Particle simulation of fluid-solid interactions

Author(s):
Sidler, Hansjörg

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Semester Thesis

**Particle Simulation of Fluid-Solid Interactions**

Hansjörg Sidler  
Riedtlistrasse 72  
8006 Zürich  
sidlerha@student.ethz.ch

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Advisor: Simone E. Hieber  
Group: Prof. Petros Koumoutsakos
Abstract

We present a particle solution of a unified mathematical formulation of fluid-solid interactions in one and two dimensions. Our mathematical model involves the Navier-Stokes equation of compressible viscous fluid and the constitutive law of a linear elastic solid. The governing equations coupling the fluid and the solid model are derived from first principles of continuum mechanics. The resulting Partial Differential Equations are solved using a Lagrangian particle method. To demonstrate the performance of the particle simulation we consider various cases testing the behavior of the fluid-solid interface exposed to normal and shear stresses. The particle solution compares well with the analytic solution in all cases which are analytically solvable.
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1 Introduction

Most biological flows are characterized by

- irregular geometry
- instationary flow conditions
- relatively low Reynolds numbers
- non-Newtonian fluid properties

Analytical solutions are usually not possible for such problems, therefore numerical methods have to be used. Describing fluid-solid interactions with Smoothed Particle Hydrodynamics (SPH) is a promising way in order to avoid dealing with two different methods, one describing the fluid (CFD), the other the solid part (FEA), both requiring adaptive meshing.

Describing fluid-solid interactions is extremely important in medical applications for example virtual surgery, since approximately eighty percent of the human body consists of water and most vitals are filled with fluid. G.H. Cottet [1] developed a way to describe a one dimensional fluid-solid interaction with a single equation by making use of the force equilibrium at the interface. The goal of my Semester Thesis is to carry on the work by Cottet and Hieber [2]. Adding a time dependent density and a pressure term to the one dimensional model is the first step. Expanding the model to two dimensions is the main goal of this work. These techniques may be used in the future to simulate virtual surgery. Up to now most virtual surgery tools are based on mass-spring models. Algorithms based on mass-spring models are very fast, and can be used for real-time applications. These algorithms are physically based but calculations are reduced to a minimum in order to save computing time.

It is also quite difficult to describe a physically correct interaction between fluid and solid with a mass-spring model. By using the Navier-Stokes equations to describe the fluid and expecting the solid to be linear elastic, we try to correctly model the interactions in the multi-physical system.
2 Remeshed Smoothed Particle Hydrodynamics (rSPH)

2.1 Smoothed Particle Hydrodynamics (SPH)

In Remeshed Smoothed Particle Hydrodynamics, the flow field is described by a finite number of particles, which are characterized by a smooth scalar kernel \( W(r, h) \) with finite support (core radius \( h \)). Any quantity, \( A \) (mass, velocity, density) is interpolated at location \( r \) by a weighted sum of contributions from all particles.

\[
A_S(r) = \sum_j m_j \frac{A_j}{\rho_j} W(x - x_j, h) \tag{2.1}
\]

\( m_j \)=mass of particle \( j \) ; \( \rho_j \)=density ; \( x_j \)=position ; \( A_j \)=field quantity at location \( x_j \) ; subscript \( S \) denotes a ”smoothed” quantity.

In this study, a quartic spline kernel of second order of accuracy is used where \( h \) is also the particle spacing \([2]\).

\[
W(x, h) = \frac{2}{h} \begin{cases} 
-\frac{s^4}{6} & 0 \leq s < \frac{1}{2}, \\
\frac{s^4}{4} - \frac{5s^2}{6} - \frac{115}{192} \frac{s^2}{4} + \frac{5s}{24} + \frac{55}{96} & \frac{1}{2} \leq s < \frac{3}{2}, \\
\frac{s^2}{2} & \frac{3}{2} \leq s < \frac{5}{2}, \\
0 & s \geq \frac{5}{2}.
\end{cases}
\]

2.2 Remeshing

To secure the convergence of the method, the particle map must remain regular throughout the simulation. This is achieved by a reinitialization scheme (remeshing) in which the momentum of the particle is redistributed onto a uniform set of particles with spacing \( h \). In our simulations remeshing is only applied on the fluid particles.
3 One dimensional Model

3.1 Model by Cottet

Cottet’s [1] model describes the fluid with the Burger’s equation and the solid with a one-dimensional wave equation. The Burger’s equation is a one-dimensional approximation of the Navier Stokes equations which neglects pressure gradients. From a more engineering point of view Cottet’s governing equations look as follows. The fluid velocity $u$ and the solid displacement $d$ are governed by

Fluid described by Burger’s equation

$$\frac{\partial u}{\partial t} + \frac{3}{2} u \frac{\partial u}{\partial x} - \frac{\mu}{\rho \frac{\partial^2 u}{\partial x^2}} = 0 \quad \text{for } x \in [-0.5, \gamma(t)] \quad \rho = 1$$

(3.1)

Solid described by one-dimensional wave equation

$$\frac{\partial^2 d}{\partial t^2} - E \frac{\partial^2 d}{\partial \xi^2} = 0 \quad \text{for } \xi \in [0, 0.5]$$

(3.2)

Interface condition: equilibrium of stress at $\gamma(t) = d(0, t)$

$$\mu \frac{\partial u(\gamma(t), t)}{\partial x} = E \frac{\partial d(0, t)}{\partial \xi}$$

(3.3)

Where $\mu$ is the fluid viscosity coefficient and $E$ stands for the coefficient of elasticity. $\xi$ is the Langrangian variable for the solid which is related to the Eulerian coordinate by $x(\xi, t) = \xi + d(\xi, t)$.

