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Computations in Groups Acting on a Product of Trees: Normal Subgroup Structures and Quaternion Lattices

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Abstract

Motivated by the work of Burger-Mozes and Wise, we study groups in a class of cocompact lattices in $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, the product of automorphism groups of two regular trees. From a geometric viewpoint, these groups are fundamental groups of certain finite square complexes, and therefore infinite, finitely presented and torsion-free. We are interested in their normal subgroup structures and construct examples of such groups without non-trivial normal subgroups of infinite index, groups which are non-residually finite, groups without proper subgroups of finite index, and simple groups. Moreover, we generalize a construction of quaternion cocompact lattices in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$, where p, l are two distinct odd prime numbers. To generate and analyze all these groups, we have written several computer programs with GAP.

Kurzfassung

Motiviert durch Arbeiten von Burger-Mozes und Wise untersuchen wir Gruppen innerhalb einer Klasse von kokompakten Gittern in $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, dem Produkt der Automorphismengruppen zweier regulärer Bäume. Diese Gruppen sind aus geometrischer Sicht Fundamentalgruppen von gewissen endlichen Quadratkomplexen, und deshalb unendlich, endlich präsentiert und torsionsfrei. Wir interessieren uns für die Struktur ihrer Normalteiler und konstruieren Beispiele von solchen Gruppen ohne nicht-triviale Normalteiler von unendlichem Index, Gruppen die nicht residuell endlich sind, Gruppen ohne echte Untergruppen von endlichem Index, und einfache Gruppen. Ausserdem verallgemeinern wir eine Konstruktion von quaternionischen kokompakten Gittern in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$, wobei p, l zwei verschiedene ungerade Primzahlen sind. Um all diese Gruppen zu erzeugen und analysieren, haben wir mehrere Computerprogramme mit GAP geschrieben.

Introduction

Our main goal is to study aspects related to the structure of fundamental groups of finite square complexes covered by a product of two regular trees of even degrees $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. These groups can be seen as cocompact lattices in the product $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ of automorphism groups of the trees. The original motivation for Burger, Mozes and Zimmer to study such groups was the expected analogy to the rich structure theory of irreducible lattices in higher rank semisimple Lie groups, where one has for example the remarkable (super-)rigidity and arithmeticity results of Margulis. Note that in the rank one case, a similar analogy to lattices in certain simple Lie groups led to the extensive development of the theory of tree lattices by Bass, Lubotzky and others in the last 15 years. Besides many analogies, there are also some fascinating new phenomena. We want to mention one of them, since it has a strong influence on this work. It is the construction by Burger-Mozes of an infinite family of cocompact lattices in $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ (for sufficiently large m and n), which are the first infinite groups being simultaneously finitely presented, torsion-free and simple. Moreover, these groups are CAT(0) and bi-automatic, have finite cohomological dimension, and are decomposable as amalgamated free products of finitely generated non-abelian free groups, hence are very interesting objects from many different viewpoints of infinite group theory.

We proceed now with an outline of the chapters and explain our main results and methods. Chapter 1 serves as a preparation for the following three main chapters. After giving some general preliminaries, we define a certain class of finite 2-dimensional cell complexes, called $(2m, 2n)$–complexes. Under different names, they have already been used by Burger-Mozes and Wise for many interesting constructions. These $(2m, 2n)$–complexes X have only one vertex, and the 2-cells are squares with boundary consisting of alternating horizontal and vertical edges, such that the universal cover of X is the product of two regular trees $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. Equivalently, the link of the single vertex in X is the complete bipartite graph $K_{2m,2n}$ induced by the subdivision of the edges in the 1-skeleton into m horizontal and n vertical geometric loops. We call the fundamental group $\Gamma = \pi_1(X)$ a $(2m, 2n)$–group. By construction, it is an infinite, finitely presented, torsion-free group, and a cocompact lattice in $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, where the group $\mathrm{Aut}(\mathcal{T})$ is equipped with some natural topology. Moreover, Γ acts freely and transitively on the vertices of $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. Following Burger-Mozes, we associate to Γ certain finite permutation groups. They describe the local actions of vertex stabilizers, if one projects Γ to a factor of $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$. These local groups can be easily read off from the complex X and play an important role in constructing groups Γ with interesting properties. Having in mind some analogy to lattices in higher rank semisimple Lie groups, it is not surprising that irreducibility is another important notion. We recall the definition for irreducible lattices in a product of trees and some criteria proposed by Burger-Mozes. In the remaining sections of Chapter 1, we discuss some other useful properties of $(2m, 2n)$–groups, for example the existence of amalgam decompositions, the behaviour under embeddings, or normal forms associated to a word in Γ. This has some applications to the structure of centralizers.

Groups acting on a product of trees are a rich source for examples of interesting infinite groups. The highlight was certainly the construction of finitely presented torsion-free simple groups by Burger-Mozes some years ago, thereby answering several long-standing open questions in group theory. These groups occur as index 4 subgroups of certain $(2m, 2n)$–groups. Unfortunately, since m and n have to be quite big in the given constructions, the presentations of those simple groups turn out to be very large; any of them would require more than 360000 relators. Therefore, one aim at the beginning of this work was to understand the construction of Burger-Mozes, and then to construct smaller finitely presented torsion-free simple groups, refining their methods or developing new methods. This is done in Chapter 2. Since finite index subgroups of $(2m, 2n)$–groups are already finitely presented and torsion-free, the difficult part is to find simple ones. The most natural strategy to prove that an infinite group is simple, is to show that (I) there are no non-trivial normal subgroups of infinite index, and (II) there are no proper normal subgroups of finite index. In the context of irreducible lattices in higher rank semisimple Lie groups, part (I) is true by a famous result of Margulis. He proved proper quotients Γ/N to be finite by showing that they are at the same time amenable and satisfy Kazhdan's property (T). This ingenious proof has been successfully adapted by Burger-Mozes to a class of irreducible lattices in products of trees, having highly transitive local groups, and we have constructed many explicit examples where this "normal subgroup theorem" applies. A necessary condition for part (II) is that the group is non-residually finite, i.e. the intersection of all finite index subgroups is not the trivial group. We know of two sources for non-residually finite $(2m, 2n)$–groups. One is a sufficient criterium of Burger-Mozes, the other is a concrete example of Wise. However, Wise's example has non-trivial normal subgroups of infinite index, and also all non-residually finite groups coming from the Burger-Mozes criterion have non-trivial normal subgroups of infinite index by construction. Since subgroups of residually finite groups are again residually finite, we follow the strategy of Burger-Mozes to inject a non-residually finite group into a group satisfying the normal subgroup theorem. The π_1-injection is obtained geometrically, using an appropriate embedding of the corresponding finite square complexes.

Now, such a non-residually finite group G without non-trivial infinite index normal subgroups has a subgroup H of finite index satisfying condition (II), namely the intersection of all finite index subgroups of G. If one can moreover guarantee that H still satisfies the normal subgroup theorem, then H is a simple group. Nevertheless, a major problem in general is to determine explicitly this simple subgroup H, given G. We were able to do this in some examples by taking an appropriate embedding of Wise's non-residually finite $(8, 6)$–group and using the fact that an explicit non-trivial element is known, which belongs to any finite index subgroup. This idea of construction led to a finitely presented torsion-free simple subgroup of index 4 of a $(10, 10)$–group, and to many more simple groups. Along the way, we have constructed new small $(2m, 2n)$–groups without non-trivial normal subgroups of infinite index, and new non-residually finite examples. They can be used as building blocks to improve lower bounds on m and n in several theorems of Burger-Mozes about infinite families of groups with interesting normal subgroup structures. By a slight variation of the above construction of simple groups, we also have produced a group with non-trivial normal subgroups of infinite index, but without proper finite index subgroups. Moreover, using an idea of Wise, we give an example of a finitely presented group which is not virtually torsion-free. The search for all these groups has been enormously simplified, and even made possible to some extent, by several GAP-programs we have written, in particular one which generates all $(2m, 2n)$–groups for given $m, n \in \mathbb{N}$. The same program can also be used to generate all possible embeddings of a given $(2m, 2n)$–group. We have written many more programs related to $(2m, 2n)$–group, for example one which computes local groups. They are described in Appendix B. In the remaining sections of Chapter 2, we study on the one hand an example which almost satisfies the normal subgroup theorem, give ideas how to construct and how not to construct an explicit proper infinite quotient, and on the other hand we present several other groups that are candidates for being finitely presented torsion-free simple groups, including some very small ones. According to several computer experiments, it seems reasonable to hope that some of them indeed are simple, but proofs appear to be challenging.

Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers. Using a construction based on the multiplication of Hamilton quaternions, Mozes has associated to any such pair (p, l) a cocompact lattice $\Gamma_{p,l}$ in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$, which is moreover an irreducible $(p + 1, l + 1)$–group, induced by the actions of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and $\mathrm{PGL}_2(\mathbb{Q}_l)$ on their Bruhat-Tits trees $\mathcal{T}_{p+1}$ and $\mathcal{T}_{l+1}$, respectively. Mozes originally used the groups $\Gamma_{p,l}$ to define certain tiling systems, so-called two-dimensional subshifts of finite type, and to study a resulting dynamical system. Later, the group $\Gamma_{13,17}$ appears as a building block in the construction of a non-residually finite $(196, 324)$–group and in a construction of an infinite family of finitely presented torsion-free virtually simple groups by Burger-Mozes. In Chapter 3, we first recall the definition of $\Gamma_{p,l}$. The fact that $\Gamma_{p,l}$ is a $(p+1, l+1)$–group can almost be deduced from an old result of Dickson about the existence and uniqueness of the factorization of integer quaternions. Inspired by the construction and properties of a certain cocompact lattice in $SO_3(\mathbb{R}) \times PGL_2(\mathbb{Q}_p)$ in Lubotzky's book, which was used there to generate Ramanujan graphs and to solve the Banach-Ruziewicz problem, we prove that $\Gamma_{p,l}$ is a normal subgroup of index 4 of the group (modulo its center) of invertible elements in the Hamilton quaternion algebra over the ring $\mathbb{Z}[1/p, 1/l] < \mathbb{Q}$. The same idea using overrings gives explicit realizations of $\Gamma_{p,l}$ as a subgroup of $SO_3(\mathbb{Q})$ and $PGL_2(\mathbb{C})$. Moreover, we explicitly define for each odd prime number q different from p and l, a homomorphism from $\Gamma_{p,l}$ to the finite group $PGL_2(\mathbb{Z}/q\mathbb{Z})$ and determine its image. Recently, Kimberley-Robertson have formulated a very simple conjecture for the abelianization of the groups $\Gamma_{p,l}$, based on computations in many examples. We do not know how to prove this conjecture, but can express it in terms of the number of commuting quaternions in certain generating sets. This could shed some light on the hidden nature of this conjecture. The general assumption $p, l \equiv 1 \pmod 4$ is made to guarantee the existence of a square root of -1 in the fields $\mathbb{Q}_p$ and $\mathbb{Q}_l$, respectively, which is needed in the explicit definition of $\Gamma_{p,l}$. However, by adapting several parts in the definition of $\Gamma_{p,l}$, we are able to generalize it to the case of prime numbers $p, l \equiv 3 \pmod 4$ and to the mixed case $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$. Those new groups, also called $\Gamma_{p,l}$, are subgroups of $PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_l)$, and we prove that they are $(p+1, l+1)$–group, too. In some subcases for p and l, there is a second possible definition of $\Gamma_{p,l}$, which leads to a different but similar group. The Kimberley-Robertson conjecture can be extended to all these generalized groups. They have a certain normal subgroup of index 4, a cocompact lattice in $PSL_2(\mathbb{Q}_p) \times PSL_2(\mathbb{Q}_l)$. It seems that the abelianization of this subgroup does not depend on p and l, provided that $p, l \geq 5$. Let now Γ be any $(2m, 2n)$–group. We say that the horizontal element $a \in \Gamma$ and the vertical element $b \in \Gamma$ generate the anti-torus $\langle a, b\rangle$ in Γ, if a and b have no commuting non-trivial powers. This notion was introduced by Wise, and essentially used in his constructions of the first examples of non-residually finite groups in the following three important classes: finitely presented small cancellation groups, automatic groups, and groups acting properly discontinuously and cocompactly on CAT(0)-spaces. Only few examples and no general criterion for the existence of anti-tori are known. We observe that in a commutative transitive $(2m, 2n)$–group, a and b generate an anti-torus if and only if they do not commute, in particular either $\langle a, b\rangle$ is isomorphic to the abelian group $\mathbb{Z} \times \mathbb{Z}$, or $\langle a, b\rangle$ is an anti-torus. Then we prove that the groups $\Gamma_{p,l}$ are commutative transitive, using a similar property for integer quaternions, and we therefore get plenty of anti-tori. Combining this with results on centralizers for general $(2m, 2n)$–groups, we get some interesting statements on commuting elements and anti-tori in $\Gamma_{p,l}$, as well as for integer quaternions after a transformation from $\Gamma_{p,l}$ back to $\mathbb{H}(\mathbb{Z})$. We also discuss the existence of *free* anti-tori in $\Gamma_{p,l}$, related to free subgroups in the group of invertible rational quaternions, and to free subgroups of $SO_3(\mathbb{Q})$. As a corollary, we can prove that certain pairs of integer quaternions, for example $1 + 2i$ and $1 + 4k$, do *not* generate a free group. All results and constructions of groups $\Gamma_{p,l}$ in this chapter are illustrated by many examples and very explicit computations.

In Chapter 4, we discuss miscellaneous topics related to $(2m, 2n)$–groups Γ. First, we naturally associate to Γ a finite set of unit squares, so-called Wang tiles, and prove that there always exists a doubly periodic tiling of the Euclidean plane with these tiles. As a consequence, Γ has a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. This is not clear in general for groups acting cocompactly and properly discontinuously on a CAT(0)-space. In a second section, we illustrate a result of Burger-Mozes by constructing certain examples of irreducible non-linear $(2m, 2n)$–groups. Then, we study possible connections between irreducibility, finite abelianization, and transitivity properties of the local groups, illustrated for small groups Γ. In a further section, we recall Mozes' definition of two infinite families of finite regular graphs associated to Γ. In the case of the groups $\Gamma_{p,l}$, these graphs are Ramanujan. Afterwards, we compute the growth of Γ. Although $(2m, 2n)$–groups can be algebraically very different, from a geometric viewpoint they all look the same, and therefore this computation is easy. Finally, we show that any $(2m, 2n)$–group Γ is efficient and has deficiency $m + n - mn$.

Appendix A is a big reservoir of supplementary examples. In addition, we describe explicit amalgam decompositions for several important examples of the preceding chapters.

Appendix B contains the ideas and the GAP-code for the main computer programs which led to the constructions of most examples in this work.

In Appendix C, we first compile some known lists of finite (quasi-)primitive permutation groups and then give classifications of $(2m, 2n)$–groups with respect to certain easily computable properties. It can be seen that even for small m and n there is an enormous diversity of such groups.

Starting with the question of Kuroš in 1944 on the existence of finitely generated infinite simple groups, we list in Appendix D in chronological order some important developments in the area of finitely presented simple groups and amalgams of free groups. The second part of this appendix is devoted to a review of the topology of the group of automorphisms of a regular tree.

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Chapter 1

Preliminaries, notations, definitions

In Section 1.1, we fix some general notations and provide some basic definitions, mainly concerning groups and graphs, for the convenience of the reader. Most terms should be standard and well-known. In Sections 1.2 to 1.10, we introduce some terminology and several concepts which will be extensively used in the subsequent chapters. Many ideas have been taken from the work of Burger-Mozes ([16, 17]), to some extent with modified notations. Most statements in these sections are reformulations or direct consequences of results given in [16, 17] or Wise's Ph.D. thesis ([68]), only a few results are new.

1.1 Basic definitions and notations

We divide this section into subsections on numbers, groups, permutation groups, graphs, groups acting on trees and lattices.

Numbers

We denote by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{Q}_p$ (where p is a prime number) the positive integer, non-negative integer, integer, rational, real and p-adic numbers, respectively.

Groups

The trivial group as well as the identity element in a group are denoted by "1". In the following, let G be a group, $S \subset G$ a subset, $H < G$ a subgroup, $N \lhd G$ a normal subgroup, $g, g_1, g_2, g_3 \in G$ elements and $k \in \mathbb{N}$ a positive integer. Note that all the signs $\subset$, $<$, $\lhd$ do not exclude equality here, and elsewhere in this work.

We write G/N for the quotient group, G^k for the direct product $G \times \ldots \times G$ of k copies of G and G^{*k} for the free product $G * \ldots * G$ of k copies of G. The finitely generated free group isomorphic to $\mathbb{Z}^{*k}$ is denoted by F_k.

Let $\langle S\rangle_G$ be the subgroup of G generated by the set S, and let $\langle\langle S\rangle\rangle_G$ be the *normal closure* of S in G, i.e. the smallest normal subgroup of G containing S. For a finite subset $S = \{g_1, \ldots, g_k\}$, we usually drop the brackets and write $\langle g_1, \ldots, g_k\rangle_G$ or $\langle\langle g_1, \ldots, g_k\rangle\rangle_G$. Also the subscript "$G$" is often omitted if the ambient group G is evident. We denote by $[g_1, g_2] := g_1g_2g_1^{-1}g_2^{-1}$ the *commutator* of g_1 and g_2. A group G is called *commutative transitive*, if $[g_1, g_2] = [g_2, g_3] = 1, g_1, g_2, g_3 \neq 1$, always implies $[g_1, g_3] = 1$, i.e. if the relation of commutativity is transitive on the non-trivial elements of G. The expressions $[g_1, g_2]$, where $g_1, g_2 \in G$, generate the *commutator subgroup* $[G, G]$. We write $G^{ab} := G/[G, G]$ for the *abelianization* of G. A group G is *perfect* if $G = [G, G]$, it is *simple* if 1 and G are the only normal subgroups of G and it is *residually finite* if the intersection of all normal subgroups of finite index of G is the trivial group 1. We denote by $Z(G)$ or ZG the *center* of G, i.e. the normal subgroup $\{x \in G : xg = gx \text{ for all } g \in G\}$, by $Z_G(g)$ the *centralizer* $\{x \in G : xg = gx\}$ of g and by $N_G(H)$ the *normalizer* $\{x \in G : xHx^{-1} = H\}$ of H. A subgroup H is called *proper*, if $H \neq G$, the quotient G/N is called proper if $G/N \neq G$. We write $[G : H]$ for the index of H in G, and $|G|$ for the order (if it is finite). A group is *torsion-free* if any non-trivial element has infinite order. We say that G has virtually some property (P), or *is virtually* (P), if G has a subgroup of finite index with this property (P). The groups of automorphisms, inner automorphisms and outer automorphisms of G are denoted by $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$ and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, respectively. For a finitely generated group G, let $d(G)$ be the minimal number of generators of G. If we write

$$G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l\rangle, \quad G = \langle x_1, \ldots, x_k \mid r_1 = 1, \ldots, r_l = 1\rangle$$

or $G = \langle x_1, \ldots, x_k \mid S\rangle$, where $S = \{r_1, \ldots, r_l\}$ is a finite set of freely reduced words in $F_k \cong \langle x_1, \ldots, x_k\rangle$, then the three expressions are *finite presentations* of G, and we have $G \cong F_k/\langle\langle S\rangle\rangle_{F_k}$.

Let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z} = \{0 + n\mathbb{Z}, 1 + n\mathbb{Z}, \ldots, (n-1) + n\mathbb{Z}\}$ be the cyclic group of order n (not to confuse with "n-adic integers" which will never appear in this work). We write D_n for the dihedral group of order $2n$.

Permutation groups

A very good introduction to permutation groups is the book of Dixon-Mortimer [25]. Let Ω be a non-empty set. The group of all bijections of Ω under composition of mappings is denoted by $\mathrm{Sym}(\Omega)$. If $n \in \mathbb{N}$, we write $S_n := \mathrm{Sym}(\{1, \ldots, n\})$ for the symmetric group on n letters and A_n for the alternating group, the index 2 subgroup of S_n consisting of even permutations. Let G be a *permutation group*, i.e. a subgroup $G < \mathrm{Sym}(\Omega)$. The *degree* of $G < \mathrm{Sym}(\Omega)$ is the cardinality of the set Ω. For $k \in \mathbb{N}$, the permutation group G is said to be *k-transitive* if for every pair $(\omega_1, \ldots, \omega_k)$,

$(\xi_1, \ldots, \xi_k)$ of k-tuples of distinct points in Ω, there exists an element $g \in G$ such that $g(\omega_1) = \xi_1, \ldots, g(\omega_k) = \xi_k$. Let $G < \mathrm{Sym}(\Omega)$ be a transitive (i.e. 1-transitive, according to the definition above) permutation group. A non-empty subset $\Delta \subset \Omega$ is called a *block* for G, if for each $g \in G$ either $g(\Delta) = \Delta$, or $g(\Delta) \cap \Delta$ is the empty set $\emptyset$. We say that G is *primitive* if it has no non-trivial blocks on Ω, i.e. no blocks except Ω itself and the one-element subsets $\{\omega\}$ of Ω. See Appendix C.1 for a list of all finite primitive permutation groups of even degree up to 14. A non-trivial permutation group $G < \mathrm{Sym}(\Omega)$ of a set Ω is called *quasi-primitive*, if every non-trivial normal subgroup of G (in particular G itself) acts transitively on Ω. See Appendix C.2 for a list of all quasi-primitive subgroups of S_{2n}, which are not 2-transitive, $n \leq 8$. Observe that primitive groups are quasi-primitive, and that quasi-primitive groups are transitive by definition.

Two permutation groups $G < \mathrm{Sym}(\Omega)$ and $H < \mathrm{Sym}(\Omega')$ are called *permutation isomorphic* if there exists a bijection $f : \Omega \to \Omega'$ and an isomorphism of groups $\psi : G \to H$ such that the following diagram commutes for each $g \in G$

$$\begin{tikzcd} \Omega \arrow[r, "g"] \arrow[d, "f"'] & \Omega \arrow[d, "f"] \\ \Omega' \arrow[r, "\psi(g)"] & \Omega' \end{tikzcd}$$

Graphs

For the definition of a graph, we follow the viewpoint of Serre ([64, Section 2.1]): A *graph* X is a pair of sets $(V(X), E(X))$, consisting of the *vertex* set $V(X) \neq \emptyset$ and the *edge* set $E(X)$, equipped with *origin* and *terminus* maps $o, t : E(X) \to V(X)$ and an *inverse* map $\overline{} : E(X) \to E(X)$ such that for each edge $e \in E(X)$ we have $\overline{e} \neq e$, $\overline{\overline{e}} = e$ and $o(e) = t(\overline{e})$. An edge $e \in E(X)$ is called a *loop* if $o(e) = t(e)$. A *geometric edge* is a set $\{e, \overline{e}\}$, consisting of an edge $e \in E(X)$ and its inverse edge $\overline{e}$. Let $x_1, x_2 \in V(X)$ be two vertices and let $k \in \mathbb{N}$ be a number. A *path* (of length k from x_1 to x_2) in the graph X is a sequence $(e_1, \ldots, e_k)$ of edges such that $o(e_1) = x_1$, $t(e_k) = x_2$ and $t(e_i) = o(e_{i+1})$ for each $1 \leq i < k$. The path is *without backtracking* or *reduced* if always $e_{i+1} \neq \overline{e_i}$. The graph X is said to be *connected* if given any two vertices $x_1, x_2 \in V(X)$, there is a path from x_1 to x_2. Two distinct vertices x_1 and x_2 are *neighbours*, if there is a path of length 1 from x_1 to x_2. A *circuit* (of length k) is a path $(e_1, \ldots, e_k)$ without backtracking such that $t(e_1), \ldots, t(e_k)$ are distinct vertices and $t(e_k) = o(e_1)$. Note that a circuit of length 1 is a loop. A *tree* is a connected graph without circuits. The *valency* of a vertex $x \in V(X)$ is the number of edges $e \in E(X)$ such that $o(e) = x$. A graph is called *k-regular* if each vertex has valency k. We denote by $\mathcal{T}_\ell$ the ℓ-regular tree. It has infinitely many vertices if $\ell \geq 2$. There is an obvious distance function (the *combinatorial distance*) on the set of vertices $V(\mathcal{T}_\ell)$, such that neighbours have distance 1. For a vertex $x \in \mathcal{T}_\ell$ and a number $k \in \mathbb{N}$, let $S(x,k)$ be the k-sphere, i.e. the set of vertices in $\mathcal{T}_\ell$ of combinatorial distance k from x. A *geodesic ray* in $\mathcal{T}_\ell$ is an infinite sequence $(e_1, e_2, \ldots)$ of edges $e_i \in E(\mathcal{T}_\ell)$ such that for each $i \in \mathbb{N}$ we have $t(e_i) = o(e_{i+1})$ and $e_{i+1} \neq \overline{e_i}$. Two geodesic rays are said to be equivalent if their intersection (as set of edges) is infinite. The *boundary at infinity* $\partial_\infty \mathcal{T}_\ell$ is defined as the set of equivalence classes of geodesic rays.

Let $m, n \in \mathbb{N}$. The *complete bipartite graph* $X = K_{m,n}$ is a graph where $V(X)$ is divided into two disjoint subsets $V_1(X)$ and $V_2(X)$ of cardinality m and n respectively, such that for each $e \in E(X)$ the origin $o(e)$ and the terminus $t(e)$ are in different sets $V_i(X)$ and such that given any two vertices $x_1 \in V_1(X)$, $x_2 \in V_2(X)$, there is a unique edge $e \in E(X)$ from x_1 to x_2.

Groups acting on trees

An *automorphism* ϕ of a graph X is a pair of bijective maps $\phi_1 : V(X) \to V(X)$, $\phi_2 : E(X) \to E(X)$ such that for each edge $e \in E(X)$ we have $\phi_1(o(e)) = o(\phi_2(e))$, $\phi_1(t(e)) = t(\phi_2(e))$ and $\phi_2(\overline{e}) = \overline{\phi_2(e)}$. The group of automorphism of X is denoted by $\mathrm{Aut}(X)$. Note that an element ϕ of $\mathrm{Aut}(\mathcal{T}_\ell)$ is already determined by the bijection $\phi_1 : V(\mathcal{T}_\ell) \to V(\mathcal{T}_\ell)$, so we usually understand an element in $\mathrm{Aut}(\mathcal{T}_\ell)$ as a bijective map on the vertices $V(\mathcal{T}_\ell)$ which respects the edges. We endow the set $\mathrm{Aut}(\mathcal{T}_\ell)$ with the *topology of pointwise convergence*. See Appendix D.2 for a precise definition. Informally, two elements in $\mathrm{Aut}(\mathcal{T}_\ell)$ are close with respect to this topology, if they do the same on a large set of vertices of $\mathcal{T}_\ell$. It is well-known that $\mathrm{Aut}(\mathcal{T}_\ell)$ is a locally compact, totally disconnected, second countable, metrizable Hausdorff space and a topological group (see Proposition D.1 for elementary proofs of these facts).

A group G *acts* on the regular tree $\mathcal{T}_\ell$ if there is a homomorphism $G \to \mathrm{Aut}(\mathcal{T}_\ell)$. Let $H < \mathrm{Aut}(\mathcal{T}_\ell)$ be a subgroup, $x \in V(\mathcal{T}_\ell)$ a vertex and S a subset of vertices of $\mathcal{T}_\ell$. We write $H(S)$ to denote the *pointwise stabilizer*

$$H(S) := \mathrm{Stab}_H(S) = \{h \in H : h(x) = x \text{ for each } x \in S\},$$

and use the notation $H(x) := H(\{x\})$. We say that H is *locally transitive, locally quasi-primitive, locally primitive,* or *locally 2-transitive,* if for each vertex $x \in V(\mathcal{T}_\ell)$ the stabilizer $H(x)$ induces a transitive, quasi-primitive, primitive, or 2-transitive permutation group, respectively, on the 1-sphere $S(x,1)$ (equivalently, on the set of edges with origin x). Moreover, we call H *locally ∞-transitive,* if $H(x)$ acts transitively on $S(x,k)$ for each $k \in \mathbb{N}$ and each vertex x of $\mathcal{T}_\ell$.

We recall now the definition of the universal group $U(F)$ from [16, Section 3.2] or [17, Chapter 5]. Let $\ell \geq 3$ and write here E_x for the set of edges in $\mathcal{T}_\ell$ with origin $x \in V(\mathcal{T}_\ell)$. A *legal edge coloring* is a map $i : E(\mathcal{T}_\ell) \to \{1, \ldots, \ell\}$ such that $i(e) = i(\overline{e})$ for each $e \in E(\mathcal{T}_\ell)$, and such that the restriction $i|_{E_x} : E_x \to \{1, \ldots, \ell\}$ is bijective for each $x \in V(\mathcal{T}_\ell)$. Given a permutation group $F < S_\ell$, the group

$$U(F) := \{g \in \mathrm{Aut}(\mathcal{T}_\ell) : i \circ g \circ (i|_{E_x})^{-1} \in F \text{ for each } x \in V(\mathcal{T}_\ell)\}$$

is up to conjugation in $\mathrm{Aut}(\mathcal{T}_\ell)$ independent of the legal edge coloring i, and is called the *universal group*. See [16, Section 3.2] for some properties of $U(F)$.

Lattices

Let G be any locally compact group. A subgroup $\Gamma < G$ is called a *lattice* if it is discrete and G/Γ carries a finite G-invariant measure. If moreover G/Γ is compact then Γ is a *cocompact lattice*. Our main examples for G will be $G = \mathrm{Aut}(\mathcal{T}_\ell)$ with the topology mentioned above and $G = \mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ with the product topology. Note that a subgroup $H < \mathrm{Aut}(\mathcal{T}_\ell)$ is discrete if and only if the stabilizer $H(x)$ is finite for each vertex $x \in V(\mathcal{T}_\ell)$, see Proposition D.2 for a proof.

1.2 Square complexes and $(2m, 2n)$–groups

On an intuitive level, a *square complex* is a 2-dimensional cell complex, such that the 2-cells are "squares". We want to study square complexes which have additional quite restrictive properties. They are called 1-vertex VH-T-square complexes in [17] or complete squared VH-complexes with one vertex in [68]. We will just call them $(2m, 2n)$–complexes to emphasize the parameters m and n. Before giving the precise definition, we need some preparation. Fix two numbers $m, n \in \mathbb{N}$ and let $(\{x\}, E)$ be the graph with one vertex x and $m+n$ geometric loops. We use the following notation for the edges: $E = E_h \sqcup E_v$, where

$$E_h := \{a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}\}, \ E_v := \{b_1, \ldots, b_n, b_n^{-1}, \ldots, b_1^{-1}\}$$

and $^{-1}$ stands here for the inverse map $\bar{\ }$ in a graph. The advantage of this notation will become clear when we define corresponding groups and $^{-1}$ will be the inversion in the group. We call any set $\{a_i, a_i^{-1}\}$, $i = 1, \ldots, m$, a *horizontal* geometric loop and $\{b_j, b_j^{-1}\}$, $j = 1, \ldots, n$, a *vertical* geometric loop. A *square* is an expression $aba'b'$ such that $\{a, a'\} \subset E_h$, $\{b, b'\} \subset E_v$. We visualize it as a 2-dimensional cell with oriented boundary as in Figure 1.1 (left hand side).

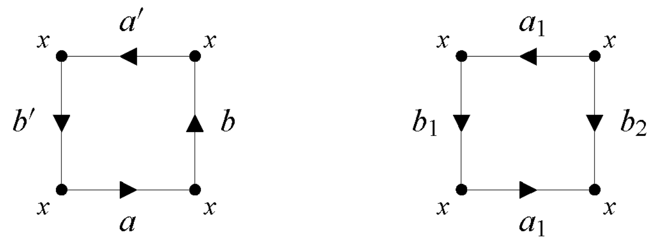


Figure 1.1: The squares $aba'b'$ and $a_1 b_2^{-1} a_1 b_1$

See the right hand side of Figure 1.1 for an explicit example of a square. If it does not matter where to start to read off the edges of the boundary, or if we identify squares that are reflected along an edge, then we are automatically led to the following definition. A *geometric square* is a set

$$\{aba'b',\ a'b'ab,\ a^{-1}b'^{-1}a'^{-1}b^{-1},\ a'^{-1}b^{-1}a^{-1}b'^{-1}\} =: [aba'b'],$$

where $\{a, a'\} \subset E_h$, $\{b, b'\} \subset E_v$. Note that

$$[aba'b'] = [a'b'ab] = [a^{-1}b'^{-1}a'^{-1}b^{-1}] = [a'^{-1}b^{-1}a^{-1}b'^{-1}].$$

Any of the four squares in the set $\{aba'b',\ a'b'ab,\ a^{-1}b'^{-1}a'^{-1}b^{-1},\ a'^{-1}b^{-1}a^{-1}b'^{-1}\}$ *represents* the geometric square $[aba'b']$. Given a non-empty set S of geometric squares, the *link* $Lk(S)$ is defined as the graph with vertex set $E = E_h \sqcup E_v$ and an edge set, where each square $aba'b'$ represented in S contributes an edge s such that $o(s) = a$, $t(s) = b'^{-1}$, and its inverse $\overline{s}$ to this edge set of $Lk(S)$. In other words, each geometric square $[aba'b']$ in S contributes four geometric edges to $Lk(S)$, corresponding to the four "corners" in any of the four squares representing $[aba'b']$. A *$(2m, 2n)$–complex* is a set X consisting of exactly mn geometric squares such that the link $Lk(X)$ is the complete bipartite graph $K_{2m,2n}$ (where the bipartite structure is induced by the decomposition $E = E_h \sqcup E_v$). This *link condition* means that given any $a \in E_h$ and $b \in E_v$, there are unique $a' \in E_h$ and $b' \in E_v$ such that $[aba'b'] \in X$. Note that this definition automatically excludes geometric squares of the form $[abab]$ (so-called *projective planes*) in a $(2m, 2n)$–complex X.

We usually think of X as a finite 2-dimensional cell complex which is built by attaching mn squares of the form $aba'b'$ to the 1-skeleton $(\{x\}, E)$, according to the labels a, b, a', b' in the squares. By the link condition, the universal covering space $\tilde{X}$ of X is the product of two regular trees $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. In fact, both conditions are equivalent, see [17, Proposition 1.1] or [68, Theorem II.1.10]. By construction, the fundamental group $\Gamma := \pi_1(X, x) < \mathrm{Aut}(\mathcal{T}_{2m} \times \mathcal{T}_{2n})$ of a $(2m, 2n)$–complex X is a finitely presented torsion-free cocompact lattice, acting freely and transitively on the vertices of $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. The decomposition $E_h \sqcup E_v$ of E guarantees that Γ does not interchange the factors of $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$, i.e. Γ is in fact a subgroup of the direct product $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n}) < \mathrm{Aut}(\mathcal{T}_{2m} \times \mathcal{T}_{2n})$. Such a group Γ will be called a *$(2m, 2n)$–group*. A finite presentation of Γ can be directly read off from X:

$$\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid aba'b' = 1,\ \text{if } [aba'b'] \in X \rangle .$$

Note that all four representatives of a geometric square $[aba'b'] \in X$ give the same relation in Γ, in particular we get a presentation of Γ with $m + n$ generators and only mn relators. We write $R_{m \cdot n}$ for such a set of mn relators. This presentation is optimal in some sense, see Section 4.6. If we give explicit examples of $(2m, 2n)$–groups Γ, we usually specify only the set $R_{m \cdot n}$, since it completely determines Γ. Observe that $\langle a_1, \dots, a_m \rangle_\Gamma$ and $\langle b_1, \dots, b_n \rangle_\Gamma$ are free subgroups of Γ, see Corollary 1.11(1).

Given a $(2m, 2n)$–group Γ by its presentation $\langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m\cdot n}\rangle$, we can always define the surjective homomorphism of groups

$$\begin{aligned} \Gamma &\to \mathbb{Z}_2^2 \\ a_i &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}), \quad i = 1, \dots, m \\ b_j &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}), \quad j = 1, \dots, n\,. \end{aligned}$$

Obviously, the kernel of this homomorphism is a normal subgroup of Γ of index 4. We always denote this subgroup by Γ_0. Geometrically, it can be seen as the fundamental group of a corresponding finite square complex X_0 with 4 vertices, a 4-fold regular covering space of X.

We define an *automorphism* of a $(2m, 2n)$–complex X as a graph automorphism of the 1-skeleton $(\{x\}, E)$ which induces a permutation on the set of geometric squares of X. The group of all such maps is denoted by $\mathrm{Aut}(X)$.

1.3 Projections and quasi-center

Let Γ be a $(2m, 2n)$–group. Since Γ is a subgroup of $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, we have two canonical projections, the homomorphisms of groups

$$\mathrm{pr}_1 : \Gamma \to \mathrm{Aut}(\mathcal{T}_{2m}) \text{ and } \mathrm{pr}_2 : \Gamma \to \mathrm{Aut}(\mathcal{T}_{2n})\,.$$

We define the two groups $H_i := \overline{\mathrm{pr}_i(\Gamma)}$, $i = 1, 2$, where the closure of $\mathrm{pr}_i(\Gamma)$ is taken with respect to the topology of $\mathrm{Aut}(\mathcal{T}_\ell)$ described in Section 1.1 or Appendix D.2. Let

$$\mathrm{QZ}(H_i) := \{h \in H_i : Z_{H_i}(h) \text{ is open in } H_i\}$$

be the *quasi-center* of H_i. See [16] for some properties and examples of this group.

Recall that Γ acts freely on the vertices of $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$, but in general, it is possible that non-trivial elements of Γ act trivially on (exactly) one factor of $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. Therefore, we define the group

$$\Lambda_1 := \mathrm{pr}_1(\Gamma \cap (H_1 \times \{1\})) = \mathrm{pr}_1(\Gamma \cap (\mathrm{Aut}(\mathcal{T}_{2m}) \times \{1\})) < \mathrm{Aut}(\mathcal{T}_{2m})$$

and similarly

$$\Lambda_2 := \mathrm{pr}_2(\Gamma \cap (\{1\} \times H_2)) = \mathrm{pr}_2(\Gamma \cap (\{1\} \times \mathrm{Aut}(\mathcal{T}_{2n}))) < \mathrm{Aut}(\mathcal{T}_{2n})\,.$$

Observe that

$$\Lambda_i = \mathrm{pr}_i(\ker(\mathrm{pr}_{3-i})) \cong \ker(\mathrm{pr}_{3-i}) \lhd \Gamma$$

and note that $\Lambda_i \lhd \mathrm{QZ}(H_i)$, since every discrete normal subgroup of H_i is contained in $\mathrm{QZ}(H_i)$, as explained in [16]. In particular, we conclude that $\mathrm{QZ}(H_i) = 1$ implies an isomorphism $\Gamma \cong \mathrm{pr}_{3-i}(\Gamma)$ and in this case we can naturally see Γ as a subgroup of $\mathrm{Aut}(\mathcal{T}_{2m})$, if $i = 2$, or as a subgroup of $\mathrm{Aut}(\mathcal{T}_{2n})$, if $i = 1$.

1.4 Local groups

Let X be a $(2m, 2n)$–complex and Γ its fundamental group. We turn now to the definition of their finite "local groups" P_h and P_v, which will play a major role in the construction of interesting examples. Let $E_v^{(k)}$ be the set of reduced paths of combinatorial length $k \in \mathbb{N}$ in the vertical 1-skeleton $X_v^{(1)} := (\{x\}, E_v)$ of X. We identify elements in $E_v^{(k)}$ with freely reduced words of length k in the fundamental group $\pi_1(X_v^{(1)}, x) = \langle b_1, \ldots, b_n \rangle = F_n$. The set $E_h^{(k)}$ is defined analogously and identified with the set of reduced words of length k in the free group $\langle a_1, \ldots, a_m \rangle = F_m$. Note that $E_v^{(1)} = E_v$ and $E_h^{(1)} = E_h$.

There is a family of homomorphisms

$$\rho_h^{(k)} : F_m = \langle a_1, \ldots, a_m \rangle \to \mathrm{Sym}(E_v^{(k)}) \cong S_{2n \cdot (2n-1)^{k-1}}$$

and a family of homomorphisms

$$\rho_v^{(k)} : F_n = \langle b_1, \ldots, b_n \rangle \to \mathrm{Sym}(E_h^{(k)}) \cong S_{2m \cdot (2m-1)^{k-1}} \, .$$

We denote their images by

$$\begin{aligned} P_v^{(k)} &:= \mathrm{im}(\rho_h^{(k)}) = \langle \rho_h^{(k)}(a_1), \ldots, \rho_h^{(k)}(a_m) \rangle \\ P_h^{(k)} &:= \mathrm{im}(\rho_v^{(k)}) = \langle \rho_v^{(k)}(b_1), \ldots, \rho_v^{(k)}(b_n) \rangle \, . \end{aligned}$$

If $k = 1$, we omit the superscript "(1)" and simply write

$$\rho_h : \langle a_1, \ldots, a_m \rangle \twoheadrightarrow \langle \rho_h(a_1), \ldots, \rho_h(a_m) \rangle = P_v < \mathrm{Sym}(E_v) \cong S_{2n} \, ,$$

where for the isomorphism $\mathrm{Sym}(E_v) \cong S_{2n}$ we always use the explicit identification

$$\begin{aligned} E_v &\cong \{1, \ldots, 2n\} \\ b_j &\leftrightarrow j \\ b_j^{-1} &\leftrightarrow 2n + 1 - j \, , \end{aligned}$$

$j = 1, \ldots, n$, and

$$\rho_v : \langle b_1, \ldots, b_n \rangle \twoheadrightarrow \langle \rho_v(b_1), \ldots, \rho_v(b_n) \rangle = P_h < \mathrm{Sym}(E_h) \cong S_{2m} \, ,$$

via the identification (for $i = 1, \ldots, m$)

$$\begin{aligned} E_h &\cong \{1, \ldots, 2m\} \\ a_i &\leftrightarrow i \\ a_i^{-1} &\leftrightarrow 2m + 1 - i \, . \end{aligned}$$

Now, it is time to give the definition of $\rho_h^{(k)}$ and $\rho_v^{(k)}$. First, we take $k = 1$. The two homomorphisms ρ_h and ρ_v are explicitly constructed as follows: each geometric square $[aba'b']$ of X defines

$$\begin{aligned} \rho_h(a)(b'^{-1}) &:= b \\ \rho_h(a')(b^{-1}) &:= b' \\ \rho_v(b)(a^{-1}) &:= a' \\ \rho_v(b')(a'^{-1}) &:= a\,, \end{aligned}$$

as visualized in Figure 1.2.

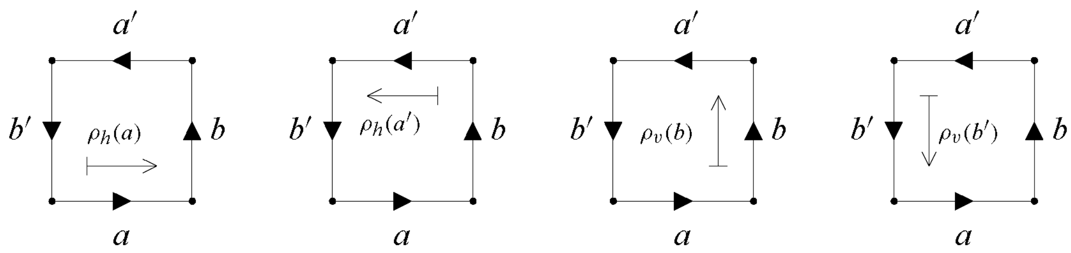


Figure 1.2: Visualizing the definition of ρ_h, ρ_v

By the link condition in X, these $4mn$ expressions (going through all mn geometric squares of X) indeed uniquely determine ρ_h and ρ_v. If $k \geq 2$, the homomorphisms $\rho_h^{(k)}$ and $\rho_v^{(k)}$ are defined in a similar way, see [17, Chapter 1]. We give an inductive definition of $\rho_h^{(k)}$, the homomorphism $\rho_v^{(k)}$ can be defined analogously: Let $a \in E_h$ and $b = b' \cdot b'' \in E_v^{(k)}$, where we write a dot for the concatenation of paths and where $b' \in E_v$, $b'' \in E_v^{(k-1)}$. Then

$$\rho_h^{(k)}(a)(b) := \rho_h(a)(b') \cdot \rho_h^{(k-1)}\left(\rho_v(b')(a)\right)(b'')\,,$$

see Figure 1.3 for an illustration.

Starting with a $(2m, 2n)$–complex X, the finite permutation groups $P_v^{(k)}$ and $P_h^{(k)}$ can be effectively computed, see Appendix B.4 for an implementation in GAP ([29]) for $k = 1$ and $k = 2$. These groups describe the local actions of the projections of Γ on k-spheres in $\mathcal{T}_{2n}$ and $\mathcal{T}_{2m}$, respectively. More precisely, let x_v be any vertex in $\mathcal{T}_{2n}$ and let $S(x_v, k)$ be the k-sphere around x_v, then the two groups

$$P_v^{(k)} < \mathrm{Sym}(E_v^{(k)}) \text{ and } H_2(x_v)/H_2(S(x_v, k)) < \mathrm{Sym}(S(x_v, k))$$

are permutation isomorphic (see [17, Chapter 1]). The analogous statement holds for $P_h^{(k)}$ and $H_1(x_h)/H_1(S(x_h, k))$, where x_h is any vertex in $\mathcal{T}_{2m}$.

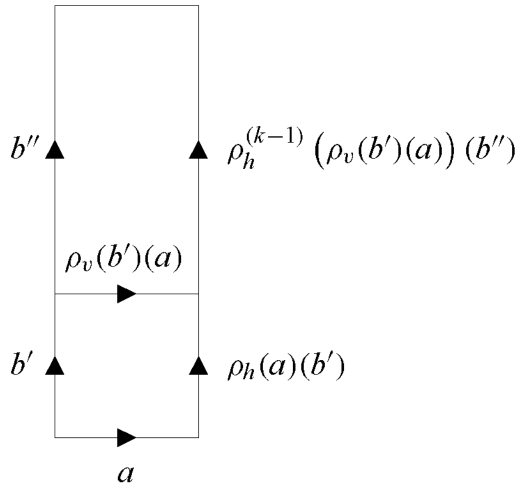


Figure 1.3: Inductive definition of $\rho_h^{(k)}$, $k \geq 2$

Taking this identification for $k = 2$

$$P_h^{(2)} \cong H_1(x_h)/H_1(S(x_h, 2)) < \mathrm{Sym}(S(x_h, 2)),$$

we define the subgroup

$$K_h := \mathrm{Stab}_{P_h^{(2)}}(S(x_h, 1) \cup S(y_h, 1)) < P_h^{(2)},$$

where y_h is any neighbouring vertex of x_h in $\mathcal{T}_{2m}$. In our applications, the definition of K_h will be independent of the choice of y_h (up to permutation isomorphism). See Appendix B.4 for the GAP-program ([29]) computing K_h if $m = 3$. Analogously, one defines the group $K_v < P_v^{(2)}$.

For each $k \in \mathbb{N}$, there is a commutative diagram

$$\begin{tikzcd}
\langle a_1, \ldots, a_m \rangle \arrow[r, two heads, "\rho_h^{(k+1)}"] \arrow[dr, two heads, "\rho_h^{(k)}"'] & P_v^{(k+1)} < \mathrm{Sym}(E_v^{(k+1)}) \arrow[d, two heads, "p_k"] \\
& P_v^{(k)} < \mathrm{Sym}(E_v^{(k)})
\end{tikzcd}$$

where p_k is the homomorphism restricting the action of $P_v^{(k+1)}$ on the $(k+1)$–sphere $S(x_v, k+1)$ to the k-sphere $S(x_v, k)$. In particular, the order $|P_v^{(k)}|$ divides $|P_v^{(k+1)}|$. Note that

$$\bigcap_{k \in \mathbb{N}} \ker \rho_h^{(k)} \cong \Lambda_1 \quad \text{and} \quad \bigcap_{k \in \mathbb{N}} \ker \rho_v^{(k)} \cong \Lambda_2 .$$

Lemma 1.1. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$ *be a* $(2m, 2n)$*–group.*

(1a) *Let* $A \subset \langle a_1, \ldots, a_m \rangle$*. If for each* $a \in A$ *and* $b \in E_v$ *we have* $\rho_h(a)(b) = b$ *and* $\rho_v(b)(a) \in A$*, then* $A \subset \Lambda_1$.

(1b) *Let* $B \subset \langle b_1, \ldots, b_n \rangle$*. If for each* $b \in B$ *and* $a \in E_h$ *we have* $\rho_v(b)(a) = a$ *and* $\rho_h(a)(b) \in B$*, then* $B \subset \Lambda_2$.

Proof. The assumptions made in (1a) directly imply

$$A \subset \bigcap_{k \in \mathbb{N}} \ker \rho_h^{(k)} \cong \Lambda_1 \, .$$

(1b) follows similarly. □

Because of the importance of the local groups P_h and P_v in our study of X, we will sometimes call X a (P_h, P_v)*–complex* and the corresponding fundamental group Γ a (P_h, P_v)*–group*.

1.5 Irreducibility

An important notion in the theory of lattices in higher rank semisimple Lie groups is "irreducibility". In our situation, we adopt the generalized definition given in [17]. A $(2m, 2n)$–group Γ is called *reducible* if $\mathrm{pr}_1(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2m})$ is discrete. Otherwise, Γ is called *irreducible*. A $(2m, 2n)$–complex X is said to be reducible (irreducible) if and only if $\Gamma = \pi_1(X, x)$ is reducible (irreducible).

Remarks. (1) Recall that a subgroup of $\mathrm{Aut}(\mathcal{T}_\ell)$ is discrete if and only if its vertex stabilizers are all finite, see Proposition D.2 for a proof.

(2) It is shown in [17, Proposition 1.2] that $\mathrm{pr}_1(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2m})$ is discrete if and only if $\mathrm{pr}_2(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2n})$ is discrete.

(3) Note that $\mathrm{pr}_1(\Gamma)$ is never dense in $\mathrm{Aut}(\mathcal{T}_{2m})$, i.e. $H_1 \subsetneqq \mathrm{Aut}(\mathcal{T}_{2m})$, in contrast to the behaviour of "irreducible" lattices in higher rank semisimple Lie groups.

(4) In terms of orders of the local groups $P_h^{(k)}$ and $P_v^{(k)}$, the group Γ is reducible if and only if the set $\{|P_h^{(k)}|\}_{k \in \mathbb{N}}$ is bounded, if and only if $\{|P_v^{(k)}|\}_{k \in \mathbb{N}}$ is bounded.

In geometric terms, the $(2m, 2n)$–complex X is reducible if and only if X admits a finite covering which is a product of two graphs (see [17, Chapter 1]). Therefore, a reducible $(2m, 2n)$–group Γ is virtually a direct product of two finitely generated free groups, in particular Γ is residually finite. As a consequence, a non-residually finite $(2m, 2n)$–group Γ has to be irreducible. In general, no algorithm is known to determine whether a given Γ is reducible or not. However, a useful sufficient criterion for irreducibility, based on the Thompson-Wielandt theorem (see e.g. [16, Theorem 2.1.1] for a formulation of this theorem), is presented in [17, Proposition 1.3].

We will strongly use the criteria (1) and (2), divided into (1a), (1b), (2a) and (2b), of the following proposition which is based on results in [16, 17]. The third criterion, i.e. part (3a) and (3b), will only be used in Theorem 2.27, where (1) does not apply.

Proposition 1.2. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ *be a* $(2m, 2n)$*–group.*

(1a) *Suppose that* $m \geq 3$ *and* $P_h = A_{2m}$*. Then* Γ *is irreducible if and only if*

$$\left|P_h^{(2)}\right| = |A_{2m}| \left(\frac{|A_{2m}|}{2m}\right)^{2m} = \frac{(2m)!}{2} \left(\frac{(2m-1)!}{2}\right)^{2m}.$$

(1b) *Suppose that* $P_v = A_{2n}$*,* $n \geq 3$*. Then* Γ *is irreducible if and only if*

$$\left|P_v^{(2)}\right| = |A_{2n}| \left(\frac{|A_{2n}|}{2n}\right)^{2n} = \frac{(2n)!}{2} \left(\frac{(2n-1)!}{2}\right)^{2n}.$$

(2a) *The group* Γ *is reducible if and only if* $|P_h^{(k+1)}| = |P_h^{(k)}|$ *for some* $k \in \mathbb{N}$.

(2b) *The group* Γ *is reducible if and only if* $|P_v^{(k+1)}| = |P_v^{(k)}|$ *for some* $k \in \mathbb{N}$.

(3a) *Let* $P_h < S_{2m}$ *be transitive and suppose that for each* $k \in \mathbb{N}$ *there exist freely reduced words* $b \in \langle b_1, \dots, b_n \rangle$ *and* $a \in \langle a_1, \dots, a_m \rangle$ *with* $|a| = k$ *such that* $\rho_v^{(k)}(b)(a) = a$*, and* $\rho_v(\tilde{b})$ *acts transitively on* $E_h \setminus \{a''^{-1}\}$*, where* $\tilde{b} := \rho_h^{(|b|)}(a)(b)$ *and* $a = a' \cdot a''$ *is the decomposition of* a *with* $a' \in E_h^{(k-1)}$*,* $a'' \in E_h$ *(see Figure 1.4). Then* $\mathrm{pr}_1(\Gamma)$ *is locally* ∞*-transitive, in particular* Γ *is irreducible.*

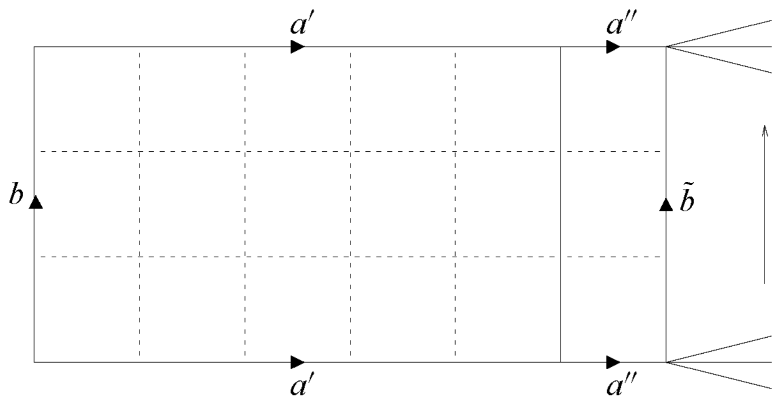


Figure 1.4: Notations in Proposition 1.2(3a)

(3b) *Let $P_v < S_{2n}$ be transitive and suppose that for each $k \in \mathbb{N}$ there exist freely reduced words $a \in \langle a_1, \dots, a_m \rangle$ and $b \in \langle b_1, \dots, b_n \rangle$ with $|b| = k$ such that $\rho_h^{(k)}(a)(b) = b$, and such that $\rho_h(\tilde{a})$ acts transitively on $E_v \setminus \{b''^{-1}\}$, where $\tilde{a} := \rho_v^{(|a|)}(b)(a)$ and $b = b' \cdot b''$ with $b' \in E_v^{(k-1)}$, $b'' \in E_v$. Then $\mathrm{pr}_2(\Gamma)$ is locally ∞-transitive, in particular Γ is irreducible.*

Proof. We only prove part a) of each statement, since part b) is completely analogous.

(1a) The statement follows directly from [16, Proposition 3.3.1].

(2a) Obviously, $|P_h^{(k+1)}| = |P_h^{(k)}|$ for some $k \in \mathbb{N}$ is a necessary condition, since $\{|P_h^{(k)}|\}_{k\in\mathbb{N}}$ is bounded for a reducible Γ. We want to prove now, that it is also sufficient for the reducibility of Γ. It is enough to show $|P_h^{(k+2)}| = |P_h^{(k+1)}|$. First observe that for all vertices $x_h \in \mathcal{T}_{2m}$ we have

$$H_1(S(x_h, k+1)) = H_1(S(x_h, k)) < H_1(x_h)\,, \tag{1.1}$$

since

$$1 = \left|P_h^{(k+1)}\right| / \left|P_h^{(k)}\right| = \left|H_1(S(x_h,k)) / H_1(S(x_h,k+1))\right| \,.$$

Assume now that

$$|P_h^{(k+2)}| > |P_h^{(k+1)}|\,.$$

It follows that there is an element $g \in H_1(S(x_h, k+1)) \setminus H_1(S(x_h, k+2))$. But then, for at least one neighbouring vertex y_h of x_h,

$$g \in H_1(S(y_h, k)) \setminus H_1(S(y_h, k+1))\,,$$

contradicting equation (1.1).

(3a) We have to show that $\mathrm{pr}_1(\Gamma)(x_h)$ acts transitively on $S(x_h, k)$ for each $k \in \mathbb{N}$. This is done by induction on k using the identification (see [17, Chapter 1])

$$\langle b_1, \dots, b_n \rangle \cong \{\gamma \in \Gamma : \mathrm{pr}_1(\gamma)(x_h) = x_h\}\,.$$

For $k = 1$, the statement is obvious since P_h is transitive by assumption. To prove the induction step $k \to k+1$, note that $\mathrm{pr}_1(\Gamma)(x_h)$ acts by induction hypothesis transitively on $S(x_h, k)$, hence we have at most $2m - 1$ orbits in $S(x_h, k+1)$. But now, the assumptions, in particular the transitivity of $\rho_v(\tilde{b})$ on $E_h \setminus \{a''^{-1}\}$, exactly guarantee that there is in fact only one orbit.

Since $P_h^{(k)}$ is transitive for each $k \geq 1$, the set $\{|P_h^{(k)}|\}_{k\in\mathbb{N}}$ is not bounded and therefore Γ is irreducible.

□

Remark. Observe that Proposition 1.2(1a) cannot be generalized to the case where $P_h = A_4$ (i.e. to $m = 2$), because there are for example irreducible (A_4, A_{10})–groups such that

$$|P_h^{(2)}| = 324 < |A_4| \left(\frac{|A_4|}{4}\right)^4 = 972$$

(cf. Appendix C.6).

