = x$. If we write $T_e(\infty)$ the boundary of the subtree T_{e_i}, seen as subset of $T(\infty)$, the family

$\{T_e(\infty) \mid e \in Y\}$ is a base of topology for which $T(\infty)$ is compact.

If g is an automorphism and $(e_i)_{i\in\mathbb{N}}$ an infinite chain of the tree T, then $g(e_i)_{i\in\mathbb{N}} := (ge_i)_{i\in\mathbb{N}}$ is also an infinite chain of T. This defines an action of aut (T) on the infinite chains of T which, as it is easily seen, preserves the equivalence relation on infinite chains, and hence induces a natural action on the boundary of T.

We now discuss briefly some properties of the group of automorphisms aut (T) of T.

An automorphism $g \in$ aut (T) is called a *rotation* if g fixes some vertex x (i.e. $gx = x$) and an *inversion* if there exists an edge e such that $ge = \bar{e}$. Suppose g is neither a rotation nor an inversion, then for every vertex x we have $d(x, gx) \geq 1$. Let x_0 be a vertex such as $d(x_0, gx_0) = n$ is minimal and let $e_1, e_2, \ldots, e_n$ be the chain linking x_0 to gx_0. We construct the doubly infinite path $\gamma = (e_k)_{k\in\mathbb{Z}}$ by setting $e_{k+ln} = g^l e_k$ for all $k \in \{1, \ldots, n\}$ and $l \in \mathbb{Z}$. It is easily seen that this path is a chain which is g-stable. The automorphism g is called *translation of step n along the path γ*.

We have therefore the following proposition which is quite classical (See for example [Tits]).

Proposition 1. *Every automorphism of a tree is either a rotation, an inversion or a translation.*

For a subgroup G of aut (T) and for every subtree S, we write the *S-fixing group*

$$G(S) = \{g \in G \mid gx = x, \text{ for all vertices } x \text{ of } S\}$$

and the *S-stabilizing group*

$$\widetilde{G}(S) = \{g \in G \mid gS = S\}.$$

In particular, if S is contains only one edge e, we write $G(e) = G(S)$ as well as $\widetilde{G}(e) = \widetilde{G}(S)$, and if S contains only one vertex x, we write $G(x) = G(S)(= \widetilde{G}(x))$.

Since a tree is connected and has no cycles, two (distict) vertices can be linked by exactly one finite chain and their distance is the length of this chain. The set of vertices X endowed with this distance is a discrete metric space.

From now on and for the rest of this work, we suppose all our trees locally finite if not indicated otherwise. The image by an automorphism of a chain is again a chain, and automorphisms can be seen as isometries on the space of vertices. In fact aut (T) is isomorphic to the group of isometries on the metric space X, and via this isomorphism the topology of uniform convergence on the compact (i.e. finite) subsets of X makes $\mathrm{aut}\,(T)$ a topological group under which the action on X is equicontinuous. If G is a closed subgroup of $\mathrm{aut}\,(T)$, the $G(S)$ are open for all finite subtrees S and the set $\{G(S) \mid S \text{ is a finite subtree of } T\}$ is a base of neighbourhoods of the identity. Since $G(S)$ is equicontinuous as set of isometries, and since each of its orbit is finite in X, the subgroup $G(S)$ is compact. Therefore G is locally compact. Moreover the maximal compact subgroups of G are of the form $G(x)$, for some vertex x, or $\widetilde{G}(e)$, for some edge e. If G acts transitively on the vertices then $G(x)$ is maximal compact, for every vertex x, and if G acts transitively on the set $\{\{e, \bar{e}\} \mid e \in Y\}$ then $\widetilde{G}(e)$ is maximal compact, for every edge e

Lemma 2. *If a closed subgroup G of* $\mathrm{aut}\,(T)$ *contains no translations, then G is either compact or fixes a point of the boundary of T.*

Proof. Cf. [F-T; N] Theorem 8.1, page 20. □

Definition 3. *A tree is* homogeneous of degree d *or* d-regular *if all its vertices have same index d.*

A tree is semi-homogeneous of degree (r, s) *if the degree function $d : X \to \mathbb{N}$ takes exactly two values r and s, and the level sets $d^{-1}(r)$ and $d^{-1}(s)$ give a bipartite structure on T.*

Proposition 4. *Suppose that the boundary of the tree T has at least three elements. Let G be a closed non compact subgroup of* $\mathrm{aut}\,(T)$. *Suppose G acts transitively on the boundary of T. Then $G(x)$ acts transitively on the boundary of T for every vertex x.*

Moreover, if the above holds, the tree T is homogeneous or semi homogeneous and G has at most two orbits on X. The group G acts transitively on the set of vertices of T if and only if G contains for every edge of T an inversion, otherwise the G-orbits on the vertices are $\{z \mid d(x, z) \text{ is even}\}$ and $\{z \mid d(x, z) \text{ is odd}\}$ for some vertex x.

Proof. By Lemma 2 we can suppose that G contains a translation. Let $\tau \in G$ be a translation along the doubly infinite chain γ, let x be a vertex of γ, and let $\epsilon \in T(\infty)$. Since $G(x)$ is open in G and has countable index, $T(\infty) = G \cdot \epsilon = \bigcup_{i \in \mathbb{N}} h_i G(x) \cdot \epsilon$ where $\{h_i \mid i \in \mathbb{N}\}$ is a complete set of coset representatives and $G(x) \cdot \epsilon$ the $G(x)$-orbit of ϵ. Since the boundary $T(\infty)$ is complete metrizable, and $h_i G(x) \cdot \epsilon$ is compact, $h_i G(x) \cdot \epsilon$ has an interior point. Therefore the orbit $G(x) \cdot \epsilon$ has an interior point and hence is open.

Let now ϵ' and ϵ'' be the two ends of the chain γ. Since $G(x) \cdot \epsilon'$ and $G(x) \cdot \epsilon''$ are open, there are two edges e' and e'' of γ such that $T_{\bar{e}'}(\infty) \subseteq G(x) \cdot \epsilon'$ and $T_{e''}(\infty) \subseteq G(x) \cdot \epsilon''$. Suppose τ moves x in direction to ϵ' and let $\epsilon \in T(\infty) \setminus \{\epsilon', \epsilon''\}$. If e is the edge of the infinite chain starting at x and corresponding to ϵ such that $o(e)$ is a vertex of γ but $t(e)$ is not, then there exists a natural number n such that $\tau^n e$ is an edge of $T_{\bar{e}'}$, which means that $\tau^n \epsilon \in T_{\bar{e}'}(\infty) \subseteq G(x) \cdot \epsilon'$. Therefore $T(\infty) \setminus \{\epsilon''\} \subseteq G(x) \cdot \epsilon'$. Similarily we have also $T(\infty) \setminus \{\epsilon'\} \subseteq G(x) \cdot \epsilon''$. Since the boundary of T has at least three elements, $T(\infty) \setminus \{\epsilon'\}$ and $T(\infty) \setminus \{\epsilon''\}$ intersect. Hence $G(x)$ acts transitively on the boundary for every vertex x of γ. Moreover $G(x)$ acts also for every n transitively on the sets $S_{x,n} = \{y \in X \mid d(x,y) = n\}$.

Since the boundary of T has at least three elements, T has at least one vertex with index greater than 2, and one of such has to be a vertex of the doubly infinite path γ. Now take the vertex x of γ at distance 1 of this vertex. Then all vertices at distance 1 of x have the index greater than 2. If y is a vertex at distance n of x, then $S_{x,n} = G(x) \cdot y \subseteq G \cdot x$. Therefore the G-orbit of x is $G \cdot x = \{x\} \cup \bigcup_{i \in E} S_{x,i}$ where E is a set of positive numbers. Let $k = \min E$ and take $y \in S_{x,k}$. Then $y \in G \cdot x$; furthermore $G(y)$ is a conjugate of $G(x)$ in G, hence $G(y)$ acts also transitively on the boundary and on the sets $S_{y,n} = \{z \in X \mid d(y,z) = n\}$. This means that every element having distance k from y belongs to $G \cdot x$. It follows that E must contain all multiples of k. But k is at most 2. Indeed, suppose that $k > 1$. Let y be a vertex at distance k of x and z a vertex at distance k from y and 2 from x. Then $z \in G(y) \cdot x \subseteq G \cdot x$ and therefore $k = 2$ and finally $G \cdot x = \{z \mid d(x,z) \text{ is even}\}$. Let $z \in \{z \mid d(x,z) \text{ is odd}\}$, then there exists $g \in G$ such that $d(g^{-1}x, z) = 1$ and therefore $d(x, gz) = 1$ and since $G(x)$ is transitive on $S_{x,1}$, $\{z \mid d(x,z) \text{ is odd}\}$ is the second orbit of G.

Finally, if $k = 1$ then G acts transitively on the vertices of T which has to be homogeneous. Let e be an edge and $g \in G$ with $go(e) = t(e)$, then, since $G\big(t(e)\big)$ acts transitively on $t^{-1}\big(t(e)\big)$, there exists $h \in G\big(t(e)\big)$ with $hge = \bar{e}$, i.e. hg is an inversion of e. Inversely, if G contains an inversion for some edge e, then clearly $k = 1$ and G acts transitively on the vertices of T. □

Observe that, if the boundary of T has only two elements, the above proposition is false. Indeed consider a chain on which we number the vertices so that the vertex k and $k+1$ are adjacent. To each vertex whose number is a multiple of $n \geq 1$ we ad one edge. The boundary of this tree T has only two elements and the automorphism group $\mathrm{aut}\,(T)$ is non compact since it contains a translation of step n. One sees easily, that if n is odd and $n \geq 3$ then for every vertex x of degree 2, the stabilizer group $G(x)$ is trivial, and if moreover $n \geq 5$, the group G has more than two orbits.

We also observe that a group satisfying the hypothesis of the above proposition acts minimaly.

Definition 5. *An automorphism group G is said* locally n-transitive *if for every vertex x of the tree T, the stabilizer $G(x)$ of x acts transitively on the set $\{y \in X \mid d(x,y) = n\}$. If $n = 1$, we say just that G is* locally transitive.

Obviously a locally n-transitive automorphism group is also locally k-transitive for every $k \leq n$.

Proposition 6. *Any locally transitive closed subgroup G is unimodular.*

Proof. At first, we notice that by the transitivity of the action of $G(x)$ on $t^{-1}(x)$, we have for every vertex x of T and every edge e of $t^{-1}(x)$,

$$G(x) = \bigsqcup_{\varepsilon \in t^{-1}(x)} h_\varepsilon G(e),$$

where $h_\varepsilon \in G(x)$ with $h_\varepsilon(e) = \varepsilon$.

Thus, if $\mathrm{m}(E)$ denotes the left Haar measure of a subset E of G,

$$\mathrm{m}(G(x)) = \sum_{\varepsilon \in t^{-1}(x)} \mathrm{m}(h_\varepsilon G(e)) = \left|t^{-1}(x)\right| \mathrm{m}(G(e)).$$

It follows from the equality $G(e) = G(\bar{e})$, that for every vertex x of T and every edges $e, e' \in t^{-1}(x) \cup o^{-1}(x)$,

$$\mathrm{m}(G(e)) = \frac{1}{|t^{-1}(x)|}\mathrm{m}(G(x)) = \mathrm{m}(G(e')),$$

and by induction on the distance between two edges, one sees that $\mathrm{m}(G(e)) = \mathrm{m}(G(e'))$ for every edges e, e' of T.

Let now $g \in G$ and e an edge of T. Obviously we have $G(ge) = gG(e)g^{-1}$. Let $\mathbf{1}_E$ denote the characteristic function of the subset E of G and Δ the modular function of G. Then

$$\begin{aligned}\mathrm{m}(G(e)) = \mathrm{m}(G(ge)) &= \int \mathbf{1}_{G(ge)}(h)\, dh = \int \mathbf{1}_{G(e)}(g^{-1}hg)\, dh \\ &= \int \mathbf{1}_{G(e)}(hg)\, dh = \Delta(g^{-1}) \int \mathbf{1}_{G(e)}(h)\, dh \\ &= \Delta(g^{-1})\mathrm{m}(G(e)).\end{aligned}$$

Thus $\Delta(g) = 1$ and the group G is therefore unimodular. □

Refer also to the unimodularity criterium of Bass-Kulkarni in [B; K].

Observe that, if the hypothesis of the above proposition is not satisfied, the group needs not be unimodular, as the example of the stabilizer of a boundary point shows (cf. [Nebbia]).

Recall that, for a locally compact group G and a compact subgroup K of G, the pair $\big(G, K\big)$ is called a *Gel'fand pair*, if the convolution algebra $\mathbf{C}_{\circ\circ}(G)^{\natural}$ of complex valued K-bi-invariant functions (i.e. functions f satisfying $f(kgk') = f(g)$ for all $k, k' \in K$) with compact support on G is commutative.

Proposition 7. *Suppose that the boundary of the tree T has at least three elements. Let G be a closed non compact subgroup of* aut (T) *which acts transitively on the boundary of T. Then $\big(G, G(x)\big)$ is a Gel'fand pair for every vertex x.*

Proof. By Proposition 4, $G(x)$ acts transitively on the boundary of the tree T, and since for each $g \in G$, $d(x, gx) = d(x, g^{-1}x)$, there exists $k \in G(x)$ with $kgx = g^{-1}x$, i.e. $gkgx = x$. Therefore there exists $k' \in G(x)$ such that $k' = gkg$, and hence $g^{-1} \in G(x)gG(x)$.

This means that for every $G(x)$-bi-invariant function f, we have $f(g^{-1}) = f(g)$. Hence for $f_1, f_2 \in \mathbf{C}_{\circ\circ}(G)^{\natural}$ we have

$$\begin{aligned} f_1 * f_2(h) &= \int_G f_1(g) f_2(g^{-1}h)\, dg = \int_G f_1(g) f_2\big((h^{-1}g)^{-1}\big)\, dg \\ &= \int_G f_1(hg) f_2(g^{-1})\, dg = \int_G f_1(g^{-1}h^{-1}) f_2(g)\, dg \\ &= \int_G f_2(g) f_1(g^{-1}h^{-1})\, dg = f_2 * f_1(h^{-1}) = f_2 * f_1(h) \end{aligned}$$

for every $h \in G$. □

1.2 Groups with the Independence Property

Consider a locally finite tree T (i.e. every vertex of T has finite index).

First we notice the following obvious fact.

Lemma 8. *If S_1 and S_2 are two subtrees of T such that the union of their sets of vertices is the set of all vertices of T, then for all automorphism group G, the elements of $G(S_1)$ and $G(S_2)$ commute.*

Definition 9. *We say that the group G has the* independence property *if for every edge e of the tree T we have the equality*

$$G(e) = G(T_e)G(T_{\bar{e}}).$$

The group aut (T) of all automorphisms has of course the independence property. We shall further see in Chapter 3 a large class of groups as the universal group which has also this property.

If the subtree S has at least one edge, we define for each vertex x of S, the subtree $F_{S,x} = \cup_e T_e$ where e runs along the set of edges of S having x as their terminal vertex.

Lemma 10. *A group G of automorphisms of a tree T has the independence property if and only if for every finite subtree S with at least one edge, the equality*

$$G(S) = \prod_x G(F_{S,x}),$$

with x running along the set of vertices of S, holds.

Proof. Suppose that G has the independence property. We shall prove inductively on the number N of vertices of S that $G(S) = \prod_x G(F_{S,x})$.

For $N = 2$ it is the independence property. Let $N > 2$ and suppose that G is S'-independent for all subtrees S' with $N - 1$ vertices. Take a subtree S with N vertices and $h \in G(S)$. Take also a terminal edge f; since $h \in G(f)$, there exist $h_1 \in G(T_f)$ and $h_2 \in G(T_{\bar{f}})$ with $h = h_1 h_2$. If S' denotes the subtree obtained from S by cutting off the edge f, then $h_2 = h_1^{-1} h \in G(T_f)G(S) \subseteq G(S) \subseteq G(S')$. Write X' the set of vertices of S'. By induction hypothesis $h_2 = \prod_{y \in X'} h_y$ for some $h_y \in G(F_{S',y})$. But we can easily see that for all $y \in X' \setminus \{o(f)\}$, $T_{\bar{f}}$ is a subtree of $F_{S',y}$, so $h_y \in G(T_{\bar{f}})$ for all these y, and therefore $h_{o(f)} \in G(T_{\bar{f}})$ too. Finally

$$\begin{aligned} h = h_1 h_2 \in G(T_f) \cdot \prod_{y \in X'} \big(G(F_{S',y}) \cap G(T_{\bar{f}})\big) &= G(F_{S,t(f)}) \cdot \prod_{y \in X'} G(F_{S,y}) \\ &= \prod_y G(F_{S,y}), \text{ with } y \text{ over all vertices of } S. \end{aligned}$$

□

If G is closed in aut (T), we notice that, using an approximation argument, it is possible to prove such a statement for infinite subtrees S. Therefore one sees that for closed groups the independence property is equivalent to Property (P) by J. Tits as stated in [Tits] (4.2, p. 197), just by taking for S all chains C of length at least one. The Property $(*)$ introduced in [Nebbia] (p. 346) however, is somewhat more restrictive, since it implies that the fixing group of a subtree Δ is equal the fixing group of the minimal complete subtree containing Δ — it would have been enough to require this condition for the complete subtrees only.

For a complete subtree S with at least one edge, and for each vertex x of S, we have $F_{S,x} = T$, if x is not a leaf, and $F_{S,x} = T_e$, if x is the terminal vertex of the terminal edge e of S. Thus we have the following proposition:

Proposition 11. *The group G has the independence property if and only if for every finite complete subtree S of T we have*

$$G(S) = \prod_e G(T_e)$$

where e is to be taken among all terminal edges of S.

Lemma 12. *Suppose the automorphism group G of $T = (X, Y)$ has the independence property. Let $x \in X$ be a vertex of T and for each edge e of T write $X_{\bar{e}} \subseteq X$ the set of vertices of the subtree $T_{\bar{e}}$. If $O \subseteq X$ is a $G(x)$-orbit, then for each $e \in o^{-1}(x)$, $O \cap X_{\bar{e}}$ is a $G(T_e)$-orbit. If further $G(x)$ acts transitively on $o^{-1}(x)$, then $O = \bigsqcup_{e \in o^{-1}(x)} O \cap X_{\bar{e}}$.*

Proof. It is enough to show that $G(T_e)$ acts transitively on $O \cap X_{\bar{e}}$. Take two vertices y and y' in $O \cap X_{\bar{e}}$. Since the edge e is common to the geodesic segments joining $x = o(e)$ with y respectively joining $o(e)$ to y', an automorphism $h \in G(x)$ such that $hy = y'$, fixes e. By the independence property of G, there exists $h_e \in G(T_e)$ and $h_{\bar{e}} \in G(T_{\bar{e}})$ with $h = h_e h_{\bar{e}}$, and therefore $h_e y = h h_{\bar{e}}^{-1} y = hy = y'$. □

We write G^+ for the subgroup of G generated by the edge-fixing automorphisms.

Definition 13. *We say that G* acts minimaly *on T, if there is no G-invariant proper subtree and no element of the boundary of T fixed by G.*

If the tree T is homogeneous or semi-homogeneous the group aut (T) acts minimaly on T. We shall also see in Chapter 3, that the universal group $U(F)$ does so. But for example the stabilizer of a horicycle does not acts minimaly on T.

Proposition 14. *Assume that G has the independence property and acts minimaly on T. Then the following statements are equivalent.*

1. *$G^+ \neq \{id\}$.*

2. *$G(T_e) \neq \{id\}$, for every edge e.*

If these conditions are satisfied, then $G(S) \neq \{id\}$ for every finite subtree S of T and therefore, G is not discrete.

Proof. That statement *2.* implies statement *1.* is clear.

Conversely if we show that for every edge e, the group $G(e)$ is trivial if and only if $G(T_e)$ is trivial, then supposing that $G^+ \neq \{id\}$, take an edge e with $G(e) \neq \{id\}$ and let ε be any edge. As $G(e) = G(\bar{e})$ and $G(\varepsilon) = G(\bar{\varepsilon})$, one can suppose that T_ε is a subtree of T_e, thus

$$G(\varepsilon) \supseteq G(T_\varepsilon) \supseteq G(T_e) \neq \{id\}$$

and $G(T_\varepsilon)$ and $G(T_{\bar{\varepsilon}})$ are non-trivial.

We show now that for every edge e, the group $G(e)$ is trivial if and only if $G(T_e)$ is trivial. The direct sense is trivial, so we show the reciprocal sense. If the tree T is homogeneous of degree 2, then $G(e)$ and $G(T_e)$ are trivial anyway. So we suppose that T has at least one vertex of degree greater than 2.

Let e be an edge of T and suppose that $G(T_e) = \{id\}$. Then $G(e) = G(T_{\bar{e}})$, by the independence property of G. If we show that there exists a translation $g \in G$ along a doubly infinite chain C which is contained in $T_{\bar{e}}$, then the subtree $T_{ge} = gT_e$ would be a subtree of $T_{\bar{e}}$ and therefore we would have

$$G(e) = G(T_{\bar{e}}) \subseteq G(gT_e) = gG(T_e)g^{-1} = \{id\}\,.$$

The group G is supposed acting minimaly on T, so, by Lemma 2, there exists a translation $h_1 \in G$ along a doubly-infinite chain C_1. Now we see that for every vertex x of the chain, the orbit $G \cdot x$ has a nontrivial intersection with the set of vertices of the interior of $T_{\bar{e}}$ (This is the subtree $T_{\bar{e}}$ without the edges e and $\bar{e}$ and the vertex $o(e)$). Indeed, otherwise the minimal subtree of T containing $G \cdot x$ would be G-invariant and a subtree of T_e and hence a proper G- invariant subtree of T, which is contrary to our hypothesis. Therefore there exists $h_2 \in G$ such that the chain h_2C_1 has at least one vertex in the interior of $T_{\bar{e}}$. The automorphism $h_2h_1h_2^{-1}$ is a translation along h_2C.

Let $\epsilon, \epsilon' \in T(\infty)$ be the ends of the chain h_2C_1. Then at least one of them, say ϵ, is in $T_{\bar{e}}(\infty)$. The end ϵ' is not G-invariant, as well as the set $\{\epsilon, \epsilon'\}$, otherwise the chain

h_2C_1 would be G-invariant and T must be homogeneous of degree 2, contrary to our supposition at the beginning. Hence there exists $h_3 \in G$ with $h_3\epsilon' \notin \{\epsilon, \epsilon'\}$, i.e. with $\epsilon' \notin \left\{h_3^{-1}\epsilon, h_3^{-1}\epsilon'\right\}$. Consider the set M of the vertices of the chain h_2C_1 which are the closest to the chain $h_3^{-1}h_2C_1$. Since ϵ' is not an end of $h_3^{-1}h_2C_1$, there are only a finite number of vertices of T_e in M. This means that there exists a relative number n such that $(h_2h_1h_2^{-1})^n M$ is completely included in the set of vertices of the interior of $T_{\bar{e}}$. But then, the chain $C = (h_2h_1h_2^{-1})^n(h_3^{-1}h_2C_1)$ is also completely contained in the interior of $T_{\bar{e}}$ and the automorphism $g = (h_2h_1h_2^{-1})^n h_3^{-1}h_2h_1h_2^{-1}h_3(h_2h_1h_2^{-1})^{-n}$ is a translation along C. □

Proposition 15. *Suppose the boundary of the tree T has at least three elements. Let G be a closed non compact automorphism group of T with the independence property. Then G acts transitively on the boundary if and only if G acts locally 2-transitively on T.*

Moreover, if the above holds, the tree T is homogeneous or semi homogeneous and G has at most two orbits on X. The group G acts transitively on the set of vertices of T if and only if G contains for every edge of T an inversion, otherwise the G-orbits on the vertices are $\{z \mid d(x,z) \text{ is even}\}$ and $\{z \mid d(x,z) \text{ is odd}\}$ for some vertex x.

Proof. Let us write $\underline{G(x)}$ the permutation group on $t^{-1}(x)$ induced by the stabilizer $G(x)$ of x. By Proposition 4 it is enough to show that for every vertex x, $G(x)$ acts transitively on the boundary if and only if for every vertex x, $\underline{G(x)}$ is 2-transitive.

Write for every edge e, $\underline{G(e)}$ respectively $\underline{G(T_e)}$ the group of permutations on $t^{-1}\big(t(e)\big)$ induced by the stabilizer $G(e)$ respectively $G(T_e)$. Then, if G has the independence property, we have the equality $\underline{G(e)} = \underline{G(T_e)}$ and $\underline{G\big(t(e)\big)}$ is 2-transitive if and only if $\underline{G\big(t(e)\big)}$ is transitive and $\underline{G(e)}$ is transitive on $t^{-1}\big(t(e)\big) \setminus \{e\}$.

Suppose now that $\underline{G(x)}$ is 2-transitive for every vertex x. Fix now a vertex x. We show that $G(x)$ acts transitively on the boundary of T. Let ϵ and ϵ' be two ends of T and take the unique infinite chains $\gamma \in \epsilon$ and $\gamma' \in \epsilon'$ starting from x. We show by induction that for every natural number $n \in \mathbb{N}$, there exists $g_n \in G(x)$ such that $g_n\gamma(n) = \gamma'(n)$: for $n = 0$, by transitivity of $\underline{G(x)}$, there exists $g_0 \in G(x)$ with $g_0\gamma(0) = \gamma'(0)$. If we have $g_n \in G(x)$ such that $g_n\gamma(n) = \gamma'(n)$, then by transitivity of $\underline{G(T_{\gamma'(n)})}$ on $t^{-1}\big(t(\gamma'(n))\big) \setminus \{e\}$, there exists $h \in G(T_{\gamma'(n)})$ with $g^{n+1} := hg_n\gamma(n+1) = \gamma'(n+1)$. By closedness of $G(x)$, we have the transitivity of the action of $G(x)$ on the boundary.

The inverse statement is immediate. □

Chapter 2

Unitary Continuous Representations of Tree-Automorphism Groups

In this chapter we shall use the independence property to discuss the existence of some irreducible unitary continuous representations of closed tree-automorphism groups.

We shall further give a complete classification of all unitary continuous representations of the closed automorphism groups which have the independence property and act transitively on the boundary of the tree. Using Proposition 15, we see that such groups acts in fact on the homogeneous or semi-homogeneous tree. The classification presented here is a generalization of these given by G. I. Ol'shanski in [Ol'sh] and by Alessandro Figa-Talamanca and Claudio Nebbia in [F-T; N]. Ol'shanski considers only the group aut (T) for T homogeneous and semi-homogeneous, and in [F-T; N] only the homogeneous tree is treated and in the case of the supercuspidal representations, only aut (T) is considered. Our classification applies to all closed non compact automorphism groups which have the independence property and act transitively on the boundary (or equivalently: which are locally 2-transitive) of the homogeneous or semi-homogeneous tree.