As boundary conditions we assume that both fluid and solid stay at rest at their outer limits. Therefore

$$u(-0.5, t) = d(0.5, t) = 0$$

(3.4)

The single equation Cottet [1] derived from the equations (2.1-3) describing both materials over the Intervall $[-0.5,0.5]$ is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{2} u \frac{\partial u}{\partial x} \chi^F + \left( \chi^F + \frac{\partial x}{\partial \xi} \chi^S \right) \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\rho \frac{\partial x}{} \chi^F} + E \frac{\partial d}{\partial \xi} \chi^S \right) \quad ; \quad \rho = 1$$

(3.5)

where $\chi^F$ and $\chi^S$ are either one or zero depending on the position where we evaluate the function. In the fluid domain $\chi^F$ is one and $\chi^S$ is zero and vice versa. The initial conditions of our problem are set as follows.

$$d(\xi, 0) = 0 \quad \text{for } \xi \in [0, 0.5]$$

$$u(x, 0) = \frac{\partial d(\xi, 0)}{\partial t} = -0.1(\cos(2\pi x) + 1) \quad \text{for } x \in [-0.5, 0.5]$$

(3.6)
3.2 Expanding Burger’s equation to a one dimensional Navier-Stokes equation

We now extend the Burger’s equation used by Cottet to describe the fluid with a time dependent density and a pressure gradient, which leads us to the full one dimensional Navier-Stokes equation.

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{4}{3} \nu \frac{\partial^2 u}{\partial x^2} \quad \text{for } x \in [-0.5, \gamma(t)]
\] (3.7)

Where \( \nu \) is the kinematic viscosity (\( \nu = \frac{\mu}{\rho} \)). The factor \( \frac{4}{3} \) comes from Stokes’ Postulate [4], since we simulate a compressible fluid. Although equation (3.7) is formally only valid for monatomic gases, it is generally assumed valid in most analytical and computational research [4].

The solid displacement is still described by a one-dimensional wave equation.

\[
\rho \frac{\partial^2 d}{\partial t^2} - E \frac{\partial}{\partial x} \frac{\partial d}{\partial \xi} = 0 \quad \text{for } \xi \in [0, 0.5]
\] (3.8)

The equilibrium of stress equation at the interface has now a further term, since the pressure in the fluid always acts on the interface.

\[-p(\rho(\gamma(t), t)) + \mu \frac{\partial u(\gamma(t), t)}{\partial x} = E \frac{\partial d(0, t)}{\partial \xi}\]

(3.9)

Thus it appears that the solid has to be pre-stressed in order to satisfy the equilibrium of stress equation at the interface. The derivative \( \frac{\partial u}{\partial x} \) is zero at \( t = 0 \), therefore an initial displacement is needed which will balance the pressure in the fluid.

We derive a single equation describing both fluid and solid with the same approach Cottet did.

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \frac{4}{3} \mu \frac{\partial u}{\partial x} - p \right) \chi^F + E \frac{\partial d}{\partial \xi} \chi^S
\]

(3.10)

The initial conditions are now different from the case Cottet [1] and Hieber [2] stated, since the solid has to be pre-stressed at \( t = 0 \). Pre-stressing the solid is described in detail in section 3.2.1.

\[
d(\xi, 0) = -\frac{p}{E} \xi + \frac{1}{2} \frac{p}{E} \quad \text{for } \xi \in [0, 0.5]
\]

\[
u(x, 0) = \frac{\partial d(\xi, 0)}{\partial t} = -0.1(\cos (2\pi x) + 1) \quad \text{for } x \in [-0.5, 0.5]
\] (3.11)

The boundary conditions remain the same.

\[
u(-0.5, t) = d(0.5, t) = \frac{\partial d(0.5, t)}{\partial t} = 0 \quad \text{for } t \geq 0
\]

(3.12)
### 3.2.1 Calculating the initial displacement

As stated in equation (3.9) we have a new term $p(\rho(\gamma(t), t))$ in the stress-balance at the interface. In order to satisfy this condition, we have to press-stress the solid material to obtain an equilibrium of stress at the interface from the very start ($t = 0$) Fig. 3.1.

From the ideal gas equation we can derive that the pressure is linearly dependent on the density:

\[ pV = nRT \]  \hspace{1cm} (3.13)

\[ p = \frac{RT}{MW \rho} \]  \hspace{1cm} (3.14)

\[ p = k\rho \quad k=\text{const} \]  \hspace{1cm} (3.15)

T=Temperature, R=Ideal Gas Constant, MW=Molecular Weight, $n=$Number of moles.