1.6 Amalgam decompositions

Let A, B, C be groups. By writing an expression of the form $A *_C B$, we mean that there is given a commutative diagram of injective group homomorphisms

$$\begin{tikzcd} C \arrow[r, "i_B"] \arrow[d, "i_A"'] & B \arrow[d, "j_B"] \\ A \arrow[r, "j_A"'] & A *_C B \end{tikzcd}$$

(in particular C can be seen as a subgroup of A and B via the injections i_A and i_B, respectively), and the group $A *_C B$ is uniquely determined by the following universal property: Given any group G and any homomorphisms $j'_A : A \to G$, $j'_B : B \to G$ such that $j'_A \circ i_A = j'_B \circ i_B$, there is a unique homomorphism $\rho : A *_C B \to G$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "i_B"] \arrow[d, "i_A"'] & B \arrow[d, "j_B"] \arrow[ddr, bend left, "j'_B"] & \\ A \arrow[r, "j_A"'] \arrow[drr, bend right, "j'_A"'] & A *_C B \arrow[dr, dotted, "\rho"] & \\ & & G \end{tikzcd}$$

The group $A *_C B$ is called the *amalgamated free product* of the groups A and B amalgamating the "subgroup" C, or simply an *amalgam*.

In most of our examples of amalgams, the three groups A, B, C will be finitely generated non-abelian free groups, i.e. we will have amalgams of the form $F_k *_{F_m} F_l$ for some $k, l, m \geq 2$. Moreover, $i_A(F_m)$ and $i_B(F_m)$ will have finite index in F_k and F_l, respectively, where $i_A : F_m \to F_k$, $i_B : F_m \to F_l$ denote the given injective homomorphisms. Note that k, l, m are then related by the index formulae (see e.g. [49, Proposition I.3.9])

$$[F_k : F_m] = \frac{m-1}{k-1} \text{ and } [F_l : F_m] = \frac{m-1}{l-1}.$$

If F_k is generated by $a_1, \ldots, a_k$, F_l by $b_1, \ldots, b_l$ and F_m by $c_1, \ldots, c_m$, then $F_k *_{F_m} F_l$ has the finite presentation

$$\langle a_1, \ldots, a_k, b_1, \ldots, b_l \mid i_A(c_1) = i_B(c_1), \ldots, i_A(c_m) = i_B(c_m) \rangle$$

and is torsion-free (this follows from [49, Theorem IV.2.7]).

A $(2m, 2n)$–group Γ splits by a result of Wise ([68, Theorem I.1.18]) in two ways as a fundamental group of a finite graph of finitely generated free groups (using the terminology of the Bass-Serre theory). We are mainly interested in amalgamated free products of free groups, i.e. fundamental groups of edges of free groups. This case happens if the local groups are transitive:

Proposition 1.3. *Let* Γ *be a* $(2m, 2n)$*–group.*

(1a) *If* $P_h < S_{2m}$ *is a transitive permutation group, then* Γ *can be written as an amalgamated free product of finitely generated free groups as follows:*

$$\Gamma \cong F_n *_{F_{1-2m+2mn}} F_{1-m+mn} \, .$$

We call it the vertical decomposition *of* Γ.

(1b) *If* $P_v < S_{2n}$ *is transitive, then we have a* horizontal decomposition

$$\Gamma \cong F_m *_{F_{1-2n+2mn}} F_{1-n+mn} \, .$$

Proof. The two statements follow directly from [68, Theorem I.1.18] after a vertical subdivision of the cell complex X in (1a), and a horizontal subdivision of X in (1b). □

Note that the indices in the inclusions of the splitting in Proposition 1.3(1a) are

$$[F_n : F_{1-2m+2mn}] = 2m \text{ and } [F_{1-m+mn} : F_{1-2m+2mn}] = 2 \, .$$

The tree on which Γ naturally acts is the first barycentric subdivision of $\mathcal{T}_{2m}$, the "bi-regular" tree of valencies 2 and $2m$. Note that F_n is identified with the free subgroup $\langle b_1, \ldots, b_n \rangle$ of Γ. Furthermore, the second factor F_{1-m+mn} is the fundamental group of a graph with m vertices (one for each geometric edge $\{a_i, a_i^{-1}\}$) and mn geometric edges (one for each geometric square in X). Finally, the amalgamated group $F_{1-2m+2mn}$ is the fundamental group of a graph having $2m$ vertices (one for each edge in E_h) and $2mn$ geometric edges (one for each geometric square in the vertically subdivided complex X'). The two injections in the amalgamated free product are induced by immersions (i.e. local injections, see [68, Definition I.1.16]) in X'. Analogous statements hold for the second splitting of Γ.

The following proposition describes amalgam decompositions for the important subgroup $\Gamma_0 < \Gamma$.

Proposition 1.4. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ *be a* $(2m, 2n)$*–group. We denote by* $F_n^{(2)}$ *the subgroup of* $F_n = \langle b_1, \dots, b_n \rangle$ *of index* 2 *consisting of elements with even length. Analogously, we define* $F_m^{(2)} \lhd F_m = \langle a_1, \dots, a_m \rangle$. *If* $\rho_v(F_n^{(2)}) < S_{2m}$ *is transitive (which holds if for example* P_h *is a quasi-primitive permutation group and* $m \geq 2$*), then there is an amalgam decomposition of* Γ_0*, the so-called* vertical decomposition *of* Γ_0*,*

$$\Gamma_0 \cong F_{2n-1} *_{F_{1-4m+4mn}} F_{2n-1} \, .$$

Similarly, if $\rho_h(F_m^{(2)}) < S_{2n}$ *is transitive (which holds if for example* P_v *is quasi-primitive and* $n \geq 2$*), then we get a* horizontal decomposition

$$\Gamma_0 \cong F_{2m-1} *_{F_{1-4n+4mn}} F_{2m-1} \, .$$

In particular, if $m = n \geq 2$ *and* P_h*,* P_v *both are quasi-primitive, then we have two decompositions of* Γ_0 *as*

$$F_{2n-1} *_{F_{(2n-1)^2}} F_{2n-1} \, .$$

Proof. Again, this can be immediately deduced from the more general result of Wise [68, Theorem I.1.18]. Note that the indices are

$$[F_{2n-1} : F_{1-4m+4mn}] = 2m \text{ and } [F_{2m-1} : F_{1-4n+4mn}] = 2n \, .$$

To see why $\rho_v(F_n^{(2)})$ is transitive if $P_h < S_{2m}$ ($m \geq 2$) is quasi-primitive, first observe that in general $\rho_v(F_n^{(2)})$ is a normal subgroup of $P_h = \rho_v(F_n)$ of index at most $[F_n : F_n^{(2)}] = 2$. If we assume that P_h is quasi-primitive, then $\rho_v(F_n^{(2)})$ is trivial or transitive, but $\rho_v(F_n^{(2)}) = 1$ would imply $|P_h| = 2$ and $m = 1$. □

We call a $(2m, 2n)$–group Γ *horizontally directed*, if a_i is not in the same orbit as a_i^{-1} in the natural action of P_h on E_h for all $i \in \{1, \dots, m\}$. The term *vertically directed* can be defined analogously. These definitions are equivalent to those given in [68, Definition I.1.10]. We formulate in Proposition 1.5 another interesting special case of [68, Theorem I.1.18] concerning HNN-extensions. In general, if a group G is given by a presentation $\langle S \mid R \rangle$, and A, B are isomorphic subgroups of G, then the *HNN-extension* (Higman-Neumann-Neumann extension) of G with associated subgroups A and B via the isomorphism $\phi : A \to B$ is the group with presentation

$$\langle S, t \mid R, t^{-1}at = \phi(a), \text{ if } a \in A \rangle \, .$$

Proposition 1.5. *Let* $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ *be a* $(2m, 2n)$*–group.*

(1a) *If* Γ *is horizontally directed and* P_h *has exactly two orbits in its natural action on* E_h*, then* Γ *is a HNN-extension of the free group* $F_n = \langle b_1, \dots, b_n \rangle$ *associating subgroups* F_{1-m+mn} *of index* m*.*

(1b) *If Γ is vertically directed and P_v has exactly two orbits in its natural action on E_v, then Γ is a HNN-extension of the free group $F_m = \langle a_1, \dots, a_m \rangle$ associating subgroups F_{1-n+mn} of index n.*

Remark. Horizontally (or vertically) directed $(2m, 2n)$–groups Γ have an infinite abelianization Γ^{ab}, in particular they have a proper infinite quotient. To see this, let $\mathcal{O}_1$ be the orbit of a_1 under the natural action of P_h on E_h. Define a surjective homomorphism $\Gamma \to \mathbb{Z}$ by mapping all $b_1, \dots, b_n$ to the trivial element 0 in $\mathbb{Z}$, and all elements in $\mathcal{O}_1$ to the generator 1 of $\mathbb{Z}$. If both a_i and a_i^{-1} are not in $\mathcal{O}_1$, then we map a_i to $0 \in \mathbb{Z}$, $i = 2, \dots, m$.

1.7 Double cosets

Given a group G and a subgroup $H < G$, the corresponding set of *double cosets* is defined as

$$H \backslash G / H := \{HgH : g \in G\},$$

where $HgH := \{h_1 g h_2 : h_1, h_2 \in H\}$ is as usual. The cardinalities of the two sets of double cosets corresponding to the two amalgam decompositions of a $(2m, 2n)$–group Γ are related to transitivity properties of its local groups, as seen in the following proposition (as always, similar statements can be made for P_v).

Proposition 1.6. *Let Γ be a $(2m, 2n)$–group. Suppose that $P_h < S_{2m}$ is transitive. Then there is a bijection between the set of orbits of the diagonal action of P_h on $\{1, \dots, 2m\} \times \{1, \dots, 2m\}$ and the set $F_{1-2m+2mn} \backslash F_n / F_{1-2m+2mn}$ of double cosets, where*

$$\Gamma \cong F_n *_{F_{1-2m+2mn}} F_{1-m+mn}$$

is the vertical decomposition given by Proposition 1.3(1a). In particular, the number $|F_{1-2m+2mn} \backslash F_n / F_{1-2m+2mn}|$ is the rank *of P_h (in the terminology of [25, p.67]) and can be easily computed knowing the finite group P_h, but without knowing the explicit amalgam decomposition, for example using the* GAP*-command ([29])*

```
1 + Size(OrbitLengths(Ph,
Arrangements([1..2*m],2),OnTuples));
```

where `Ph` *describes the group P_h. Another consequence is that*

$$|F_{1-2m+2mn} \backslash F_n / F_{1-2m+2mn}| = 2,$$

if and only if P_h is a 2-transitive permutation group.

Proof. We define $B := F_n$ and $C := F_{1-2m+2mn}$. Let $\mathcal{T}'_{2m}$ be the bi-regular Bass-Serre tree on which the amalgam $\Gamma \cong B *_C F_{1-m+mn}$ naturally acts and let x_h be the vertex of $\mathcal{T}'_{2m}$ such that $B = \mathrm{Stab}_\Gamma(x_h)$. Denote by Ω the set of edges in $\mathcal{T}'_{2m}$ with origin x_h and let $\omega \in \Omega$ be the edge such that $\mathrm{Stab}_\Gamma(\omega) = C$. Note that

$$|\Omega| = [B : C] = [F_n : F_{1-2m+2mn}] = 2m \,.$$

By construction, the action of P_h on $\{1, \dots, 2m\} \cong E_h$ is equivalent (permutation isomorphic) to the action of B on Ω. We want to define a bijection

$$\varphi : \{\text{Orbits of } B \curvearrowright \Omega \times \Omega\} \longrightarrow C\backslash B/C \,.$$

Let $(\omega_1, \omega_2) \in \Omega \times \Omega$. We denote by $[(\omega_1, \omega_2)]$ its B-orbit under the diagonal left action, in particular $[(\omega_1, \omega_2)] = [(b\omega_1, b\omega_2)]$ for each $b \in B$. Since B acts transitively on Ω, we can choose $b_1, b_2 \in B$ such that $\omega = b_1\omega_1 = b_2\omega_2$. Now we define

$$\varphi\left([(\omega_1, \omega_2)]\right) := Cb_1b_2^{-1}C \in C\backslash B/C \,.$$

We first show that φ is independent of the choice of b_1, b_2. Take $\tilde{b}_1, \tilde{b}_2 \in B$ such that $\omega = \tilde{b}_1\omega_1 = \tilde{b}_2\omega_2$. Then $b_i\tilde{b}_i^{-1}\omega = b_i\omega_i = \omega$, $(i = 1, 2)$, hence $b_i\tilde{b}_i^{-1} \in C$, i.e. $Cb_1 = C\tilde{b}_1$ and $b_2^{-1}C = \tilde{b}_2^{-1}C$ which implies

$$C\tilde{b}_1\tilde{b}_2^{-1}C = Cb_1b_2^{-1}C \,.$$

Next we show that φ is independent of the representative of $[(\omega_1, \omega_2)]$. Any representative of $[(\omega_1, \omega_2)]$ has the form $(b\omega_1, b\omega_2)$ for some $b \in B$. But then

$$\omega = b_1b^{-1}(b\omega_1) = b_2b^{-1}(b\omega_2)$$

and

$$\varphi\left([(b\omega_1, b\omega_2)]\right) = Cb_1b^{-1}(b_2b^{-1})^{-1}C = Cb_1b_2^{-1}C \,.$$

This proves that φ is well-defined.

Note that $\varphi([(\omega, b\omega)]) = CbC$ for each $b \in B$, hence φ is surjective. To show the injectivity of φ, assume that

$$\varphi\left([(\omega_1, \omega_2)]\right) = Cb_1b_2^{-1}C = C\tilde{b}_1\tilde{b}_2^{-1}C = \varphi\left([(\tilde{\omega}_1, \tilde{\omega}_2)]\right) \,,$$

such that $\omega = b_1\omega_1 = b_2\omega_2 = \tilde{b}_1\tilde{\omega}_1 = \tilde{b}_2\tilde{\omega}_2$. The assumption $Cb_1b_2^{-1}C = C\tilde{b}_1\tilde{b}_2^{-1}C$ implies that there is some $c \in C$ such that

$$\begin{aligned} cb_1b_2^{-1} &\in \tilde{b}_1\tilde{b}_2^{-1}C \\ \tilde{b}_2\tilde{b}_1^{-1}cb_1b_2^{-1} &\in C \\ \tilde{b}_2\tilde{b}_1^{-1}cb_1b_2^{-1}\omega &= \omega \\ cb_1b_2^{-1}\omega &= \tilde{b}_1\tilde{b}_2^{-1}\omega \,, \end{aligned}$$

hence

$$[(\omega_1, \omega_2)] = [(\omega, b_1b_2^{-1}\omega)] = [(c\omega, cb_1b_2^{-1}\omega)] = [(\omega, \tilde{b}_1\tilde{b}_2^{-1}\omega)] = [(\tilde{\omega}_1, \tilde{\omega}_2)] \,.$$

$\square$

1.8 SQ-universal groups

A countable group G is called *SQ-universal*, if every countable group can be embedded in a quotient of G. According to [56], this term was suggested by Graham Higman. The following result of Ilya Rips is mentioned in the book of Bass-Lubotzky [3, Section 9.15].

Proposition 1.7. *(Rips) Let $G = A *_C B$ be an amalgam such that $C \neq B$ and $|C\backslash A/C| \geq 3$. Then G is SQ-universal.*

There seems to be no published proof of this proposition, but the main idea is explained in [3, p.149]: "Rips' explanation uses Small Cancellation Theory, as in [62]. Explicitly, let CaC and $Ca'C$ be distinct non-trivial double cosets in $C\backslash A/C$ and $b \in B \setminus C$. Consider words in G of the form

$$w = a^{n_1} b a'^{n'_1} b a^{n_2} b a'^{n'_2} b a^{n_3} b a'^{n'_3} b \cdots .$$

When the exponents n_i, n'_i are suitably large one can apply Small Cancellation Theory to conclude that adding the relation $w = 1$ does not kill G, whence G is not simple."

Corollary 1.8. *Let Γ be a $(2m, 2n)$–group. If the local group $P_h < S_{2m}$ is transitive, but not 2-transitive, or if $P_v < S_{2n}$ is transitive, but not 2-transitive, then the group Γ is SQ-universal, in particular it has "many" normal subgroups of infinite index.*

Proof. Combine Proposition 1.3, 1.6 and 1.7. □

1.9 Embeddings

The constructions of many interesting groups in the subsequent chapters will be based on certain embedding techniques. In the following proposition, we give some elementary general consequences for the case that a $(2m, 2n)$–complex is embedded in a "bigger" complex, using the following definition: Let X be a $(2m, 2n)$–complex and let Y be a $(2\tilde{m}, 2\tilde{n})$–complex, where $\tilde{m} \geq m$ and $\tilde{n} \geq n$. We say that X is *embedded* in Y, if the $\tilde{m}\tilde{n}$ geometric squares of Y contain all mn geometric squares of X.

Proposition 1.9. *Let $\tilde{m} \geq m$ and $\tilde{n} \geq n$. Suppose that the $(2m, 2n)$–complex X is embedded in the $(2\tilde{m}, 2\tilde{n})$–complex Y. Then*

(1) *The fundamental groups inject: $\pi_1 X < \pi_1 Y$.*

(2) *The order $|P_h^{(k)}(X)|$ divides $|P_h^{(k)}(Y)|$ and the order $|P_v^{(k)}(X)|$ divides $|P_v^{(k)}(Y)|$ for each $k \in \mathbb{N}$.*

(3) *If X is irreducible, then also Y is irreducible. The converse is not true in general.*

Proof. (1) See [9, Proposition II.4.14(1)].

(2) To take into account the two involved complexes X, Y, we write here $P_h^{(k)}(X)$, $P_h^{(k)}(Y)$, $P_v^{(k)}(X)$, $P_v^{(k)}(Y)$, $\rho_{v,X}$, $\rho_{v,Y}$ instead of $P_h^{(k)}$, $P_v^{(k)}$, ρ_v. We prove now that $|P_h(X)|$ divides $|P_h(Y)|$. The other statements are proved similarly. Let G be the subgroup of $S_{2\tilde{m}}$

$$G := \langle \rho_{v,Y}(b_1), \ldots, \rho_{v,Y}(b_n) \rangle_{S_{2\tilde{m}}}$$

and Δ the subset of $\{1, \ldots, 2\tilde{m}\}$ with $2m$ elements

$$\Delta := \{1, \ldots, m\} \sqcup \{2\tilde{m} - m + 1, \ldots, 2\tilde{m}\}\,.$$

Because of the embedding assumption and the link conditions in X and Y, the set Δ is G-invariant and the restriction of G to Δ is permutation isomorphic to

$$P_h(X) = \langle \rho_{v,X}(b_1), \ldots, \rho_{v,X}(b_n) \rangle_{S_{2m}}$$

via the inclusion

$$\begin{aligned} \{1, \ldots, 2m\} &\to \{1, \ldots, 2\tilde{m}\} \\ i &\mapsto i \\ 2m + 1 - i &\mapsto 2\tilde{m} + 1 - i\,, \end{aligned}$$

$i = 1, \ldots, m$, hence $|G| = |P_h(X)| \cdot l$, where l is the order of the pointwise stabilizer of Δ in G (cf. [25, p.17]). The claim follows now, since G is obviously a subgroup of

$$\langle \rho_{v,Y}(b_1), \ldots, \rho_{v,Y}(b_n), \ldots, \rho_{v,Y}(b_{\tilde{n}}) \rangle_{S_{2\tilde{m}}} = P_h(Y)\,.$$

(3) The set $\{|P_h^{(k)}(X)|\}_{k\in\mathbb{N}}$ is unbounded since X is irreducible by assumption, hence by part (2) also $\{|P_h^{(k)}(Y)|\}_{k\in\mathbb{N}}$ is unbounded, i.e. Y is irreducible, too.

To see that the converse is not true in general, we can take for example any irreducible $(2\tilde{m}, 2\tilde{n})$–complex Y having a pair of commuting generators $\{a_i, b_j\}$ (hence having an embedded reducible $(2, 2)$–complex). An explicit example is described in Example 2.2, where $a_1 b_1 = b_1 a_1$.

□

1.10 Normal form and applications

Due to the link condition in a $(2m, 2n)$–complex X, every element $\gamma \in \Gamma = \pi_1(X)$ can be brought in a unique normal form, where "the a's are followed by the b's". The idea is to successively replace length 2 subwords of γ of the form ba by $a'b'$, if $[a'b'a^{-1}b^{-1}]$ is a geometric square in X. Analogously, there is a unique normal form, where "the b's are followed by the a's". Here is the precise statement of Bridson-Wise:

Proposition 1.10. *(Bridson-Wise [10, Normal Form Lemma 4.3]) Let γ be any element in a $(2m, 2n)$–group $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$. Then γ can be written as*

$$\gamma = \sigma_a \sigma_b = \sigma'_b \sigma'_a$$

where σ_a, σ'_a are freely reduced words in the subgroup $\langle a_1, \ldots, a_m \rangle_\Gamma$ and σ_b, σ'_b are freely reduced words in $\langle b_1, \ldots, b_n \rangle_\Gamma$. The words $\sigma_a, \sigma'_a, \sigma_b, \sigma'_b$ are uniquely determined by γ. Moreover, $|\sigma_a| = |\sigma'_a|$ and $|\sigma_b| = |\sigma'_b|$, where $|\cdot|$ is the word length with respect to the standard generators $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}^{\pm 1}$.

Proof. See [10]. For an implementation of the algorithm in GAP ([29]) to compute the two normal forms of a given element in Γ, see Appendix B.6. $\square$

If $\gamma = \sigma_a \sigma_b = \sigma'_b \sigma'_a$ as in Proposition 1.10, then we call $\sigma_a \sigma_b$ the *ab-normal form* and $\sigma'_b \sigma'_a$ the *ba-normal form* of γ. The *length* of γ is by definition

$$|\gamma| := |\sigma_a| + |\sigma_b| = |\sigma'_b| + |\sigma'_a| \,.$$

Note that $|1| = 0$. It takes at most $k^2/4$ switches to bring a word of length k from its ba-normal form to its ab-normal form.

Proposition 1.10 has direct consequences for the structure of a $(2m, 2n)$–group:

Corollary 1.11. *Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$ be a $(2m, 2n)$–group. Then*

(1) *The two groups $\langle a_1, \ldots, a_m \rangle_\Gamma$ and $\langle b_1, \ldots, b_n \rangle_\Gamma$ are free subgroups of Γ of rank m and n, respectively.*

(2) *The group Γ is virtually abelian or contains a non-abelian free subgroup.*

(3) *The center $Z\Gamma$ is trivial if $m, n \geq 2$.*

(4) *The group Γ is residually finite if and only if $\mathrm{Aut}(\Gamma)$ is residually finite.*

Proof. (1) This follows directly from the uniqueness of the normal forms described in Proposition 1.10.

(2) If $m \geq 2$ or $n \geq 2$, then Γ contains a non-abelian free subgroup by part (1). If $m = n = 1$, then either

$$\Gamma \cong \langle a_1, b_1 \mid a_1 b_1 = b_1 a_1 \rangle \cong \mathbb{Z}^2$$

is abelian, or

$$\Gamma \cong \langle a_1, b_1 \mid a_1 b_1 a_1 = b_1 \rangle \,,$$

which has the abelian group $\langle a_1, b_1^2 \rangle_\Gamma \cong \mathbb{Z}^2$ as a subgroup of index 2.

(3) Assume that there is an element $\gamma \in Z\Gamma \setminus \{1\}$ and let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} ,$$

$a^{(1)}, \dots, a^{(k)} \in E_h$, $b^{(1)}, \dots, b^{(l)} \in E_v$, be its ab-normal form, where we can assume without loss of generality that $k \geq 1$ and $l \geq 0$. Take any element

$$a \in E_h \setminus \{a^{(1)}, a^{(1)^{-1}}\} \neq \emptyset .$$

Then, we have

$$aa^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} a .$$

The left hand side of this equation is already in ab-normal form, hence by uniqueness of the ab-normal form, we can conclude from the right hand side that $a = a^{(1)}$, but this is a contradiction to the choice of a, and it follows $Z\Gamma = 1$.

(4) By a result of Baumslag ([5], or see [49, Theorem IV.4.8]) the group $\mathrm{Aut}(\Gamma)$ is residually finite, if Γ is a finitely generated residually finite group. For the other direction, first note that if $m = 1$, then

$$P_h^{(k)} < S_{2m\cdot(2m-1)^{k-1}} = S_2 ,$$

hence $|P_h^{(k)}| \leq 2$ for each $k \in \mathbb{N}$, and Γ is reducible. The same holds if $n = 1$. In particular, Γ is residually finite, if $m = 1$ or $n = 1$. Assume now that Γ is non-residually finite. Then $m, n \geq 2$, and by part (3) we have $Z\Gamma = 1$, hence $\Gamma \cong \mathrm{Inn}(\Gamma) < \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ is non-residually finite.

□

Remark. The group $\mathbb{Z} \times F_n$ is a $(2, 2n)$–group with a non-trivial (infinite) center ($\mathbb{Z} \times \{1\}$ if $n \geq 2$, $\mathbb{Z} \times \mathbb{Z}$ if $n = 1$).

Using Proposition 1.10, we are able to compute certain centralizers of generators, and their normalizers. The sufficient conditions in part (1) of the following proposition can easily be checked by hand, given a $(2m, 2n)$–group Γ. If they are satisfied, also part (2) applies.

Proposition 1.12. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m\cdot n}\rangle$ be a $(2m, 2n)$–group.*

(1a) *Assume that there is an element $a_i \in \{a_1, \dots, a_m\}$ such that $\rho_h(a_i)(b) \neq b$ for all $b \in E_v$ (i.e. $R_{m\cdot n}$ has no relator representing a geometric square of the form $[a_i bab^{-1}]$, where $a \in E_h$, $b \in E_v$). Then $Z_\Gamma(a_i) = \langle a_i \rangle \cong \mathbb{Z}$.*

(1b) *Assume that there is an element $b_j \in \{b_1, \dots, b_n\}$ such that $\rho_v(b_j)(a) \neq a$ for all $a \in E_h$ (i.e. $R_{m\cdot n}$ has no relator representing a geometric square of the form $[a^{-1}b_j ab]$, where $a \in E_h$, $b \in E_v$). Then $Z_\Gamma(b_j) = \langle b_j \rangle \cong \mathbb{Z}$.*

(2a) *Assume that* $Z_\Gamma(a_i) = \langle a_i \rangle$ *for some* $a_i \in \{a_1, \dots, a_m\}$. *Then the normalizer of* $\langle a_i \rangle$ *is* $N_\Gamma(\langle a_i \rangle) = Z_\Gamma(a_i) = \langle a_i \rangle$.

(2b) *Assume that* $Z_\Gamma(b_j) = \langle b_j \rangle$ *for some* $b_j \in \{b_1, \dots, b_n\}$. *Then the normalizer of* $\langle b_j \rangle$ *is* $N_\Gamma(\langle b_j \rangle) = Z_\Gamma(b_j) = \langle b_j \rangle$.

Proof. We prove (1b) and (2b), the proofs of (1a) and (2a) are similar.

(1b) Obviously, $\langle b_j \rangle < Z_\Gamma(b_j)$. We have to show $Z_\Gamma(b_j) < \langle b_j \rangle$. Let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} \in Z_\Gamma(b_j)$$

be in ab-normal form, $a^{(1)}, \dots, a^{(k)} \in E_h$, $b^{(1)}, \dots, b^{(l)} \in E_v$, $k, l \geq 0$. Then

$$a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} b_j = b_j a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} .$$

Assume first that $k \geq 1$. The ab-normal form of γb_j starts with $a^{(1)} \dots a^{(k)}$. Bringing also $b_j a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)}$ to this normal form, we must have in a first step $b_j a^{(1)} = a^{(1)} b$ for some $b \in E_v$, i.e. $\rho_v(b_j)(a^{(1)}) = a^{(1)}$, which is impossible by assumption, hence $k = 0$. This means $\gamma = b^{(1)} \dots b^{(l)}$ and

$$b^{(1)} \dots b^{(l)} b_j = b_j b^{(1)} \dots b^{(l)} .$$

By uniqueness of the ab-normal form of

$$b_j = b^{(l)^{-1}} \dots b^{(1)^{-1}} b_j b^{(1)} \dots b^{(l)}$$

we have $l = 0$ or $b^{(1)}, \dots, b^{(l)} \in \{b_j, b_j^{-1}\}$ and hence $\gamma = b^{(1)} \dots b^{(l)} \in \langle b_j \rangle$.

(2b) Obviously, we have $\langle b_j \rangle < N_\Gamma(\langle b_j \rangle)$. It remains to show that $N_\Gamma(\langle b_j \rangle) < \langle b_j \rangle$. Let $\gamma \in N_\Gamma(\langle b_j \rangle)$, then in particular $\gamma^{-1} b_j \gamma \in \langle b_j \rangle$, i.e. b_j is conjugate to a power of itself, hence by a result of Bridson-Haefliger (see Proposition 2.13) we conclude $\gamma^{-1} b_j \gamma \in \{b_j, b_j^{-1}\}$. If $\gamma^{-1} b_j \gamma = b_j$, then $\gamma \in Z_\Gamma(b_j) = \langle b_j \rangle$ and we are done. So from now on let us suppose that $\gamma^{-1} b_j \gamma = b_j^{-1}$ (we will see in the proof that this case is in fact not possible under the assumption $Z_\Gamma(b_j) = \langle b_j \rangle$), then

$$\gamma^{-2} b_j \gamma^2 = \gamma^{-1}(\gamma^{-1} b_j \gamma)\gamma = \gamma^{-1} b_j^{-1} \gamma = (\gamma^{-1} b_j \gamma)^{-1} = (b_j^{-1})^{-1} = b_j ,$$

i.e. $\gamma^2 \in Z_\Gamma(b_j) = \langle b_j \rangle$ (which however does *not* directly imply $\gamma \in \langle b_j \rangle$ in general). Let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)},$$

$k, l \geq 0$, be the ab-normal form of γ. We first assume that $k \geq 1$, in particular $\gamma \neq 1$. Then

$$\gamma^2 = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} = b_j^s \tag{1.2}$$

for some $s \in \mathbb{Z} \setminus \{0\}$ (we know that $s \neq 0$, since $\gamma \neq 1$ and Γ is torsion-free). Note that it follows $l \geq 1$, otherwise we would have the contradiction $(a^{(1)} \dots a^{(k)})^2 = b_j^s$. The expression $b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)}$ is in ba-normal form, let $\tilde{a}^{(k)} \dots \tilde{a}^{(1)} \tilde{b}^{(1)} \dots \tilde{b}^{(l)}$ be its ab-normal form, i.e.

$$b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)} = \tilde{a}^{(k)} \dots \tilde{a}^{(1)} \tilde{b}^{(1)} \dots \tilde{b}^{(l)}. \tag{1.3}$$

Then, putting (1.3) into (1.2) gives

$$\gamma^2 = a^{(1)} \dots a^{(k)} \tilde{a}^{(k)} \dots \tilde{a}^{(1)} \tilde{b}^{(1)} \dots \tilde{b}^{(l)} b^{(1)} \dots b^{(l)} = b_j^s . \tag{1.4}$$

The right hand side b_j^s of equation (1.4) is in ab-normal form, hence the a's on the left hand side have to cancel (i.e. $\tilde{a}^{(k)} = {a^{(k)}}^{-1}, \dots, \tilde{a}^{(1)} = {a^{(1)}}^{-1}$, because $a^{(1)} \dots a^{(k)}$ and $\tilde{a}^{(k)} \dots \tilde{a}^{(1)}$ are freely reduced words in $\langle a_1, \dots, a_m \rangle$), so we have

$$b^{(1)} \dots b^{(l)} a^{(1)} \dots a^{(k)} = {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} \tilde{b}^{(1)} \dots \tilde{b}^{(l)} \tag{1.5}$$

from equation (1.3) and

$$\gamma^2 = \tilde{b}^{(1)} \dots \tilde{b}^{(l)} b^{(1)} \dots b^{(l)} = b_j^s \tag{1.6}$$

from equation (1.4). Moreover, since $b^{(1)} \dots b^{(l)}$ and $\tilde{b}^{(1)} \dots \tilde{b}^{(l)}$ are freely reduced words in $\langle b_1, \dots, b_n \rangle$, we conclude from equation (1.6) that s is even,

$$b^{(1)} \dots b^{(l)} = b^{(1)} \dots b^{(r)} b_j^t \tag{1.7}$$

and

$$\tilde{b}^{(1)} \dots \tilde{b}^{(l)} = b_j^t {b^{(r)}}^{-1} \dots {b^{(1)}}^{-1}, \tag{1.8}$$

where $t = s/2$ and $0 \leq r < l$ is the number of cancellations in

$$\tilde{b}^{(1)} \dots \tilde{b}^{(l)} b^{(1)} \dots b^{(l)},$$

i.e. $\tilde{b}^{(l)} b^{(1)} = 1, \dots, \tilde{b}^{(l-r+1)} b^{(r)} = 1$. Note that $|t| = l - r \geq 1$, in particular also the right hand sides of (1.7) and (1.8) are in normal form. First, we assume $r \geq 1$. Putting (1.7) and (1.8) into (1.5), we get

$$b^{(1)} \dots b^{(r)} b_j^t a^{(1)} \dots a^{(k)} = {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j^t {b^{(r)}}^{-1} \dots {b^{(1)}}^{-1}. \tag{1.9}$$

Since both sides of equation (1.9) are in normal form, we have (looking at the right ends)

$$b_j^{\pm 1} a^{(1)} \dots a^{(k)} = w_k(a) {b^{(1)}}^{-1} \tag{1.10}$$

and (looking at the left ends)

$$ {a^{(k)}}^{-1} \dots {a^{(1)}}^{-1} b_j^{\pm 1} = b^{(1)} \tilde{w}_k(a) , \tag{1.11}$$

where $w_k(a)$ and $\tilde{w}_k(a)$ are freely reduced words of length k in $\langle a_1, \ldots, a_m \rangle$, and the sign of b_j in (1.10) and (1.11) is according to the sign of t, i.e. we have b_j, if t is positive, and b_j^{-1}, if t is negative. Now, equation (1.11) gives

$$a^{(1)} \ldots a^{(k)} = b_j^{\pm 1} \tilde{w}_k^{-1}(a) b^{(1)^{-1}}. \tag{1.12}$$

Putting (1.12) into (1.10) gives

$$b_j^{\pm 2} \tilde{w}_k^{-1}(a) b^{(1)^{-1}} = w_k(a) b^{(1)^{-1}}, \tag{1.13}$$

i.e. the contradiction $b_j^{\pm 2} = w_k(a)\tilde{w}_k(a) \in \langle a_1, \ldots, a_m \rangle$. Thus, we have to study the remaining case $r = 0$, i.e. $|t| = l = |s|/2$ and

$$\gamma = a^{(1)} \ldots a^{(k)} b_j^t \,.$$

Then equation (1.5) or (1.9) is

$$b_j^t a^{(1)} \ldots a^{(k)} = a^{(k)^{-1}} \ldots a^{(1)^{-1}} b_j^t \,, \tag{1.14}$$

which is equivalent to

$$a^{(k)^{-1}} \ldots a^{(1)^{-1}} b_j = b_j^t a^{(1)} \ldots a^{(k)} b_j^{1-t}. \tag{1.15}$$

The equation $\gamma^{-1} b_j \gamma = b_j^{-1}$ is equivalent to

$$b_j^{-t} a^{(k)^{-1}} \ldots a^{(1)^{-1}} b_j a^{(1)} \ldots a^{(k)} b_j^t = b_j^{-1}. \tag{1.16}$$

Putting (1.15) into (1.16) gives

$$b_j^{-t} b_j^t a^{(1)} \ldots a^{(k)} b_j^{1-t} a^{(1)} \ldots a^{(k)} b_j^t = b_j^{-1} \tag{1.17}$$

or equivalently

$$a^{(1)} \ldots a^{(k)} b_j^{1-t} = b_j^{-1-t} a^{(k)^{-1}} \ldots a^{(1)^{-1}}, \tag{1.18}$$

which is a contradiction, since both sides of the equation are in normal form, but $t = s/2 \neq 0$ and hence

$$|b_j^{1-t}| = |1-t| \neq |-1-t| = |b_j^{-1-t}| \,.$$

This means that the case $k \geq 1$ is impossible. It remains to consider the case $k = 0$, i.e. $\gamma = b^{(1)} \ldots b^{(l)}$ for some $l \geq 0$. But then, $\gamma^{-1} b_j \gamma = b_j^{-1}$ gives a non-trivial relation in the free group $\langle b_1, \ldots, b_n \rangle$.

$\square$

Remark. The assumptions made in Proposition 1.12(1a),(1b) are sufficient but not necessary, as shown in Theorem 2.3(10).

Chapter 2

Normal subgroup structure, simplicity

The main goal of this chapter is to construct explicit examples of finitely presented torsion-free simple groups (Section 2.5). We choose a step-by-step approach by which we explain the main ingredients of the proof and produce other interesting groups, e.g. a non-residually finite (non-simple) group. In a first step, we apply the important "normal subgroup theorem" of Burger-Mozes and thus get in Section 2.1 for example an (A_6, A_6)–group without non-trivial normal subgroups of infinite index. The same holds for an (A_6, M_{12})–group and an $(A_6, \mathrm{ASL}_3(2))$–group constructed in that section. We believe that these three groups are non-residually finite and have a simple subgroup of index 4, but a proof seems to be hard. Instead of that, we construct in Section 2.2 a non-residually finite $(4, 12)$–group, applying another criterion of Burger-Mozes. This group has non-trivial normal subgroups of infinite index by construction, but we can embed it as a subgroup for example in an (A_6, A_{16})–group where the normal subgroup theorem applies. Consequently, this $(6, 16)$–group is virtually simple (Section 2.3). We think that it has a simple subgroup of index 4, but again it is not clear how to prove it. We evade this problem by taking another non-residually finite group (Section 2.4) constructed by Wise, using completely different ideas than those used in the Burger-Mozes criterion. Explicitly knowing a non-trivial element in the intersection of all finite index normal subgroups of Wise's $(8, 6)$–group, we are able to prove that this group can be embedded for example in an (A_{10}, A_{10})–group which has a simple subgroup of index 4 (Section 2.5). We give other examples of virtually simple $(2m, 2n)$–groups where the simple subgroup has index 4, among those an (M_{12}, A_8)–group, or where the simple subgroup has index bigger than 4, like another (A_{10}, A_{10})–group which has a simple subgroup of index 40. A slight variation of these techniques leads in Section 2.6 to an index 4 subgroup of a $(10, 10)$–group which has non-trivial normal subgroups of infinite index but no proper finite index subgroups. Following Wise, we construct in Section 2.7 a finitely presented group which is not virtually torsion-free, i.e. each finite index subgroup has a non-trivial element of finite order. In Section 2.8, we study what can happen if we replace in the normal subgroup theorem the 2-transitivity condition for the local group P_v by the slightly weaker condition that P_v is primitive. Comparing an (A_6, P_v)–group, where P_v is primitive but not 2-transitive, with the (A_{2m}, A_{2n})–groups constructed before, we observe that they seem to share the properties on the finite index normal subgroups but not on the infinite index normal subgroups. We discuss several ideas how to construct an explicit non-trivial normal subgroup of infinite index. Finally, we give in Section 2.9 smaller candidates for being finitely presented torsion-free simple groups; "smaller" in the sense that they have very short presentations. The example of Proposition 2.78 has a presentation with two generators and only three relations.

See Table 2.1 for an overview of some properties of several irreducible examples constructed in this chapter. The groups in Example 2.2, 2.30, 2.43 and the groups in Example 2.26, 2.52, 2.58, respectively, seem to have the same properties in the list. They are completely proved for Example 2.43 and Example 2.52. We have included in the table an example of Chapter 3 which has no non-trivial normal subgroups of infinite index, but behaves completely differently than the examples in Chapter 2, for example it is linear, hence residually finite. The following abbreviations are used in the table: "tr", "prim", "q-prim", "Y" and "N" stand for "transitive", "primitive", "quasi-primitive", "yes" and "no", respectively. Moreover, the $(2m, 2n)$–groups are always called Γ, and Γ^* denotes the normal subgroup of Γ

$$\Gamma^* := \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N \,,$$

where "f.i." stands for "finite index".

Example Γ	2.2	2.30	2.43	2.26	2.52	2.58	3.26
P_h	2-tr	2-tr	2-tr	2-tr	tr	2-tr	2-tr
P_v	2-tr	2-tr	2-tr	q-prim	2-tr	prim	2-tr
irreducible	Y	Y	Y	Y	Y	Y	Y
not linear	Y	Y	Y	Y	Y	Y	N
Γ_0 perfect	Y	Y	Y	Y	Y	Y	N
$\Gamma_0 = [\Gamma, \Gamma]$	Y	Y	Y	Y	Y	Y	N
non-residually finite	Y ?	Y	Y	Y	Y	Y ?	N
all proper quotients finite	Y	Y	Y	N	N	N	Y
$H_b^2(\Gamma; \mathbb{R}) = 0$	Y	Y	Y	N	N	N	Y
$\Gamma^* = \Gamma_0$	Y ?	Y ?	Y	Y ?	Y	Y ?	N
Γ_0 simple	Y ?	Y ?	Y	N	N	N	N

Table 2.1: Subgroup properties for some examples of Chapter 2

2.1 Normal subgroup theorem

We construct examples of $(2m, 2n)$–groups without non-trivial normal subgroups of infinite index, applying the crucial "normal subgroup theorem" due to Burger-Mozes (see [15, Theorem 4], [17, Theorem 4.1, Corollary 5.1, Corollary 5.3]). Here is an adapted special version of it:

Proposition 2.1. *(Burger-Mozes, see [17, Chapter 4 and 5]) Let Γ be an irreducible $(2m, 2n)$–group such that P_h, P_v are 2-transitive, and $\mathrm{Stab}_{P_h}(\{1\})$, $\mathrm{Stab}_{P_v}(\{1\})$ are non-abelian simple groups. Then any non-trivial normal subgroup of Γ has finite index in Γ.*

Proof. Combine [17, Corollary 5.1, Proposition 5.2, Corollary 5.3]. □

Concretely, we will apply Proposition 2.1 to irreducible $(2m, 2n)$–groups such that (P_h, P_v) belongs to the set

$$\{(A_{2m}, A_{2n}), (A_{2m}, M_{12}), (A_{2m}, \mathrm{ASL}_3(2)), (M_{12}, A_{2n}), (\mathrm{ASL}_3(2), A_{2n})\},$$

where $2m \geq 6$, $2n \geq 6$, $M_{12} < S_{12}$ and $\mathrm{ASL}_3(2) < S_8$. In particular, we will construct in this section two (A_6, A_6)–groups (Example 2.2 and Example 2.15), an (A_6, M_{12})–group (Example 2.18) and an $(A_6, \mathrm{ASL}_3(2))$–group (Example 2.21) without non-trivial normal subgroups of infinite index. See [16, Section 3.3] for a list of finite permutation groups satisfying the assumptions made on the local groups P_h and P_v in Proposition 2.1.

Note that the smallest groups without non-trivial normal subgroups of infinite index appearing in [15, 16, 17], are an (A_{30}, A_{38})–group ([17, Theorem 6.3]) and a certain $(14, 18)$–group (see also Example 3.26), to which Proposition 2.1 does not apply but the more general original result [17, Theorem 4.1].

All examples of $(2m, 2n)$–groups will be given only in terms of the set of mn relators $R_{m \cdot n}$. The corresponding presentation of Γ is

$$\langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle,$$

and it determines the groups P_h, P_v, Γ_0, H_1, H_2, Λ_1, Λ_2 and the complex X as explained in Chapter 1.

Example: (A_6, A_6)–group

We give a first small example to which Proposition 2.1 can be applied.

Example 2.2.

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2b_2^{-1}, \\ a_1b_3^{-1}a_3^{-1}b_2, & a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, \\ a_2b_3a_3^{-1}b_1, & a_2b_3^{-1}a_3b_2, & a_2b_1^{-1}a_3^{-1}b_1^{-1} \end{array} \right\}.$$

Theorem 2.3. *Let Γ be the $(6,6)$–group defined by $R_{3\cdot 3}$ in Example 2.2. Then*

(1) $P_h = A_6$, $P_v = A_6$.

(2) Γ *is irreducible.*

(3) *Any non-trivial normal subgroup of Γ has finite index.*

(4) $[\Gamma, \Gamma] = \Gamma_0$ *and Γ_0 is perfect.*

(5) Γ *is not linear over any field.*

(6) Γ *can be decomposed in two ways as an amalgamated free product of finitely generated free groups $\Gamma \cong F_3 *_{F_{13}} F_7$. Its subgroup Γ_0 has two amalgam decompositions $F_5 *_{F_{25}} F_5$.*

(7) $\Gamma \cong \mathrm{pr}_i(\Gamma) \lneqq H_i = \overline{\mathrm{pr}_i(\Gamma)}$, $i = 1, 2$.

(8) $H_b^2(\Gamma; \mathbb{R}) = 0$, *i.e. the second bounded cohomology of Γ with $\mathbb{R}$-coefficients vanishes.*

(9) $\mathrm{Aut}(X) \cong \mathbb{Z}_2$ *and* $\mathrm{Out}(\Gamma) \neq 1$.

(10) *We have $Z_\Gamma(a_i) = N_\Gamma(\langle a_i \rangle) = \langle a_i \rangle$, if $a_i \in \{a_2, a_3\}$ and $Z_\Gamma(b_j) = N_\Gamma(\langle b_j \rangle) = \langle b_j \rangle$, if $b_j \in \{b_2, b_3\}$.*

Proof. (1) We only list the generators of P_h and P_v. It can easily be checked for example with GAP ([29]), that these permutations indeed generate A_6.

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (1,5,4,2,3), \\ \rho_v(b_3) &= (2,3,5,4,6), \text{ generating } P_h = A_6. \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,3)(4,5), \\ \rho_h(a_2) &= (1,6,3,2)(4,5), \\ \rho_h(a_3) &= (1,4,5,6)(2,3), \text{ generating } P_v = A_6. \end{aligned}$$

(2) We compute $|P_h^{(2)}| = 360 \cdot 60^6$ and apply Proposition 1.2(1a).

(3) We apply Proposition 2.1 or [17, Corollary 5.3], using the facts that P_h and P_v are 2-transitive (in fact 4-transitive), that the stabilizers

$$\mathrm{Stab}_{P_h}(\{1\}) = \langle (2,3)(4,5), (2,3,5,4,6) \rangle \cong A_5\,,$$
$$\mathrm{Stab}_{P_v}(\{1\}) = \langle (2,3)(4,5), (2,4,5), (4,5,6) \rangle \cong A_5$$

are non-abelian simple groups and that Γ is irreducible by part (2).

(4) These are easy computations using GAP ([29]). To see by hand that Γ_0 is perfect, one first computes a presentation of Γ_0 by the Reidemeister-Schreier method (see e.g. [49, Section II.4]) and then adds commutators to the relators to simplify the presentation.

(5) It follows from [17, Theorem 1.4], see also Proposition 4.4 in Section 4.2.

(6) Use Proposition 1.3 and Proposition 1.4. Explicit amalgam decompositions of Γ and Γ_0 are described in Appendix A.2.

(7) By [16, Proposition 3.1.2, 1)], the quasi-center $\mathrm{QZ}(H_i)$ is trivial for $i = 1, 2$, hence the homomorphism pr_{3-i} is injective, which shows that $\Gamma \cong \mathrm{pr}_{3-i}(\Gamma)$. The group H_i is by [16, Proposition 3.3.1] isomorphic to the universal group $U(A_6)$, which is not torsion-free, thus $\mathrm{pr}_i(\Gamma) \cong \Gamma \neq H_i$.

(8) We have noticed in the proof of part (7) that $H_i \cong U(A_6)$, $i = 1, 2$. Hence, by [16, Chapter 3], H_1 and H_2 act transitively on the boundary at infinity $\partial_\infty \mathcal{T}_6$ of their corresponding trees $\mathcal{T}_{2m} = \mathcal{T}_6$ and $\mathcal{T}_{2n} = \mathcal{T}_6$, respectively. The claim follows now from [14, Corollary 26]. As pointed out there, this result has some applications to Γ-actions on the circle S^1 (see [14, Corollary 22]).

(9) Checking all of the $2^6 6! = 46080$ candidates (using the GAP-program of Appendix B.7), we have found exactly one non-trivial automorphism given by $a_i \mapsto a_i^{-1}$, $i = 1, 2, 3$, $b_1 \mapsto b_1^{-1}$, $b_2 \mapsto b_3$, $b_3 \mapsto b_2$. It fixes seven of nine geometric squares. The two non-trivially permuted geometric squares of X are $[a_2 b_1 a_3^{-1} b_2^{-1}]$ and $[a_2 b_3 a_3^{-1} b_1]$. Note that this automorphism induces a non-trivial element in the group of outer automorphisms $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$, since it has order 2 but $\mathrm{Inn}(\Gamma) \cong \Gamma$ is torsion-free (the isomorphism $\mathrm{Inn}(\Gamma) \cong \Gamma$ holds because $\mathrm{Inn}(\Gamma) \cong \Gamma / Z\Gamma$ and $Z\Gamma = 1$ by Corollary 1.11(3)).

(10) The statements $Z_\Gamma(a_2) = N_\Gamma(\langle a_2 \rangle) = \langle a_2 \rangle$, $Z_\Gamma(a_3) = N_\Gamma(\langle a_3 \rangle) = \langle a_3 \rangle$, $N_\Gamma(\langle b_2 \rangle) = \langle b_2 \rangle$ and $N_\Gamma(\langle b_3 \rangle) = \langle b_3 \rangle$ follow from Proposition 1.12. We prove $Z_\Gamma(b_3) = \langle b_3 \rangle$. Similarly, one can prove $Z_\Gamma(b_2) = \langle b_2 \rangle$. Let

$$\gamma = a^{(1)} \dots a^{(k)} b^{(1)} \dots b^{(l)} \in Z_\Gamma(b_3)$$

be in ab-normal form, such that $a^{(1)}, \ldots, a^{(k)} \in E_h$, $b^{(1)}, \ldots, b^{(l)} \in E_v$ and $k, l \geq 0$. Then

$$a^{(1)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} b_3 = b_3 a^{(1)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} .$$

Assume first that $k = 1$, thus

$$a^{(1)} b^{(1)} \ldots b^{(l)} b_3 = b_3 a^{(1)} b^{(1)} \ldots b^{(l)} .$$

The ab-normal form of $a^{(1)} b^{(1)} \ldots b^{(l)} b_3$ starts with $a^{(1)}$. Bringing also the right hand side $b_3 a^{(1)} b^{(1)} \ldots b^{(l)}$ to this normal form, we must have in a first step $b_3 a^{(1)} = a^{(1)} b$ for some $b \in E_v$. Checking all elements in $R_{3 \cdot 3}$, the only possibility is $a^{(1)} = a_1$, $b = b_2$, hence

$$a_1 b^{(1)} \ldots b^{(l)} b_3 = a_1 b_2 b^{(1)} \ldots b^{(l)}$$

or equivalently

$$b^{(1)} \ldots b^{(l)} b_3 = b_2 b^{(1)} \ldots b^{(l)} ,$$

but this gives a non-trivial relation in the free group $\langle b_1, b_2, b_3 \rangle$.

Assume now that $k \geq 2$. As in the case $k = 1$, we conclude $a^{(1)} = a_1$ and $b_3 a^{(1)} = a_1 b_2$, i.e.

$$a_1 a^{(2)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} b_3 = a_1 b_2 a^{(2)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)}$$

hence

$$a^{(2)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} b_3 = b_2 a^{(2)} \ldots a^{(k)} b^{(1)} \ldots b^{(l)} .$$

The ab-normal form of the left hand side of the last equation starts with $a^{(2)}$. Bringing the right hand side to this normal form, we must have $b_2 a^{(2)} = a^{(2)} b$ for some $b \in E_v$. Here, the only possibility is $a^{(2)} = a_1^{-1}$, $b = b_3$, but this contradicts the fact that $a^{(1)} a^{(2)} \ldots a^{(k)} = a_1 a_1^{-1} \ldots a^{(k)}$ is freely reduced.

It follows that $k = 0$, and we conclude $\gamma \in \langle b_3 \rangle$ exactly as in the proof of Proposition 1.12(1b).

Note that $Z_\Gamma(a_1) = Z_\Gamma(b_1) = \langle a_1, b_1 \rangle_\Gamma \cong \mathbb{Z}^2$.

□

The $(6, 6)$–group Γ of Example 2.2 can be used to simplify certain constructions of infinite families made in [17], see also Proposition 2.29.

Proposition 2.4. *(See [17, Theorem 6.3] for the same statement but with lower bounds $m \geq 15$, $n \geq 19$) For every $m \geq 7$ and $n \geq 7$, there exists a torsion-free cocompact lattice $\Lambda < U(A_{2m}) \times U(A_{2n})$ with dense projections. Any non-trivial normal subgroup $N \lhd \Lambda$ is of finite index in Λ.*

Proof. We follow the proof of [17, Theorem 6.3]. The only difference is that we can replace the $(\mathrm{PSL}_2(13), \mathrm{PSL}_2(17))$–complex ${}^{(0)}X = \mathcal{A}_{13,17}$ used there (see also Example 3.26 and Proposition 3.27 for a description of that $(14, 18)$–complex) by our (A_6, A_6)–complex X of Example 2.2. An illustration of this construction is given in Appendix A.3 for the smallest values $m = 7$, $n = 7$ of Proposition 2.4. □

We believe that apart from having no non-trivial normal subgroups of infinite index, the group Γ of Example 2.2 also has only very few normal subgroups of finite index. More precisely, we think that Γ is non-residually finite, virtually simple, and that its subgroup Γ_0 is simple.

Conjecture 2.5. *Let Γ be the $(6, 6)$–group defined in Example 2.2. Then Γ_0 is a finitely presented torsion-free simple group.*

The following elementary lemmas lead to Proposition 2.10 which could be useful in a proof of Conjecture 2.5.

Lemma 2.6. *Let G be a group and $H < G$ a subgroup of finite index. Then there is a group $N < H$ such that $N \lhd G$ and $[G : N] \leq [G : H]! < \infty$, in particular*

$$\bigcap_{M \overset{f.i.}{<} G} M = \bigcap_{L \overset{f.i.}{\lhd} G} L\,.$$

Proof. (Probably due to Hall Jr. [31]) Let k be the finite index $[G : H]$ and write G as a disjoint finite union of left cosets

$$G = \bigsqcup_{i=1}^{k} g_i H\,.$$

Left multiplication $g_i H \mapsto g g_i H$ induces a homomorphism $\phi : G \to S_k$ such that $N := \ker \phi < H$ and $[G : N] \leq |S_k| = [G : H]! < \infty$. Note that

$$N = \bigcap_{g \in G} g H g^{-1}.$$

□

Lemma 2.7. *Let G be a group and $H \lhd G$ a normal subgroup of finite index. Assume that there is an element $h \in H$ such that $\langle\langle h^k \rangle\rangle_G > H$ for each $k \in \mathbb{N}$. Then every proper normal subgroup of H has infinite index.*

Proof. Let $N \lhd H$ be a normal subgroup of finite index. By Lemma 2.6, there is a group $M < N$ such that $M \lhd G$ and $[G : M] < \infty$. Looking at the left cosets of the form $h^k M$, $k \in \mathbb{N}$, we see that at least two of them are equal, in particular $h^i \in M$ for some $i \in \mathbb{N}$, thus $\langle\langle h^i \rangle\rangle_G < M$. By assumption, we have $H < \langle\langle h^i \rangle\rangle_G$, hence $H < M$ and $M = N = H$. □

Lemma 2.8. *Let G be a group and let H, M be two subgroups of G such that M has finite index in G. Then $[H : (M \cap H)] \leq [G : M] < \infty$.*

Proof. Let k be the finite index $[G : M]$ and write

$$G = \bigsqcup_{i=1}^{k} Mg_i \,.$$

Then, intersecting with H, we get

$$H = G \cap H = \bigsqcup_{i=1}^{k} (Mg_i \cap H)\,.$$

Fix $i \in \{1, \ldots, k\}$. If $Mg_i \cap H \neq \emptyset$, take any element $mg_i = h \in Mg_i \cap H$. Then $Mg_i \cap H = Mmg_i \cap H = Mh \cap H = Mh \cap Hh = (M \cap H)h$ and we are done. □

Lemma 2.9. *Let G be a group and $H < G$ a subgroup of finite index. Then*

$$\bigcap_{N \overset{f.i.}{\lhd} H} N = \bigcap_{N \overset{f.i.}{\lhd} G} N\,.$$

In particular, H is residually finite if and only if G is residually finite.

Proof.

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} H} N = \bigcap_{M \overset{\text{f.i.}}{<} H} M = \bigcap_{M \overset{\text{f.i.}}{<} G} M = \bigcap_{N \overset{\text{f.i.}}{\lhd} G} N\,,$$

where the first and third equalities follow from Lemma 2.6. The inclusion "$\supseteq$" in the second equality is obvious, whereas "$\subseteq$" in the second equality directly follows from Lemma 2.8. □

Proposition 2.10. *Suppose that Γ satisfies the assumptions of the normal subgroup theorem (Proposition 2.1). Let $H \lhd \Gamma$ be a non-trivial normal subgroup of Γ and assume that there is an element $h \in H$ such that $\langle\langle h^k \rangle\rangle_\Gamma > H$ holds for each $k \in \mathbb{N}$. Then H is a finitely presented torsion-free simple group.*

Proof. First note that by assumption H has finite index in Γ. By Lemma 2.7

$$H = \bigcap_{N \overset{\text{f.i.}}{\lhd} H} N$$

and hence by Lemma 2.9

$$H = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N\,.$$

In particular, Γ is non-residually finite and [17, Corollary 5.4] shows that H is simple. It is obvious that H is finitely presented and torsion-free, since it is a finite index subgroup of the finitely presented torsion-free group Γ. □

Corollary 2.11. *Let Γ be as in Example 2.2. Assume that there is an element $\gamma_0 \in \Gamma_0$ such that $\langle\langle \gamma_0^k \rangle\rangle_\Gamma = \Gamma_0$ for each $k \in \mathbb{N}$. Then Γ_0 is a finitely presented torsion-free simple group.*

Proof. This follows directly from Proposition 2.10 using the fact (see Theorem 2.3(3)) that any non-trivial normal subgroup of Γ has finite index. $\square$

One step towards the proof of Conjecture 2.5 (or an application of Corollary 2.11) could be the following proposition.