2.1 Definitions and Notations

Recall that the set of vertices of a tree has a natural metric and that aut (T), the group of all automorphisms of T acting on this space by isometries, with the topology of uniform convergence on compacts, is a locally compact topological group. Any closed subgroup of aut (T) is also locally compact and admits therefore a left invariant Haar measure dg.

We shall write $m(E)$ for the measure of every measurable subset E of G.

Let G be a closed subgroup of aut (T) and let S be a complete finite subtree of T. Set $Q(S) = \widetilde{G}(S)/G(S)$ (which is in fact a group of automorphisms of S) and $p_S : \widetilde{G}(S) \to Q(S)$ the canonical projection. Consider the maximal proper complete subtrees $S_1, \ldots, S_n$ of S. One has $G(S) \subseteq G(S_i) \subseteq \widetilde{G}(S)$. We set $A_i := p_S\big(G(S_i)\big)$. If S is neither a point nor an edge, the A_i's commute and satisfy $A_j \cap \prod_{i \neq j} A_i = \{id\}$. Since the inner automorphisms of the group $Q(S)$ interchange the A_i's, the group $N := A_1 \times \cdots \times A_n$ is a normal subgroup of $Q(S)$.

Definition 16. *We call the complete subtree S* non degenerate *if $A_i \neq \{id\}$, for all i, or, which is equivalent, if $G(S_i) \neq G(S)$ for all S_i's.*

Definition 17. *A unitary representation $(\omega, \mathcal{K})$ of $Q(S)$ is said to be* non degenerate *if it has no non zero A_i-invariant vectors for all i.*

Let $(\pi, \mathcal{H})$ be a unitary, continuous representation of G, that is, the map $g \mapsto \langle \pi(g)v, w \rangle$ is continuous for every $v, w \in \mathcal{H}$. From now on, all representations will be assumed unitary and continuous, nevertheless we'll call them just "representations". We write $\mathcal{H}^K$ for the subspace of K-invariant vectors, and $\mathcal{H}^{(\infty)} = \bigcup_K \mathcal{H}^K$, where K runs over the set of open compact subgroups of G. Since $\{G(S) \mid S \subseteq T \text{ finite complete}\}$ is a base for the filter of neighbourhoods of $id \in G$, we know that

$$\mathcal{H}^{(\infty)} = \bigcup_S \mathcal{H}^{G(S)},$$

where the union is over all complete finite subtrees S of T, is a subspace of $\mathcal{H}$. Moreover $\mathcal{H}^{(\infty)}$ is dense in $\mathcal{H}$ by continuity of the representation π.

Definition 18. *Let A_π be the set of complete finite subtrees S for which $\mathcal{H}^{G(S)}$ is non trivial. This set is non empty, G-invariant and is ordered by inclusion. Let M_π be the set of minimal elements of A_π*

We call the irreducible representation $(\pi, \mathcal{H})$

1. super cuspidal *if there exists an element of M_π which is neither a point nor an edge;*

2. special *if there exists an element of M_π which is an edge;*

3. spherical *if there exists an element of M_π which has exactly one vertex.*

2.2 Super Cuspidal Representations

Theorem 1. *Let G be a closed automorphism group of a locally finite tree T with the independence property.*

1. (a) *If $(\pi, \mathcal{H})$ is a super cuspidal representation of G, then*
   i. *the group G acts transitively on M_π;*
   ii. *all coefficients of π with vectors in $\mathcal{H}^{(\infty)}$ have compact support;*
   iii. *if $S \in M_\pi$ and ω is the representation of $Q(S)$ defined by the action of $\widetilde{G}(S)$ on $\mathcal{H}^{G(S)}$, then ω is irreducible, non degenerate and π is equivalent to the representation $\mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S)$ induced to G by $\omega \circ p_S$.*

   (b) *If S is a finite complete non degenerate subtree of T and ω a non degenerate irreducible representation of $Q(S)$, then the representation*
   $$T(S,\omega) := \mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S)$$
   induced on G by ω is irreducible and super-cuspidal.
   Moreover, the representation $T(S,\omega)$ is equivalent to another such representation $T(S',\omega')$ if and only if there exists $g \in G$ with $S' = gS$ and ω' corresponds to ω via the isomorphism $Q(S) \to Q(S')$ induced by g.

   (c) *If G is unimodular, then the formal degree of $T(S,\omega)$ is equal to*
   $$\frac{\dim \omega}{\mathrm{m}\left(\widetilde{G}(S)\right)},$$
   where $\mathrm{m}\left(\widetilde{G}(S)\right)$ is the measure of $\widetilde{G}(S)$.

2. *If G acts minimaly and if G^+ is not trivial, then G has a super cuspidal representation.*

3. *If G acts transitively on the boundary of T, i.e. if G is locally 2-transitive, then there exists for every complete finite subtree S, with at least one vertex which is not a leaf, a super cuspidal representation π with $S \in M_\pi$.*

Proof of the Theorem

The following fact, due to G. I. Ol'shanski (see [Ol'sh]), is of great importance in the proof of this theorem:

Lemma 19. *Suppose that G has the independence property. Let U be a complete finite subtree of T with at least one vertex which is not a leaf and let V be a complete subtree not containing U. Then there exists a proper complete subtree W of U such that*

$$G(W) \subseteq G(V)G(U).$$

Proof. Since U is not contained in V, one can take a leaf x of U which is not in V; now pick the adjacent vertex y in U. The tree U is complete and y is one of its vertices which are not leaves, therefore all edges of T with y as terminal vertex are in U. Moreover for one of those edges e, the tree V is contained in T_e, because V is complete. Let $W = T_e \cap U$; this is a complete finite subtree containing the edge e. As G has the independence property, by Proposition 11, we have $G(W) = G(T_e) \prod_f G(T_f)$ where f are all terminal edges of W which are terminal edges of U. For these edges one has $U \subseteq T_f$, so $G(T_f) \subseteq G(U)$; on the other hand, $G(T_e) \subseteq G(V)$, since $V \subseteq T_e$. It follows that $G(W) \subseteq G(V)G(U)$. □

From now on and for the rest of this section, let us suppose that G is a closed subgroup of $\operatorname{aut}(T)$ having the independence property.

Lemma 20. *Let S and S' be complete subtrees, where S is finite and contains at least one vertex which is not a leaf. If f is a $G(S)$-right-invariant and $G(S')$-left-invariant complex function on G such that $\int_{G(U)} f(gk)dk$ is defined and equals zero for all complete proper subtrees U of S and for all $g \in G$, then it is supported in*

$$\{g \in G \mid gS \subseteq S'\}.$$

Proof. If $g \in G$ satisfies $gS \not\subseteq S'$, *ie* $S \not\subseteq g^{-1}S'$, then by Lemma 19 we can take a complete proper subtree U of S such that $G(U) \subseteq G(g^{-1}S')G(S) = g^{-1}G(S')gG(S)$. Therefore $gG(U) \subseteq G(S')gG(S)$ and thus for all $h \in G(U)$, $f(g) = f(gh)$. Hence $f(g) = \frac{1}{\mathrm{m}(G(U))} \int_{G(U)} f(gk)dk = 0$. □

For finite complete subtrees S, let $\mathcal{H}(S)$ denote the subspace of $L^2(G)$ consisting of all $G(S)$-right-invariant functions satisfying $\int_{G(S')} f(gk)dk = 0$, for all proper complete subtrees S' of S and for all $g \in G$. This subspace is closed and invariant under the action of the left-regular representation. Thus let T_S denote the unitary continuous representation of G obtained by restricting the left-regular representation to $\mathcal{H}(S)$.

From now on S will denote a finite complete subtree with at least one vertex which is not a leaf.

Lemma 20 gives the following corollaries.

Proposition 21. *The subspace $H(S)^{G(S)}$ of $G(S)$-bi-invariant functions in $L^2(G)$ has finite dimension.*

Proposition 22. *If $(\pi, \mathcal{H})$ is a super cuspidal representation, then for all $u, v \in \mathcal{H}^{(\infty)}$ the coefficient $g \mapsto \langle u, \pi(g)v \rangle$ has compact support.*

Which proves the statement *1.(a)ii.* of Theorem 1.

Lemma 23. *If $(\pi, \mathcal{H})$ is a super cuspidal representation of G, then G acts transitively on M_π.*

Proof. Since π is super cuspidal, there exists $S \in M_\pi$ whose vertices are not all leaves. Take $U \in M_\pi$ and pick $u \in \mathcal{H}^{G(U)}$, $v \in \mathcal{H}^{G(S)}$, both non zero, and set $f(g) = \langle u, \pi(g)v \rangle$ for all $g \in G$. Then $f \in \mathcal{H}(S)^{G(U)}$, is continuous and non trivial because v is cyclic. It follows that $\varnothing \neq \operatorname{supp}(\mathrm{f}) \subseteq \{g \in G \mid gS \subseteq U\}$ and there exists $g \in G$ such that $gS \subseteq U$. Therefore the vertices of U are also not all leaves. Now we can set $f'(g) = \langle v, \pi(g)u \rangle$ for all $g \in G$ and obtain a non trivial continuous function supported in $\{g \in G \mid gU \subseteq S\}$. Hence there exists $h \in G$ with $hU \subseteq S$. Because g and h are automorphisms and S and U are finite, this implies that $gS = U$ and $hU = S$. □

This proves statement *1.(a)i.*

Lemma 24. *Every non trivial $T_S(G)$-invariant closed subspace of $\mathcal{H}(S)$ contains a non trivial $G(S)$-bi-invariant function.*

Proof. Suppose that M is a non-trivial closed $T_S(G)$-invariant subspace of $\mathcal{H}(S)$ and take $u \in M$ and $g \in G$ such that $u(g) \neq 0$. The function $v = \int_{G(S)} T_S(kg^{-1})u \, dk$ is $G(S)$-bi-invariant and, since M is closed and $T_S(G)$-invariant, $v \in M$. Moreover

$$\begin{aligned} v(id) &= \int_{G(S)} u(gk^{-1}) \, dk = \int_{G(S)} u(g) \, dk \\ &= \mathrm{m}\left(G(S)\right) u(g) \neq 0, \ \ ie\ v \neq 0. \end{aligned}$$

□

Proposition 21 and Lemma 24 imply that the representation $\big(T_S, \mathcal{H}(S)\big)$ is a finite sum of irreducible representations. By Lemma 20, $\mathcal{H}(S)$ does not contain nonzero $G(S')$-invariant vectors if S' is a proper complete subtree of S. Therefore T_S is finite sum of super cuspidal representations.

Lemma 25. *Let $(\pi, \mathcal{H})$ be a super cuspidal representation of G and $S \in M_\pi$. Then π is equivalent to a sub-representation of T_S.*

Proof. Fix $\xi \in \mathcal{H}^{G(S)} \setminus \{0\}$. The linear operator $U : \mathcal{H}^{(\infty)} \to \mathcal{H}(S)$ defined by $U(\eta)(g) = \langle \eta, \pi(g)\xi \rangle$ for all $g \in G$ is continuous and intertwining π and T_S; consequently its continuous extension on $\mathcal{H}$ is an intertwining operator and is, by Schur's lemma, a positive multiple of an isometry. □

Therefore we have the following corollary.

Proposition 26. *The representation T_S is the finite direct sum of all super cuspidal representation π having $S \in M_\pi$.*

Lemma 27. *If $(\omega, \mathcal{K})$ is an irreducible non-degenerate representation of $Q(S)$ then $\omega \circ p_S$ is equivalent to a sub-representation of $\widetilde{G}(S)$ defined by the action of $T_S\big(\widetilde{G}(S)\big)$ over $\mathcal{H}(S)^{G(S)}$.*

Proof. The subspace $H(S)^{G(S)}$ is clearly $\widetilde{G}(S)$-invariant. Now fix $\xi \in \mathcal{K} \setminus \{0\}$. The linear operator $U : \mathcal{K} \to \mathcal{H}(S)^{G(S)}$, where $U\eta(g) = \langle \eta, \omega \circ p_S(g)\xi \rangle$, if $g \in \widetilde{G}(S)$, and $U\eta(g) = 0$, if $g \in G \setminus \widetilde{G}(S)$, for all $\eta \in \mathcal{K}$ is continuous and interlacing $\omega \circ p_S$ and T_S. □

Now we consider an irreducible non degenerate representation $(\omega, \mathcal{K})$ of $Q(S)$. By Lemma 27 we can suppose that $(\omega \circ p_S, \mathcal{K})$ is a sub-representation of $T_S|_{\widetilde{G}(S)}$ restricted to $\mathcal{H}(S)^{G(S)}$. Recall that the induced representation $T(S, \omega)$ of $\omega \circ p_S$ acts as the left-regular representation on the Hilbert space $\mathcal{L}$ of functions $f : G \to \mathcal{K}$ satisfying $f(gh) = \omega \circ p_S(h^{-1}) f(g)$, for every $h \in \widetilde{G}(S)$ and $g \in G$, and $\int_G \|f(g)\|^2 \, dg < \infty$. The scalar product is given by

$$\langle f_1, f_2 \rangle = \int_G \langle f_1(g), f_2(g) \rangle \, dg \quad (\text{with} f_1, f_2 \in \mathcal{L})$$

and moreover

$$\|f\|^2 = \int_G \|f(g)\|_2^2 \, dg = \sum_{\dot{g} \in G/\widetilde{G}(S)} \|f(g)\|_2^2 \, \mathrm{m}\left(\widetilde{G}(S)\right)$$

where $g \in \dot{g}$. Therefore, if $\mathcal{L}'$ designs the closed and non-trivial subspace of $\mathcal{L}$ formed by the functions supported in $\widetilde{G}(S)$, the subspace

$$\bigoplus_{\dot{g} \in G/\widetilde{G}(S)} T(S, \omega)(g)\mathcal{L}'$$

is dense in $\mathcal{L}$. Recall that we consider $\mathcal{K}$ as a subspace of $\mathcal{H}(S)^{G(S)}$.

For $f \in T(S,\omega)(g)\mathcal{L}'$ ($g \in G$) we see, that for all $k \in \widetilde{G}(S)$ following equalities hold:

$$\begin{aligned} T_S(gk)\big(f(gk)\big) &= T_S(g) \circ T_S(k)\big(f(gk)\big) \\ &= T_S(g)\Big(\omega \circ p_S(k)\big(f(gk)\big)\Big) \\ &= T_S(g)\big(f(g)\big). \end{aligned}$$

Therefore we can define the function

$$Uf = \mathrm{m}\left(\widetilde{G}(S)\right)\left(T_S(g)\big(f(g)\big)\right),$$

which is obviously an element of the subspace of $G(gS)$-left-invariant functions

$$\mathcal{H}(S)^{G(gS)} = T_S(g)\mathcal{H}(S)^{G(S)}.$$

Since the Haar measure is left-invariant,

$$\begin{aligned} \|Uf\|_2^2 &= \mathrm{m}\left(\widetilde{G}(S)\right) \int_G \left|T_S(gk)\big(f(gk)\big)(h)\right|^2 dh \\ &= \mathrm{m}\left(\widetilde{G}(S)\right) \int_G \left|\big(f(gk)\big)\big((gk)^{-1}h\big)\right|^2 dh \\ &= \mathrm{m}\left(\widetilde{G}(S)\right) \int_G \left|\big(f(gk)\big)(h)\right|^2 dh \\ &= \mathrm{m}\left(\widetilde{G}(S)\right) \|f(gk)\|_2^2 . \end{aligned}$$

Hence the function $k \mapsto \|f(k)\|_2^2$ is supported and constant on $g\widetilde{G}(S)$, so

$$\begin{aligned} \|f\|^2 &= \int_G \|f(h)\|_2^2\, dh = \int_{\widetilde{G}(S)} \|f(h)\|_2^2\, dh \\ &= \mathrm{m}\left(\widetilde{G}(S)\right) \|f(g)\|_2^2 \end{aligned}$$

and finally

$$\|Uf\|_2 = \|f\| .$$

The linear map

$$U : \bigoplus_{\dot{g} \in G/\widetilde{G}(S)} T(S,\omega)(g)\mathcal{L}' \to \mathcal{H}(S)$$

is an isometry, extensible to $\mathcal{L}$ and interlacing $T(S,\omega)$ and T_S, since, if $f \in T(S,\omega)(g)\mathcal{L}'$, then $T(S,\omega)(h)f \in T(S,\omega)(hg)\mathcal{L}'$, and so

$$\begin{aligned} U\big(T(S,\omega)(h)f\big) &= T_S(hg)\Big(\big(T(S,\omega)(h)f\big)(hg)\Big) \\ &= T_S(h)T_S(g)\big(f(g)\big) = T_S(h)Uf. \end{aligned}$$

We now show that $T(S,\omega)$ is irreducible. Let $\mathcal{M}$ be a closed G-invariant subspace of $\mathcal{L}$. The pre-image of $\mathcal{H}(S)^{G(S)}$ by U is $\mathcal{L}'$, hence by Lemma 23, $\mathcal{M}$ has a non-trivial element ϕ of $\mathcal{L}'$. Moreover $\phi(id) \neq 0$ because $\|\phi(id)\|_2 = \|\phi(k)\|_2$ for all $k \in \widetilde{G}(S)$. As $\big(\omega \circ p_s(k)\big)(id) = \big(T(S,\omega)(k)f\big)(id)$ for all $k \in \widetilde{G}(S)$ and for every $f \in M \cap \mathcal{L}'$, the set $\mathcal{K}' = \{f(id) \mid f \in \mathcal{M} \cap \mathcal{L}'\}$ is a non-trivial $\widetilde{G}(S)$-invariant subspace of $\mathcal{K}$. But $\omega \circ p_S$ is irreducible, so $\mathcal{K}' = \mathcal{K}$. It then follows that for each $g \in \mathcal{L}'$, there exists $f \in \mathcal{M} \cap \mathcal{L}'$ with $g(id) = f(id)$. Moreover

$$g(k) = \omega \circ p_S(k^{-1})g(id) = \omega \circ p_S(k^{-1})f(id) = f(k)$$

for all $k \in \widetilde{G}(S)$, *ie* $g = f$. Consequently $\mathcal{L}' \subseteq \mathcal{M}$, hence $\mathcal{M} = \mathcal{L}$ because $\mathcal{M}$ is closed and G-invariant and $\bigoplus_{\dot{g} \in G/\widetilde{G}(S)} T(S,\omega)(g)\mathcal{L}'$ is dense in $\mathcal{L}$.

We have shown the following lemma:

Lemma 28. *If ω is a non degenerate irreducible representation of $Q(S)$, then the representation*

$$T(S,\omega) = \mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S)$$

is equivalent to an irreducible sub-representation of T_S, therefore $T(S,\omega)$ is super cuspidal.

Next lemma proves the statement *1.(a)iii.*

Lemma 29. *Let $(\pi,\mathcal{H})$ be a super cuspidal representation of G, $S \in M_\pi$ and ω the representation of $Q(S)$ defined by the action of $\widetilde{G}(S)$ on $\mathcal{H}^{G(S)}$, then ω is irreducible non degenerate and π is equivalent to $T(S,\omega)$.*

Proof. Let $\mathcal{M}$ be a non-trivial, closed and $Q(S)$-invariant subspace of $\mathcal{H}^{G(S)}$. Take a non zero vector v of $\mathcal{H}^{G(S)}$ and a non zero vector u of $\mathcal{M}$. Consider the function $f : g \mapsto \langle v, \pi(g)u\rangle$. It's a non trivial element of $\mathcal{H}(S)^{G(S)}$, hence, by Lemma 20, it is supported by $\widetilde{G}(S)$. Therefore there exists $g \in \widetilde{G}(S)$ with $\langle v, \omega \circ p_S(g)u\rangle = \langle v, \pi(g)u\rangle = f(g) \neq 0$. But $\mathcal{M}$ being $Q(S)$-invariant, $\omega\big(p_S(g)\big)u \in \mathcal{M}$. It follows that the orthogonal space of $\mathcal{M}$ in $\mathcal{H}^{G(S)}$ is trivial, *ie* $M = \mathcal{H}^{G(S)}$.

The representation ω is obviously non degenerate.

Since $\omega \circ p_S$ is equivalent to a sub-representation of the reduction to $\widetilde{G}(S)$ of the representation $\mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S) = T(S,\omega)$, it follows from Lemma 25 and Lemma 28 that π is equivalent to the representation $\mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S)$. □

For an irreducible non degenerate representation $(\omega, \mathcal{K})$ of $Q(S)$, let, for $\xi \in \mathcal{K}$, be $f_\xi(g) = \omega \circ p_S(g^{-1})\xi$ if $g \in \widetilde{G}(S)$ and $f_\xi(g) = 0$ if $g \in G \setminus \widetilde{G}(S)$. Then f_ξ is an element of H, the Hilbert space on which $T(S, \omega)$ acts, and we have for all $\eta \in \mathcal{K}$, the scalar product

$$\begin{aligned} \langle f_\eta, T(S,\omega)(g) f_\xi \rangle &= \int_G \big\langle f_\eta(h), f_\xi(g^{-1}h) \big\rangle \, dh \\ &= \int_{\widetilde{G}(S)} \big\langle \omega(h^{-1})\eta, \omega(h^{-1}g)\xi \big\rangle \, dh \\ &= \mathrm{m}\left(\widetilde{G}(S)\right) \langle \eta, \omega(g)\xi \rangle \,, \end{aligned}$$

if $g \in \widetilde{G}(S)$, and $= 0$ if $g \notin \widetilde{G}(S)$. Therefore every coefficient of ω is a coefficient of $T(S, \omega)$ and with the preceding we have following lemma.

Lemma 30. *Let $(\omega, \mathcal{K})$ and $(\omega', \mathcal{K}')$ be two irreducible non degenerate representations of $Q(S)$ respectively $Q(S')$. Then $T(S, \omega)$ and $T(S', \omega')$ are equivalent if and only if there exists $g \in G$ such that $S' = gS$ and ω' corresponds to ω via the isomorphism $\widetilde{G}(S) \to \widetilde{G}(S')$ induced by g.*

Lemma 31. *If S is a non degenerate subtree which is neither a point nor an edge, then $Q(S)$ has a non degenerate irreducible unitary representation.*

Proof. Let π_i be an irreducible not one dimensional representation of A_i and set $\pi = \pi_1 \otimes \ldots \otimes \pi_n$ and $\omega = \mathrm{ind}_{A_1 \times \ldots \times A_n}^{Q(S)}(\pi)$, then π has no non-zero A_i-invariant vectors and, using the formula for the restriction to $A_1 \times \ldots \times A_n$ of ω, one sees that ω is non-degenerate. □

For each edge e we write $\underline{G(T_e)}$ the group of permutations on $t^{-1}\big(t(e)\big)$ induced by the stabilizer $G(T_e)$. The following lemma is an immediate consequence of the definition of non degenerate subtrees:

Lemma 32. *If for every edge e ot T, the permutation group $\underline{G(T_e)}$ is not trivial, then every complete subtree which is neither a point nor an edge is non degenerate.*

Lemma 33. *Let x be a vertex and S the minimal complete subtree of T containing all vertices adjacent to x. If the stabilizer $G(x)$ acts 2-transitively on the leaves of the subtree S, then $Q(S)$ has a non degenerate irreducible unitary representation.*

Proof. If we identifie the leaves of S with the elements of the set of numbers $\{1, \ldots, d\}$, where d is the degree of x, then $Q = Q(S)$ is a 2-transitive permutation group of

$\{1,\ldots,d\}$. We define for every $g \in Q$ define the $d \times d$-matrix $P_g = (\delta_{i,g(j)})_{i,j=1}^d$, where $\delta_{k,l}$ is the Kronecker symbol. The map $g \mapsto P_g$ is a unitary representation of Q onto the space $M_{d,1}(\mathbb{C})$ of $d \times 1$-matrices. Write $e_j = (\delta_{i,j})_{i=1}^n$ the $d \times 1$-matrix with only zeros except at the line j where is the number 1. The subspace $\mathbb{C} \cdot (\sum_{i=1}^d e_i)$ is Q-invariant. Let $\mathcal{M}$ be the orthogonal complement of $\mathbb{C} \cdot (\sum_{i=1}^d e_i)$. Then $\mathcal{M}$ is also Q-invariant. But $\mathcal{M}$ is also irreducible, since $\mathcal{M}^{Q(1)} = \mathbb{C} \cdot (e_1 - \frac{1}{d-1} \sum_{i=2}^d e_i)$ is of dimension one and the vector $e_1 - \frac{1}{d-1} \sum_{i=2}^d e_i$ is cyclic. □

The statement *1.(b)* follows therefore from Lemma 28 and Lemma 30 and the statement *2.* follows from statement *1.(b)* and Lemma 31. The statement *3.* follows from *2.* and Lemma 33.

Now, assume furthermore that G is unimodular. Take $\xi \in \mathcal{K}$, such that $\|\xi\| = 1$, and let f_ξ be as before. One has, if $d_{T(S,\omega)}$ designs the formal degree of $T(S,\omega)$,

$$\begin{aligned}
\frac{1}{d_{T(S,\omega)}} \mathrm{m}\left(\widetilde{G}(S)\right)^2 &= \frac{1}{d_{T(S,\omega)}} \langle f_\xi, f_\xi \rangle^2 \\
&= \int_G |\langle f_\xi, T(S,\omega)(g) f_\xi \rangle|^2 \, dg \\
&= \frac{\mathrm{m}\left(\widetilde{G}(S)\right)^3}{\mathrm{m}\left(\widetilde{G}(S)\right)} \int_{\widetilde{G}(S)} |\langle f_\xi, T(S,\omega)(g) f_\xi \rangle|^2 \, dg \\
&= \mathrm{m}\left(\widetilde{G}(S)\right)^3 \frac{1}{\dim \omega}
\end{aligned}$$

and finally

$$d_{T(S,\omega)} = \frac{\dim \omega}{\mathrm{m}\left(\widetilde{G}(S)\right)}.$$

We have now proved the theorem.

2.3 Classification of the Representations

In Proposition 15 we have seen that, if on a locally finite tree a closed non compact group acts transitively on the boundary, then the tree is homogeneous or semi-homogeneous and the action of the group on the vertices of the tree is either transitive or has two orbits. We shall now give a classification of the continuous unitary representations of a closed non compact locally 2-transitive automorphism group with the independence property of a homogeneous or a semi-homogeneous tree. This classification applies on a larger class of groups than the classifications known before (c.f. [Ol'sh] and [F-T; N]).