Knowing the actual value of the pressure at $t = 0$ we can now calculate the needed initial displacement in order to fulfill the equilibrium of stress condition at the interface.

We derive the proper initial displacement by solving the following equations.

\[ x = \xi + d(\xi) \]

\[ \sigma = E \frac{\partial d(\xi)}{\partial \xi} = -p \]

ODE: \[ \frac{\partial d(\xi)}{\partial \xi} = - \frac{p}{E} \]
solution: \( d(\xi) = -\frac{p}{E} \xi + C \quad C=\text{const} \)

BC: \( d(0.5, 0) = 0 \)

\[
\begin{align*}
d(\xi, 0) & = -\frac{p}{E} \xi + \frac{1}{2} \frac{p}{E} \\
x(\xi) & = \xi - \frac{p}{E} \xi + \frac{1}{2} \frac{p}{E} \\
\xi & = \frac{x}{1 - \frac{p}{E}}
\end{align*}
\]

Assuming that \( \rho=1; \ k=0.1 \) therefor \( p=0.1 \) and \( E=1 \) the initial displacement looks as follows.

![Figure 3.2: pre-stressed solid](image)

### 3.3 Remeshed Smoothed Particle Hydrodynamics

#### 3.3.1 Particle discretization of governing equations

Particle position \( x_p \), density \( \rho_p \) and velocity \( u_p \) evolve by the following ordinary differential equations derived from equation (3.10):
\[ \text{position: } \frac{dx_p}{dt} = u_p \quad (3.16) \]

\[ \text{density: } \frac{d\rho_p}{dt} = -\rho \left( \frac{\partial u}{\partial x} \right)_p \quad (3.17) \]

\[ \text{velocity: } \rho \frac{d\mathbf{u}_p}{dt} = D_p \quad (3.18) \]

\[ \text{one sided d.: } D_p = \left[ \frac{\partial}{\partial x} \left( \frac{4}{3} \mu \left( \frac{\partial u}{\partial x} \right)_p - p \right) \chi^F + E \left( \frac{\partial d}{\partial \xi} \right)_p \chi^S \right]_p \quad (3.19) \]

\[ \text{aniso. diff.: } D_p = \chi^F + \left( \frac{\partial x}{\partial \xi} \right)_p \chi^S \]

\[ \left[ \left( \frac{\partial}{\partial x} \chi^F \left( \frac{4}{3} \mu \left( \frac{\partial u}{\partial x} \right)_p - p \right) \right)_p + \left( \frac{\partial \xi}{\partial x} \right)_p \left( \frac{\partial E \chi^S \frac{\partial d}{\partial \xi}}{\partial \xi} \right)_p \right] \quad (3.20) \]

There are two different approaches how equation (3.10) can be solved. One approach uses first derivatives only, as shown in equation (3.19), where \[ \left( \frac{\partial u}{\partial x} \right)_p \chi^F \] and \[ \left( \frac{\partial d}{\partial \xi} \right)_p \chi^S \] are one-sided derivatives within the appropriate domain. Using one-sided derivatives an additional interface condition is mandatory, thus \[ u(\gamma(t), t) = \partial d(\gamma(t), t) \quad (3.20) \]. The interface velocity and force are approximated by the average of the two adjacent particles to the interface and assigned to these particles. The advantage of the \textbf{rSPH with One Sided Differentiation} is the ease of handling the interface. With this approach we generated promising results.

The other approach is called \textbf{rSPH with anisotropic diffusion} shown in equation (3.20) which uses second derivatives. With the anisotropic approach it is necessary to include the Jacobian \[ \left( \frac{\partial x}{\partial \xi} \right)_p \]. As stated in [2] the Jacobian of the flow mapping is approximated by \[ \left( \frac{\partial x}{\partial \xi} \right)_p = \frac{v_a}{h} \]. In this approach the anisotropic diffusion terms are solved using the method of particle strength exchange (PSE) for anisotropic diffusion [1, 2]. This approach does not work as good as the one-sided differentiation approach, mainly because the extrapolation of the displacement is to imprecise to gain a good solution.

\subsection*{3.3.2 Boundary Conditions}

The no-slip boundary condition at \( x = -0.5 \) is obtained by using ghost particles with an extrapolated velocity, using the theorem of intersecting lines. The solid boundary conditions at \( x = 0.5 \) is complied with using ghost particles with an extrapolated displacement.
3.4 Results 1D

The motivation in beginning with a one dimensional problem was to get a feel for the problem and to figure out, whether to use anisotropic diffusion or one-sided differentiation for the two dimensional case, which will follow the one dimensional analysis. Another objective was to implement the density evolution and the full Navier Stokes equation, since [1, 2] describe the Fluid with the Burger’s equation, which is a one dimensional approximation of the Navier Stokes equation.