Proposition 2.12. *For Γ as defined in Example 2.2, we have $\langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma = \Gamma_0$ for each $k \in \mathbb{N}_0$.*

Proof. We first prove two auxiliary results: The first one says that for each $k \in \mathbb{N}_0$

$$b_3^{-1} b_2 a_1^{6(1+2k)} b_2^{-1} b_3 = a_2^{-6(1+2k)} .$$

Since $a_1^{6(1+2k)}$ and $a_2^{-6(1+2k)}$ are claimed to be conjugate, we only have to show it for $k = 0$, i.e. $b_3^{-1} b_2 a_1^6 b_2^{-1} b_3 = a_2^{-6}$. But this follows bringing the left hand side of the equation to its ab-normal form.

The second result needed is the following: For each $k \in \mathbb{N}_0$

$$a_2 b_3 b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} = a_2^{6(1+2k)} b_2 b_1 .$$

This proof is by induction on k. If $k = 0$,

$$a_2 b_3 b_2 b_3^{-1} a_1^6 b_3 b_2^{-1} b_3^{-1} a_2^{-1} = a_2^6 b_2 b_1$$

again follows by computing the ab-normal form of the left hand side. For the induction step $k \to k + 1$, we get

$$\begin{aligned}
& a_2 b_3 b_2 b_3^{-1} a_1^{6(1+2(k+1))} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2 b_3 b_2 b_3^{-1} a_1^{12} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \\
&= a_2^{12} a_2 b_3 b_2 b_3^{-1} a_1^{6(1+2k)} b_3 b_2^{-1} b_3^{-1} a_2^{-1} \quad (\text{using } b_3 b_2 b_3^{-1} a_1^{12} = a_2^{12} b_3 b_2 b_3^{-1}) \\
&= a_2^{12} a_2^{6(1+2k)} b_2 b_1 \quad (\text{by the induction hypothesis}) \\
&= a_2^{6(1+2(k+1))} b_2 b_1
\end{aligned}$$

as required. Now we are ready to prove the proposition. Since $a_1^2 \in \Gamma_0$, one inclusion is obvious:

$$\langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma < \Gamma_0 .$$

For the other inclusion we have by our first auxiliary result

$$a_2^{-6(1+2k)} \in \langle\langle a_1^{6(1+2k)} \rangle\rangle_\Gamma ,$$

and by the second one

$$a_2^{6(1+2k)} b_2 b_1 \in \langle\!\langle a_1^{6(1+2k)} \rangle\!\rangle_\Gamma ,$$

hence together

$$b_2 b_1 \in \langle\!\langle a_1^{6(1+2k)} \rangle\!\rangle_\Gamma . \tag{2.1}$$

Next, we observe that $b_1^2 \in \langle\!\langle b_2 b_1 \rangle\!\rangle_\Gamma$ since

$$(a_1 a_2^{-1} b_2 b_1 a_2 a_1^{-1})(a_1^2 b_2 b_1 a_1^{-2}) = b_1^2 .$$

Moreover, $a_1 a_3^{-1} \in \langle\!\langle b_1^2 \rangle\!\rangle_\Gamma < \langle\!\langle b_2 b_1 \rangle\!\rangle_\Gamma$, since

$$(a_1 a_2^{-1} b_1^{-2} a_2 a_1^{-1})(a_1^{-1} a_2^{-1} b_1^2 a_2 a_1) = a_1 a_3^{-1} .$$

It is easy to check that Γ_0 is generated (as a subgroup of Γ) by $\{a_1 a_3^{-1}, b_1^2\}$ and we conclude that

$$\Gamma_0 = \langle a_1 a_3^{-1}, b_1^2 \rangle_\Gamma < \langle\!\langle b_2 b_1 \rangle\!\rangle_\Gamma \overset{(2.1)}{<} \langle\!\langle a_1^{6(1+2k)} \rangle\!\rangle_\Gamma .$$

□

Remark. A calculation with MAGNUS ([50]) shows, that moreover

$$\langle\!\langle a_1^{12} \rangle\!\rangle_\Gamma = \langle\!\langle a_1^{24} \rangle\!\rangle_\Gamma = \Gamma_0 .$$

See Table 2.2 for the orders of some quotients of Γ, illustrating that Conjecture 2.5 could be reasonable.

$\|\Gamma/\langle\!\langle w^k \rangle\!\rangle_\Gamma\|$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1, a_2, a_3$	2	4	2	4	2	4	2	4	2	4	2	4
b_1, b_2, b_3	2	4	2	4	2	4	2	4	2	4	2	4

Table 2.2: Some orders of $\Gamma/\langle\!\langle w^k \rangle\!\rangle_\Gamma$ in Example 2.2

In order to prove that Γ_0 has no proper finite index subgroups, it could be useful to have a non-trivial element $\gamma \in \Gamma$ such that γ^k and γ^l are conjugate for some $k, l \in \mathbb{Z}$, where $|k| \neq |l|$. As an illustration, we mention that Bhattacharjee has constructed in [7] an amalgam without non-trivial finite quotients, essentially using in the proof that there is a non-trivial element a such that a^2 and a^5 are conjugate. However, this technique is not possible for $(2m, 2n)$–groups by the following proposition which is a special case of a result of Bridson-Haefliger ([9]):

Proposition 2.13. *(Bridson-Haefliger [9]) Let Γ be a $(2m, 2n)$–group and let $\gamma \in \Gamma$ be a non-trivial element. Then γ^k can only be conjugate to γ^l if $|k| = |l|$.*

Proof. (Sketch, following Bridson-Haefliger [9]) Assume that γ^k and γ^l are conjugate for some $k, l \in \mathbb{Z}$. Then by [9, Proposition II.6.2(2)], γ^k and γ^l have the same translation length, and by [9, Theorem II.6.8(1)] we have $|k| = |l|$, using the fact that the element γ acts as a hyperbolic isometry on the CAT(0)-space $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$. □

By results of Wiegold-Wilson given in [67], the observation that Γ_0 has no proper subgroups of small index is somehow reflected in the next proposition on the slow growth of the number of generators of direct powers. Recall that we denote by $d(G)$ the minimal number of elements needed to generate the group G and by G^k the direct product of k copies of G.

Proposition 2.14. *Let Γ be the group of Example 2.2 and l a positive even integer. Suppose that $\langle\langle w \rangle\rangle_{\Gamma_0} = \Gamma_0$ for all words $w \in \Gamma_0$ of even length $2, 4, \ldots, 2l$. Let*

$$b(l) := \frac{1}{2}|\{w \in \Gamma_0 : 2 \leq |w| \leq l\}| .$$

Then $d(\Gamma_0^k) \leq 3$ for each $k \leq b(l)$.

Proof. (Adapted from [67, Proof of Theorem 4.2]) Since $w \neq w^{-1}$ and $|w| = |w^{-1}|$ for any non-trivial element $w \in \Gamma$, we can choose a subset

$$S = \{\gamma_1, \ldots, \gamma_{b(l)}\} \subset \Gamma_0$$

of cardinality $b(l)$ such that $S \cap S^{-1} = \emptyset$, and $2 \leq |\gamma_i| \leq l$ for all $\gamma_i \in S$. It follows that $|\gamma_{i_1}\gamma_{i_2}^{-1}| \in \{2, 4, \ldots, 2l\}$ whenever $\gamma_{i_1}, \gamma_{i_2}$ are different elements of S. By assumption $\langle\langle \gamma_{i_1}\gamma_{i_2}^{-1} \rangle\rangle_{\Gamma_0} = \Gamma_0$. Note that Γ_0 is generated by two elements, for example by $\{a_1^2, b_2 b_1^{-1}\}$. We want to show by induction that Γ_0^k is for each $k \leq b(l)$ generated by the element $(\gamma_1, \ldots, \gamma_k)$ and the diagonal subgroup of Γ_0^k (which is for example generated by the two diagonal elements $(a_1^2, \ldots, a_1^2)$ and $(b_2 b_1^{-1}, \ldots, b_2 b_1^{-1})$ in Γ_0^k). For $k = 1$, this is obviously true. We assume that $2 \leq k \leq b(l)$ is fixed and that Γ_0^{k-1} is generated by its diagonal subgroup and $(\gamma_1, \ldots, \gamma_{k-1})$. Let H be the subgroup of Γ_0^k generated by the diagonal subgroup of Γ_0^k and $(\gamma_1, \ldots, \gamma_k)$. Our goal is to show that $H = \Gamma_0^k$. If we think Γ_0^{k-1} embedded in Γ_0^k as a subgroup $\Gamma_0^{k-1} \times \{1\} < \Gamma_0^{k-1} \times \Gamma_0 = \Gamma_0^k$, then for any $\gamma \in \Gamma_0$ the group H contains by assumption $k - 1$ elements of the form

$$(\gamma, 1, \ldots, 1, *), \ \ldots \ , (1, \ldots, 1, \gamma, *) ,$$

where "$*$" are certain elements in Γ_0 we do not have to care about. By construction, H also contains the element

$$(\gamma_1\gamma_k^{-1}, \ldots, \gamma_{k-1}\gamma_k^{-1}, 1) = (\gamma_1, \ldots, \gamma_k) \cdot (\gamma_k^{-1}, \ldots, \gamma_k^{-1}) .$$

Computing the $k-1$ commutators

$$[(\gamma, 1, \ldots, 1, *), (\gamma_1\gamma_k^{-1}, \ldots, \gamma_{k-1}\gamma_k^{-1}, 1)],$$
$$\vdots$$
$$[(1, \ldots, 1, \gamma, *), (\gamma_1\gamma_k^{-1}, \ldots, \gamma_{k-1}\gamma_k^{-1}, 1)],$$

we see that H contains the $k-1$ elements

$$([\gamma, \gamma_1\gamma_k^{-1}], 1, \ldots, 1), \ \ldots\ , (1, \ldots, 1, [\gamma, \gamma_{k-1}\gamma_k^{-1}], 1).$$

For $j = 1, \ldots, k-1$, let N_j be the subgroup of Γ_0

$$N_j := \langle [\gamma, \gamma_j\gamma_k^{-1}] : \gamma \in \Gamma_0 \rangle < \Gamma_0 .$$

Then N_j is a normal subgroup of Γ_0, since for each $g \in \Gamma_0$

$$g[\gamma, \gamma_j\gamma_k^{-1}]g^{-1} = [g\gamma, \gamma_j\gamma_k^{-1}] \cdot [g, \gamma_j\gamma_k^{-1}]^{-1} \in N_j .$$

Note that $\gamma_j\gamma_k^{-1}N_j \in Z(\Gamma_0/N_j)$, by definition of N_j. Since $\langle\langle \gamma_j\gamma_k^{-1} \rangle\rangle_{\Gamma_0} = \Gamma_0$, we have $\langle\langle \gamma_j\gamma_k^{-1}N_j \rangle\rangle_{\Gamma_0/N_j} = \Gamma_0/N_j$ and $Z(\Gamma_0/N_j) = \Gamma_0/N_j$, i.e. Γ_0/N_j is abelian. But then $N_j = \Gamma_0$, because Γ_0 is perfect. In particular, Γ_0 is generated by the elements $[\gamma, \gamma_j\gamma_k^{-1}]$ and H contains therefore the j-th direct factor of Γ_0^k. Since

$$(1, \ldots, 1, \gamma) = (\gamma, \ldots, \gamma) \cdot (\gamma^{-1}, 1, \ldots, 1) \cdot \ldots \cdot (1, \ldots, 1, \gamma^{-1}, 1),$$

H also contains the k-th direct factor of Γ_0^k, therefore $H = \Gamma_0^k$ and Γ_0^k is generated by three elements. □

Remark. We have used GAP ([29]) to check that $\langle\langle w \rangle\rangle_{\Gamma_0} = \Gamma_0$, whenever $w \in \Gamma_0$ has length 2, 4, or 6. Note that $b(2) = 30$, $b(4) = 1230$, $b(6) = 42480$, $b(8) = 1354980$.

Another example of an (A_6, A_6)–group

In most of our main examples (e.g. Example 2.2, 2.18, 2.21, 2.26, 2.30, 2.33, 2.43, 2.46, 2.52 and 2.58) of this chapter, we always have $[\Gamma, \Gamma] = \Gamma_0$, where in addition Γ_0 is perfect. The next example is different in this regard (see also Appendix C.6 for more such groups), but it shares many other properties with Example 2.2.

Example 2.15.

$$R_{3.3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_1, & a_1b_3a_1^{-1}b_3, \\ a_1b_2^{-1}a_2b_1^{-1}, & a_2b_1a_3^{-1}b_3^{-1}, & a_2b_2a_3^{-1}b_3, \\ a_2b_3a_3^{-1}b_2, & a_2b_3^{-1}a_3^{-1}b_1, & a_3b_1a_3b_2 \end{array} \right\}.$$

Theorem 2.16. *Let* Γ *be the* $(6, 6)$*–group defined in Example 2.15.*

(1) *The statements of Theorem 2.3(1)-(3) and (5)-(8) also hold for this* Γ.

(2) $[\Gamma, \Gamma]$ *is not perfect, of index* 32 *in* Γ*, and* Γ_0 *is not perfect either.*

Proof. (1) We can use the same arguments as in the proof of Theorem 2.3, of course with different generators of P_h and P_v:

$$\begin{aligned}
\rho_v(b_1) &= (1, 5, 4, 3, 2),\\
\rho_v(b_2) &= (2, 6, 5, 4, 3),\\
\rho_v(b_3) &= (2, 3)(4, 5),\\
\rho_h(a_1) &= (1, 5, 6, 2)(3, 4),\\
\rho_h(a_2) &= (1, 5, 3)(2, 6, 4),\\
\rho_h(a_3) &= (1, 3, 5)(2, 4, 6).
\end{aligned}$$

(2) It is easy to check that $[\Gamma, \Gamma]$ is the kernel of the surjective homomorphism

$$\begin{aligned}
\Gamma &\to \mathbb{Z}_2^2 \times \mathbb{Z}_8\\
a_1 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 0 + 8\mathbb{Z})\\
a_2 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 6 + 8\mathbb{Z})\\
a_3 &\mapsto (0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 1 + 8\mathbb{Z})\\
b_1 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}, 3 + 8\mathbb{Z})\\
b_2 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}, 3 + 8\mathbb{Z})\\
b_3 &\mapsto (1 + 2\mathbb{Z}, 1 + 2\mathbb{Z}, 0 + 8\mathbb{Z})\,.
\end{aligned}$$

Note that the commutator subgroup of $[\Gamma, \Gamma]$ has index 6 in $[\Gamma, \Gamma]$ and that $\langle\!\langle a_1^2 \rangle\!\rangle_\Gamma$ is a perfect normal subgroup of Γ of index 192. See Table 2.3 for the orders of some other quotients. Moreover, $[\Gamma_0, \Gamma_0]$ has index $64 = 4 \cdot 16$ in Γ, more precisely $\Gamma_0^{ab} \cong \mathbb{Z}_{16}$.

□

Conjecture 2.17. *Let* Γ *be the* (A_6, A_6)*–group defined in Example 2.15. Then* Γ *is non-residually finite such that*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = [[\Gamma, \Gamma], [\Gamma, \Gamma]] = \langle\!\langle a_1^{2k} \rangle\!\rangle_\Gamma$$

for each $k \in \mathbb{N}$*, and this subgroup of index* 192 *is simple.*

$\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w=a_1$	48	192	48	192	48	192	48	192	48	192	48	192
a_2	8	16	24	32	8	48	8	64	24	16	8	96
a_3	4	24	4	48	4	24	4	96	4	24	4	48
b_1, b_2	4	8	12	16	4	24	4	32	12	8	4	48
b_3	16	96	16	192	16	96	16	192	16	96	16	192

Table 2.3: Some orders of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$ in Example 2.15

Example: (A_6, M_{12})–group

The famous group M_{12} was discovered by Emile Mathieu in 1861. It can be described as a 5-transitive subgroup of A_{12} of order 95040 and belongs together with the other Mathieu groups M_{11}, M_{22}, M_{23} and M_{24} to the list of 26 sporadic finite simple groups. With the exception of symmetric and alternating groups, M_{12} and M_{24} are the only finite 5-transitive groups. See [25] for the relation to Steiner systems and more background information on Mathieu groups.

Example 2.18.

$$R_{3\cdot 6} := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_1^{-1}, & a_1b_3a_1^{-1}b_3^{-1}, \\ a_1b_4a_1^{-1}b_4^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_1^{-1}a_2b_2, & a_2b_1a_2b_3^{-1}, & a_2b_3a_2b_4^{-1}, \\ a_2b_4a_3^{-1}b_5^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_5^{-1}a_3b_4, & a_3b_1a_3^{-1}b_2^{-1}, & a_3b_2a_3^{-1}b_1^{-1}, \\ a_3b_3a_3b_6^{-1}, & a_3b_5a_3^{-1}b_4^{-1}, & a_3b_6a_3b_3^{-1} \end{array} \right\}.$$

Theorem 2.19. *Let Γ be the $(6, 12)$–group of Example 2.18. Then*

(1) $P_h = A_6$, $P_v \cong M_{12}$.

(2) *Any non-trivial normal subgroup of Γ has finite index.*

(3) *Γ is not linear over any field, in particular irreducible.*

(4) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (2, 6, 5),\\
\rho_v(b_2) &= (1, 2, 5),\\
\rho_v(b_3) &= (2, 5)(3, 4),\\
\rho_v(b_4) &= (2, 5, 4),\\
\rho_v(b_5) &= (2, 3, 5),\\
\rho_v(b_6) &= (2, 5)(3, 4),\\
\rho_h(a_1) &= (1, 2)(5, 6)(7, 8)(11, 12),\\
\rho_h(a_2) &= (1, 2, 7, 5, 4, 3)(6, 11, 12, 10, 9, 8),\\
\rho_h(a_3) &= (1, 2)(3, 6)(4, 5)(7, 10)(8, 9)(11, 12).
\end{aligned}$$

Observe that $P_v \cong M_{12}$ is already generated by $\rho_h(a_1) =: \sigma$ and $\rho_h(a_2) =: \tau$, since

$$\rho_h(a_3) = \sigma\tau^3\sigma\tau\sigma\tau^2\sigma\tau^2\sigma\tau\sigma\tau^3\sigma\ .$$

As a by-product, we get the following short finite presentation of M_{12} with two generators and six relators:

$$M_{12} \cong \langle \sigma, \tau \mid \sigma^2, \tau^6, (\sigma\tau)^5, (\sigma\tau\sigma\tau^5)^4, (\sigma\tau^2)^6, (\sigma\tau\sigma\tau^4)^5 \rangle\ .$$

(2) We apply Proposition 2.1 or [17, Corollary 5.3], using the fact that the stabilizer $\mathrm{Stab}_{P_v}(\{1\})$ is the group generated by the three permutations

$$\begin{gathered}
(2, 8, 10, 12, 5)(3, 4, 7, 6, 9),\\
(2, 3, 6, 9)(5, 10, 7, 12),\\
(5, 8)(6, 7)(9, 10)(11, 12),
\end{gathered}$$

which is isomorphic to the non-abelian simple group M_{11} of order 7920.

(3) It follows from [17, Theorem 1.4], see also Proposition 4.4 in Section 4.2.

(4) This is a short computation.

□

Conjecture 2.20. *Let* Γ *be the group defined in Example 2.18. Then its subgroup* Γ_0 *is simple.*

Remark. By analyzing many (4, 12)–groups, we have observed that $P_v \cong M_{12}$ can be generated in several ways by $\{\rho_h(a_1), \rho_h(a_2)\}$. We have found seven different cycle structures for $\{\rho_h(a_1), \rho_h(a_2)\}$ generating M_{12}. They are listed in Table 2.4:

$\rho_h(a_1)$	$\rho_h(a_2)$
(3, 4)(5, 6)(7, 8)(9, 10)	(1, 7, 5, 3, 2)(6, 12, 11, 10, 8)
(3, 4)(5, 6)(7, 8)(9, 10)	(1, 6, 5, 9, 3, 2)(4, 8, 7, 12, 11, 10)
(3, 6, 5, 4)(7, 8, 9, 10)	(1, 4, 2)(3, 8, 6)(5, 10, 7)(9, 11, 12)
(3, 6, 5, 4)(7, 8, 9, 10)	(1, 6, 3, 2)(4, 8)(5, 9)(7, 12, 11, 10)
(3, 6, 5, 4)(7, 8, 9, 10)	(1, 7, 3, 2)(6, 12, 11, 10)
(3, 6, 5, 4)(7, 8, 9, 10)	(1, 9, 6, 3, 2)(4, 12, 11, 10, 7)
(3, 6, 5, 4)(7, 8, 9, 10)	(1, 5, 9, 6, 3, 2)(4, 8, 12, 11, 10, 7)

Table 2.4: Several pairs which generate M_{12}

Example: $(A_6, \mathrm{ASL}_3(2))$–group

See [25, p.55] for the definition of the *affine special linear group* $\mathrm{ASL}_3(2)$. It can be realized as a non-simple 3-transitive subgroup of A_8 of order 1344.

Example 2.21.

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_3^{-1}, & a_1b_4^{-1}a_2^{-1}b_3, & a_2b_1a_2^{-1}b_2^{-1}, \\ a_2b_2a_3^{-1}b_1, & a_2b_3a_2^{-1}b_4, & a_2b_2^{-1}a_3b_1^{-1}, \\ a_3b_1a_3b_3^{-1}, & a_3b_2a_3b_4^{-1}, & a_3b_3a_3b_4 \end{array} \right\}.$$

Theorem 2.22. *Let Γ be the $(6, 8)$–group defined in Example 2.21. Then*

(1) $P_h = A_6$, $P_v \cong \mathrm{ASL}_3(2) < S_8$.

(2) *Any non-trivial normal subgroup of Γ has finite index.*

(3) *Γ is not linear over any field, in particular irreducible.*

(4) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= (2, 4, 3), \\ \rho_v(b_2) &= (3, 5, 4), \\ \rho_v(b_3) &= (1, 2)(3, 4), \\ \rho_v(b_4) &= (3, 4)(5, 6), \end{aligned}$$

$$\rho_h(a_1) = (3,4)(5,6),$$
$$\rho_h(a_2) = (1,7,8,2)(3,4,6,5),$$
$$\rho_h(a_3) = (1,7,5,3)(2,8,6,4).$$

(2) Note that

$$\mathrm{Stab}_{P_v}(\{1\}) = \langle (3,4)(5,6), (3,5,7)(4,6,8), (2,7,6,3)(4,8)\rangle \cong \mathrm{PSL}_3(2)$$

is a non-abelian simple group. The statement follows now either from Proposition 2.1, or from [16, Proposition 3.3.3] together with [17, Theorem 4.1], or directly from [17, Corollary 5.3].

(3) The claim is a consequence of [17, Theorem 1.4], see also Proposition 4.4 in Section 4.2.

(4) This is a short computation.

□

Conjecture 2.23. *Let Γ be the group defined in Example 2.21. Then its subgroup Γ_0 is simple.*

Question 2.24. *Let Γ be a $(2m, 2n)$–group such that any non-trivial normal subgroup of Γ has finite index. Assume that $\Lambda \lhd \Gamma$ is a non-trivial perfect normal subgroup (of finite index). Is Λ simple?*

2.2 A non-residually finite group

Non-residually finite $(2m, 2n)$–groups have been constructed by Burger-Mozes in [15, 16, 17] for $2m = 196 = 14^2$, $2n = 324 = 18^2$ and independently by Wise in [68] for $2m = 8$, $2n = 6$ using completely different techniques. See Example 2.39 in Section 2.4 for the non-residually finite example of Wise. We present in this section an irreducible (A_4, P_v)–group Γ with $P_v < S_{12}$ quasi-primitive but such that the quasi-center $\mathrm{QZ}(H_2)$ is not trivial. Applying a result of Burger-Mozes ([17]), this shows that Γ is non-residually finite (Example 2.26).

We first restate a special case of the criterium for non-residual finiteness taken from [17, Section 2.1] and adapted to our situation:

Proposition 2.25. *(Burger-Mozes, [17, Proposition 2.1, Corollary 2.3]) Let Γ be an irreducible $(2m, 2n)$–group. If $P_v < S_{2n}$ is a quasi-primitive permutation group and $\Lambda_2 \neq 1$, then Γ is non-residually finite. (Similarly, if $P_h < S_{2m}$ is a quasi-primitive permutation group and $\Lambda_1 \neq 1$, then Γ is non-residually finite.)*

Example 2.26.

$$R_{2\cdot 6} := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, \\ a_1b_3a_1^{-1}b_4^{-1}, & a_1b_4a_1^{-1}b_5^{-1}, \\ a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_2^{-1}, \\ a_1b_2^{-1}a_2b_3, & a_2b_1a_2^{-1}b_5^{-1}, \\ a_2b_2a_2b_3^{-1}, & a_2b_4a_2^{-1}b_4, \\ a_2b_5a_2^{-1}b_1^{-1}, & a_2b_6a_2^{-1}b_6 \end{array} \right\}.$$

Theorem 2.27. *Let* Γ *be the* $(4, 12)$*–group defined in Example 2.26. Then*

(1) $P_h = A_4$, $P_v \cong \mathrm{PSL}_2(5) < S_{12}$, $|P_v| = 60$.

(2) Γ *is irreducible.*

(3) P_v *is quasi-primitive, but not primitive.*

(4) $\Lambda_2 \neq 1$, *in particular* $\mathrm{QZ}(H_2) \neq 1$.

(5) Γ *is non-residually finite.*

(6) $[\Gamma, \Gamma] = \Gamma_0$ *is perfect, but not simple.*

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= (), \\ \rho_v(b_2) &= (2, 4, 3), \\ \rho_v(b_3) &= (1, 2, 3), \\ \rho_v(b_4) &= (), \\ \rho_v(b_5) &= (), \\ \rho_v(b_6) &= (), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2, 6, 5, 4, 3)(7, 8, 9, 10, 11), \\ \rho_h(a_2) &= (1, 5)(2, 3)(4, 9)(6, 7)(8, 12)(10, 11). \end{aligned}$$

(2) Figure 2.1 shows that we can apply Proposition 1.2(3a) using the fact that $a_1b_1 = b_1a_1$ and that $\rho_v(b_3) = (1, 2, 3)$ acts transitively on the set

$$\{1, 2, 3\} \cong E_h \setminus \{a_1^{-1}\} = \{a_1, a_2, a_2^{-1}\}.$$

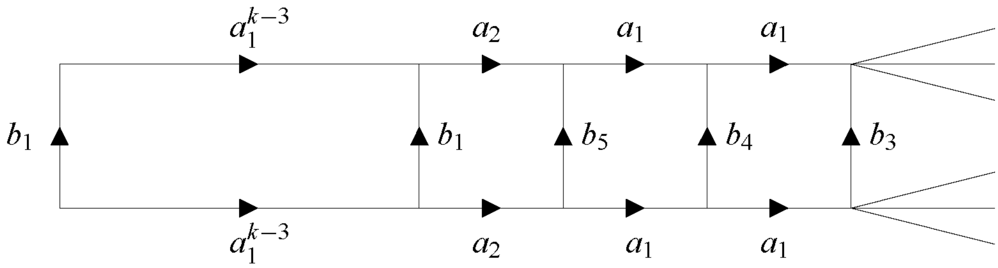


Figure 2.1: Illustration to the proof of Theorem 2.27(2)

Note that the irreducibility criterion [17, Proposition 1.3] cannot be applied here, since P_v is not primitive and K_h is a 3-group ($|K_h| = 27$).

(3) The group P_v is quasi-primitive, since it is simple and transitive. It has the non-trivial blocks $\{1, 12\}$, $\{5, 8\}$, $\{4, 9\}$, $\{3, 10\}$, $\{2, 11\}$, $\{6, 7\}$, and is therefore not primitive.

(4) The set $B := \{b_1^3, b_2^3, b_3^3, b_4^3, b_5^3, b_6^3\}^{\pm 1}$ is a subset of Λ_2 by Lemma 1.1(1b), since for each $b \in B$ and $a \in E_h$ we have $\rho_v(b)(a) = a$ and $\rho_h(a)(b) \in B$.

(5) We can apply Proposition 2.25.

(6) The first part of the statement is an easy computation. The group Γ_0 is not simple, since $\Gamma_0 \cap \mathrm{QZ}(H_2)$ is a non-trivial normal subgroup of Γ_0 of infinite index, using part (4).

$\square$

See Table 2.5 for the orders of some quotients of Γ. The infinite quotients in this list, denoted by "∞", correspond to elements in Λ_2.

$\lvert\Gamma/\langle\langle w^k\rangle\rangle_\Gamma\rvert$	$k = 1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1, a_2$	2	4	2	4	2	4	2	4	2	4	2	4
$b_1, \ldots, b_6$	2	4	∞	4	2	∞	2	4	∞	4	2	∞

Table 2.5: Some orders of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$ in Example 2.26

Conjecture 2.28. *Let Γ be the group defined in Example 2.26. Then*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = \Gamma_0 .$$

Note that by [17, Proposition 2.1], we have

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N > (\{1\} \times [H_2^{(\infty)}, \Lambda_2]) \neq 1\,,$$

where $H_2^{(\infty)}$ is the intersection of all closed finite index subgroups of $H_2 < \mathrm{Aut}(\mathcal{T}_{12})$, but we do not know how to determine explicitly a non-trivial element in $H_2^{(\infty)}$.

Substituting the non-residually finite $(196, 324)$–group $\pi_1(\mathcal{A}_{13,17} \boxtimes \mathcal{A}_{13,17})$ of Burger-Mozes ([17]) by the non-residually finite (4, 12)–group of Example 2.26, we can simplify some constructions made in [17]:

Proposition 2.29. (1) *(See [17, Theorem 6.4] for the same statement but with lower bounds $m \geq 109$, $n \geq 175$. Note that the number 150 in [17, Theorem 6.4] is a misprint and has to be replaced by 175) For every $m \geq 9$ and $n \geq 13$, there exists a torsion-free cocompact lattice $\Lambda < U(A_{2m}) \times U(A_{2n})$ which is virtually simple and has dense projections.*

(2) *(cf. [17, Theorem 6.5]) Any $(2m, 2n)$–group injects for any even natural numbers $k \geq 4$, $l \geq 4$ in a virtually simple $(A_{4m+14+k}, A_{4n+22+l})$–group.*

(3) *(cf. [17, Theorem 6.5]) Any $(2m, 2n)$–group such that $P_h < A_{2m}$ and $P_v < A_{2n}$ are even permutation groups, injects for any even natural numbers $k \geq 4$, $l \geq 4$ in a virtually simple $(A_{2m+14+k}, A_{2n+22+l})$–group.*

Proof. (1) We essentially imitate the proof of [17, Theorem 6.4], but replace the $(14, 18)$–complex ${}^{(0)}X = \mathcal{A}_{13,17}$ (which is also described in Example 3.26) by the (A_6, A_6)–complex of Example 2.2, and replace the $(196, 324)$–complex ${}^{(1)}X = \mathcal{A}_{13,17} \boxtimes \mathcal{A}_{13,17}$ by the non-residually finite (4, 12)–complex of Example 2.26. Note that we use in the proof that $\mathrm{PSL}_2(5) < S_{12}$ is *even*, i.e. a subgroup of A_{12}.

(2) We embed the given corresponding $(2m, 2n)$–complex by [17, Proposition 6.2] in a $(4m, 4n)$–complex Y with even local permutation groups. Then we apply [17, Proposition 6.1] to the case where ${}^{(0)}X$ is the (A_6, A_6)–complex of Example 2.2, ${}^{(1)}X$ is the non-residually finite (4, 12)–complex of Example 2.26 and ${}^{(2)}X = Y$.

(3) Same proof as in part (2), but without embedding the given $(2m, 2n)$–complex in a $(4m, 4n)$–complex, since the local groups are already even by assumption.

□

2.3 Virtually simple groups

We embed in this section the non-residually finite $(4, 12)$–group Γ of Example 2.26 into an (A_6, A_{16})–group (Example 2.30), into an (A_8, A_{14})–group (described in Example A.26), and into an $(\mathrm{ASL}_3(2), A_{14})$–group (Example 2.33). All three examples turn out to be virtually simple by results of Burger-Mozes. Therefore, their minimal normal subgroup of finite index (in other words, the normal subgroup of maximal finite index) is a finitely presented torsion-free simple group. We believe that this index is 4 in our three given examples.

A virtually simple (A_6, A_{16})–group

Example 2.30.

$$R_{3\cdot 8} := \left\{ \begin{array}{lll} \underline{a_1 b_1 a_1^{-1} b_1^{-1}}, & \underline{a_1 b_2 a_2^{-1} b_3^{-1}}, & \underline{a_1 b_3 a_1^{-1} b_4^{-1}}, \\ \underline{a_1 b_4 a_1^{-1} b_5^{-1}}, & \underline{a_1 b_5 a_1^{-1} b_6^{-1}}, & \underline{a_1 b_6 a_1^{-1} b_2^{-1}}, \\ a_1 b_7 a_2 b_8^{-1}, & a_1 b_8 a_2 b_8, & a_1 b_8^{-1} a_2 b_7^{-1}, \\ a_1 b_7^{-1} a_3^{-1} b_7, & \underline{a_1 b_2^{-1} a_2 b_3}, & \underline{a_2 b_1 a_2^{-1} b_5^{-1}}, \\ \underline{a_2 b_2 a_2 b_3^{-1}}, & \underline{a_2 b_4 a_2^{-1} b_4}, & \underline{a_2 b_5 a_2^{-1} b_1^{-1}}, \\ \underline{a_2 b_6 a_2^{-1} b_6}, & a_2 b_7 a_3 b_7^{-1}, & a_3 b_1 a_3^{-1} b_8, \\ a_3 b_2 a_3^{-1} b_2, & a_3 b_3 a_3^{-1} b_4^{-1}, & a_3 b_4 a_3^{-1} b_1, \\ a_3 b_5 a_3^{-1} b_3, & a_3 b_6 a_3^{-1} b_6, & a_3 b_8 a_3^{-1} b_5 \end{array} \right\}.$$

Theorem 2.31. *Let Γ be the $(6, 16)$–group of Example 2.30. Then*

(1) $P_h = A_6$, $P_v = A_{16}$.

(2) Γ *is non-residually finite.*

(3) Γ *is a finitely presented torsion-free virtually simple group, in particular the minimal normal subgroup of finite index in* Γ

$$\bigcap_{N \overset{f.i.}{\triangleleft} \Gamma} N$$

is a finitely presented torsion-free simple group.

(4) *We have amalgam decompositions*

$$F_8 *_{F_{43}} F_{22} \cong \Gamma \cong F_3 *_{F_{33}} F_{17}$$

and

$$\mathrm{Aut}(\mathcal{T}_6) > F_{15} *_{F_{85}} F_{15} \cong \Gamma_0 \cong F_5 *_{F_{65}} F_5 < \mathrm{Aut}(\mathcal{T}_{16})\,.$$

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= \rho_v(b_4) = \rho_v(b_5) = \rho_v(b_6) = (),\\
\rho_v(b_2) &= (2, 6, 5),\\
\rho_v(b_3) &= (1, 2, 5),\\
\rho_v(b_7) &= (1, 5, 3)(2, 4, 6),\\
\rho_v(b_8) &= (1, 5)(2, 6),\\
\rho_h(a_1) &= (2, 6, 5, 4, 3)(7, 9, 8)(11, 12, 13, 14, 15),\\
\rho_h(a_2) &= (1, 5)(2, 3)(4, 13)(6, 11)(8, 10, 9)(12, 16)(14, 15),\\
\rho_h(a_3) &= (1, 13, 14, 5, 9)(2, 15)(3, 12, 8, 16, 4)(6, 11).
\end{aligned}$$

(2) The embedding of the $(A_4, \mathrm{PSL}_2(5) < S_{12})$–complex of Example 2.26 into X (indicated by the twelve underlined relators in $R_{3\cdot 8}$) induces a π_1-injection by Proposition 1.9(1). Since the (4, 12)–group of Example 2.26 is non-residually finite, Γ is also non-residually finite.

(3) Apply [17, Corollary 5.4].

(4) Use Proposition 1.3 and Proposition 1.4.

(5) These are easy computations.

□

Conjecture 2.32. *Let* Γ *be the* (6, 16)*–group of Example 2.30. Then* Γ_0 *is a finitely presented torsion-free simple group. Equivalently,*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = \Gamma_0\,.$$

A virtually simple (A_8, A_{14})–group

See Appendix A.4 for the definition of a finitely presented, non-residually finite, torsion-free, virtually simple (A_8, A_{14})–group. It behaves as the (A_6, A_{16})–group of Example 2.30.

Remark. It seems to be impossible to embed the (4, 12)–complex X of Example 2.26 into a virtually simple (A_6, A_{14})–complex. However, it seems to be easy to embed X into a virtually simple (A_{2m}, A_{2n})–complex, if $m \geq 3$, $n \geq 8$ or if $m \geq 4$, $n \geq 7$.

A virtually simple $(\mathrm{ASL}_3(2), A_{14})$–group

Example 2.33.

$$R_{4\cdot 7} := \left\{ \begin{array}{llll}
\underline{a_1 b_1 a_1^{-1} b_1^{-1}}, & \underline{a_1 b_2 a_2^{-1} b_3^{-1}}, & \underline{a_1 b_3 a_1^{-1} b_4^{-1}}, & \underline{a_1 b_4 a_1^{-1} b_5^{-1}}, \\
\underline{a_1 b_5 a_1^{-1} b_6^{-1}}, & \underline{a_1 b_6 a_1^{-1} b_2^{-1}}, & a_1 b_7 a_2^{-1} b_7^{-1}, & a_1 b_7^{-1} a_3 b_7, \\
\underline{a_1 b_2^{-1} a_2 b_3}, & \underline{a_2 b_1 a_2^{-1} b_5^{-1}}, & \underline{a_2 b_2 a_2 b_3^{-1}}, & \underline{a_2 b_4 a_2^{-1} b_4}, \\
\underline{a_2 b_5 a_2^{-1} b_1^{-1}}, & \underline{a_2 b_6 a_2^{-1} b_6}, & a_2 b_7 a_4^{-1} b_7^{-1}, & a_3 b_1 a_4 b_4, \\
a_3 b_2 a_3^{-1} b_3^{-1}, & a_3 b_3 a_4^{-1} b_2^{-1}, & a_3 b_4 a_4 b_7, & a_3 b_5 a_4 b_6^{-1}, \\
a_3 b_6 a_4 b_1^{-1}, & a_3 b_7^{-1} a_4 b_1, & a_3 b_6^{-1} a_4 b_5, & a_3 b_5^{-1} a_4 b_6, \\
a_3 b_4^{-1} a_4 b_5^{-1}, & a_3 b_3^{-1} a_4 b_2, & a_3 b_1^{-1} a_4 b_4^{-1}, & a_4 b_3 a_4 b_2^{-1}
\end{array} \right\}.$$

Theorem 2.34. *Let* Γ *be the* (8, 14)*–group defined in Example 2.33. Then*

(1) $P_h \cong \mathrm{ASL}_3(2) < S_8$, $P_v = A_{14}$.

(2) Γ *is non-residually finite.*

(3) Γ *is a finitely presented torsion-free virtually simple group.*

(4) *There are amalgam decompositions*

$$F_7 *_{F_{49}} F_{25} \cong \Gamma \cong F_4 *_{F_{43}} F_{22}$$

and

$$\mathrm{Aut}(\mathcal{T}_8) > F_{13} *_{F_{97}} F_{13} \cong \Gamma_0 \cong F_7 *_{F_{85}} F_7 < \mathrm{Aut}(\mathcal{T}_{14}) .$$

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) = \rho_v(b_4) = \rho_v(b_5) = \rho_v(b_6) &= (3,5)(4,6),\\
\rho_v(b_2) &= (2,8,7)(3,4,5),\\
\rho_v(b_3) &= (1,2,7)(4,6,5),\\
\rho_v(b_7) &= (1,2,4,6)(3,8,7,5),\\
\rho_h(a_1) &= (2,6,5,4,3)(9,10,11,12,13),\\
\rho_h(a_2) &= (1,5)(2,3)(4,11)(6,9)(10,14)(12,13),\\
\rho_h(a_3) &= (1,6,5,11)(2,3)(4,14,8)(9,10)(12,13),\\
\rho_h(a_4) &= (1,11,7)(2,3)(4,10,9,14)(5,6)(12,13).
\end{aligned}$$

(2) The embedding of the $(A_4, \mathrm{PSL}_2(5) < S_{12})$–complex of Example 2.26 into the $(8,14)$–complex X (indicated by the twelve underlined relators in $R_{4\cdot 7}$) induces a π_1-injection by Proposition 1.9(1).

(3) Apply [17, Corollary 5.3] (cf. Example 2.21 for the role of $\mathrm{ASL}_3(2)$).

(4) Use Proposition 1.3 and Proposition 1.4.

(5) These are easy computations.

□

Conjecture 2.35. *Let* Γ *be the* $(8,14)$*–group defined in Example 2.33. Then the subgroup* Γ_0 *is a finitely presented torsion-free simple group.*

2.4 Two examples of Wise

We recall in this section two interesting groups of Wise ([68]).

Example 2.36. *(See [68, Section II.2.1], the transition from Wise's notations to ours is given by* $x \mapsto a_1$, $y \mapsto a_2$, $a \mapsto b_1$, $b \mapsto b_2$, $c \mapsto b_3$.*)*

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_2 a_1^{-1} b_1^{-1}, & a_2 b_2 a_2^{-1} b_1^{-1}, \\ \\ a_1 b_3 a_2^{-1} b_3^{-1}, & a_1 b_1 a_2^{-1} b_2^{-1}, \\ \\ a_2 b_1 a_1^{-1} b_3^{-1}, & a_2 b_3 a_1^{-1} b_2^{-1} \end{array} \right\}.$$

Theorem 2.37. *(Wise [68]) The* $(4,6)$*–group* Γ *of Example 2.36 is irreducible and not* $\langle b_1, b_2, b_3 \rangle$*-separable.*

Proof. See [68]. Let G be a group and $H < G$ a subgroup. Recall that G is said to be *H-separable,* if for each element $g \in G \setminus H$, there is a homomorphism $\psi : G \to Q$ onto a finite group Q such that $\psi(g) \notin \psi(H)$. It is shown in [68, Corollary II.4.4] that $\psi(a_1 a_2^{-1}) \in \psi(\langle b_1, b_2, b_3\rangle)$ for every homomorphism $\psi : \Gamma \to Q$ with finite Q. $\square$

Remark. The proof of Theorem 2.37 given in [68] is based on the fact that the two elements a_2, b_3 have no commuting non-trivial powers (this phenomenon is called *anti-torus* and is proved in [68, Proposition II.3.8]. Much more about anti-tori can be found in Section 3.6). Note however, that $\langle a_2, b_3\rangle$ is not a free subgroup of Γ since we have for example the non-trivial relation $b_3^{-2} a_2^{-3} b_3^2 a_2 b_3^{-1} a_2 b_3 a_2 = 1$ in Γ.

Using the separability property of the $(4, 6)$–group Γ described in Theorem 2.37 and the following lemma of Long-Niblo ([44]), a doubling of Γ along its subgroup $\langle b_1, b_2, b_3\rangle$ (geometrically, doubling X along its vertical 1-skeleton $(\{x\}, E_v)$) leads to the non-residually finite $(8, 6)$–group of Example 2.39. (By a *double* or a *doubling* of a group G along a subgroup H, we mean the amalgamated free product $G *_{H=\bar{H}} \bar{G}$, where $\bar{G} \hookrightarrow \bar{H}$ is an isomorphic copy of $G \hookrightarrow H$.)

Lemma 2.38. *(Long-Niblo, see [44, Lemma, p.211]) Let $\theta : G \to G$ be an automorphism of a residually finite group G. Then G is* $\mathrm{Fix}(\theta)$*–separable, where*

$$\mathrm{Fix}(\theta) := \{g \in G : \theta(g) = g\}$$

is the subgroup of elements fixed by the homomorphism θ.

More precisely, if $\theta : G \to G$ is an automorphism and G is not $\mathrm{Fix}(\theta)$*–separable, then*

$$x^{-1}\theta(x) \in \bigcap_{N \overset{f.i.}{\lhd} G} N,$$

where $x \in G \setminus \mathrm{Fix}(\theta)$ is any element such that $\psi(x) \in \psi(\mathrm{Fix}(\theta))$ for all homomorphisms $\psi : G \to Q$ onto finite groups Q.

Proof. See [44]. Note that the same result is true for endomorphisms $\theta : G \to G$ of finitely generated residually finite groups G, see [68, Theorem II.5.2]. $\square$

Example 2.39. *(See [68, Section II.5], where this example is called D)*

$$R_{4\cdot 3} := \left\{ \begin{array}{cccc} \underline{a_1 b_2 a_1^{-1} b_1^{-1}}, & \underline{a_2 b_2 a_2^{-1} b_1^{-1}}, & \underline{a_1 b_3 a_2^{-1} b_3^{-1}}, & \underline{a_1 b_1 a_2^{-1} b_2^{-1}}, \\ \underline{a_2 b_1 a_1^{-1} b_3^{-1}}, & \underline{a_2 b_3 a_1^{-1} b_2^{-1}}, & a_3 b_2 a_3^{-1} b_1^{-1}, & a_4 b_2 a_4^{-1} b_1^{-1}, \\ a_3 b_3 a_4^{-1} b_3^{-1}, & a_3 b_1 a_4^{-1} b_2^{-1}, & a_4 b_1 a_3^{-1} b_3^{-1}, & a_4 b_3 a_3^{-1} b_2^{-1} \end{array} \right\}.$$

The six underlined relators are the relators of Example 2.36 which is embedded in Example 2.39.

Theorem 2.40. *(Wise [68, Main Theorem II.5.5]) The* $(8,6)$*–group* Γ *of Example 2.39 is non-residually finite.*

Proof. By [68], we have for example

$$a_2 a_1^{-1} a_3 a_4^{-1} \in \bigcap_{\substack{\text{f.i.} \\ N \lhd \Gamma}} N .$$

$\square$

2.5 Constructing simple groups

Using an appropriate embedding of Wise's non-residually finite group described in Example 2.39 above, we construct in this section a virtually simple (A_{10}, A_{10})–group (Example 2.43). Moreover, we are able to prove in Theorem 2.45 that its index 4 subgroup Γ_0 is a simple group. Therefore, we get an explicit description of a finitely presented torsion-free simple group in $\mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$, which moreover has the form $F_9 *_{F_{81}} F_9$.

At first, we give two very elementary but crucial lemmas used in the proof of Theorem 2.45.

Lemma 2.41. *Let* G *be a group,* $H < G$ *a non-residually finite subgroup of* G *and* $h \in H$ *an element such that*

$$1 \neq h \in \bigcap_{\substack{f.i. \\ M \lhd H}} M .$$

Then

$$h \in \bigcap_{\substack{f.i. \\ N \lhd G}} N ,$$

in particular G *is also non-residually finite.*

Proof. Let $N \lhd G$ be any normal subgroup of finite index in G. Obviously,

$$N \cap H \lhd G \cap H = H .$$

Moreover

$$[H : (N \cap H)] \leq [G : N]$$

is finite by Lemma 2.8, hence

$$h \in N \cap H < N$$

and we are done. $\square$

Lemma 2.42. *Let G be a non-residually finite group and $g \in G$ an element such that*

$$1 \neq g \in \bigcap_{\substack{\text{f.i.} \\ N \lhd G}} N\,.$$

Moreover, assume that the normal subgroup $\langle\langle g \rangle\rangle_G$ has finite index in G. Then

$$\langle\langle g \rangle\rangle_G = \bigcap_{\substack{\text{f.i.} \\ N \lhd G}} N\,.$$

Proof. By assumption, $\langle\langle g \rangle\rangle_G$ is a normal subgroup of G of finite index, hence

$$\langle\langle g \rangle\rangle_G \supseteq \bigcap_{\substack{\text{f.i.} \\ N \lhd G}} N\,.$$

The other inclusion follows directly from

$$g \in \bigcap_{\substack{\text{f.i.} \\ N \lhd G}} N \lhd G\,,$$

by definition of the normal closure of g. ☐

Now, we are ready to describe one of our main examples:

Example 2.43. *Let $R_{5.5}$ be the set of 25 relators*

$$\left\{\begin{array}{lllll}
\underline{\underline{a_1 b_1 a_2^{-1} b_2^{-1}}}, & \underline{\underline{a_1 b_2 a_1^{-1} b_1^{-1}}}, & \underline{\underline{a_1 b_3 a_2^{-1} b_3^{-1}}}, & a_1 b_4 a_2 b_5^{-1}, & a_1 b_5 a_5^{-1} b_4, \\
a_1 b_5^{-1} a_3 b_4^{-1}, & a_1 b_4^{-1} a_3 b_5, & \underline{\underline{a_1 b_3^{-1} a_2^{-1} b_2}}, & \underline{\underline{a_1 b_1^{-1} a_2^{-1} b_3}}, & \underline{\underline{a_2 b_2 a_2^{-1} b_1^{-1}}}, \\
a_2 b_4 a_2^{-1} b_5, & a_2 b_5 a_4 b_4^{-1}, & \underline{a_3 b_1 a_4^{-1} b_2^{-1}}, & \underline{a_3 b_2 a_3^{-1} b_1^{-1}}, & \underline{a_3 b_3 a_4^{-1} b_3^{-1}}, \\
a_3 b_4 a_4 b_5, & a_3 b_5^{-1} a_4 b_4, & \underline{a_3 b_3^{-1} a_4^{-1} b_2}, & \underline{a_3 b_1^{-1} a_4^{-1} b_3}, & \underline{a_4 b_2 a_4^{-1} b_1^{-1}}, \\
a_4 b_5^{-1} a_5^{-1} b_4^{-1}, & a_5 b_1 a_5^{-1} b_3, & a_5 b_2 a_5^{-1} b_5^{-1}, & a_5 b_3 a_5^{-1} b_1^{-1}, & a_5 b_4 a_5^{-1} b_2^{-1}
\end{array}\right\}.$$

Proposition 2.44. *Let Γ be the $(10, 10)$–group of Example 2.43. Then*

(1) $P_h = A_{10}$, $P_v = A_{10}$.

(2) Γ *is non-residually finite.*

(3) Γ *is a finitely presented torsion-free virtually simple group.*

(4) *There are two amalgam decompositions*

$$\Gamma \cong F_5 *_{F_{41}} F_{21}$$

and two amalgam decompositions

$$\Gamma_0 \cong F_9 *_{F_{81}} F_9 < \mathrm{Aut}(\mathcal{T}_{10}) .$$

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

(6) *The number of generators* $d(\Gamma^k)$ *grows linearly to infinity for* $k \to \infty$, *but* $d(\Gamma_0^k) \leq 3$ *for all* $k \in \mathbb{N}$.

(7) $Z_\Gamma(a_5) = N_\Gamma(\langle a_5 \rangle) = \langle a_5 \rangle$.

(8) $b_1 \in Z_\Gamma(a_5^4)$, *in particular* Γ *is not commutative transitive.*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (7, 8)(9, 10),\\
\rho_v(b_2) &= (1, 2)(3, 4),\\
\rho_v(b_3) &= (1, 2)(3, 4)(7, 8)(9, 10),\\
\rho_v(b_4) &= (1, 8, 4, 5)(2, 7, 3, 10),\\
\rho_v(b_5) &= (1, 9, 4, 8)(3, 10, 6, 7),\\
\rho_h(a_1) &= (1, 2)(4, 6, 7, 5)(8, 10, 9),\\
\rho_h(a_2) &= (1, 2, 3)(4, 5, 7, 6)(9, 10),\\
\rho_h(a_3) &= (1, 2)(4, 5, 7, 6)(8, 10, 9),\\
\rho_h(a_4) &= (1, 2, 3)(4, 6, 7, 5)(9, 10),\\
\rho_h(a_5) &= (1, 3, 10, 8)(2, 4, 6, 9, 7, 5).
\end{aligned}$$

(2) The embedding of the non-residually finite $(8, 6)$–complex of Example 2.39 into the $(10, 10)$–complex X, indicated by the twelve (single or double) underlined relators in $R_{5.5}$, induces a π_1-injection by Proposition 1.9(1). The six relators coming from Example 2.36 (which is embedded in Example 2.39) are doubly underlined.

(3) Apply [17, Corollary 5.4].

(4) We use Proposition 1.3 and Proposition 1.4.

(5) These are easy computations.

(6) We apply results of Wiegold-Wilson ([67]). First note that $d(\Gamma) = 2$, since for example $\Gamma = \langle a_1, b_4 \rangle$, and that $d(\Gamma_0) = 2$, since $\Gamma_0 = \langle a_1^2, b_5 b_1^{-1} \rangle$ (this can be easily checked with GAP ([29])). By [67, Theorem 2.2], we have $d(\Gamma^k) = 2k$, if $k \geq 18$. However, using the simplicity of Γ_0 which is shown in the following Theorem 2.45, the result [67, Theorem 4.3] implies $d(\Gamma_0^k) \leq d(\Gamma_0) + 1 = 3$ for all $k \in \mathbb{N}$.

(7) This follows from Proposition 1.12.

(8) We compute $a_5^4 b_1 = b_1 a_5^4$. Obviously, a_5 and a_5^4 commute. Part (7) shows that a_5 and b_1 do not commute and we conclude that Γ is not commutative transitive.

□

Theorem 2.45. *Let Γ be the $(10, 10)$–group of Example 2.43. Then the subgroup Γ_0 is a finitely presented torsion-free simple group.*

Proof. Using Proposition 2.44, we "only" have to show that

$$\Gamma_0 = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N \,.$$

Take $w := a_2 a_1^{-1} a_3 a_4^{-1} \in \Gamma_0$. Then by Theorem 2.40 and Lemma 2.41 we have

$$w \in \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N \,,$$

hence by Lemma 2.42, using the fact that every non-trivial normal subgroup of Γ has finite index in Γ (applying Proposition 2.1), we have

$$\langle\langle w \rangle\rangle_\Gamma = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N \,.$$

A computer algebra system like GAP ([29]) immediately checks that

$$[\Gamma : \langle\langle w \rangle\rangle_\Gamma] = |\langle a_1, \ldots, a_5, b_1, \ldots, b_5 \mid R_{5 \cdot 5}, w \rangle| = 4 \,.$$

Since $[\Gamma : \Gamma_0] = 4$ and $w \in \Gamma_0$, we conclude that

$$\bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N = \langle\langle w \rangle\rangle_\Gamma = \Gamma_0 \,.$$

Alternatively and more explicitly, one proves $\langle\langle w \rangle\rangle_\Gamma = \Gamma_0$ by checking that

$$\Gamma_0 = \langle a_1 a_2^{-1}, b_3 b_1^{-1}, b_3 b_5^{-1} \rangle$$

and

$$\begin{aligned} a_1 a_2^{-1} &= (b_2 b_5 w b_5^{-1} b_2^{-1})(b_5 w^{-1} b_5^{-1}) \in \langle\langle w \rangle\rangle_\Gamma \\ b_3 b_1^{-1} &= (b_1^{-1} b_5 w^{-1} b_5^{-1} b_1)(b_1 b_5 w b_5^{-1} b_1^{-1}) \in \langle\langle w \rangle\rangle_\Gamma \\ b_3 b_5^{-1} &= (b_1^{-1} b_4^{-1} w b_4 b_1)(b_5 b_4^{-1} w^{-1} b_4 b_5^{-1}) \in \langle\langle w \rangle\rangle_\Gamma . \end{aligned}$$

□

A finite presentation of the simple group Γ_0 is given as follows: We take the 37 generators $s_1, \dots, s_{37}$ and the 100 relators of Table 2.6.

$$\left\{ \begin{array}{lllll}
s_{24}s_{34}, & s_{10}s_{23}s_{33}, & s_{11}s_{24}s_{35}, & s_{12}s_{19}s_{37}, & s_{13}s_{27}s_{31}, \\
s_{18}s_{20}s_{36}, & s_{17}s_{20}s_{32}, & s_{16}s_{24}s_{29}, & s_{14}s_{24}s_{30}, & s_{1}s_{10}s_{24}s_{33}, \\
s_{1}s_{12}s_{24}s_{32}, & s_{1}s_{13}s_{21}s_{36}, & s_{2}s_{26}s_{34}, & s_{2}s_{10}s_{25}s_{33}, & s_{2}s_{11}s_{26}s_{35}, \\
s_{2}s_{12}s_{21}s_{32}, & s_{2}s_{18}s_{21}s_{31}, & s_{2}s_{16}s_{26}s_{29}, & s_{2}s_{14}s_{26}s_{30}, & s_{3}s_{10}s_{26}s_{33}, \\
s_{3}s_{18}s_{27}s_{36}, & s_{4}s_{27}s_{30}, & s_{4}s_{10}s_{27}s_{37}, & s_{4}s_{11}s_{27}s_{33}, & s_{4}s_{12}s_{27}s_{34}, \\
s_{5}s_{10}s_{19}s_{33}, & s_{5}s_{34}, & s_{5}s_{11}s_{19}s_{35}, & s_{5}s_{13}s_{24}s_{36}, & s_{5}s_{17}s_{22}s_{37}, \\
s_{5}s_{12}s_{25}s_{32}, & s_{5}s_{18}s_{25}s_{31}, & s_{5}s_{15}s_{19}s_{30}, & s_{5}s_{16}s_{19}s_{28}, & s_{6}s_{19}s_{34}, \\
s_{6}s_{18}s_{19}s_{36}, & s_{6}s_{12}s_{26}s_{37}, & s_{7}s_{10}s_{21}s_{33}, & s_{7}s_{20}s_{34}, & s_{7}s_{11}s_{21}s_{35}, \\
s_{7}s_{18}s_{26}s_{36}, & s_{7}s_{17}s_{26}s_{32}, & s_{7}s_{15}s_{21}s_{30}, & s_{7}s_{16}s_{21}s_{28}, & s_{8}s_{21}s_{34}, \\
s_{8}s_{12}s_{22}s_{32}, & s_{9}s_{16}s_{22}s_{33}, & s_{9}s_{13}s_{22}s_{34}, & s_{9}s_{22}s_{35}, & s_{9}s_{10}s_{22}s_{36}, \\
s_{6}s_{15}s_{28}, & s_{5}s_{14}s_{29}, & s_{6}s_{16}s_{30}, & s_{1}s_{18}s_{31}, & s_{9}s_{12}s_{32}, \\
s_{2}s_{17}s_{37}, & s_{2}s_{13}s_{36}, & s_{6}s_{10}s_{35}, & s_{6}s_{11}s_{33}, & s_{6}s_{14}s_{19}s_{29}, \\
s_{6}s_{13}s_{19}s_{31}, & s_{3}s_{17}s_{19}s_{32}, & s_{8}s_{15}s_{20}s_{28}, & s_{7}s_{14}s_{20}s_{29}, & s_{8}s_{16}s_{20}s_{30}, \\
s_{3}s_{13}s_{20}s_{31}, & s_{3}s_{12}s_{20}s_{37}, & s_{8}s_{10}s_{20}s_{35}, & s_{8}s_{11}s_{20}s_{33}, & s_{8}s_{14}s_{21}s_{29}, \\
s_{9}s_{17}s_{21}s_{37}, & s_{9}s_{11}s_{22}s_{28}, & s_{9}s_{18}s_{22}s_{29}, & s_{9}s_{14}s_{22}s_{30}, & s_{9}s_{15}s_{22}s_{31}, \\
s_{1}s_{14}s_{23}s_{29}, & s_{15}s_{23}s_{28}, & s_{1}s_{16}s_{23}s_{30}, & s_{6}s_{17}s_{23}s_{32}, & s_{4}s_{18}s_{23}s_{36}, \\
s_{7}s_{13}s_{23}s_{31}, & s_{7}s_{12}s_{23}s_{37}, & s_{1}s_{11}s_{23}s_{34}, & s_{1}s_{23}s_{35}, & s_{1}s_{15}s_{24}s_{28}, \\
s_{1}s_{17}s_{24}s_{37}, & s_{8}s_{18}s_{24}s_{31}, & s_{3}s_{14}s_{25}s_{29}, & s_{2}s_{15}s_{25}s_{28}, & s_{3}s_{16}s_{25}s_{30}, \\
s_{8}s_{17}s_{25}s_{37}, & s_{8}s_{13}s_{25}s_{36}, & s_{3}s_{11}s_{25}s_{34}, & s_{3}s_{25}s_{35}, & s_{3}s_{15}s_{26}s_{28}, \\
s_{4}s_{13}s_{26}s_{31}, & s_{4}s_{14}s_{27}s_{35}, & s_{4}s_{15}s_{27}s_{32}, & s_{4}s_{16}s_{27}s_{28}, & s_{4}s_{17}s_{27}s_{29}
\end{array} \right\} .$$

Table 2.6: Relators of the simple group of Theorem 2.45

Of course, this presentation can be slightly simplified, for example using the identities $s_5 = s_{24} = s_{34}^{-1}$. Applying the GAP-command ([29])

```
SimplifiedFpGroup(G);
```

we get a presentation of Γ_0 with 3 generators and 66 relators of lengths between 18 and 113. Note that the deficiency of Γ_0 is -63, cf. Section 4.6.