Theorem 2. *Suppose the closed non compact automorphism group G has the independence property and acts transitively on the boundary of T, where $T = (X, Y)$ is a homogeneous tree of degree $d \geq 3$ or a semi-homogeneous tree of degree (r, s), with $r, s \geq 2$ and $r \geq 3$ or $s \geq 3$.*

Then a unitary continuous irreducible representation $(\pi, \mathcal{H})$ of G is of exactly one of the three types: super cuspidal, special or spherical.

1. (a) *If $(\pi, \mathcal{H})$ is a super cuspidal representation of G, then*
   - i. *the group G acts transitively on M_π;*
   - ii. *all coefficients of π with vectors in $\mathcal{H}^{(\infty)}$ have compact support;*
   - iii. *if $S \in M_\pi$ and ω is the representation of $Q(S)$ defined by the action of $\widetilde{G}(S)$ on $\mathcal{H}^{G(S)}$, then ω is irreducible, non degenerate and π is equivalent to the representation $\mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S)$ induced on G by ω.*

   (b) *If S is a finite complete subtree which is neither a point nor an edge, then there exists at least one non degenerate irreducible representation ω of $Q(S)$, and the representation*

   $$T(S, \omega) := \mathrm{ind}_{\widetilde{G}(S)}^{G}(\omega \circ p_S)$$

   induced on G by ω is irreducible and super cuspidal, and $S \in M_{T(S,\omega)}$. Moreover, the representation $T(S, \omega)$ is equivalent to another such representation $T(S', \omega')$ if and only if there exists $g \in G$ with $S' = gS$ and ω' corresponds to ω via the isomorphism $Q(S) \to Q(S')$ induced by g.

   (c) *The formal degree of $T(S, \omega)$ is equal to*

   $$\frac{\dim \omega}{\mathrm{m}\left(\widetilde{G}(S)\right)},$$

   where $\mathrm{m}\left(\widetilde{G}(S)\right)$ is the measure of $\widetilde{G}(S)$.

2. (a) *If G acts transitively on X, there exist precisely two equivalence classes of special representations $\left(\lambda_{-1}, \mathcal{H}(e)_{-1}\right)$ and $\left(\lambda_1, \mathcal{H}(e)_1\right)$.*

   (b) *If G has two orbits on X, there exists precisely one equivalence class of special representations $\left(\lambda_1, \mathcal{H}(e)_1\right)$.*

(c) *The group G acts transitively on the sets $M_{\lambda_{-1}}$ and M_{λ_1}.*

(d) *The representations $\big(\lambda_{-1}, \mathcal{H}(e)\big)$ and $\big(\lambda_1, \mathcal{H}(e)\big)$ are square integrable and defined as follows:*

Let e be an edge of T, set $\mathcal{H}(e)$ the subspace of $\mathrm{L}^2(G)$ consisting of the $G(e)$-right-invariant functions f satisfying $\int_{G(x)} f(gk)\,dk = 0$ for all $g \in G$ and $x \in \{o(e), t(e)\}$. Further let σ be the linear involution on $\mathcal{H}(e)$ defined by $\sigma(f)(g) = f(g\tilde{g})$ where $\tilde{g}$ is an inversion of the edge e, if there exist such in G, or $\tilde{g} = id$ otherwise (This involution does not depend on the choice of $\tilde{g}$). Write $\mathcal{H}(e)_\kappa$ the eigenspace of σ corresponding to the eigenvalue $\kappa \in \{-1, 1\}$ and λ_κ the restriction on $\mathcal{H}(e)_\kappa$ of the left regular representation of G.

3. (a) *If π is spherical, then following holds:*

   i. *The set M_π corresponds to the set X of vertices of T,*

   ii. *For every $x \in M_\pi$ and $u \in \mathcal{H}^{G(x)}$ with $\|u\| = 1$, the function $g \mapsto \varphi_\pi(g) = \langle u, \pi(g)u\rangle$ is a zonal spherical function with respect to the compact group $G(x)$;*

   (b) *Let x be a vertex and a the degree of x.*

   i. *If G acts transitively on X, the equivalence classes of spherical representations π with $x \in M_\pi$ are in one to one correspondance with the interval $[-1, 1]$ via the map $\pi \mapsto \varphi_\pi(\bar{g})$ where $\bar{g} \in G$ such that $d(\bar{g}x, x) = 1$.*

   ii. *If G has two orbits on X, the equivalence classes of spherical representations π with $x \in M_\pi$ are in one to one correspondance with the interval $[-\frac{2+(a-2)(r+s-a-1)}{a(r+s-a-1)}, 1]$ via the map $\pi \mapsto \varphi_\pi(\bar{g})$ where $d(\bar{g}x, x) = 2$.*

   iii. *The spherical representation corresponding to $\varphi_\pi(\bar{g}) = 1$ is the trivial character.*

   iv. *If G acts transitively on X, the spherical representation corresponding to $\varphi_\pi(\bar{g}) = -1$ is the character $g \mapsto (-1)^{d(x,gx)}$ and writing π_λ the spherical representation corresponding to $\lambda \in [-1, 1]$ we have $\pi_{-\lambda} = \pi_{-1} \otimes \pi_\lambda|_G$, the inner tensor product of π_{-1} and π_λ.*

   (c) *Let x, y be two vertices. Let a be the degree of x and b the degree of y. Let $\pi_{x,\lambda}$ be a spherical representation with $x \in M_{\pi_{x,\lambda}}$ and $\lambda = \varphi_{\pi_{x,\lambda}}(\bar{g})$ and let $\pi_{y,\mu}$ be a spherical representation with $y \in M_{\pi_{y,\mu}}$ and $\mu = \varphi_{\pi_{y,\mu}}(\bar{g})$ with $\bar{g}$ as above. Then $\pi_{x,\lambda}$ and $\pi_{y,\mu}$ are equivalent if and only if*

$$\mu = \frac{a(b-1)}{b(a-1)}\lambda + \frac{a-b}{b(r+s-b-1)}.$$

2.3.1 Super Cuspidal Representations

If M_π contains a complete finite subtree S with at least two edges, then Theorem 1 applies immediately.

2.3.2 Special Representation

Let e be an edge of T. Recall that there exists an inversion $\tilde{g} \in G$ if and only if G acts transitively on X. We write $\mathcal{H}(e)$ the set of $G(e)$-right-invariant L^2-functions f satisfying

$$\int_{G(x)} f(gk)\, dk = 0$$

for all $g \in G$ and $x \in \{o(e), t(e)\}$.

Lemma 34. *Let S be a finite complete subtree and f a continuous $G(e)$-right-invariant and $G(S)$-left-invariant function satisfying*

$$\int_{G(x)} f(gk)\, dk = 0$$

for all $g \in G$ and $x \in \{o(e), t(e)\}$. Then the function f is completely determined by the values it takes on the set $\{g \in G \mid ge$ is an edge of $S\}$. More precisely, if $g' \in G$ is such that $g'e$ or $g'\bar{e}$ is a terminal edge e' of S, then for every $g \in G$ such that ge is an edge of $T_{e'}$ we have

$$f(g) = \frac{(-1)^{n+1}}{\big((a-1)(b-1)\big)^{\left[\frac{n+1}{2}\right]}(a-1)^{n+1-2\left[\frac{n+1}{2}\right]}} f(g'\tilde{g}),$$

where $(a,b) = \left(\left|t^{-1}\big(t(e')\big)\right|, \left|t^{-1}\big(t(e')\big)\right|\right)$, $[q]$ is the integer part of the real number q, $\tilde{g}$ is an inversion of e if $d\big(g't(e), gt(e)\big)$ is odd and the identity otherwise (observe that if G does not contain any inversion, this distance is always even), and $n = \min\left\{d\big(g't(e), gt(e)\big), d\big(g't(e), go(e)\big)\right\}$. In particular, f is square integrable.

Proof. Let $f \in \mathcal{H}(e)$ be a $G(S)$-left-invariant function and take $g \in G$ be such that ge is not an edge of S. Let e' be the terminal edge of S which is the closest to ge. By replacing e by $\bar{e}$ if necessary, we can take $g' \in G$ such that $e' = g'e$. Let $e_0 = e', e_1, \dots, e_{n+1}$ be a chain with $e_{n+1} = ge$ or $g\bar{e}$ and pick for each $i \in \{0, 1, \dots, n\}$ a $k_i \in G\big(t(e_i)\big)$ such that $k_i e_i = \bar{e}_{i+1}$. Then $ge = k_n \cdot \ldots \cdot k_0 g'\tilde{g}e$, where $\tilde{g}$ is an inversion of e if $d\big(g't(e), gt(e)\big)$ is odd and the identity otherwise.

Let $0 \leq i \leq n$. Since by Lemma 15 the permutation group $\underline{G(x)}$ is 2-transitive for every vertex x, there exists for each $\varepsilon \in t^{-1}(t(e_i)) \setminus \{e_i\}$ a $h^\varepsilon \in G(T_{e_i}) \subseteq G(S)$ with $h^\varepsilon \bar{e}_{i+1} = \varepsilon$ and hence for every $\gamma \in G$ with $\gamma e \in \{e_i, \bar{e}_i\}$ we have

$$\begin{aligned}
\big(\big|t^{-1}\big(t(e_i)\big)\big| - 1\big) f(k_i\gamma) &= \sum_{\varepsilon \in t^{-1}(t(e_i)) \setminus \{e_i\}} f(h^\varepsilon k_i \gamma) \\
&= \frac{1}{\mathrm{m}\,(G(e_i))} \sum_{\varepsilon \in t^{-1}(t(e_i))} \int_{G(e_i)} f(h^\varepsilon k_i \gamma)\, dk - f(\gamma) \\
&= \frac{1}{\mathrm{m}\,(G(e_i))} \sum_{\varepsilon \in t^{-1}(t(e))} \int_{G(e)} f(\gamma h^\varepsilon k_i)\, dk - f(\gamma) \\
&= \frac{1}{\mathrm{m}\,(G(e_i))} \int_{G(x)} f(\gamma k)\, dk - f(\gamma) = -f(\gamma),
\end{aligned}$$

where $x \in \{o(e), t(e)\}$, and therefore by induction on n,

$$f(g) = \frac{(-1)^n}{\prod_{i=0}^{n} \big(\big|t^{-1}\big(t(e_i)\big)\big| - 1\big)} f(g'\tilde{g}).$$

The square integrability of f follows. □

As a corollary to this lemma the dimension in $\mathcal{H}(e)$ of the subspace of $G(e)$-left-invariant functions is equal to the index $[\widetilde{G}(e) : G(e)]$.

Now consider the left-regular representation λ of G in $\mathcal{H}(e)$, i.e. the representation defined by $\lambda(g)f(\gamma) := f(g^{-1}\gamma)$ for every $\gamma \in G$. This representation is known to be unitary.

Lemma 35. *Every non-trivial closed λ-invariant subspace of $\mathcal{H}(e)$ contains a non-trivial $G(e)$-left-invariant function.*

Proof. Let M be a non-trivial closed λ-invariant subspace of $\mathcal{H}(e)$ and let $u \in M$ and $g \in G$ with $u(g) \neq 0$. The function $f := \int_{G(e)} \lambda(k)\lambda(g^{-1})u\, dk$ is $G(e)$-left-invariant. But f is also non-trivial, since $f(id) = \int_{G(e)} \lambda(kg^{-1})u(id)\, dk = \int_{G(e)} u(gk^{-1})\, dk = \mathrm{m}\,(G(e))\, u(g) \neq 0$. □

Therefore if G has two orbits on X, the representation $\big(\lambda, \mathcal{H}(e)\big)$ is irreducible.

If G contains an inversion and hence an inversion of e, we consider on $\mathcal{H}(e)$ the linear involution σ defined by $\sigma(f)(g) := f(g\tilde{g})$ for every $g \in G$, were $\tilde{g} \in G$ is an inversion of e. This involution does not depend on the inversion $\tilde{g}$ since, if $\tilde{g}_1$ is an other inversion of the edge e, the automorphism $\tilde{g}_1^{-1}\tilde{g}$ fixes e and we have $f(g\tilde{g}) = f(g\tilde{g}_1\tilde{g}_1^{-1}\tilde{g}) = f(g\tilde{g}_1)$.

We write $\mathcal{H}(e)^+$ and $\mathcal{H}(e)^-$ the eigenspaces of σ corresponding to the eigenvalues 1 respectively -1. These subspaces are closed in $\mathcal{H}(e)$ and λ-invariant, which implies that the restricted representations $\big(\lambda^+, \mathcal{H}(e)^+\big)$ and $\big(\lambda^-, \mathcal{H}(e)^-\big)$ are irreducible.

Lemma 36. *The representations* $\big(\lambda^+, \mathcal{H}(e)^+\big)$ *and* $\big(\lambda^-, \mathcal{H}(e)^-\big)$ *are inequivalent.*

Proof. The formula given in Lemma 34 shows that if $\tilde{g}$ is an inversion of the edge e and f is a $G(e)$-left-invariant function, then $\lambda^+(\tilde{g})f = f$ if $f \in \mathcal{H}(e_1)^+$ and $\lambda^-(\tilde{g})f = -f$ if $f \in \mathcal{H}(e_1)^-$. Suppose there exists an unitary operator $T : \mathcal{H}(e_1)^+ \to \mathcal{H}(e_2)^-$ interlacing λ^+ and λ^-. Then, if $f \in \mathcal{H}(e_1)^+$ is a non trivial function, $Tf = T\lambda^+(\tilde{g})f = \lambda^-(\tilde{g})Tf = -Tf$ and hence $Tf = 0$ which contradicts the injectivity of T. $\square$

Let $\big(\pi, \mathcal{H}\big)$ be a continuous unitary irreducible representation of G. Suppose M_π contains a subtree with exactly one edge e. If v is a $\pi\big(G(e)\big)$-invariant vector and $g \in G$, then $\pi(g)v$ is $\pi\big(G(ge)\big)$-invariant, hence $ge \in A_\pi$. Since $G\big(o(e)\big)$ and $G\big(t(e)\big)$ are conjugated in G the edge ge is also minimal and thus element of M_π. Therefore, since $G(\bar{e}) = G(e)$, the set M_π contains all edges of T.

Fix now an edge e of T. Let $u \in \mathcal{H}^{G(e)}$ and $v \in \mathcal{H}^\infty$ two non trivial vectors. Then the function $g \mapsto f_{v,u}(g) := \langle v, \pi(g)u\rangle$ is continuous $G(e)$-right-invariant and $G(S)$-left-invariant for some complete finite subtree S and satisfies

$$\int_{G(x)} f(gk)\, dk = 0$$

for all $g \in G$ and $x \in \{o(e), t(e)\}$. Therefore $f \in \mathcal{H}(e)$ and the representation π is square integrable. For fixed non trivial $G(e)$-invariant vector $u \in \mathcal{H}$, the set $\mathcal{D}_u \{v \in \mathcal{H} \mid f_{v,u} \in \mathcal{H}(e)\}$ is stable under the action of π and contains $\mathcal{H}^\infty$ and hence is dense in $\mathcal{H}$. The linear operator $T_u : \mathcal{D}_u \to \mathcal{H}(e), v \mapsto f_{v,u}$ is closed. Indeed, if $v_n \to v$, we have $Tv_n(g) = f_{v_n,u}(g) \to \langle v, \pi(g)u\rangle =: f_{v,u}$ by continuity of the scalar product. Moreover, if $\|Tv_n - f\|_2 \to 0$ for some $f \in \mathcal{H}(e)$, then $Tv_n \to f$ allmost everywhere and hence in $\mathcal{H}(e)$ the $f = f_{v,u} =: Tv$. We have also $T\pi(g)v(\gamma) = f_{\pi(g)v,u} = \langle \pi(g)v, \pi(\gamma)u, =\rangle \langle v, \pi(g^{-1}\gamma)u\rangle = f_{v,u}(g^{-1}\gamma) = \lambda(g)f_{v,u}(\gamma)$ for all $g, \gamma \in G$. It follows that T is a non-zero multiple of an isometry (c.f. for example [Gaal], p. 160, proposition 10) and therefore the representation $\big(\pi, \mathcal{H}\big)$ is equivalent to a subrepresentation of $\big(\lambda, \mathcal{H}(e)\big)$.

2.3.3 Spherical Representations

Fix a vertex x of T. Without limiting the generality we write r the index of the vertex x and hence all vertices at even distance to x and s the index of the vertices at odd distance to x. If $r = s$, we write $d = r = s$. We also set $\kappa = 1$ if G acts transitively on X and $\kappa = 2$ otherwise.

In the following, dg represents the left invariant Haar measure $\mathrm{m}\,(.)$ with $\mathrm{m}\,(G(x)) = 1$. We write $\mathbf{C}_{\circ\circ}(G)^\natural$ the space of continuous complex valued functions f with compact support which are $G(x)$-bi-invariant, i.e. which satisfy $f(kgk') = f(g)$ for all $k, k' \in G(x)$. This space endowed with the convolution product $(f_1, f_2) \mapsto f_1 * f_2$ defined by $f_1 * f_2(h) = \int_G f_1(g) f_2(h^{-1}g)\, dg$ is an algebra.

By proposition 7, $\big(G, G(x)\big)$ is a Gel'fand pair. Therefore the convolution algebra $\mathbf{C}_{\circ\circ}(G)^\natural$ is commutative. By Proposition 6 the group G is unimodular.

A continuous complex valued function φ on G is a *spherical function* if it is $G(x)$-bi-invariant and such that the map $f \mapsto \chi(f) = \int_G f(g)\varphi(g^{-1})\, dg$ is a non trivial character of the convolution algebra $\mathbf{C}_{\circ\circ}(G)^\natural$.

We write also $\mathbf{L}^1(G)^\natural$ the space of the (classes of) complex valued integrable $G(x)$-bi-invariant functions on G. This space is also a commutative convolution algebra.

In [Far], page 320, we have Theorem I.5:

Lemma 37. *Let φ be a bounded spherical function. Then the map $f \mapsto \chi(f) = \int_G f(g)\varphi(g^{-1})\, dg$ is a character of $\mathbf{L}^1(G)^\natural$, and all characters of $\mathbf{L}^1(G)^\natural$ are of this form.*

A complex valued function φ on G is *positive definite* if for every choice of $c_1, \ldots, c_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in G$ we have $\sum_{i,j=1}^n c_i \bar{c}_j \varphi(g_i g_j^{-1}) \geq 0$. For such a function we have of course $\varphi(g^{-1}) = \overline{\varphi(g)}$.

By Proposition II.1 and Theorem III.2 in [Far], pages 323 and 331, we have following result:

Lemma 38. 1. *For each positive definite spherical function φ, there exists an irreducible unitary representation $\big(\pi_\varphi, \mathcal{H}_\varphi\big)$ accepting a $G(x)$-invariant vector u such that $\varphi(g) = \langle u, \pi_\varphi(g)u\rangle$ for all $g \in G$.*

2. *Let $\big(\pi, \mathcal{H}\big)$ be a unitary representation accepting a cyclic $G(x)$-invariant vector u' such that $\varphi(g) = \langle u', \pi(g)u'\rangle$ for all $g \in G$, then there exists a unitary isomorphism*

$U : \mathcal{H}_\varphi \to \mathcal{H}$ such that $U \circ \pi_\varphi(g) = \pi(g) \circ U$ for all $g \in G$ and $Uu = u'$. In particular the representation $\big(\pi, \mathcal{H}\big)$ is irreductible.

We have also Theorem III.1 in [Far], page 330:

Lemma 39. *If (G, K) is a Gel'fand pair and $(\pi, \mathcal{H})$ an irreducible unitary representation of G, then $\mathcal{H}^K$ is of dimension at most one.*

Therefore for each spherical representation $(\pi, \mathcal{H})$ corresponds exactly one positive definite spherical function φ_π. This function φ_π is the matrix coefficient $\langle u, \pi(g)u\rangle = \varphi_\pi(g)$ with $u \in \mathcal{H}^{G(x)}$ and $\|u\| = 1$. Conversely, every positive definite spherical function is a matrix coefficient of a spherical representation. In particular different sherical functions are coefficients of inequivalent representations.

In order to study the spherical representations π of G with $x \in M_\pi$ we are therefore interested in studying the bounded positive definite spherical functions of the Gel'fand pair $\big(G, G(x)\big)$.

Since $G(x)$ acts transitively on all spheres $S_{x,n} = \{z \in X \mid d(x, z) = n\}$ for every radius n, the convolution algebra $\mathbf{C}_{\circ\circ}(G)^\natural$ can be identified to the set $\mathbf{C}_{\circ\circ}(\kappa\mathbb{N})$ of complex functions with finite support on $\kappa\mathbb{N}$ via the isomorphism $f \mapsto \dot{f}$ defined by $f(g) = \dot{f}\big(d(x, gx)\big)$ for all $g \in G$. For the integral we have

$$\int f(g)\, dg = \mathrm{m}\left(G(x)\right) \sum_{z \in G\cdot x} \dot{f}\big(d(x, z)\big) = \mathrm{m}\left(G(x)\right) \sum_{r \in \kappa\mathbb{N}} |S_{x,r}|\, \dot{f}(r).$$

The convolution of two $G(x)$-bi-invariant functions f_1 and f_2 is therefore

$$f_1 * f_2(g) = \sum_{z \in G\cdot x} \dot{f}_1\big(d(x, z)\big) \dot{f}_2\big(d(gx, z)\big).$$

Observe that for every $g \in G$, $G(x)gG(x) = \{g' \in G \mid d(x, g'x) = d(x, gx)\}$. Therefore if we set for $i \in \kappa\mathbb{N}$,

$$\phi_i = \frac{1}{|S_{x,i}|} \mathbf{1}_{G(x)g_iG(x)},$$

where $g_i \in G$ with $d(x, g_ix) = i$, we have for every $m, n \in \kappa\mathbb{N}$, with $m \geq n > 0$,

$$\phi_n * \phi_m(g) = \phi_m * \phi_n(g) = \frac{|S_{x,m} \cap S_{gx,n}|}{|S_{x,m}|\, |S_{x,n}|}.$$

This gives

$$\phi_n * \phi_m = \frac{d-1}{d}\Big(\frac{1}{(d-1)^n}\phi_{m-n} + \sum_{k=1}^{n-1} \frac{d-2}{(d-1)^{n+1-k}}\phi_{m-n+2k} + \phi_{m+n}\Big)$$

if G acts transitively on X,

$$\begin{aligned}\phi_n * \phi_m = \frac{r-1}{r}\Big(& \frac{1}{\big((r-1)(s-1)\big)^{\frac{n}{2}}}\phi_{m-n} \\ & + \sum_{l=1}^{\frac{n}{2}-1}\Big(\frac{s-2}{\big((r-1)(s-1)\big)^{\frac{n}{2}+1-l}}\phi_{m-n+4l-2} \\ & \qquad + \frac{r-2}{(r-1)\big((r-1)(s-1)\big)^{\frac{n}{2}-l}}\phi_{m-n+4l}\Big) \\ & + \frac{s-2}{(r-1)(s-1)}\phi_{m+n-2} + \phi_{m+n}\Big)\end{aligned}$$

otherwise, and hence

$$\phi_1 * \phi_m = \begin{cases} \phi_1 & \text{if } m = 0 \\ \frac{1}{d}\phi_{m-1} + \frac{d-1}{d}\phi_{m+1} & \text{if } m \geq 1 \end{cases}$$

if G acts transitively on X and

$$\phi_2 * \phi_m = \begin{cases} \phi_2 & \text{if } m = 0 \\ \frac{1}{r(s-1)}\phi_{m-2} + \frac{s-2}{r(s-1)}\phi_m + \frac{r-1}{r}\phi_{m+2} & \text{if } m \geq 2 \end{cases}$$

otherwise. This means that ϕ_0 and ϕ_κ generate $\mathrm{C}_{\circ\circ}(G)^\natural$.

Let χ be a character of $\mathrm{C}_{\circ\circ}(G)^\natural$, then the suite $\big(p_i(\lambda)\big)_i := \big(\chi(\phi_i)\big)_i$ is determined as soon as we know $\chi(\phi_\kappa) = \lambda$, since $\chi(\phi_0) = 1$. We have

$$\begin{cases} p_0(\lambda) = 1 \\ p_1(\lambda) = \lambda \\ p_n(\lambda) = \frac{d}{d-1}\lambda p_{n-1}(\lambda) - \frac{1}{d-1}p_{n-2}(\lambda), \quad n \geq 2 \end{cases}$$

if G acts transitively on X and

$$\begin{cases} p_0(\lambda) = 1 \\ p_2(\lambda) = \lambda \\ p_n(\lambda) = \frac{r(s-1)\lambda-(s-2)}{(r-1)(s-1)}p_{n-2}(\lambda) - \frac{1}{(r-1)(s-1)}p_{n-4}(\lambda), \quad n \geq 4 \end{cases}$$

otherwise.

Lemma 40. *A $G(x)$-bi-invariant function φ on G is spherical with respect to the compact subgroup $G(x)$ if and only if we have*

$$\int_{G(x)} \varphi(gkg') = \varphi(g)\varphi(g')$$

for every $g, g' \in G$.

Proof. See for example Proposition I.3 in [Far], page 319, or Theorem 10 in [Gaal], page 451. □

Lemma 41. *The function $g \mapsto \varphi_\lambda(g) = p_{d(x,gx)}(\lambda)$ is spherical and all spherical functions of G are of this form. In other words, a $G(x)$-bi-invariant function φ on G is spherical if and only if it satisfies following conditions:*

$$\begin{cases} \dot\varphi(0) = 1 \\ \dot\varphi(1) = \lambda \\ \dot\varphi(n) = \frac{d}{d-1}\lambda\dot\varphi(n-1) - \frac{1}{d-1}\dot\varphi(n-2), \quad n \geq 2 \end{cases}$$

if G acts transitively on X and

$$\begin{cases} \dot\varphi(0) = 1 \\ \dot\varphi(2) = \lambda \\ \dot\varphi(n) = \frac{r(s-1)\lambda-(s-2)}{(r-1)(s-1)}\dot\varphi(n-2) - \frac{1}{(r-1)(s-1)}\dot\varphi(n-4), \quad n \geq 4 \end{cases}$$

otherwise, where λ is some complex number.