3.4.1 Numerical Aspects

The viscosity $\mu$ describes the character of the fluid while the Young’s modulus $E$ specifies the solid. The viscosity in the simulation is set to $\mu = 0.01$ while the Young’s modulus is defined to be $E = 1$. 300 Particles are initialized to describe the fluid-solid domain. A fourth order Runge-Kutta scheme with a time-step of 0.00033 is used for time integration of equation (3.18).

Remeshing of the fluid takes place after every time-integration.

3.4.2 rSPH with one-sided differentiation

It is quite difficult to assess the simulation quality, since we neither have a analytical solution to compare our results with nor a benchmark simulation. The interface is handled as stated in section 3.3.1 by averaging the velocity and force terms at the interface.

**Initial Velocity $u = 0$**

Setting the initial velocity to zero the fluid and solid have to remain still since the solid is pre-stressed in order to equal out the pressure of the fluid. This simulation runs robust and correctly as shown in Figure 3.3. Therefor we can assume that the force balance of our simulation is implemented correctly.

**Initial Velocity $u = -0.1(\cos (2\pi x) + 1)$**

see Figure 3.4 for detail. The simulation behaves as expected and all boundary conditions are met. As the fluid moves to the left towards the wall, the density increases also towards the wall. On the other side, the solid becomes first unstressed and later stressed again as it overshoots its equilibrium. This results in pressure waves which are always reflected by the walls.
Figure 3.3: Velocity Profile at $t=0 ; 0.5 ; 1 ; 2$
3 One dimensional Model

Figure 3.4: Velocity Profile at $t=0; 0.5; 1; 2$ (left to right, top to bottom)
4 Two dimensional Model

We now expand the model to two dimensions. Schematically the two dimensional model looks as shown in Figure 4.1. We first verify our results obtained with the one dimensional model, and then in a further step letting the fluid flow from left to right in order to apply shear stress on the solid.

![Figure 4.1: 2D Model](image-url)
4 Two dimensional Model

4.1 Governing equations for the fluid

We already used the standard Navier Stokes equation for a monatomic gas. In vector form, the equation without the body force meaning gravitational force looks as follows.

\[ \rho \frac{du}{dt} = -\nabla p + \nabla (2\mu[S] + \zeta \nabla u[\delta]) \]  

(4.1)

The 2D strain-rate tensor is defined by

\[ \nabla [S] = \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] \begin{bmatrix} \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) & \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{bmatrix} \]  

(4.2)

The second viscosity is related to \( \mu \) by

\[ \zeta = -\frac{2}{3} \mu \]  

(4.3)

The matrix \([\delta]\) is known as the identity matrix or identity tensor

\[ [\delta] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]  

(4.4)

Therefore we can write the 2D Navier-Stokes equation (momentum equation) for compressible flow without the body force term in component form for Cartesian coordinates for each particle \( p \).

\[ \langle \rho \frac{Du}{Dt} \rangle_p = -\langle \frac{\partial p}{\partial x} \rangle_p + \frac{4}{3} \langle \frac{\partial}{\partial x} \mu \frac{\partial u}{\partial x} \rangle_p - \frac{2}{3} \langle \frac{\partial}{\partial x} \mu \frac{\partial v}{\partial y} \rangle_p + \langle \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} \rangle_p + \langle \frac{\partial}{\partial y} \mu \frac{\partial v}{\partial x} \rangle_p \]  

(4.5)

\[ \langle \rho \frac{Dv}{Dt} \rangle_p = -\langle \frac{\partial p}{\partial y} \rangle_p + \frac{4}{3} \langle \frac{\partial}{\partial y} \mu \frac{\partial v}{\partial y} \rangle_p - \frac{2}{3} \langle \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial x} \rangle_p + \langle \frac{\partial}{\partial x} \mu \frac{\partial u}{\partial x} \rangle_p + \langle \frac{\partial}{\partial x} \mu \frac{\partial v}{\partial x} \rangle_p \]  

(4.6)

4.2 Governing equations for the solid

The solid is described through linear elastic behavior in plane stress. All its properties can be derived through the basic equations of continuum mechanics.

continuums equilibrium condition 2D:

\[ \sigma_{11,1} + \tau_{12,2} + f_1 = \rho \frac{Du}{Dt} \]  

(4.7)

\[ \tau_{21,1} + \sigma_{22,2} + f_2 = \rho \frac{Dv}{Dt} \]  

(4.8)
constitutive model 2D:

\[ \sigma_{ij} = 2\mu_s\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \quad (4.9) \]

\[ \mu_s = G = \frac{E}{2(1 + \nu)} \quad (4.10) \]

\[ \lambda = \frac{\nu E}{(1 - 2\nu)(1 + \nu)} \quad (4.11) \]

kinematic relations 2D:

\[ \varepsilon_{11} = d_{1,1} \quad (4.12) \]

\[ \varepsilon_{22} = d_{2,2} \quad (4.13) \]

\[ \varepsilon_{12} = \frac{1}{2}(d_{1,2} + d_{2,1}) \quad (4.14) \]