Remark. The smallest finitely presented torsion-free simple group coming from the construction given in [17, Section 6.5] either has amalgam decompositions

$$\mathrm{Aut}(\mathcal{T}_{48}) > F_{7919} *_{F_{380065}} F_{7919} \cong F_{47} *_{F_{364321}} F_{47} < \mathrm{Aut}(\mathcal{T}_{7920}),$$

if we take $k = 3$, $l = 44$, $P_h = A_6$, $P_v = A_{88}$, or has amalgam decompositions

$$\mathrm{Aut}(\mathcal{T}_{48}) > F_{8279} *_{F_{397345}} F_{8279} \cong F_{47} *_{F_{380881}} F_{47} < \mathrm{Aut}(\mathcal{T}_{8280}),$$

if we take $k = 3$, $l = 45$ and $Y = \mathcal{A}_{5,89}$, using the notation of [17]. Observe that both groups need more than 360000 relators in any finite presentation. Also the smallest candidate for being a finitely presented torsion-free simple group in the construction leading to [17, Theorem 6.4] has complicated amalgam decompositions

$$\mathrm{Aut}(\mathcal{T}_{218}) > F_{349} *_{F_{75865}} F_{349} \cong F_{217} *_{F_{75601}} F_{217} < \mathrm{Aut}(\mathcal{T}_{350}),$$

needing more than 75000 relators. Obviously, it would be an enormous work to write down a presentation of such a group.

More simple groups

Using exactly the same ideas as in Theorem 2.45, we embed now the non-residually finite $(8, 6)$–complex of Example 2.39 into several $(2m, 2n)$–complexes with virtually simple fundamental groups Γ. See the following list (Table 2.7) for examples with

$$(2m, 2n) \in \{(10, 10), (10, 12), (12, 8), (12, 10), (12, 12)\}.$$

As before, the group

$$\Gamma^* := \bigcap_{\substack{\text{f.i.} \\ N \triangleleft \Gamma}} N = \langle\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\rangle_\Gamma$$

is finitely presented, torsion-free and simple. In the list, we use the following notation: In the third column, $[2, 2]$ stands for $\mathbb{Z}_2^2$ etc. and in the last column, for example $(9, 81, 9)$ means an amalgam decomposition $F_9 *_{F_{81}} F_9$. Note that Γ_0 and Γ^* always have two amalgam decompositions, a horizontal and a vertical one. Since $\Gamma^* < \Gamma_0$ is a subgroup, the index $[\Gamma : \Gamma^*]$ is a multiple of 4. In most (but not all) examples listed below, we have $[\Gamma, \Gamma] = \Gamma^*$, in particular for these examples $\left|\Gamma^{ab}\right| = [\Gamma : \Gamma^*]$ and $[\Gamma, \Gamma]$ is simple. In all examples (in particular for those with $\Gamma^* \lneqq [\Gamma, \Gamma]$), we compute that Γ^* is the group

$$\langle\langle [a_1, a_2], [a_1, b_1], [a_1, b_2], [a_1, b_3], [a_2, b_1],$$
$$[a_2, b_2], [a_2, b_3], [b_1, b_2], [b_1, b_3], [b_2, b_3] \rangle\rangle_\Gamma .$$

If $[\Gamma : \Gamma^*] > \left|\Gamma^{ab}\right|$, we give the non-abelian quotient Γ / Γ^*.

Ex	Γ	Γ^{ab}	Γ/Γ^*	$\lvert\Gamma^{ab}\rvert$	$\Gamma^* = \langle\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\rangle_\Gamma$
2.43	(A_{10}, A_{10})	$[2, 2]$		4	$(9, 81, 9) = (9, 81, 9)$
	(A_{10}, A_{10})	$[2, 2, 2]$		8	$(17, 161, 17) = (17, 161, 17)$
	(A_{10}, A_{10})	$[2, 4]$		8	$(17, 161, 17) = (17, 161, 17)$
	(A_{10}, A_{10})	$[2, 6]$		12	$(25, 241, 25) = (25, 241, 25)$
	(A_{10}, A_{10})	$[2, 2, 4]$		16	$(33, 321, 33) = (33, 321, 33)$
	(A_{10}, A_{10})	$[2, 8]$		16	$(33, 321, 33) = (33, 321, 33)$
	(A_{10}, A_{10})	$[2, 10]$		20	$(41, 401, 41) = (41, 401, 41)$
	(A_{10}, A_{10})	$[2, 2, 6]$		24	$(49, 481, 49) = (49, 481, 49)$
	(A_{10}, A_{10})	$[2, 12]$		24	$(49, 481, 49) = (49, 481, 49)$
	(A_{10}, A_{10})	$[2, 2, 8]$		32	$(65, 641, 65) = (65, 641, 65)$
2.50	(A_{10}, A_{10})	$[2, 20]$		40	$(81, 801, 81) = (81, 801, 81)$
	(A_{10}, A_{12})	$[2, 2]$		4	$(11, 101, 11) = (9, 97, 9)$
2.48	(A_{10}, A_{12})	$[2, 2]$	D_6	4	$(31, 301, 31) = (25, 289, 25)$
	(A_{10}, A_{12})	$[2, 2, 2]$		8	$(21, 201, 21) = (17, 193, 17)$
	(A_{10}, A_{12})	$[2, 2, 2]$	$S_3 \times \mathbb{Z}_2^2$	8	$(61, 601, 61) = (49, 577, 49)$
	(A_{10}, A_{12})	$[2, 4]$		8	$(21, 201, 21) = (17, 193, 17)$
	$2m = 12$				
	(A_{12}, A_8)	$[2, 2]$		4	$(7, 73, 7) = (11, 81, 11)$
	(A_{12}, A_8)	$[2, 4]$		8	$(13, 145, 13) = (21, 161, 21)$
2.46	(M_{12}, A_8)	$[2, 2]$		4	$(7, 73, 7) = (11, 81, 11)$
	(A_{12}, A_{10})	$[2, 2]$		4	$(9, 97, 9) = (11, 101, 11)$
	(A_{12}, A_{10})	$[2, 2]$	D_6	4	$(25, 289, 25) = (31, 301, 31)$
	(A_{12}, A_{10})	$[2, 2]$	$D_5 \times \mathbb{Z}_2$	4	$(41, 481, 41) = (51, 501, 51)$
	(A_{12}, A_{10})	$[2, 2, 2]$		8	$(17, 193, 17) = (21, 201, 21)$
	(A_{12}, A_{10})	$[2, 4]$		8	$(17, 193, 17) = (21, 201, 21)$
	(A_{12}, A_{10})	$[2, 2, 2]$	$D_4 \times \mathbb{Z}_2$	8	$(33, 385, 33) = (41, 401, 41)$
	(A_{12}, A_{10})	$[2, 6]$		12	$(25, 289, 25) = (31, 301, 31)$
	(A_{12}, A_{10})	$[2, 8]$		16	$(33, 385, 33) = (41, 401, 41)$
	(A_{12}, A_{10})	$[2, 10]$		20	$(41, 481, 41) = (51, 501, 51)$
	(A_{12}, A_{10})	$[2, 2, 6]$		24	$(49, 577, 49) = (61, 601, 61)$
	(M_{12}, A_{10})	$[2, 2]$		4	$(9, 97, 9) = (11, 101, 11)$
	(A_{12}, A_{12})	$[2, 2]$		4	$(11, 121, 11) = (11, 121, 11)$
	(A_{12}, A_{12})	$[2, 2, 2]$		8	$(21, 241, 21) = (21, 241, 21)$
	(A_{12}, A_{12})	$[2, 6]$		12	$(31, 361, 31) = (31, 361, 31)$

Table 2.7: Many simple groups Γ^*

Three more examples appearing in Table 2.7 (namely Example 2.46, Example 2.48 and Example 2.50) will be described now. We have chosen these three examples for the following reasons:

- In Example 2.46, $P_h \cong M_{12}$, the fascinating Mathieu group.
- In Example 2.48, $\Gamma^* \lneqq [\Gamma, \Gamma]$.
- In Example 2.50, $[\Gamma : \Gamma^*] = 40$ is the largest such index in Table 2.7.

Here is the description of a (M_{12}, A_8)–group:

Example 2.46.

$$R_{6\cdot 4} := \left\{ \begin{array}{llll}
\underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_3b_4, \\
a_1b_4^{-1}a_2b_4^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}}, & \underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, \\
a_2b_4a_5b_4, & \underline{a_3b_1a_4^{-1}b_2^{-1}}, & \underline{a_3b_2a_3^{-1}b_1^{-1}}, & \underline{a_3b_3a_4^{-1}b_3^{-1}}, \\
a_3b_4^{-1}a_4^{-1}b_4^{-1}, & \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, & \underline{a_4b_2a_4^{-1}b_1^{-1}}, \\
a_4b_4^{-1}a_5b_4^{-1}, & a_5b_1a_6^{-1}b_2, & a_5b_2a_6^{-1}b_2^{-1}, & a_5b_3a_5^{-1}b_3^{-1}, \\
a_5b_2^{-1}a_6^{-1}b_1^{-1}, & a_5b_1^{-1}a_6^{-1}b_1, & a_6b_3a_6^{-1}b_4^{-1}, & a_6b_4a_6^{-1}b_3
\end{array} \right\} .$$

Theorem 2.47. *Let Γ be the $(12, 8)$–group defined in Example 2.46. Then*

(1) $P_h \cong M_{12}$, $P_v = A_8$.

(2) Γ *is non-residually finite.*

(3) Γ *is a finitely presented torsion-free virtually simple group.*

(4) *There are amalgam decompositions*

$$F_4 *_{F_{37}} F_{19} \cong \Gamma \cong F_6 *_{F_{41}} F_{21}$$

and

$$\mathrm{Aut}(\mathcal{T}_{12}) > F_7 *_{F_{73}} F_7 \cong \Gamma_0 \cong F_{11} *_{F_{81}} F_{11} < \mathrm{Aut}(\mathcal{T}_8) .$$

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

(6) Γ_0 *is a finitely presented torsion-free simple group.*

Proof. (1) We compute

$$
\begin{aligned}
\rho_v(b_1) &= (5,6)(7,8)(9,10)(11,12),\\
\rho_v(b_2) &= (1,2)(3,4)(5,6)(7,8),\\
\rho_v(b_3) &= (1,2)(3,4)(9,10)(11,12),\\
\rho_v(b_4) &= (1,11,5,9,10)(2,12,3,4,8),\\
\rho_h(a_1) &= \rho_h(a_3) = (1,2)(4,5)(6,8,7),\\
\rho_h(a_2) &= \rho_h(a_4) = (1,2,3)(4,5)(7,8),\\
\rho_h(a_5) &= (1,7)(4,5),\\
\rho_h(a_6) &= (2,8)(3,5,6,4).
\end{aligned}
$$

(2) The embedding of the non-residually finite $(8,6)$–complex of Example 2.39 into the $(12,8)$–complex X (indicated by the twelve underlined relators in $R_{6\cdot 4}$) induces a π_1-injection by Proposition 1.9(1).

(3) We use [17, Corollary 5.3] and conclude as in [17, Corollary 5.4].

(4) Use Proposition 1.3 and Proposition 1.4.

(5) These are easy computations.

(6) The proof is in the same spirit as the proof of Theorem 2.45.

□

Our next example is an (A_{10}, A_{12})–group Γ with a simple subgroup Γ^* of index 12 such that Γ/Γ^* is non-abelian:

Example 2.48. *Let $R_{5\cdot 6}$ be the set of relators*

$$
\left\{
\begin{array}{lllll}
\underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_2b_4^{-1}, & a_1b_5a_2b_5^{-1},\\
a_1b_6a_4^{-1}b_4, & a_1b_6^{-1}a_4b_6, & a_1b_5^{-1}a_2^{-1}b_5, & a_1b_4^{-1}a_4^{-1}b_6^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}},\\
\underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, & a_2b_4a_3^{-1}b_6^{-1}, & a_2b_6a_3^{-1}b_4^{-1}, & a_2b_6^{-1}a_3b_6,\\
\underline{a_3b_1a_4^{-1}b_2^{-1}}, & \underline{a_3b_2a_3^{-1}b_1^{-1}}, & \underline{a_3b_3a_4^{-1}b_3^{-1}}, & a_3b_4a_5b_5, & a_3b_5a_4^{-1}b_4^{-1},\\
a_3b_5^{-1}a_4^{-1}b_5^{-1}, & \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, & \underline{a_4b_2a_4^{-1}b_1^{-1}}, & a_4b_4^{-1}a_5b_5^{-1},\\
a_5b_1a_5^{-1}b_1^{-1}, & a_5b_2a_5^{-1}b_2, & a_5b_3a_5^{-1}b_5, & a_5b_4a_5^{-1}b_3^{-1}, & a_5b_6a_5^{-1}b_6
\end{array}
\right\}
$$

Theorem 2.49. *Let* Γ *be the* $(10, 12)$*–group defined in Example 2.48 and let*

$$\Gamma^* := \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N .$$

Then

(1) $P_h = A_{10}$, $P_v = A_{12}$.

(2) *The group* Γ^* *is finitely presented, torsion-free and simple.*

(3) *The finite index subgroups of* Γ *and the normal subgroups of* Γ *are completely known (and explicitly described below).*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (7, 8)(9, 10),\\
\rho_v(b_2) &= (1, 2)(3, 4),\\
\rho_v(b_3) &= (1, 2)(3, 4)(7, 8)(9, 10),\\
\rho_v(b_4) &= (1, 9, 8, 5, 7, 10, 2, 3, 4),\\
\rho_v(b_5) &= (1, 9, 10, 2)(3, 4, 6)(7, 8),\\
\rho_v(b_6) &= (1, 4, 10, 7)(2, 3, 9, 8),\\
\rho_h(a_1) &= (1, 2)(6, 9)(10, 12, 11),\\
\rho_h(a_2) &= (1, 2, 3)(4, 6)(11, 12),\\
\rho_h(a_3) &= (1, 2)(4, 5, 8)(7, 9)(10, 12, 11),\\
\rho_h(a_4) &= (1, 2, 3)(4, 7)(5, 9, 8)(11, 12),\\
\rho_h(a_5) &= (2, 11)(3, 4, 8)(5, 10, 9)(6, 7).
\end{aligned}$$

(2) Same proof as in the previous theorems.

(3) We have used GAP ([29]) for the computations. Look at the following diagram (Figure 2.2), which describes all subgroups of Γ of finite index (Γ has no non-trivial normal subgroups of infinite index by Proposition 2.1).

Here are some explanations: N_1, N_2, N_3, N_4 are normal subgroups of Γ. The subgroups H_1, H_2, H_3 are not normal. The index in Γ is given on the left hand side of the diagram. All arrows are inclusions. The subgroups of Γ are defined as follows:

$$\begin{aligned}
N_1 &:= \ker(\Gamma \to S_2),\ a_i \mapsto (),\ b_j \mapsto (1, 2)\\
N_2 &:= \ker(\Gamma \to S_2),\ a_i \mapsto (1, 2),\ b_j \mapsto ()\\
N_3 &:= \ker(\Gamma \to S_2),\ a_i \mapsto (1, 2),\ b_j \mapsto (1, 2) .
\end{aligned}$$

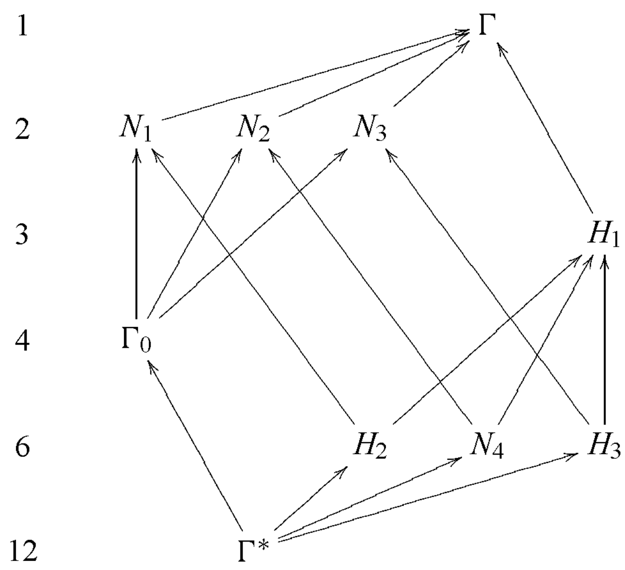


Figure 2.2: Subgroups of Example 2.48

$$\begin{aligned} N_4 := \ker(\Gamma &\to S_3) \\ a_1, a_2 &\mapsto (1,2)(3,5)(4,6) \\ a_3, a_4, a_5 &\mapsto (1,3)(2,4)(5,6) \\ b_1, b_2, b_3, b_4, b_5 &\mapsto () \\ b_6 &\mapsto (1,4,5)(2,3,6)\,. \end{aligned}$$

$$\begin{aligned} H_1 &:= \langle a_1, a_5 a_3^{-1}, b_1 \rangle \\ H_2 &:= \langle a_1, a_5 a_3^{-1}, b_2 b_1^{-1} \rangle \\ H_3 &:= \langle a_5 a_3^{-1}, b_1 a_1^{-1}, b_2 a_1^{-1} \rangle\,. \end{aligned}$$

We have

$$\begin{gathered} \Gamma/\Gamma^* \cong D_6,\ \Gamma/N_4 \cong S_3,\ H_1/\Gamma^* \cong \mathbb{Z}_2^2\,, \\ N_1/\Gamma^* \cong S_3,\ N_2/\Gamma^* \cong \mathbb{Z}_6,\ N_3/\Gamma^* \cong S_3\,, \\ [\Gamma,\Gamma] = [N_1, N_1] = [N_3, N_3] = \Gamma_0\,, \\ [\Gamma_0,\Gamma_0] = [N_2, N_2] = [N_4, N_4] = [H_1, H_1] = [H_2, H_2] = [H_3, H_3] = \Gamma^*\,. \end{gathered}$$

The following commutators are not in Γ^*:

$$\begin{gathered} [a_1, a_3],\ [a_1, a_4],\ [a_1, a_5],\ [a_1, b_6]\,, \\ [a_2, a_3],\ [a_2, a_4],\ [a_2, a_5],\ [a_2, b_6],\ [a_3, b_6],\ [a_4, b_6],\ [a_5, b_6]\,. \end{gathered}$$

$\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1, \dots, a_5$	2	12	2	12	2	12	2	12	2	12	2	12
$b_1, \dots, b_5$	6	12	6	12	6	12	6	12	6	12	6	12
b_6	2	4	6	4	2	12	2	4	6	4	2	12

Table 2.8: Some orders of $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$ in Example 2.48

In addition, see Table 2.8 for the orders of some quotients of Γ.

□

Here is an example of an (A_{10}, A_{10})–group with a simple subgroup of index 40:

Example 2.50. *Let $R_{5.5}$ be the set*

$$\left\{ \begin{array}{lllll} \underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_3b_4, & a_1b_5a_1^{-1}b_5^{-1}, \\ a_1b_4^{-1}a_2b_4^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}}, & \underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, & a_2b_4a_4b_4, \\ a_2b_5a_5^{-1}b_5^{-1}, & a_2b_5^{-1}a_5^{-1}b_5, & \underline{a_3b_1a_4^{-1}b_2^{-1}}, & \underline{a_3b_2a_3^{-1}b_1^{-1}}, & \underline{a_3b_3a_4^{-1}b_3^{-1}}, \\ a_3b_5a_4b_4^{-1}, & a_3b_5^{-1}a_4b_5^{-1}, & a_3b_4^{-1}a_4b_5, & \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, \\ \underline{a_4b_2a_4^{-1}b_1^{-1}}, & a_5b_1a_5^{-1}b_3^{-1}, & a_5b_2a_5^{-1}b_2^{-1}, & a_5b_3a_5^{-1}b_4, & a_5b_4a_5^{-1}b_1 \end{array} \right\}.$$

Theorem 2.51. *Let Γ be the $(10, 10)$–group of Example 2.50 and define*

$$\Gamma^* := \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N \,.$$

Then

(1) $P_h = A_{10}$, $P_v = A_{10}$.

(2) Γ^* *is a finitely presented torsion-free simple group.*

(3) *All finite index subgroups of Γ are normal. They are visualized in the following diagram (Figure 2.3), where all arrows are inclusions.*

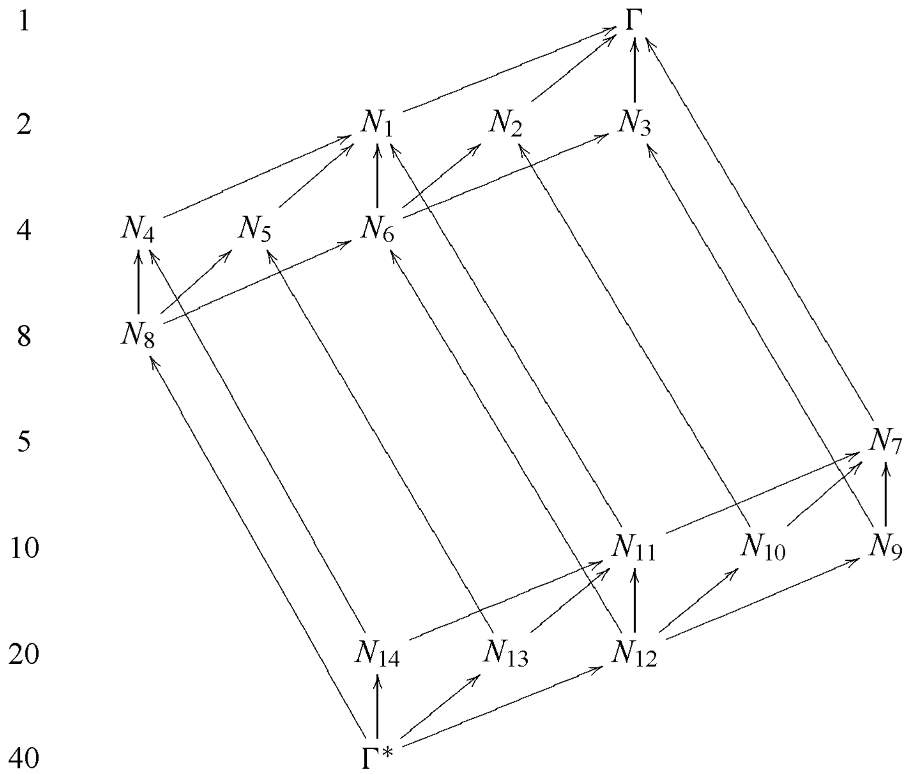


Figure 2.3: Subgroups of Example 2.50

Proof. (1) We compute

$$
\begin{aligned}
\rho_v(b_1) &= (7, 8)(9, 10),\\
\rho_v(b_2) &= (1, 2)(3, 4),\\
\rho_v(b_3) &= (1, 2)(3, 4)(7, 8)(9, 10),\\
\rho_v(b_4) &= (1, 9, 4, 8)(2, 10, 3, 7),\\
\rho_v(b_5) &= (2, 5)(3, 7)(4, 8)(6, 9),\\
\\
\rho_h(a_1) &= (1, 2)(4, 7)(8, 10, 9),\\
\rho_h(a_2) &= (1, 2, 3)(4, 7)(9, 10),\\
\rho_h(a_3) &= (1, 2)(4, 5, 6, 7)(8, 10, 9),\\
\rho_h(a_4) &= (1, 2, 3)(4, 5, 6, 7)(9, 10),\\
\rho_h(a_5) &= (1, 7, 3)(4, 8, 10).
\end{aligned}
$$

(2) We apply the same strategy as in the previous theorems.

(3) Using GAP ([29]), we have computed

$$
\begin{array}{rlrl}
N_1 &= \langle\!\langle a_1^2, a_1b_1 \rangle\!\rangle_\Gamma & \Gamma/N_1 &\cong \mathbb{Z}_2 \\
N_2 &= \langle\!\langle b_1 \rangle\!\rangle_\Gamma & \Gamma/N_2 &\cong \mathbb{Z}_2 \\
N_3 &= \langle\!\langle a_1 \rangle\!\rangle_\Gamma & \Gamma/N_3 &\cong \mathbb{Z}_2 \\
N_4 &= \langle\!\langle a_1b_4 \rangle\!\rangle_\Gamma & \Gamma/N_4 &\cong \mathbb{Z}_4 \\
N_5 &= \langle\!\langle a_1b_5 \rangle\!\rangle_\Gamma & \Gamma/N_5 &\cong \mathbb{Z}_4 \\
N_6 &= \langle\!\langle a_1^2 \rangle\!\rangle_\Gamma = \Gamma_0 & \Gamma/N_6 &\cong \mathbb{Z}_2^2 \\
N_7 &= \langle\!\langle a_1^5, b_1^5 \rangle\!\rangle_\Gamma & \Gamma/N_7 &\cong \mathbb{Z}_5 \\
N_8 &= \langle\!\langle a_1^4 \rangle\!\rangle_\Gamma & \Gamma/N_8 &\cong \mathbb{Z}_2 \times \mathbb{Z}_4 \\
N_9 &= \langle\!\langle a_1^2a_3^{-1} \rangle\!\rangle_\Gamma & \Gamma/N_9 &\cong \mathbb{Z}_{10} \\
N_{10} &= \langle\!\langle a_1^2b_5^{-1} \rangle\!\rangle_\Gamma & \Gamma/N_{10} &\cong \mathbb{Z}_{10} \\
N_{11} &= \langle\!\langle a_1^{10}, a_1b_1 \rangle\!\rangle_\Gamma & \Gamma/N_{11} &\cong \mathbb{Z}_{10} \\
N_{12} &= \langle\!\langle a_1^{10} \rangle\!\rangle_\Gamma & \Gamma/N_{12} &\cong \mathbb{Z}_2 \times \mathbb{Z}_{10} \\
N_{13} &= \langle\!\langle a_1b_1 \rangle\!\rangle_\Gamma & \Gamma/N_{13} &\cong \mathbb{Z}_{20} \\
N_{14} &= \langle\!\langle b_5a_3^{-1} \rangle\!\rangle_\Gamma & \Gamma/N_{14} &\cong \mathbb{Z}_{20}
\end{array}
$$

$$
\begin{array}{rlrl}
\Gamma^* &= [\Gamma, \Gamma] & \Gamma/\Gamma^* &\cong \mathbb{Z}_2 \times \mathbb{Z}_{20}\,. \\
&= \langle\!\langle a_1a_2^{-1} \rangle\!\rangle_\Gamma & & \\
&= \langle\!\langle a_1^{20} \rangle\!\rangle_\Gamma & & \\
&= \langle\!\langle b_1^{20} \rangle\!\rangle_\Gamma & &
\end{array}
$$

□

See Table 2.9 for the orders of some quotients of Γ:

$\lvert\Gamma/\langle\!\langle w^k \rangle\!\rangle_\Gamma\rvert$	$k = 1$	2	3	4	5	6	7	8	9	10	11	12	20
$w = a_1, \dots, a_5$	2	4	2	8	10	4	2	8	2	20	2	8	40
$b_1, \dots, b_5$	2	4	2	8	10	4	2	8	2	20	2	8	40

Table 2.9: Some orders of $\Gamma/\langle\!\langle w^k \rangle\!\rangle_\Gamma$ in Example 2.50

See Appendix C.7 for a long list of other embeddings of the non-residually finite (8, 6)–complex of Example 2.39 into (10, 10)–complexes X such that P_h and P_v are both primitive permutation groups.

2.6 A non-simple group without finite quotients

We use an embedding of the non-residually finite $(8, 6)$–complex of Example 2.39 into a $(10, 10)$–complex to get a non-simple group $\Gamma_0 < \mathrm{Aut}(\mathcal{T}_{10}) \times \mathrm{Aut}(\mathcal{T}_{10})$ without proper subgroups of finite index.

Example 2.52. *Let $R_{5\cdot 5}$ be the set of relators*

$$\left\{\begin{array}{lllll}
\underline{\underline{a_1b_1a_2^{-1}b_2^{-1}}}, & \underline{\underline{a_1b_2a_1^{-1}b_1^{-1}}}, & \underline{\underline{a_1b_3a_2^{-1}b_3^{-1}}}, & a_1b_4a_1b_5, & a_1b_5^{-1}a_2b_5^{-1}, \\
a_1b_4^{-1}a_4^{-1}b_4^{-1}, & \underline{\underline{a_1b_3^{-1}a_2^{-1}b_2}}, & \underline{\underline{a_1b_1^{-1}a_2^{-1}b_3}}, & \underline{\underline{a_2b_2a_2^{-1}b_1^{-1}}}, & a_2b_4a_2b_5, \\
a_2b_4^{-1}a_3^{-1}b_4^{-1}, & \underline{a_3b_1a_4^{-1}b_2^{-1}}, & \underline{a_3b_2a_3^{-1}b_1^{-1}}, & \underline{a_3b_3a_4^{-1}b_3^{-1}}, & a_3b_5a_4b_4^{-1}, \\
a_3b_5^{-1}a_5^{-1}b_5^{-1}, & a_3b_4^{-1}a_4b_5, & \underline{a_3b_3^{-1}a_4^{-1}b_2}, & \underline{a_3b_1^{-1}a_4^{-1}b_3}, & \underline{a_4b_2a_4^{-1}b_1^{-1}}, \\
a_4b_5^{-1}a_5b_5^{-1}, & a_5b_1a_5b_4, & a_5b_2a_5^{-1}b_3, & a_5b_3a_5^{-1}b_2, & a_5b_4^{-1}a_5b_1^{-1}
\end{array}\right\}.$$

Proposition 2.53. *Let Γ be the $(10, 10)$–group defined in Example 2.52. Then*

(1) *$P_h < S_{10}$ is transitive, but not quasi-primitive; $P_v = S_{10}$.*

(2) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(3) *There are two amalgam decompositions*

$$\Gamma \cong F_5 *_{F_{41}} F_{21}$$

and two amalgam decompositions

$$\Gamma_0 \cong F_9 *_{F_{81}} F_9 < \mathrm{Aut}(\mathcal{T}_{10})\,.$$

(4) *Γ is non-residually finite, in particular not linear over any field and irreducible.*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (5, 6)(7, 8)(9, 10),\\
\rho_v(b_2) &= (1, 2)(3, 4),\\
\rho_v(b_3) &= (1, 2)(3, 4)(7, 8)(9, 10),\\
\rho_v(b_4) &= (1, 4, 8, 9, 2, 3, 7, 10)(5, 6),\\
\rho_v(b_5) &= (1, 9, 2, 10)(3, 5, 7)(4, 6, 8).
\end{aligned}$$

These permutations generate a transitive group $P_h < S_{10}$ of order 3840 which is not quasi-primitive, since P_h has a normal subgroup of order 2 generated by the element $(1,2)(3,4)(5,6)(7,8)(9,10) = \rho_v(b_1)\rho_v(b_2)$.

$$\begin{aligned}
\rho_h(a_1) &= (1,2)(4,7,5,6)(8,10,9),\\
\rho_h(a_2) &= (1,2,3)(4,7,5,6)(9,10),\\
\rho_h(a_3) &= (1,2)(4,5,6,7)(8,10,9),\\
\rho_h(a_4) &= (1,2,3)(4,5,6,7)(9,10),\\
\rho_h(a_5) &= (1,7)(2,8)(3,9)(4,10)(5,6).
\end{aligned}$$

(2) These are easy computations.

(3) We use Proposition 1.3 and Proposition 1.4. To apply Proposition 1.4, the only thing to check is that $\rho_v(F_n^{(2)}) < S_{2m}$ is transitive, but here we have

$$P_h = \langle \rho_v(b_1^2), \rho_v(b_1 b_2), \rho_v(b_1 b_4), \rho_v(b_5^2)\rangle ,$$

in particular $\rho_v(F_n^{(2)}) = \rho_v(F_n) = P_h$ in the notation of Proposition 1.4.

(4) We use the fact that the non-residually finite $(8,6)$–complex of Example 2.39 embeds into the $(10,10)$–complex X, see the twelve underlined relators in $R_{5\cdot 5}$. □

Theorem 2.54. *Let* Γ *be the* $(10,10)$*–group defined in Example 2.52. Then*

(1) *The subgroup* Γ_0 *has no proper subgroups of finite index.*

(2) Γ_0 *is not simple.*

Proof. (1) By construction, the non-residually finite complex of Example 2.39 is embedded into X. Take $w := a_2 a_1^{-1} a_3 a_4^{-1}$ and

$$\Gamma^* := \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N .$$

As in Theorem 2.45, we observe that $\langle\langle w\rangle\rangle_\Gamma = \Gamma_0$, in particular $\langle\langle w\rangle\rangle_\Gamma > \Gamma^*$. Since $w \in \Gamma^*$, using Theorem 2.40 and Lemma 2.41, we conclude that

$$\langle\langle w\rangle\rangle_\Gamma = \Gamma^* = \Gamma_0 .$$

Assume now that M is a finite index subgroup of Γ_0. Then M also has finite index in Γ and therefore

$$M > \bigcap_{L \overset{\text{f.i.}}{<} \Gamma} L = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N = \Gamma^* = \Gamma_0 ,$$

using Lemma 2.6, hence $M = \Gamma_0$.

(2) $\mathrm{QZ}(H_1) \cap \Gamma_0$ is a non-trivial normal subgroup of infinite index in Γ_0. More precisely, let A be the set

$$A := \{(a_1 a_2^{-1})^2, (a_2^{-1} a_1)^2, (a_3 a_4^{-1})^2, (a_4^{-1} a_3)^2, a_5^4\}^{\pm 1}.$$

Then $A \subset \Lambda_1 \cap \Gamma_0 < \mathrm{QZ}(H_1) \cap \Gamma_0$, since for each $a \in A$ and $b \in E_v$ we have $\rho_h(a)(b) = b$ and $\rho_v(b)(a) \in A$, using Lemma 1.1(1a).

Note that we have $|F_{81} \backslash F_9 / F_{81}| = 3$ for the vertical amalgam decomposition of $\Gamma_0 \cong F_9 *_{F_{81}} F_9$ (more than 2 by Proposition 1.6, since P_h is not 2-transitive), and Γ_0 is therefore even SQ-universal, according to Proposition 1.7.

□

Remarks. (see Appendix D.1 for much more history)

(1) Higman's group

$$H = \langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2, a^{-1}da = d^2 \rangle$$

introduced in [34], has no proper subgroup of finite index. There is another similarity to the group Γ_0 of Example 2.52: Using small cancellation theory, Schupp proved in [62] that H is SQ-universal. By the way, H was used to show the existence of a finitely generated infinite simple group (one takes the quotient of H by a maximal normal subgroup of H), thus answering a question posed by Kuroš ([42]).

(2) Bhattacharjee has constructed in [7] an amalgam $F_3 *_{F_{13}} F_3$ without non-trivial finite quotients. It is not clear if it has proper infinite quotients.

(3) In [68], Wise gave a construction of a square complex, whose fundamental group has no non-trivial finite quotients.

As usual, we give in Table 2.10 orders of some quotients of the group Γ defined in Example 2.52. The infinite quotients in the table correspond to elements in Λ_1.

$\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$	$k = 1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1, \ldots, a_4$	2	4	2	4	2	4	2	4	2	4	2	4
a_5	2	4	2	∞	2	4	2	∞	2	4	2	∞
$b_1, \ldots, b_5$	2	4	2	4	2	4	2	4	2	4	2	4

Table 2.10: Some orders of $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$ in Example 2.52

2.7 A group which is not virtually torsion-free

Using an idea of Wise ([68, Section II.6]), we construct a finitely presented infinite quotient Q of an $(8, 8)$–group such that Q is not virtually torsion-free, i.e. each subgroup of Q of finite index has non-trivial elements of finite order.

Lemma 2.55. *(Wise, cf. [68, Easy Lemma II.6.1]) Let G be a non-residually finite group and $g \in G$ a non-trivial element such that*

$$g \in \bigcap_{N \overset{f.i.}{\lhd} G} N$$

and assume that $g \notin \langle\langle g^n \rangle\rangle_G$ for some $n \geq 2$ (equivalently: $\langle\langle g^n \rangle\rangle_G \subsetneqq \langle\langle g \rangle\rangle_G$). Then the quotient $G/\langle\langle g^n \rangle\rangle_G$ is non-residually finite and not virtually torsion-free.

Proof. (cf. [68, Proof of Easy Lemma II.6.1]) Let $H < G/\langle\langle g^n \rangle\rangle_G =: Q$ be a subgroup of finite index (say of index k). Let $\psi = \phi \circ \pi$ be the composition homomorphism

$$\psi : G \xrightarrow{\pi} Q \xrightarrow{\phi} S_k\,,$$

where π is the canonical projection and ϕ is induced by left multiplication on left cosets in Q/H, i.e. $\phi(q)(q_i H) := q q_i H$ (cf. proof of Lemma 2.6). Since $\ker \psi \lhd G$ and $[G : \ker \psi] \leq |S_k| = k!$ is finite, we have $g \in \ker \psi$, hence

$$\pi(g) = g\langle\langle g^n \rangle\rangle_G \in \ker \phi < H\,.$$

By assumption $g \notin \langle\langle g^n \rangle\rangle_G$, which implies $g\langle\langle g^n \rangle\rangle_G \neq 1_Q$. We conclude that Q is non-residually finite.

H is not torsion-free, since $(g\langle\langle g^n \rangle\rangle_G)^n = \langle\langle g^n \rangle\rangle_G = 1_H$. ☐

Example 2.56.

$$R_{4\cdot 4} := \left\{ \begin{array}{llll} \underline{\underline{a_1 b_1 a_2^{-1} b_2^{-1}}}, & \underline{\underline{a_1 b_2 a_1^{-1} b_1^{-1}}}, & \underline{\underline{a_1 b_3 a_2^{-1} b_3^{-1}}}, & a_1 b_4 a_2^{-1} b_4, \\ a_1 b_4^{-1} a_2^{-1} b_4^{-1}, & \underline{\underline{a_1 b_3^{-1} a_2^{-1} b_2}}, & \underline{\underline{a_1 b_1^{-1} a_2^{-1} b_3}}, & \underline{\underline{a_2 b_2 a_2^{-1} b_1^{-1}}}, \\ \underline{a_3 b_1 a_4^{-1} b_2^{-1}}, & \underline{a_3 b_2 a_3^{-1} b_1^{-1}}, & \underline{a_3 b_3 a_4^{-1} b_3^{-1}}, & a_3 b_4 a_3^{-1} b_4, \\ \underline{a_3 b_3^{-1} a_4^{-1} b_2}, & \underline{a_3 b_1^{-1} a_4^{-1} b_3}, & \underline{a_4 b_2 a_4^{-1} b_1^{-1}}, & a_4 b_4 a_4^{-1} b_4^{-1} \end{array} \right\}.$$

Theorem 2.57. *Let Γ be the $(8,8)$–group defined in Example 2.56 and let w be the element $a_2a_1^{-1}a_3a_4^{-1}$. Then $Q := \Gamma/\langle\langle w^2\rangle\rangle_\Gamma$ is non-residually finite and not virtually torsion-free. More precisely, the element*

$$w\langle\langle w^2\rangle\rangle_\Gamma \in \bigcap_{N \overset{f.i.}{\lhd} Q} N < Q$$

has order 2 in Q.

Proof. The non-residually finite $(8,6)$–complex of Example 2.39 embeds into the $(8,8)$–complex of Example 2.56 and induces a π_1-injection by Proposition 1.9(1), in particular

$$w \in \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N$$

by Lemma 2.41. Note that $w \notin \Lambda_1$, since $\rho_h(w)(b_4) = b_4^{-1} \neq b_4$ (see Figure 2.4).

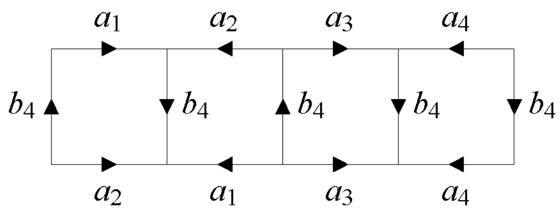


Figure 2.4: Illustrating $\rho_h(w)(b_4) = b_4^{-1}$ in Example 2.56

However, by Lemma 1.1(1a), the set

$$A := \{w^2, (a_1a_2^{-1}a_4a_3^{-1})^2, (a_1a_2^{-1}a_3a_4^{-1})^2, (a_2a_1^{-1}a_4a_3^{-1})^2\}$$

is a subset of Λ_1, since for each $a \in A$ and $b \in E_v$ we have $\rho_h(a)(b) = b$ and $\rho_v(b)(a) \in A$. Using $w^2 \in \Lambda_1 \lhd \Gamma$, we conclude that $\langle\langle w^2\rangle\rangle_\Gamma < \Lambda_1$ and therefore $w \notin \langle\langle w^2\rangle\rangle_\Gamma$. Now apply Lemma 2.55 to the quotient $\Gamma/\langle\langle w^2\rangle\rangle_\Gamma$. □

Remark. Let Γ be a $(2m,2n)$–group such that every non-trivial normal subgroup of Γ has finite index, for example by Proposition 2.1. Then every quotient of Γ is either torsion-free (if the quotient is $\Gamma/1 \cong \Gamma$) or finite, in particular virtually torsion-free.

2.8 Locally primitive, not 2-transitive

To guarantee that an irreducible $(2m,2n)$–group has no non-trivial normal subgroup of infinite index, it is required in Proposition 2.1 that both local groups P_h and P_v are 2-transitive. We construct now an irreducible (A_6, P_v)–group, where $P_v < S_{10}$ is primitive, but not 2-transitive. All primitive permutation groups are 2-transitive in degree 2, 4, 6, 8, 12 and 14, see Table C.1.

Example 2.58.

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_1, \\ a_1b_4a_2^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_5, & a_1b_5^{-1}a_2^{-1}b_4^{-1}, \\ a_1b_4^{-1}a_2b_1^{-1}, & a_1b_3^{-1}a_2^{-1}b_3, & a_1b_2^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_2, & a_2b_2a_3^{-1}b_1, & a_3b_1a_3b_2, \\ a_3b_3a_3^{-1}b_3^{-1}, & a_3b_4a_3b_4^{-1}, & a_3b_5a_3^{-1}b_5 \end{array} \right\}.$$

Theorem 2.59. *Let* Γ *be the* $(6, 10)$*–group defined in Example 2.58. Then*

(1) $P_h = A_6$*;* $P_v \cong S_5 < S_{10}$ *is primitive, not 2-transitive.*

(2) *There are two amalgam decompositions of* Γ*:*

$$F_5 *_{F_{25}} F_{13} \cong \Gamma \cong F_3 *_{F_{21}} F_{11}\,.$$

There is a vertical decomposition of Γ_0

$$\Gamma_0 \cong F_9 *_{F_{49}} F_9\,,$$

acting locally like A_6 *(but possibly not effectively) on the tree* $\mathcal{T}_{2m} = \mathcal{T}_6$*, and a horizontal decomposition*

$$\Gamma_0 \cong F_5 *_{F_{41}} F_5 < \mathrm{Aut}(\mathcal{T}_{10})\,,$$

where the (effective) action on $\mathcal{T}_{10}$ *is locally like* $S_5 < S_{10}$*, in particular locally primitive, but not locally 2-transitive.*

(3) $H_b^2(\Gamma; \mathbb{R})$ *is infinite dimensional as* $\mathbb{R}$*-vector space (cf. Theorem 2.3(8)).*

(4) Γ *is SQ-universal, in particular not virtually simple.*

(5) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

(6) Γ *is not linear over any field, in particular irreducible.*

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= (1, 5, 4, 3, 2), \\ \rho_v(b_2) &= (2, 6, 5, 4, 3), \\ \rho_v(b_3) &= \rho_v(b_5) = (1, 2)(5, 6), \\ \rho_v(b_4) &= (1, 2, 6, 5)(3, 4), \end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1, 7, 9, 10, 3, 2)(4, 6, 5),\\
\rho_h(a_2) &= (1, 8, 9)(2, 4, 10)(5, 6, 7),\\
\rho_h(a_3) &= (1, 9)(2, 10)(5, 6).
\end{aligned}$$

The action of $P_v^{(2)}$ on the sphere $S(x_v, 2)$ has two orbits of size 60 and 30, respectively. Observe that in general the action of $P_v^{(2)}$ on $S(x_v, 2)$ is transitive if and only if P_v is a 2-transitive permutation group. Note that P_v acts like S_5 on the set of 2-element subsets of $\{1, 2, 3, 4, 5\}$.

(2) Use Proposition 1.3 and Proposition 1.4. The explicit horizontal decomposition of Γ_0 can be found in Appendix A.5.

(3) In the horizontal amalgam decomposition $\Gamma \cong F_3 *_{F_{21}} F_{11}$ we have

$$|F_{21} \backslash F_3 / F_{21}| = 3 \text{ and } |F_{11}/F_{21}| = 2\,.$$

See Proposition 1.6 for an easy method to compute $|F_{21}\backslash F_3/F_{21}|$. Now we apply a result of Fujiwara ([28, Theorem 1.1]), which states that $H_b^2(A *_C B; \mathbb{R})$ is an infinite dimensional $\mathbb{R}$-vector space if $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$.

Note that the assumptions of Fujiwara's theorem are not fulfilled in the two $(F_3 *_{F_{13}} F_7)$–decompositions of Example 2.2, since $|F_{13}\backslash F_3/F_{13}| = 2$ due to the 2-transitivity of P_h and P_v in Example 2.2.

(4) Apply Proposition 1.7 to $\Gamma \cong F_3 *_{F_{21}} F_{11}$. Observe that Γ does not satisfy the assumptions of the normal subgroup theorem [17, Theorem 4.1], since H_2 is not locally 2-transitive and consequently not locally ∞-transitive.

(5) This is a short computation.

(6) It follows from [17, Theorem 1.4], see also Proposition 4.4 in Section 4.2.

□

Proposition 2.60. *Let Γ be as in Example 2.58. Then*

$$\langle\langle a_1^k \rangle\rangle_\Gamma = \Gamma_0,\ \textit{if } k \in \{2 + 6l, 4 + 6l\},\ l \in \mathbb{N}_0\,.$$

Moreover, $\langle\langle a_1^6 \rangle\rangle_\Gamma = \langle\langle a_1^{12} \rangle\rangle_\Gamma = \langle\langle a_1^{18} \rangle\rangle_\Gamma = \Gamma_0$.

Proof. For the first part, we only give the idea of the proof, which is essentially the same as in the proof of Proposition 2.12: show that $\langle\langle b_4 b_5 \rangle\rangle_\Gamma = \Gamma_0$ and $\langle\langle b_5^2 \rangle\rangle_\Gamma = \Gamma_0$, then show that for $l \in \mathbb{N}_0$

$$a_1^{-k}(b_5^{-1} b_3 a_1^k b_3^{-1} b_5) = \begin{cases} b_4 b_5\,, & k = 2 + 6l \\ b_5^2\,, & k = 4 + 6l\,. \end{cases}$$

We have checked the second part of the proposition with MAGNUS ([50]). □

Conjecture 2.61. *The group* Γ *of Example 2.58 is non-residually finite and*

$$\bigcap_{N \overset{f.i.}{\triangleleft} \Gamma} N = \Gamma_0 .$$

See Table 2.11 for the orders of some quotients of Γ.

$\lvert\Gamma/\langle\langle w^k\rangle\rangle_\Gamma\rvert$	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$w = a_1, a_2, a_3$	2	4	2	4	2	4	2	4	2	4	2	4
$b_1, \ldots, b_5$	2	4	2	4	2	4	2	4	2	4	2	4

Table 2.11: Some orders of $\Gamma/\langle\langle w^k\rangle\rangle_\Gamma$ in Example 2.58

We also would like to construct an explicit non-trivial infinite index normal subgroup of Γ, for example given as normal closure of one element or of several elements, but we did not manage to do this. What follows is a mix of ideas to achieve this goal, a possible application to Kazhdan's property (T), and some remarks on SQ-universality.

Conjecture 2.62. *Let* Γ *be the group defined in Example 2.58 and* x_v *a vertex in* $\mathcal{T}_{10}$*. Then every orbit of the* $H_2(x_v)$*–action on* $\partial_\infty \mathcal{T}_{10}$ *is uncountable.*

"Proof". Studying the orbits of the local action of H_2 on finite spheres $S(x_v, k)$, we believe that the orbit of any boundary point $\omega \in \partial_\infty \mathcal{T}_{10}$ under the $H_2(x_v)$–action contains the uncountable boundary at infinity $\partial_\infty \mathcal{T}_{10;4,7}$ of a certain infinite subtree $\mathcal{T}_{10;4,7} \subset \mathcal{T}_{10}$. This subtree contains $S(x_v, 1)$ and the valency of any vertex $y_v \neq x_v$ is either 4 or 7 (depending on ω), but constant on finite spheres $S(x_v, k)$.

More precisely, we imagine reduced paths in $\mathcal{T}_{10}$ originating at x_v to be labelled by freely reduced words in the free group $\langle b_1, \ldots, b_5\rangle$. Using the explicit isomorphism $E_v \cong \{1, \ldots, 10\}$ described in Section 1.4, we identify the sphere $S(x_v, k)$ with the set of k-tuples

$$\{(e_1, \ldots, e_k) \in \{1, \ldots, 10\}^k : e_i + e_{i+1} \neq 11 \text{ for each } i \in \{1, \ldots, k-1\}\} .$$

For each $k \geq 2$, we define an equivalence relation $\sim_k$ on $S(x_v, k)$ as follows. First, $\sim_2$ gives a partition of $S(x_v, 2)$ into two equivalence classes consisting of 30 and 60 elements, respectively. The equivalence class with 30 elements is the set $\{(1,3),$ $(1,5),$ $(1,9),$ $(2,6),$ $(2,7),$ $(2,10),$ $(3,4),$ $(3,5),$ $(3,6),$ $(4,1),$ $(4,4),$ $(4,9),$ $(5,2),$ $(5,8),$ $(5,9),$ $(6,1),$ $(6,8),$ $(6,10),$ $(7,3),$ $(7,7),$ $(7,8),$ $(8,2),$ $(8,4),$ $(8,10),$ $(9,1),$ $(9,3),$ $(9,6),$ $(10,2),$ $(10,5),$ $(10,7)\}$. For $k \geq 3$ we define

$$(e_1, \ldots, e_k) \sim_k (f_1, \ldots, f_k) :\Longleftrightarrow (e_i, e_{i+1}) \sim_2 (f_i, f_{i+1}) \ \forall i \in \{1, \ldots, k-1\} .$$

Note that we have 2^{k-1} equivalence classes on $S(x_v, k)$ with respect to $\sim_k$, where the number of elements in each class is $10 \cdot 6^j \cdot 3^{k-1-j}$ for some $j \in \{0, \ldots, k-1\}$. We have checked that the $H_2(x_v)$–action induces exactly the equivalence relation $\sim_k$ on $S(x_v, k)$ for $k = 2, 3, 4$. □

As a “corollary” of Conjecture 2.62, we have

Conjecture 2.63. *Let* Γ *be the group of Example 2.58. Then* $\mathrm{QZ}(H_2) = 1$.

“Proof”. If Conjecture 2.62 holds, then we follow verbatim the proof of [16, Proposition 3.1.2, 1)]: Let $S \subset \partial_\infty \mathcal{T}_{10}$ be the set of fixed points of hyperbolic elements in $\mathrm{QZ}(H_2)$. Then S is countable, since $\mathrm{QZ}(H_2)$ is countable, which follows directly from the fact that $\mathrm{QZ}(H_2)$ is discrete (see [16, Proposition 1.2.1, 2)]). Moreover, S is H_2-invariant, since $\mathrm{QZ}(H_2)$ is a normal subgroup of H_2. We could conclude by Conjecture 2.62 that S is empty, in other words $\mathrm{QZ}(H_2)$ has no hyperbolic elements. On the other hand, $\mathrm{QZ}(H_2)$ acts by [16, Proposition 1.2.1, 2)] freely on the vertices of $\mathcal{T}_{10}$ (in particular, there are no elliptic elements in $\mathrm{QZ}(H_2) \setminus \{1\}$), hence $|\mathrm{QZ}(H_2)| \le 2$. But then, $\mathrm{QZ}(H_2) \subseteq Z(H_2) = 1$. $\square$

See the subsequent Table 2.12 to check that small powers of $b_1, \ldots, b_5$ are not in the group $\Lambda_2 < \mathrm{QZ}(H_2)$ (see also Appendix A.5 for a computation of $|\rho_v^{(k)}(w)|$ for all words w of length 2 and $k \le 5$).

$\lvert\rho_v^{(k)}(w)\rvert$	$k = 1$	2	3	4	5
$w = b_1, b_2$	5	10	100	600	3000
b_3	2	10	50	100	1000
b_4	4	8	40	200	1000
b_5	2	4	20	40	1200

Table 2.12: Order of $\rho_v^{(k)}(w)$ in Example 2.58

For instance, it follows from this table that $b_1^j \notin \Lambda_2$, if $1 \le j < 3000$, using the following general lemma.

Lemma 2.64. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$ *be a* $(2m, 2n)$*–group and* $b \in \langle b_1, \ldots, b_n \rangle$ *an element such that* $b^j \in \Lambda_2$ *for some* $j \in \mathbb{N}$. *Then* $|\rho_v^{(k)}(b)| \le j$ *for each* $k \in \mathbb{N}$.

Proof. Fix any $k \in \mathbb{N}$. Using the identification

$$\Lambda_2 \cong \bigcap_{k \in \mathbb{N}} \ker \rho_v^{(k)}$$

we get

$$\left(\rho_v^{(k)}(b)\right)^j = \rho_v^{(k)}(b^j) = 1_{\mathrm{Sym}(E_h^{(k)})}$$

hence $|\rho_v^{(k)}(b)| \le j$. $\square$

$\lvert\rho_h^{(k)}(w)\rvert$	$k=1$	2	3	4
$w=a_1$	6	12	72	432
a_2	3	6	12	72
a_3	2	4	8	16

Table 2.13: Order of $\rho_h^{(k)}(w)$ in Example 2.58

Compare Table 2.12 to Table 2.13, where we already know that $\mathrm{QZ}(H_1)$ is trivial by [16, Proposition 3.1.2, 1)].