Proof. Let φ be a $G(x)$-bi-invariant function. Pick $g, g' \in G$ and set $m = d(x, gx)$ as well as $n = d(x, g'x)$. Since $\varphi(gkg') = \varphi(g'^{-1}k^{-1}g^{-1})$, we can suppose $m \geq n$. We compute

$$\begin{aligned} \int_{G(x)} \varphi(gkg')\,dk &= \int_{G(x)} \dot\varphi\big(d(g^{-1}x, kg'x)\big)\,dk \\ &= \int_{G(x)/G(T_{x,n})} \int_{G(T_{x,n})} \dot\varphi\big(d(g^{-1}x, khg'x)\big)\,d\dot k\,dh \\ &= \int_{G(x)/G(T_{x,n})} \int_{G(T_{x,n})} \dot\varphi\big(d(g^{-1}x, kg'x)\big)\,d\dot k\,dh \\ &= \mathrm{m}\,(G(T_{x,n})) \sum_{z\in S_{x,n}} \dot\varphi\big(d(g^{-1}x, z)\big) \\ &= \frac{1}{|S_{x,n}|} \sum_{k=m-n}^{m+n} |S_{x,k} \cap S_{gx,n}|\,\dot\varphi(k) \\ &= \frac{1}{|S_{x,n}|} \sum_{k=0}^{n} |S_{x,m-n+2k} \cap S_{gx,n}|\,\dot\varphi(m-n+2k). \end{aligned}$$

This gives

$$\begin{aligned} \int_{G(x)} \varphi(gkg')\,dk = \frac{d-1}{d}\Big(&\frac{1}{(d-1)^n}\dot\varphi(m-n) \\ &+ \sum_{k=1}^{n-1} \frac{d-2}{(d-1)^{n+1-k}}\dot\varphi(m-n+2k) + \dot\varphi(m+n)\Big) \end{aligned}$$

if G acts transitively on X, and

$$\int_{G(x)} \varphi(gkg')\,dk = \frac{r-1}{r}\Bigg(\frac{1}{\big((r-1)(s-1)\big)^{\frac{n}{2}}}\dot\varphi(m-n) + \sum_{l=1}^{\frac{n}{2}-1}\Big(\frac{s-2}{\big((r-1)(s-1)\big)^{\frac{n}{2}+1-l}}\dot\varphi(m-n+4l-2) + \frac{r-2}{(r-1)\big((r-1)(s-1)\big)^{\frac{n}{2}-l}}\dot\varphi(m-n+4l)\Big) + \frac{s-2}{(r-1)(s-1)}\dot\varphi(m+n-2) + \dot\varphi(m+n)\Bigg)$$

otherwise.

We have to show that φ_λ is spherical. Since for every natural number i the function $\dot\varphi_\lambda(i) = p_i(\lambda) = \chi(\phi_i)$, we have, using above computations,

$$\varphi_\lambda(g)\varphi_\lambda(g') = \dot\varphi_\lambda(m)\dot\varphi_\lambda(n) = \chi(\phi_m)\chi(\phi_m) = \chi(\phi_m * \phi_n) = \int_{G(x)} \varphi(gkg'),$$

hence φ_λ is spherical by Lemma 40.

In the other sense, if φ is spherical, using above formulas with $n = \kappa$, we see that $\dot\varphi(k) = p_k(\lambda)$ for every natural number k, where $\lambda = \dot\varphi(1)$. □

Therefore all spherical functions of G with respect to the compact subgroup $G(x)$ are given in a unique manner by φ_λ with $\lambda \in \mathbb{C}$. We shall now give another characterisation of those spherical functions.

For the vertices $y \neq z$, we define the open compact subsets $O_y(z) = T_e(\infty)$ of the boundary $T(\infty)$ of the tree T, where $e \in o^{-1}(z)$ such that $d\big(y,t(e)\big) < d\big(y,o(e)\big)$. We have for every vertex y and for every natural number n, the equality $T(\infty) = \bigsqcup_{z\in S_{y,n}} O_y(z)$. Since $|S_{y,n}| = \frac{a}{a-1}(a-1)^{n-[\frac{n}{2}]}(b-1)^{[\frac{n}{2}]}$, if $n \geq 1$, and $|S_{y,0}| = 1$ (where a is the degree of y, $b \in \{r,s\} \setminus \{a\}$ and $[\alpha]$ denotes the entire part of $\alpha \in \mathbb{R}$), the only $G(y)$-invariant Borel probability measure μ_y on $T(\infty)$ satisfies $\mu_y\big(O_y(z)\big) = \frac{a-1}{a}(a-1)^{[\frac{n}{2}]-n}(b-1)^{-[\frac{n}{2}]}$, if $n = d(y,z) \geq 1$.

Let $\epsilon \in T(\infty)$ and y a vertex. We write $\epsilon_y \in \epsilon$ the unique infinite chain with $o\big(\epsilon_y(0)\big) = y$ representing the end ϵ. If y and z are two vertices, there exists a unique relative number k such that $\epsilon_y(n+k) = \epsilon_z(n)$ for $n \geq \frac{1}{2}\big(d(y,z)-k\big)$. This number k defines a function $k(y,z,\epsilon) = k$.

Pick now a vertex y, a $g \in G$, $\epsilon \in T(\infty)$ and set $k = k(x, gx, \epsilon)$. Observe that

$$O_y\Big(o\big(\epsilon_y(n+k)\big)\Big) = O_{gy}\Big(o\big(\epsilon_{gy}(n)\big)\Big)$$

for $n \geq \frac{1}{2}\big(d(y, gy) - k\big)$. Therefore

$$\begin{aligned}\mu_{gy}\Big(O_y\big(o(\epsilon_y(n+k))\big)\Big) &= \mu_{gy}\Big(O_{gy}\big(o(\epsilon_{gy}(n))\big)\Big)\\ &= \frac{a-1}{a}(a-1)^{[\frac{n}{2}]-n}(b-1)^{-[\frac{n}{2}]}\end{aligned}$$

(where a is the degree of y, $b \in \{r, s\} \setminus \{a\}$) while

$$\mu_y\Big(O_y\big(o(\epsilon_y(n+k))\big)\Big) = \frac{a-1}{a}(a-1)^{[\frac{n+k}{2}]-n-k}(b-1)^{-[\frac{n+k}{2}]},$$

and hence for $n \geq \frac{1}{2}\big(d(y, gy) - k\big)$ we have

$$\mu_{gy}\Big(O_y\big(o(\epsilon_y(n+k))\big)\Big) = (a-1)^{k+[\frac{n}{2}]-[\frac{n+k}{2}]}(b-1)^{[\frac{n+k}{2}]-[\frac{n}{2}]}\mu_y\Big(O_y\big(o(\epsilon_y(n+k))\big)\Big).$$

Since the set of $O_y\big(o(\epsilon_y(n+k))\big)$ form a base of neighbourhood of ϵ, and observing that if G has two orbits on X, k is an even number, we get the formula for the Radon-Nikodim derivate

$$\frac{d\mu_{gy}}{d\mu_y}(\epsilon) = \begin{cases}(d-1)^k & \text{if } G \text{ acts transitively on } X \text{ and}\\ \big((r-1)(s-1)\big)^{\frac{k}{2}} & \text{else.}\end{cases}$$

In fact, one observes that we can write in both cases $\frac{d\mu_{gy}}{d\mu_y}(\epsilon) = \big((r-1)(s-1)\big)^{\frac{k}{2}}$. We shall hence set $\alpha = \sqrt{(r-1)(s-1)}$.

Set for $g \in G$ and $\epsilon \in T(\infty)$, the number

$$P_y(g, \epsilon) = \frac{d\mu_{gy}}{d\mu_y}(\epsilon) = \alpha^k.$$

For fixed g, the function $\epsilon \mapsto P_y(g, \epsilon)$ takes only finitely many values, because for all $\epsilon \in T(\infty)$, we have $-d(y, gy) \leq k(y, gy, \epsilon) \leq d(y, gy)$. This function is eventually (for fixed g) a linear combination of characteristic functions of the sets $O_y(z)$ and is therefore integrable.

Lemma 42. *For every $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) \geq 0$, set*

$$\lambda_x(\nu) = \begin{cases}\frac{1}{d}\big((d-1)^\nu + (d-1)^{1-\nu}\big) & \text{if } G \text{ acts transitively on } X\\ \frac{1}{r(s-1)}\Big(\big((r-1)(s-1)\big)^\nu + \big((r-1)(s-1)\big)^{1-\nu} + s - 2\Big) & \text{else,}\end{cases}$$

then the function $\mathbb{R} + i\mathbb{R} \to \mathbb{C}, \nu \mapsto \lambda_x(\nu)$ is surjective and for every $g \in G$ we have

$$\varphi_{\lambda_x(\nu)}(g) = \int_{T(\infty)} \big(P_x(g, \epsilon)\big)^\nu \, d\mu_x(\epsilon).$$

Proof. Set $\psi_{x,\nu}(g) = \int_{T(\infty)} \big(P_x(g,\epsilon)\big)^\nu \, d\mu_x(\epsilon)$.

The function $g \mapsto P_x(g,\epsilon)$ is obviously $G(x)$-right-invariant, hence the function $\psi_{x,\nu}$ is also $G(x)$-right-invariant. Let $h \in G(x)$ and $g \in G$, then, since $k(x, hgx, \epsilon) = k(hx, hgx, hh^{-1}\epsilon) = k(x, gx, h^{-1}\epsilon)$, we have also $P_x(hg,\epsilon) = P_x(g, h^{-1}\epsilon)$ and finally

$$\begin{aligned}\psi_{x,\nu}(hg) &= \int_{T(\infty)} \big(P_x(hg,\epsilon)\big)^\nu \, d\mu_x(\epsilon) \\ &= \int_{T(\infty)} \big(P_x(g,h^{-1}\epsilon)\big)^\nu \, d\mu_x(\epsilon) = \psi_{x,\nu}(g),\end{aligned}$$

because $d\mu_x$ is $G(x)$-invariant. Therefore $\psi_{x,\nu}$ is $G(x)$-bi-invariant.

Pick $g \in G$, $g \neq id$, and set $m = d(x, gx)$. We want to compute $\dot{\psi}_{x,\nu}(m) = \psi_{x,\nu}(g)$. It is easy to see that if $k \in \{0, 1, \ldots, m\}$ and $y \in S_{x,m} \cap S_{gx,2k}$, then for all $\epsilon \in O_x(y)$, the number $k(x, gx, \epsilon) = m - 2k$. Hence

$$P_x(g,\epsilon) = \sum_{k=0}^{m} \sum_{y \in S_{x,m} \cap S_{gx,2k}} \big((r-1)(s-1)\big)^{\frac{m}{2}-k} \mathbf{1}_{O_x(y)}(\epsilon),$$

where $\mathbf{1}_E$ is the characteristic function of the subset E of $T(\infty)$, and

$$\begin{aligned}\dot{\psi}_{x,\nu}(m) = \frac{1}{d(d-1)^{m-1}} \Big(&(d-1)^{m\nu} + (d-2)\big(\sum_{k=1}^{m-1} (d-1)^{(m-2k)\nu+k-1}\big) \\ &+ (d-1)^{(m-1)\nu} \Big)\end{aligned}$$

if G acts transitively on X and

$$\begin{aligned}\dot{\psi}_{x,\nu}(m) = \frac{r-1}{r\alpha^m} \Big(&\alpha^{m\nu} + \big((s-2)\alpha^{2\nu} + (r-2)(s-1)\big)\big(\sum_{l=1}^{\frac{m}{2}-1} \alpha^{(m-4l)\nu+2l-2}\big) \\ &+ (s-2)\alpha^{(m-2)(1-\nu)} + \alpha^{m(1-\nu)} \Big)\end{aligned}$$

else, where $\alpha = \sqrt{(r-1)(s-1)}$. A calculus shows then that

$$\begin{cases} \dot{\psi}_{x,\nu}(0) = 1 \\ \dot{\psi}_{x,\nu}(1) = \lambda_x(\nu) \\ \dot{\psi}_{x,\nu}(n) = \frac{d}{d-1}\lambda_x(\nu)\dot{\psi}_{x,\nu}(n-1) - \frac{1}{d-1}\dot{\psi}_{x,\nu}(n-2), \quad n \geq 2 \end{cases}$$

if G acts transitively on X and

$$\begin{cases} \dot{\psi}_{x,\nu}(0) = 1 \\ \dot{\psi}_{x,\nu}(2) = \lambda_x(\nu) \\ \dot{\psi}_{x,\nu}(n) = \frac{r(s-1)\lambda_x(\nu)-(s-2)}{(r-1)(s-1)}\dot{\psi}_{x,\nu}(n-2) - \frac{1}{(r-1)(s-1)}\dot{\psi}_{x,\nu}(n-4), \quad n \geq 4 \end{cases}$$

otherwise. By Lemma 41 we have $\psi_{x,\nu} = \varphi_{\lambda_x(\nu)}$. □

Recall that we are interested to study the positive definite spherical functions of G with respect to the subgroup $G(x)$. Since the spherical functions φ_λ of G are characterized by λ, we want to know for which λ the function φ_λ is positive definite.

First we observe that if φ_λ is positive definite, since φ_λ is $G(x)$-bi-invariant and since $g^{-1} \in G(x)gG(x)$, for every $g \in G$, we have $\overline{\varphi_\lambda(g)} = \varphi_\lambda(g^{-1}) = \varphi_\lambda(g)$, i.e. $\varphi_\lambda(g) \in \mathbb{R}$ and in particular $\lambda \in \mathbb{R}$. Moreover we have $0 \leq \varphi_\lambda(gg^{-1}) + \varphi_\lambda(g) + \varphi_\lambda(g^{-1}) + \varphi_\lambda(id)$, and hence $\varphi_\lambda(g) \geq -1$, as well as $0 \leq \varphi_\lambda(gg^{-1}) - \varphi_\lambda(g) - \varphi_\lambda(g^{-1}) + \varphi_\lambda(id)$, and hence $\varphi_\lambda(g) \leq 1$. Therefore λ has to be between -1 and 1.

We further observe that the map $\nu \mapsto \lambda(\nu)$ is a bijection between the set $\nu \in \left(\left\{\frac{1}{2}\right\} + i[0, \frac{\pi}{\kappa \ln \alpha}]\right) \sqcup \left((-\infty, \frac{1}{2}) + i\left\{0, \frac{\pi}{\kappa \ln \alpha}\right\}\right)$ and $\mathbb{R}$, where $\alpha = \sqrt{(r-1)(s-1)}$ and κ is the number of orbits of G on X.

We shall now show that $\varphi_{\lambda(\nu)}$ is positive definite if and only if $\nu \in \left(\left\{\frac{1}{2}\right\} + i[0, \frac{\pi}{\kappa \ln \alpha}]\right) \sqcup \left([0, \frac{1}{2}) + i\left\{0, \frac{\pi}{\kappa \ln \alpha}\right\}\right)$. In the case where G acts transitively on X this corresponds to the interval $[-1, 1]$; in the other case this corresponds to the interval $[-\frac{2+(r-2)(s-1)}{r(s-1)}, 1]$.

Suppose G has has two orbits on X. The equation $\dot{\varphi}_\lambda(n) = \frac{r(s-1)\lambda-(s-2)}{(r-1)(s-1)}\dot{\varphi}_\lambda(n-2) - \frac{1}{(r-1)(s-1)}\dot{\varphi}_\lambda(n-4)$ has the solution $\dot{\varphi}_\lambda(2n) = A\beta_1^{2n} + B\beta_2^{2n}$ where $\beta_1^2 = \frac{1}{2\alpha^2}\left(r(s-1)\lambda-(s-2)+\sqrt{\Delta}\right)$ and $\beta_2^2 = \frac{1}{2\alpha^2}\left(r(s-1)\lambda-(s-2)-\sqrt{\Delta}\right)$ with $\Delta = \left(r(s-1)\lambda-(s-2)\right)^2 - 4\alpha^2$. The constants A and B are determined by the initial conditions $\dot{\psi}_{x,\nu}(0) = 1$ and $\dot{\psi}_{x,\nu}(2) = \lambda(\nu)$. We have $A = \frac{1}{4\sqrt{\Delta}}\left((2\alpha^2 - r(s-1))\lambda + (s-2) + \sqrt{\Delta}\right)$ and $B = 1 - A$. Suppose now $-1 \leq \lambda < -\frac{2+(r-2)(s-1)}{r(s-1)}$. It is easy to check that then $\Delta > 0$ Further we have $-1 < \beta_1^2 < 1$ and $\beta_2^2 < -1$. This means that for such λ the sequence $\dot{\varphi}_\lambda(n)$ is not bounded and that therefore φ_λ cannot be positive definite.

Hence for $\varphi_{\lambda(\nu)}$ being positive definite, we must have $\nu \in \left(\left\{\frac{1}{2}\right\} + i[0, \frac{\pi}{\kappa \ln \alpha}]\right) \sqcup \left([0, \frac{1}{2}) + i\left\{0, \frac{\pi}{\kappa \ln \alpha}\right\}\right)$.

We show now that this condition is sufficient.

Write $\mathcal{K}$ the linear space of continuous functions on $T(\infty)$ taking a finite number of values. This space is spanned by the characteristic functions of the sets $O_x(y)$ with $y \in X$. Further let $\mathcal{K}_n$ be the subspace of $\mathcal{K}$ spanned by the characteristic functions of the sets $O_x(y)$ with $y \in S_{x,n}$. Obviously $\{O_x(y) \mid y \in S_{x,n}\}$ is a basis of $\mathcal{K}_n$ and we have $\mathcal{K}_n \subset \mathcal{K}_{n+1}$ and $\mathcal{K} = \bigcup_n \mathcal{K}_n$.

For $f \in \mathcal{K}$ and $g \in G$ define

$$\pi_{x,\lambda_x(\nu)}(g)f(\epsilon) = P_x(g,\epsilon)^\nu f(g^{-1}\epsilon).$$

Observe that the identity $P_x(gh,\epsilon) = P_x(g,\epsilon)P_{gx}(gh,\epsilon) = P_x(g,\epsilon)P_x(h,g^{-1}\epsilon)$ which follows directly from the definition of $P_x(g,\epsilon)$ as Radon-Nikodym derivate, implies that $\pi_{x,\lambda_x(\nu)}(gh)f = \pi_{x,\lambda_x(\nu)}(g)\big(\pi_{x,\lambda_x(\nu)}(h)f\big)$.

Suppose $\nu \in \left\{\frac{1}{2}\right\} + i[0,\frac{\pi}{\kappa \ln \alpha}]$. Then we have

$$\begin{aligned}\int \left|\pi_{\lambda(\nu)}(g)f(\epsilon)\right|^2 \, d\mu_x(\epsilon) &= \int P_x(g,\epsilon)\left|f(g^{-1}\epsilon)\right|^2 \, d\mu_x(\epsilon)\\ &= \int P_{g^{-1}x}(id,g^{-1}\epsilon)\left|f(g^{-1}\epsilon)\right|^2 \, d\mu_x(\epsilon) = \int |f(\epsilon)|^2 \, P_{g^{-1}x}(id,\epsilon)\, d\mu_x(g\epsilon)\\ &= \int |f(\epsilon)|^2 \, P_{g^{-1}x}(id,\epsilon)\, d\mu_{g^{-1}x}(\epsilon) = \int |f(\epsilon)|^2 \, d\mu_x(\epsilon)\end{aligned}$$

Therefore $\pi_{x,\lambda_x(\nu)}$ is a unitary representation of G in the Hilbert-space $\mathcal{H}_{x,\lambda_x(\nu)} = \mathrm{L}^2\big(T(\infty),\mu_x\big)$ and $\varphi_{\lambda_x(\nu)}(g) = \big\langle \mathbf{1}, \pi_{x,\lambda_x(\nu)}(g)\mathbf{1}\big\rangle$ is therefore positive definite.

Now suppose $\nu \in [0,\frac{1}{2}) + i\left\{0,\frac{\pi}{\kappa\ln\alpha}\right\}$.

For $\epsilon,\epsilon' \in T(\infty)$ with $\epsilon \neq \epsilon'$ and $y \in X$ we define the number

$$(\epsilon,\epsilon')_y = \max\left\{n \in \mathbb{N} \mid \epsilon_y(n) = \epsilon'_y(n)\right\}.$$

If $n \geq (\epsilon,\epsilon')_y$, we have $2(\epsilon,\epsilon')_y = d\big(y,\epsilon_y(n)\big) + d\big(y,\epsilon'_y(n)\big) - d\big(\epsilon_y(n),\epsilon'(n)\big)$. Moreover, if $g \in G$ and $n \geq \max\{(\epsilon,\epsilon')_y, d(y,gy)\}$, then

$$\begin{aligned}2(g\epsilon,g\epsilon')_y &= d\big(y,(g\epsilon)_y(n)\big) + d\big(y,(g\epsilon')_y(n)\big) - d\big((g\epsilon)_y(n),(g\epsilon')_y(n)\big)\\ &= d\big(g^{-1}y,\epsilon_{g^{-1}y}(n)\big) - d\big(y,\epsilon_{g^{-1}y}(n)\big) + d\big(g^{-1}y,\epsilon'_{g^{-1}y}(n)\big)\\ &\quad - d\big(y,\epsilon'_{g^{-1}y}(n)\big) + d\big(y,\epsilon_{g^{-1}y}(n)\big) + d\big(y,\epsilon'_{g^{-1}y}(n)\big)\\ &\quad - d\big(\epsilon_{g^{-1}y}(n),\epsilon'_{g^{-1}y}(n)\big) = 2(\epsilon,\epsilon')_y - k(y,g^{-1}y,\epsilon) - k(y,g^{-1}y,\epsilon')\end{aligned}$$

Further we have

$$(\epsilon,\epsilon')_y = \sum_{k=0}^{\infty} k\Big(\sum_{z\in S_{y,k}} \mathbf{1}_{O_y(z)}(\epsilon)\mathbf{1}_{O_y(z)}(\epsilon') - \sum_{z'\in S_{y,k+1}} \mathbf{1}_{O_y(z')}(\epsilon)\mathbf{1}_{O_y(z')}(\epsilon')\Big).$$

Let $g \in G \setminus \{id\}$ and $\epsilon \in T(\infty)$. Set $m = d(x,gx)$, take $\delta \in T(\infty)$ with $\delta(m) = gx$ and set $y_k = o\big(\delta_x(k)\big)$ for every $k \geq 0$. Using the calculus for $(\epsilon, \epsilon')_x$ above, we have

$$\begin{aligned}
\int & \alpha^{2\nu(\epsilon,\epsilon')_x} \mathbf{1}_{O_x(gx)}(\epsilon')\, d\mu_x(\epsilon') \\
&= \sum_{k=0}^{\infty} \alpha^{2k\nu} \Big(\sum_{y \in S_{x,k}} \mu_x\big(O_x(y) \cap O_x(gx)\big) \mathbf{1}_{O_x(y)}(\epsilon) \\
&\qquad\qquad - \sum_{y \in S_{x,k+1}} \mu_x\big(O_x(y) \cap O_x(gx)\big) \mathbf{1}_{O_x(y)}(\epsilon) \Big) \\
&= \sum_{k=0}^{m-1} \alpha^{2k\nu} \mu_x\big(O_x(gx)\big) \Big(\mathbf{1}_{O_x(y_k)}(\epsilon) - \mathbf{1}_{O_x(y_{k+1})}(\epsilon) \Big) \\
&+ \sum_{k=m}^{\infty} \alpha^{2k\nu} \Big(\mu_x\big(O_x(y_k)\big) - \mu_x\big(O_x(y_{k+1})\big) \Big) \mathbf{1}_{O_x(gx)}(\epsilon) \\
&= \sum_{k=0}^{m-1} \alpha^{2k\nu} \mu_x\big(O_x(gx)\big) \sum_{y \in S_{x,m} \cap S_{gx,2(m-k)}} \mathbf{1}_{O_x(y)}(\epsilon) \\
&+ \sum_{l=0}^{\infty} \Big(\alpha^{2(2l+m)\nu} \frac{\alpha^{-2l-m}}{r} (r-2) \\
&\qquad + \alpha^{2(2l+1+m)\nu} \frac{\alpha^{-2l-m}}{r} \frac{s-2}{s-1} \Big) \mathbf{1}_{O_x(gx)}(\epsilon) \\
&= \sum_{k=0}^{m-1} \alpha^{2k\nu} \frac{r-1}{r} \alpha^{-m} \sum_{y \in S_{x,m} \cap S_{gx,2(m-k)}} \mathbf{1}_{O_x(y)}(\epsilon) \\
&+ \frac{r-1}{r} \Big(\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)} \Big) \alpha^{m(2\nu-1)} \sum_{l=0}^{\infty} \alpha^{2(2\nu-1)l} \mathbf{1}_{O_x(gx)}(\epsilon) \\
&= \frac{r-1}{r} \alpha^{-m} \sum_{k=0}^{m-1} \alpha^{2k\nu} \sum_{y \in S_{x,m} \cap S_{gx,2(m-k)}} \mathbf{1}_{O_x(y)}(\epsilon) \\
&+ \frac{r-1}{r} \cdot \frac{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}}{1 - \alpha^{2(2\nu-1)}} \alpha^{m(2\nu-1)} \mathbf{1}_{O_x(gx)}(\epsilon)
\end{aligned}$$

and

$$\int \alpha^{2\nu(\epsilon,\epsilon')_x} \mathbf{1}(\epsilon')\, d\mu_x(\epsilon') = \frac{r-1}{r} \cdot \frac{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}}{1 - \alpha^{2(2\nu-1)}} \mathbf{1}(\epsilon)$$

For $f \in \mathcal{K}$ and $\epsilon \in T(\infty)$, we define

$$I_{x,\nu} f(\epsilon) = \frac{r}{r-1} \cdot \frac{1 - \alpha^{2(2\nu-1)}}{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}} \int \alpha^{2\nu(\epsilon,\epsilon')_x} f(\epsilon')\, d\mu_x(\epsilon').$$

We want to know for which ν the relation $\langle f, f' \rangle_{x,\nu} = \int I_{x,\nu} f(\epsilon) \overline{f'(\epsilon)}\, d\mu_x(\epsilon)$ defines an inner product on $\mathcal{K}$.

A function (kernel) $\psi : E \times E \to \mathbb{C}$ is called a *conditionally negative definite kernel* if ψ is hermitian (i.e. $\psi(a,b) = \overline{\psi(b,a)}$) and if for all $b_1, \ldots, b_n \in E$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\sum_{k=1}^n \lambda_k = 0$, we have $\sum_{i=1}^n \sum_{j=1}^n \lambda_i \overline{\lambda_j} \psi(b_i, b_j) \leq 0$.