4.3 Equilibrium of stress at the interface

The stress at the fluid interface is expressed by

\[ \sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla u \quad (4.15) \]

\[ A = \frac{2}{3}\mu \left( 2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \quad (4.16) \]

\[ \sigma_{yy} = -p + \frac{2}{3}\mu \left( 2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \quad (4.17) \]

\[ \tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (4.18) \]

Whereas the stress at the solid interface is defined by

\[ \sigma_{\xi\xi} = 2\mu_s\varepsilon_{\xi\xi} + \lambda(\varepsilon_{\xi\xi} + \varepsilon_{\eta\eta}) \quad (4.19) \]

\[ \sigma_{\eta\eta} = 2\mu_s\varepsilon_{\eta\eta} + \lambda(\varepsilon_{\xi\xi} + \varepsilon_{\eta\eta}) \quad (4.20) \]

\[ \tau_{\xi\eta} = 2\mu_s\varepsilon_{\xi\eta} \quad (4.21) \]

Making use of the kinematic relations leads to

\[ \sigma_{\xi\xi} = 2\mu_s \frac{\partial d_{\xi}}{\partial \xi} + \lambda \left( \frac{\partial d_{\xi}}{\partial \xi} + \frac{\partial d_{\eta}}{\partial \eta} \right) \quad (4.22) \]

\[ \sigma_{\eta\eta} = 2\mu_s \frac{\partial d_{\eta}}{\partial \xi} + \lambda \left( \frac{\partial d_{\xi}}{\partial \xi} + \frac{\partial d_{\eta}}{\partial \eta} \right) \quad (4.23) \]

\[ \tau_{\xi\eta} = \mu_s \left( \frac{\partial d_{\xi}}{\partial \eta} + \frac{\partial d_{\eta}}{\partial \xi} \right) \quad (4.24) \]
4 Two dimensional Model

4.3.1 Interface condition

Figure 4.2: Stresses at Interface

If we cut the interface vertically we can apply a simple force balance and get the following results.

\[
-p + \frac{2}{3} \mu \left( -\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) = 2 \mu_S \frac{\partial d_\eta}{\partial \xi} + \lambda \left( \frac{\partial d_\xi}{\partial \xi} + \frac{\partial d_\eta}{\partial \eta} \right)
\]

(4.25)

\[
\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu_S \left( \frac{\partial d_\xi}{\partial \eta} + \frac{\partial d_\eta}{\partial \xi} \right)
\]

(4.26)

Cutting the interface horizontally is, as shown in Figure 4.3, a little more complicated

\[
-p + \frac{2}{3} \mu \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) = 2 \mu_S \frac{\partial d_\xi}{\partial \xi} + \lambda \left( \frac{\partial d_\xi}{\partial \xi} + \frac{\partial d_\eta}{\partial \eta} \right) + \text{const}(y)
\]

(4.27)

\[
\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu_S \left( \frac{\partial d_\xi}{\partial \eta} + \frac{\partial d_\eta}{\partial \xi} \right)
\]

(4.28)

Conservation of angular momentum shows that \([\tau]\) is symmetric, which provides the following two equations \(\tau_{xy} = \tau_{yx}\), \(\tau_{\xi\eta} = \tau_{\eta\xi}\).
4. Two dimensional Model

Next we observe that

\[ \sigma_{xx} \chi^F + \sigma_{x\xi} \chi^S \text{ smooth for } x \in [-0.5, 0.5] \] (4.29)

\[ \sigma_{yy} \chi^F + \sigma_{\eta} \chi^S \text{ smooth for } y \in [-0.5, 0.5] \] (4.30)

\[ \tau_{xy} \chi^F + \tau_{x\eta} \chi^S \text{ smooth for } x \text{ and } y \in [-0.5, 0.5] \] (4.31)

4.4. Rewriting system as one equation in each direction

Applying equations (4.7) and (4.8) over the fluid-side interface leads to

\[ \rho \frac{Du}{Dt} = \frac{\partial}{\partial x} \left[ \sigma_{xx} \chi^F + \sigma_{x\xi} \chi^S \right] + \frac{\partial}{\partial y} \left[ \tau_{yx} \chi^F + \tau_{y\eta} \chi^S \right] \] (4.32)

\[ \rho \frac{Dv}{Dt} = \frac{\partial}{\partial y} \left[ \sigma_{yy} \chi^F + \sigma_{\eta} \chi^S \right] + \frac{\partial}{\partial x} \left[ \tau_{xy} \chi^F + \tau_{x\eta} \chi^S \right] \] (4.33)