Conjecture 2.63 implies another conjecture:

Conjecture 2.65. *Let Γ be the group of Example 2.58 and let $N \lhd \Gamma$ be a non-trivial normal subgroup of infinite index. Then Γ/N is an infinite group having property* (T) *of Kazhdan.*

"Proof". We know that $\mathrm{QZ}(H_1) = 1$ (see [16, Proposition 3.1.2, 1)]) and assume that $\mathrm{QZ}(H_2) = 1$ (see Conjecture 2.63). For $1 \neq N \lhd \Gamma$ and $i = 1, 2$, we have $1 \neq \overline{\mathrm{pr}_i(N)} \lhd H_i$. By [16, Proposition 1.2.1] $H_i/\overline{\mathrm{pr}_i(N)}$ is compact. We can apply [17, Proposition 3.1] to conclude that Γ/N has property (T).

Note that there are uncountably many non-isomorphic infinite quotients Γ/N, since Γ is SQ-universal by Theorem 2.59(4) (see [56], the proof is based on the fact that there are uncountably many non-isomorphic finitely generated groups, but each quotient Γ/N, being countable, has only countably many finitely generated subgroups). □

A homomorphism of B. H. Neumann

Proposition 2.66. *(Neumann, see [55]) Let A, B, C be groups, $i_A : C \to A$ and $i_B : C \to B$ two injective homomorphisms and assume that $A \neq 1$. Then there is a surjective homomorphism*

$$\rho : A *_C B \twoheadrightarrow P < \mathrm{Sym}(A \times B)\,,$$

such that $P \neq 1$. In particular, if ρ is not injective, we get a non-trivial proper quotient $P \cong (A *_C B)/\ker\rho$ of $A *_C B$, and if ρ is injective, then $A *_C B < \mathrm{Sym}(A \times B)$.

Proof. (cf. [55]) We fix right coset representatives $S_A := \{a_1 = 1, a_2, a_3, \ldots\}$ and $S_B := \{b_1 = 1, b_2, b_3, \ldots\}$ of C in A and B, respectively, i.e.

$$A = \bigsqcup_i Ca_i \text{ and } B = \bigsqcup_j Cb_j\,.$$

We will define two homomorphisms

$$\rho_A : A \to \mathrm{Sym}(A \times B) \text{ and } \rho_B : B \to \mathrm{Sym}(A \times B)$$

as follows. Let $(x, y) \in A \times B$, then $\rho_A(a)(x, y) := (ax, y)$. Obviously, ρ_A is a homomorphism:

$$\rho_A(a\tilde{a})(x, y) = (a\tilde{a}x, y) = \rho_A(a)(\tilde{a}x, y) = \rho_A(a)\rho_A(\tilde{a})(x, y)\,.$$

To define $\rho_B(b)(x, y)$, note that with respect to the chosen (fixed) right coset representatives, we have unique decompositions

$$x = c_x a_x,\ y = c_y b_y,\ bc_x b_y = c_z b_z \quad (c_x, c_y, c_z \in C,\ a_x \in S_A,\ b_y, b_z \in S_B)\,.$$

Now we define $\rho_B(b)(x, y) := (c_z a_x, c_y b_z)$ and check that ρ_B is a homomorphism:

$$\rho_B(b\tilde{b})(x, y) = (c_t a_x, c_y b_t)\,,$$

where $b\tilde{b}c_x b_y = c_t b_t$ ($c_t \in C$, $b_t \in S_B$) is the unique decomposition. We have

$$\rho_B(\tilde{b})(x, y) = (c_r a_x, c_y b_r)\,,$$

where $\tilde{b}c_x b_y = c_r b_r$ ($c_r \in C$, $b_r \in S_B$) is the unique decomposition. Hence,

$$\rho_B(b)\rho_B(\tilde{b})(x, y) = \rho_B(b)(c_r a_x, c_y b_r) = (c_t a_x, c_y b_t) = \rho_B(b\tilde{b})(x, y)\,,$$

since $bc_r b_r = b\tilde{b}c_x b_y = c_t b_t$. Let $c \in C$, then

$$\rho_B(c)(x, y) = (cc_x a_x, c_y b_y) = (cx, y) = \rho_A(c)(x, y)\,,$$

in other words, $\rho_A \circ i_A = \rho_B \circ i_B$. By the universal property of $A *_C B$, the desired homomorphism $\rho : A *_C B \twoheadrightarrow P$ exists (see the following diagram), where the group $P < \mathrm{Sym}(A \times B)$ is generated by $\{\rho_A(A), \rho_B(B)\} \subseteq \mathrm{Sym}(A \times B)$. Obviously, $P \neq 1$, since $A \neq 1$ (by assumption) and $\rho_A(a)(1_A, 1_B) = (a, 1_B)$.

$$\begin{tikzcd}
C \arrow[r, "i_B"] \arrow[d, "i_A"'] & B \arrow[d] \arrow[ddr, bend left, "\rho_B"] & \\
A \arrow[r] \arrow[drr, bend right, "\rho_A"'] & A *_C B \arrow[dr, dotted, two heads, "\rho"] & \\
 & & P
\end{tikzcd}$$

□

Question 2.67. *Let Γ be the group defined in Example 2.58. Is there an amalgam decomposition $A *_C B$ of Γ (or of its subgroup Γ_0) such that the homomorphism ρ of Proposition 2.66 is not injective?*

A result of Lyndon

Perhaps useful in the construction of infinite quotients of amalgamated free products could be the following proposition of Lyndon:

Proposition 2.68. *(Lyndon [48, Proposition 1.3]) Let $G = A *_C B$ be an amalgamated free product. Let $N_A \lhd A$, $N_B \lhd B$ be normal subgroups such that $N_A \cap C = N_B \cap C$. Then*

$$G/N \cong A/N_A *_{C/N_C} B/N_B\,,$$

where $N_C := N_A \cap C = N_B \cap C$ and $N := \langle\langle N_A \cup N_B \rangle\rangle_G$.

Proof. See [48] or [22]. $\square$

Blocking pairs

One method to prove the SQ-universality of an amalgamated free product is a criterion of Schupp ([62]) using the notion of a blocking pair. The following definition is taken from [62]: Let $C < A$ be groups. A pair $\{x_1, x_2\}$ of distinct elements in $A \setminus C$ is called a *blocking pair* for $C < A$ if

i) $x_i^\epsilon x_j^\delta \notin C \setminus \{1\}$, for all $i, j = 1, 2$; $\epsilon, \delta = \pm 1$.

ii) $x_i^\epsilon c x_j^\delta \notin C$, if $c \in C \setminus \{1\}$; $i, j = 1, 2$; $\epsilon, \delta = \pm 1$.

Proposition 2.69. (1) *(Schupp [62]) If there is a blocking pair for $C < A$ or a blocking pair for $C < B$, then the amalgam $A *_C B$ is SQ-universal.*

(2) *If there is a blocking pair for $C < A$, then $|C\backslash A/C| \geq 3$.*

(3) *Let Γ be a $(2m, 2n)$–group. Suppose that $P_h < S_{2m}$ is transitive. Then there is no blocking pair for $C < B$ and no blocking pair for $C < A$, where*

$$B *_C A := F_n *_{F_{1-2m+2mn}} F_{1-m+mn} \cong \Gamma$$

is the vertical decomposition given by Proposition 1.3(1a).

Proof. (1) See [62], the proof uses small cancellation theory.

(2) Let $\{x_1, x_2\}$ be a blocking pair for $C < A$. Obviously $Cx_1C \neq C \neq Cx_2C$. Assume that $Cx_1C = Cx_2C$, thus there exist $c_1, c_2 \in C$ such that $x_1 = c_1x_2c_2$. If $c_1 = 1$ and $c_2 = 1$, then $x_1 = x_2$, a contradiction. If $c_1 \neq 1$, then we get the contradiction $x_1^{-1}c_1x_2 = c_2^{-1} \in C$. If $c_2 \neq 1$, then $x_2c_2x_1^{-1} = c_1^{-1} \in C$, again a contradiction to the blocking pair assumption.

(3) By part (2), there is no blocking pair for $C < A$, since

$$|C \backslash A/C| \leq |A/C| = 2 < 3\,.$$

Let x_1 be in a blocking pair for $C < B$. Let b be a non-trivial element in $\ker(\rho_v : \langle b_1, \ldots, b_n \rangle \twoheadrightarrow P_h)$. Since $[B : C] = 2m$ is finite, there is an integer $k \in \mathbb{N}$ such that $b^k \in C$. Let $c := b^k$, then $c \in \ker \rho_v \setminus \{1\}$ fixes the 1-sphere around the vertex "B" in the corresponding Bass-Serre tree (see Figure 2.5), in particular c fixes the edge "Cx_1", hence $Cx_1c = Cx_1$, but then $x_1cx_1^{-1} \in C$ is a contradiction to the assumption that x_1 is in a blocking pair for $C < B$.

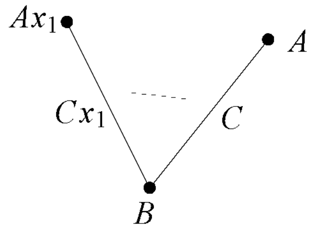


Figure 2.5: Illustration in the proof of Proposition 2.69(3)

□

2.9 Three candidates for simplicity

So far, we have presented many simple groups and many candidates. In this section, we give three more candidates for simplicity coming from three different constructions. The third one (Example 2.77) has very small finite presentations and is therefore particularly suitable for computer experiments.

A non-linear $(4, 6)$–group

Let Γ be the $(4, 6)$–group defined by

$$R_{2.3} := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_1^{-1}, \\ a_1 b_3 a_2^{-1} b_1, & a_1 b_3^{-1} a_2 b_3, \\ a_1 b_2^{-1} a_2^{-1} b_3^{-1}, & a_2 b_1 a_2^{-1} b_2 \end{array} \right\}.$$

Some properties of Γ will be described in Section 4.2, in particular Γ is not linear.

Question 2.70. *Let Γ be as above. Is Γ_0 simple?*

Embedding the $(4,6)$–group of Wise

Recall Wise's $(4,6)$–group of Example 2.36:

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_2 a_1^{-1} b_1^{-1}, & a_2 b_2 a_2^{-1} b_1^{-1}, \\ a_1 b_3 a_2^{-1} b_3^{-1}, & a_1 b_1 a_2^{-1} b_2^{-1}, \\ a_2 b_1 a_1^{-1} b_3^{-1}, & a_2 b_3 a_1^{-1} b_2^{-1} \end{array} \right\}.$$

Lemma 2.71. *Let Γ be the group defined in Example 2.36 and let $\theta : \Gamma \to \Gamma$, $\gamma \mapsto b_3 \gamma b_3^{-1}$ be the conjugation by b_3. Then* $\mathrm{Fix}(\theta) = \langle b_3 \rangle$.

Proof. Note that $\mathrm{Fix}(\theta) = \{\gamma \in \Gamma : b_3 \gamma b_3^{-1} = \gamma\}$ is the centralizer of b_3 in Γ. The statement follows now from Proposition 1.12(1b). $\square$

Proposition 2.72. *Let Γ be the $(4,6)$–group defined in Example 2.36 and let S be the subset*

$$S := \bigcap_{k \in \mathbb{N}} \langle b_3 \rangle \langle\!\langle b_3^{2k} \rangle\!\rangle_{\Gamma} \setminus \langle b_3 \rangle \subset \Gamma \,.$$

(1) *If S is non-empty, then Γ is not $\langle b_3 \rangle$-separable.*

(2) *If $\gamma \in S$ for some $\gamma \in \Gamma$, then Γ is non-residually finite such that*

$$\gamma^{-1} \theta(\gamma) = [\gamma^{-1}, b_3] \in \bigcap_{N \overset{f.i.}{\lhd} \Gamma} N \,.$$

(3) *If $a_1 a_2^{-1} \in S$, then the index 4 subgroup $\hat{\Gamma}_0$ of the (A_8, A_8)–group $\hat{\Gamma}$ which is given by*

$$R_{4\cdot 4} := \left\{ \begin{array}{llll} \underline{a_1 b_1 a_2^{-1} b_2^{-1}}, & \underline{a_1 b_2 a_1^{-1} b_1^{-1}}, & \underline{a_1 b_3 a_2^{-1} b_3^{-1}}, & a_1 b_4 a_4^{-1} b_4, \\ a_1 b_4^{-1} a_2 b_4^{-1}, & \underline{a_1 b_3^{-1} a_2^{-1} b_2}, & \underline{a_1 b_1^{-1} a_2^{-1} b_3}, & \underline{a_2 b_2 a_2^{-1} b_1^{-1}}, \\ a_2 b_4 a_3 b_4, & a_3 b_1 a_3 b_2, & a_3 b_3 a_4^{-1} b_3^{-1}, & a_3 b_4^{-1} a_4^{-1} b_3, \\ a_3 b_3^{-1} a_4^{-1} b_2^{-1}, & a_3 b_2^{-1} a_4^{-1} b_4^{-1}, & a_3 b_1^{-1} a_4 b_1^{-1}, & a_4 b_1 a_4 b_2^{-1} \end{array} \right\},$$

is a finitely presented torsion-free simple group isomorphic to an amalgam of the form $F_7 *_{F_{49}} F_7$.

Proof. (1) Let γ be an element in S, let $\psi : \Gamma \to Q$ be a homomorphism onto a finite group Q and let k be the order of $\psi(b_3)$ in Q. Then $b_3^k \in \ker(\psi)$ and ψ can be written as a composition

$$\Gamma \xrightarrow{\psi_1} \Gamma/\langle\langle b_3^{2k}\rangle\rangle_\Gamma \xrightarrow{\psi_2} \Gamma/\langle\langle b_3^{k}\rangle\rangle_\Gamma \xrightarrow{\psi_3} Q\,.$$

Hence

$$\psi(\gamma) = \psi_3\psi_2(\gamma\langle\langle b_3^{2k}\rangle\rangle_\Gamma) \in \psi_3\psi_2(\langle b_3\rangle\langle\langle b_3^{2k}\rangle\rangle_\Gamma) = \psi_3(\langle b_3\rangle\langle\langle b_3^{k}\rangle\rangle_\Gamma) = \psi(\langle b_3\rangle)$$

and Γ is not $\langle b_3\rangle$-separable.

(2) It follows from Lemma 2.38, using part (1) of this proposition and Lemma 2.71.

(3) Using part (2) of this proposition, the claim follows as in Section 2.5, because the (4, 6)–complex corresponding to Γ embeds into the (8, 8)–complex corresponding to $\hat{\Gamma}$, and $\langle\langle [a_2a_1^{-1}, b_3]\rangle\rangle_{\hat{\Gamma}}$ has index 4 in $\hat{\Gamma}$.

□

Lemma 2.73. *Let Γ be the group of Example 2.36. Then $[\Gamma,\Gamma] = \langle\langle a_1a_2^{-1}\rangle\rangle_\Gamma$ and $\Gamma/[\Gamma,\Gamma] \cong \langle a_1, b_1 \mid a_1b_1 = b_1a_1\rangle \cong \mathbb{Z}^2$.*

Proof. The inclusion $[\Gamma,\Gamma] > \langle\langle a_1a_2^{-1}\rangle\rangle_\Gamma$ follows from $a_1a_2^{-1} = [a_1, b_3^{-1}] \in [\Gamma,\Gamma]$. Let $N \lhd \Gamma$ be any normal subgroup containing $a_1a_2^{-1}$, for example $N = \langle\langle a_1a_2^{-1}\rangle\rangle_\Gamma$. Then $a_1N = a_2N$, hence

$$a_2b_1N = a_1b_1N = b_2a_2N = b_2a_1N = a_2b_3N\,,$$

and

$$b_2a_2N = a_1b_1N = a_2b_1N = b_3a_1N = b_3a_2N\,,$$

which implies $b_1N = b_2N = b_3N$. Moreover, $b_1a_1N = a_1b_2N = a_1b_1N$, in particular, the group Γ/N is generated by $\{a_1N, b_1N\}$ and abelian, therefore $[\Gamma,\Gamma]$ is a subgroup of N. □

Lemma 2.74. *Let Γ be the (4, 6)–group defined in Example 2.36. Then*

$$\langle\langle [a_2a_1^{-1}, b_3]\rangle\rangle_\Gamma = \langle\langle a_1a_2^{-1}\rangle\rangle_\Gamma\,.$$

Proof. We have checked the statement using MAGNUS ([50]). The inclusion

$$\langle\langle [a_2a_1^{-1}, b_3]\rangle\rangle_\Gamma < \langle\langle a_1a_2^{-1}\rangle\rangle_\Gamma$$

is obvious, since $[a_2a_1^{-1}, b_3] \in [\Gamma,\Gamma] = \langle\langle a_1a_2^{-1}\rangle\rangle_\Gamma$ by Lemma 2.73. □

Conjecture 2.75. *Let* Γ *be the group of Example 2.36. Then for each* $k \in \mathbb{N}$

$$a_1 a_2^{-1} \in \langle b_3 \rangle \langle\langle b_3^{2k} \rangle\rangle_\Gamma \, ,$$

in particular Proposition 2.72 can be applied.

Conjecture 2.76. *Let* Γ *be the group of Example 2.36. Then*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = [\Gamma, \Gamma] \, .$$

Remarks. Let Γ be the group of Example 2.36. Then

(1) $\langle\langle b_3^i \rangle\rangle_\Gamma \neq \langle\langle b_3^j \rangle\rangle_\Gamma$, if $i \neq j$ and $i, j \in \mathbb{N}$, since $(\Gamma / \langle\langle b_3^i \rangle\rangle_\Gamma)^{ab} \cong \mathbb{Z} \times \mathbb{Z}_i$.

(2) It follows from Lemma 2.73 that $a_1 a_2^{-1} \in \langle\langle b_3^{2k} \rangle\rangle_\Gamma$ if and only if $\Gamma / \langle\langle b_3^{2k} \rangle\rangle_\Gamma$ is abelian. Using MAGNUS ([50]), we see that $\Gamma / \langle\langle b_3^{8} \rangle\rangle_\Gamma$ is not abelian, in other words $a_1 a_2^{-1} \notin \langle\langle b_3^{8} \rangle\rangle_\Gamma$.

(3) If $k \leq 10$, then the number of subgroups of index k is the same for the group Γ and the group $\mathbb{Z}^2$.

A 4-vertex construction

A $(2m, 2n)$–group Γ is never simple, since Γ_0 is a normal subgroup of index 4. However, we have conjectured Γ_0 to be simple in Example 2.2, 2.18, 2.21, 2.30, A.26 and 2.33, and proved it to be simple in Example 2.43 and in many more examples listed in Table 2.7. The corresponding square complex X_0 has 4 vertices and $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$ as universal covering space. In this section, we *directly* construct a 4-vertex square complex Y, which is not a 4-fold covering of a $(2m, 2n)$–complex. Its universal covering space $\tilde{Y}$ is $\mathcal{T}_3 \times \mathcal{T}_4$. Observe that due to this more general construction, the valencies of the regular trees in $\tilde{Y}$ are not necessarily even. As a consequence, the number of geometric squares in Y is only 12 (this is small, compared to the 36 geometric squares of X_0 in Example 2.2 or the 100 geometric squares of X_0 in Example 2.43) and we get therefore relatively short presentations of $\pi_1 Y$. The construction of Y is done in such a way that Y is irreducible, all the "local groups" are at least 2-transitive and $\pi_1 Y$ is perfect. This seems to give some reasons to hope that $\pi_1 Y$ is a simple group.

Note that we have introduced the local groups and the notion of link in Section 1.2 only for $(2m, 2n)$–complexes, but they can also be defined similarly, now depending on the vertices, for more general square complexes, see [17, Chapter 1]. In the following, we denote these local groups by $P_h^{(k)}(\alpha)$, $P_v^{(k)}(\alpha)$, $P_h^{(k)}(\beta)$, $P_v^{(k)}(\beta)$, $P_h^{(k)}(\gamma)$, $P_v^{(k)}(\gamma)$, $P_h^{(k)}(\delta)$, $P_v^{(k)}(\delta)$, and the links by $Lk(\alpha)$, $Lk(\beta)$, $Lk(\gamma)$, $Lk(\delta)$, where α, β, γ, δ are the four vertices of Y and $k \in \mathbb{N}$.

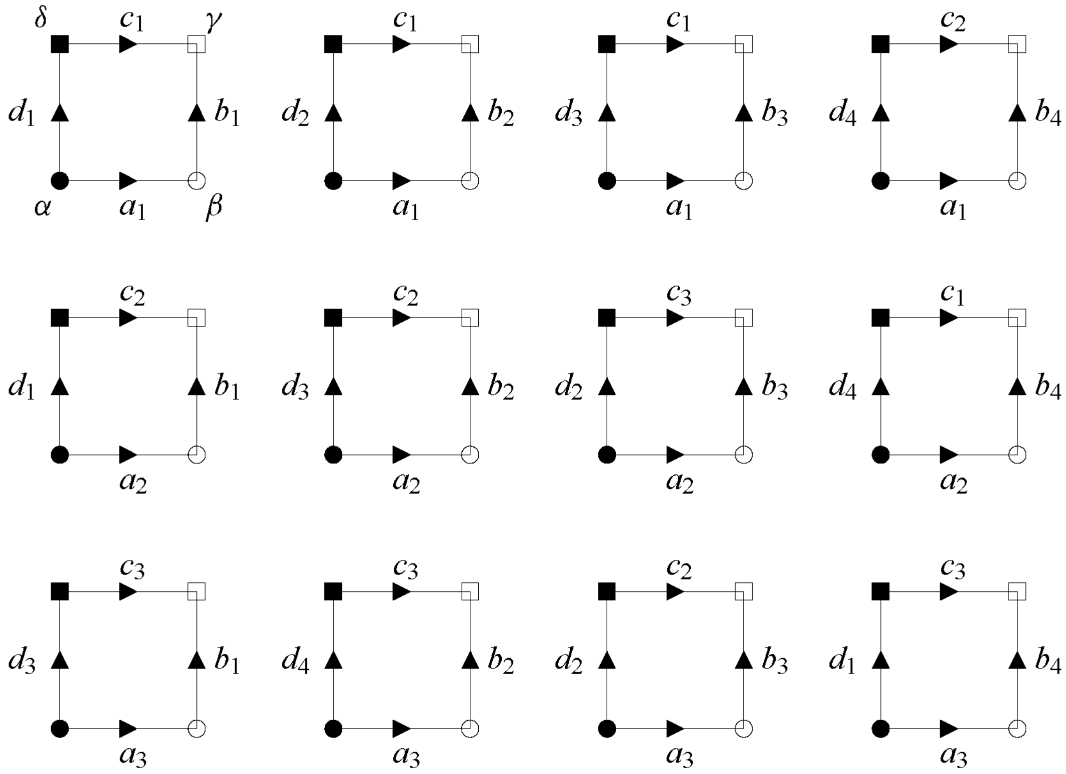


Figure 2.6: The 4-vertex square complex Y of Example 2.77

Example 2.77. *Let Y be the 4-vertex square complex illustrated in Figure 2.6.*

Proposition 2.78. *Let Y be the 2-dimensional cell complex of Figure 2.6 with four vertices α, β, γ and δ. Then*

(1) *The links are $Lk(\alpha) \cong Lk(\beta) \cong Lk(\gamma) \cong Lk(\delta) \cong K_{3,4}$ (complete bipartite graph), the universal covering space of Y is $\tilde{Y} = \mathcal{T}_3 \times \mathcal{T}_4$.*

(2) *We have local groups*

$$P_h(\alpha) \cong P_h(\delta) \cong S_3\,, \quad P_h(\beta) \cong P_h(\gamma) \cong S_3$$
$$P_v(\alpha) \cong P_v(\beta) \cong S_4\,, \quad P_v(\gamma) \cong P_v(\delta) \cong S_4\,.$$

(3) *The complex Y is irreducible.*

(4) *The fundamental group $\pi_1 Y$ is a perfect group.*

(5) *There are amalgam decompositions $F_3 *_{F_7} F_3 \cong \pi_1 Y \cong F_2 *_{F_5} F_2$.*

Proof. (1) It can be directly read off from Figure 2.6.

(2) This follows from the definitions (see [17, Chapter 1]) and Figure 2.6. Note that for example $P_h(\alpha)$ and $P_h(\beta)$ could a priori be different, since α and β are not in the same connected component of the vertical 1-skeleton of Y. For an example where indeed $P_h(\alpha) \not\cong P_h(\beta)$, see Example A.29.

(3) We compute

$$\left|P_v^{(2)}(\alpha)\right| = \left|P_v^{(2)}(\beta)\right| = \left|P_v^{(2)}(\gamma)\right| = \left|P_v^{(2)}(\delta)\right| = 24 \cdot 6^4 .$$

The claim follows now from an obvious generalization of [17, Proposition 1.3] to the case where the horizontal 1-skeleton is not connected.

(4) This follows directly from any of the explicit presentations of $\pi_1 Y$ given in the proof of part (5).

(5) We give three presentations of $\pi_1 Y$ and the corresponding isomorphisms between them. If we choose the vertex α as base point and the edges a_1, b_1, d_1 as "spanning tree" in the 1-skeleton of Y, we immediately get the following finite presentation of $\pi_1(Y, \alpha)$:

$$\begin{aligned}\pi_1(Y,\alpha) \cong \langle a_2,\ a_3, b_2,\ b_3,\ b_4,\ c_2,\ c_3,\ d_2,\ d_3,\ d_4 \mid \qquad& \\ b_2 = d_2,\ b_3 = d_3,\ b_4 = d_4c_2,& \\ a_2 = c_2,\ a_2b_2 = d_3c_2,\ a_2b_3 = d_2c_3,\ a_2b_4 = d_4,& \\ a_3 = d_3c_3,\ a_3b_2 = d_4c_3,\ a_3b_3 = d_2c_2,\ a_3b_4 = c_3\rangle\,,&\end{aligned}$$

and after replacing c_2, d_2, d_3 by a_2, b_2 and b_3, respectively, we get

$$\begin{aligned}\pi_1(Y,\alpha) \cong \langle a_2,\ a_3, b_2,\ b_3,\ b_4,\ c_3,\ d_4 \mid \qquad& \\ b_4 = d_4a_2,\ a_2b_2 = b_3a_2,\ a_2b_3 = b_2c_3,\ a_2b_4 = d_4,& \\ a_3 = b_3c_3,\ a_3b_2 = d_4c_3,\ a_3b_3 = b_2a_2,\ a_3b_4 = c_3\rangle\,.&\end{aligned}$$

Using the GAP-commands ([29])

`GG := SimplifiedFpGroup(G);` and `RelatorsOfFpGroup(GG);`

where `G` describes the group $\pi_1(Y, \alpha)$ as given above, and writing a_2, b_3 as x and y, respectively, we get a presentation of $\pi_1 Y$ with two generators x, y and three relators

$$\begin{aligned}&xy^2x^{-2}y^{-1}xyx^{-1}y^{-1}x\,,\\ &xyx^{-2}y^{-2}x^2yxy^{-1}x^{-2}y^2x^2y^{-1}\,,\\ &x^{-1}yxy^{-1}x^{-2}yx^2y^{-1}x^{-2}y^2xy^{-1}x^2y\,.\end{aligned}$$

The two decompositions of $\pi_1 Y$ as amalgamated free products of free groups follow from [68, Theorem I.1.18].

$$\begin{aligned} F_3 *_{F_7} F_3 = \langle b_2,\, b_3,\, b_4,\, d_2,\, d_3,\, d_4 \mid d_2 = b_2,\ d_3 = b_3,\ d_4^2 = b_4^2,\\ d_4 d_3 d_4^{-1} = b_4 b_2 b_4^{-1},\\ d_4 d_2^2 d_4^{-1} = b_4 b_3 b_4^{-1} b_3 b_4^{-1},\\ d_4 d_2^{-1} d_4 d_2 d_4^{-1} = b_4 b_3^{-1} b_2 b_4^{-1} b_3 b_4^{-1},\\ d_4 d_2^{-1} d_3 d_2 d_4^{-1} = b_4 b_3^{-1} b_4^{-1} b_3 b_4^{-1} \rangle\,. \end{aligned}$$

$$\begin{aligned} F_2 *_{F_5} F_2 = \langle a_2,\, a_3,\, c_2,\, c_3 \mid a_2 = c_2,\ a_3^4 = c_3 c_2^{-1} c_3 c_2 c_3,\\ a_3^{-1} a_2 a_3^{-2} = c_3^{-1} c_2 c_3^{-1} c_2 c_3^{-1},\\ a_3 a_2 a_3^{-1} = c_3 c_2^{-1} c_3^{-1},\ a_3^2 a_2 a_3 = c_3 c_2^{-1} c_3^3 \rangle\,. \end{aligned}$$

Isomorphisms between these three groups are given as follows:

$$\begin{array}{rcccl} \mathcal{T}_4 \curvearrowleft F_2 *_{F_5} F_2 & \stackrel{\cong}{\longleftrightarrow} & \pi_1(Y,\alpha) & \stackrel{\cong}{\longleftrightarrow} & F_3 *_{F_7} F_3 \curvearrowright \mathcal{T}_3 \\ \\ a_2 & \longleftrightarrow & a_2 & \longleftrightarrow & d_4 b_4^{-1} \\ a_3 & \longleftrightarrow & a_3 & \longleftrightarrow & d_2 d_4^{-1} b_4 b_3^{-1} \\ a_2^{-1} a_3 c_3^{-1} c_2 & \longleftrightarrow & b_2 & \longleftrightarrow & b_2 \\ a_3 c_3^{-1} & \longleftrightarrow & b_3 & \longleftrightarrow & b_3 \\ a_3^{-1} c_3 & \longleftrightarrow & b_4 & \longleftrightarrow & b_4 \\ c_2 & \longleftrightarrow & c_2 & \longleftrightarrow & d_4 b_4^{-1} \\ c_3 & \longleftrightarrow & c_3 & \longleftrightarrow & d_2^{-1} d_4^{-1} b_4 b_3 \\ a_2^{-1} a_3 c_3^{-1} c_2 & \longleftrightarrow & d_2 & \longleftrightarrow & d_2 \\ a_3 c_3^{-1} & \longleftrightarrow & d_3 & \longleftrightarrow & d_3 \\ a_2 a_3^{-1} c_3 & \longleftrightarrow & d_4 & \longleftrightarrow & d_4\,. \end{array}$$

□

Question 2.79. *Let Y be as in Example 2.77.*

(1) *Is it true that $\pi_1 Y$ does not have proper subgroups of finite index?*

(2) *Is $\pi_1 Y$ a non-residually finite group?*

(3) *Does every non-trivial normal subgroup of $\pi_1 Y$ have finite index?*

(4) *Is $\pi_1 Y$ a simple group?*

Remark. We have checked with GAP ([29]) that $\langle\!\langle w^k \rangle\!\rangle_{\pi_1 Y} = \pi_1 Y$, where w is any generator of $\pi_1(Y,\alpha)$ in the first presentation given in the proof of Proposition 2.78(5), and $k = 1, \ldots, 8$.

Chapter 3

Quaternion lattices in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$

In Section 3.1, we provide some concepts which will be used throughout this chapter, in particular we study Hamilton quaternion algebras over commutative rings. To any pair of distinct prime numbers $p, l \equiv 1 \pmod 4$, Mozes has associated a $(p+1, l+1)$–group $\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$. There is a strong interplay between properties of quaternions and the group $\Gamma_{p,l}$, for example $\Gamma_{p,l}$ turns out to be commutative transitive. We recall the definition of $\Gamma_{p,l}$ in Section 3.2 and prove that it is a normal subgroup of index 4 of the group of invertible elements of the Hamilton quaternion algebra over the ring $\mathbb{Z}[1/p, 1/l]$, modulo its center, adapting some ideas from Lubotzky's book [45]. These ideas are also useful to realize $\Gamma_{p,l}$ as a subgroup of $\mathrm{SO}_3(\mathbb{Q})$ or $\mathrm{PGL}_2(\mathbb{C})$, and to construct homomorphisms onto finite groups $\mathrm{PGL}_2(\mathbb{Z}_q)$ or $\mathrm{PSL}_2(\mathbb{Z}_q)$ for each odd prime number q different from p and l. These and other results are illustrated by concrete examples. In Section 3.3 and 3.4, we generalize and adapt the construction of $\Gamma_{p,l}$ to the other cases of prime numbers $p, l \equiv 3 \pmod 4$ and $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$, prove that these groups are also $(p+1, l+1)$–groups, and again give many examples. In total, we have made computations in 130 examples. They lead to some conjectures in Section 3.5, in particular about the abelianization of $\Gamma_{p,l}$, generalizing a conjecture of Kimberley-Robertson given for the classical case. It also seems that the abelianization of the subgroup $(\Gamma_{p,l})_0$ is independent of p and l, except if $p = 3$ or $l = 3$. The notion of an anti-torus was introduced by Wise, and only very few examples are known. We give in Section 3.6 an easy criterion for the existence of anti-tori in commutative transitive $(2m, 2n)$–groups and combine it with earlier results on centralizers. In particular, these results can be applied to the groups $\Gamma_{p,l}$, and can therefore also be expressed in terms of integer quaternions. It turns out that the groups $\Gamma_{p,l}$ have many anti-tori. Then we study relations between free anti-tori in $\Gamma_{p,l}$, free subgroups of $\mathrm{SO}_3(\mathbb{Q})$ and quaternions generating a free group. As an application, we prove for example that the two quaternions $1 + 2i$ and $1 + 4k$ do not generate a free group, which is quite surprising. In Section 3.7, we give a different construction for $p = 2$, $l = 5$, also based on quaternion multiplication.

3.1 Some notations and preliminaries

At first, we define quaternions over a commutative ring, following [23, Section 2.5]: Let R be a commutative ring with unit. Then the *Hamilton quaternion algebra* over R, denoted by $\mathbb{H}(R)$, is the associative unital algebra defined as follows:

- $\mathbb{H}(R) = \{x_0 + x_1 i + x_2 j + x_3 k : x_0, x_1, x_2, x_3 \in R\}$ is the free R-module with basis $1, i, j, k$.
- $1 = 1 + 0i + 0j + 0k$ is the multiplicative unit.
- $i^2 = j^2 = k^2 = -1$.
- $ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$.

This gives the multiplication rule in $\mathbb{H}(R)$

$$\begin{aligned}(x_0 + x_1 i + x_2 j + x_3 k)&(y_0 + y_1 i + y_2 j + y_3 k)\\ = \ & x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3\\ &+ (x_0 y_1 + x_1 y_0 + x_2 y_3 - x_3 y_2)\, i\\ &+ (x_0 y_2 - x_1 y_3 + x_2 y_0 + x_3 y_1)\, j\\ &+ (x_0 y_3 + x_1 y_2 - x_2 y_1 + x_3 y_0)\, k\,.\end{aligned}$$

For a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(R)$, let $\bar{x} := x_0 - x_1 i - x_2 j - x_3 k$ be its *conjugate*, $|x|^2 := x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2 \in R$ its *norm*, and $\Re(x) := x_0$ its "R-part". Note that $|xy|^2 = |x|^2|y|^2$.

We divide quaternions $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$ with odd norm $|x|^2$ into eight classes (and say that these quaternions have type o_0, o_1, o_2, o_3, e_0, e_1, e_2 or e_3) according to Table 3.1.

This terminology of types is not standard, but useful to simplify some definitions and statements. Moreover, we say that x has type o if it has type o_0, o_1, o_2 or o_3. Note that x has type o if and only if $|x|^2 \equiv 1 \pmod 4$. Finally, we say that x has type e if it has type e_0, e_1, e_2 or e_3, which happens if and only if $|x|^2 \equiv 3 \pmod 4$.

If R is a ring with unit (denoted by 1), let $U(R)$ be the group of (left and right) invertible elements in R, i.e. elements $x \in R$ such that there are $y_1, y_2 \in R$ satisfying $y_1 x = x y_2 = 1$. Observe that then $y_1 = y_2$. This element is uniquely determined by $x \in U(R)$ and is usually written as x^{-1}.

The following elementary lemmas characterize invertible and central elements in the Hamilton quaternion algebra $\mathbb{H}(R)$.

x	x_0	x_1	x_2	x_3
type o_0	odd	even	even	even
o_1	even	odd	even	even
o_2	even	even	odd	even
o_3	even	even	even	odd
e_0	even	odd	odd	odd
e_1	odd	even	odd	odd
e_2	odd	odd	even	odd
e_3	odd	odd	odd	even

Table 3.1: Types of integer quaternions x with odd norm $|x|^2$.

Lemma 3.1. *Let R be a commutative ring with unit. Then*

$$U(\mathbb{H}(R)) = \{x \in \mathbb{H}(R) : |x|^2 \in U(R)\}\,.$$

Proof. "$\supseteq$" Take $x^{-1} = (|x|^2)^{-1}\bar{x}$.
"$\subseteq$" Let $x \in U(\mathbb{H}(R))$ and $y := x^{-1}$, then $1 = |xy|^2 = |x|^2|y|^2 = |y|^2|x|^2$, and it follows $|x|^2 \in U(R)$. □

Lemma 3.2. *Let R be a commutative ring with unit and let $x = x_0 + x_1 i + x_2 j + x_3 k$, $y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}(R)$. Then $xy = yx$ if and only if the following three equations hold:*

$$\begin{aligned} 2(x_2 y_3 - x_3 y_2) &= 0 \\ 2(x_3 y_1 - x_1 y_3) &= 0 \\ 2(x_1 y_2 - x_2 y_1) &= 0\,. \end{aligned}$$

Proof. This is an elementary computation. We only use the multiplication rule for quaternions in $\mathbb{H}(R)$. □

Lemma 3.3. *Let R be a commutative ring with unit.*

(1) *The central elements in $\mathbb{H}(R)$ are*

$$\{x \in \mathbb{H}(R) : xy = yx,\ \forall y \in \mathbb{H}(R)\} = \{x \in \mathbb{H}(R) : x = \bar{x}\}\,.$$

(2) $ZU(\mathbb{H}(R)) = \{x \in U(\mathbb{H}(R)) : x = \overline{x}\}$.

Proof. (1) Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(R)$. The condition $x = \overline{x}$ is equivalent to the condition

$$2x_1 = 2x_2 = 2x_3 = 0\,,$$

thus if $x = \overline{x}$, then $xy = yx$ for each $y \in \mathbb{H}(R)$ by Lemma 3.2. Conversely, suppose that $xy = yx$ for each $y \in \mathbb{H}(R)$. Taking $y = i$ gives $xi = ix$, which is

$$-x_1 + x_0 i + x_3 j - x_2 k = -x_1 + x_0 i - x_3 j + x_2 k\,,$$

hence $2x_2 = 0$, $2x_3 = 0$. Moreover, taking $y = j$, we conclude in a similar way $2x_1 = 2x_3 = 0$ and get $x = \overline{x}$.

(2) We can use the same proof as in part (1), since $i(-i) = j(-j) = 1$, which shows that $i, j \in U(\mathbb{H}(R))$.

□

Remark. If R is a subring of $\mathbb{C}$ with unit, then

$$\{x \in \mathbb{H}(R) : xy = yx,\ \forall y \in \mathbb{H}(R)\} = \{x \in \mathbb{H}(R) : x = \Re(x)\}$$

and

$$ZU(\mathbb{H}(R)) = \{x \in U(\mathbb{H}(R)) : x = \Re(x)\} = U(\mathbb{H}(R)) \cap ZU(\mathbb{H}(\mathbb{C}))\,.$$

However, for example the case $R = \mathbb{Z}_2$ is different, since $\mathbb{H}(\mathbb{Z}_2)$ is commutative and

$$ZU(\mathbb{H}(\mathbb{Z}_2)) = U(\mathbb{H}(\mathbb{Z}_2)) \neq \{x \in U(\mathbb{H}(\mathbb{Z}_2)) : x = \Re(x)\} = \{1\}\,.$$

The following lemma, especially part (3), will be very useful in Section 3.2.

Lemma 3.4. *Let R be a commutative ring with unit and let $x = x_0 + x_1 i + x_2 j + x_3 k$, $y = y_0 + y_1 i + y_2 j + y_3 k$ and $z = z_0 + z_1 i + z_2 j + z_3 k$ be three quaternions in $\mathbb{H}(R)$. Then*

(1) *$xy = -yx$ if and only if the following four equations hold:*

$$\begin{aligned} 2(x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) &= 0 \\ 2(x_0y_1 + x_1y_0) &= 0 \\ 2(x_0y_2 + x_2y_0) &= 0 \\ 2(x_0y_3 + x_3y_0) &= 0\,. \end{aligned}$$

(2) *Suppose that R is a subring of $\mathbb{R}$ with unit, $x_0 \neq 0$ and $xy = -yx$. Then $y = 0$.*

(3) *Let R be a subring of $\mathbb{C}$ with unit, $x \neq x_0$, $xy = yx$ and $xz = zx$. Then $yz = zy$, in particular $U(\mathbb{H}(\mathbb{C}))$ is commutative transitive on non-central elements.*

Proof. (1) This is an elementary computations using the multiplication rule for quaternions in $\mathbb{H}(R)$.

(2) Using part (1), we have $x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 = 0$ and

$$y_1 = \frac{-x_1y_0}{x_0}\,, \quad y_2 = \frac{-x_2y_0}{x_0}\,, \quad y_3 = \frac{-x_3y_0}{x_0}\,.$$

It follows

$$x_0y_0 + \frac{x_1^2y_0}{x_0} + \frac{x_2^2y_0}{x_0} + \frac{x_3^2y_0}{x_0} = 0\,,$$

and therefore $y_0|x|^2 = 0$. Since $|x|^2 \geq x_0^2 > 0$, we conclude $y_0 = 0$ which implies $y_1 = 0$, $y_2 = 0$ and $y_3 = 0$, in other words $y = 0$.

(3) By Lemma 3.2, we have to prove $y_2z_3 = y_3z_2$, $y_3z_1 = y_1z_3$ and $y_1z_2 = y_2z_1$. We only prove here $y_1z_2 = y_2z_1$, the other two computations are completely analogous: If $x_2 = 0$, then using the assumption $xy = yx$ and Lemma 3.2, we have $x_1y_2 = x_2y_1 = 0$ and $x_3y_2 = x_2y_3 = 0$. This implies $y_2 = 0$ (otherwise $x_1 = x_3 = 0$ and $x = x_0$). Moreover, using $xz = zx$, we have $x_1z_2 = x_2z_1 = 0$ and $x_3z_2 = x_2z_3 = 0$, which implies $z_2 = 0$. So, we conclude that $y_1z_2 = 0 = y_2z_1$. Assume now that $x_2 \neq 0$, then $y_1z_2 = \frac{x_1}{x_2}y_2z_2 = y_2z_1$, using $x_2y_1 = x_1y_2$ and $x_2z_1 = x_1z_2$.

□

Remark. The statement of Lemma 3.4(2) is not true in $\mathbb{H}(\mathbb{C})$. Take for example $x = 1 + i_{\mathbb{C}}i$, $y = i_{\mathbb{C}} + i$, where $i_{\mathbb{C}}$ denotes the imaginary unit in $\mathbb{C}$, and check that $xy = -yx = 0$.

Throughout this chapter, let p, l be two distinct odd prime numbers. Then the ring

$$\mathbb{Z}[1/p, 1/l] := \{0\} \cup \{tp^rl^s : r, s, t \in \mathbb{Z};\ t \neq 0;\ t \text{ is relatively prime to } p \text{ and } l\}$$

is a subring of $\mathbb{Q}$, containing $\mathbb{Z}$. Note that with this definition, any non-zero element in $\mathbb{Z}[1/p, 1/l]$ uniquely determines a triple (t, r, s) having the properties required in the definition, and vice versa. Of course $\mathbb{Z}[1/p, 1/l]$ could also be defined as

$$\{\frac{t}{p^rl^s} : t \in \mathbb{Z}; r, s \in \mathbb{N}_0\}\,.$$

Let $\left(\frac{p}{l}\right)$ be the *Legendre symbol*. This means that $\left(\frac{p}{l}\right) := 1$, if p is a quadratic residue modulo l, i.e. if the equation $x^2 \equiv p \pmod{l}$ has an integer solution, and $\left(\frac{p}{l}\right) := -1$, otherwise. See Table 3.2 for some small examples, where "+" and "−" stand for 1 and −1, respectively. The definition of the Legendre symbol can be generalized to non-prime numbers, but we do not need it here.

Let K be a field, $K^\times = K \setminus \{0\} = U(K)$ the group of invertible elements and $\mathrm{GL}_2(K)$ the group of invertible (2×2)–matrices with coefficients in K. We denote by $\mathrm{PGL}_2(K)$ the quotient group

$$\mathrm{PGL}_2(K) = \mathrm{GL}_2(K)/\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in K^\times\right\} = \mathrm{GL}_2(K)/Z\mathrm{GL}_2(K)\,.$$

$\left(\frac{p}{l}\right)$	$l = 3$	5	7	11	13	17	19	23	29	31	37	41	43	47
$p =$														
3		$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
5	$-$		$-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$	$-$
7	$+$	$-$		$-$	$-$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$+$
11	$-$	$+$	$+$		$-$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$+$	$-$
13	$+$	$-$	$-$	$-$		$+$	$-$	$+$	$+$	$-$	$-$	$-$	$+$	$-$
17	$-$	$-$	$-$	$-$	$+$		$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
19	$+$	$+$	$-$	$-$	$-$	$+$		$-$	$-$	$+$	$-$	$-$	$-$	$-$
23	$-$	$-$	$+$	$+$	$+$	$-$	$+$		$+$	$-$	$-$	$+$	$+$	$-$
29	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$		$-$	$-$	$-$	$-$	$-$
31	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$-$		$-$	$+$	$+$	$-$
37	$+$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$		$+$	$-$	$+$
41	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$		$+$	$-$
43	$+$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$		$-$
47	$-$	$-$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$-$	$+$	

Table 3.2: Legendre symbol $\left(\frac{p}{l}\right)$ for small distinct odd prime numbers p, l

If A is a matrix in $\mathrm{GL}_2(K)$, we write

$$[A] := A\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in K^\times\right\} \in \mathrm{PGL}_2(K)$$

for the image of A under the quotient map $\mathrm{GL}_2(K) \to \mathrm{PGL}_2(K)$. We denote by $\mathrm{SL}_2(K)$ the kernel of the determinant map $\det : \mathrm{GL}_2(K) \to K^\times$ and by $\mathrm{PSL}_2(K)$ the quotient group

$$\mathrm{PSL}_2(K) = \mathrm{SL}_2(K)/\left\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} : \epsilon^2 = 1\right\} = \mathrm{SL}_2(K)/Z\mathrm{SL}_2(K)\,.$$

The group $\mathrm{PSL}_2(K)$ can be seen as a (normal) subgroup of $\mathrm{PGL}_2(K)$ via the injective homomorphism

$$\begin{aligned} \theta : \mathrm{PSL}_2(K) &\to \mathrm{PGL}_2(K) \\ A\left\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} : \epsilon^2 = 1\right\} &\mapsto [A]\,, \end{aligned}$$

where $A \in \mathrm{SL}_2(K) < \mathrm{GL}_2(K)$.

For q a prime number, we write $\mathrm{GL}_2(q)$, $\mathrm{PGL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$ instead of $\mathrm{GL}_2(\mathbb{Z}_q)$, $\mathrm{PGL}_2(\mathbb{Z}_q)$, $\mathrm{SL}_2(\mathbb{Z}_q)$, $\mathrm{PSL}_2(\mathbb{Z}_q)$. Recall that $\mathbb{Z}_q$ stands for the finite ring (field) $\mathbb{Z}/q\mathbb{Z}$ and *not* for the "q-adic integers".

Lemma 3.5. *Let K be a field and $B \in \mathrm{GL}_2(K)$. Then $[B] \in \theta(\mathrm{PSL}_2(K)) \cong \mathrm{PSL}_2(K)$ if and only if* $\det B \in (K^\times)^2 := \{\lambda^2 : \lambda \in K^\times\}$.

Proof. By definition, $[B] \in \theta(\mathrm{PSL}_2(K))$ if and only if there is a matrix $A \in \mathrm{SL}_2(K)$ such that $[A] = [B] \in \mathrm{PGL}_2(K)$, i.e. if and only if there is a matrix $A \in \mathrm{SL}_2(K)$ and an element $\lambda \in K^\times$ such that

$$B^{-1}A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

To prove the statement of the lemma, we first assume that $[B] \in \theta(\mathrm{PSL}_2(K))$. Then (with A and λ as above)

$$\det B = \det A \cdot \lambda^{-2} = \lambda^{-2} \in (K^\times)^2.$$

To show the other direction, assume that $\det B = \lambda^2$ for some $\lambda \in K^\times$. If we choose

$$A := B \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

then $A \in \mathrm{SL}_2(K)$, since $\det A = \lambda^2 \cdot \lambda^{-2} = 1$, and we have

$$B^{-1}A = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

□

From now on, we will see $\mathrm{PSL}_2(K)$ as a subgroup of $\mathrm{PGL}_2(K)$ without mention of the homomorphism θ.

Lemma 3.6. *Let p, l be two distinct odd prime numbers. Then $p + l\mathbb{Z} \in (\mathbb{Z}_l^\times)^2$ if and only if $\left(\frac{p}{l}\right) = 1$.*

Proof. We have the following equivalences:

$$\begin{aligned} p + l\mathbb{Z} \in (\mathbb{Z}_l^\times)^2 &\iff \exists\, x + l\mathbb{Z} \in \mathbb{Z}_l^\times \text{ such that } (x + l\mathbb{Z})^2 = p + l\mathbb{Z} \\ &\iff \exists\, x \in \{1, \dots, l-1\} \text{ such that } x^2 + l\mathbb{Z} = p + l\mathbb{Z} \\ &\iff \exists\, x \in \{1, \dots, l-1\} \text{ such that } x^2 \equiv p \pmod{l} \\ &\iff \exists\, x \in \mathbb{Z} \text{ such that } x^2 \equiv p \pmod{l} \\ &\iff \left(\frac{p}{l}\right) = 1. \end{aligned}$$

□

The next lemma gives a selection of results about the decomposability of prime numbers as certain sums of squares of integers. They are all well-known in number theory.

Lemma 3.7. *Let p be an odd prime number.*

(1) *(Fermat, Euler) p is a sum of two squares if and only if $p \equiv 1 \pmod 4$.*

(2) *(Gauss) Assume that $p \equiv 3 \pmod 4$. Then p is a sum of three squares if and only if $p \equiv 3 \pmod 8$. More generally, an odd natural number s is a sum of three squares if and only if $s \not\equiv 7 \pmod 8$.*

(3) *(Jacobi) There are exactly $8(p+1)$ representations of p as a sum of four squares $p = x_0^2 + x_1^2 + x_2^2 + x_3^2$; $x_0, x_1, x_2, x_3 \in \mathbb{Z}$. For each such representation, three integers in $\{x_0, x_1, x_2, x_3\}$ are even, if $p \equiv 1 \pmod 4$, and three integers are odd, if $p \equiv 3 \pmod 4$. It follows that for $p \equiv 1 \pmod 4$*

$$|\{x \in \mathbb{H}(\mathbb{Z}) : |x|^2 = p\}| = 8(p+1)\,,$$
$$|\{x \in \mathbb{H}(\mathbb{Z}) : |x|^2 = p,\ x \text{ has type } o_0\}| = 2(p+1)\,,$$
$$|\{x \in \mathbb{H}(\mathbb{Z}) : |x|^2 = p,\ x \text{ has type } o_0,\ \Re(x) > 0\}| = p+1\,.$$

Let p be an odd prime number. The following lemma applies for example to the finite field $\mathbb{Z}_p$, the field of p-adic numbers $\mathbb{Q}_p$ and algebraically closed fields of characteristic different from 2 like $\mathbb{C}$, but not to $\mathbb{Z}_2$ or subfields of $\mathbb{R}$.

Lemma 3.8. *(see [23, Proposition 2.5.2]) Let K be a field of characteristic different from 2, and assume that there exist $c, d \in K$ such that $c^2 + d^2 + 1 = 0$. Then $\mathbb{H}(K)$ is isomorphic to the algebra $M_2(K)$ of (2×2)–matrices over K. An isomorphism of algebras is given by the map*

$$\begin{aligned} \mathbb{H}(K) &\to M_2(K) \\ x_0 + x_1 i + x_2 j + x_3 k &\mapsto \begin{pmatrix} x_0 + x_1 c + x_3 d & -x_1 d + x_2 + x_3 c \\ -x_1 d - x_2 + x_3 c & x_0 - x_1 c - x_3 d \end{pmatrix}. \end{aligned}$$

In particular, if $c^2 + 1 = 0$ in K, i.e. if we can choose $d = 0$, then the isomorphism above is given by

$$\begin{aligned} \mathbb{H}(K) &\to M_2(K) \\ x_0 + x_1 i + x_2 j + x_3 k &\mapsto \begin{pmatrix} x_0 + x_1 c & x_2 + x_3 c \\ -x_2 + x_3 c & x_0 - x_1 c \end{pmatrix}. \end{aligned}$$

Proof. See [23, Proof of Proposition 2.5.2]. □

Note that the determinant of the image of x

$$\det \begin{pmatrix} x_0 + x_1 c + x_3 d & -x_1 d + x_2 + x_3 c \\ -x_1 d - x_2 + x_3 c & x_0 - x_1 c - x_3 d \end{pmatrix}$$

equals $x_0^2 - x_1^2(c^2 + d^2) + x_2^2 - x_3^2(c^2 + d^2) = |x|^2$, i.e. the norm of x.

3.2 Standard case $p, l \equiv 1 \pmod 4$

The following construction of the group $\Gamma_{p,l}$ is taken from [54], see also [53], [17] and [41]. Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers. We first define the map ψ (a monoid homomorphism, as we will see):

$$\psi : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$
$$x \mapsto \left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

where $x = x_0 + x_1 i + x_2 j + x_3 k$, and $i_p \in \mathbb{Q}_p$, $i_l \in \mathbb{Q}_l$ satisfy the conditions

$$i_p^2 + 1 = 0 \text{ and } i_l^2 + 1 = 0 .$$

The assumption $p, l \equiv 1 \pmod 4$ guarantees the existence of such elements i_p, i_l. Note that ψ is not injective, but (for $x, y \in \mathbb{H}(\mathbb{Z}) \setminus \{0\}$) we have $\psi(x) = \psi(y)$ if and only if $y = \lambda x$ for some $\lambda \in \mathbb{Q}^\times$. Moreover,

$$\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \begin{pmatrix} y_0 + y_1 i_p & y_2 + y_3 i_p \\ -y_2 + y_3 i_p & y_0 - y_1 i_p \end{pmatrix}$$
$$= \begin{pmatrix} z_0 + z_1 i_p & z_2 + z_3 i_p \\ -z_2 + z_3 i_p & z_0 - z_1 i_p \end{pmatrix},$$

where z_0, z_1, z_2, z_3 are determined by the quaternion multiplication

$$z_0 + z_1 i + z_2 j + z_3 k = (x_0 + x_1 i + x_2 j + x_3 k)(y_0 + y_1 i + y_2 j + y_3 k),$$

in particular $\psi(xy) = \psi(x)\psi(y)$ and

$$\begin{aligned} \ker(\psi) &:= \{x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\} : \psi(x) = 1_{\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)}\} \\ &= \{x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\} : x = \overline{x}\} \\ &= \mathbb{H}(\mathbb{Z}) \cap ZU(\mathbb{H}(\mathbb{Q})), \end{aligned}$$

where

$$1_{\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)} = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right).$$

This implies that $\psi(x)^{-1} = \psi(\bar{x})$ if $x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\}$, since

$$\psi(x)\psi(\bar{x}) = \psi(x\bar{x}) = \psi(|x|^2)$$

and $|x|^2 \in \ker(\psi)$. Finally, let

$$\begin{aligned} \Gamma_{p,l} &:= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\} \\ &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \Re(x) > 0,\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\} \end{aligned}$$

be our desired subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$.

Mozes has proved the following result:

Proposition 3.9. *(Mozes, [54, Section 3]) If $p, l \equiv 1 \pmod 4$ are two distinct prime numbers, then*

$$\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l) < \mathrm{Aut}(\mathcal{T}_{p+1}) \times \mathrm{Aut}(\mathcal{T}_{l+1})$$

is a $(p+1, l+1)$–group.

See for example [45, Section 5.3] or [64] for the description of the tree (the "Bruhat-Tits building") $\mathcal{T}_{p+1}$ corresponding to $\mathrm{PGL}_2(\mathbb{Q}_p)$ and its action on $\mathcal{T}_{p+1}$.

The fact that $\Gamma_{p,l}$ is a $(p+1, l+1)$–group is mainly based on a factorization property for integer quaternions, first proved by Dickson ([24]). However, it does not follow that $\Gamma_{p,l}$ is torsion-free; this is shown in [54, Proposition 3.6]. It is also known that the groups $\Gamma_{p,l}$ are irreducible (see Corollary 3.59(3)).

See [40] for an alternative proof that $\Gamma_{p,l}$ is a $(p+1, l+1)$–group.

Proposition 3.10. *(Dickson [24, Theorem 8]) Let $x \in \mathbb{H}(\mathbb{Z})$ be of odd norm and let $|x|^2 = p_1 \dots p_r$ be the prime decomposition of $|x|^2$, where the factors p_ι are arranged in an arbitrary but definite order. Then x can be decomposed as $x = x^{(1)} \dots x^{(r)}$ such that $x^{(\iota)} \in \mathbb{H}(\mathbb{Z})$ and $|x^{(\iota)}|^2 = p_\iota$, $\iota = 1, \dots, r$. This decomposition is uniquely determined up to multiplication of the factors $x^{(\iota)}$ with a unit $\pm 1, \pm i, \pm j, \pm k \in \mathbb{H}(\mathbb{Z})$ (if there is no prime number dividing x; this is somehow missing in Dickson's original statement, as noted and corrected by Kimberley [40]).*

Before applying Proposition 3.10, we define the two subsets of $\Gamma_{p,l}$

$$\begin{aligned} E_h &:= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p\} \\ &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \Re(x) > 0,\ |x|^2 = p\}, \\ E_v &:= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l\} \\ &= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \Re(y) > 0,\ |y|^2 = l\}. \end{aligned}$$

If $\psi(x) \in E_h$ then also $\psi(\bar{x}) = \psi(x)^{-1} \in E_h$. By Lemma 3.7(3), the set E_h has exactly $p+1$ elements. For these reasons, we write

$$E_h = \{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1}$$

and similarly

$$E_v = \{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1}.$$

As probably expected, all these definitions of E_h, E_v, $a_1, \dots, a_{\frac{p+1}{2}}$, $b_1, \dots, b_{\frac{l+1}{2}}$ will be compatible with the original ones for general $(2m, 2n)$–groups given in Section 1.2 (here, we have $2m = p+1$ and $2n = l+1$).