In [H; V] Chapter 5 & 6 we find following facts:

Lemma 43. *(*SCHOENBERG*) Let $\psi : E \times E \to \mathbb{C}$ be a kernel which is zero on the diagonal. Following properties are equvalent:*

1. *the kernel ψ is conditionally negative definite.*
2. *For every $t > 0$, the real kernel $e^{-t\psi}$ is positive definite.*

Proof. Cf. [H; V], Theorem 16, page 66. □

Lemma 44. *For every tree T, the function "distance" is a negative definite kernel on the set of vertices of T.*

Proof. Cf. [H; V], Proposition 2, page 69. □

We shall now prove that the function $(f, f') \mapsto \langle f, f' \rangle_{x,\nu} = \int I_{x,\nu} f(\epsilon) f'(\epsilon)\, d\mu_x(\epsilon)$ is a inner product on $\mathcal{K}$ if $\nu \in [0, \frac{1}{2}) + i\left\{0, \frac{\pi}{\kappa \ln \alpha}\right\}$.

The only thing which is not obvious and which we shall show, is that $\langle f, f \rangle_{x,\nu} > 0$ if $f \neq 0$. Since $\mathcal{K} = \bigcup_n \mathcal{K}_n$, it is enough to show it on $\mathcal{K}_m$ with $m > 0$.

Take $y, z \in S_{x,m}$. We have

$$\begin{aligned}
\left\langle \mathbf{1}_{O_x(y)}, \mathbf{1}_{O_x(z)} \right\rangle_{x,\nu} &= \int I_\nu \mathbf{1}_{O_x(y)}(\epsilon) \mathbf{1}_{O_x(z)}(\epsilon)\, d\mu_x(\epsilon) = \frac{r}{r-1} \cdot \frac{1-\alpha^{2(2\nu-1)}}{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}} \\
&\quad \Big(\frac{r-1}{r} \alpha^{-m} \sum_{k=0}^{m-1} \alpha^{2k\nu} \sum_{y' \in S_{x,m} \cap S_{y,2(m-k)}} \mu_x\big(O_x(y') \cap O_x(z)\big) \\
&\quad + \frac{r-1}{r} \cdot \frac{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}}{1-\alpha^{2(2\nu-1)}} \alpha^{m(2\nu-1)} \mu_x\big(O_x(y) \cap O_x(z)\big) \Big) \\
&= \frac{r}{r-1} \cdot \frac{1-\alpha^{2(2\nu-1)}}{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}} \\
&\quad \Big(\frac{r-1}{r} \alpha^{-m} \alpha^{(2m-d(y,z))\nu} \frac{r-1}{r} \alpha^{-m} \\
&\quad + \frac{r-1}{r} \cdot \frac{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}}{1-\alpha^{2(2\nu-1)}} \alpha^{m(2\nu-1)} \frac{r-1}{r} \alpha^{-m} \delta_{y,z} \Big) \\
&= \frac{r-1}{r} \alpha^{2m(\nu-1)} \Big(\frac{1-\alpha^{2(2\nu-1)}}{\frac{r-2}{r-1} + (s-2)\alpha^{2(\nu-1)}} e^{-\nu \ln \alpha \cdot d(y,z)} + \delta_{y,z} \Big).
\end{aligned}$$

By the preceding two lemmas, the kernel

$$(y,z) \mapsto \psi(y,z) = \frac{1-\alpha^{2(2\mathrm{Re}(\nu)-1)}}{\frac{r-2}{r-1}+(s-2)\alpha^{2(\mathrm{Re}(\nu)-1)}} e^{-2\mathrm{Re}(\nu)\ln\alpha\cdot d(y,z)}$$

is positive definite if $0 \le \mathrm{Re}(\nu) < \frac{1}{2}$. Moreover if $f = \sum_{k=1}^{n} \lambda_k \mathbf{1}_{O_x(y_k)} \neq 0$ with $y_1, \ldots, y_n \in S_{x,m}$, then

$$\langle f, f\rangle_{x,\nu} = \frac{r-1}{r}\alpha^{-2m(1-\nu)}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\overline{\lambda}_j\psi(y_i,y_j) + \sum_{i=1}^{n}|\lambda_i|^2\Big) > 0.$$

Let $\nu \in [0,\frac{1}{2}) + i\left\{0, \frac{\pi}{\kappa\ln\alpha}\right\}$.

Then we have

$$\begin{aligned}
&\left\langle \pi_{\lambda(\nu)}(g)f, \pi_{\lambda(\nu)}(g)f\right\rangle_{x,\nu} = \int I_\nu \pi_{\lambda(\nu)}(g)f(\epsilon)\overline{\pi_{\lambda(\nu)}(g)f(\epsilon)}\, d\mu_x(\epsilon)\\
&= C_\nu \int\int \alpha^{\nu 2(\epsilon,\epsilon')_x}\alpha^{(1-\nu)k(x,gx,\epsilon')} f(g^{-1}\epsilon')\alpha^{(1-\nu)k(x,gx,\epsilon)}\overline{f(g^{-1}\epsilon)}\, d\mu_x(\epsilon')d\mu_x(\epsilon)\\
&= C_\nu \int\int \alpha^{\nu 2(g\epsilon,g\epsilon')_x}\alpha^{(1-\nu)k(x,gx,g\epsilon')} f(\epsilon')\alpha^{(1-\nu)k(x,gx,g\epsilon)}\overline{f(\epsilon)}\, d\mu_x(g\epsilon')d\mu_x(g\epsilon)\\
&= C_\nu \int\int \alpha^{\nu 2(g\epsilon,g\epsilon')_x}\alpha^{(1-\nu)k(g^{-1}x,x,\epsilon')} f(\epsilon')\alpha^{(1-\nu)k(g^{-1}x,x,\epsilon)}\overline{f(\epsilon)}\, d\mu_{g^{-1}x}(\epsilon')d\mu_{g^{-1}x}(\epsilon)\\
&= C_\nu \int\int \alpha^{\nu 2(g\epsilon,g\epsilon')_x}\alpha^{-(1-\nu)k(x,g^{-1}x,\epsilon')} f(\epsilon')\alpha^{-(1-\nu)k(x,g^{-1}x,\epsilon)}\overline{f(\epsilon)}\\
&\qquad \alpha^{k(x,g^{-1}x,\epsilon')}\alpha^{k(x,g^{-1}x,\epsilon)}\, d\mu_x(\epsilon')d\mu_x(\epsilon)\\
&= C_\nu \int\int \alpha^{\nu(2(g\epsilon,g\epsilon')_x + k(x,g^{-1}x,\epsilon') + k(x,g^{-1}x,\epsilon))} f(\epsilon')\overline{f(\epsilon)}\, d\mu_x(\epsilon')d\mu_x(\epsilon)\\
&= C_\nu \int\int \alpha^{\nu 2(\epsilon,\epsilon')_x} f(\epsilon')\overline{f(\epsilon)}\, d\mu_x(\epsilon')d\mu_x(\epsilon)\\
&= \langle f, f\rangle_{x,\nu}
\end{aligned}$$

where $C_\nu = \frac{r}{r-1}\frac{1-\alpha^{2(2\nu-1)}}{\frac{r-2}{r-1}+(s-2)\alpha^{2(\nu-1)}}$.

Therefore $\pi_{x,\lambda_x(\nu)}$ is a unitary representation of G in the completion $\mathcal{H}_{x,\lambda_x(\nu)}$ of $\mathcal{K}$ with respect to the inner product $\langle .,.\rangle_{x,\nu}$. Moreover $g \mapsto \varphi_{\lambda_x(\nu)}(g) = \left\langle \mathbf{1}, \pi_{x,\lambda_x(\nu)}(g)\mathbf{1}\right\rangle_{x,\nu}$ and hence is positive definite.

Lemma 45. *Take ν such that $\alpha^\nu \neq \pm 1$, then the set $\{P_x(g,.)^\nu \mid g \in G\}$ spans $\mathcal{K}$.*

Proof. For $y \in X$, $y \neq x$ and $\epsilon \in T(\infty)$, we write $P(y,\epsilon) = \frac{d\mu_y}{d\mu_x}(\epsilon)$. We have

$$P(y,\epsilon) = \begin{cases} \alpha^{-k(x,y,\epsilon)}, & \text{if } d(x,y) \text{ is even}\\ r\alpha^{\frac{s-1}{s}}\alpha^{-k(x,y,\epsilon)}, & \text{if } d(x,y) \text{ is odd.}\end{cases}$$

If $n = d(x,y) \geq 1$, we shall write $P(y,\epsilon) = C_n\alpha^{-k(x,y,\epsilon)}$ where $C_n = 1$ if n is even and $C_n = \left(r\alpha^{\frac{s-1}{s}}\right)$ if n is odd. We have $P(x,.) = \mathbf{1}_{T(\infty)}$.

Since for every vertex y we have $\mathbf{1}_{O_x(y)} = \sum_{y'\in S_{x,d(x,y)+1}\cap S_{y,1}} \mathbf{1}_{O_x(y')}$, it is enough to show that for every $m \geq 1$ and $y \in S_{x,2(m-1)}$, the function $\mathbf{1}_{O_x(y)}$ is a linear combination of elements in the set $\left\{P(y,.)^\nu \mid \frac{1}{2}d(x,y) \in \{0,1,\ldots,m\}\right\}$.

For $n \geq 1$, take a vertex y_n with $d(x,y_n) = n$, write a_n the index of y. We have

$$\begin{aligned}
\sum_{y\in S_{x,n+1}\cap S_{y_n,1}} P(y,.)^\nu &= C_{n+1}^\nu \sum_{y\in S_{x,n+1}\cap S_{y_n,1}} \sum_{y'\in S_{x,n+1}} \alpha^{\nu(n+1-d(y',y))}\mathbf{1}_{O_x(y')} \\
&= C_{n+1}^\nu\Big(\sum_{y'\in S_{x,n+1}\setminus S_{y_n,1}} \sum_{y\in S_{x,n+1}\cap S_{y_n,1}} \alpha^{\nu(n+1-d(y',y))}\mathbf{1}_{O_x(y')} \\
&\quad + \sum_{y'\in S_{x,n+1}\cap S_{y_n,1}} \big(\alpha^{\nu(n+1-d(y',y'))}\mathbf{1}_{O_x(y')} \\
&\qquad + \sum_{y\in (S_{x,n+1}\cap S_{y_n,1})\setminus\{y'\}} \alpha^{\nu(n+1-d(y',y))}\mathbf{1}_{O_x(y')}\big)\Big) \\
&= C_{n+1}^\nu\Big(\sum_{y'\in S_{x,n+1}\setminus S_{y_n,1}} \alpha^\nu\alpha^{\nu(n+1-d(y',y_n))}(a_n-1)\mathbf{1}_{O_x(y')} \\
&\quad + \sum_{y'\in S_{x,n+1}\cap S_{y_n,1}} \big(\alpha^{\nu(n+1)}\mathbf{1}_{O_x(y')} + (a_n-1)\alpha^{\nu(n-1)}\mathbf{1}_{O_x(y')}\big)\Big) \\
&= C_{n+1}^\nu\Big(\frac{\alpha^\nu(a_n-1)}{C_n^\nu}\big(C_n^\nu\alpha^{\nu(n-d(y_n,y_n))}\mathbf{1}_{O_x(y_n)} \\
&\qquad + \sum_{y'\in S_{x,n}\setminus\{y_n\}} C_n^\nu\alpha^{\nu(n-d(y',y_n))}\mathbf{1}_{O_x(y')}\big) \\
&\quad - \alpha^{\nu(n-1)}\alpha^{2\nu}(a_n-1)\mathbf{1}_{O_x(y_n)} + \alpha^{\nu(n-1)}\big(\alpha^{2\nu}+(a_n-1)\big)\mathbf{1}_{O_x(y_n)}\Big) \\
&= \alpha^\nu(a_n-1)\frac{C_{n+1}^\nu}{C_n^\nu}P(y_n,.)^\nu + C_{n+1}^\nu\alpha^{\nu(n-1)}\big((a_n-2)(1-\alpha^{2\nu})\big)\mathbf{1}_{O_x(y_n)} \\
&= K_nP(y_n,.) + L_n\mathbf{1}_{O_x(y_n)}
\end{aligned}$$

with $K_n = \alpha^\nu(a_n-1)\frac{C_{n+1}^\nu}{C_n^\nu}$ and $L_n = C_{n+1}^\nu\alpha^{\nu(n-1)}\big((a_n-2)(1-\alpha^{2\nu})\big)$, and for $n=0$ we have

$$\begin{aligned}
\sum_{y\in S_{x,1}} P(y,.)^\nu &= C_1^\nu\sum_{y\in S_{x,1}}\sum_{y'\in S_{x,1}} \alpha^{\nu(1-d(y',y))}\mathbf{1}_{O_x(y')} \\
&= C_1^\nu\big((r-1)+\alpha^\nu\big)\mathbf{1}_{T(\infty)} = K_0P(y_0,.)^\nu + L_0\mathbf{1}_{O_x(y_0)}
\end{aligned}$$

with $K_0 = C_1\big((r-1)+\alpha^\nu\big)$, $L_0 = 0$ and $y_0 = x$.

Finaly

$$\begin{aligned}\sum_{y\in S_{x,2m}\cap S_{y_{2(m-1)},2}} P(y,.)^{\nu} &= \sum_{y'\in S_{x,2m-1}\cap S_{y_{2(m-1)},1}} \sum_{y\in S_{x,2m}\cap S_{y',1}} P(y,.)^{\nu} \\ &= \sum_{y'\in S_{x,2m-1}\cap S_{y_{2(m-1)},1}} \big(K_{2m-1}P(y',.)+L_{2m-1}\mathbf{1}_{O_x(y')}\big) \\ &= K_{2m-1}K_{2(m-1)}P(y_{2(m-1)},.) \\ &+ \big(K_{2m-1}L_{2(m-1)}+L_{2m-1}\big)\mathbf{1}_{O_x(y_{2(m-1)})}\end{aligned}$$

hence

$$\begin{aligned}\mathbf{1}_{O_x(y_{2(m-1)})} = \big(K_{2m-1}L_{2(m-1)}+L_{2m-1}\big)^{-1}\Big(&\sum_{y\in S_{x,2m}\cap S_{y_{2(m-1)},2}} P(y,.)^{\nu} \\ &- K_{2m-1}K_{2(m-1)}P(y_{2(m-1)},.)^{\nu}\Big).\end{aligned}$$

□

The space of $G(x)$-invariant vectors $\mathcal{H}^{G(x)}_{x,\lambda_x(\nu)} = \mathbb{C}\mathbf{1}$, because for $h \in G(x)$ we have $\pi_{x,\lambda_x(\nu)}f(\epsilon) = f(h^{-1}\epsilon) = f(\epsilon)$ for all $\epsilon \in T(\infty)$ if and only if f is constant, since $G(x)$ acts transitively on $T(\infty)$.

Moreover, if $\nu \in \big(\{\frac{1}{2}\}+i[0,\frac{\pi}{\kappa\ln\alpha}]\big)\sqcup\big((0,\frac{1}{2})+i\left\{0,\frac{\pi}{\kappa\ln\alpha}\right\}\big)$, the vector $\mathbf{1}$ is cyclic in $\mathcal{H}_{x,\lambda_x(\nu)}$ because $\left\{\pi_{x,\lambda_x(\nu)}(g)\mathbf{1} = P_x(g,.)^{\nu} \mid g \in G\right\}$ spans $\mathcal{K}$ which is dense in $\mathcal{H}_{x,\lambda_x(\nu)}$. Hence $(\pi_{x,\lambda_x(\nu)},\mathcal{H}_{x,\lambda_x(\nu)})$ is spherical (irreducible) if $\nu \in \big(\{\frac{1}{2}\}+i[0,\frac{\pi}{\kappa\ln\alpha}]\big)\sqcup\big((0,\frac{1}{2})+i\left\{0,\frac{\pi}{\kappa\ln\alpha}\right\}\big)$.

We observe that the spherical representation $(\pi_{x,1},\mathcal{H}_{x,1})$ corresponding to $\lambda = 1$ is the trivial character of G in $\mathbb{C}$.

If G acts transitively on X the spherical representation $(\pi_{x,-1},\mathcal{H}_{x,-1})$ corresponding to $\lambda = -1$ is the character $g \mapsto (-1)^{d(x,gx)}$. We have to show that $(-1)^{d(x,ghx)} = (-1)^{d(x,gx)}(-1)^{d(x,hx)}$. Let $\epsilon,\epsilon' \in T(\infty)$ such that ϵ_{gx} passes through x and ϵ'_{gx} passes through ghx. Then $2(\epsilon,\epsilon')_{gx} = d(x,gx) + d(ghx,gx) - d(x,ghx)$ and $(-1)^{d(x,ghx)} = (-1)^{d(x,gx)+d(ghx,gx)-2(\epsilon,\epsilon')_{gx}} = (-1)^{d(x,gx)}(-1)^{d(x,hx)}$.

The inner tensor product$\pi_{x,-1}\otimes\pi_{x,\lambda}|_G$ (i.e. the restriction to the diagonal subgroup $G \leq G\times G$ of the tensor product $\pi_{x,-1}\otimes\pi_{x,\lambda}$) of the spherical representaions $(\pi_{x,-1},\mathcal{H}_{x,-1})$ and $(\pi_{x,\lambda},\mathcal{H}_{x,\lambda})$ satisfies $\pi_{x,-1}\otimes\pi_{x,\lambda}|_G(g) = (-1)^{d(x,gx)}\pi_{x,\lambda}(g)$ and acts on the space $\mathcal{H} = \mathbb{C}\otimes\mathcal{H}_{x,\lambda} \simeq \mathcal{H}_{x,\lambda}$. Obviously $\mathcal{H}^{G(x)} = \mathcal{H}^{G(x)}_{x,\lambda}$ and the non trivial $G(x)$-invariant vectors are cyclic. Therefore $\pi_{x,-1}\otimes\pi_{x,\lambda}|_G$ is irreducible and hence $\pi_{x,-1}\otimes\pi_{x,\lambda}|_G = \pi_{x,-\lambda}$.

Now let x and y be two edges with respective degrees a and b. Then all the constructions and formulas above are also valid by replacing r by a respectively b and s by $r+s-a$

respectively $r+s-b$. In particular we have the representations $(\pi_{x,\lambda_x(\nu)}, \mathcal{H}_{x,\lambda_x(\nu)})$ and $(\pi_{y,\lambda_y(\nu)}, \mathcal{H}_{y,\lambda_y(\nu)})$ which are spherical (irreducible) for every $\nu \in \big(\{\frac{1}{2}\} + i[0, \frac{\pi}{\kappa \ln \alpha}]\big) \sqcup \big((0, \frac{1}{2}) + i\{0, \frac{\pi}{\kappa \ln \alpha}\}\big)$.

But $(\pi_{x,\lambda_x(\nu)}, \mathcal{H}_{x,\lambda_x(\nu)})$ and $(\pi_{y,\lambda_y(\nu)}, \mathcal{H}_{y,\lambda_y(\nu)})$ are indeed equivalent: Let $T_{x,y,\nu} : \mathcal{K} \to \mathcal{K}$ be the linear operator defined by $\big(T_{x,y,\nu} f\big)(\epsilon) = \big(\frac{d\mu_x}{d\mu_y}(\epsilon)\big)^{\nu} f(\epsilon)$ for every $\epsilon \in T(\infty)$. Moreover $T_{x,y,\nu}$ is a intertwining of $\pi_{x,\lambda_x(\nu)}$ and $\pi_{y,\lambda_y(\nu)}$, because for every $g \in G$ and $\epsilon \in T(\infty)$ we have

$$\begin{aligned}
\pi_{y,\lambda_y(\nu)}(g)\big(T_{x,y,\nu} f\big)(\epsilon) &= P_y(g,\epsilon)^{\nu}\big(\frac{d\mu_x}{d\mu_y}(g^{-1}\epsilon)\big)^{\nu} f(g^{-1}\epsilon) \\
&= \big(\frac{d\mu_{gy}}{d\mu_y}(\epsilon)\big)^{\nu}\big(\frac{d\mu_x}{d\mu_y}(g^{-1}\epsilon)\big)^{\nu} f(g^{-1}\epsilon) \\
&= \big(\frac{d\mu_{gy}}{d\mu_y}(\epsilon)\big)^{\nu}\big(\frac{d\mu_{gx}}{d\mu_{gy}}(\epsilon)\big)^{\nu} f(g^{-1}\epsilon) \\
&= \big(\frac{d\mu_{gx}}{d\mu_y}(\epsilon)\big)^{\nu} f(g^{-1}\epsilon) \\
&= \big(\frac{d\mu_x}{d\mu_y}(\epsilon)\big)^{\nu}\big(\frac{d\mu_{gx}}{d\mu_x}(\epsilon)\big)^{\nu} f(g^{-1}\epsilon) \\
&= T_{x,y,\nu}\big(\pi_{x,\lambda_x(\nu)}(g) f\big)(\epsilon).
\end{aligned}$$

Therefore $T_{x,y,\nu}$ is a multiple of an isometry intertwining $\pi_{x,\lambda_x(\nu)}$ and $\pi_{y,\lambda_y(\nu)}$ (cf. for example [Gaal], Proposition 10, Chapter IV, Section 3, p. 160).

Considering the bijection

$$\lambda_x : \big(\left\{\frac{1}{2}\right\} + i[0, \frac{\pi}{\kappa \ln \alpha}]\big) \sqcup \big([0, \frac{1}{2}) + i\left\{0, \frac{\pi}{\kappa \ln \alpha}\right\}\big) \to [-\frac{2+(a-2)(r+s-a-1)}{a(r+s-a-1)}, 1],$$

a short computation shows that

$$\lambda_y\big(\lambda_x^{-1}(\lambda)\big) = \frac{a(b-1)}{b(a-1)}\lambda + \frac{a-b}{b(r+s-b-1)},$$

which proves the last statement of Theorem 2.

Chapter 3

The Universal Group $U(F)$

In this chapter we shall study an example of group which has the independence property: the universal group $U(F)$ of the homogeneous tree T of finite degree $d \geq 3$ as defined by Marc Burger and Shahar Mozes in [B; M].

Let $T = (X, Y)$ be the homogeneous tree with degree d, $d \geq 3$.

We call *legal colouring of* T the map $\iota : Y \to \{1, \dots, d\}$ with following properties:

1. for each edge e, $\iota(e) = \iota(\bar{e})$
2. for each vertex x, the restriction to the set $t^{-1}(x)$ of the map ι is a bijection.

Now, let F be a permutation group of the set $\{1, \dots, d\}$, and let ι be a legal colouring of the tree T.

Definition 46. *We define the set*

$$U_{(\iota)}(F) = \left\{ g \in \operatorname{aut}(T) \mid \text{ For every } x \in X, \iota \circ g \circ \left(\iota|_{t^{-1}(x)}\right)^{-1} \in F \right\}.$$

As it is easily seen, this set is a closed subgroup of $\operatorname{aut}(T)$.

If $F = \operatorname{Perm}\{1, \dots, d\}$, then $U_{(\iota)}(F) = \operatorname{aut}(T)$.

If F is trivial, then $U_{(\iota)}(F)$ is the free product of d copies of the group of ordre 2. Indeed the homogeneous tree of degree d is the Cayley graph Γ of the free product G of d copies of the group of ordre 2, which has the presentation $(s_1, \dots, s_d \mid s_i^2 = e, \forall i)$. The set of vertices of the Cayley graph Γ is $X' = G$ and the set of edges is $Y' = \left\{(g_1, g_2) \mid g_1, g_2 \in G, g_1^{-1} g_2 \in \{s_1, \dots, s_d\}\right\}$. The group G acts on Γ by left multiplication. Now let $\iota : Y \to \{1, \ldots, d\}$ be such that $s_{\iota((g_1,g_2))} = g_1^{-1}g_2$. Then ι is a legal colouring of Γ which is invariant for the action of G. Therefore, using Lemma 49 on page 47, we have $G = U_{(\iota)}(F)$.

In Section 3.1 we shall study some properties of the universal group and in Section 3.2 we shall study the nature of the maximal compact subgroups in the case where F is transitive. Further we shall see that all closed edge transitive automorphism groups on T which have the independence property are universal groups with a permutation group F acting transitively on $\{1, \ldots, d\}$. Precisely, we shall show this (cf. Lemma 58):

Proposition 47. *Let G be a closed edge-transitive automorphism group of the d-regular tree T which has the independence property. Set*

$$F = \left\{\phi \circ h \circ \phi^{-1} \mid h \in G(x)\right\},$$

where x is some vertex and $\phi : t^{-1}(x) \to \{1, \ldots, d\}$ some bijection. Then there exists a legal colouring ι such that $G = U_{(\iota)}(F)$.

Finally, in Sections 3.3 and 3.4 we show following result:

Theorem 3. *For transitive permutation group F with non trivial stabilizer group F_1 fixing $1 \in \{1, \ldots, d\}$, the three following conditions are equivalent:*

1. *the stabilizer group F_1 is perfect and equal to its normalizer,*
2. *the vertex stabilizing subgroup $U(F)(x)$ of the universal group $U(F)$ is topologically finitely generated,*
3. *for every real positive number M, the group $U(F)$ has finitely many equivalence classes of super cuspidal representations with formal degree less than M.*

The equivalence between the first two statements has first been announced in [Mozes], p. 574.

3.1 Some Properties

Proposition 48. *The group $U_{(\iota)}(F)$ has the independence property.*

Proof. Take an element h of the stabilizer $U_{(\iota)}(F)(e)$ of an edge e of T. Since the group $G = \text{aut}\,(T)$ has the independence property, there exist $h_1 \in G(T_e)$ and $h_2 \in G(T_{\bar{e}})$ such that $h = h_1 h_2$. For every edge ε of $T_{\bar{e}}$ we have $h_1(\varepsilon) = h(\varepsilon)$ and $h_2(\varepsilon) = \varepsilon$. Therefore, if x is an interior vertex of the subtree $T_{\bar{e}}$, then $\iota \circ h_1 \circ \left(\iota|_{t^{-1}(x)}\right)^{-1} \in F$ and $\iota \circ h_2 \circ \left(\iota|_{t^{-1}(x)}\right)^{-1}$ is the itentity on $t^{-1}(x)$ and hence is also an element of F. By a similar argument, if x is an interior vertex of the subtree T_e, then $\iota \circ h_2 \circ \left(\iota|_{t^{-1}(x)}\right)^{-1} \in F$ and $\iota \circ h_1 \circ \left(\iota|_{t^{-1}(x)}\right)^{-1}$ is the identity on $t^{-1}(x)$ and hence is also an element of F. Finally, $h_1 \in U_{(\iota)}(F)(T_e)$ and $h_2 \in U_{(\iota)}(F)(T_{\bar{e}})$, which shows that $U_{(\iota)}(F)$ has the independence property. □

We write $T_{x,n}$ respectively $T_{e,n}$, where $x \in X, e \in Y$ and n is a natural number, the minimal subtree containing the vertices at distance less or equal n from x respectively e.