4.4.1 Simplification of the model by setting the poisson ratio \( \nu = 0 \)

As a first approximation we set the poisson ratio to zero (\( \nu = 0 \)), which simplifies the equations used to describe the stresses in the solid. Rewriting equations (4.32)/(4.33) for a two dimensional flow with a poisson ratio equal to zero can be written for each particle p as
4 Two dimensional Model

x-direction:

\[
\left\langle \rho \frac{D u}{D t} \right\rangle_p = \left( \frac{\partial}{\partial x} \left\langle \left[ \sigma_{xx} \chi^F + \sigma_{\xi \xi} \chi^S \right] \right\rangle_p \right) + \\
\left( \frac{\partial}{\partial y} \left\langle \left[ \tau_{xy} \chi^F + \tau_{\eta \xi} \chi^S \right] \right\rangle_p \right)
\]

\[
\sigma_{xx} = -p + \frac{2}{3} \mu \left( 2 \left\langle \frac{\partial u}{\partial x} \right\rangle_p - \left\langle \frac{\partial v}{\partial y} \right\rangle_p \right) \tag{4.35}
\]

\[
\sigma_{\xi \xi} = E \left\langle \frac{\partial d_{\xi}}{\partial \xi} \right\rangle_p + \text{const}(y) \tag{4.36}
\]

\[
\tau_{yx} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{4.37}
\]

\[
\tau_{\eta \xi} = E \left( \frac{\partial d_{\xi}}{\partial \eta} \right)_p + \frac{\partial d_{\eta}}{\partial \xi} \tag{4.38}
\]

y-direction:

\[
\left\langle \rho \frac{D v}{D t} \right\rangle_p = \left( \frac{\partial}{\partial y} \left\langle \left[ \sigma_{yy} \chi^F + \sigma_{\eta \eta} \chi^S \right] \right\rangle_p \right) + \\
\left( \frac{\partial}{\partial x} \left\langle \left[ \tau_{xy} \chi^F + \tau_{\xi \eta} \chi^S \right] \right\rangle_p \right) \tag{4.39}
\]

\[
\sigma_{yy} = -p + \frac{2}{3} \mu \left( 2 \left\langle \frac{\partial v}{\partial y} \right\rangle_p - \left\langle \frac{\partial u}{\partial x} \right\rangle_p \right) \tag{4.40}
\]

\[
\sigma_{\eta \eta} = E \left\langle \frac{\partial d_{\eta}}{\partial \eta} \right\rangle_p \tag{4.41}
\]

\[
\tau_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{4.42}
\]

\[
\tau_{\xi \eta} = E \left( \frac{\partial d_{\xi}}{\partial \eta} \right)_p + \frac{\partial d_{\eta}}{\partial \xi} \tag{4.43}
\]

4.5 Remeshed Smoothed Particle Hydrodynamics solution

We solve the following equation by smoothed particles. Particle position \( \mathbf{r}_p \), density \( \rho_p \) and velocity \( \mathbf{u}_p \) evolve from the following ordinary differential equations.
Two dimensional Model

position: \( \frac{d\mathbf{r}_p}{dt} = \mathbf{u}_p \quad \mathbf{r}_p = \begin{pmatrix} x_p \\ y_p \end{pmatrix} \) \hspace{1cm} (4.44)

density: \( \frac{d\rho_p}{dt} = -\rho \nabla \mathbf{u}_p \quad \mathbf{u}_p = \begin{pmatrix} u_p \\ v_p \end{pmatrix} \) \hspace{1cm} (4.45)

velocity: \( \rho \frac{d\mathbf{u}_p}{dt} = \text{force}_p \) \hspace{1cm} (4.46)

The \textbf{force} terms are calculated in equations (4.34) (x-direction) and (4.39) (y-direction).

4.6 Results 2D

In two dimensions two cases were examined. The shear and normal stresses in a fluid are related to the time-rate-of-change of the deformation of the fluid element. As a result, both shear and normal stresses depend on velocity gradients in the flow. In most viscous flows, normal stresses are much smaller than shear stresses, and many times are neglected. Normal stresses become important when the normal velocity gradients (say \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial y} \)) are very large, such inside a shock wave. Therefore two cases have to be accounted for, one being normal stress the other shear stress driven.

4.6.1 First case

velocityvector: \( \mathbf{u}_p = \begin{pmatrix} u_p \\ v_p \end{pmatrix} \)

Figure 4.4: First case computed, \( u_p=0 \); \( v_p = -0.1(\cos 2\pi y + 1) \)
The first case is an extension of the one dimensional case (normal stress driven). Figure 4.4 shows the initial setup. We define a solid in the positive y-domain and a fluid in the negative y-domain. The same cosine velocity profile is used as in the one dimensional case, in order to compare the two cases. Like in the one-dimensional case it is rather difficult to evaluate the physical quality of the simulation since neither an analytical nor a benchmark simulation is at our disposal. One indication for correct results is if shear stresses in both fluid and solid remain zero for this case over the period of time. The only forces acting on the particles are normal forces in y-direction. The simulations showed that this constraint is met. Also if the initial velocity is set to zero, the particles will not move in the simulation, which is another indication that the simulation is implemented correctly.

Boundary Conditions

For a viscous fluid, the boundary condition on a surface assumes no relative velocity between the surface and the gas immediately at the surface. Therefore we have a no-slip conditions at $y = -0.5$. This boundary condition is obtained by using ghost particles with an extrapolated velocity, using again the theorem of intersecting lines. At the solid boundary $y = 0.5$ ghost particles are also used with an extrapolated displacement, in order to get a zero displacement at the solid-wall boundary.