Corollary 3.11. *Let $p, l \equiv 1 \pmod 4$ be distinct odd prime numbers and recall that*

$$\Gamma_{p,l} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\}\,.$$

(1) *Let $x \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|x|^2 = pl$. Then there are integer quaternions $y, \tilde{y}, z, \tilde{z} \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|y|^2 = |\tilde{y}|^2 = p$, $|z|^2 = |\tilde{z}|^2 = l$ and $yz = x = \tilde{z}\tilde{y}$. The quaternions $y, \tilde{y}, z, \tilde{z}$ are uniquely determined by x up to sign.*

(2) *Let $a \in E_h$, $b \in E_v$. Then there are unique elements $\tilde{a} \in E_h$, $\tilde{b} \in E_v$ such that $ab = \tilde{b}\tilde{a}$ in $\Gamma_{p,l}$.*

(3) *The group $\Gamma_{p,l}$ is generated by $\{a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}\}$.*

(4) *Let $\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}$ be the set of quaternions*

$$\{x \in \mathbb{H}(\mathbb{Z}) : x \text{ has type } o_0,\ \Re(x) > 0,\ |x|^2 = p\}$$

and let $x \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|x|^2 = p^r$ for some $r \in \mathbb{N}_0$. Then there is a unique representation

$$x = \pm p^{r_1} w_{r_2}(\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1})\,,$$

where $r_1, r_2 \in \mathbb{N}_0$, $2r_1 + r_2 = r$ and

$$w_{r_2}(\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1})$$

denotes a reduced word of length r_2 in

$$\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}$$

(reduced means here that there are no subwords of the form $\alpha_i\overline{\alpha_i}$ or $\overline{\alpha_i}\alpha_i$).

(5) *There are two non-abelian free groups*

$$\langle a_1, \dots, a_{\frac{p+1}{2}}\rangle_{\Gamma_{p,l}} \cong F_{\frac{p+1}{2}} \text{ and } \langle b_1, \dots, b_{\frac{l+1}{2}}\rangle_{\Gamma_{p,l}} \cong F_{\frac{l+1}{2}}\,.$$

Proof. We define a map $u : \{x \in \mathbb{H}(\mathbb{Z}) : x \text{ has type } o\} \to \{1, i, j, k\}$ by

$$u(x) := \begin{cases} 1\,, & \text{if } x \text{ has type } o_0 \\ i\,, & \text{if } x \text{ has type } o_1 \\ j\,, & \text{if } x \text{ has type } o_2 \\ k\,, & \text{if } x \text{ has type } o_3\,. \end{cases}$$

Note that $u(1) = 1$, $u(i) = i$, $u(j) = j$, $u(k) = k$ and that $xu(x)$ always has type o_0.

(1) By Proposition 3.10 there are $\hat{y}, \hat{z} \in \mathbb{H}(\mathbb{Z})$ such that $|\hat{y}|^2 = p$, $|\hat{z}|^2 = l$ and $x = \hat{y}\hat{z}$. Since $p, l \equiv 1 \pmod 4$, the quaternions $\hat{y}$ and $\hat{z}$ have type o. They have both the same type since $x = \hat{y}\hat{z}$ has type o_0. If $\hat{y}$ and $\hat{z}$ have type o_0, we take $y := \hat{y}$, $z := \hat{z}$ and are done. If $\hat{y}$ and $\hat{z}$ have type o_1, o_2 or o_3, we take $y := -\hat{y}u(\hat{y})$, $z := u(\hat{z})\hat{z}$ and get $yz = -\hat{y}u(\hat{y})u(\hat{z})\hat{z} = -\hat{y}(-1)\hat{z} = x$. The uniqueness of y and z up to sign follows from the uniqueness statement in Proposition 3.10. Analogously, one proves $x = \tilde{z}\tilde{y}$.

(2) The given elements a and b uniquely determine $y, z \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $\Re(y) > 0$, $\Re(z) > 0$, $|y|^2 = p$, $|z|^2 = l$ and $\psi(y) = a$, $\psi(z) = b$. It follows that yz has type o_0 and $|yz|^2 = pl$. By part (1), there are $\tilde{y}, \tilde{z} \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|\tilde{y}|^2 = p$, $|\tilde{z}|^2 = l$ and $yz = \tilde{z}\tilde{y}$. Moreover, $\tilde{y}, \tilde{z}$ are uniquely determined up to sign. In particular, there are unique $\tilde{y}, \tilde{z} \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|\tilde{y}|^2 = p$, $|\tilde{z}|^2 = l$, $\Re(\tilde{y}) > 0$, $\Re(\tilde{z}) > 0$ and $\tilde{z}\tilde{y} \in \{yz, -yz\}$. Now take $\tilde{b} := \psi(\tilde{z}) \in E_v$ and $\tilde{a} := \psi(\tilde{y}) \in E_h$. The claim follows, since $ab = \psi(y)\psi(z) = \psi(yz) = \psi(-yz) = \psi(\tilde{z}\tilde{y}) = \psi(\tilde{z})\psi(\tilde{y}) = \tilde{b}\tilde{a}$.

(3) Fix any element $x \in \mathbb{H}(\mathbb{Z})$ of type o_0 such that $|x|^2 \in \{p^r l^s : r, s \in \mathbb{N}_0\}$ and $\Re(x) > 0$. We may assume that $r > 0$ or $s > 0$. By Proposition 3.10, there is a decomposition
$$x = y^{(1)} \dots y^{(r)} z^{(1)} \dots z^{(s)}$$
such that $y^{(1)}, \dots, y^{(r)} \in \mathbb{H}(\mathbb{Z})$ have norm p, and $z^{(1)}, \dots, z^{(s)} \in \mathbb{H}(\mathbb{Z})$ have norm l. Note that the quaternions $y^{(1)}, \dots, y^{(r)}, z^{(1)}, \dots, z^{(s)}$ all have type o, since $p, l \equiv 1 \pmod 4$. Our goal is to have a decomposition
$$x = \hat{y}^{(1)} \dots \hat{y}^{(r)} \hat{z}^{(1)} \dots \hat{z}^{(s)}$$
such that $\hat{y}^{(1)}, \dots, \hat{y}^{(r)}$ and $\hat{z}^{(1)}, \dots, \hat{z}^{(s)}$ have norm p and l, respectively, and are moreover of type o_0. To achieve this, we define the following algorithm:
$$\begin{array}{lll}
\tilde{y}^{(1)} & := y^{(1)} & \\
\tilde{y}^{(\iota)} & := u(\tilde{y}^{(\iota-1)})y^{(\iota)}\,, & \iota = 2, \dots, r \\
\hat{y}^{(\iota)} & := \tilde{y}^{(\iota)}u(\tilde{y}^{(\iota)})\,, & \iota = 1, \dots, r-1 \\
\hat{y}^{(r)} & := \tilde{y}^{(r)}u(\tilde{y}^{(r)})\,, & \text{if } s \geq 1 \\
\hat{y}^{(r)} & := \tilde{y}^{(r)}\,, & \text{if } s = 0 \\
\tilde{z}^{(1)} & := u(\tilde{y}^{(r)})z^{(1)}\,, & \text{if } r \geq 1 \\
\tilde{z}^{(1)} & := z^{(1)}\,, & \text{if } r = 0 \\
\tilde{z}^{(\kappa)} & := u(\tilde{z}^{(\kappa-1)})z^{(\kappa)}\,, & \kappa = 2, \dots, s \\
\hat{z}^{(\kappa)} & := \tilde{z}^{(\kappa)}u(\tilde{z}^{(\kappa)})\,, & \kappa = 1, \dots, s-1 \\
\hat{z}^{(s)} & := \tilde{z}^{(s)}\,. &
\end{array}$$

By construction, $\hat{y}^{(1)}, \dots, \hat{y}^{(r-1)}, \hat{z}^{(1)}, \dots, \hat{z}^{(s-1)}$ have type o_0 and

$$|\hat{y}^{(\iota)}|^2 = |\tilde{y}^{(\iota)}|^2 = |y^{(\iota)}|^2 = p, \quad \iota = 1, \dots, r\,,$$
$$|\hat{z}^{(\kappa)}|^2 = |\tilde{z}^{(\kappa)}|^2 = |z^{(\kappa)}|^2 = l, \quad \kappa = 1, \dots, s\,.$$

Moreover,

$$\begin{aligned}
x &= y^{(1)}y^{(2)}y^{(3)}\dots y^{(r)}z^{(1)}\dots z^{(s)} \\
&= \pm\underbrace{y^{(1)}u(y^{(1)})}_{=\hat{y}^{(1)}}\underbrace{u(y^{(1)})y^{(2)}}_{=\tilde{y}^{(2)}}y^{(3)}\dots y^{(r)}z^{(1)}\dots z^{(s)} \\
&= \pm\hat{y}^{(1)}\underbrace{\tilde{y}^{(2)}u(\tilde{y}^{(2)})}_{=\hat{y}^{(2)}}\underbrace{u(\tilde{y}^{(2)})y^{(3)}}_{=\tilde{y}^{(3)}}\dots\cdot y^{(r)}z^{(1)}\dots z^{(s)} \\
&= \dots \\
&= \pm\hat{y}^{(1)}\dots\hat{y}^{(r)}\underbrace{u(\tilde{y}^{(r)})z^{(1)}}_{=\tilde{z}^{(1)}}\dots z^{(s)} \\
&= \pm\hat{y}^{(1)}\dots\hat{y}^{(r)}\underbrace{\tilde{z}^{(1)}u(\tilde{z}^{(1)})}_{=\hat{z}^{(1)}}\underbrace{u(\tilde{z}^{(1)})z^{(2)}}_{=\tilde{z}^{(2)}}\dots z^{(s)} \\
&= \dots \\
&= \pm\hat{y}^{(1)}\dots\hat{y}^{(r)}\hat{z}^{(1)}\dots\hat{z}^{(s-1)}\underbrace{u(\tilde{z}^{(s-1)})z^{(s)}}_{=\tilde{z}^{(s)}} \\
&= \pm\hat{y}^{(1)}\dots\hat{y}^{(r)}\hat{z}^{(1)}\dots\hat{z}^{(s)}\,.
\end{aligned}$$

It follows that also $\hat{y}^{(r)}$ and $\hat{z}^{(s)}$ have type o_0. After replacing those $\hat{y}^{(\iota)}$ and $\hat{z}^{(\kappa)}$ satisfying $\Re(\hat{y}^{(\iota)}) < 0$ and $\Re(\hat{z}^{(\kappa)}) < 0$ by $-\hat{y}^{(\iota)}$ and $-\hat{z}^{(\kappa)}$, respectively, we can assume that moreover

$$\Re(\hat{y}^{(1)}) > 0, \dots, \Re(\hat{y}^{(r)}) > 0,\ \Re(\hat{z}^{(1)}) > 0, \dots, \Re(\hat{z}^{(s)}) > 0$$

and still $x = \pm\hat{y}^{(1)}\dots\hat{y}^{(r)}\hat{z}^{(1)}\dots\hat{z}^{(s)}$. But now,

$$\psi(x) = \psi(\pm\hat{y}^{(1)}\dots\hat{y}^{(r)}\hat{z}^{(1)}\dots\hat{z}^{(s)}) = \psi(\hat{y}^{(1)})\dots\psi(\hat{y}^{(r)})\psi(\hat{z}^{(1)})\dots\psi(\hat{z}^{(s)})\,,$$

where $\psi(\hat{y}^{(1)}), \dots, \psi(\hat{y}^{(r)}) \in E_h$ and $\psi(\hat{z}^{(1)}), \dots, \psi(\hat{z}^{(s)}) \in E_v$, and we are done.

A shorter proof of part (3) would be to generalize part (4) as in Theorem 3.30(1) and apply it as in Theorem 3.30(2).

(4) See [46, Corollary 3.2] or [45, Corollary 2.1.10]. The existence proof is based on Proposition 3.10, the uniqueness follows from a counting argument; we will reproduce it in a more general context in Theorem 3.30.

(5) The first isomorphism $\langle a_1, \dots, a_{\frac{p+1}{2}} \rangle_{\Gamma_{p,l}} \cong F_{\frac{p+1}{2}}$ is implied by the uniqueness statement of part (4), using

$$E_h = \psi(\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}) .$$

The second isomorphism $\langle b_1, \dots, b_{\frac{l+1}{2}} \rangle_{\Gamma_{p,l}} \cong F_{\frac{l+1}{2}}$ follows analogously.

$\square$

To summarize, we can see $\Gamma_{p,l}$ as a $(p+1, l+1)$–group with a finite presentation

$$\Gamma_{p,l} = \langle a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}} \rangle ,$$

where the $\frac{p+1}{2} \cdot \frac{l+1}{2}$ relators in $R_{\frac{p+1}{2}\cdot\frac{l+1}{2}}$ come from Corollary 3.11(2), and as the subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$

$$\Gamma_{p,l} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\} .$$

For certain important subsets or subgroups of $\Gamma_{p,l}$, we thus get the following characterizations:

$$\begin{aligned}
\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p\} \\
\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} &= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l\} \\
F_{\frac{p+1}{2}} \cong \langle a_1, \dots, a_{\frac{p+1}{2}} \rangle &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r;\ r \in \mathbb{N}_0\} \\
F_{\frac{l+1}{2}} \cong \langle b_1, \dots, b_{\frac{l+1}{2}} \rangle &= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l^s;\ s \in \mathbb{N}_0\}
\end{aligned}$$

and

$$\begin{aligned}
(\Gamma_{p,l})_0 &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r} l^{2s};\ r, s \in \mathbb{N}_0\} \\
&< \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l) .
\end{aligned}$$

We can see $\mathrm{PSL}_2(\mathbb{Q}_p)$ as a subgroup of $\mathrm{PGL}_2(\mathbb{Q}_p)$ of index $4 = |\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2|$. With the identification from above, we have

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} \subset \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l) < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

if and only if $\left(\frac{p}{l}\right) = 1$, and

$$\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} \subset \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l) < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

if and only if $\left(\frac{l}{p}\right) = 1$. This follows from Lemma 3.5 (and Hensel's Lemma), see also [16, p.134]. Note that our assumption $p, l \equiv 1 \pmod 4$ implies $\left(\frac{p}{l}\right) = \left(\frac{l}{p}\right)$ by the famous law of quadratic reciprocity, see e.g. [23, Theorem 2.3.2 (iii)].

The following theorem is motivated by Lubotzky's book [45], and some parts are obvious generalizations of results appearing there; nevertheless, we try to give very detailed proofs here.

Theorem 3.12. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers. Let $G_{p,l}$ be the group $U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))$. Then*

(1) *The group $\Gamma_{p,l}$ is (isomorphic to) a normal subgroup of $G_{p,l}/ZG_{p,l}$ of index 4 such that $(G_{p,l}/ZG_{p,l})/\Gamma_{p,l} \cong \mathbb{Z}_2^2$.*

(2) *The group $\Gamma_{p,l}$ can be realized as a rational matrix group. More precisely, there is a chain of subgroups*

$$\Gamma_{p,l} < \mathrm{SO}_3(\mathbb{Q}) < \mathrm{SO}_3(\mathbb{R}) < \mathrm{PGL}_2(\mathbb{C}),$$

in particular $\Gamma_{p,l}$ is residually finite.

(3) *If q is an odd prime number different from p and l, then there is a non-trivial homomorphism $\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$.*

(4) *Let $\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ be the homomorphism constructed in part (3), where q is an odd prime number different from p and l. Then its image is*

$$\tau(\Gamma_{p,l}) = \begin{cases} \mathrm{PSL}_2(q), & \text{if } \left(\frac{p}{q}\right) = \left(\frac{l}{q}\right) = 1 \\ \mathrm{PGL}_2(q), & \text{else}. \end{cases}$$

Moreover, $\tau(a_1^2) \in \tau(\langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle)$.

Proof. (1) To simplify the notation, we write $G_p := U(\mathbb{H}(\mathbb{Q}_p))$. Since

$$ZG_{p,l} = G_{p,l} \cap ZG_p = G_{p,l} \cap ZG_l,$$

and $\mathbb{Z}[1/p, 1/l]$ is a subring of $\mathbb{Q}_p$ and $\mathbb{Q}_l$ (which implies $G_{p,l} \subset G_p$ and $G_{p,l} \subset G_l$), there is an injective diagonal homomorphism

$$\begin{aligned} G_{p,l}/ZG_{p,l} &\to G_p/ZG_p \times G_l/ZG_l \\ xZG_{p,l} &\mapsto (xZG_p, xZG_l). \end{aligned}$$

The isomorphism $\mathbb{H}(\mathbb{Q}_p) \to M_2(\mathbb{Q}_p)$ of Lemma 3.8 (with $i_p^2 + 1 = 0$) induces an isomorphism

$$G_p = U(\mathbb{H}(\mathbb{Q}_p)) \to U(M_2(\mathbb{Q}_p)) = \mathrm{GL}_2(\mathbb{Q}_p)$$

and consequently an isomorphism

$$\begin{aligned} G_p/ZG_p &\to \mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)/Z\mathrm{GL}_2(\mathbb{Q}_p) \\ xZG_p &\mapsto \left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right]. \end{aligned}$$

Let ρ be the injective composition homomorphism

$$G_{p,l}/ZG_{p,l} \hookrightarrow G_p/ZG_p \times G_l/ZG_l \xrightarrow{\cong} \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)\,,$$

explicitly given by mapping $xZG_{p,l} \in G_{p,l}/ZG_{p,l}$ to

$$\tilde{\psi}(x) = \left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

where $x = x_0 + x_1 i + x_2 j + x_3 k \in G_{p,l}$ and $\tilde{\psi}$ is the natural extension of ψ from $\mathbb{H}(\mathbb{Z}) \setminus \{0\}$ to $\mathbb{H}(\mathbb{Z}[1/p, 1/l]) \setminus \{0\}$.

Note that

$$U(\mathbb{Z}[1/p, 1/l]) = \{\pm p^r l^s : r, s \in \mathbb{Z}\}\,,$$

hence by Lemma 3.1

$$G_{p,l} = \{x \in \mathbb{H}(\mathbb{Z}[1/p, 1/l]) : |x|^2 = p^r l^s;\ r, s \in \mathbb{Z}\}$$

and by Lemma 3.3(2)

$$ZG_{p,l} = \{x \in \mathbb{H}(\mathbb{Z}[1/p, 1/l]) : x = \overline{x} = \pm p^r l^s;\ r, s \in \mathbb{Z}\}\,.$$

Now let $x \in \mathbb{H}(\mathbb{Z})$ be an integer quaternion such that $|x|^2 = p^r l^s$ for some $r, s \in \mathbb{N}_0$, then $x \in G_{p,l}$ and $\psi(x) = \tilde{\psi}(x) = \rho(xZG_{p,l}) \in \rho(G_{p,l}/ZG_{p,l})$, hence $\Gamma_{p,l} < \rho(G_{p,l}/ZG_{p,l}) \cong G_{p,l}/ZG_{p,l}$.

Note that each element in $G_{p,l}/ZG_{p,l}$ has a representative $xZG_{p,l}$ such that $x \in \mathbb{H}(\mathbb{Z})$ and $|x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0$, by multiplying with large enough positive powers of p and l, however $\Gamma_{p,l} \neq \rho(G_{p,l}/ZG_{p,l})$ since x must have type o_0 in the definition of $\Gamma_{p,l}$. More precisely, we can write

$$\rho(G_{p,l}/ZG_{p,l}) = g_0\Gamma_{p,l} \sqcup g_1\Gamma_{p,l} \sqcup g_2\Gamma_{p,l} \sqcup g_3\Gamma_{p,l} < \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

where for each $\iota \in \{0, 1, 2, 3\}$ we choose any element $g_\iota = \psi(x)$, such that $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$ has type o_ι and norm $|x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0$. For example, the simplest choice is to take $r = s = 0$ (i.e. $|x|^2 = 1$) and consequently

$$g_0 := \psi(1) = \left(\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \right)$$

$$g_1 := \psi(i) = \left(\left[\begin{pmatrix} i_p & 0 \\ 0 & -i_p \end{pmatrix} \right], \left[\begin{pmatrix} i_l & 0 \\ 0 & -i_l \end{pmatrix} \right] \right)$$

$$g_2 := \psi(j) = \left(\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] \right)$$

$$g_3 := \psi(k) = \left(\left[\begin{pmatrix} 0 & i_p \\ i_p & 0 \end{pmatrix} \right], \left[\begin{pmatrix} 0 & i_l \\ i_l & 0 \end{pmatrix} \right] \right).$$

To see the decomposition of $\rho(G_{p,l}/ZG_{p,l})$ given above, we first observe that $p^r l^s \equiv 1 \pmod 4$, since $p, l \equiv 1 \pmod 4$. Therefore, each decomposition of $|x|^2 = p^r l^s$ as a sum of four squares is a sum of squares of three even numbers and one odd number (cf. Lemma 3.7(3)). If we take the quaternion multiplication on the four classes of quaternions of type o_0, o_1, o_2 and o_3 respectively, then we get a group structure, where the class of type o_0 quaternions is the identity element. The group is isomorphic to $\mathbb{Z}_2^2$, as it is seen in the following multiplication table.

$\cdot$	type o_0	type o_1	type o_2	type o_3
type o_0	type o_0	type o_1	type o_2	type o_3
type o_1	type o_1	type o_0	type o_3	type o_2
type o_2	type o_2	type o_3	type o_0	type o_1
type o_3	type o_3	type o_2	type o_1	type o_0

Table 3.3: Multiplication table for quaternions of type o

Because of $\psi(xy) = \psi(x)\psi(y)$, this group structure carries over to the cosets

$$\{g_0\Gamma_{p,l}, g_1\Gamma_{p,l}, g_2\Gamma_{p,l}, g_3\Gamma_{p,l}\}$$

in $\rho(G_{p,l}/ZG_{p,l})$ and we are done. To summarize, we have shown that

$$\begin{aligned} \Gamma_{p,l} &\overset{4}{\lhd} \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\} \\ &= \rho(G_{p,l}/ZG_{p,l}) \\ &\cong G_{p,l}/ZG_{p,l}\,. \end{aligned}$$

(2) If G is a group, we denote here by G/Z the quotient group G/ZG of G by its center ZG. We study the following diagram of group homomorphisms:

$$\begin{tikzcd} \Gamma_{p,l} \arrow[r] & G_{p,l}/Z \arrow[r] & U(\mathbb{H}(\mathbb{Q}))/Z \arrow[r] \arrow[d, "\cong"] & U(\mathbb{H}(\mathbb{R}))/Z \arrow[r] \arrow[d, "\cong"] & U(\mathbb{H}(\mathbb{C}))/Z \arrow[d, "\cong"] \\ & & \mathrm{SO}_3(\mathbb{Q}) & \mathrm{SO}_3(\mathbb{R}) & \mathrm{PGL}_2(\mathbb{C}) \end{tikzcd}$$

The homomorphisms in the top line are all injective: the first of them is described in part (1) of this theorem. The other three homomorphisms are induced by the natural injective group homomorphisms (which are induced themselves by the chain of the corresponding subrings $\mathbb{Z}[1/p, 1/l]) \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$)

$$U(\mathbb{H}(\mathbb{Z}[1/p, 1/l])) \hookrightarrow U(\mathbb{H}(\mathbb{Q})) \hookrightarrow U(\mathbb{H}(\mathbb{R})) \hookrightarrow U(\mathbb{H}(\mathbb{C}))\,, \tag{3.1}$$

since

$$ZU(\mathbb{H}(\mathbb{Z}[1/p, 1/l])) \subset ZU(\mathbb{H}(\mathbb{Q})) \subset ZU(\mathbb{H}(\mathbb{R})) \subset ZU(\mathbb{H}(\mathbb{C})) . \tag{3.2}$$

Assertion (3.2) follows directly from (3.1) using the fact, see Lemma 3.3(2),

$$ZU(\mathbb{H}(R)) = U(\mathbb{H}(R)) \cap \{x \in U(\mathbb{H}(\mathbb{C})) : x = \overline{x}\} ,$$

which holds if $R \in \{\mathbb{Z}[1/p, 1/l], \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

The homomorphisms

$$G_{p,l}/Z \longrightarrow U(\mathbb{H}(\mathbb{Q}))/Z \longrightarrow U(\mathbb{H}(\mathbb{R}))/Z \longrightarrow U(\mathbb{H}(\mathbb{C}))/Z$$

are injective, since (3.1) directly implies

$$U(\mathbb{H}(R_1)) \cap ZU(\mathbb{H}(R_2)) < ZU(\mathbb{H}(R_1)) ,$$

whenever $(R_1, R_2) \in \{(\mathbb{Z}[1/p, 1/l], \mathbb{Q}), (\mathbb{Q}, \mathbb{R}), (\mathbb{R}, \mathbb{C})\}$. In fact, the equality $U(\mathbb{H}(R_1)) \cap ZU(\mathbb{H}(R_2)) = ZU(\mathbb{H}(R_1))$ holds by (3.2).

To get $U(\mathbb{H}(\mathbb{Q}))/Z \cong \mathrm{SO}_3(\mathbb{Q})$, first note that $U(\mathbb{H}(\mathbb{Q})) = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$. Now define $\vartheta : U(\mathbb{H}(\mathbb{Q})) \to \mathrm{SO}_3(\mathbb{Q})$ by mapping x to the (3×3)–matrix

$$\frac{1}{|x|^2} \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix} ,$$

where $x = x_0 + x_1i + x_2j + x_3k \in U(\mathbb{H}(\mathbb{Q}))$. Note that this is the matrix which represents the $\mathbb{Q}$-linear map $\mathbb{Q}^3 \to \mathbb{Q}^3$, $y \mapsto xyx^{-1}$ with respect to the standard basis of $\mathbb{Q}^3$, where $y = (y_1, y_2, y_3)^T \in \mathbb{Q}^3$ is identified with the "purely imaginary" quaternion $y_1i + y_2j + y_3k \in \mathbb{H}(\mathbb{Q})$. It is well-known that ϑ is a surjective group homomorphism. Even the restricted map

$$\vartheta|_{\mathbb{H}(\mathbb{Z})\setminus\{0\}} : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{SO}_3(\mathbb{Q})$$

is surjective, since $\vartheta(ax) = \vartheta(x)$, if $a \in \mathbb{Q}^\times$ and $x \in U(\mathbb{H}(\mathbb{Q}))$. For an elementary proof of the surjectivity of $\vartheta|_{\mathbb{H}(\mathbb{Z})\setminus\{0\}}$, see [43]. Moreover, it is easy to check by solving a system of equations that

$$\ker(\vartheta) = \{x \in \mathbb{H}(\mathbb{Q}) \setminus \{0\} : x = \overline{x}\} = ZU(\mathbb{H}(\mathbb{Q})) .$$

Seeing $\vartheta(x)$ as $\mathbb{Q}$-linear map $y \mapsto xyx^{-1}$ as described above, it is even very easy to determine the kernel:

$$\begin{aligned} \ker(\vartheta) &= \{x \in U(\mathbb{H}(\mathbb{Q})) : xyx^{-1} = y,\ \forall y \in \mathbb{H}(\mathbb{Q}) \text{ such that } \Re(y) = 0\} \\ &= \{x \in U(\mathbb{H}(\mathbb{Q})) : xy = yx,\ \forall y \in \mathbb{H}(\mathbb{Q}) \text{ such that } \Re(y) = 0\} \\ &= \{x \in U(\mathbb{H}(\mathbb{Q})) : x = \overline{x}\} . \end{aligned}$$

Observe that if $x \in U(\mathbb{H}(\mathbb{Q})) \setminus ZU(\mathbb{H}(\mathbb{Q}))$, then the axis of the rotation $\vartheta(x)$ is the line $(x_1, x_2, x_3)^T \cdot \mathbb{Q}$, and the rotation angle ω satisfies

$$\cos\omega = \frac{x_0^2 - x_1^2 - x_2^2 - x_3^2}{|x|^2}.$$

Equivalently,

$$\cos\frac{\omega}{2} = \frac{x_0}{\sqrt{|x|^2}}.$$

To prove $U(\mathbb{H}(\mathbb{R}))/Z \cong \mathrm{SO}_3(\mathbb{R})$, replace $\mathbb{Q}$ by $\mathbb{R}$ above.

The isomorphism $U(\mathbb{H}(\mathbb{C}))/Z \cong \mathrm{PGL}_2(\mathbb{C})$ follows from Lemma 3.8.

Note that the injective composition homomorphism $\Gamma_{p,l} \to \mathrm{SO}_3(\mathbb{Q})$ can be explicitly constructed as follows: if $\gamma \in \Gamma_{p,l}$ is given as $\gamma = \psi(x)$, where $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$ has type o_0 and $|x|^2 = p^r l^s$; $r, s \in \mathbb{N}_0$, then the image of γ in $\mathrm{SO}_3(\mathbb{Q})$ is $\vartheta(x)$, independent of the possible choice of x. In the same way, the image of $\gamma = \psi(x)$ in $\mathrm{PGL}_2(\mathbb{C})$ is

$$\left[\begin{pmatrix} x_0 + x_1 i_{\mathbb{C}} & x_2 + x_3 i_{\mathbb{C}} \\ -x_2 + x_3 i_{\mathbb{C}} & x_0 - x_1 i_{\mathbb{C}} \end{pmatrix}\right].$$

By a result of Malcev ([51]), finitely generated linear groups (over a field of characteristic zero) are residually finite.

(3) Let q be an odd prime number different from p, l and let

$$G_{q,p,l} := U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l])).$$

As in the proof of part (2), we denote by G/Z the quotient G/ZG of a group G by its center ZG. We want to define the desired homomorphism

$$\tau : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$$

as composition of the homomorphisms

$$\Gamma_{p,l} \hookrightarrow G_{p,l}/Z \to G_{q,p,l}/Z \xrightarrow{\cong} U(\mathbb{H}(\mathbb{Z}_q))/Z \xrightarrow{\cong} \mathrm{PGL}_2(q).$$

We describe now separately these four homomorphisms.

The injection $\Gamma_{p,l} \hookrightarrow G_{p,l}/Z$ is given by part (1) of this theorem.

The unital (quotient) ring homomorphism

$$\mathbb{Z}[1/p, 1/l] \to \mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l]$$

extends to a unital ring homomorphism

$$\mathbb{H}(\mathbb{Z}[1/p, 1/l]) \to \mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l])$$

mapping $1, i, j, k$, to $1, i, j, k$, respectively (see [23, Section 2.5]), and induces a group homomorphism of the invertible elements $G_{p,l} \to G_{q,p,l}$. Since

$$ZG_{p,l} = \{x \in G_{p,l} : x = \overline{x}\}$$

by Lemma 3.3(2), it is not difficult to see that the image of $ZG_{p,l}$ under the homomorphism $G_{p,l} \to G_{q,p,l}$ is contained in $ZG_{q,p,l}$. This gives the second homomorphism

$$G_{p,l}/Z \to G_{q,p,l}/Z \, .$$

Now we attack the third one $G_{q,p,l}/Z \xrightarrow{\cong} U(\mathbb{H}(\mathbb{Z}_q))/Z$. The map

$$\begin{aligned} \phi : \mathbb{Z}_q &\to \mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l] \\ v + q\mathbb{Z} &\mapsto v + q\mathbb{Z}[1/p, 1/l] \, , \end{aligned}$$

$v \in \mathbb{Z}$, is an isomorphism of rings (even of fields, since q is a prime number), and ϕ^{-1} therefore induces isomorphisms

$$\mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l]) \xrightarrow{\cong} \mathbb{H}(\mathbb{Z}_q) \, ,$$

$$G_{q,p,l} = U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l])) \xrightarrow{\cong} U(\mathbb{H}(\mathbb{Z}_q))$$

and finally an isomorphism $G_{q,p,l}/Z \to U(\mathbb{H}(\mathbb{Z}_q))/Z$. The only non-trivial thing to check is the surjectivity of ϕ: First, we have

$$\phi(0 + q\mathbb{Z}) = 0 + q\mathbb{Z}[1/p, 1/l] \, .$$

Now, take any element

$$tp^r l^s + q\mathbb{Z}[1/p, 1/l] \in \mathbb{Z}[1/p, 1/l]/q\mathbb{Z}[1/p, 1/l] \, ,$$

where $t \in \mathbb{Z} \setminus \{0\}$ is relatively prime to p and l. To simplify matters, we assume that $r, s < 0$ (if $r, s \geq 0$, then $\phi^{-1}(tp^r l^s + q\mathbb{Z}[1/p, 1/l]) = tp^r l^s + q\mathbb{Z}$; in the cases $r \geq 0, s < 0$ and $r < 0, s \geq 0$ the proofs are similar to the proof for the case $r, s < 0$ given now). Then $\gcd(p^{-r} l^{-s}, q)$ is 1 and therefore obviously divides t, hence (see e.g. [36, Proposition 3.3.1]) there is an integer u such that $p^{-r} l^{-s} u \equiv t \pmod q$, i.e. $t - p^{-r} l^{-s} u \in q\mathbb{Z}$ and

$$tp^r l^s - u = p^r l^s (t - p^{-r} l^{-s} u) \in q\mathbb{Z}[1/p, 1/l] \, .$$

This implies

$$tp^r l^s + q\mathbb{Z}[1/p, 1/l] = u + q\mathbb{Z}[1/p, 1/l] = \phi(u + q\mathbb{Z}) .$$

The isomorphism $U(\mathbb{H}(\mathbb{Z}_q))/Z \cong \mathrm{PGL}_2(q)$ follows from Lemma 3.8, since there exist elements c and d in the field $\mathbb{Z}_q$ such that $c^2 + d^2 + 1 = 0$ in $\mathbb{Z}_q$, see [23, Proposition 2.5.3].

Therefore, if $\gamma \in \Gamma_{p,l}$ is given by $\gamma = \psi(x_0 + x_1 i + x_2 j + x_3 k)$ (where we require as in the definition of $\Gamma_{p,l}$ that $x \in \mathbb{H}(\mathbb{Z})$ has type o_0 and norm $|x|^2 = p^r l^s; r, s \in \mathbb{N}_0$), and we have chosen $c, d \in \mathbb{Z}$ such that $c^2 + d^2 + 1 \equiv 0 \pmod q$, then $\tau = \tau_{c,d} : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ is explicitly constructed as

$$\tau_{c,d}(\gamma) = \left[\begin{pmatrix} x_0 + x_1 c + x_3 d + q\mathbb{Z} & -x_1 d + x_2 + x_3 c + q\mathbb{Z} \\ -x_1 d - x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c - x_3 d + q\mathbb{Z} \end{pmatrix}\right] .$$

If for example $q \equiv 1 \pmod 4$, we can choose $d = 0$ and $c \in \{1, \dots, q-1\}$, such that $c^2 + 1 \equiv 0 \pmod q$, and $\tau = \tau_{c,0}$ then simplifies to

$$\gamma \mapsto \left[\begin{pmatrix} x_0 + x_1 c + q\mathbb{Z} & x_2 + x_3 c + q\mathbb{Z} \\ -x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c + q\mathbb{Z} \end{pmatrix}\right] .$$

What happens if we take $q = 2$?

The group $G_{2,p,l} \cong U(\mathbb{H}(\mathbb{Z}_2)) \cong \mathbb{Z}_2^3$ is abelian, hence

$$G_{2,p,l}/Z \cong U(\mathbb{H}(\mathbb{Z}_2))/Z = 1 \neq \mathrm{PGL}_2(2) \cong S_3 .$$

Note that the field $\mathbb{Z}_2$ is excluded in the assumptions of Lemma 3.8.

(4) At first, we show that $\tau(\Gamma_{p,l}) < \mathrm{PSL}_2(q)$ if and only if $\left(\frac{p}{q}\right) = \left(\frac{l}{q}\right) = 1$. The group $\Gamma_{p,l}$ is generated by the set $\{a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}\}$, hence we have $\tau(\Gamma_{p,l}) < \mathrm{PSL}_2(q)$ if and only if

$$\{\tau(a_1), \dots, \tau(a_{\frac{p+1}{2}}), \tau(b_1), \dots, \tau(b_{\frac{l+1}{2}})\} \subset \mathrm{PSL}_2(q) .$$

Since the elements $\tau(a_1), \dots, \tau(a_{\frac{p+1}{2}})$ are represented by matrices in $\mathrm{GL}_2(q)$ with determinant $p + q\mathbb{Z} \in \mathbb{Z}_q$ and $\tau(b_1), \dots, \tau(b_{\frac{l+1}{2}})$ are represented by matrices in $\mathrm{GL}_2(q)$ with determinant $l + q\mathbb{Z} \in \mathbb{Z}_q$, the condition $\tau(\Gamma_{p,l}) < \mathrm{PSL}_2(q)$ is by Lemma 3.5 equivalent to the condition $\{p + q\mathbb{Z}, l + q\mathbb{Z}\} \subset (\mathbb{Z}_q^\times)^2$. But this is equivalent to $\left(\frac{p}{q}\right) = \left(\frac{l}{q}\right) = 1$ by Lemma 3.6.

By [45, Lemma 7.4.2] or [46, Proposition 3.3], we have

$$\mathrm{PSL}_2(q) < \tau(\langle a_1, \dots, a_{\frac{p+1}{2}} \rangle) \text{ and } \mathrm{PSL}_2(q) < \tau(\langle b_1, \dots, b_{\frac{l+1}{2}} \rangle) ,$$

in particular $\mathrm{PSL}_2(q) < \tau(\Gamma_{p,l}) < \mathrm{PGL}_2(q)$.

This determines the image of τ, since $[\mathrm{PGL}_2(q) : \mathrm{PSL}_2(q)] = 2$.

Exactly as above, we can show that

$$\tau(\langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle) = \begin{cases} \mathrm{PSL}_2(q)\,, & \text{if } \left(\frac{l}{q}\right) = 1 \\ \mathrm{PGL}_2(q)\,, & \text{if } \left(\frac{l}{q}\right) = -1\,. \end{cases}$$

Since the element $\tau(a_1^2) = \tau(a_1)^2$ is represented by a matrix in $\mathrm{GL}_2(q)$ with determinant $(p+q\mathbb{Z})^2 = p^2+q\mathbb{Z} \in \mathbb{Z}_q$, we have $\tau(a_1^2) \in \mathrm{PSL}_2(q)$ by Lemma 3.5 and consequently $\tau(a_1^2) \in \tau(\langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle)$.

□

See Table 3.4 for some information about groups $U(\mathbb{H}(R))/ZU(\mathbb{H}(R))$, where R is a commutative ring with unit, $p, l \equiv 1 \pmod 4$ are distinct prime numbers and q is an odd prime number.

R	$U(\mathbb{H}(R))/ZU(\mathbb{H}(R))$
$\mathbb{Z}[1/p, 1/l]$	contains $\Gamma_{p,l}$ as index 4 subgroup
$\mathbb{Z}[1/p]$	important in [45], virtually $F_{\frac{p+1}{2}}$
$\mathbb{Z}$	$\mathbb{Z}_2^2$
$\mathbb{Z}_q$	$\mathrm{PGL}_2(q)$
$\mathbb{Z}_2$	1
$\mathbb{Q}$	$\mathrm{SO}_3(\mathbb{Q})$
$\mathbb{R}$	$\mathrm{SO}_3(\mathbb{R})$
$\mathbb{C}$	$\mathrm{PGL}_2(\mathbb{C})$
$\mathbb{Q}_q$	$\mathrm{PGL}_2(\mathbb{Q}_q)$

Table 3.4: The group $U(\mathbb{H}(R))/ZU(\mathbb{H}(R))$ for some rings R

The following result is also mentioned in [59, Example 5.12] and [30, Proposition 3.2, Proof of Theorem 4.1]. It is a very special case of Proposition 4.2(3), where we prove that *all* $(2m, 2n)$–groups contain $\mathbb{Z}^2$–subgroups.

Proposition 3.13. *The group $\Gamma_{p,l}$ contains a subgroup isomorphic to $\mathbb{Z}^2$.*

Proof. By Lemma 3.7(1), we can choose $x = x_0+x_1 i, y = y_0+y_1 i \in \mathbb{H}(\mathbb{Z})$ such that x_0, y_0 are odd, x_1, y_1 are even and non-zero, $|x|^2 = x_0^2 + x_1^2 = p$, $|y|^2 = y_0^2 + y_1^2 = l$. Obviously, we have $xy = yx$, hence $\psi(x)\psi(y) = \psi(y)\psi(x)$, where $\psi(x)$, $\psi(y)$ are non-trivial. The subgroup $\langle \psi(x), \psi(y) \rangle$ of $\Gamma_{p,l}$ is isomorphic to $\mathbb{Z}^2$ by the following general lemma. □

Lemma 3.14. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m\cdot n}\rangle$ be a $(2m, 2n)$–group and let $a \in \langle a_1, \dots, a_m\rangle$, $b \in \langle b_1, \dots, b_n\rangle$ be two non-trivial elements. If a and b commute, then $\langle a, b\rangle \cong \mathbb{Z}^2$.*

Proof. Since Γ is torsion-free, the subgroup $\langle a, b\rangle$ is a finitely generated abelian torsion-free quotient of $\mathbb{Z}^2$. Using $a, b \neq 1$ and the uniqueness of the ab-normal forms (see Proposition 1.10) of powers of a and b, we conclude that $\langle a, b\rangle$ is not cyclic, but itself isomorphic to $\mathbb{Z}^2$. □

Kimberley-Robertson have computed presentations of $\Gamma_{p,l}$ for many pairs (p, l). They conjecture for the abelianization $\Gamma^{ab}_{p,l}$

Conjecture 3.15. *(Kimberley-Robertson [41, Section 6]) Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers and let*

$$r := \gcd\left(\frac{p-1}{4}, \frac{l-1}{4}, 6\right).$$

Then

$$\Gamma^{ab}_{p,l} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4^3, & \text{if } r = 1 \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2, & \text{if } r = 2 \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3, & \text{if } r = 3 \\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \text{if } r = 6. \end{cases}$$

Note that the smallest pairs (p, l) such that $r = 1, 2, 3, 6$ are $(5, 13)$, $(17, 41)$, $(13, 37)$ and $(73, 97)$, respectively. Conjecture 3.15 is equivalent to the following conjecture (see Section 3.5 for generalizations of Conjecture 3.16):

Conjecture 3.16. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers.*
If $p, l \equiv 1 \pmod 8$, then

$$\Gamma^{ab}_{p,l} \cong \begin{cases} \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \text{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2, & \text{else}. \end{cases}$$

If $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$, then

$$\Gamma^{ab}_{p,l} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3, & \text{if } p, l \equiv 1 \pmod 3 \\ \mathbb{Z}_2 \times \mathbb{Z}_4^3, & \text{else}. \end{cases}$$

Proof of the equivalence of Conjecture 3.15 and Conjecture 3.16. First, observe that $r \in \{1, 2, 3, 6\}$ in Conjecture 3.15 and that all possibilities for (p, l) are treated in the four cases of Conjecture 3.16.

If $r = 6$, then $(p-1)/4 = 6s$ and $(l-1)/4 = 6t$ for some $s, t \in \mathbb{N}$, i.e. $p = 24s+1$ and $l = 24t + 1$. It follows $p, l \equiv 1 \pmod 8$ and $p, l \equiv 1 \pmod 3$.

If $r = 3$, then $(p-1)/4 = 3s$ and $(l-1)/4 = 3t$, where s or t is odd (otherwise r would be 6). Consequently, we have $p = 12s + 1$ and $l = 12t + 1$, in particular $p, l \equiv 1 \pmod 3$. If s is odd, then $p \equiv 5 \pmod 8$. If t is odd, then $l \equiv 5 \pmod 8$.

If $r = 2$, then $(p-1)/4 = 2s$ and $(l-1)/4 = 2t$, i.e. $p = 8s + 1$ and $l = 8t + 1$, hence $p, l \equiv 1 \pmod 8$. Moreover, $s \not\equiv 0 \pmod 3$ or $t \not\equiv 0 \pmod 3$ (otherwise r would be 6). In the first case, we have $p \not\equiv 1 \pmod 3$, in the second case $l \not\equiv 1 \pmod 3$.

If $r = 1$, then $(p-1)/4 = 2s - 1$ or $(l-1)/4 = 2t - 1$ (otherwise r would be even), hence $p = 8s - 3$ or $l = 8t - 3$, i.e. $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$. Moreover: $(p-1)/4 = 3s + 1$ or $(p-1)/4 = 3s + 2$ or $(l-1)/4 = 3t + 1$ or $(l-1)/4 = 3s + 2$ for some $s, t \in \mathbb{N}_0$ (otherwise r would be a multiple of 3), hence $p = 12s + 5$ or $p = 12s + 9$ or $l = 12t + 5$ or $l = 12t + 9$, in particular $p \not\equiv 1 \pmod 3$ or $l \not\equiv 1 \pmod 3$. $\square$

The structure of $\Gamma^{ab}_{p,l}$ also seems to depend only on the number of commuting quaternions whose ψ-images generate $\Gamma_{p,l}$. To make this precise, if $l \equiv 1 \pmod 4$ is a prime number, let $Y_l \subset \mathbb{H}(\mathbb{Z})$ be any set of cardinality $\frac{l+1}{2}$, such that $\langle \psi(Y_l) \rangle \cong F_{\frac{l+1}{2}}$ and each element $y \in Y_l$ has type o_0 and satisfies $\Re(y) > 0$, $|y|^2 = l$. We think of $Y_l = \{\psi^{-1}(b_1), \ldots, \psi^{-1}(b_{\frac{l+1}{2}})\}$ and $Y_p = \{\psi^{-1}(a_1), \ldots, \psi^{-1}(a_{\frac{p+1}{2}})\}$, where

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2} \cdot \frac{l+1}{2}} \rangle .$$

Then, let

$$c_{p,l} := |\{(x, y) : x \in Y_p,\ y \in Y_l,\ xy = yx\}| .$$

Note that the definition of $c_{p,l}$ is independent of the explicit choice of elements in Y_p and Y_l. Obviously,

$$c_{p,l} \le \min\left\{ \frac{p+1}{2}, \frac{l+1}{2} \right\} .$$

Moreover, $c_{p,l} \ge 3$, since Y_p contains by Lemma 3.7(1) elements of the form $x_0 + x_1 i$, $x_0 + x_2 j$, $x_0 + x_3 k$ and Y_l contains elements of the form $y_0 + y_1 i$, $y_0 + y_2 j$, $y_0 + y_3 k$, and for example $x_0 + x_1 i$ commutes with $y_0 + y_1 i$.

Conjecture 3.17. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers, and*

$$r = \gcd\left(\frac{p-1}{4}, \frac{l-1}{4}, 6 \right)$$

as in Conjecture 3.15. Then

$$c_{p,l} \equiv \begin{cases} 3 \pmod{12}, & \text{if } r = 1 \\ 9 \pmod{12}, & \text{if } r = 2 \\ 7 \pmod{12}, & \text{if } r = 3 \\ 1 \pmod{12}, & \text{if } r = 6 . \end{cases}$$

We have checked Conjecture 3.17 for all possible $p, l < 1000$. The following values for $c_{p,l}$ appear in this range:

$$c_{p,l} \in \begin{cases} \{3, 15, 27, 39, 51, 63, 75, 87, 99\}, & \text{if } r = 1 \\ \{9, 21, 33, 45, 57, 69, 81, 93, 105, 117, 129, 153\}, & \text{if } r = 2 \\ \{7, 19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 151\}, & \text{if } r = 3 \\ \{37, 49, 61, 73, 85, 97, 109, 121, 133\}, & \text{if } r = 6 . \end{cases}$$

See Table 3.5 for the frequencies of the values of $c_{p,l}$, where $p, l \equiv 1 \pmod 4$ are prime numbers such that $p < l < 1000$.

$c_{p,l}$	3	15	27	39	51	63	75	
#	1242	449	143	56	34	17	7	
	87	99						
	5	2						1955
$c_{p,l}$	9	21	33	45	57	69	81	
#	178	158	84	57	40	21	8	
	93	105	117	129	141	153		
	9	12	5	2		1		575
$c_{p,l}$	7	19	31	43	55	67	79	
#	236	130	79	42	18	8	12	
	91	103	115	127	139	151		
	6	1	4	2		1		539
$c_{p,l}$	1	13	25	37	49	61	73	
#				26	15	15	16	
	85	97	109	121	133			
	7	4	3	2	3			91
								3160

Table 3.5: $c_{p,l}$ and its frequency, $p < l < 1000$

Combining Conjecture 3.17 with Conjecture 3.15, we get another conjecture:

Conjecture 3.18. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers, then*

$$\Gamma_{p,l}^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_4^3, & \textit{if } c_{p,l} \equiv 3 \pmod{12} \\ \mathbb{Z}_2^3 \times \mathbb{Z}_8^2, & \textit{if } c_{p,l} \equiv 9 \pmod{12} \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3, & \textit{if } c_{p,l} \equiv 7 \pmod{12} \\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2, & \textit{if } c_{p,l} \equiv 1 \pmod{12} . \end{cases}$$

Now, we want to prove that the groups $\Gamma_{p,l}$ are commutative transitive. This has for example applications to centralizers of powers of elements, and a nice application which allows to detect "anti-tori" in $\Gamma_{p,l}$ (see Proposition 3.53 in Section 3.6).

Lemma 3.19. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers. Let $x, y \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|x|^2, |y|^2 \in \{p^r l^s : r, s \in \mathbb{N}_0\}$. Then $xy = yx$ if and only if $\psi(x)\psi(y) = \psi(y)\psi(x)$.*

Proof. Obviously $xy = yx$ implies $\psi(x)\psi(y) = \psi(y)\psi(x)$. Assume now that $\psi(x)\psi(y) = \psi(y)\psi(x)$. Then $\psi(xy) = \psi(yx)$ and $xy = \lambda yx$ for some $\lambda \in \mathbb{Q}^\times$. Taking the norm $|\cdot|^2$ of $xy = \lambda yx$, we conclude $|\lambda|^2 = \lambda^2 = 1$, hence $\lambda = 1$ or $\lambda = -1$. If $\lambda = 1$, then $xy = yx$ and we are done. The case $\lambda = -1$ is impossible since $xy = -yx$ together with $\Re(x) \neq 0$ implies by Lemma 3.4(2) the contradiction $y = 0$. □

Proposition 3.20. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers. Then $\Gamma_{p,l}$ is commutative transitive, i.e. the relation of commutativity is transitive on the set of non-trivial elements of $\Gamma_{p,l}$.*

Equivalently, this means that if $x, y, z \in \mathbb{H}(\mathbb{Z})$ are of type o_0 such that

$$x \neq \Re(x),\ y \neq \Re(y),\ z \neq \Re(z)\,,$$
$$|x|^2,\ |y|^2,\ |z|^2 \in \{p^r l^s : r, s \in \mathbb{N}_0\}\,,$$
$$\psi(x)\psi(y) = \psi(y)\psi(x) \ \text{ and } \ \psi(x)\psi(z) = \psi(z)\psi(x)\,,$$

then also $\psi(y)\psi(z) = \psi(z)\psi(y)$.

Proof. Note that for x of type o_0 we have $x \neq \Re(x)$, if and only if $\psi(x) \neq 1$. By Lemma 3.19, we have $xy = yx$ and $xz = zx$. Moreover, again by Lemma 3.19, $\psi(y)\psi(z) = \psi(z)\psi(y)$ if and only if $yz = zy$. But $yz = zy$ follows now directly by Lemma 3.4(3). □

Corollary 3.21. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers, $\Gamma = \Gamma_{p,l}$ and $\gamma \in \Gamma$ a non-trivial element.*

(1) *If $k \in \mathbb{N}$, then $Z_\Gamma(\gamma^k) = Z_\Gamma(\gamma)$.*

(2) *The centralizer $Z_\Gamma(\gamma)$ is abelian.*

(3) *The center $Z\Gamma$ is trivial.*

Proof. (1) Since γ and γ^k commute, the statement follows from Proposition 3.20, using the fact that Γ is torsion-free.

(2) Again, this is a direct consequence of Proposition 3.20.

(3) Of course, the statement follows from the more general result Corollary 1.11(3) for $(2m, 2n)$–groups. Here, it follows directly from Proposition 3.20, since the existence of a non-trivial element in $Z\Gamma$ would imply that Γ is abelian.

$\square$

Using the following result of Mozes ([54]) together with Proposition 1.12 about centralizers, we give some applications to number theory, illustrated for two concrete examples in Proposition 3.23:

Proposition 3.22. *(Mozes [54, Proposition 3.15]) Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers,*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}} \rangle$$

and let $z \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $z \neq \Re(z)$ and $|z|^2 = l^s$ for some $s \in \mathbb{N}$. Take $c_1, c_2, c_3 \in \mathbb{Z}$ relatively prime such that $c := c_1 i + c_2 j + c_3 k \in \mathbb{H}(\mathbb{Z})$ commutes with z. Then there exists a non-trivial element $a \in \langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle \subset \Gamma$ commuting with $\psi(z)$ if and only if there are integers $x, y \in \mathbb{Z}$ such that

$$\gcd(x, y) = \gcd(x, pl) = \gcd(y, pl) = 1$$

and $x^2 + 4|c|^2 y^2 \in \{p^r l^s : r, s \in \mathbb{N}\}$.

Proposition 3.23. (1) *There are no pairs of integers $x, y \in \mathbb{Z}$ such that*

$$\gcd(x, y) = \gcd(x, 65) = \gcd(y, 65) = 1$$

and

$$x^2 + 12y^2 \in \{5^r 13^s : r, s \in \mathbb{N}\}.$$

(2) *There are no pairs $x, y \in \mathbb{Z}$ such that*

$$\gcd(x, y) = \gcd(x, 221) = \gcd(y, 221) = 1$$

and

$$x^2 + 8y^2 \in \{13^r 17^s : r, s \in \mathbb{N}\}.$$

Proof. (1) For $b_1 = \psi(1+2i+2j+2k) \in \Gamma_{5,13} =: \Gamma$ we have $Z_\Gamma(b_1) = \langle b_1 \rangle$, see Proposition 3.29(7) below. In particular, b_1 does not commute with any element in $\langle a_1, a_2, a_3 \rangle \setminus \{1\}$. The statement follows now by Proposition 3.22, taking $c = i + j + k$.

(2) Proposition 3.27(4) below shows that $Z_\Gamma(b_4) = \langle b_4 \rangle$, where

$$b_4 = \psi(3 + 2i + 2j) \in \Gamma_{13,17} =: \Gamma.$$

Taking $c = i + j$, we can again apply Proposition 3.22.

$\square$

The results on centralizers in $\Gamma_{p,l}$ used in the proof of the preceding proposition can also be applied to give statements about non-commuting quaternions. We first illustrate it again for $(p, l) \in \{(5, 13), (13, 17)\}$ and generalize it in Proposition 3.25.

Proposition 3.24. (1) *Let $y = 1 + 2i + 2j + 2k$. Then there is no $x \in \mathbb{H}(\mathbb{Z})$, $x \neq \Re(x)$, of type o_0 such that $|x|^2 \in \{5^r : r \in \mathbb{N}\}$ and $xy = yx$.*

(2) *Let $y = 3 + 2i + 2j$. Then there is no $x \in \mathbb{H}(\mathbb{Z})$, $x \neq \Re(x)$, of type o_0 such that $|x|^2 \in \{13^r : r \in \mathbb{N}\}$ and $xy = yx$.*

Proof. (1) Let $\Gamma = \Gamma_{5,13}$ and $b_1 = \psi(y) \in \Gamma$. Assume that $x \in \mathbb{H}(\mathbb{Z})$ is of type o_0 such that $|x|^2 \in \{5^r : r \in \mathbb{N}\}$ and $xy = yx$, where $x \neq \Re(x)$. This implies $\psi(x) \in \langle a_1, a_2, a_3 \rangle \setminus \{1\}$ and $\psi(x) \in Z_\Gamma(b_1)$, contradicting $Z_\Gamma(b_1) = \langle b_1 \rangle$ (which holds by Proposition 3.29(7)).

(2) Same proof as in part (1) taking $p = 13$, $l = 17$, $b_4 = \psi(y) \in \Gamma = \Gamma_{13,17}$ and using $Z_\Gamma(b_4) = \langle b_4 \rangle$ (which holds by Proposition 3.27(4)).

□

Proposition 3.25. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers and*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2} \cdot \frac{l+1}{2}} \rangle .$$

Assume that $\rho_v(b_j)(a) \neq a$ for some $b_j \in \{b_1, \ldots, b_{\frac{l+1}{2}}\}$ and all elements $a \in E_h$. Let $y \in \mathbb{H}(\mathbb{Z})$ be of type o_0 such that $|y|^2 = l$ and $b_j = \psi(y)$. Then there is no $x \in \mathbb{H}(\mathbb{Z})$, $x \neq \Re(x)$, of type o_0 such that $|x|^2 \in \{p^r : r \in \mathbb{N}\}$ and $xy = yx$.

Proof. As in the proof of Proposition 3.24 the claim follows directly from the fact $Z_\Gamma(b_j) = \langle b_j \rangle$ which is a consequence of Proposition 1.12(1b). □

Now, we want to study the two examples $\Gamma_{13,17}$ and $\Gamma_{5,13}$.