Following lemma can be found in [B; M] and in [B; M; Z].

Lemma 49. *For any two legal colourings ι_1 and ι_2 and two vertices x_1 and x_2 of T, there exists a unique automorphism $g \in \text{aut}\,(T)$ such that $x_2 = gx_1$ and $\iota_2 = \iota_1 \circ g$.*

Proof. Define for every natural number n the compact subset

$$G_n = \left\{g \in \text{aut}\,(T) \mid x_2 = gx_1 \text{ and } \iota_2|_{T_{x_1,n}} = \iota_1 \circ g|_{T_{x_1,n}}\right\}.$$

Clearly, $G_1 = \left\{g \in \text{aut}\,(T) \mid x_2 = gx_1 \text{ and } \iota_2|_{t^{-1}(x_1)} = \iota_1 \circ g|_{t^{-1}(x_1)}\right\}$ is not an empty set. Now suppose $n \geq 1$ and assume as induction hypothesis, that G_n is nonempty. Pick $g \in G_n$ and chose for any terminal edge e of $T_{x_1,n}$ an element $h_e \in G(T_e)$ such that $\iota \circ g \circ h_e|_{t^{-1}(t(e))} = \iota_2|_{t^{-1}(t(e))}$. We have, as follows, $g \circ \prod_e h_e \in G_{n+1}$, and hence $G_{n+1} \neq \varnothing$. Therefore $\bigcap_{n\geq 1} G_n \neq \varnothing$. But $\bigcap_{n\geq 1} G_n = \{g \in \text{aut}\,(T) \mid x_2 = gx_1 \text{ and } \iota_2 = \iota_1 \circ g\}$, and we have shown the existence of an automorphism $g \in \text{aut}\,(T)$ such that $x_2 = gx_1$ and $\iota_2 = \iota_1 \circ g$.

Now, let g_1 and g_2 be automorphisms belonging to $\bigcap_{n\geq 1} G_n$. Then we have

$$g_1|_{t^{-1}(x_1)} = \left(i_1|_{t^{-1}(x_2)}\right)^{-1} \circ \iota_2|_{t^{-1}(x_1)} = g_2|_{t^{-1}(x_1)}.$$

Suppose $n \geq 1$. If we assume that $g_1|_{T_{x_1,n}} = g_2|_{T_{x_1,n}}$, then, for every terminal edge e of $T_{x_1,n}$,

$$\begin{aligned} g_1|_{t^{-1}(t(e))} &= \left(i_1|_{t^{-1}(t(g_1 e))}\right)^{-1} \circ \iota_2|_{t^{-1}(t(e))} \\ &= \left(i_1|_{t^{-1}(t(g_2 e))}\right)^{-1} \circ \iota_2|_{t^{-1}(t(e))} = g_2|_{t^{-1}(t(e))}, \end{aligned}$$

and hence $g_1|_{T_{x_1,n+1}} = g_2|_{T_{x_1,n+1}}$. This proves the unicity. □

Proposition 50. *The group $U_{(\iota)}(F)$ acts transitively on the set of vertices of T.*

Proof. Let x_1 and x_2 be two vertices. By the preceding lemma, there exists an automorphism $g \in \text{aut}\,(T)$ such that $x_2 = gx_1$ and $\iota = \iota \circ g$. If x is any vertex of T, then $\iota|_{t^{-1}(x)} = \iota \circ g|_{t^{-1}(x)}$ and thus $\iota \circ g \circ \left(\iota|_{t^{-1}(x)}\right)^{-1}$ is the identity and, hence, element of the group F. Therefore $g \in U_{(\iota)}(F)$. □

Proposition 51. *The group $U_{(\iota)}(F)$ acts minimaly on T.*

Proof. Since $U_{(\iota)}(F)$ acts transitively on the set of vertices, there exists no $U_{(\iota)}(F)$-invariant proper subtree of T. Now, obviously, $U_{(\iota)}(\{id\}) \leq U_{(\iota)}(F)$ and $U_{(\iota)}(\{id\})$ acts also transitively on the set of vertices and contains therefore an inversion (The automorphism g with $go(e) = t(e)$ for some edge e.). Hence $U_{(\iota)}(\{id\})$, and eventually $U_{(\iota)}(F)$, does not fix any end of T. □

Proposition 52. *For any two legal colourings ι_1 and ι_2 and vertex x of T, the groups $U_{(\iota_1)}(F)$ and $U_{(\iota_2)}(F)$ are conjugated in* aut (T) *by an automorphism fixing x.*

Proof. By the preceding lemma we have an automorphism g fixing x and satisfying $\iota_2 = \iota_1 \circ g$. And obviously, we have also $U_{(\iota_2)}(F) = g^{-1}U_{(\iota_1)}(F)g$. □

Therefore the nature of the universal group $U_{(\iota)}(F)$ depends only on the permutation group F and not on the colouring ι. Therefore we shall write in the following $U(F)$ instead of $U_{(\iota)}(F)$.

If G is a subgroup of aut (T) and x a vertex of the tree T, we write $\underline{G(x)}$ the group of permutations on $t^{-1}(x)$ induced by the stabilizer $G(x)$.

Since $U(F)$ acts transitively on the vertices, the stabilizers of a vertex are all conjugated in $U(F)$. Therefore we have for every vertex x, $\underline{\big(U(F)\big)(x)} \simeq F$

Proposition 53. *The universal group $U(F)$ acts transitively on the edges of T if and only if the permutation group F acts transitively on the set $\{1, \ldots, d\}$.*

Proof. If $U(F)$ acts transitively on the edges of T, then, for each vertex x, the group $\underline{\big(U(F)\big)(x)} \simeq F$ acts transitively.

If F acts transitively on the set $\{1, \ldots, d\}$, then, for every vertex x, the stabilizer group $\underline{\big(U(F)\big)(x)}$ acts transitively on $t^{-1}(x)$. Let e_1 and e_2 be two edges. Since $U(F)$ acts transitively on the vertices of T, there exists $g \in U(F)$ such that $o(e_2) = go(e_1)$. Further, as we have seen, there exists a $h \in \big(U(F)\big)(o(e_2))$ such that $e_2 = h(g(e_1)) = h \circ g(e_1)$. Therefore $U(F)$ acts transitively on the edges of T. □

By Proposition 6, we have following fact:

Proposition 54. *If the permutation group F acts transitively, then the group $U(F)$ is unimodular.*

As corrollary of Proposition 15 we have also the proposition

Proposition 55. *The group $U(F)$ acts transitively on the boundary of T, if and only if the permutation group F is 2-transitive.*

Following lemma can also be found in [B; M]:

Lemma 56. *Let G be a group of automorphisms of T_d acting transitively on the edges. Set $F = \{\phi \circ h \circ \phi^{-1} \mid h \in G(x)\}$, where x is some vertex and $\phi : t^{-1}(x) \to \{1, 2, \ldots, d\}$ a bijection. Then there exists a legal colouring ι such that G is a subgroup of $U_{(\iota)}(F)$.*

Proof. We construct the colouring ι on the edges of $T_{x,n}$ by induction over n . If $n = 1$, set for each edge e of $T_{x,1}$, $\iota(e) = \phi(e)$ if $e \in t^{-1}(x)$ and $\iota(e) = \phi(\bar{e})$ if $e \in o^{-1}(x)$. If $n \geq 1$, take a terminal edge f of $T_{x,n}$. By induction hypothesis, ι is defined on $T_{x,n}$ and by transitivity of the action of G on the edges, one can take $\eta_f \in G$ such that $\eta_f f = \phi^{-1} \circ \iota(f)$. Hence $\eta_f t(f) = x$ and $\phi \circ \eta_f(f) = \iota(f)$. Set for $e \in t^{-1}(t(f))$, $\iota(\bar{e}) = \iota(e) = \phi \circ \eta_f(e)$, which defines ι on $T_{x,n+1}$.

Now, if $h \in G$ and y a vertex, take an edge f between x and y such that $t(f) = y$. Then $\iota \circ h \circ \left(\iota|_{t^{-1}(x)}\right)^{-1} = \phi \circ \eta_{hf} \circ h \circ \eta_f^{-1} \circ \phi^{-1} \in F$, because $\eta_{hf} \circ h \circ \eta_f^{-1} \in G(x)$. Therefore G is a subgroup of $U_{(\iota)}(F)$. □

Since the universal group $U(F)$ is closed and has the independence property, it has also Tits' property (P). Using 4.5 in [Tits], p.198, we get for the subgroup $U(F)^+$ generated by the edge stabilizing elements of $U(F)$, the following proposition (cf. also [B; M], Proposition 3.2.1.)

Proposition 57. 1. *The group $U(F)^+$ is trivial or simple.*

2. *The group $U(F)^+$ is of finite index in $U(F)$ if and only if $F <$ Perm$1, \ldots, d$ is transitive and generated by its point stabilizers; in this case, $U(F)^+ = U(F) \cap (\operatorname{aut}(T))^+$ and is of index 2 in $U(F)$.*

3.2 The Maximal Compact Subgroups as Projective Limit

We shall use the notion of wreath product for our further investigations. For doing this, let us quickly recall that for two finite groups X and B, where X is acting on a set Ω, the *wreath product* of X with B, sometimes written $B \wr X$, is the semi-direct product $B^\Omega \rtimes X$ on which the composition law is defined by $(f, x) \cdot (g, y) := (fg_x, xy)$ for all $(f, x), (g, y) \in B^\Omega \rtimes X$, where $g_x : \omega \mapsto g(x^{-1}\omega)$ and $fg : \omega \mapsto f(\omega)g(\omega)$.

It is easy to see that for example for all $f \in B^\Omega$ and $x, y \in X$, $(f_x)_y = f_{yx}$. If we write 1_X and 1_B the neutral element of X respectively of B as well as $1 : \omega \mapsto 1_B$ the neutral element of B^Ω and $f^{-1} : \omega \mapsto \big(f(\omega)\big)^{-1}$, the inverse element of $f \in B^\Omega$, then the neutral element of $B^\Omega \rtimes X$ is $(1, 1_X)$ and the inverse element of $(f, x) \in B^\Omega \rtimes X$ is $(f, x)^{-1} = (f^{-1}_{x^{-1}}, x^{-1})$. (Observe that $(f_{x^{-1}})^{-1} = (f^{-1})_{x^{-1}} =: f^{-1}_{x^{-1}}$.)

If moreover the group B acts on a set Θ, then there exists a natural action of $B \wr X$ on the set $\Theta \times \Omega$ defined by $(f, x)(\theta, \omega) = \big(f(x\omega)\theta, x\omega\big)$ for every $(f, x) \in B \wr X$ and every $(\theta, \omega) \in \Theta \times \Omega$. Indeed, for $(f, x), (g, y) \in B \wr X$ and $(\theta, \omega) \in \Theta \times \Omega$, one has

$$\begin{aligned} (f, x)\big((g, y)(\theta, \omega)\big) &= (f, x)\big(g(y\omega)\theta, y\omega\big) = \big(f(x(y\omega))g(y\omega)\theta, x(y\omega)\big) \\ &= \big(f((xy)\omega)g_x((xy)\omega)\theta, (xy)\omega\big) \\ &= \big(fg_x((xy)\omega)\theta, (xy)\omega\big) = (fg_x, xy)(\theta, \omega) \\ &= \big((f, x) \cdot (g, y)\big)(\theta, \omega). \end{aligned}$$

In this section we shall prove the following lemma.

Lemma 58. *Let G be a closed edge-transitive automorphism group of the d-regular tree T which has the independence property. Let a be an edge or a vertex of T. Set*

$$F = \left\{\phi \circ h \circ \phi^{-1} \mid h \in G(x)\right\},$$

where x is some vertex and $\phi : t^{-1}(x) \to \{1, 2, \ldots, d\}$ a bijection. We write B for the stabilizer in F of $1 \in \{1, \ldots, d\}$ and $D = \{2, 3, \ldots, d\}$, and we define recursively

if a is an edge,	$\Delta_0 = \{1, 2\}$,	$\Delta_1 = D \times \Delta_0$,
or if a is a vertex,	$\Delta_0 = \{1\}$,	$\Delta_1 = \{1, \ldots, d\}$,
and for $n \geq 1$,	$\Delta_{n+1} = \{2, \ldots, d\} \times \Delta_n$;	

as well as the following groups:

if a is an edge,	$G^{(0)} = \mathrm{Perm}(\{1,2\})$,	$G^{(1)} = B^{\Delta_0} \rtimes G^{(0)}$
or if a is a vertex	$G^{(0)} = (e)$,	$G^{(1)} = F$,
and for $n \geq 1$,	$G^{(n+1)} = B^{\Delta_n} \rtimes G^{(n)}$,	

with their respective action as described before. The projections $\rho_n : G^{(n+1)} \to G^{(n)}$, $(f,h) \mapsto h$ define a projective system $G^{(1)} \overset{\rho_1}{\leftarrow} G^{(2)} \overset{\rho_1}{\leftarrow} \cdots$ of groups.

Then we have:

1. *for every natural number n the group $G_n := \widetilde{G}(T_{a,n})/G(T_{a,n})$ is isomorphic to $G^{(n)}$ (Where $T_{a,n}$ is the minimal subree of T containing the vertices at distance n of a.),*
2. *the a stabilizing subgroup $\widetilde{G}(a)$ of G is isomorphic to the projective limit $\lim_{\leftarrow} G^{(n)}$, and*
3. *there exists a legal colouring ι such that $G = U_{(\iota)}(F)$.*

Proof. Let G be an automorphism group of the d-regular tree T.

In the following, the letter a denotes a fixed edge or a fixed vetex of the tree T. We write for every natural positive number n the quotient group

$$G_n = \widetilde{G}(T_{a,n})/G(T_{a,n}).$$

These groups can be considered as automorphism groups of $T_{a,n}$. Together with the restriction homomorphisms $R_n : G_{n+1} \to G_n$, they constitute a projective system $G_1 \overset{R_1}{\longleftarrow} G_2 \overset{R_2}{\longleftarrow} \cdots$ and one verifies easily that, if G is closed in aut (T), its projective limit is

$$\lim_{\leftarrow} G_n = \widetilde{G}(a).$$

Now assume that the group G acts transitively on the edges of the tree T and let ι be the legal colouring and $U(F) = U_{(\iota)}(F)$, with $F = \{\phi \circ h \circ \phi^{-1} \mid h \in G(x)\}$, as in Lemma 56. Then G is a subgroup of $U(F)$.

If a is an edge, we can assume $\iota(a) = 1$. We shall now build for every positive natural number n an injective homomorphism $\varphi_n : G_n \to G^{(n)}$, such that $\varphi_n \circ R_n = \rho_n \circ \varphi_{n+1}$.

For this, consider first for each natural number n the set $E_{a,n}$ of the terminal edges of $T_{a,n}$ (Observe that $E_{a,0}$ is the empty set if a is a vertex and equal to $\{a, \bar{a}\}$ if a

is an edge.). For every $e \in E_{a,n+1}$, we write $P_n(e)$ the unique edge in $E_{a,n}$ satisfying $t\big(P_n(e)\big) = o(e)$. This gives us surjective maps

$$P_n : E_{a,n+1} \to E_{a,n}, n \geq 0.$$

On the other side, we define the projections

$$\pi_n : \Delta_{n+1} \to \Delta_n, (k, \delta) \mapsto \delta \text{ for } n \geq 1.$$

Using the group G and especially the fact that F acts transitively, we define inductively on n bijections $\psi_n : E_{a,n} \to \Delta_n$:

$$\psi_0(e) = \begin{cases} 1 & \text{if } e = a \\ 2 & \text{if } e = \bar{a} \end{cases} \quad \text{if } a \text{ is an edge}$$

$$\psi_1(e) = \iota(e), \quad \text{if } a \text{ is a vertex}$$

$$\psi_{n+1}(e) = \big(\tau_{P_n(e)} \circ \iota(e), \psi_n(P_n(e))\big), \text{ where } \tau_e \in F \text{ such that } \tau_e \circ \iota(e) = 1.$$

By construction, $\psi_n \circ P_n = \pi_n \circ \psi_{n+1}$ for every n. Further we notice that for $n \geq 1$ ($n \geq 0$ if a is an edge) the inverse map $\psi_{n+1}^{-1}(k, \delta) = \big(\iota|_{t^{-1}(\psi_n^{-1}(\delta))}\big)^{-1} \circ \tau_{\psi_n^{-1}(\delta)}^{-1}(k)$.

Now, for each $h \in G_n$, set $\varphi_n(h) = \psi_n \circ h \circ \psi_n^{-1}$. We have then for $(f, h) \in G_{n+1}$ and $(k, \delta) \in \Delta_{n+1}$,

$$\begin{aligned} \varphi_{n+1}(h)(k, \delta) &= \psi_{n+1} \circ h \circ \big(\iota|_{t^{-1}(\psi_n^{-1}(\delta))}\big)^{-1} \circ \tau_{\psi_n^{-1}(\delta)}^{-1}(k) \\ &= \Big(\tau_{R_n(h)\psi_n^{-1}(\delta)} \circ \iota \circ h \circ \big(\iota|_{t^{-1}(\psi_n^{-1}(\delta))}\big)^{-1} \circ \tau_{\psi_n^{-1}(\delta)}^{-1}(k), \varphi_n(R_n(h))\Big) \\ &= \big(f^h(\delta)k, \varphi_n(R_n(h))\big) \end{aligned}$$

where we have set

$$f^h(\delta) = \tau_{R_n(h)\psi_n^{-1}(\delta)} \circ \iota \circ h \circ \big(\iota|_{t^{-1}(\psi_n^{-1}(\delta))}\big)^{-1} \circ \tau_{\psi_n^{-1}(\delta)}^{-1}.$$

Since $\iota \circ h \circ \big(\iota|_{t^{-1}(\psi_n^{-1}(\delta))}\big)^{-1} \in F$, for all $\delta \in \Delta_n$, it follows $f^h \in B^{\Delta_n}$ and hence $\varphi_{n+1}(h) \in B^{\Delta_n} \rtimes G^{(n)} = G^{(n+1)}$. Therefore we have built for each n a homomorphism $\varphi_n : G_n \to G^{(n)}$ which is clearly injective.

Moreover we have

$$\begin{aligned} \big((\rho_n \circ \varphi_{n+1})(h)\big) \circ \pi_n &= \rho_n\big(\varphi_{n+1}(h)\big) \circ \pi_n = \pi_n \circ \varphi_{n+1}(h) \\ &= \pi_n \circ \psi_{n+1} \circ h \circ \psi_{n+1}^{-1} = \psi_n \circ P_n \circ h \circ \psi_{n+1}^{-1} \\ &= \psi_n \circ R_n(h) \circ P_n \circ \psi_{n+1}^{-1} = \psi_n \circ R_n(h) \circ \psi_n^{-1} \circ \pi_n \\ &= \varphi_n\big(R_n(h)\big) \circ \pi_n = \big((\varphi_n \circ R_n)(h)\big) \circ \pi_n; \end{aligned}$$

and since π_n is surjective, $\rho_n \circ \varphi_{n+1} = \varphi_n \circ R_n$.

Now, we suppose moreover, that G has the independence property. Then, the homomorphisms φ_n are isomorphisms. Indeed:

If a is an edge, φ_0 is clearly a isomorphism.

If a is a vertex, the homomorphism φ_1 is an isomorphism between $\underline{G(a)}$ and F, as Lemma 56 shows. Therefore the subgroup B is the image by φ_1 of the subgroup $\{h \in G_1 \mid he = e\}$, where e is an edge with $o(e) = a$ and $\iota(e) = 1$.

Hence by the independence property of G, we have for every edge e the equalities among sets

$$\begin{aligned}&\left\{\tau_{\iota(e)} \circ \iota \circ h \circ \left(\iota|_{t^{-1}(t(e))}\right)^{-1} \circ \tau_{\iota(e)}^{-1} \mid h \in G(T_e)\right\} \\ = &\left\{\tau_{\iota(e)} \circ \iota \circ h \circ \left(\iota|_{t^{-1}(t(e))}\right)^{-1} \circ \tau_{\iota(e)}^{-1} \mid h \in G(e)\right\} \quad = B.\end{aligned}$$

Now if a is an edge or a vetex, for appropriate n and for $(f, \eta) \in G^{(n+1)} = B^{\Delta_n} \rtimes G^{(n)}$, if we take as induction hypothesis, that there exists an element $h \in G_n$ with $\varphi_n(h) = \eta$ and if we pick a corresponding $h' \in G_{n+1}$ with $R_n(h') = h$, then we can choose for every $\delta \in \Delta_n$ a $h_\delta \in G(T_{e_\delta})$ such that $h_\delta|_{t^{-1}(t(e_\delta))} = h'^{-1} \circ \iota|_{t^{-1}(t(he_\delta))} \circ \tau_{\iota(he_\delta)}^{-1} \circ f\left(\varphi_n(h)\delta\varphi\right) \circ \tau_{\iota(he_\delta)} \circ \iota|_{t^{-1}(t(h^{-1}e_\delta))}$ and it is easy to see that $\varphi_{n+1}\left(h' \prod_{\delta \in \Delta_n} \mathrm{Res}_{T_{a,n+1}}(h_\delta)\right) = (f, \eta)$ (Where $\mathrm{Res}_{T_{a,n+1}}(h_\delta)$ means the restriction to $T_{a,n+1}$ of the map h_δ.)

Therefore we have shown the first two statements of the lemma.

To prove the last statement, it is enough to show that $U_{(\iota)}(F) < G$. This is easily done since we have now seen that $U_{(\iota)}(F)(x) = G(x)$ for every vertex x. So, if $u \in U_{(\iota)}(F)$, take $h \in G$ with $h(ux) = x$ (G is transitive on the vertices). Therefore $k := hu \in U_{(\iota)}(F)(x) = G(x)$ and $u = h^{-1}k \in G$.

□

3.3 Topologically Finite Generation of the Maximal Compact Subgroups

Let $F < S_d$ be a transitive permutation group acting on $\{1, 2, \ldots, d\}$ and F_δ the stabilizing group in F of $\delta \in \{1, \ldots, d\}$. Suppose that F_δ is not trivial. We shall show the following proposition:

Proposition 59. *Let x be a vertex of the tree T. The group $U(F)(x)$ is topologically finitely generated if and only if F_δ is perfect and equal to its normalizer.*

For the proof, we shall begin with following remark:

For a finite group G and k a natural number, set

$$p_k(G) = \frac{1}{|G|^k} \left|\left\{(x_1, \ldots, x_k) \in G^k \mid \langle x_1, \ldots, x_k \rangle = G\right\}\right|.$$

For a surjective homomorphism of finite groups $\pi : Y \twoheadrightarrow X$, define

$$\begin{aligned}\zeta_{Y/X}(k) &= \sum_{\substack{M<Y \text{ maximal,} \\ \pi(M)=X}} \frac{1}{[Y:M]^{k+1}} \\ &= \sum_{\substack{C(M):M<Y \text{ maximal,} \\ \pi(M)=X}} \frac{|C(M)|}{[Y:M]^{k+1}},\end{aligned}$$

where $C(M)$ is the set of conjugates of M.

We have the lemma:

Lemma 60.

$$p_k(Y) \geq \big(1 - \zeta_{Y/X}(k-1)\big)p_k(X).$$

Proof. The reader has certainly already noticed that $p_k(G)$ is the probability with which a k-uple $(x_1, \ldots, x_k) \in G^k$ generates the group G. Now pick $(x_1, \ldots, x_k) \in X^k$ such that $\langle x_1, \ldots, x_k \rangle = X$ and choose $(y_1, \ldots, y_k) \in Y^k$ with $\pi(y_i) = x_i$. Then, if $\langle y_1, \ldots, y_k \rangle \neq Y$, there exists a maximal (proper) subgroup M in Y containing $y_1, \ldots, y_k$ – we have $\pi(M) = X$. Therefore $p_k(Y) \geq Q \cdot p_k(X)$, where Q is . But the probability for $y_1, \ldots, y_k$ to be all in a fixed such maximal subgroup M is $\frac{|M|^k}{|Y|^k}$, and hence the probability for them to be in any of such maximal subgroups is smaller than or equal to

$$\begin{aligned}\sum_{\substack{M<Y:M \text{ maximal,} \\ \pi(M)=X}} \frac{|M|^k}{|Y|^k} &= \sum_{\substack{C(M):M<Y \text{ maximal,} \\ \pi(M)=X}} |C(M)| \frac{|M|^k}{|Y|^k} = \sum_{\substack{C(M):M<Y \text{ maximal,} \\ \pi(M)=X}} \frac{|C(M)|}{[Y:M]^k} \\ &= \zeta_{Y/X}(k-1).\end{aligned}$$

Therefore the probability $p_k(Y)$ that $y_1, \ldots, y_k$ generate Y, i.e. the probability that $y_1, \ldots, y_k$ are not all in a maximal (proper) subgroup M of Y, satisfies $p_k(Y) \geq \big(1 - \zeta_{Y/X}(k-1)\big) \geq \big(1 - \zeta_{Y/X}(k-1)\big)p_k(X)$. □

So we obtain:

Proposition 61. *If* $\ldots \leftarrow F_{n-1} \leftarrow F_n \leftarrow \ldots$ *is an inverse system of finite groups* F_n *and if for some* $\eta > 0$ *the product* $\prod_{n=1}^{\infty}\bigl(1 - \zeta_{F_{n+1}/F_n}(\eta)\bigr) > 0$, *then the projective limit* $\lim_{\leftarrow} F_n$ *is topologically finitely generated.*

Maximal Subgroups in $B^\Omega \rtimes X$

We shall now study maximal subgroups in a wreath product $Y = B^\Omega \rtimes X$, where X and B are finite groups and X acts on the finite set Ω, in order to give an estimation of the Zeta function.

Let $\Omega = L_1 \sqcup \ldots \sqcup L_t$ be the partition into X-orbits. A normal subgroup $U \lhd B^\Omega$ is said *standard* if $U = \prod_{i=1}^{t} U_i^{L_i}$, where U_i are normal subgroups of B. Such a standard group is clearly also a normal subgroup of Y. A subgroup M of Y is called *clean* if it does not contain any non-trivial standard subgroup.

We notice that, given a subgroup M of $Y = B^\Omega \rtimes X$, there is a unique maximal standard $U \lhd B^\Omega$, contained in M. Thus, M is the inverse image of a clean $M' < Y/U$, and M is maximal if and only if M' is maximal in $Y/U = B^\Omega/U \rtimes X$. Therefore we begin to study maximal clean subgroups in $Y = B^\Omega \rtimes X$.