Since one-sided differentiation is applied an additional interface condition is needed, to avoid a shift in the velocity profile. The interface velocity and force are again approximated by the average of the interface adjacent particles and assigned to these.

Numerical Aspects

The system parameters of the simulation are $\mu = 0.01$ for the viscosity of the fluid, the E-module is set to one as in the 1D simulation. For time-integration a fourth order Runge-Kutta scheme with a timestep of 0.005 was used.

Discussion

Since the first 2D case is the expanded version of the 1D case with the same initial velocity, the velocity profile after $t = 2$ must look similar. Figure 4.5 shows the final velocity profiles for both 1D and 2D. Even though the one dimensional simulation has a much higher resolution with 300 particles used and a time-step 15 times smaller than the two dimensional simulation, the final velocity profiles look very much alike.

Calculating the 2D scenario with such a high resolution as done in the 1D case is no problem with the written C++ Code, but a problem occurs when trying to visualize the obtained data with Matlab, since data gridding is very time- and memory-consuming. In Matlab the function griddata fits a surface of the form $z = f(x,y)$ to the data in the (usually) nonuniformly spaced vectors $(x,y,z)$. The surface always passes through the data points. Since our data points move with respect to time, and therefore no meshgrid
Figure 4.5: 1D/2D comparison of final velocity profile
It was observed, that the delicate part in the 2D simulations is the remeshing procedure. Generating particles very close to the boundary can still pose a problem, since the particle velocity is extrapolated to the ghost particles over the boundary to maintain the no slip condition.

### 4.6.2 Second case

The second case is much more intuitive. The setup looks as follows (Figure 4.7). In this case we have a clear understanding of what will evolve from these conditions. The fluid will perform a classical motion described as channel flow, which is also called plane Poiseuille flow. This is a two-dimensional flow between parallel plates, with a constant pressure gradient or in our case a body force $f_x$. At the interface the passing fluid will exert shear stress on the solid which will deform the solid dependent on the shear modulus of the solid.
Boundary Conditions

No-slip condition at the fluid-wall interface \((y = -0.5)\). This is done again with ghost particles using extrapolated velocities. For solid-wall interface \((y = 0.5)\) also ghost particles are used with extrapolated displacement.

The additional fluid-solid interface condition is satisfied by assigning the average velocity and force of the interface adjacent particles to these particles.

Numerical Aspects

The system parameters of the simulation are \(\mu = 0.01\) for the viscosity of the fluid, the E-module is chosen differently for each simulation. For time-integration a fourth order Runge-Kutta scheme with a timestep of 0.005 was used.

Discussion

This simulation even works without remeshing since the particles are well distributed over the whole time-period. We can see the end positions of the particles after one time-period in Figure 4.8 for different coefficients of elasticity (E-module). The demeanor of the solid for shear stressing is governed by the E-module or shear module \(G\), respectively, since \(G = \frac{E}{2(1+\nu)}\) (see also equation 4.10) or in our simulation \(G = \frac{E}{2}\) since we set the Poisson-ratio to zero. With increasing E-module the solid gets stiffer therefore less deformable which can be seen nicely in Figure 4.8.

The displacement at the interface is plotted for various Young’s moduli in Figure 4.9. As shown in Figure 4.9, the relation between the Young’s module and the displacement is described by a hyperbola, which is correct since doubling the Young’s module causes the
Figure 4.8: displacements of particles after $t=2$ for different E-moduli (not remeshed)
displacement to halve. Plotting the relationship between Young’s module and displacement double logarithmically, a straight line is supposed to show if the data obtained by our simulations is correct. Figure 4.9 shows that our simulations run correctly since the data points derived by the simulations are not far off from a straight line if plotted double logarithmically.

We know from solving the Navier-Stokes equation analytically for channel flow conditions that the velocity profile is a parabolic function which satisfies the two no-slip boundary conditions. The Navier-Stokes equation describing our fluid with a gravitational force \( g \) acting in x-direction looks as follows.

\[
\mu \frac{\partial^2 u}{\partial y^2} + \rho g = 0 \tag{4.47}
\]

Integrating twice leads to the following equation.

\[
u(y) = -\frac{\rho \mu gy^2}{2} + Ay + B \tag{4.48}
\]

With the two no-slip conditions \( u(y = 0) = 0 \) and \( u(y = -0.5) = 0 \) integration constants \( (A, B) \) and therefore the analytical velocity profile can be derived.

\[
u(y) = -\frac{\rho \mu gy^2}{2} - \frac{1}{4 \mu} gy \tag{4.49}
\]

The maximum velocity is at \( y = -0.25 \). With \( \rho = 1, \mu = 0.01 \) and \( g = 0.03 \) the maximum velocity in the channel flow comes out to be

\[
u(y = -0.25) = 0.09375 \tag{4.50}
\]