Example: $p = 13, l = 17$

Using the explicit identification

$$\begin{aligned}
a_1 &= \psi(1 + 2i + 2j + 2k), & a_1^{-1} &= \psi(1 - 2i - 2j - 2k),\\
a_2 &= \psi(1 + 2i + 2j - 2k), & a_2^{-1} &= \psi(1 - 2i - 2j + 2k),\\
a_3 &= \psi(1 + 2i - 2j + 2k), & a_3^{-1} &= \psi(1 - 2i + 2j - 2k),\\
a_4 &= \psi(1 - 2i + 2j + 2k), & a_4^{-1} &= \psi(1 + 2i - 2j - 2k),\\
a_5 &= \psi(3 + 2i), & a_5^{-1} &= \psi(3 - 2i),\\
a_6 &= \psi(3 + 2j), & a_6^{-1} &= \psi(3 - 2j),\\
a_7 &= \psi(3 + 2k), & a_7^{-1} &= \psi(3 - 2k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1+4i), & b_1^{-1} &= \psi(1-4i),\\
b_2 &= \psi(1+4j), & b_2^{-1} &= \psi(1-4j),\\
b_3 &= \psi(1+4k), & b_3^{-1} &= \psi(1-4k),\\
b_4 &= \psi(3+2i+2j), & b_4^{-1} &= \psi(3-2i-2j),\\
b_5 &= \psi(3+2i-2j), & b_5^{-1} &= \psi(3-2i+2j),\\
b_6 &= \psi(3+2i+2k), & b_6^{-1} &= \psi(3-2i-2k),\\
b_7 &= \psi(3+2i-2k), & b_7^{-1} &= \psi(3-2i+2k),\\
b_8 &= \psi(3+2j+2k), & b_8^{-1} &= \psi(3-2j-2k),\\
b_9 &= \psi(3+2j-2k), & b_9^{-1} &= \psi(3-2j+2k),
\end{aligned}$$

we get the example $\Gamma = \Gamma_{13,17}$. The corresponding $(14, 18)$–complex X is denoted by $\mathcal{A}_{13,17}$ in [17] and essentially used there in the construction of finitely presented torsion-free (virtually) simple groups, see [17, Theorem 6.4].

Example 3.26. *Let $R_{7.9} = R_{\frac{p+1}{2}.\frac{l+1}{2}}$ be the set of 63 relators*

$$R_{7.9} := \left\{ \begin{array}{lll} a_1b_1a_3b_3, & a_1b_2a_2b_1, & a_1b_3a_4b_2, \\ \dots & \dots & \dots \\ a_7b_3a_7^{-1}b_3^{-1}, & a_7b_7a_7b_6^{-1}, & a_7b_9a_7b_8^{-1} \end{array} \right\}.$$

(The complete set of relators can be found in Appendix A.10.)

Proposition 3.27. *Let $\Gamma = \Gamma_{13,17}$ be the $(14, 18)$–group defined in Example 3.26 (actually in Appendix A.10). Then*

(1) $P_h \cong \mathrm{PSL}_2(13) < S_{14}$, $P_v \cong \mathrm{PSL}_2(17) < S_{18}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^3$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_{16}^3$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(3) *Any non-trivial normal subgroup of Γ has finite index.*

(4) $Z_\Gamma(b) = N_\Gamma(\langle b\rangle) = \langle b\rangle$, *if $b \in \{b_4, \dots, b_9\}$.*
$Z_\Gamma(a) = N_\Gamma(\langle a\rangle) = \langle a\rangle$, *if $a \in \{a_1, a_2, a_3, a_4\}$.*

(5) *Let V be the subgroup of $U(\mathbb{H}(\mathbb{Q}))$*

$$V := \langle 1+2i+2j+2k,\ 3+2i,\ 1+4j,\ 3+2i+2j\rangle .$$

Then $\Gamma \cong V/ZV$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1, 8, 13)(2, 9, 4)(3, 6, 14)(7, 12, 11),\\
\rho_v(b_2) &= (1, 10, 11)(2, 7, 14)(3, 4, 8)(5, 13, 12),\\
\rho_v(b_3) &= (1, 9, 12)(2, 3, 10)(4, 5, 14)(6, 11, 13),\\
\rho_v(b_4) &= (1, 4, 8, 3, 13, 5, 10)(2, 11, 7, 12, 14, 6, 9),\\
\rho_v(b_5) &= (1, 8, 13, 4, 9, 6, 3)(2, 12, 5, 10, 11, 14, 7),\\
\rho_v(b_6) &= (1, 2, 9, 4, 12, 7, 8)(3, 13, 6, 11, 14, 5, 10),\\
\rho_v(b_7) &= (1, 4, 5, 10, 2, 12, 9)(3, 6, 14, 13, 8, 7, 11),\\
\rho_v(b_8) &= (1, 3, 10, 2, 11, 6, 9)(4, 12, 5, 13, 14, 7, 8),\\
\rho_v(b_9) &= (1, 10, 11, 3, 8, 7, 2)(4, 13, 6, 9, 12, 14, 5),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1, 5, 17, 3, 12, 18, 2, 9, 16)(4, 14, 15, 6, 7, 13, 8, 10, 11),\\
\rho_h(a_2) &= (1, 6, 3, 2, 14, 18, 16, 11, 17)(4, 5, 15, 9, 8, 10, 7, 13, 12),\\
\rho_h(a_3) &= (1, 7, 16, 17, 15, 18, 3, 8, 2)(4, 14, 10, 11, 9, 6, 12, 13, 5),\\
\rho_h(a_4) &= (1, 3, 10, 17, 18, 13, 16, 2, 4)(5, 8, 9, 11, 12, 6, 7, 14, 15),\\
\rho_h(a_5) &= (2, 8, 3, 10, 17, 11, 16, 9)(4, 14, 6, 12, 5, 15, 7, 13),\\
\rho_h(a_6) &= (1, 7, 16, 13, 18, 12, 3, 6)(4, 5, 9, 11, 14, 15, 8, 10),\\
\rho_h(a_7) &= (1, 4, 2, 14, 18, 15, 17, 5)(6, 7, 8, 9, 12, 13, 10, 11).
\end{aligned}$$

(2) We use GAP ([29]).

(3) We can apply [17, Theorem 4.1] using the results described in [17, Section 2.4] and [16, Section 1.8]. Note that

$$\mathrm{PSL}_2(\mathbb{Q}_{13}) \lneqq H_1 \lneqq \mathrm{PGL}_2(\mathbb{Q}_{13}) \text{ and } \mathrm{PSL}_2(\mathbb{Q}_{17}) \lneqq H_2 \lneqq \mathrm{PGL}_2(\mathbb{Q}_{17})\,,$$

in particular

$$[\mathrm{PGL}_2(\mathbb{Q}_{13}) : H_1] = [H_1 : \mathrm{PSL}_2(\mathbb{Q}_{13})] = 2$$

and

$$[\mathrm{PGL}_2(\mathbb{Q}_{17}) : H_2] = [H_2 : \mathrm{PSL}_2(\mathbb{Q}_{17})] = 2\,.$$

(4) This follows from Proposition 1.12.

(5) Let $\hat{\psi} : V \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$ be the map which sends the quaternion $x = x_0 + x_1 i + x_2 j + x_3 k \in V$ to

$$\left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix}\right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix}\right]\right).$$

It is a group homomorphism such that $\hat{\psi}(x) = \psi(x)$, if $x \in \mathbb{H}(\mathbb{Z}) \cap V$. We have

$$\begin{aligned}\hat{\psi}(V) &= \langle \hat{\psi}(1+2i+2j+2k),\ \hat{\psi}(3+2i),\ \hat{\psi}(1+4j),\ \hat{\psi}(3+2i+2j)\rangle \\ &= \langle \psi(1+2i+2j+2k),\ \psi(3+2i),\ \psi(1+4j),\ \psi(3+2i+2j)\rangle \\ &= \langle a_1, a_5, b_2, b_4\rangle < \Gamma\,.\end{aligned}$$

In fact, GAP ([29]) shows that $[\Gamma : \langle a_1, a_5, b_2, b_4\rangle] = 1$, in other words

$$\langle a_1, a_5, b_2, b_4\rangle = \Gamma\,.$$

Therefore $\Gamma = \hat{\psi}(V) \cong V/\ker(\hat{\psi})$. We claim that $\ker(\hat{\psi}) = ZV$. On the one hand, we have

$$\ker(\hat{\psi}) = \{x \in V : x = \overline{x}\} = V \cap ZU(\mathbb{H}(\mathbb{Q})) < ZV\,.$$

On the other hand, if $x = x_0 + x_1 i + x_2 j + x_3 k \in V < U(\mathbb{H}(\mathbb{Q}))$ commutes both with $3+2i \in V$ and $1+4j \in V$, then $x = \overline{x} \neq 0$, hence $x \in \ker(\hat{\psi})$ and in particular $ZV < \ker(\hat{\psi})$.

□

Note that the only commuting pairs among the standard generators of $\Gamma_{13,17}$ are $\{a_5, b_1\}$, $\{a_6, b_2\}$ and $\{a_7, b_3\}$.

Example: $p = 5, l = 13$

Our second example is $\Gamma = \Gamma_{5,13}$, using the identification

$$\begin{aligned}
a_1 &= \psi(1+2i), & a_1^{-1} &= \psi(1-2i),\\
a_2 &= \psi(1+2j), & a_2^{-1} &= \psi(1-2j),\\
a_3 &= \psi(1+2k), & a_3^{-1} &= \psi(1-2k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1+2i+2j+2k), & b_1^{-1} &= \psi(1-2i-2j-2k),\\
b_2 &= \psi(1+2i+2j-2k), & b_2^{-1} &= \psi(1-2i-2j+2k),\\
b_3 &= \psi(1+2i-2j+2k), & b_3^{-1} &= \psi(1-2i+2j-2k),\\
b_4 &= \psi(1-2i+2j+2k), & b_4^{-1} &= \psi(1+2i-2j-2k),\\
b_5 &= \psi(3+2i), & b_5^{-1} &= \psi(3-2i),\\
b_6 &= \psi(3+2j), & b_6^{-1} &= \psi(3-2j),\\
b_7 &= \psi(3+2k), & b_7^{-1} &= \psi(3-2k).
\end{aligned}$$

Example 3.28.

$$R_{3\cdot 7} := \left\{ \begin{array}{lll} a_1b_1a_3b_6^{-1}, & a_1b_2a_2b_7, & a_1b_3a_2^{-1}b_7^{-1}, \\ a_1b_4a_1b_1^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_6a_3b_3, \\ a_1b_7a_2^{-1}b_4^{-1}, & a_1b_7^{-1}a_2b_1, & a_1b_6^{-1}a_3^{-1}b_2, \\ a_1b_4^{-1}a_3^{-1}b_6, & a_1b_3^{-1}a_1b_2^{-1}, & a_2b_2a_3^{-1}b_5^{-1}, \\ a_2b_3a_2b_1^{-1}, & a_2b_4a_3b_5, & a_2b_5a_3^{-1}b_3^{-1}, \\ a_2b_6a_2^{-1}b_6^{-1}, & a_2b_5^{-1}a_3b_1, & a_2b_4^{-1}a_2b_2^{-1}, \\ a_3b_2a_3b_1^{-1}, & a_3b_7a_3^{-1}b_7^{-1}, & a_3b_4^{-1}a_3b_3^{-1} \end{array} \right\}.$$

Proposition 3.29. *Let $\Gamma = \Gamma_{5,13}$ be the $(6, 14)$–group defined in Example 3.28 and let $G = U(\mathbb{H}(\mathbb{Z}[1/5, 1/13]))/ZU(\mathbb{H}(\mathbb{Z}[1/5, 1/13]))$. Then*

(1) $P_h \cong \mathrm{PGL}_2(5) < S_6$, $P_v \cong \mathrm{PGL}_2(13) < S_{14}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^3$, $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_{16}^3$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(3) *There are finite quotients*

$$\Gamma/\langle\langle b_1^3, b_5^2, (a_1a_2)^3, (b_1b_5)^3\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(3) \cong S_4,$$
$$\textit{such that } \langle\langle b_1^3, b_5^2, (a_1a_2)^3, (b_1b_5)^3\rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}^3.$$

$$\Gamma/\langle\langle a_1^8, (a_1a_2)^3, (a_1b_1)^7, (b_1b_5)^7, (a_1b_1b_5)^6\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(7),$$
$$\textit{such that } \langle\langle a_1^8, (a_1a_2)^3, (a_1b_1)^7, (b_1b_5)^7, (a_1b_1b_5)^6\rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_{14} \times \mathbb{Z}_{56}.$$

$$\Gamma/\langle\langle b_1^4, (b_1b_5)^3, (a_1a_2)^5, (a_1b_1b_5)^5\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(11),$$
$$\Gamma/\langle\langle b_2^9, b_5^8, (a_1a_2)^9, (a_1a_3)^9, (b_2b_6)^8, (a_1b_1b_5)^2\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(17),$$
$$\Gamma/\langle\langle a_1^5, a_2^5, a_3^5, b_5^{20}\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(19),$$
$$\Gamma/\langle\langle b_4^{12}, b_5^3, b_6^3, (b_4b_5)^{11}\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(23),$$
$$\Gamma/\langle\langle a_1^{14}, b_1^5, b_5^7, b_6^7, (a_1b_1)^3\rangle\rangle_\Gamma \cong \mathrm{PSL}_2(29).$$

(4) *We get a finite presentation of G by adding to the presentation*

$$\langle a_1, a_2, a_3, b_1, \ldots, b_7 \mid R_{3\cdot 7}\rangle$$

of Γ two new generators i, j and the relations/relators

$$i^2,\ j^2,\ [i,j],$$
$$[a_1,i],\ a_2 i = i a_2^{-1},\ a_3 i = i a_3^{-1},\ a_1 j = j a_1^{-1},\ [a_2,j],\ a_3 j = j a_3^{-1},$$
$$b_1 i = i b_4^{-1},\ b_2 i = i b_3,\ b_3 i = i b_2,\ [b_5,i],\ b_6 i = i b_6^{-1},\ b_7 i = i b_7^{-1},$$
$$b_1 j = j b_3^{-1},\ b_2 j = j b_4,\ b_4 j = j b_2,\ b_5 j = j b_5^{-1},\ [b_6,j],\ b_7 j = j b_7^{-1},$$

and Γ is then the kernel of the homomorphism

$$\begin{aligned} G &\to \mathbb{Z}_2^2 \\ i &\mapsto (1+2\mathbb{Z}, 0+2\mathbb{Z}) \\ j &\mapsto (0+2\mathbb{Z}, 1+2\mathbb{Z}) \\ a_1, a_2, a_3 &\mapsto (0+2\mathbb{Z}, 0+2\mathbb{Z}) \\ b_1, \dots, b_7 &\mapsto (0+2\mathbb{Z}, 0+2\mathbb{Z})\,. \end{aligned}$$

(5) *For a group H we use the notation $H^{(1)} := [H,H]$, $H^{(2)} := [H^{(1)}, H^{(1)}]$. There is a chain of normal subgroups of G*

$$\Gamma^{(2)} \overset{64}{\lhd} G^{(2)} \overset{16}{\lhd} \Gamma_0^{(1)} \overset{12}{\lhd} \Gamma^{(1)} \overset{8}{\lhd} G^{(1)} \overset{4}{\lhd} \Gamma_0 \overset{4}{\lhd} \Gamma \overset{4}{\lhd} G$$

such that

$$G/\Gamma \cong \Gamma/\Gamma_0 \cong \Gamma_0/G^{(1)} \cong \mathbb{Z}_2^2\,,\ G^{(1)}/\Gamma^{(1)} \cong \mathbb{Z}_2^3\,,\ \Gamma^{(1)}/\Gamma_0^{(1)} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3\,,$$

$G^{ab} \cong \mathbb{Z}_2^6$ *and* $G/\Gamma_0 \cong \mathbb{Z}_2^4$. *It follows for example that $\Gamma^{(2)}$ is a normal subgroup of G of index* $6291456 = 3 \cdot 2^{21}$.

(6) $\Gamma < \mathrm{SO}_3(\mathbb{Q})$ *(illustrating Theorem 3.12(2)).*

(7) $Z_\Gamma(b) = N_\Gamma(\langle b \rangle) = \langle b \rangle$, *if $b \in \{b_1, b_2, b_3, b_4\}$.*

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= (1,6,3,4,2,5), \\ \rho_v(b_2) &= (1,6,2,5,4,3), \\ \rho_v(b_3) &= (1,6,5,2,3,4), \\ \rho_v(b_4) &= (1,2,5,3,4,6), \\ \rho_v(b_5) &= (2,3,5,4), \\ \rho_v(b_6) &= (1,4,6,3), \\ \rho_v(b_7) &= (1,2,6,5), \\ \rho_h(a_1) &= (1,4,7,3,13,9,11,14,8,2,12,6), \\ \rho_h(a_2) &= (1,3,5,2,11,8,12,14,10,4,13,7), \\ \rho_h(a_3) &= (1,2,6,4,12,10,13,14,9,3,11,5). \end{aligned}$$

(2) We use GAP ([29]).

(3) We have used quotpic ([58]) to compute the abelianizations

$$\langle\langle b_1^3, b_5^2, (a_1a_2)^3, (b_1b_5)^3 \rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}^3$$

and

$$\langle\langle a_1^8, (a_1a_2)^3, (a_1b_1)^7, (b_1b_5)^7, (a_1b_1b_5)^6 \rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_{14} \times \mathbb{Z}_{56} .$$

The other statements about the finite quotients of the group Γ are computed by GAP ([29]).

To illustrate Theorem 3.12(3) and (4), the homomorphism $\tau_{2,3} : \Gamma \twoheadrightarrow \mathrm{PGL}_2(7)$ with kernel

$$\langle\langle a_1^8, (a_1a_2)^3, (a_1b_1)^7, (b_1b_5)^7, (a_1b_1b_5)^6 \rangle\rangle_\Gamma$$

is given by

$$a_1 \mapsto \left[\begin{pmatrix} 5+7\mathbb{Z} & 1+7\mathbb{Z} \\ 1+7\mathbb{Z} & 4+7\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 1+7\mathbb{Z} & 2+7\mathbb{Z} \\ 5+7\mathbb{Z} & 1+7\mathbb{Z} \end{pmatrix}\right]$$
$$a_3 \mapsto \left[\begin{pmatrix} 0+7\mathbb{Z} & 4+7\mathbb{Z} \\ 4+7\mathbb{Z} & 2+7\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 4+7\mathbb{Z} & 0+7\mathbb{Z} \\ 3+7\mathbb{Z} & 5+7\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 6+7\mathbb{Z} & 6+7\mathbb{Z} \\ 2+7\mathbb{Z} & 3+7\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 4+7\mathbb{Z} & 3+7\mathbb{Z} \\ 0+7\mathbb{Z} & 5+7\mathbb{Z} \end{pmatrix}\right]$$
$$b_4 \mapsto \left[\begin{pmatrix} 3+7\mathbb{Z} & 5+7\mathbb{Z} \\ 1+7\mathbb{Z} & 6+7\mathbb{Z} \end{pmatrix}\right]$$
$$b_5 \mapsto \left[\begin{pmatrix} 0+7\mathbb{Z} & 1+7\mathbb{Z} \\ 1+7\mathbb{Z} & 6+7\mathbb{Z} \end{pmatrix}\right]$$
$$b_6 \mapsto \left[\begin{pmatrix} 3+7\mathbb{Z} & 2+7\mathbb{Z} \\ 5+7\mathbb{Z} & 3+7\mathbb{Z} \end{pmatrix}\right]$$
$$b_7 \mapsto \left[\begin{pmatrix} 2+7\mathbb{Z} & 4+7\mathbb{Z} \\ 4+7\mathbb{Z} & 4+7\mathbb{Z} \end{pmatrix}\right] .$$

We observe that this homomorphism $\tau_{2,3} : \Gamma \to \mathrm{PGL}_2(7)$ corresponds to the permutation representation in S_8 found by quotpic ([58]):

$$\begin{aligned}
a_1 &\mapsto (1, 5, 7, 2, 4, 6, 3, 8),\\
a_2 &\mapsto (1, 5, 6, 4, 8, 3, 7, 2),\\
a_3 &\mapsto (1, 5, 3, 8, 2, 7, 6, 4),\\
\\
b_1 &\mapsto (2, 6, 4, 3, 8, 7),\\
b_2 &\mapsto (1, 5, 4, 6, 8, 3),\\
b_3 &\mapsto (1, 5, 2, 7, 4, 6),\\
b_4 &\mapsto (1, 5, 8, 3, 2, 7),\\
b_5 &\mapsto (1, 6, 7, 8, 4, 5, 3, 2),\\
b_6 &\mapsto (1, 3, 6, 2, 8, 5, 7, 4),\\
b_7 &\mapsto (1, 7, 3, 4, 2, 5, 6, 8).
\end{aligned}$$

For $q = 29$, we have $\tau_{12,0}(\Gamma) = \mathrm{PSL}_2(29) < \mathrm{PGL}_2(29)$, given by

$$\begin{aligned}
a_1 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 0+29\mathbb{Z} \\ 0+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right]\\
a_2 &\mapsto \left[\begin{pmatrix} 1+29\mathbb{Z} & 2+29\mathbb{Z} \\ 27+29\mathbb{Z} & 1+29\mathbb{Z} \end{pmatrix}\right]\\
a_3 &\mapsto \left[\begin{pmatrix} 1+29\mathbb{Z} & 24+29\mathbb{Z} \\ 24+29\mathbb{Z} & 1+29\mathbb{Z} \end{pmatrix}\right]\\
\\
b_1 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 26+29\mathbb{Z} \\ 22+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right]\\
b_2 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 7+29\mathbb{Z} \\ 3+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right]\\
b_3 &\mapsto \left[\begin{pmatrix} 25+29\mathbb{Z} & 22+29\mathbb{Z} \\ 26+29\mathbb{Z} & 6+29\mathbb{Z} \end{pmatrix}\right]\\
b_4 &\mapsto \left[\begin{pmatrix} 6+29\mathbb{Z} & 26+29\mathbb{Z} \\ 22+29\mathbb{Z} & 25+29\mathbb{Z} \end{pmatrix}\right]\\
b_5 &\mapsto \left[\begin{pmatrix} 27+29\mathbb{Z} & 0+29\mathbb{Z} \\ 0+29\mathbb{Z} & 8+29\mathbb{Z} \end{pmatrix}\right]\\
b_6 &\mapsto \left[\begin{pmatrix} 3+29\mathbb{Z} & 2+29\mathbb{Z} \\ 27+29\mathbb{Z} & 3+29\mathbb{Z} \end{pmatrix}\right]\\
b_7 &\mapsto \left[\begin{pmatrix} 3+29\mathbb{Z} & 24+29\mathbb{Z} \\ 24+29\mathbb{Z} & 3+29\mathbb{Z} \end{pmatrix}\right]
\end{aligned}$$

and kernel $\langle\langle a_1^{14}, b_1^5, b_5^7, b_6^7, (a_1b_1)^3\rangle\rangle_\Gamma$. The choice $c = 17, d = 0$ gives another homomorphism

$$\tau_{17,0} : \Gamma \twoheadrightarrow \mathrm{PSL}_2(29)$$

with kernel $\ker(\tau_{17,0}) = \ker(\tau_{12,0})$.

Note that $q = 29$ is the smallest odd prime number such that $\left(\frac{5}{q}\right) = \left(\frac{13}{q}\right) = 1$, see Table 3.2 (other numbers with this property are for example 61 and 79).

(4) This follows from Theorem 3.12(1). Observe that the generators i and j in the given presentation correspond to

$$\psi(i) = \left(\left[\begin{pmatrix} i_5 & 0 \\ 0 & -i_5 \end{pmatrix}\right], \left[\begin{pmatrix} i_{13} & 0 \\ 0 & -i_{13} \end{pmatrix}\right]\right) \in \mathrm{PGL}_2(\mathbb{Q}_5) \times \mathrm{PGL}_2(\mathbb{Q}_{13})$$

and

$$\psi(j) = \left(\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\right) \in \mathrm{PGL}_2(\mathbb{Q}_5) \times \mathrm{PGL}_2(\mathbb{Q}_{13}),$$

respectively. Note that it would be enough to add the relations/relators

$$\begin{gathered} i^2,\ j^2,\ [i,j], \\ [a_1, i],\ a_1 j = j a_1^{-1},\ [a_2, j],\ a_3 j = j a_3^{-1}, \\ b_1 i = i b_4^{-1},\ [b_5, i],\ b_6 i = i b_6^{-1},\ b_1 j = j b_3^{-1} \end{gathered}$$

in order to get a presentation of the group G.

(5) We have used GAP ([29]), quotpic ([58]) and the presentation of G given in part (4).

(6) The injective group homomorphism $\Gamma \to \mathrm{SO}_3(\mathbb{Q})$ of Theorem 3.12(2) is given by

$$a_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}$$

$$a_2 \mapsto \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix}$$

$$a_3 \mapsto \begin{pmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
b_1 &\mapsto \frac{1}{13}\begin{pmatrix} -3 & 4 & 12 \\ 12 & -3 & 4 \\ 4 & 12 & -3 \end{pmatrix} \\
b_2 &\mapsto \frac{1}{13}\begin{pmatrix} -3 & 12 & -4 \\ 4 & -3 & -12 \\ -12 & -4 & -3 \end{pmatrix} \\
b_3 &\mapsto \frac{1}{13}\begin{pmatrix} -3 & -12 & 4 \\ -4 & -3 & -12 \\ 12 & -4 & -3 \end{pmatrix} \\
b_4 &\mapsto \frac{1}{13}\begin{pmatrix} -3 & -12 & -4 \\ -4 & -3 & 12 \\ -12 & 4 & -3 \end{pmatrix} \\
b_5 &\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5/13 & -12/13 \\ 0 & 12/13 & 5/13 \end{pmatrix} \\
b_6 &\mapsto \begin{pmatrix} 5/13 & 0 & 12/13 \\ 0 & 1 & 0 \\ -12/13 & 0 & 5/13 \end{pmatrix} \\
b_7 &\mapsto \begin{pmatrix} 5/13 & -12/13 & 0 \\ 12/13 & 5/13 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

(7) This follows from Proposition 1.12.

□

See Table 3.6 for the index $[\Gamma : U]$ and the abelianization U^{ab}, where U is of the form $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1, a_2, a_3\}$, $b_j \in \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ and $\Gamma = \Gamma_{5,13}$ is the $(6, 14)$–group defined in Example 3.28:

	b_1, b_2, b_3, b_4	b_5	b_6	b_7
a_1	16, [16, 32]	∞, [0, 0]	96, [16, 32]	96, [16, 32]
a_2	16, [16, 32]	96, [16, 32]	∞, [0, 0]	96, [16, 32]
a_3	16, [16, 32]	96, [16, 32]	96, [16, 32]	∞, [0, 0]

Table 3.6: Index $[\Gamma : U]$ and group U^{ab}, where $U = \langle a_i, b_j \rangle$ in Example 3.28

Observe that $\langle a_1, b_5 \rangle \cong \langle a_2, b_6 \rangle \cong \langle a_3, b_7 \rangle \cong \mathbb{Z}^2$ in $\Gamma_{5,13}$.

3.3 Generalization to $p, l \equiv 3 \pmod 4$

The main goal of this section is to generalize the construction of $\Gamma_{p,l}$ of Section 3.2 to the case where $p \equiv 3 \pmod 4$ and $l \equiv 3 \pmod 4$ are distinct prime numbers. Before giving the ultimate definitions, we discuss some possible approaches. If we just naively define Γ as set

$$\{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p^r l^s;\ r, s \in \mathbb{N}_0\},$$

then we have several problems:

(1) The condition "x has type e_0" is not preserved under quaternion multiplication (for example $(i+j+k)^2 = -3$ has type o_0), so we better define Γ just as group generated by $a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}$, where

$$\begin{aligned} \{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p\} \\ \{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} &= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |y|^2 = l\} \end{aligned}$$

or (as will be explained in (3))

$$\begin{aligned} \{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |x|^2 = p\} \\ \{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} &= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |y|^2 = l\}, \end{aligned}$$

i.e. we get

$$\begin{aligned} \Gamma = \{\psi(x) : & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_0, \text{ if } |x|^2 \equiv 3 \pmod 4, \\ & x \text{ has type } o_0, \text{ if } |x|^2 \equiv 1 \pmod 4\} \\ = \{\psi(x) : & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_0, \text{ if } r+s \text{ is odd}, \\ & x \text{ has type } o_0, \text{ if } r+s \text{ is even}\}, \end{aligned}$$

or

$$\begin{aligned} \Gamma = \{\psi(x) : & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_1, \text{ if } |x|^2 \equiv 3 \pmod 4, \\ & x \text{ has type } o_0, \text{ if } |x|^2 \equiv 1 \pmod 4\} \\ = \{\psi(x) : & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_1, \text{ if } r+s \text{ is odd}, \\ & x \text{ has type } o_0, \text{ if } r+s \text{ is even}\} \end{aligned}$$

for a suitable map ψ, see (2) below.

(2) What is a good definition for ψ? Since now $p, l \equiv 3 \pmod 4$, there are no elements $i_p \in \mathbb{Q}_p$, $i_l \in \mathbb{Q}_l$ anymore such that $i_p^2 + 1 = 0$ and $i_l^2 + 1 = 0$. We have two possibilities to generalize the map ψ of Section 3.2: Either we define

$$\psi : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{PGL}_2(K_p) \times \mathrm{PGL}_2(K_l)\,,$$

where $x = x_0 + x_1 i + x_2 j + x_3 k$ is mapped to

$$\left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

and K_p, K_l are quadratic extensions of $\mathbb{Q}_p$ and $\mathbb{Q}_l$, respectively, containing elements $i_p \in K_p$, $i_l \in K_l$ such that $i_p^2 + 1 = 0$ and $i_l^2 + 1 = 0$, or we define

$$\psi : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)\,,$$

$$\begin{aligned} x \mapsto \Bigg(& \left[\begin{pmatrix} x_0 + x_1 c_p + x_3 d_p & -x_1 d_p + x_2 + x_3 c_p \\ -x_1 d_p - x_2 + x_3 c_p & x_0 - x_1 c_p - x_3 d_p \end{pmatrix} \right], \\ & \left[\begin{pmatrix} x_0 + x_1 c_l + x_3 d_l & -x_1 d_l + x_2 + x_3 c_l \\ -x_1 d_l - x_2 + x_3 c_l & x_0 - x_1 c_l - x_3 d_l \end{pmatrix} \right] \Bigg), \end{aligned}$$

where $c_p, d_p \in \mathbb{Q}_p$, $c_l, d_l \in \mathbb{Q}_l$ are elements satisfying

$$c_p^2 + d_p^2 + 1 = 0 \text{ and } c_l^2 + d_l^2 + 1 = 0\,.$$

Such elements exist since the equation $x^2 + y^2 + 1 = 0$ has solutions in $\mathbb{Z}_p$ and $\mathbb{Z}_l$ (see [23, Proposition 2.5.3]) and then applying Hensel's Lemma. Both constructions of ψ are equivalent in the sense that they will give the same defining relations, hence isomorphic groups Γ. This mainly follows from $\psi(xy) = \psi(x)\psi(y)$ for both ψ. Therefore, we can always choose any of those two definitions of ψ in the following constructions. In practice, we will choose the second one, since we prefer to be inside $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$ as in the classical case of Section 3.2.

(3) If $p \equiv 3 \pmod 8$, then p can be written as a sum of (0 and) three odd squares (by Lemma 3.7(2),(3)). So if we take for example one generator $a_1 := \psi(x)$ such that $x = 0 + x_1 i + x_2 j + x_3 k$ and $|x|^2 = x_1^2 + x_2^2 + x_3^2 = p$, then

$$a_1 = \psi(x) = \psi(-x) = \psi(\bar{x}) = \psi(x)^{-1} = a_1^{-1},$$

i.e. $a_1^2 = 1$ in Γ, in particular the group Γ is not torsion-free and therefore certainly no $(p + 1, l + 1)$–group.

We can easily avoid this problem by changing the type from e_0 to e_1 whenever $p \equiv 3 \pmod 8$ or $l \equiv 3 \pmod 8$:

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |x|^2 = p\}$$
$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |y|^2 = l\}.$$

In the remaining case $p, l \equiv 7 \pmod 8$, we essentially (we could replace e_1 by e_2 or e_3) have two possibilities: Either we again take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |x|^2 = p\}$$
$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |y|^2 = l\},$$

or we take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p\}$$
$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |y|^2 = l\}.$$

These two constructions give different groups (we have different abelianizations in our examples, see the list in Section 3.5), but the groups are quite similar (its intersection has index 2 in both groups).

We always avoid type-mixing constructions, since if x has type e_ι, $|x|^2 = p$ and y has type $e_\kappa \neq e_\iota$, $|y|^2 = l$, then $|xy|^2 = pl \equiv 1 \pmod 4$. Hence, by Lemma 3.7(2), $|xy|^2$ can be written as a sum of three squares (one odd and two even squares). By the following multiplication table (Table 3.7), xy has type o_1, o_2 or o_3, in particular $\Re(xy)$ is even, so it can happen that $\Re(xy) = 0$, but then $xy = -\overline{xy}$, hence $(xy)^2 = xy(-\overline{xy}) \in \mathbb{Z}$ and $(\psi(xy))^2$ is the identity in Γ which implies that Γ is not torsion-free.

$\cdot$	o_0	o_1	o_2	o_3	e_0	e_1	e_2	e_3
o_0	o_0	o_1	o_2	o_3	e_0	e_1	e_2	e_3
o_1	o_1	o_0	o_3	o_2	e_1	e_0	e_3	e_2
o_2	o_2	o_3	o_0	o_1	e_2	e_3	e_0	e_1
o_3	o_3	o_2	o_1	o_0	e_3	e_2	e_1	e_0
e_0	e_0	e_1	e_2	e_3	o_0	o_1	o_2	o_3
e_1	e_1	e_0	e_3	e_2	o_1	o_0	o_3	o_2
e_2	e_2	e_3	e_0	e_1	o_2	o_3	o_0	o_1
e_3	e_3	e_2	e_1	e_0	o_3	o_2	o_1	o_0

Table 3.7: Multiplication table of quaternion types

After those preliminary considerations, we give now the final definitions for ψ and the group $\Gamma_{p,l}$ for this section: Let $p, l \equiv 3 \pmod 4$ be distinct prime numbers, and

$$\psi : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l),$$

mapping the quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$ to

$$\left(\left[\begin{pmatrix} x_0 + x_1 c_p + x_3 d_p & -x_1 d_p + x_2 + x_3 c_p \\ -x_1 d_p - x_2 + x_3 c_p & x_0 - x_1 c_p - x_3 d_p \end{pmatrix} \right], \right.$$
$$\left. \left[\begin{pmatrix} x_0 + x_1 c_l + x_3 d_l & -x_1 d_l + x_2 + x_3 c_l \\ -x_1 d_l - x_2 + x_3 c_l & x_0 - x_1 c_l - x_3 d_l \end{pmatrix} \right] \right),$$

where $c_p, d_p \in \mathbb{Q}_p$, $c_l, d_l \in \mathbb{Q}_l$ are elements such that

$$c_p^2 + d_p^2 + 1 = 0 \text{ and } c_l^2 + d_l^2 + 1 = 0 .$$

Then, we define the group

$$\begin{aligned} \Gamma_{p,l} = \{\psi(x) : \; & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_1, \text{ if } r + s \text{ is odd}, \\ & x \text{ has type } o_0, \text{ if } r + s \text{ is even}\} \\ = \{\psi(x) : \; & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_1, \text{ if } |x|^2 \equiv 3 \pmod 4, \\ & x \text{ has type } o_0, \text{ if } |x|^2 \equiv 1 \pmod 4\} , \end{aligned}$$

with subsets

$$\begin{aligned} E_h &:= \{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |x|^2 = p\} \\ E_v &:= \{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |y|^2 = l\} . \end{aligned}$$

In the subcase $p, l \equiv 7 \pmod 8$, we additionally define the group

$$\begin{aligned} \Gamma_{p,l,e_0} = \{\psi(x) : \; & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_0, \text{ if } r + s \text{ is odd}, \\ & x \text{ has type } o_0, \text{ if } r + s \text{ is even}\} \\ = \{\psi(x) : \; & x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0, \\ & x \text{ has type } e_0, \text{ if } |x|^2 \equiv 3 \pmod 4, \\ & x \text{ has type } o_0, \text{ if } |x|^2 \equiv 1 \pmod 4\} , \end{aligned}$$

with corresponding subsets

$$\begin{aligned} E_h &:= \{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p\} \\ E_v &:= \{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |y|^2 = l\} . \end{aligned}$$

Our next goal is to prove that $\Gamma_{p,l}$ and Γ_{p,l,e_0} are $(p+1, l+1)$–groups.

Theorem 3.30. *Let Γ be either the group $\Gamma_{p,l}$, where $p, l \equiv 3 \pmod 4$, or let Γ be the group Γ_{p,l,e_0}, where $p, l \equiv 7 \pmod 8$. In the first case, let*

$$\begin{aligned}
\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\} &= \{x \in \mathbb{H}(\mathbb{Z}) \text{ of type } e_1 : |x|^2 = p,\ \Re(x) > 0\}\\
\{\beta_1, \dots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \dots, \overline{\beta_1}\} &= \{y \in \mathbb{H}(\mathbb{Z}) \text{ of type } e_1 : |y|^2 = l,\ \Re(y) > 0\}\\
E_h &= \psi\big(\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}\big) = \{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1}\\
E_v &= \psi\big(\{\beta_1, \dots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \dots, \overline{\beta_1}\}\big) = \{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1}.
\end{aligned}$$

In the second case, we take the same definitions, but replace e_1 by e_0.

A word in $\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}$ is called reduced, *if it has no subword of the form $\alpha_i \overline{\alpha_i}$ or $\overline{\alpha_i}\alpha_i$. A reduced word in $\{\beta_1, \dots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \dots, \overline{\beta_1}\}$ is defined analogously. Then in both cases the following statements hold.*

(1) *Any quaternion $x \in \mathbb{H}(\mathbb{Z})$ such that $|x|^2 = p^r l^s$; $r, s \in \mathbb{N}_0$, can be uniquely expressed in the form*

$$x = \varepsilon p^{r_1} l^{s_1} w_{r_2}(\alpha) w_{s_2}(\beta),$$

where

- *$\varepsilon \in \mathbb{H}(\mathbb{Z})$ is a unit, i.e. $\varepsilon \in \{\pm 1, \pm i, \pm j, \pm k\}$*
- *$r_1, r_2, s_1, s_2 \in \mathbb{N}_0$ such that $2r_1 + r_2 = r$ and $2s_1 + s_2 = s$*
- *$w_{r_2}(\alpha)$ is a reduced word in $\{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\}$ of length r_2*
- *$w_{s_2}(\beta)$ is a reduced word in $\{\beta_1, \dots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \dots, \overline{\beta_1}\}$ of length s_2.*

(2) *The group Γ is generated by the set $\{a_1, \dots, a_{\frac{p+1}{2}}, b_1, \dots, b_{\frac{l+1}{2}}\}$, i.e. by the set $\{\psi(\alpha_1), \dots, \psi(\alpha_{\frac{p+1}{2}}), \psi(\beta_1), \dots, \psi(\beta_{\frac{l+1}{2}})\}$.*

(3) *To any pair $a \in E_h$, $b \in E_v$, there are unique elements $\tilde{a} \in E_h$, $\tilde{b} \in E_v$ such that $ba = \tilde{a}\tilde{b}$.*

(4) *The group Γ is torsion-free.*

(5) *The group Γ is a $(p+1, l+1)$–group.*

Proof. (1) We follow the strategy of the proof of [45, Lemma 2.1.9], see also the proof of [23, Theorem 2.6.13].

Existence: By Proposition 3.10, we can write

$$x = y^{(1)} \dots y^{(r)} z^{(1)} \dots z^{(s)}$$

such that $y^{(\iota)}, z^{(\kappa)} \in \mathbb{H}(\mathbb{Z})$, $|y^{(\iota)}|^2 = p$ and $|z^{(\kappa)}|^2 = l$, where $\iota = 1, \ldots, r$ and $\kappa = 1, \ldots, s$. Observe that all quaternions $y^{(\iota)}, z^{(\kappa)}$ have type e by the assumption $p, l \equiv 3 \pmod 4$. Multiplying $y^{(\iota)}, z^{(\kappa)}$ with suitable units, we can achieve that x has the form

$$x = \varepsilon y^{(1)} \ldots y^{(r)} z^{(1)} \ldots z^{(s)} ,$$

such that $\varepsilon \in \mathbb{H}(\mathbb{Z})$ is a unit, $y^{(\iota)}, z^{(\kappa)} \in \mathbb{H}(\mathbb{Z})$ have type e_1, and $\Re(y^{(\iota)}) > 0$, $\Re(z^{(\kappa)}) > 0$; or we can achieve that $y^{(\iota)}, z^{(\kappa)}$ have type e_0 instead of type e_1. We get the desired expression if we replace all subwords

$$y^{(\iota)} y^{(\iota+1)} = y^{(\iota)} \overline{y^{(\iota)}}$$

by $p = |y^{(\iota)}|^2$, and all subwords

$$z^{(\kappa)} z^{(\kappa+1)} = z^{(\kappa)} \overline{z^{(\kappa)}}$$

by $l = |z^{(\kappa)}|^2$.

Uniqueness: We adapt the counting argument given in [45, Lemma 2.1.9]. The number of reduced words $w_{r_2}(\alpha)$ is

$$\begin{cases} (p+1)p^{r_2-1}\,, & \text{if } r_2 \geq 1 \\ 1\,, & \text{if } r_2 = 0\,. \end{cases}$$

Similarly as in [45], it follows that the number of expressions

$$\varepsilon p^{r_1} l^{s_1} w_{r_2}(\alpha) w_{s_2}(\beta)$$

is

$$8\big(1 + p + \cdots + p^r\big)\big(1 + l + \cdots + l^s\big) = 8 \sum_{d \mid p^r l^s} d\,,$$

which is also the number of quaternions $x \in \mathbb{H}(\mathbb{Z})$, such that $|x|^2 = p^r l^s$ by the Jacobi Theorem (see for example [45, Theorem 2.1.8] for a formulation and a proof of the Jacobi Theorem).

(2) Let $x \in \mathbb{H}(\mathbb{Z})$ be a quaternion of norm $|x|^2 = p^r l^s$; $r, s \in \mathbb{N}_0$. By part (1), we can write

$$x = \varepsilon p^{r_1} l^{s_1} w_{r_2}(\alpha) w_{s_2}(\beta)\,.$$

Assume that we are in the first case $\Gamma = \Gamma_{p,l}$. If x has type e_1 and $r + s$ is odd, then

$$r_2 + s_2 = r + s - 2(r_1 + s_1)$$

is odd. By Table 3.7, the quaternion $w_{r_2}(\alpha)w_{s_2}(\beta)$ has type e_1, hence ε has type o_0, i.e. $\varepsilon \in \{-1, 1\}$ and it follows

$$\psi(x) = \psi(\pm p^{r_1} l^{s_1} w_{r_2}(\alpha)w_{s_2}(\beta)) = \psi(w_{r_2}(\alpha)w_{s_2}(\beta)).$$

If x has type o_0 and $r+s$ is even, then r_2+s_2 is even, $w_{r_2}(\alpha)w_{s_2}(\beta)$ has type o_0, again $\varepsilon \in \{-1, 1\}$ and $\psi(x) = \psi(w_{r_2}(\alpha)w_{s_2}(\beta))$.

The proof in the second case $\Gamma = \Gamma_{p,l,e_0}$ is completely analogous, we only have to substitute e_1 by e_0 everywhere.

(3) Write $a = \psi(\alpha)$ and $b = \psi(\beta)$ for some

$$\alpha \in \{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\} \text{ and } \beta \in \{\beta_1, \dots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \dots, \overline{\beta_1}\}.$$

The quaternion $\beta\alpha$ has type o_0 and norm $|\beta\alpha|^2 = pl$. By part (1), it can be expressed as $\beta\alpha = \varepsilon\tilde{\alpha}\tilde{\beta}$ with a uniquely determined unit ε and uniquely determined quaternions

$$\tilde{\alpha} \in \{\alpha_1, \dots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \dots, \overline{\alpha_1}\} \text{ and } \tilde{\beta} \in \{\beta_1, \dots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \dots, \overline{\beta_1}\}.$$

Since $\tilde{\alpha}\tilde{\beta}$ has type o_0, the unit ε also has type o_0, i.e. $\varepsilon \in \{-1, 1\}$ and we conclude

$$ba = \psi(\beta)\psi(\alpha) = \psi(\beta\alpha) = \psi(\varepsilon\tilde{\alpha}\tilde{\beta}) = \psi(\tilde{\alpha}\tilde{\beta}) = \psi(\tilde{\alpha})\psi(\tilde{\beta}) =: \tilde{a}\tilde{b}.$$

(4) We adapt the proof given in [54, Proposition 3.6]. Let $\psi(x)$ be a non-trivial element in Γ. Assume that $\psi(x)^k = 1$ for some $k \in \mathbb{N}$. Then there is an element $\mu \in \mathbb{Q}_p^\times$ such that

$$\begin{pmatrix} x_0 + x_1 c_p + x_3 d_p & -x_1 d_p + x_2 + x_3 c_p \\ -x_1 d_p - x_2 + x_3 c_p & x_0 - x_1 c_p - x_3 d_p \end{pmatrix}^k = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p),$$

hence $\mu = \lambda_1^k = \lambda_2^k$, where λ_1, λ_2 are the two eigenvalues

$$\lambda_{1,2} = x_0 \pm \sqrt{-x_1^2 - x_2^2 - x_3^2}$$

of the matrix

$$\begin{pmatrix} x_0 + x_1 c_p + x_3 d_p & -x_1 d_p + x_2 + x_3 c_p \\ -x_1 d_p - x_2 + x_3 c_p & x_0 - x_1 c_p - x_3 d_p \end{pmatrix},$$

using the identity $c_p^2 + d_p^2 + 1 = 0$ in $\mathbb{Q}_p$. We write

$$v := x_1^2 + x_2^2 + x_3^2 \in \mathbb{N}, \ \lambda_1 = x_0 + \sqrt{-v} \text{ and } \lambda_2 = x_0 - \sqrt{-v}.$$

By construction of $\Gamma_{p,l}$ and Γ_{p,l,e_0}, there are only three possible types for the quaternion x.

Case 1: x has type o_0, in particular x_0 is odd and v is positive even.

Case 2: x has type e_1, and again x_0 is odd and v is positive even.

Case 3: x has type e_0 such that $|x|^2 \equiv 7 \pmod 8$, in particular x_0 is non-zero even and v is positive odd.

We will use the following facts which hold in all three cases:

$$v \neq 0,\ x_0 \neq 0,\ 3x_0^2 - v \neq 0,\ x_0^2 - v \neq 0 \text{ and } x_0^2 - 3v \neq 0\,.$$

They follow directly looking at the parity. Since λ_1/λ_2 belongs to a quadratic extension of $\mathbb{Q}$, and $(\lambda_1/\lambda_2)^k = 1$, we can conclude that $k \in \{1, 2, 3, 4, 6\}$. But

- $k \neq 1$, since $\lambda_1 - \lambda_2 = 2\sqrt{-v} \neq 0$
- $k \neq 2$, since $\lambda_1^2 - \lambda_2^2 = 4x_0\sqrt{-v} \neq 0$
- $k \neq 3$, since $\lambda_1^3 - \lambda_2^3 = 2\sqrt{-v}\,(3x_0^2 - v) \neq 0$
- $k \neq 4$, since $\lambda_1^4 - \lambda_2^4 = 8x_0\sqrt{-v}\,(x_0^2 - v) \neq 0$
- $k \neq 6$, since $\lambda_1^6 - \lambda_2^6 = 4x_0\sqrt{-v}\,(x_0^2 - 3v)(3x_0^2 - v) \neq 0$.

It follows that $\psi(x)^k \neq 1$ and Γ is torsion-free.

(5) By part (2), the group Γ is generated by its subset

$$\{a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}\}\,,$$

and by part (3) there are $(p+1)(l+1)$ relators of the form $\tilde{a}\tilde{b}a^{-1}b^{-1}$, where $a, \tilde{a} \in E_h$ and $b, \tilde{b} \in E_v$. These $(p+1)(l+1)$ relators are represented by exactly $(p+1)(l+1)/4$ relators $\tilde{a}\tilde{b}a^{-1}b^{-1}$ (geometric squares $[\tilde{a}\tilde{b}a^{-1}b^{-1}]$), if and only if the four squares

$$\tilde{a}\tilde{b}a^{-1}b^{-1},\ a^{-1}b^{-1}\tilde{a}\tilde{b},\ \tilde{a}^{-1}ba\tilde{b}^{-1},\ a\tilde{b}^{-1}\tilde{a}^{-1}b$$

are always distinct, i.e. if and only if there are no $a \in E_h$, $b \in E_v$ such that $abab = 1$. We want to exclude such "projective planes", so let us assume that $abab = 1$ for some $a \in E_h$, $b \in E_v$. Since Γ is torsion-free by part (4), it follows that $ab = 1$, hence $\psi(\alpha\beta) = 1$ for some

$$\alpha \in \{\alpha_1, \ldots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \ldots, \overline{\alpha_1}\} \text{ and } \beta \in \{\beta_1, \ldots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \ldots, \overline{\beta_1}\}\,.$$

This implies (looking at the two eigenvalues λ_1, λ_2 of part (4) which have to be equal here) that $\alpha\beta = \Re(\alpha\beta) \in \mathbb{Z}$, contradicting $|\alpha\beta|^2 = pl$. We conclude that Γ is a quotient of a $(p+1, l+1)$–group

$$\langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}} \rangle\,.$$

This quotient is not proper (i.e. Γ is exactly this $(p+1, l+1)$–group), if and only if any non-trivial relation which holds in Γ is a consequence of the square relations $ba = \tilde{a}\tilde{b}$ of part (3). So we assume that w is any relator in Γ, i.e. any word in the generators

$$\{a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1}$$

which represents the identity in Γ. Then, gradually using part (3), i.e. replacing every ba by the corresponding $\tilde{a}\tilde{b}$, and cancelling all subwords of the form

$$a_i a_i^{-1},\ a_i^{-1} a_i,\ b_j b_j^{-1},\ b_j^{-1} b_j\,,$$

either w cancels to 1, which means that w is a consequence of the defining relators in $R_{\frac{p+1}{2},\frac{l+1}{2}}$ and we are done, or w is represented by an element in Γ of the form $a^{(1)} \ldots a^{(r)} b^{(1)} \ldots b^{(s)}$, where $(r, s) \neq (0, 0)$, such that $a^{(1)} \ldots a^{(r)}$ and $b^{(1)} \ldots b^{(s)}$ are freely reduced words in $\langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle$ and $\langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle$, respectively. Therefore,

$$\psi(\alpha^{(1)} \ldots \alpha^{(r)} \beta^{(1)} \ldots \beta^{(s)}) = 1$$

for some

$$\alpha^{(1)}, \ldots, \alpha^{(r)} \in \{\alpha_1, \ldots, \alpha_{\frac{p+1}{2}}, \overline{\alpha_{\frac{p+1}{2}}}, \ldots, \overline{\alpha_1}\}$$

and

$$\beta^{(1)}, \ldots, \beta^{(s)} \in \{\beta_1, \ldots, \beta_{\frac{l+1}{2}}, \overline{\beta_{\frac{l+1}{2}}}, \ldots, \overline{\beta_1}\}\,,$$

where $\alpha^{(1)} \ldots \alpha^{(r)}$ and $\beta^{(1)} \ldots \beta^{(s)}$ are reduced words. This implies

$$\alpha^{(1)} \ldots \alpha^{(r)} \beta^{(1)} \ldots \beta^{(s)} = \Re(\alpha^{(1)} \ldots \alpha^{(r)} \beta^{(1)} \ldots \beta^{(s)}) =: x_0 \in \mathbb{Z}\,.$$

Taking the norm of the last expression, we get $p^r l^s = x_0^2$, hence r, s are even and

$$x_0 = \pm p^{r/2} l^{s/2},$$

which contradicts the uniqueness statement of part (1) for the quaternion

$$\alpha^{(1)} \ldots \alpha^{(r)} \beta^{(1)} \ldots \beta^{(s)} = \pm p^{r/2} l^{s/2}\,.$$

□

In both constructions of $\Gamma = \Gamma_{p,l}$ and $\Gamma = \Gamma_{p,l,e_0}$, we have

$$\Gamma_0 = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r} l^{2s}; r, s \in \mathbb{N}_0\} < \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l)$$

as in Section 3.2. Note that in the case $p, l \equiv 7 \pmod 8$, the common subgroup Γ_0 has index 2 in $\Gamma_{p,l} \cap \Gamma_{p,l,e_0}$.

We describe now (or in Appendix A) several explicit examples for the three cases $p, l \equiv 7 \pmod 8$, $p, l \equiv 3 \pmod 8$ and $p \equiv 3 \pmod 8$, $l \equiv 7 \pmod 8$, where the first case is again divided into the type e_1 and type e_0 subcase:

Case $p, l \equiv 7 \pmod 8$, type e_1

Let $p, l \equiv 7 \pmod 8$ be distinct prime numbers. Here, we take $\{a_1, \ldots, a_{\frac{p+1}{2}}\}$ to be the set

$$\{\psi(x) : x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ x_0, x_1 > 0,\ |x|^2 = p\},$$

and take $\{b_1, \ldots, b_{\frac{l+1}{2}}\}$ to be the set

$$\{\psi(y) : y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ y_0, y_1 > 0,\ |y|^2 = l\}.$$

See Appendix A.7 for the explicit definition of the group $\Gamma = \Gamma_{7,23}$. It has for example the following properties:

$$P_h \cong \mathrm{PSL}_2(7) < S_8,\ P_v \cong \mathrm{PGL}_2(23) < S_{24}.$$
$$\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2,\ [\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64},\ \Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2.$$

In Appendix A.8 is the explicit definition of $\Gamma = \Gamma_{7,31}$. We have computed

$$P_h \cong \mathrm{PGL}_2(7) < S_8,\ P_v \cong \mathrm{PSL}_2(31) < S_{32}.$$
$$\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ [\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64},\ \Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2.$$

Case $p, l \equiv 7 \pmod 8$, type e_0

Again, let $p, l \equiv 7 \pmod 8$ be distinct prime numbers, but now we take

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p\}$$

and

$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |y|^2 = l\}.$$

As an example, the group $\Gamma = \Gamma_{7,23,e_0}$ is explicitly defined in Appendix A.9, and we have

$$P_h \cong \mathrm{PSL}_2(7) < S_8,\ P_v \cong \mathrm{PGL}_2(23) < S_{24}.$$
$$\Gamma^{ab} \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4,\ [\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2,\ \Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2.$$

Note that $(\Gamma_{7,23,e_0})^{ab} \neq (\Gamma_{7,23})^{ab}$, in particular the groups $\Gamma_{7,23,e_0}$ and $\Gamma_{7,23}$ are not isomorphic.

Case $p, l \equiv 3 \pmod 8$

Let $p, l \equiv 3 \pmod 8$ be distinct prime numbers. We give the example $\Gamma_{3,11}$, taking

$$\begin{aligned}
a_1 &= \psi(1+j+k), & a_1^{-1} &= \psi(1-j-k),\\
a_2 &= \psi(1+j-k), & a_2^{-1} &= \psi(1-j+k),\\[1em]
b_1 &= \psi(1+j+3k), & b_1^{-1} &= \psi(1-j-3k),\\
b_2 &= \psi(1+j-3k), & b_2^{-1} &= \psi(1-j+3k),\\
b_3 &= \psi(1+3j+k), & b_3^{-1} &= \psi(1-3j-k),\\
b_4 &= \psi(1+3j-k), & b_4^{-1} &= \psi(1-3j+k),\\
b_5 &= \psi(3+j+k), & b_5^{-1} &= \psi(3-j-k),\\
b_6 &= \psi(3+j-k), & b_6^{-1} &= \psi(3-j+k).
\end{aligned}$$

Example 3.31.

$$R_{2\cdot 6} := \left\{ \begin{array}{ll}
a_1b_1a_1b_6^{-1}, & a_1b_2a_1b_4^{-1},\\[1em]
a_1b_3a_1b_6, & a_1b_4a_2^{-1}b_3^{-1},\\[1em]
a_1b_5a_1^{-1}b_5^{-1}, & a_1b_3^{-1}a_2^{-1}b_4,\\[1em]
a_1b_2^{-1}a_2b_1^{-1}, & a_1b_1^{-1}a_2b_2^{-1},\\[1em]
a_2b_1a_2b_3^{-1}, & a_2b_2a_2b_5^{-1},\\[1em]
a_2b_4a_2b_5, & a_2b_6a_2^{-1}b_6^{-1}
\end{array} \right\}.$$

Proposition 3.32. *Let $\Gamma = \Gamma_{3,11}$ be the $(4, 12)$–group defined in Example 3.31. Then*

(1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v \cong \mathrm{PSL}_2(11) < S_{12}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_8^2 \times \mathbb{Z}_{64}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) = \rho_v(b_2) &= (1,3,2,4),\\
\rho_v(b_3) &= (1,2,3,4),\\
\rho_v(b_4) &= (1,4,3,2),\\
\rho_v(b_5) &= (2,3),\\
\rho_v(b_6) &= (1,4),
\end{aligned}$$

$$\rho_h(a_1) = (1, 11, 9, 10, 6)(2, 12, 7, 3, 4),$$
$$\rho_h(a_2) = (1, 11, 8, 4, 3)(2, 12, 10, 9, 5).$$

(2) GAP ([29]).