Proposition 62. *Let* $Y = B^\Omega \rtimes X$ *with* B *perfect. Let* M *be a maximal clean subgroup of* Y *such that its canonical projection onto* X *is* X. *Then one of the following holds.*

1. *The action of* X *on* Ω *is transitive.*
   (a) *Up to conjugacy* $M = T^\Omega \rtimes X$ *where* T *is a non-normal maximal subgroup of* B.
   (b) *We have* $M \cap B^\Omega = (e)$, *the group* B *is simple (non-abelian) and, if we write* M° *the intersection of all conjugates of* M, *then we have either*
      i. *the subgroup* M° *contains the centralizer* $C_X(B^\Omega)$ *in* X *of* B^Ω, *or*
      ii. *the quotient* $C_X(B^\Omega)/\bigl(M^\circ \cap C_X(B^\Omega)\bigr)$ *is isomorph to* B^Ω, *and via this isomorphism, the* X*-action by conjugation acts transitively on the factors of* B^Ω.
   (c) *The group* B *admits a unique minimal normal subgroup* N; *the subgroup* N *is non-abelian,* M *is the normalizer of* $M \cap N^\Omega$, $pr_\omega(M \cap B^\Omega) = B$ *and* $pr_\omega(M \cap N^\Omega) = N$ *for all* $\omega \in \Omega$.

(d) *The group B is the direct product $N_1 \times N_2$ of simple non-abelian groups $N_1 \simeq N_2$; M is the normalizer in Y of $M \cap B^{\Omega}$, the projections $pr_{\omega}(M \cap B^{\Omega}) = B$ for every $\omega \in \Omega$, and $M \cap N_1^{\Omega} = M \cap N_2^{\Omega} = (e)$.*

2. *The action of X on Ω is not transitive.*

   Then B is simple, $M \cap B^{L_i} = (e)$ and $pr_{B^{L_i}}(M \cap B^{\Omega}) = B^{L_i}$ for all $i = \{1, \ldots, t\}$.

Proof. Let for all $i \in \{1, \ldots, t\}$ be $T_i := pr_{B^{L_i}}(M \cap B^{\Omega})$. Then M normalizes $T_1 \times \ldots \times T_t$. We distinguish the following cases:

I. Some $T_i \neq B^{L_i}$;

II. For all $i \in \{1, \ldots, t\}$, we have $T_i = B^{L_i}$.

I. Some $T_i \neq B^{L_i}$.

Without lost of generality, assume $T_1 \neq B^{L_1}$. Then $M \cap B^{\Omega} \subset T_1 \times B^{L_2} \times \ldots \times B^{L_t}$, M normalizes $T_1 \times B^{L_2} \times \ldots \times B^{L_t}$ and hence we have $M \subseteq (T_1 \times B^{L_2} \times \ldots \times B^{L_t}) \rtimes X \neq Y$. Since M is maximal, M contains $T_1 \times B^{L_2} \times \ldots \times B^{L_t}$, and hence $t = 1$ since M is clean. Thus X is transitive on Ω.

Let for every $\omega \in \Omega$, $S_{\omega} = pr_{\omega}(M \cap B^{\Omega})$. Since $pr_X(M) = X$, all S_{ω} are conjugate in B to a fixed $T < B$. As we are interested in maximal subgroups up to conjugacy, we may assume $T = pr_{\omega}(M \cap B^{\Omega})$ and we have therefore $M \subseteq T^{\Omega} \rtimes X$.

There are three subcases:

(a) T is not normal in B;

(b) T is normal in B but not equal to B;

(c) $T = B$.

(a) T is not normal in B.

As before, M normalizes T^{Ω} in Y, and since T is not normal in B, the normalizer of T^{Ω} in Y is not all Y; i.e. we have $M \subseteq N_Y(T^{\Omega}) \subseteq N_B(T^{\Omega}) \rtimes X \neq Y$ as well as $M \subseteq T^{\Omega} \rtimes X \subseteq N_B(T^{\Omega}) \rtimes X \neq Y$. Hence, by maximality of M, $M = T^{\Omega} \rtimes X$ where T is maximal and not normal in B. This leads to case *1a* of the proposition.

(b) T is normal in B but not equal to B.

Take an normal strict subgroup N of B containing T. Then $M \subseteq N^{\Omega} \rtimes X \neq Y$ and thus, by maximality, M contains N^{Ω}. Since M is clean, $N = (e)$. On the other hand, N^{Ω} contains $M \cap B^{\Omega}$, thus $M \cap B^{\Omega} = (e)$ as well as T. The group B is therefore simple. Since B is perfect, B is simple non-abelian and hence the centralizer $C_Y(B^{\Omega}) = C_X(B^{\Omega})$ is the kernel of the X-action on Ω. Let $p : Y \to Y/M^{\circ}$ be the canonical projection. Then Y/M° acts primitively and faithfully on Y/M. Then, either $p\big(C_X(B^{\Omega})\big) = (e)$, that is M° contains $C_X(B^{\Omega})$ and we are in the case *1(b)i*, or $(e) \neq p\big(C_X(B^{\Omega})\big) = p\big(C_Y(B^{\Omega})\big) \subset C_{Y/M^{\circ}}\big(p(B^{\Omega})\big)$. Since $M \cap B^{\Omega} = (e)$, the projection $p(B^{\Omega}) \simeq B^{\Omega}$ and is a normal regular subgroup of Y/M°, and therefore minimal. The centralizer $C_{Y/M^{\circ}}\big(p(B^{\Omega})\big)$ is not trivial, and is hence also minimal regular. Thus $p\big(C_Y(B^{\Omega})\big) = C_{Y/M^{\circ}}\big(p(B^{\Omega})\big)$ and are isomorph to $B^{|\Omega|}$ and the Y/M°-conjugation permutes transitively the simple factors. Thus $C_X(B^{\Omega})/\big(M^{\circ} \cap C_X(B^{\Omega})\big)$ is isomorph to $B^{|\Omega|}$ and the X-action by conjugation permutes transitively the simple factors. We are therefore in the case *1(b)ii* of the proposition.

(c) $T = B$.

This subcase and the case

II. For all $i \in \{1, \dots, t\}$, we have $T_i = B^{L_i}$

will be treated by analyzing the situation where $M < B^{\Omega} \rtimes X$ is maximal, clean and such that $pr_X(M) = X$ and $pr_{\omega}(M \cap B^{\Omega}) = B$ for all $\omega \in \Omega$.

As before, $\Omega = L_1 \sqcup \dots \sqcup L_t$ is the partition into X-orbits of Ω and B is perfect.

Let N be a non trivial minimal normal subgroup of B. Then

(A) if for some $i \in \{1, \dots, t\}$, the intersection $M \cap N^{L_i}$ is trivial, the group B is a direct product $T \times N$, where T is the projection onto the first factor, the B-factor, of $M \cap (B \times N^{|L_i|-1})$.

(B) the subgroup N is not abelian, and

To prove the point **(A)**, we observe that since M is maximal and clean, $B^{\Omega} = (M \cap B^{\Omega}) \cdot N^{L_i}$, and thus $B \times N^{|L_i|-1} = \big(M \cap (B \times N^{|L_i|-1})\big) \cdot N^{L_i}$. Projecting it onto the first factor, we get $B = T \cdot N$. Since $M \cap N^{L_i}$ is trivial, $T \cap N = \varnothing$. Moreover $M \cap (B \times N^{|L_i|-1}) \lhd M \cap B^{\Omega}$, and thus $T \lhd B$ an finally $B = T \times N$.

To prove the point **(B)**, we assume that N is abelian, and see first that N is not central in B. Indeed, we have $B^{\Omega} = (M \cap B^{\Omega}) \cdot N^{\Omega}$; if now N is central in B, then

$B^\Omega = [B^\Omega, B^\Omega] = [M \cap B^\Omega, M \cap B^\Omega] \subset M \cap B^\Omega$ which is a contradiction.

Then we show that $M^\circ \cap N^{L_i}$ is trivial: Since N is a minimal normal and, as assumed, abelian subgroup of B, it is a $\mathbb{F}_p$-vector space. The B-conjugation on N gives an irreducible representation of B, which is non-trivial since N is not central in B. The B^{L_i}-action on N^{L_i} is therefore a direct sum of $|L_i| - 1$ irreducible and inequivalent representations of B^{L_i}. Thus, any B^{L_i}-invariant subspace is a sum of factors of N^{L_i}. In particular, if $M^\circ \cap N^{L_i}$ is non-trivial, it contains a N-factor. But $M^\circ \cap N^{L_i}$ is normal in Y, thus M° contains N^{L_i}. This contradicts the assumption that M is clean.

Then we show that $M \cap N^{L_i}$ is trivial: Since $M^\circ \cap N^{L_i}$ is trivial, the Y-action on Y/M is primitive, and N^{L_i} acts faithfully on Y/M. So N^{L_i} acts transitively and faithfully. Being abelian, it must act regularly. Thus $M \cap N^{L_i}$ is trivial.

Finally, we apply point **(A)** and get that $B = T \times N$ with N abelian, which contradicts the assumption that B is perfect. This proves point **(B)**.

Now we have two cases.

First: There exist a non-trivial normal minimal subgroup N of B and a $i \in \{1, \ldots, t\}$ with $M \cap N^{L_i}$ non-trivial.

We first observe that $M \cap N^{L_i}$ is not normal in Y. Indeed, for some $\omega \in L_i$, $pr_\omega(M \cap N^{L_i}) \neq (e)$ but $pr_\omega(M \cap N^{L_i}) \triangleleft pr_\omega(M \cap B^\Omega) = B$ and, since N is minimal normal, we get $pr_\omega(M \cap N^{L_i}) = N$. Applying M-conjugation and using $pr_X(M) = X$, we get the latter equality for all $\omega \in L_i$. But if now $M \cap N^{L_i}$ is normal in Y, it is normal in N^{L_i}. But N is non-abelian and product of simple factors, which, together with above, leads to the contradiction that M contains N^{L_i}. Thus, M is the normalizer in Y of $M \cap N^{L_i}$. This normalizer obviously contains all B^{L_j} for $j \neq i$, and since M is clean, this forces t to be 1, that is, the action of X on Ω is transitive. Therefore M is the normalizer in Y of $M \cap N^\Omega$ and $pr_\omega(M \cap N^\Omega) = N$ for all $\omega \in \Omega$. If now N' is another non-trivial minimal normal subgroup of B, then N'^Ω centralizes N^Ω and hence M contains N'^Ω. Since M is clean, this shows that N' is trivial, and we are in case *1c*.

Second: For any non-trivial minimal normal subgroup N of B and for any $i \in \{1, \ldots, t\}$, the intersection $M \cap N^{L_i}$ is trivial.

It follows then from point **(A)** that any such minimal normal subgroup N of B is a direct factor of B, in particular simple. Hence B is a direct product of simple non-abelian groups $N_1, \ldots, N_m$. Now assume $m \geq 2$. Then, since $N_1^{L_1}, \ldots, N_m^{L_1}$ act all faithfully transitively on Y/M, they act also regularly, which forces $m = 2$. Now,

$N_1^{L_1}, N_2^{L_1}, N_1^{L_2}, N_2^{L_2} \ldots, N_1^{L_t}, N_2^{L_t}$ act faithfully transitively on Y/M, which forces $t = 1$, and hence X acts transitively on Ω. Thus we are in case *1d*.

Finally, if $m = 1$, B is simple, and since $M \cap N^{L_i} = M \cap B^{L_i} = (e)$, we must have $t \geq 2$. Thus, we are in the situation II. and hence in case *2* of the proposition.

This proves the proposition. □

Estimation of the Zeta Function

We have still $Y = B^\Omega \rtimes X$ and, since we have treated now the maximal clean subgroups of Y, we define the relative Zeta function

$$\zeta^{\mathrm{cl}}_{Y/X}(\eta) = \sum_{\substack{M<Y \text{ maximal, clean,} \\ pr_X(M)=X}} \frac{1}{[Y:M]^\eta} = \sum_{\substack{C(M):M<Y \text{ maximal, clean,} \\ pr_X(M)=X}} \frac{|C(M)|}{[Y:M]^\eta}.$$

We first begin by estimating $\zeta^{\mathrm{cl}}_{Y/X}(\eta)$. We have

$$\begin{aligned}\zeta^{\mathrm{cl}}_{Y/X}(\eta) &= \Sigma(\mathit{1a})(\eta) + \Sigma(\mathit{1(b)i})(\eta) + \Sigma(\mathit{1(b)ii})(\eta) \\ &\quad + \Sigma(\mathit{1c})(\eta) + \Sigma(\mathit{1d})(\eta) + \Sigma(\mathit{2})(\eta),\end{aligned}$$

where $\Sigma(x)(\eta)$ is the sum over the subgroups M corresponding to the case x of proposition 62.

Case *1a*:

The sum is over $M = T^\Omega \rtimes X$ with T a maximal non normal subgroup of B. Thus

$$\Sigma(\mathit{1a})(\eta) = \zeta^{\mathrm{cl}}_{Y/X}(\eta) = \sum_{\substack{C(T): \\ T<B \text{ maximal,}}} \frac{1}{[Y:T^\Omega \rtimes X]^\eta} = \sum_{\substack{C(T): \\ T<B \text{ maximal,}}} \frac{1}{[B:T]^{|\Omega|\eta}}.$$

So, if C_1 is the number of maximal subgroups in B and n_1 the lower bound on their indices, we get

$$\Sigma(\mathit{1a})(\eta) \leq C_1 n_1^{-|\Omega|\eta} = C_1 \left(\frac{1}{n_1^\eta}\right)^{|\Omega|}.$$

Case *1(b)i*:

Here we do not have an estimate in the general case; we shall study it in the next subsection.

Cases *1(b)ii* and *1d* can be treated together by considering the following situation: let G be a finite group containing $J_1 \times J_2$ where $J_1 \simeq J_2 \simeq S^N$ are minimal normal and non-abelian, and we are looking at maximal subgroups M of G with $M \cap J_1 = M \cap J_2 = (e)$

and the M-action by conjugation is transitive on the factors S of $J_1 \times J_2$. So $M \cap (J_1 \times J_2)$ is the graph of an isomorphism $\psi : J_1 \to J_2$, and M is the normalizer in G of $M \cap (J_1 \times J_2)$. Thus ψ determines M. Now ψ is of the form $\psi(b_1, \dots, b_N) = \big(\varphi_1(b_{\sigma(1)}), \dots, \varphi_N(b_{\sigma(N)})\big)$, where φ_i is an automorphism of S and σ a permutation of $\{1, \dots, N\}$. But, since M acts transitively on the simple factors of $J_1 \times J_2$, the permutation σ is completely determined by $\sigma(1)$. Thus, there are at most $|\mathrm{aut}\,(S)|^N \cdot N$ such isomorphisms ψ.

Now to the case *1(b)ii*: Let $n_2(Y)$ be the number of subgroups U of the centralizer $C_X(B^\Omega)$ in X of B^Ω such that U is normal in Y and the X-action by conjugation on $C_X(B^\Omega)/U \simeq B^{|\Omega|}$ is transitive. Then

$$\Sigma\big(\mathit{1(b)ii}\big)(\eta) \leq n_2(Y)\,|\Omega|\,\frac{|\mathrm{aut}\,(B)|^{|\Omega|}}{|B|^{|\Omega|\eta}} = n_2(Y)\,|\Omega|\left(\frac{|\mathrm{aut}\,(B)|}{|B|^{\eta}}\right)^{|\Omega|}.$$

We shall give an estimation of $n_2(Y)$ in the next subsection.

In case *1d*, with the notation used in the proposition, we have

$$\Sigma(\mathit{1d})(\eta) \leq |\Omega|\,\frac{|\mathrm{aut}\,(N_1)|^{|\Omega|}}{|N_1|^{|\Omega|\eta}} = |\Omega|\left(\frac{|\mathrm{aut}\,(N_1)|}{|N_1|^{\eta}}\right)^{|\Omega|}.$$

Now we turn to the case *1c*:

M is determined by $M \cap N^\Omega$ as being its normalizer. Now $N \simeq S^r$ where S is simple non-abelian. Let $\Omega' = \Omega \sqcup \ldots \sqcup \Omega$ the disjoint union of r copies of Ω , so that $N^\Omega \simeq S^{\Omega'}$. Since $pr_\omega(M \cap B^\Omega) = B$ for all $\omega \in \Omega$, M acts transitively on Ω'. Now $M \cap N^\Omega = M \cap S^{\Omega'}$ is a product $D_1 \times \ldots \times D_s$ of subdiagonals corresponding to a bloc decomposition $\Omega' = \Omega_1 \sqcup \ldots \sqcup \Omega_s$ for the transitive M-action. Let $n = |\Omega'|$; we have $s < n$ and s divides n, thus $s \leq \frac{n}{2}$. There are at most $\binom{n}{s}$ possibilities for Ω_1. By transitivitiy this determines then the bloc decomposition. Such a partition being fixed, there are then $|\mathrm{aut}\,(S)|^{n-s}$ such products $D_1 \times \ldots \times D_s$, each with index $|S|^{n-s}$ in $S^{\Omega'}$. Finally we get

$$\Sigma\big(\mathit{1c}\big)(\eta) \leq \sum_{1 \leq s \leq \frac{n}{2}} \binom{n}{s} \frac{|\mathrm{aut}\,(S)|^{n-s}}{|S|^{(n-s)\eta}} \leq 2^n \left(\frac{|\mathrm{aut}\,(S)|}{|S|^{\eta}}\right)^{\frac{n}{2}} = \left(2\sqrt{\frac{|\mathrm{aut}\,(S)|}{|S|^{\eta}}}\right)^{r|\Omega|},$$

where $N \simeq S^r$ and η is chosen such that $\frac{\mathrm{aut}(S)}{|S|^{\eta}} < 1$.

Now the estimation of $\Sigma(\mathit{2})(\eta)$:

Again, M is the normalizer in Y of $M \cap B^\Omega$, and since in this case B is simple $M \cap B^\Omega = D_1 \times \ldots \times D_s$ a product of subdiagonals corresponding to a bloc decomposition $\Omega = \Omega_1 \sqcup \ldots \sqcup \Omega_s$. Now assume that there is a $k \in \{1, \dots, s\}$ such that $\Omega^{(1)} =$

$\Omega_1 \sqcup \ldots \sqcup \Omega_k$ and $\Omega^{(2)} = \Omega_{k+1} \sqcup \ldots \sqcup \Omega_s = L_{i_1} \sqcup \ldots \sqcup L_{i_{s-k}}$ are X-invariant. Then $M \cap B^{\Omega} \subset (D_1 \times \ldots \times D_s) \cdot (B^{L_{i_1}} \times \ldots \times B^{L_{i_{s-k}}})$ hence $(B^{L_{i_1}} \times \ldots \times B^{L_{i_{s-k}}}) \cdot M \neq Y$, which implies that M contains $B^{L_{i_1}} \times \ldots \times B^{L_{i_{s-k}}}$. But this is not possible since M is clean. Therefore X acts transitively on the set of blocs $\{\Omega_1, \ldots, \Omega_s\}$. Moreover, since $pr_{B^{L_i}}(M \cap B^{\Omega}) = B^{L_i}$, we have $|\Omega_i \cap L_j| \leq 1$ for all i, j and hence by transitivity of X on the blocs, $|\Omega_i \cap L_j| = 1$ for all i and j.

Thus $|L_j| = s$ and $|\Omega_i| = t$ for all i and j. Now, there are at most s^t possibilities for Ω_1, which then determines the partition. For each partition we get $|\mathrm{aut}\,(B)|^{|\Omega_1|-1} \times \ldots \times |\mathrm{aut}\,(B)|^{|\Omega_s|-1}$ subdiagonal subgroups, each of index $|B|^{|\Omega_1|-1} \cdot \ldots \cdot |B|^{|\Omega_s|-1}$. Thus

$$\Sigma(\mathit{2})(\eta) \leq s^t \frac{|\mathrm{aut}\,(B)|^{|\Omega|-s}}{|B|^{(|\Omega|-s)\eta}},$$

where $\Omega = L_1 \sqcup \ldots \sqcup L_t$, $t \geq 2$, and $s = |L_i|$ for all i.

Later we shall only need following crude estimate: since for natural numbers $s^t \leq 2^{st}$, we have $|\Omega| - s = ts - s \geq \frac{ts}{2} = \frac{|\Omega|}{2}$, for $t \geq 2$. Thus choosing η with $\frac{\mathrm{aut}(B)}{|B|^{\eta}} < 1$, we get

$$\Sigma(\mathit{2})(\eta) \leq \left(4 \frac{|\mathrm{aut}\,(B)|}{|B|^{\eta}}\right)^{\frac{|\Omega|}{2}} = \left(2\sqrt{\frac{|\mathrm{aut}\,(B)|}{|B|^{\eta}}}\right)^{|\Omega|}.$$

Estimation for the Cases *1(b)i* and *1(b)ii*

We shall now give an estimation for $n_2(Y)$ of the case *1(b)ii* and for the case *1(b)i*, for which we had no general result, in the special case where Y is one of the groups $G^{(n)}$ of the projective system $G^{(1)} \leftarrow G^{(2)} \leftarrow \ldots \leftarrow G^{(n)} \leftarrow \ldots$ as given in Lemma 58. Recall, we have a transitive permutation group F on the set $\{1, \ldots, d\}$ and set $B < F$ the stabilizer of 1. We write $D := \{2, 3, \ldots, d\}$ and have $\Delta_1 = D \times \{1, 2\}$, respectively $\Delta_1 = \{1, \ldots, d\}$. Further we define $G^{(1)} = B^{\{1,2\}} \rtimes \mathrm{Perm}(\{1, 2\})$, respectively $G^{(1)} = F$ acting on Δ_1 and recursively, for $n \geq 1$, $G^{(n+1)} = B^{\Delta_n} \rtimes G^{(n)}$ acting on the set $\Delta_{n+1} = D^n \times \Delta_1$.

In the following we assume that B is perfect and equal to its normalizer. Write $D = W_1 \sqcup \ldots \sqcup W_r$ the decomposition of D into B-orbits Note that the condition for B of being equal to its normalizer is equivalent to the statement that for all i, $|W_i| \geq 2$.

The $G^{(n)}$-orbits on Δ_n are given by $W_{i_{n-1}} \times \ldots \times W_{i_0} \times \{1, 2\}$ for all choices of $(i_0, \ldots, i_{n-1}) \in \{1, \ldots, r\}^n$, respectively $W_{i_{n-1}} \times \ldots \times W_{i_1} \times \Delta_1$ for all choices of $(i_1, \ldots, i_{n-1}) \in \{1, \ldots, r\}^{n-1}$, and thus there are r^n respectively r^{n-1} orbits.

Here we fix a notation: If a group H acts on a set E and if $F \subseteq E$, then we write $H(F) = \{h \in G \mid hf = f, \text{ for all } f \in F\}$ in analogy to the notation used earlier.

Given an $G^{(n)}$-orbit $O_n = W^{(n-1)} \times \ldots \times W^{(0)} \times \{1,2\}$, respectively $O_n = W^{(n-1)} \times \ldots \times W^{(1)} \times \Delta_1$ in Δ_n, we compute the O_{n+1} fixing subgroup $G^{(n+1)}(O_{n+1})$ of $G^{(n+1)}$:

$$\begin{aligned} G^{(n+1)}(O_{n+1}) &= \left\{(f,g) \in B^{\Delta_n} \rtimes G^{(n)} \mid g|_{O_n} = id, f(p) \in B(W^{(n)}), \forall p \in O_n\right\} \\ &= \left(B(W^{(n)})^{O_n} \times B^{\Delta_n \setminus O_n}\right) \rtimes G^{(n)}(O_n) \end{aligned}$$

and we have $G^{(1)}(O_1) = B(W^{(0)})^{W^{(0)} \times \{1,2\}} \times B^{(D \setminus W^{(0)}) \times \{1,2\}}$, respectively $G^{(1)}(O_1) = G^{(1)}(\Delta_1) = (e)$ (and hence $G^{(2)}(O_2) = B(W^{(1)})^{\Delta_1}$).

We study now the case *1(b)ii* of the proposition 62:

We write the set of fixed points

$$FP_n(O_n) = \left\{p \in \Delta_n \setminus O_n \mid p \text{ is fixed by } G^{(n)}(O_n)\right\}$$

and we have $FP_1(O_1) = \Delta$, $FP_2(O_2) \subset \Delta_2 \setminus O_2$. For $n \geq 3$, we have $\Delta_n \setminus O_n = \left((D \setminus W^{(n-1)}) \times O_{n-1}\right) \sqcup \left(D \times (\Delta_{n-1} \setminus O_{n-1})\right)$. Since B has no fixed point in D, one dets $FP_m(O_n) \subset (D \setminus W^{(n-1)}) \times O_{n-1}$ for $n \geq 3$. We observe also that $FP_m(O_n)$ is $G^{(n)}$-invariant and thus consists of at most r, respectively $(r-1)$ orbits.