In the simulations we achieve fully-developed conditions within 15 seconds (t=15). This velocity profile is shown in Figure 4.10. The velocity at \( y = -0.25 \) derived from the simulation is \( u(y = -0.25) = 0.0953 \). This value is acceptable since there are some differences between the analytical and numerical solution. On the one hand we have a deformable solid in our simulation and not a stiff wall, on the other hand we average the velocity and force at the interface.
Young's Modulus vs. Displacements

![Young's Modulus vs. Displacements Graph](image)

Young's Modulus vs. Displacement (double logarithmic)

![Young's Modulus vs. Displacement Graph (double logarithmic)](image)

Figure 4.9: displacements at the interface for different Young’s moduli
Figure 4.10: velocity profile for fully developed channel flow

\[ u(y=-0.25)=0.0953 \]
5 Possible applications

5.1 Shear stresses acting on the vascular wall [3]

In blood vessels, shear stresses act on the vascular wall, i.e., on the endothelium. They have been shown to exhibit a decisive influence on the metabolism of endothelial cells, vascular remodeling, development of atherosclerosis, restenosis after dilatation, etc. Likewise, nerve cells which are exposed to fluid shear stresses resulting from accidental loading (e.g., during a car crash) may be injured. Accordingly, shear stress analysis is of importance in cellular biomechanics.

Under the assumption of a Newtonian fluid, the shear stress acting on a surface is given by

\[ \tau = \mu \frac{\partial v}{\partial x} \]

Figure 5.1: velocity profile in a large vessel

In a large vessel with a radius of 3mm and a typical flow velocity of 20 cm/sec, say, the shear stress acting at the vessel wall can therefore be approximated as \( \tau \approx 0.2Pa \) (assuming a viscosity \( \mu = 0.0035 \text{ kg m}^{-1}\text{sec} \)).

Numerical solutions of the Navier-Stokes equation (taking into account also non-Newtonian effects) were found for specific geometries and boundary conditions utilising CFD methods. Separation, reattachment and stagnation points were identified and wall shear stresses determined. Under pulsatile flow conditions, these points are not stationary, however. CFD methods can not take the solid movement in account, the developed rSPH method is capable of both covering the fluid and solid motion, and therefor better suited to describe shear stresses in a blood vessel, and other flows were fluid and solid interact.
5 Possible applications

Figure 5.2: left: no shear stress / right: after applying $\tau = 0.2 Pa$ for 24 hours

Figure 5.2 shows cells in the endothelium layer on the left with no shear stress applied and on the right after 24 hours continuously applying shear stress.[1]

Low shear stresses have generally been found to be associated with the formation of atherosclerosis. Atherosclerosis is a process in which deposits of fatty substances, cholesterol, cellular waste products, calcium and other substances build up in the inner lining of an artery. This buildup is called plaque. It usually affects large and medium-sized arteries.

Plaques can grow large enough to significantly reduce the blood’s flow through an artery. Plaques that rupture cause blood clots to form that can block blood flow or break off and travel to another part of the body. If either happens and blocks a blood vessel that feeds the heart, it causes a heart attack. If it blocks a blood vessel that feeds the brain, it causes a stroke.

In Figure 5.3 we can see atherosclerosis in the thoracic aorta in different stages.[2]

5.2 Fluid-Solid interactions during whiplash [3]

In low speed rear-end automobile impacts causing a sudden extension or flexion of the neck, a soft tissue injury to the neck called whiplash may occur. One of the hypothesis put forward to explain the occurrence of minor injury in such cases is that relative motions between fluid and solid components lead to high shear stresses.

Studies with pigs who were perfused with Evans blue which is bound by albumin and does therefor not pass the cellular membrane. After whiplash-type artificial loading, there is evidence of fluorescence (Evans blue) in some of the nerve cells harvested from ganglia of the cervical spine demonstrating pathological membrane leakage. This effect

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[1] Courtesy R.M. Nerem, Georgia Inst. of Technology
[2] Courtesy Institute of Biomedical Engineering (Prof.Dr.P.Niederer)
5 Possible applications

Figure 5.3: arteriosclerosis at different stages in the thoracic aorta

This effect is attributed to shear stresses due to relative motions of fluid and solid components in the ganglion.

Figure 5.4: pathologic membrane leakage

3Oertengren et al. (1996)
6 Conclusions

Describing fluid-solid interactions with a unified formulation, composed of the Navier-Stokes equation for the fluid and continuum mechanics equations for the solid, is a promising way, to obtain physically correct data without time-consuming and complex adaptive meshing. The one-dimensional lacked of proof quality wise. But the second case from the two dimensional analyses could be assessed since since the analytical solution of a Poiseuille flow is known, and turned out to be a good approximation.

After working with the one dimensional case it turned out that the one sided differentiation had many advantages. There is no Jacobian to integrate in the calculations, the interface is easy to handle and the results turned out to be more accurate.

The remeshing procedure in the two dimensional case is still not as sophisticated as it should be. It is definitely worth another effort to carry on this work and maybe introduce visco-elastic solids to the simulation.
Bibliography


[3] Peter Niederer, Institute of Biomedical Engineering University and ETH Zurich, Script ”Biofluidmechanics 2003/2004”