Lemma 63. *Let $H = (H_1^{L_1} \times \ldots \times H_l^{L_l}) \rtimes A$, where L_i are A-orbits, and let N be a normal subgroup of G with simple non-abelian quotient C. Then, either N contains $(H_1^{L_1} \times \ldots \times H_l^{L_l})$ and $A/pr_A(N) \simeq C$, or $N = (H_1^{L_1} \times \ldots \times U^{L_i} \times \ldots \times H_l^{L_l}) \rtimes A$ with $|L_i| = 1$, U a normal subgroup of H_i and $H_i/U \simeq C$.*

Proof. If $pr_A(N) \neq A$, then, $A/pr_A(N)$ being a non-trivial quotient of C, the subgroup N contains $H_1^{L_1} \times \ldots \times H_l^{L_l}$. If $pr_A(N) = A$, then $(H_1^{L_1} \times \ldots \times H_l^{L_l})/(H_1^{L_1} \times \ldots \times H_l^{L_l} \cap N) \simeq C$, and since C is simple non-abelian, there exists $i \in \{1, \ldots, l\}$ and a normal subgroup U of H_i with $H_i/U \simeq C$, and $N \cap (H_1^{L_1} \times \ldots \times H_l^{L_l}) = H_1^{L_1} \times \ldots \times H_i^{|L_i|-1} \times U^{L_i} \times \ldots \times H_l^{L_l}$. But N is normal in G, so $|L_i| - 1 = 0$. □

Lemma 64. *Let C be a simple non-abelian quotient of B, S a set and $U \subset G^{(m+1)}(O_{m+1})$ such that U is normal in $G^{(m+1)}$, $G^{(m+1)}(O_{m+1})/U \simeq C^S$, the $G^{(m+1)}$-conjugation acting transitively on simple factors of C^S, and $|S| > |O_m|$. Then either U contains $B(W^{(m)})^{O_m} \times B^{\Delta_m \setminus O_m}$, or there is an $G^{(m)}$-orbit S' in FP_m ($|S'| = |S|$) and a normal subgroup U_1 of B with $B/U_1 \simeq C$, such that $U = \left(B(W^{(m)})^{O_m} \times {U_1}^{S'} \times B^{\Delta_m \setminus (O_m \sqcup S')}\right) \rtimes G^{(m)}(O_m)$.*

Proof. Since $|S| > |O_m|$, we have $U = \bigcap_{s\in S} N_s$ where N_s are some normal subgroup of $G^{(m+1)}(O_{m+1})$ with quotient C, and the $G^{(m+1)}$-action by conjugation permutes transitively the set $\{N_s \mid s \in S\}$. From the preceeding lemma, we have, either N_s contains $B(W^{(m)})^{O_m} \times B^{\Delta_m \setminus O_m}$ for some and hence every s – thus U contains $B(W^{(m)})^{O_m} \times B^{\Delta_m \setminus O_m}$, or $N_s = \big(B(W^{(m)})^{O_m} \times U_s \times B^{\Delta_m \setminus (O_m \sqcup \{s\})}\big) \rtimes G^{(m)}(O_m)$ with $s \in FP_m(O_m)$ and we are done, or $N_s = \big(B(W^{(m)})^{O_m \setminus \{s\}} \times U_s \times B^{\Delta_m \setminus O_m}\big) \rtimes G^{(m)}(O_m)$. But then the $G^{(m+1)}$-action by conjugation shows that $U = \big(\prod_{s\in O_m} U_s \times B^{\Delta_m \setminus O_m}\big) \rtimes G^{(m)}(O_m)$ and $B(W^{(m)})/U_1 \simeq C$. But this is not possible since $|S| > |O_m|$. □

Now, let U be a subgroup of $G^{(n)}(O_n)$ which is normal in $G^{(n)}$ and is such that $G^{(n)}(O_n)/U \simeq C^{|O_n|}$ with transitive $G^{(n)}$-action on simple factors of $C^{|O_n|}$. As before, C is simple non-abelian. Let $m < n$ be maximal such that $pr_{G^{(n)} \to G^{(m)}}(U) = G^{(m)}(O_m)$, and set $U' = pr_{G^{(n)} \to G^{(m+1)}}(U)$. Then $U = pr^{-1}(U')$ and $U' = \big(B(W^{(m)})^{O_m} \times {U_1}^{S'} \times B^{\Delta_m \setminus (O_m \sqcup S')}\big) \rtimes G^{(m)}(O_m)$ where $S' \subset FP_m(O_m)$ is an $G^{(m)}$-orbit satisfying $|S'| = |O_n|$. Now $S' = W \times O_{m-1}$ for some $W \in \{W_1, \ldots, W_r\}$ and thus $|W| \cdot |O_{m-1}| = |O_n|$. Therefore $|O_n| = |O_{m-1}| \left|W^{(m-1)}\right| \ldots \left|W^{(n)}\right| \geq |O_{m-1}| \cdot 2n - m + 2$. Since $|O_n| \leq (d-1)\,|O_{m-1}|$, we have $2^{n-m} \leq \frac{d-1}{4}$, i.e. $m \geq n - \frac{d-1}{4 \ln 2} = n - C_1$.

Let C_2 be the number of normal subgroups of B. We have following result:

Lemma 65. *Let $C_3 = r\frac{d-1}{4\ln 2}C_2$, then for all n and all $G^{(n)}$-orbits O_n in Δ_n, there are at most C_3 subgroups U in $G^{(n)}(O_n)$, which are normal in $G^{(n)}$ and such that $G^{(n)}(O_n)/U$ is isomorph to a product of $|O_n|$ simple non-abelian groups where the $G^{(n)}$-action by conjugation acts transitively on factors.*

Therefore $n_1\big(G^{(n+1)}\big) \leq C_3$ and

$$\Sigma\big(1(b)ii\big)(\eta) \leq C_3\,|\Delta_n| \left(\frac{|\mathrm{aut}\,(B)|}{|B|^\eta}\right)^{|\Delta_n|}.$$

Now we treat the case *1(b)i*:

Let $B_{(i)} := B/B(W^{(m)})$. Then the group $\overline{G}^{(n)} := G^{(n)}/G^{(n)}(O_n)$ is described inductively by $\overline{G}^{(1)} = B_{(0)}^{O_1} \rtimes \mathrm{Perm}(\{1,2\})$, respectively $\overline{G}^{(1)} = F$ and $\overline{G}^{(m)} = B_{(m-1)}^{O_{m-1}} \rtimes \overline{G}^{(m-1)}$ acting on $W^{(m-1)} \times O_{m-1} = O_m$.

Let C be some simple non-abelian quotient of B. We need to bound the number of sections of $\overline{G}^{(n)}$ in $C^{O_n} \rtimes \overline{G}^{(n)}$; that is, the number of homomorphisms $\phi : \overline{G}^{(n)} \to C^{O_n} \rtimes \overline{G}^{(n)}, x \mapsto \big(\varphi(x), x\big)$. For $k \leq n$ we write $a(k)$ the number of such sections $\overline{G}^{(k)} \to C^{O_k} \rtimes \overline{G}^{(k)}$.

Such a section $\phi : \overline{G}^{(n)} = B_{(n-1)}^{O_{n-1}} \rtimes \overline{G}^{(n-1)} \to C^{O_n} \rtimes \big(B_{(n-1)}^{O_{n-1}} \rtimes \overline{G}^{(n-1)}\big), (x,y) \mapsto \big(\varphi(x,y),(x,y)\big)$ is determined by $\phi|_{\overline{G}^{(n-1)}}$ and $\phi|_{B_{(n-1)}^{O_{n-1}}}$. First, $\phi|_{\overline{G}^{(n-1)}}$ can be seen as a homomorphism $\overline{G}^{(n-1)} \to (C^{O_{n-1}} \times \ldots \times C^{O_{n-1}}) \rtimes \overline{G}^{(n-1)}$, where the $\left|W^{(n-1)}\right|$ factors of $C^{O_{n-1}}$ are not permuted. Hence $\phi|_{\overline{G}^{(n-1)}}$ is determined by $\left|W^{(n-1)}\right|$ sections $\overline{G}^{(n-1)} \to C^{O_{n-1}} \ltimes \overline{G}^{(n-1)}$, thus, by at most $a(n-1)^{\left|W^{(n-1)}\right|}$ possibilities for $\phi|_{\overline{G}^{(n-1)}}$. Now, we may assume $\phi|_{\overline{G}^{(n-1)}}$ fixed. Using the isomorphism $C^{O_n} \rtimes B_{(n-1)}^{O_{n-1}} \simeq C^{W^{(n-1)}} \rtimes B_{(n-1)}^{O_{n-1}}$, the homomorphism $\phi|_{B_{(n-1)}^{O_{n-1}}}$ is given by $B_{(n-1)}^{O_{n-1}} \to C^{W^{(n-1)}} \rtimes B_{(n-1)}^{O_{n-1}}, y \mapsto \big(\phi_{|O_{n-1}|}(y), \ldots, \phi_1(y)\big)$, where we see, using the action of $\overline{G}^{(n-1)}$ by conjugation, that ϕ_i are all determined by $\phi|_{\overline{G}^{(n-1)}}$ and ϕ_1. This means that we have to bound $\mathrm{Hom}\big(B_{(n-1)}^{O_{n-1}}, C^{W^{(n-1)}} \rtimes B_{(n-1)}\big)$. Let K_1 be an upper bound on the cardinality of the set of all subgroups of $C^{W^{(n-1)}} \rtimes B_{(n-1)}$, for all n and simple quotient C of B. (Since $\left|C^{W^{(n-1)}} \rtimes B_{(n-1)}\right| \leq |B|^d$, we get $K_1 \leq 2^{|B|^d}$.) But B, and hence all $B_{(i)}$, are perfect, therefore any homomorphism $\phi : B_{(n-1)}^{O_{n-1}} \to C^{W^{(n-1)}} \rtimes B_{(n-1)}$ factorizes via projection on $B_{(n-1)}^{K_1}$. Thus, if K_2 is a constant bigger than $\left|\mathrm{Hom}\big(B_{(n-1)}^{O_{n-1}}, C^{W^{(n-1)}} \rtimes B_{(n-1)}\big)\right|$, then there are at most $\binom{|O_{n-1}|}{K_1} K_2$ possibilities for $\phi|_{B_{(n-1)}^{O_{n-1}}}$. All in all we get

$$a(n) \leq a(n-1)^{|W^{(n-1)}|} \binom{|O_{n-1}|}{K_1} K_2,$$

where K_1 and K_2 are absolute constants.

We remark here that $\binom{|O_{n-1}|}{K_1} = 1$ if $|O_{n-1}| \leq K_1$.

We shall use the cruder estimate

$$a(n) \leq a(n-1)^{\left|W^{(n-1)}\right|} |O_{n-1}|^K, \quad \text{forall } n \geq 2,$$

where K is an absolute constant.

Now let $r_i := \left|W^{(i)}\right|$, for $i \in \{1, \ldots, n\}$. From the last inequality we get inductively $a(n) \leq a(1)^{r_{n-1} \cdot \ldots \cdot r_1} |O_{n-1}|^K |O_{n-2}|^{Kr_{n-1}} |O_{n-3}|^{Kr_{n-1}r_{n-2}} \cdot \ldots \cdot |O_1|^{Kr_{n-1} \cdot \ldots \cdot r_2} = a(1)^{|O_{n-1}|} t(n-1)$, with $\ln t(n-1) = K r_{n-1} \cdot \ldots \cdot r_2 \ln |O_1| + \ldots + K r_{n-1} r_{n-2} \ln |O_{n-3}| + K r_{n-1} \ln |O_{n-2}| + K \ln |O_{n-1}|$ i.e. $\frac{\ln t(n-1)}{|O_{n-1}|} = K\big(\frac{\ln|O_1|}{|O_1|} + \ldots + \frac{\ln|O_{n-1}|}{|O_{n-1}|}\big)$. Since for $x \geq 1$, $\ln x \leq 2\sqrt{x}$, we have $\frac{\ln t(n-1)}{|O_{n-1}|} \leq 2K \sum_{i=1}^{n-1} \frac{1}{|O_i|^{1/2}}$, and since for all i, the number $|W_i| \geq 2$, $|O_i| \geq 2^i$, thus $\frac{\ln t(n-1)}{|O_{n-1}|} \leq 4K$ for all n. We have the corollary:

Corollary 66. *There is an absolute constant K such that the number of sections of $\overline{G}^{(n)} \to C^{O_n} \rtimes \overline{G}^{(n)}$ is bounded by $K^{|O_n|}$.*

Under the same hypothesis than at the begining of this section we show now following proposition.

Proposition 67. *Given $\lambda \in]0,1[$, there is $\eta \geq 2$, such that for any natural number n,*

$$\zeta_{G^{(n+1)}/G^{(n)}}(\eta) \leq \lambda^{2^n}.$$

This leads to:

Corollary 68. *Let F be a transitive permutation group of the set $\{1,\ldots,d\}$ and $B = \mathrm{Stab}_F(1)$ perfect and self normalizing, then the projective limit $\varprojlim_n G^{(n)}$ is topologically finitely generated.*

Proof. Pick $\lambda \in (0,1)$ and D such that $\zeta_{G^{(n+1)}/G^{(n)}}(D) \leq \lambda^{2^n}$ for all n. Then $\prod_n \big(1 - \zeta_{G^{(n+1)}/G^{(n)}}(D)\big) \geq \prod_{n=1}^{\infty}(1-\lambda^{2^n}) > 0$. □

Now we prove the proposition.

Let M be a maximal subgroup of $G^{(n+1)} = B^{\Delta_n} \rtimes G^{(n)}$, where $\Delta_n = \sqcup_{i=1}^t L_i$ and $pr_{G^{(n)}}(M) = G^{(n)}$. If U is a maximal normal standard subgroup of B^{Δ_n} contained in M, then:

1. There exist $i \in \{1,\ldots,t\}$ and a strict normal subgroup U_i of B such that $U = B^{L_1} \times \ldots \times U_i^{L_i} \times \ldots \times B^{L_t}$ and M is the inverse image of a clean maximal subgroup in $\big(B/U_i\big)^{L_i} \rtimes G^{(n)}$. Using the results obtained so far, the contribution is bounded by $a(\eta)^{|L_i|}$, where $a(\eta) \to 0$ for $\eta \to \infty$, and $a(\eta)$ is a function which only depends on the permutation group F. Letting m be the number of normal subgroups of B, $t := r^{n-1}$ and $|L_i| \geq 2^{n-1}$, we get that the sum of the contributions of *1.*, for all $i \in \{1,\ldots,t\}$ and a strict normal subgroup U_i of B, is bounded by

$$mr^{n-1}a(\eta)^{2^n}.$$

2. There is a subset P of $\{1,\ldots,t\}$ and a strict normal subgroup U_i of B such thatB/U_i is isomorph to a simple group C and $U = \prod_{i\notin P} B^{L_i} \times \prod_{j\in P} U_j^{L_j}$. For fixed P and U, the contribution is bounded by $b(\eta)^{\sum_{j\in P}|L_j|}$. Again, $b(\eta) \to 0$ for $\eta \to \infty$. Thus the total contribution coming from *2.* is bounded by:

$$\begin{aligned}\sum_{\substack{P\subset\{1,\ldots,t\}\\|P|\geq 2}} m^{|P|}b(\eta)^{\sum_{j\in P}|L_j|} &\leq \sum_{\substack{P\subset\{1,\ldots,t\}\\|P|\geq 2}} \big(mb(\eta)\big)^{\sum_{j\in P}|L_j|} \leq \prod_{i=1}^{t}\Big(1+\big(mb(\eta)\big)^{|L_j|}\Big) - 1\\ &\leq e^{\sum_{i=1}^{t}\big(mb(\eta)\big)^{|L_j|}} - 1 \leq e^{r^{n-1}\big(mb(\eta)\big)^{2^n}} - 1\\ &\leq 2r^{n-1}\big(mb(\eta)\big)^{2^n},\end{aligned}$$

for η big enough. Summing up everything we get the proposition.

Concerning the reciprocal implication of Proposition 59, we already know that if B is not perfect, $\lim_{\leftarrow} G^{(n)}$ is not topologically finitely generated. Assume now that B is not self normalizing, or which is equivalent, that there exist a B-fixed point $\alpha \in \{2, \ldots, d\}$. Then, for all natural number n, $G^{(n)}_{(1)} := \ker(G^{(n)} \twoheadrightarrow G^{(1)})$ has a fixed point, say $\alpha_n \in \Delta_n$. So $G^{(n+1)}_{(1)} = B^{\Delta_n} \rtimes G^{(n)}_{(1)} = B \times (B^{\Delta_n \setminus \{\alpha_n\}} \rtimes G^{(n)}_{(1)})$. Thus $G^{(n+1)}_{(1)} \twoheadrightarrow B \times G^{(n)}$, which shows that $G^{(n+1)}_{(1)}$ admits B^n as quotient. Thus, again $\lim_{\leftarrow} G^{(n)}$ is not topologically finitely generated.

3.4 Super Cuspidal Representations of $U(F)$

We have seen that $U(F)$ has the independence property and acts minimaly on T. If moreover, for each $i \in \{1, \ldots, d\}$ the i-fixing subgroup F_i of F is non trivial, then by Lemma 32, page 23, since $F_i \simeq \underline{U(F)(T_e)}$ for some edge e, every complete subtree S which is neither a point nor an edge is non degenerate. Therefore, by Theorem 1, for each such S, $U(F)$ has a super cuspidal representation, and two of such representations are equivalent if and only if the two corresponding subtrees are isomorphic by an automorphim laying in $U(F)$.

Proposition 69. *Let F be a permutation group of $\{1, 2, \ldots, d\}$ with transitive action and such that F_1 is not trivial. Then for every real positive number M, the group $U(F)$ has finitely many equivalence classes of super cuspidal representations with formal degree less than M, if and only if F_1 is perfect and equal to its normalizer.*

Proof. Suppose that F_1 is not equal to its normalizer. It is easy to see that hence F_1 fixes a point. Then, fixing an edge e, one can build, using the fact that $U(F)$ has the independence property, a doubly infinite chain c such that $U(F)(e) = U(F)(c)$. Let S be the minimal complete subtree containing c and for all natural numbers $k \geq 2$ set $S_k = T_{e,k} \cap S$ and $c_k = T_{e,k} \cap c$. We have $U(F)(e) = U(F)(c_{k-1}) = U(F)(c) \subseteq \widetilde{U(F)}(S_k) \subseteq \widetilde{U(F)}(e)$. Therefore the subgroup $U(F)(c_{k-1})$ has index at most 2 in $\widetilde{U(F)}(S_k)$. The subtree S_k has exactly two maximal proper complete subtrees S_k^1 and S_k^2. Thus, by the independence property, we have (see Lemma 10, p. 10)

$$
\begin{aligned}
U(F)(c_{k-1}) &= \prod_{y \in \mathrm{som}(c_{k-1})} U(F)(F_{c_{k-1},y}) \\
&\simeq U(F)(T_{e_1}) \times H' \times U(F)(T_{e_2}),
\end{aligned}
$$

where e_1 and e_2 are the terminal edges of S_k^1 respectively S_k^2 which are not terminal edges of S_k and where

$$H' = \prod_{y \in \mathrm{som}(c_{k-2})} U(F)(F_{c_{k-1},y}).$$

By the preceding remarks,

$$\begin{aligned} U(F)(T_{e_1})/(U(F)(S_k) \cap U(F)(T_{e_1})) &\simeq U(F)(T_{e_2})/(U(F)(S_k) \cap U(F)(T_{e_2})) \\ &\simeq F_1. \end{aligned}$$

Set

$$H = U(F)(c_{k-1})/U(F)(S_k);$$

this is a normal subgroup of

$$Q(S_k) = \widetilde{U(F)}(S_k)/U(F)(S_k)$$

with index at most 2 and which we identify to

$$F_1 \times H'/(U(F)(S_k) \cap H') \times F_1.$$

In accordance with this identification, set

$$\pi = \pi_1 \otimes \pi' \otimes \pi_2$$

($\otimes$ represents the (outer) tensor product), where π_1 and π_2 are non-trivial irreducible representations of F_1 and π' is the trivial representation of H'. Take a non-degenerate irreducible sub-representation ω of $\mathrm{ind}_H^{Q(S_k)}(\pi)$. Then

$$\begin{aligned} \dim \omega &\leq [Q(S_k) : H] \cdot \dim \pi \leq 2 \dim \pi_1 \dim \pi_2 \\ &\leq 2 \left| F_1 \right|^2. \end{aligned}$$

Therefore for all $k > 1$ the representation

$$T(S_k, \omega) = \mathrm{ind}_{\widetilde{U(F)}(S_k)}^{U(F)} (\omega \circ p_{S_k})$$

is cuspidal with formal dimension

$$d = \frac{1}{\mathrm{m}\left(\widetilde{U(F)}(S_k)\right)} \dim \omega \leq \frac{2}{\mathrm{m}\left(\widetilde{U(F)}(e)\right)} \left| F_1 \right|^2.$$

Suppose that F_1 is not perfect. Then it has a non trivial character. Let χ be a such. Set $Q_n = \widetilde{U(F)}(T_{a,n})/U(F)(T_{a,n})$, the automorphism group of $T_{a,n}$ induced by $U(F)(a)$, where a is a fixed vertex. Since F is transitive, we know by Lemma 58 that for $n \geq 1$, $Q_{n+1} = F_1^{\Delta_n} \rtimes Q_n$ for the action of Q_n on Δ_n, where Δ_n denotes the set of terminal edges of the subtree $T_{a,n}$. As $Q_{k+1} = F_1^{\Delta_k} \rtimes Q_k$ and $|\Delta_k| = d(d-1)^{k-1}$, the map $(f,s) \mapsto \omega(f,s) = \prod_{\delta \in \Delta_n} \chi\big(f(\delta)\big)$ is a non trivial character of Q_{k+1}. Therefore the formal degree of the super cuspidal representation $T(T_{a,k+1}, \omega)$ is $d = \frac{1}{\mathrm{m}(U(F)(x))}$.

To show the reverse sense of the statement, let us suppose that F_1 is perfect and equal to its normalizer, i.e. that it does not fix any point and has no non-trivial characters.

Let S be a complete finite subtree. Recall that for an edge or a vertex a and a natural number m, we write $T_{a,m}$ the minimal subtree containing the vertices at distance m of a; these subtrees are complete, as it is obvious to see.

Let $T_{a,k}$ be the minimal one among all $T_{a,m}$'s containing S. It is enough to show that for $k \geq 2$ the cuspidal representation π of $G = U(F)$ with minimal subtree S has formal degree greater equal $\frac{k}{\mathrm{m}(G(x))}$.

Let $x_1, \ldots, x_n$ be the terminal vertices of S which are also terminal vertices of $T_{a,k}$, and associate to each of them the maximal complete subtree S_i of S for which x_i is not a terminal vertex. If A_i denotes the group $p_S\big(G(S_i)\big) = G(S_i)/G(S)$, then we have for every $i \in \{1, \ldots, n\}$, $A_i = F_1$.

Let π be a super cuspidal representation of $G = U(F)$ with minimal subtree S. By Theorem 1, the representation π is induced from a non degenerate irreducible representation ω of $Q(S)$, i.e. $\pi = \mathrm{ind}_{\widetilde{G}(a)}^{G}\left(\mathrm{ind}_{G(S)}^{\widetilde{G}(a)}(\omega \circ p_S)\right)$. The restriction $\mathrm{Res}_{A_1 \times \ldots \times A_n}(\omega)$ of ω on the normal subgroup $A_1 \times \ldots \times A_n$ has irreducible factors which are tensor products $\rho_1 \otimes \ldots \otimes \rho_n$, where ρ_i are irreducible representations of A_i. Since the representation ω is non degenerate, there exists for every i at least one of these factors in which the corresponding ρ_i is non trivial and hence, since $F_1 \simeq A_1$ has no non trivial characters, is of dimension at least 2. Therefore $\dim \omega = \dim\big(\mathrm{Res}_{A_1 \times \ldots \times A_n}(\omega)\big) \geq 2n$ and $\dim\big(\mathrm{ind}_{G(S)}^{\widetilde{G}(a)}(\omega \circ p_S)\big) \geq 2n[\widetilde{G}(a) : \widetilde{G}(S)]$ and finally

$$\dim \pi \geq \frac{2n}{\mathrm{m}\left(\widetilde{G}(a)\right)}[\widetilde{G}(a) : \widetilde{G}(S)].$$

Let us examine the index $[\widetilde{G}(a) : \widetilde{G}(S)]$.

We write Δ_k for the terminal vertices of the subtree $T_{a,k}$ and fix $x \in \{x_1, \ldots, x_n\}$. Let

$O_x = \widetilde{G}(S)x$ be the orbit by $\widetilde{G}(S)$ and $\Omega_x = \widetilde{G}(a)x$ the orbit by $\widetilde{G}(a)$ of x. We have the inclusions $O_x \subseteq \widetilde{G}(a)x = \Omega_x \subseteq \Delta_k$.

Define the distance of a vertex s to a set of vertices E by

$$\mathrm{d}\,(s, E) = \min\{\mathrm{d}\,(s, E) \mid e \in E\}.$$

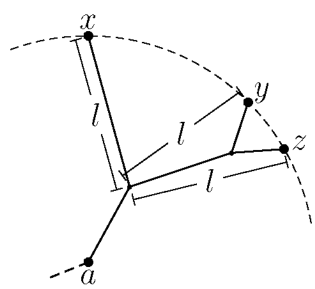
For $l \in \{0, \ldots, k\}$ we consider now the set $\{z \in \Omega_x \mid \mathrm{d}\,(z, O_x) = 2l\}$.

Let $l_1, \ldots, l_r \in \{0, 1, \ldots, k\}$ be such that $\{z \in \Omega_x \mid \mathrm{d}\,(z, O_x) = 2l_i\}$ is not empty (notice that $r \geq 1$, since $\mathrm{d}\,(x, O_x) = 0$.). Pick $z_i \in \{z \in \Omega_x \mid \mathrm{d}\,(z, O_x) = 2l\}$ and $g_i \in \widetilde{G}(a)$ with $g_i x = z_i$. Then we have $\widetilde{G}(a)x = \Omega_x \supseteq \bigsqcup_{i=1}^r \widetilde{G}(S)z_i = \bigsqcup_{i=1}^r \widetilde{G}(S)g_i x$, which implies the inclusion $\widetilde{G}(a) \supseteq \bigsqcup_{i=1}^r \widetilde{G}(S)g_i$. Therefore

$$[\widetilde{G}(a) : \widetilde{G}(S)] \geq r \geq 1.$$

On the other hand, suppose $l \in \{0, 1, \ldots, k\}$ such that $\{z \in \Omega_x \mid \mathrm{d}\,(z, O_x) = 2l\}$ is empty. Since F_1 does not fix any point, the set $\{y \in \Omega_x \mid \mathrm{d}\,(y, x) = 2l\}$ is non empty. Therefore we can take a $z \in \Omega_x$ such that $\mathrm{d}\,(z, x) = 2l$. We have $\mathrm{d}\,(z, O_x) < 2l$ which means that there exists a $y \in O_x$ with $\mathrm{d}\,(z, y) < 2l$.

But it is clear (see figure) that in this case $\mathrm{d}\,(x, y) = 2l$. Therefore we have found for each $l \in \{0, 1, \ldots, k\} \backslash \{l_1, \ldots, l_r\}$, a vertex $y \in O_x$ with $\mathrm{d}\,(x, y) = 2l$, and hence the orbit O_x, which is a subset of $\{x_1, \ldots, x_n\}$, has at least $k + 1 - r$ elements, i.e. $n \geq k + 1 - r$.



Finally, we have, since $1 \leq r \leq k$,

$$\dim \pi \geq \frac{2n}{\mathrm{m}\left(\widetilde{G}(a)\right)}[\widetilde{G}(a) : \widetilde{G}(S)] \geq \frac{2}{\mathrm{m}\left(\widetilde{G}(a)\right)}(k + 1 - r)r \geq \frac{k}{\mathrm{m}\,(G(x))}.$$

This proves the proposition. □

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