□

See Table 3.8 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient Γ/U (if U is normal in Γ), where $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1, a_2\}$ and $b_j \in \{b_1, \dots, b_6\}$.

	b_1, b_3	b_2, b_4	b_5	b_6
a_1	$2, [8, 8], \mathbb{Z}_2$	$8, [8, 32], -$	$\infty, [0, 0], -$	$2, [8, 8], \mathbb{Z}_2$
a_2	$8, [8, 32], -$	$2, [8, 8], \mathbb{Z}_2$	$2, [8, 8], \mathbb{Z}_2$	$\infty, [0, 0], -$

Table 3.8: $[\Gamma : U]$, U^{ab} and Γ/U in Example 3.31, where $U = \langle a_i, b_j \rangle$

Case $p \equiv 3 \pmod 8$, $l \equiv 7 \pmod 8$

Let $p \equiv 3 \pmod 8$, $l \equiv 7 \pmod 8$ be prime numbers, We construct the group $\Gamma_{3,7}$ as follows:

$$\begin{aligned} a_1 &= \psi(1 + j + k), & a_1^{-1} &= \psi(1 - j - k), \\ a_2 &= \psi(1 + j - k), & a_2^{-1} &= \psi(1 - j + k), \end{aligned}$$

$$\begin{aligned} b_1 &= \psi(1 + 2i + j + k), & b_1^{-1} &= \psi(1 - 2i - j - k), \\ b_2 &= \psi(1 + 2i + j - k), & b_2^{-1} &= \psi(1 - 2i - j + k), \\ b_3 &= \psi(1 + 2i - j + k), & b_3^{-1} &= \psi(1 - 2i + j - k), \\ b_4 &= \psi(1 + 2i - j - k), & b_4^{-1} &= \psi(1 - 2i + j + k). \end{aligned}$$

Example 3.33.

$$R_{2\cdot 4} := \left\{ \begin{array}{ll} a_1b_1a_2^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_3, \\ a_1b_3a_2^{-1}b_4^{-1}, & a_1b_4a_1b_1^{-1}, \\ a_1b_4^{-1}a_2b_2, & a_1b_3^{-1}a_2b_1, \\ a_2b_3a_2b_2^{-1}, & a_2b_4a_2^{-1}b_1 \end{array} \right\}.$$

Proposition 3.34. *Let $\Gamma = \Gamma_{3,7}$ be the $(4, 8)$–group defined in Example 3.33. Then*

(1) $P_h \cong \mathrm{PSL}_2(3) \cong A_4$, $P_v \cong \mathrm{PGL}_2(7) < S_8$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.

(3) *We have a quotient* $\Gamma/\langle\langle a_1^6, b_1^4, (a_1b_1)^5, (b_1b_2)^5\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(5) \cong S_5$, *such that* $\langle\langle a_1^6, b_1^4, (a_1b_1)^5, (b_1b_2)^5\rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}^3$, *and quotients*

$$\Gamma/\langle\langle a_1^5, (a_1b_1)^{12}, (b_1b_2)^5\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(11),$$
$$\Gamma/\langle\langle a_1^7, (a_1b_1)^{14}, (b_1b_2)^3\rangle\rangle_\Gamma \cong \mathrm{PGL}_2(13).$$

(4) *The group* $U(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))$ *has a presentation with generators* $a_1, a_2, b_1, b_2, b_3, b_4, i, j$ *and relators*

$$R_{2\cdot 4},\ a_1ia_1i^{-1},\ a_1ja_2^{-1}j^{-1},\ b_1ib_4^{-1}i^{-1},\ b_1jb_3j^{-1},\ i^2,\ j^2,\ [i, j].$$

(5) $(U(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/7])))^{ab} \cong \mathbb{Z}_2^4$.

(6) $(U(\mathbb{H}(\mathbb{Z}[1/3, 1/7]))/ZU(\mathbb{H}(\mathbb{Z}[1/3, 1/7])))/\Gamma_0 \cong \mathbb{Z}_2^4$.

(7) $\mathrm{Aut}(X) \cong D_4$.

(8) $\langle a_2^2a_1^2, b_2^{-1}b_3b_4b_1^{-1}\rangle \cong \mathbb{Z}^2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1, 4, 3),\\
\rho_v(b_2) &= (1, 2, 3),\\
\rho_v(b_3) &= (2, 4, 3),\\
\rho_v(b_4) &= (1, 2, 4),\\
\rho_h(a_1) &= (1, 4, 3, 7, 5, 8, 6, 2),\\
\rho_h(a_2) &= (1, 5, 6, 7, 8, 4, 2, 3).
\end{aligned}$$

(2) GAP ([29]).

(3) Let q be an odd prime number distinct from p and l, and choose $c, d \in \mathbb{Z}$ such that $c^2 + d^2 + 1 \equiv 0 \pmod q$, then we can define exactly as described in Theorem 3.12(3) a homomorphism $\tau = \tau_{c,d} : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ by

$$\tau_{c,d}(\gamma) = \left[\begin{pmatrix} x_0 + x_1c + x_3d + q\mathbb{Z} & -x_1d + x_2 + x_3c + q\mathbb{Z} \\ -x_1d - x_2 + x_3c + q\mathbb{Z} & x_0 - x_1c - x_3d + q\mathbb{Z} \end{pmatrix}\right],$$

where $\gamma = \psi(x_0 + x_1i + x_2j + x_3k)$.

For $q = 5$ we have $\tau_{0,2} : \Gamma_{3,7} \twoheadrightarrow \mathrm{PGL}_2(5)$ given by

$$a_1 \mapsto \left[\begin{pmatrix} 3+5\mathbb{Z} & 1+5\mathbb{Z} \\ 4+5\mathbb{Z} & 4+5\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 4+5\mathbb{Z} & 1+5\mathbb{Z} \\ 4+5\mathbb{Z} & 3+5\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 3+5\mathbb{Z} & 2+5\mathbb{Z} \\ 0+5\mathbb{Z} & 4+5\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 4+5\mathbb{Z} & 2+5\mathbb{Z} \\ 0+5\mathbb{Z} & 3+5\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 3+5\mathbb{Z} & 0+5\mathbb{Z} \\ 2+5\mathbb{Z} & 4+5\mathbb{Z} \end{pmatrix}\right]$$
$$b_4 \mapsto \left[\begin{pmatrix} 4+5\mathbb{Z} & 0+5\mathbb{Z} \\ 2+5\mathbb{Z} & 3+5\mathbb{Z} \end{pmatrix}\right].$$

We have used quotpic ([58]) to show that

$$\langle\!\langle a_1^6, b_1^4, (a_1b_1)^5, (b_1b_2)^5 \rangle\!\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}^3.$$

In the same way $\tau_{1,3} : \Gamma_{3,7} \twoheadrightarrow \mathrm{PGL}_2(11)$ is defined by

$$a_1 \mapsto \left[\begin{pmatrix} 4+11\mathbb{Z} & 2+11\mathbb{Z} \\ 0+11\mathbb{Z} & 9+11\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 9+11\mathbb{Z} & 0+11\mathbb{Z} \\ 9+11\mathbb{Z} & 4+11\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 6+11\mathbb{Z} & 7+11\mathbb{Z} \\ 5+11\mathbb{Z} & 7+11\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 0+11\mathbb{Z} & 5+11\mathbb{Z} \\ 3+11\mathbb{Z} & 2+11\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 6+11\mathbb{Z} & 5+11\mathbb{Z} \\ 7+11\mathbb{Z} & 7+11\mathbb{Z} \end{pmatrix}\right]$$
$$b_4 \mapsto \left[\begin{pmatrix} 0+11\mathbb{Z} & 3+11\mathbb{Z} \\ 5+11\mathbb{Z} & 2+11\mathbb{Z} \end{pmatrix}\right]$$

and $\tau_{0,5} : \Gamma_{3,7} \twoheadrightarrow \mathrm{PGL}_2(13)$ by

$$a_1 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 1+13\mathbb{Z} \\ 12+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 1+13\mathbb{Z} \\ 12+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 4+13\mathbb{Z} \\ 2+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 4+13\mathbb{Z} \\ 2+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 6+13\mathbb{Z} & 2+13\mathbb{Z} \\ 4+13\mathbb{Z} & 9+13\mathbb{Z} \end{pmatrix}\right]$$
$$b_4 \mapsto \left[\begin{pmatrix} 9+13\mathbb{Z} & 2+13\mathbb{Z} \\ 4+13\mathbb{Z} & 6+13\mathbb{Z} \end{pmatrix}\right].$$

(4) Same idea as in Proposition 3.29(4) using that the group

$$U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))/ZU(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))$$

can be described as

$$\{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0\}.$$

(5) and (6) follow from part (4) using GAP ([29]).

(7) GAP ([29]). The group $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$(a_1, a_2, b_1, b_2, b_3, b_4) \mapsto (a_1, a_2^{-1}, b_4^{-1}, b_2^{-1}, b_3^{-1}, b_1^{-1}),$$
$$(a_1, a_2, b_1, b_2, b_3, b_4) \mapsto (a_2, a_1^{-1}, b_2, b_4, b_1, b_3).$$

(8) This follows from Lemma 3.14, since the two elements $a_2^2 a_1^2 = \psi(1 + 8i - 4j)$ and $b_2^{-1} b_3 b_4 b_1^{-1} = \psi(41 - 24i + 12j)$ commute.

□

See Table 3.9 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient Γ/U, where $U = \langle a_i, b_j \rangle$, $a_i \in \{a_1, a_2\}$, $b_j \in \{b_1, b_2, b_3, b_4\}$.

	b_1, b_4	b_2, b_3
a_1	4, [8, 16], $\mathbb{Z}_4$	2, [8, 8], $\mathbb{Z}_2$
a_2	2, [8, 8], $\mathbb{Z}_2$	4, [8, 16], $\mathbb{Z}_4$

Table 3.9: $[\Gamma : U]$, U^{ab} and Γ/U in Example 3.33, where $U = \langle a_i, b_j \rangle$

3.4 Mixed examples: $p \equiv 3, l \equiv 1 \pmod 4$

Let $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$ be two prime numbers. Similarly as in Section 3.2 or Section 3.3, we define a map

$$\psi : \mathbb{H}(\mathbb{Z}) \setminus \{0\} \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)\,,$$

which sends $x = x_0 + x_1 i + x_2 j + x_3 k$ to

$$\left(\left[\begin{pmatrix} x_0 + x_1 c_p + x_3 d_p & -x_1 d_p + x_2 + x_3 c_p \\ -x_1 d_p - x_2 + x_3 c_p & x_0 - x_1 c_p - x_3 d_p \end{pmatrix} \right], \right.$$
$$\left. \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

where $c_p, d_p \in \mathbb{Q}_p$, $i_l \in \mathbb{Q}_l$ are elements such that $c_p^2 + d_p^2 + 1 = 0$ and $i_l^2 + 1 = 0$. Then we construct groups $\Gamma_{p,l}$ generated by

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ |x|^2 = p\}$$
$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l\}\,,$$

that is

$$\begin{aligned} \Gamma_{p,l} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0,& \\ x \text{ has type } e_1, \text{ if } |x|^2 \equiv 3 \pmod 4,& \\ x \text{ has type } o_0, \text{ if } |x|^2 \equiv 1 \pmod 4\}& \\ = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0,& \\ x \text{ has type } e_1, \text{ if } r \text{ is odd},& \\ x \text{ has type } o_0, \text{ if } r \text{ is even}\}\,,& \end{aligned}$$

and, in the subcase $p \equiv 7 \pmod 8$, $l \equiv 1 \pmod 8$, also groups Γ_{p,l,e_0} generated by

$$\{a_1, \ldots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ |x|^2 = p\}$$
$$\{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l\}\,,$$

i.e. Γ_{p,l,e_0} is defined as

$$\begin{aligned} \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0,& \\ x \text{ has type } e_0, \text{ if } |x|^2 \equiv 7 \pmod 8,& \\ x \text{ has type } o_0, \text{ if } |x|^2 \equiv 1 \pmod 8\}& \\ = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0,& \\ x \text{ has type } e_0, \text{ if } r \text{ is odd},& \\ x \text{ has type } o_0, \text{ if } r \text{ is even}\}\,.& \end{aligned}$$

Note that for both constructions $\Gamma = \Gamma_{p,l}$ and $\Gamma = \Gamma_{p,l,e_0}$ we have

$$\begin{aligned}\Gamma_0 &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r} l^{2s}; r, s \in \mathbb{N}_0\} \\ &< \mathrm{PSL}_2(\mathbb{Q}_p) \times \mathrm{PSL}_2(\mathbb{Q}_l)\end{aligned}$$

as in Section 3.2 and 3.3.

Theorem 3.35. *Let Γ be either the group $\Gamma_{p,l}$, where $p \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$, or let Γ be the group Γ_{p,l,e_0}, where $p \equiv 7 \pmod 8$, $l \equiv 1 \pmod 8$. Then Γ is a $(p+1, l+1)$–group.*

Proof. It is easy to adapt the proof of Theorem 3.30. □

Now, we give some explicit constructions of $\Gamma_{p,l}$ for the two cases $p \equiv 7 \pmod 8$ and $p \equiv 3 \pmod 8$. Moreover, we illustrate the type e_0 construction in the subcase $p \equiv 7 \pmod 8$, $l \equiv 1 \pmod 8$, and explain why this restriction makes sense to avoid torsion in the group.

Case $p \equiv 7 \pmod 8$, type e_1

Let $p \equiv 7 \pmod 8$, $l \equiv 1 \pmod 4$ be prime numbers,

$$\begin{aligned}\{a_1, \ldots, a_{\frac{p+1}{2}}\} &= \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_1,\ \Re(x) > 0,\ \Re(ix) < 0,\ |x|^2 = p\} \\ \{b_1, \ldots, b_{\frac{l+1}{2}}\}^{\pm 1} &= \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \Re(y) > 0,\ |y|^2 = l\}.\end{aligned}$$

We study two examples: the group $\Gamma_{7,5}$ is generated by

$$\begin{aligned} a_1 &= \psi(1+2i+j+k), & a_1^{-1} &= \psi(1-2i-j-k), \\ a_2 &= \psi(1+2i+j-k), & a_2^{-1} &= \psi(1-2i-j+k), \\ a_3 &= \psi(1+2i-j+k), & a_3^{-1} &= \psi(1-2i+j-k), \\ a_4 &= \psi(1+2i-j-k), & a_4^{-1} &= \psi(1-2i+j+k), \end{aligned}$$

$$\begin{aligned} b_1 &= \psi(1+2i), & b_1^{-1} &= \psi(1-2i), \\ b_2 &= \psi(1+2j), & b_2^{-1} &= \psi(1-2j), \\ b_3 &= \psi(1+2k), & b_3^{-1} &= \psi(1-2k). \end{aligned}$$

Example 3.36.

$$R_{4\cdot 3} := \left\{ \begin{array}{llll} a_1b_1a_3b_3^{-1}, & a_1b_2a_4b_2^{-1}, & a_1b_3a_4^{-1}b_2, & a_1b_3^{-1}a_4b_3, \\ a_1b_2^{-1}a_2b_1, & a_1b_1^{-1}a_4b_1^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, & a_2b_3a_4b_1, \\ a_2b_3^{-1}a_3b_3, & a_2b_2^{-1}a_3b_2, & a_2b_1^{-1}a_3b_1^{-1}, & a_3b_1a_4b_2 \end{array} \right\}.$$

Proposition 3.37. *Let* $\Gamma = \Gamma_{7,5}$ *be the* $(8,6)$*–group defined in Example 3.36. Then*

(1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(3) $\mathrm{Aut}(X) \cong S_4$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,5,2,6,4,8,3,7),\\
\rho_v(b_2) &= (1,5,3,7,6,2,8,4),\\
\rho_v(b_3) &= (1,6,2,3,7,4,8,5),\\
\rho_h(a_1) &= (1,6,5,3),\\
\rho_h(a_2) &= (1,6,3,2),\\
\rho_h(a_3) &= (1,6,4,5),\\
\rho_h(a_4) &= (1,6,2,4).
\end{aligned}$$

(2) and (3) are computed with GAP ([29]). The group $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$\begin{aligned}
(a_1,a_2,a_3,a_4,b_1,b_2,b_3) &\mapsto (a_1,a_3,a_4,a_2,b_3,b_1,b_2),\\
(a_1,a_2,a_3,a_4,b_1,b_2,b_3) &\mapsto (a_2,a_4^{-1},a_1,a_3^{-1},b_1,b_3^{-1},b_2^{-1}).
\end{aligned}$$

□

See Table 3.10 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient Γ/U, where $U = \langle a_i, b_j\rangle$, $a_i \in \{a_1,a_2,a_3,a_4\}$, $b_j \in \{b_1,b_2,b_3\}$.

	b_1	b_2, b_3
a_1,a_2,a_3,a_4	4, [8, 16], $\mathbb{Z}_4$	2, [8, 8], $\mathbb{Z}_2$

Table 3.10: $[\Gamma : U]$, U^{ab} and Γ/U in Example 3.36, where $U = \langle a_i, b_j\rangle$

Our second example is the group $\Gamma_{7,13}$:

$$\begin{aligned}
a_1 &= \psi(1+2i+j+k), & a_1^{-1} &= \psi(1-2i-j-k),\\
a_2 &= \psi(1+2i+j-k), & a_2^{-1} &= \psi(1-2i-j+k),\\
a_3 &= \psi(1+2i-j+k), & a_3^{-1} &= \psi(1-2i+j-k),\\
a_4 &= \psi(1+2i-j-k), & a_4^{-1} &= \psi(1-2i+j+k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1+2i+2j+2k), & b_1^{-1} &= \psi(1-2i-2j-2k),\\
b_2 &= \psi(1+2i+2j-2k), & b_2^{-1} &= \psi(1-2i-2j+2k),\\
b_3 &= \psi(1+2i-2j+2k), & b_3^{-1} &= \psi(1-2i+2j-2k),\\
b_4 &= \psi(1-2i+2j+2k), & b_4^{-1} &= \psi(1+2i-2j-2k),\\
b_5 &= \psi(3+2i), & b_5^{-1} &= \psi(3-2i),\\
b_6 &= \psi(3+2j), & b_6^{-1} &= \psi(3-2j),\\
b_7 &= \psi(3+2k), & b_7^{-1} &= \psi(3-2k).
\end{aligned}$$

Example 3.38.

$$R_{4\cdot 7} := \left\{ \begin{array}{llll}
a_1b_1a_1b_5^{-1}, & a_1b_2a_4b_3, & a_1b_3a_1^{-1}b_2^{-1}, & a_1b_4a_4b_1^{-1},\\
a_1b_5a_2b_6, & a_1b_6a_2^{-1}b_3^{-1}, & a_1b_7a_3b_5, & a_1b_7^{-1}a_3^{-1}b_4^{-1},\\
a_1b_6^{-1}a_4^{-1}b_7^{-1}, & a_1b_4^{-1}a_2^{-1}b_6^{-1}, & a_1b_2^{-1}a_3^{-1}b_7, & a_1b_1^{-1}a_4b_4,\\
a_2b_1a_2^{-1}b_4, & a_2b_2a_2b_5^{-1}, & a_2b_3a_4^{-1}b_7, & a_2b_5a_4b_7^{-1},\\
a_2b_7a_3^{-1}b_6^{-1}, & a_2b_7^{-1}a_4^{-1}b_1^{-1}, & a_2b_4^{-1}a_3b_1, & a_2b_3^{-1}a_3b_2^{-1},\\
a_2b_2^{-1}a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_1, & a_3b_6a_4^{-1}b_2,\\
a_3b_6^{-1}a_4b_5, & a_3b_1^{-1}a_4^{-1}b_6^{-1}, & a_4b_2a_4^{-1}b_3^{-1}, & a_4b_5^{-1}a_4b_4^{-1}
\end{array} \right\}.$$

Proposition 3.39. *Let $\Gamma = \Gamma_{7,13}$ be the $(8,14)$–group defined in Example 3.38. Then*

(1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(13) < S_{14}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,5,6,2,4,8),\\
\rho_v(b_2) &= (2,6,8,4,3,7),\\
\rho_v(b_3) &= (1,2,6,3,7,5),\\
\rho_v(b_4) &= (1,3,7,8,4,5),\\
\rho_v(b_5) &= (1,8,2,7,4,5,3,6),\\
\rho_v(b_6) &= (1,2,3,4,6,5,8,7),\\
\rho_v(b_7) &= (1,4,2,5,7,6,8,3),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1, 4, 8, 13, 12, 2, 3, 6, 11, 14, 10, 7, 9, 5),\\
\rho_h(a_2) &= (1, 8, 3, 13, 10, 6, 7, 5, 2, 12, 9, 4, 14, 11),\\
\rho_h(a_3) &= (1, 11, 7, 2, 12, 10, 9, 8, 5, 3, 13, 6, 14, 4),\\
\rho_h(a_4) &= (1, 4, 10, 8, 6, 5, 11, 14, 7, 12, 13, 3, 2, 9).
\end{aligned}$$

(2) GAP ([29]).

□

Case $p \equiv 7 \pmod 8$, type e_0; $l \equiv 1 \pmod 8$

Let $p \equiv 7 \pmod 8$, $l \equiv 1 \pmod 8$ be prime numbers,

$$\{a_1, \dots, a_{\frac{p+1}{2}}\}^{\pm 1} = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } e_0,\ \Re(x) > 0,\ |x|^2 = p\}$$

and

$$\{b_1, \dots, b_{\frac{l+1}{2}}\}^{\pm 1} = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ \Re(y) > 0,\ |y|^2 = l\}\,.$$

Note that we have two major restrictions in this type e_0 case. Firstly, we exclude the case $p \equiv 3 \pmod 8$ for the same reasons explained in Section 3.3. Secondly, we exclude the case $p \equiv 7 \pmod 8$, $l \equiv 5 \pmod 8$. To motivate it, observe that if x has type e_0, $|x|^2 = p \equiv 7 \pmod 8$ and y has type o_0, $|y|^2 = l \equiv 1 \pmod 8$, then xy has type e_0 such that $|xy|^2 = pl \equiv 7 \pmod 8$, in particular $\Re(xy) \neq 0$ by Lemma 3.7(2). However, if x has type e_0, $|x|^2 = p \equiv 7 \pmod 8$ and y has type o_0, $|y|^2 = l \equiv 5 \pmod 8$, then xy has type e_0 such that $|xy|^2 = pl \equiv 3 \pmod 8$ and it can happen that $\Re(xy) = 0$. But this means that $xy = -\overline{xy}$, hence $(xy)^2 = xy(-\overline{xy}) \in \mathbb{Z}$. As a consequence, $\psi((xy)^2)$ is the identity in Γ and Γ is therefore not torsion-free (we say that x, y generate a *projective plane*). We will give an example for this phenomenon later in this section (see Example 3.42).

First, we look at the $(8, 18)$–group $\Gamma_{7,17,e_0}$ having the following generators:

$$\begin{aligned}
a_1 &= \psi(2 + i + j + k), & a_1^{-1} &= \psi(2 - i - j - k),\\
a_2 &= \psi(2 + i + j - k), & a_2^{-1} &= \psi(2 - i - j + k),\\
a_3 &= \psi(2 + i - j + k), & a_3^{-1} &= \psi(2 - i + j - k),\\
a_4 &= \psi(2 - i + j + k), & a_4^{-1} &= \psi(2 + i - j - k),
\end{aligned}$$

$$\begin{aligned}
b_1 &= \psi(1 + 4i), & b_1^{-1} &= \psi(1 - 4i),\\
b_2 &= \psi(1 + 4j), & b_2^{-1} &= \psi(1 - 4j),\\
b_3 &= \psi(1 + 4k), & b_3^{-1} &= \psi(1 - 4k),
\end{aligned}$$

$$\begin{aligned}
b_4 &= \psi(3+2i+2j), & b_4^{-1} &= \psi(3-2i-2j),\\
b_5 &= \psi(3+2i-2j), & b_5^{-1} &= \psi(3-2i+2j),\\
b_6 &= \psi(3+2i+2k), & b_6^{-1} &= \psi(3-2i-2k),\\
b_7 &= \psi(3+2i-2k), & b_7^{-1} &= \psi(3-2i+2k),\\
b_8 &= \psi(3+2j+2k), & b_8^{-1} &= \psi(3-2j-2k),\\
b_9 &= \psi(3+2j-2k), & b_9^{-1} &= \psi(3-2j+2k).
\end{aligned}$$

Example 3.40.

$$R_{4.9} := \left\{ \begin{array}{llll}
a_1b_1a_2b_4, & a_1b_2a_4b_8, & a_1b_3a_3b_6, & a_1b_4a_2b_2,\\
a_1b_5a_4b_6^{-1}, & a_1b_6a_3b_1, & a_1b_7a_3^{-1}b_2^{-1}, & a_1b_8a_4b_3,\\
a_1b_9a_3b_4^{-1}, & a_1b_9^{-1}a_4^{-1}b_1^{-1}, & a_1b_8^{-1}a_3b_5^{-1}, & a_1b_7^{-1}a_2b_8^{-1},\\
a_1b_6^{-1}a_2b_9^{-1}, & a_1b_5^{-1}a_2^{-1}b_3^{-1}, & a_1b_4^{-1}a_4b_7, & a_1b_3^{-1}a_2^{-1}b_5,\\
a_1b_2^{-1}a_3^{-1}b_7^{-1}, & a_1b_1^{-1}a_4^{-1}b_9, & a_2b_1a_4^{-1}b_7, & a_2b_6a_3^{-1}b_4^{-1},\\
a_2b_7a_4^{-1}b_3^{-1}, & a_2b_8a_3b_1^{-1}, & a_2b_9a_3^{-1}b_2, & a_2b_7^{-1}a_3^{-1}b_5,\\
a_2b_6^{-1}a_4b_2^{-1}, & a_2b_5^{-1}a_4^{-1}b_9^{-1}, & a_2b_4^{-1}a_4^{-1}b_8, & a_2b_3^{-1}a_3^{-1}b_9,\\
a_2b_2^{-1}a_4b_6, & a_2b_1^{-1}a_3b_8^{-1}, & a_3b_4a_4b_3^{-1}, & a_3b_5a_4^{-1}b_1,\\
a_3b_8a_4^{-1}b_6^{-1}, & a_3b_9a_4^{-1}b_7^{-1}, & a_3b_3^{-1}a_4b_4^{-1}, & a_3b_2^{-1}a_4^{-1}b_5
\end{array} \right\}.$$

Proposition 3.41. *Let $\Gamma = \Gamma_{7,17,e_0}$ be the $(8,18)$–group defined in Example 3.40. Then*

(1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(17) < S_{18}$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,4,3,7,5,8,2,6),\\
\rho_v(b_2) &= (1,3,2,5,6,8,4,7),\\
\rho_v(b_3) &= (1,2,4,6,7,8,3,5),\\
\rho_v(b_4) &= (1,6,4,8,2,3,5,7),
\end{aligned}$$

$$
\begin{aligned}
\rho_v(b_5) &= (1, 6, 5, 7, 8, 4, 3, 2),\\
\rho_v(b_6) &= (1, 5, 2, 8, 3, 4, 7, 6),\\
\rho_v(b_7) &= (1, 3, 4, 2, 8, 6, 7, 5),\\
\rho_v(b_8) &= (1, 7, 3, 8, 4, 2, 6, 5),\\
\rho_v(b_9) &= (1, 7, 6, 5, 8, 3, 2, 4),
\end{aligned}
$$

$$
\begin{aligned}
\rho_h(a_1) &= (1, 10, 18, 6, 5, 11, 2, 7, 17, 4, 9, 13, 3, 14, 16, 8, 12, 15),\\
\rho_h(a_2) &= (1, 8, 18, 4, 6, 10, 16, 5, 3, 7, 11, 15, 2, 13, 17, 9, 14, 12),\\
\rho_h(a_3) &= (1, 11, 18, 5, 7, 9, 3, 4, 16, 6, 8, 14, 17, 12, 2, 10, 15, 13),\\
\rho_h(a_4) &= (1, 14, 13, 11, 3, 15, 16, 12, 10, 5, 2, 6, 17, 8, 4, 7, 18, 9).
\end{aligned}
$$

(2) GAP ([29]).

□

We illustrate now, why the type e_0 construction does not work in the excluded case $p \equiv 7 \pmod 8$, $l \equiv 5 \pmod 8$. Take the smallest case $p = 7, l = 5$: if for example $a_1 = \psi(2 + i + j + k)$ and $b_1 = \psi(1 + 2i)$, then

$$
\begin{aligned}
\Re((2+i+j+k)(1+2i)) &= \Re(5i+3j-k) = 0\,,\\
a_1 b_1 = \psi(2+i+j+k)\psi(1+2i) &= \psi(5i+3j-k)\,,\\
(a_1 b_1)^2 = \psi((5i+3j-k)^2) &= \psi(-35) = 1_\Gamma\,,
\end{aligned}
$$

i.e. we have a projective plane, Γ is not torsion-free and therefore no $(8, 6)$–group. Nevertheless, we can do some computations: If we take

$$
\begin{aligned}
a_1 &= \psi(2+i+j+k), & a_1^{-1} &= \psi(2-i-j-k),\\
a_2 &= \psi(2+i+j-k), & a_2^{-1} &= \psi(2-i-j+k),\\
a_3 &= \psi(2+i-j+k), & a_3^{-1} &= \psi(2-i+j-k),\\
a_4 &= \psi(2-i+j+k), & a_4^{-1} &= \psi(2+i-j-k),
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \psi(1+2i), & b_1^{-1} &= \psi(1-2i),\\
b_2 &= \psi(1+2j), & b_2^{-1} &= \psi(1-2j),\\
b_3 &= \psi(1+2k), & b_3^{-1} &= \psi(1-2k),
\end{aligned}
$$

then we get a group Γ with generators $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and the following 18 (not 12 !) relators, where the twelve projective planes are printed bold:

Example 3.42.

$$\left\{\begin{array}{lll} \mathbf{a_1b_1a_1b_1}, & \mathbf{a_1b_2a_1b_2}, & \mathbf{a_1b_3a_1b_3}, \\ a_1b_3^{-1}a_4b_2^{-1}, & a_1b_2^{-1}a_2b_1^{-1}, & a_1b_1^{-1}a_3b_3^{-1}, \\ \mathbf{a_2b_1a_2b_1}, & \mathbf{a_2b_2a_2b_2}, & a_2b_3a_4^{-1}b_1^{-1}, \\ \mathbf{a_2b_3^{-1}a_2b_3^{-1}}, & a_2b_2^{-1}a_3^{-1}b_3, & \mathbf{a_3b_1a_3b_1}, \\ \mathbf{a_3b_3a_3b_3}, & \mathbf{a_3b_2^{-1}a_3b_2^{-1}}, & a_3b_1^{-1}a_4^{-1}b_2, \\ \mathbf{a_4b_2a_4b_2}, & \mathbf{a_4b_3a_4b_3}, & \mathbf{a_4b_1^{-1}a_4b_1^{-1}} \end{array}\right\}.$$

Note that also here, if $E_h := \{a_1, a_2, a_3, a_4\}^{\pm 1}$ and $E_v := \{b_1, b_2, b_3\}^{\pm 1}$, then given any $a \in E_h$, $b \in E_v$, there are unique $\tilde{a} \in E_h$, $\tilde{b} \in E_v$ such that $ab = \tilde{b}\tilde{a}$ by an analogon of Theorem 3.30(3). However, in strong contrast to what happens in $(2m, 2n)$–groups, we sometimes have $\tilde{a} = a^{-1}$ and $\tilde{b} = b^{-1}$, i.e. $abab = 1$.

Proposition 3.43. *Let Γ be the group with generators $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and the relators of Example 3.42. Let Γ_0 be the kernel of the homomorphism*

$$\begin{aligned} \Gamma &\to \mathbb{Z}_2^2 \\ a_i &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\ b_j &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}), \end{aligned}$$

generalizing the definition of the subgroup Γ_0 of a $(2m, 2n)$–group Γ. Then

(1) $\Gamma^{ab} \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4$, $[\Gamma, \Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2$.

(2) *Γ has the (vertical) amalgam decomposition*

$$\Gamma \cong F_3 *_{F_{17}} (\mathbb{Z}_2^{*12} * F_3).$$

(3) *Γ_0 has the (vertical) amalgam decomposition*

$$\Gamma_0 \cong F_5 *_{F_{33}} F_5,$$

in particular Γ_0 is torsion-free and Γ is virtually torsion-free.

Proof. (1) This follow from computations with GAP ([29]).

(2) and (3): See Appendix A.11 for the explicit amalgam decompositions and the isomorphisms to Γ and Γ_0, respectively.

□

Remark. Taking an obvious generalized definition of ρ_h, ρ_v, P_h, P_v, we get

$$\begin{aligned}
\rho_v(b_1) &= (1,7,2,4,5,6,3,8),\\
\rho_v(b_2) &= (1,5,4,3,6,7,2,8),\\
\rho_v(b_3) &= (1,6,3,2,7,5,4,8),\\
\rho_h(a_1) &= (1,5,2,4,3,6),\\
\rho_h(a_2) &= (1,3,4,5,2,6),\\
\rho_h(a_3) &= (1,4,3,2,5,6),\\
\rho_h(a_4) &= (1,6,4,3,5,2),
\end{aligned}$$

generating $P_h \cong \mathrm{PGL}_2(7) < S_8$ and $P_v \cong \mathrm{PGL}_2(5) < S_6$, respectively.

We can take the six relators of Γ in Example 3.42 which are not projective planes and embed them in a $(\mathrm{PGL}_2(7), \mathrm{PGL}_2(5))$–group as follows:

Example 3.44.

$$R_{4\cdot 3} := \left\{ \begin{array}{llll}
a_1b_1a_4^{-1}b_1, & a_1b_2a_3^{-1}b_2, & a_1b_3a_2^{-1}b_3, & \underline{a_1b_3^{-1}a_4b_2^{-1}}, \\
\underline{a_1b_2^{-1}a_2b_1^{-1}}, & \underline{a_1b_1^{-1}a_3b_3^{-1}}, & a_2b_1a_3b_1, & a_2b_2a_4b_2, \\
\underline{a_2b_3a_4^{-1}b_1^{-1}}, & \underline{a_2b_2^{-1}a_3^{-1}b_3}, & a_3b_3a_4b_3, & \underline{a_3b_1^{-1}a_4^{-1}b_2}
\end{array} \right\}.$$

Proposition 3.45. *Let Γ be the $(8,6)$–group defined in Example 3.44. Then*

(1) $P_h \cong \mathrm{PGL}_2(7) < S_8$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_3^3$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_9^3$, $\Gamma_0^{ab} \cong \mathbb{Z}_3^4$, *in particular Γ is not isomorphic to the group $\Gamma_{7,5}$ of Example 3.36.*

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,7,3,8,5,6,2,4),\\
\rho_v(b_2) &= (1,5,2,8,6,7,4,3),\\
\rho_v(b_3) &= (1,6,4,8,7,5,3,2),\\
\rho_h(a_1) &= (1,5,2,4,3,6),\\
\rho_h(a_2) &= (1,3,4,5,2,6),\\
\rho_h(a_3) &= (1,4,3,2,5,6),\\
\rho_h(a_4) &= (1,6,4,3,5,2).
\end{aligned}$$

(2) GAP ([29]).

□

Case $p \equiv 3 \pmod 8$

Let $p \equiv 3 \pmod 8$, $l \equiv 1 \pmod 4$ be prime numbers. The example $\Gamma_{3,5}$ is given by

$$\begin{aligned} a_1 &= \psi(1+j+k), & a_1^{-1} &= \psi(1-j-k),\\ a_2 &= \psi(1+j-k), & a_2^{-1} &= \psi(1-j+k),\\ \\ b_1 &= \psi(1+2i), & b_1^{-1} &= \psi(1-2i),\\ b_2 &= \psi(1+2j), & b_2^{-1} &= \psi(1-2j),\\ b_3 &= \psi(1+2k), & b_3^{-1} &= \psi(1-2k). \end{aligned}$$

Example 3.46.

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_1 a_2 b_2, & a_1 b_2 a_2 b_1^{-1}, \\ \\ a_1 b_3 a_2^{-1} b_1, & a_1 b_3^{-1} a_1 b_2^{-1}, \\ \\ a_1 b_1^{-1} a_2^{-1} b_3, & a_2 b_3 a_2 b_2^{-1} \end{array} \right\}.$$

See Appendix B.8 for the GAP-program ([29]) constructing $\Gamma_{3,5}$.

Proposition 3.47. *Let $\Gamma = \Gamma_{3,5}$ be the $(4,6)$–group defined in Example 3.46 and let $G = U(\mathbb{H}(\mathbb{Z}[1/3,1/5]))/ZU(\mathbb{H}(\mathbb{Z}[1/3,1/5]))$. Then*

(1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $[\Gamma,\Gamma]^{ab} \cong \mathbb{Z}_8^2 \times \mathbb{Z}_{16}$, $\Gamma_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_8^2$.

(3) *There are finite quotients*

$$\Gamma/\langle\!\langle a_1^8, (a_1b_1)^7, (b_1b_2)^3 \rangle\!\rangle_\Gamma \cong \mathrm{PGL}_2(7),$$
$$\textit{such that } \langle\!\langle a_1^8, (a_1b_1)^7, (b_1b_2)^3 \rangle\!\rangle_\Gamma^{ab} \cong \mathbb{Z}_{14} \times \mathbb{Z}_{56}^2.$$

$$\Gamma/\langle\!\langle a_1^5, a_2^5, b_1^6, (a_1b_1)^3 \rangle\!\rangle_\Gamma \cong \mathrm{PSL}_2(11),$$
$$\textit{such that } \langle\!\langle a_1^5, a_2^5, b_1^6, (a_1b_1)^3 \rangle\!\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{22} \times \mathbb{Z}_{44}^2.$$

$$\Gamma/\langle\!\langle a_1^7, a_2^7, (a_1b_1)^4 \rangle\!\rangle_\Gamma \cong \mathrm{PGL}_2(13).$$

(4) *The group G has a presentation with generators $a_1, a_2, b_1, b_2, b_3, i, j$ and 13 relators*

$$R_{2\cdot 3},\ a_1 i a_1 i^{-1},\ a_1 j a_2^{-1} j^{-1},\ [b_1, i],\ b_1 j b_1 j^{-1},\ i^2,\ j^2,\ [i,j].$$

(5) *As in Proposition 3.29(5), we use for a group H the notation $H^{(1)} := [H, H]$ and $H^{(2)} := [H^{(1)}, H^{(1)}]$. Then there is a chain of normal subgroups of G*

$$\Gamma^{(2)} \overset{64}{\lhd} G^{(2)} = \Gamma_0^{(1)} \overset{16}{\lhd} \Gamma^{(1)} \overset{8}{\lhd} G^{(1)} = \Gamma_0 \overset{4}{\lhd} \Gamma \overset{4}{\lhd} G$$

such that

$$G/\Gamma \cong \Gamma/\Gamma_0 \cong \mathbb{Z}_2^2\,,\ G^{(1)}/\Gamma^{(1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4\,,\ \Gamma^{(1)}/\Gamma_0^{(1)} \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4$$

and $G^{ab} \cong G/\Gamma_0 \cong \mathbb{Z}_2^4$.

Note that $G^{(1)} = \Gamma_0$ is the kernel of the homomorphism

$$\begin{aligned} G &\to \mathbb{Z}_2^4 \\ a_1, a_2 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\ b_1, b_2, b_3 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\ i &\mapsto (0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\ j &\mapsto (0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 0 + 2\mathbb{Z}, 1 + 2\mathbb{Z})\,. \end{aligned}$$

(6) $\mathrm{Aut}(X) \cong D_4$.

(7) Γ *is commutative transitive.*

(8) *If $a \in \{a_1, a_2\}^{\pm 1}$ and $b \in \{b_1, b_2, b_3\}^{\pm 1}$, then $\langle a, b\rangle$ is an "anti-torus" in Γ.*

(9) $\langle a_1, b_1\rangle \neq F_2$.

(10) $\Gamma < \mathrm{SO}_3(\mathbb{Q})$.

(11) $Z_\Gamma(a_i) = N_\Gamma(\langle a_i\rangle) = \langle a_i\rangle$, *if $a_i \in \{a_1, a_2\}$, and* $Z_\Gamma(b_j) = N_\Gamma(\langle b_j\rangle) = \langle b_j\rangle$, *if $b_j \in \{b_1, b_2, b_3\}$.*

(12) Γ *has amalgam decompositions $F_3 *_{F_9} F_5 \cong \Gamma \cong F_2 *_{F_7} F_4$.*

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= (1, 3, 4, 2), \\ \rho_v(b_2) &= (1, 4, 2, 3), \\ \rho_v(b_3) &= (1, 4, 3, 2), \\ \rho_h(a_1) &= (1, 2, 4, 6, 3, 5), \\ \rho_h(a_2) &= (1, 4, 5, 6, 2, 3). \end{aligned}$$

(2) GAP ([29]).

(3) Let q be an odd prime number distinct from p and l, and choose $c, d \in \mathbb{Z}$ such that $c^2 + d^2 + 1 \equiv 0 \pmod q$, then we can define exactly as described in Theorem 3.12(3) a homomorphism $\tau = \tau_{c,d} : \Gamma_{p,l} \to \mathrm{PGL}_2(q)$ by

$$\tau_{c,d}(\gamma) = \left[\begin{pmatrix} x_0 + x_1 c + x_3 d + q\mathbb{Z} & -x_1 d + x_2 + x_3 c + q\mathbb{Z} \\ -x_1 d - x_2 + x_3 c + q\mathbb{Z} & x_0 - x_1 c - x_3 d + q\mathbb{Z} \end{pmatrix}\right],$$

where $\gamma = \psi(x_0 + x_1 i + x_2 j + x_3 k) \in \Gamma_{p,l}$.

For $q = 7$ we have $\tau_{2,3} : \Gamma_{3,5} \twoheadrightarrow \mathrm{PGL}_2(7)$ given by

$$a_1 \mapsto \left[\begin{pmatrix} 4 + 7\mathbb{Z} & 3 + 7\mathbb{Z} \\ 1 + 7\mathbb{Z} & 5 + 7\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 5 + 7\mathbb{Z} & 6 + 7\mathbb{Z} \\ 4 + 7\mathbb{Z} & 4 + 7\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 5 + 7\mathbb{Z} & 1 + 7\mathbb{Z} \\ 1 + 7\mathbb{Z} & 4 + 7\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 1 + 7\mathbb{Z} & 2 + 7\mathbb{Z} \\ 5 + 7\mathbb{Z} & 1 + 7\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 0 + 7\mathbb{Z} & 4 + 7\mathbb{Z} \\ 4 + 7\mathbb{Z} & 2 + 7\mathbb{Z} \end{pmatrix}\right].$$

In the same way $\tau_{1,3} : \Gamma_{3,5} \twoheadrightarrow \mathrm{PSL}_2(11)$ is defined by

$$a_1 \mapsto \left[\begin{pmatrix} 4 + 11\mathbb{Z} & 2 + 11\mathbb{Z} \\ 0 + 11\mathbb{Z} & 9 + 11\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 9 + 11\mathbb{Z} & 0 + 11\mathbb{Z} \\ 9 + 11\mathbb{Z} & 4 + 11\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 3 + 11\mathbb{Z} & 5 + 11\mathbb{Z} \\ 5 + 11\mathbb{Z} & 10 + 11\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 1 + 11\mathbb{Z} & 2 + 11\mathbb{Z} \\ 9 + 11\mathbb{Z} & 1 + 11\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 7 + 11\mathbb{Z} & 2 + 11\mathbb{Z} \\ 2 + 11\mathbb{Z} & 6 + 11\mathbb{Z} \end{pmatrix}\right]$$

and $\tau_{0,5} : \Gamma_{3,5} \twoheadrightarrow \mathrm{PGL}_2(13)$ by

$$a_1 \mapsto \left[\begin{pmatrix} 6 + 13\mathbb{Z} & 1 + 13\mathbb{Z} \\ 12 + 13\mathbb{Z} & 9 + 13\mathbb{Z} \end{pmatrix}\right]$$
$$a_2 \mapsto \left[\begin{pmatrix} 9 + 13\mathbb{Z} & 1 + 13\mathbb{Z} \\ 12 + 13\mathbb{Z} & 6 + 13\mathbb{Z} \end{pmatrix}\right]$$

$$b_1 \mapsto \left[\begin{pmatrix} 1+13\mathbb{Z} & 3+13\mathbb{Z} \\ 3+13\mathbb{Z} & 1+13\mathbb{Z} \end{pmatrix}\right]$$
$$b_2 \mapsto \left[\begin{pmatrix} 1+13\mathbb{Z} & 2+13\mathbb{Z} \\ 11+13\mathbb{Z} & 1+13\mathbb{Z} \end{pmatrix}\right]$$
$$b_3 \mapsto \left[\begin{pmatrix} 11+13\mathbb{Z} & 0+13\mathbb{Z} \\ 0+13\mathbb{Z} & 4+13\mathbb{Z} \end{pmatrix}\right].$$

We have used quotpic ([58]) to show that

$$\langle\langle a_1^8, (a_1b_1)^7, (b_1b_2)^3 \rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_{14} \times \mathbb{Z}_{56}^2$$

and

$$\langle\langle a_1^5, a_2^5, b_1^6, (a_1b_1)^3 \rangle\rangle_\Gamma^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_{22} \times \mathbb{Z}_{44}^2 .$$

(4) Same idea as in Proposition 3.29(4) using the isomorphism between

$$U(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))/ZU(\mathbb{H}(\mathbb{Z}[1/p, 1/l]))$$

and

$$\{\psi(x) : x \in \mathbb{H}(\mathbb{Z}),\ |x|^2 = p^r l^s; r, s \in \mathbb{N}_0\} .$$

(5) We have used GAP ([29]), quotpic ([58]) and the presentation of G given in part (4).

(6) GAP ([29]). The group $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_1, a_2^{-1}, b_1^{-1}, b_3, b_2),$$
$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_2, a_1^{-1}, b_1, b_3^{-1}, b_2).$$

(7) We can adapt Lemma 3.19 and Proposition 3.20, using Lemma 3.4(2). This can be done, since $\psi(x) \in \Gamma$ implies that x has type e_1 or o_0, in particular $\Re(x) \neq 0$.

(8) See Section 3.6 for the definition of an anti-torus in Γ. The statement is an application of Proposition 3.53 in Section 3.6 using part (7) of this proposition and an adaption of Lemma 3.19.

(9) We have $b_1a_1^3b_1^2a_1b_1^{-1}a_1^{-3}b_1^{-2}a_1^{-1} = 1$ in Γ and $yx^3y^2xy^{-1}x^{-3}y^{-2}x^{-1} = 1$, where $x = 1 + j + k$, $y = 1 + 2i$. There seems to be no smaller non-trivial freely reduced relation in $\langle x, y\rangle$ than the one of length 14 given above. The statement can also be deduced from Table 3.12.

(10) A generalization of Theorem 3.12(2) gives an injective group homomorphism $\Gamma \to \mathrm{SO}_3(\mathbb{Q})$, defined by

$$a_1 \mapsto \frac{1}{3}\begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}, \quad a_2 \mapsto \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

$$b_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}$$
$$b_2 \mapsto \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix}$$
$$b_3 \mapsto \begin{pmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(11) This follows from Proposition 1.12.

(12) Use Proposition 1.3. The explicit amalgam decompositions of Γ are described in Appendix A.12.

□

See Table 3.11 for the orders of some $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$, and see Table 3.12 for the index $[\Gamma : U]$, the abelianization U^{ab} and the structure of the quotient Γ/U (if U is normal in Γ), where $U = \langle a, b \rangle$, $a \in \{a_1, a_1^2, a_2, a_2^2\}$, $b \in \{b_1, b_1^2, b_2, b_2^2, b_3, b_3^2\}$.

$\lvert\Gamma/\langle\langle w^k \rangle\rangle_\Gamma\rvert$	$k = 1$	2	3	4	5	6
$w = a_1, a_2$	8	64	8	512	10560	64
b_1, b_2, b_3	16	128	16	1024	109440	168960

Table 3.11: Some orders of $\Gamma/\langle\langle w^k \rangle\rangle_\Gamma$ in Example 3.46

	b_1	b_2, b_3	b_1^2	b_2^2, b_3^2
a_1, a_2	4, [8, 16], $\mathbb{Z}_4$	2, [8, 8], $\mathbb{Z}_2$	16, [8, 64], –	88, [8, 32], –
a_1^2, a_2^2	16, [16, 32], –	8, [16, 16], –	896, [32, 64], –	352, [32, 32], –

Table 3.12: $[\Gamma : U]$, U^{ab} and Γ/U in Example 3.46, where $U = \langle a, b \rangle$

3.5 Some conjectures

Based on computations in the 130 examples described in the following list, we give some conjectures afterwards. In this list, "G" and "S" in the column P_h stand for $\mathrm{PGL}_2(p)$ and $\mathrm{PSL}_2(p)$, respectively. Similarly, "G" and "S" in the column P_v stand for $\mathrm{PGL}_2(l)$ and $\mathrm{PSL}_2(l)$, respectively. Finally, "+" and "−" stand for 1 and -1.

p	l	types	Ex.	P_h, $(\frac{l}{p})$, P_v, $(\frac{p}{l})$	Γ^{ab}	$[\Gamma,\Gamma]^{ab}$	Γ_0^{ab}
Case $p, l \equiv 1 \pmod 4$							
5	13	(o_0, o_0)	3.28	G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	17	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	29	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	37	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	41	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	53	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	61	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	73	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	89	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
5	97	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	17	(o_0, o_0)	3.26	S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	29	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	37	(o_0, o_0)		G, −, G, −	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
13	41	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	53	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	61	(o_0, o_0)		S, +, S, +	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
13	73	(o_0, o_0)		G, −, G, −	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
13	89	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
13	97	(o_0, o_0)		G, −, G, −	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
17	29	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
17	37	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
17	41	(o_0, o_0)		G, −, G, −	$2^3, 8^2$	$3, 16^2, 64$	$2, 3, 8^2$
17	53	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
17	61	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	37	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	41	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	53	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	61	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	73	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
29	89	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$

29	97	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
37	41	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
37	53	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
37	61	(o_0, o_0)		G, −, G, −	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
37	73	(o_0, o_0)		S, +, S, +	$2, 3, 4^3$	$2^2, 16^3$	$2, 3, 8^2$
37	89	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
41	53	(o_0, o_0)		G, −, G, −	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
41	61	(o_0, o_0)		S, +, S, +	$2, 4^3$	$3, 16^3$	$2, 3, 8^2$
73	97	(o_0, o_0)		S, +, S, +	$2^3, 3, 8^2$	?	$2, 3, 8^2$
Case $p, l \equiv 7 \pmod 8$							
7	23	(e_1, e_1)	A.31	S, +, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
7	31	(e_1, e_1)	A.32	G, −, S, +	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
7	47	(e_1, e_1)		G, −, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	31	(e_1, e_1)		S, +, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	47	(e_1, e_1)		S, +, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
31	47	(e_1, e_1)		S, +, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
Case $p, l \equiv 7 \pmod 8$							
7	23	(e_0, e_0)	A.33	S, +, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
7	31	(e_0, e_0)		G, −, S, +	$2^3, 3, 4$	$2^2, 4, 16^2$	$2, 3, 8^2$
7	47	(e_0, e_0)		G, −, S, +	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	31	(e_0, e_0)		S, +, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	47	(e_0, e_0)		S, +, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
31	47	(e_0, e_0)		S, +, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
Case $p, l \equiv 3 \pmod 8$							
3	11	(e_1, e_1)	3.31	G, −, S, +	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	19	(e_1, e_1)		S, +, G, −	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	43	(e_1, e_1)		S, +, G, −	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	59	(e_1, e_1)		G, −, S, +	$2, 8^2$	$8^2, 64$	$2, 8^2$
11	19	(e_1, e_1)		G, −, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	43	(e_1, e_1)		G, −, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	59	(e_1, e_1)		S, +, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	43	(e_1, e_1)		S, +, G, −	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
19	59	(e_1, e_1)		G, −, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
Case $p \equiv 3 \pmod 8$, $l \equiv 7 \pmod 8$							
3	7	(e_1, e_1)	3.33	S, +, G, −	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	23	(e_1, e_1)		G, −, S, +	$2, 4^2$	$8^2, 16$	$2, 8^2$

3	31	(e_1, e_1)		S, +, G, −	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	47	(e_1, e_1)		G, −, S, +	$2, 4^2$	$8^2, 16$	$2, 8^2$
11	7	(e_1, e_1)		G, −, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	23	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	31	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	47	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	7	(e_1, e_1)		S, +, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	23	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	31	(e_1, e_1)		G, −, S, +	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	47	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	7	(e_1, e_1)		G, −, S, +	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	23	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	31	(e_1, e_1)		S, +, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	47	(e_1, e_1)		S, +, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
Case $p \equiv 7 \pmod 8, l \equiv 1 \pmod 4$							
7	5	(e_1, o_0)	3.36	G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
7	13	(e_1, o_0)	3.38	G, −, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
7	17	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
7	29	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
7	37	(e_1, o_0)		S, +, S, +	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
7	41	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
7	73	(e_1, o_0)		G, −, G, −	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
23	5	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	13	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	17	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	29	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	37	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
23	41	(e_1, o_0)		S, +, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
23	73	(e_1, o_0)		S, +, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
31	5	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
31	13	(e_1, o_0)		G, −, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
31	17	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
31	29	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
31	37	(e_1, o_0)		G, −, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
31	41	(e_1, o_0)		S, +, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
Case $p \equiv 7 \pmod 8, l \equiv 1 \pmod 8$							
7	17	(e_0, o_0)	3.40	G, −, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$

23	17	(e_0, o_0)		G, −, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
31	17	(e_0, o_0)		G, −, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
7	41	(e_0, o_0)		G, −, G, −	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
23	41	(e_0, o_0)		S, +, S, +	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
31	41	(e_0, o_0)		S, +, S, +	$2^3, 4$	$3, 4, 16^2$	$2, 3, 8^2$
7	73	(e_0, o_0)		G, −, G, −	$2^3, 3, 4$	$2^2, 4, 16^2$	$2, 3, 8^2$
Case $p \equiv 3 \pmod 8$, $l \equiv 1 \pmod 4$							
3	5	(e_1, o_0)	3.46	G, −, G, −	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	13	(e_1, o_0)		S, +, S, +	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	17	(e_1, o_0)		G, −, G, −	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	29	(e_1, o_0)		G, −, G, −	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	37	(e_1, o_0)		S, +, S, +	$2, 4^2$	$8^2, 16$	$2, 8^2$
3	41	(e_1, o_0)		G, −, G, −	$2, 8^2$	$8^2, 64$	$2, 8^2$
3	73	(e_1, o_0)		S, +, S, +	$2, 8^2$	$8^2, 64$	$2, 8^2$
11	5	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	13	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	17	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	29	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	37	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
11	41	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
11	73	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	5	(e_1, o_0)		S, +, S, +	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	13	(e_1, o_0)		G, −, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	17	(e_1, o_0)		S, +, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	29	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
19	37	(e_1, o_0)		G, −, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
19	41	(e_1, o_0)		G, −, G, −	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
19	73	(e_1, o_0)		S, +, S, +	$2, 3, 8^2$	$2^2, 8^2, 64$	$2, 3, 8^2$
43	5	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	13	(e_1, o_0)		S, +, S, +	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	17	(e_1, o_0)		S, +, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$
43	29	(e_1, o_0)		G, −, G, −	$2, 4^2$	$3, 8^2, 16$	$2, 3, 8^2$
43	37	(e_1, o_0)		G, −, G, −	$2, 3, 4^2$	$2^2, 8^2, 16$	$2, 3, 8^2$
43	41	(e_1, o_0)		S, +, S, +	$2, 8^2$	$3, 8^2, 64$	$2, 3, 8^2$

Table 3.13: List of properties of some $\Gamma_{p,l}$

Conjecture 3.48. *Let p, l be two odd distinct prime numbers and $\Gamma = \Gamma_{p,l}$ as in Section 3.2, 3.3 or 3.4.*

(1) *(cf. Conjecture 3.16) Assume that $p, l \equiv 1 \pmod 4$ (as in Section 3.2).*
If $p, l \equiv 1 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_{16}^2 \times \mathbb{Z}_{64}) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2^3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_3 \times \mathbb{Z}_{16}^2 \times \mathbb{Z}_{64}) & \text{else}. \end{cases}$$

If $p \equiv 5 \pmod 8$ or $l \equiv 5 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^3,\ \mathbb{Z}_2^2 \times \mathbb{Z}_{16}^3) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^3,\ \mathbb{Z}_3 \times \mathbb{Z}_{16}^3) & \text{else}. \end{cases}$$

(2) *Assume that $p, l \equiv 3 \pmod 4$ (as in Section 3.3).*
If $p \pmod 8 = l \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \text{if } p = 3 \text{ or } l = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \text{else}. \end{cases}$$

If $p \pmod 8 \neq l \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \text{if } p = 3 \text{ or } l = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \text{else}. \end{cases}$$

(3) *Assume that $p \equiv 3 \pmod 4$ and $l \equiv 1 \pmod 4$ (as in Section 3.4).*
If $l \equiv 1 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \text{if } p = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_8^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{64}) & \text{else}. \end{cases}$$

If $l \equiv 5 \pmod 8$, then

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \text{if } p = 3 \\ (\mathbb{Z}_2 \times \mathbb{Z}_4^2,\ \mathbb{Z}_3 \times \mathbb{Z}_8^2 \times \mathbb{Z}_{16}) & \text{else}. \end{cases}$$

Conjecture 3.49. *Let $\Gamma = \Gamma_{p,l,e_0}$ be as in Section 3.3 or 3.4, then*

$$(\Gamma^{ab},\ [\Gamma,\Gamma]^{ab}) \cong \begin{cases} (\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2) & \text{if } p, l \equiv 1 \pmod 3 \\ (\mathbb{Z}_2^3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}^2) & \text{else}\,. \end{cases}$$

Conjecture 3.50. *Let Γ be any group $\Gamma_{p,l}$ or Γ_{p,l,e_0} of Chapter 3, then*

$$\Gamma_0^{ab} \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_8^2\,, & \text{if } p = 3 \text{ or } l = 3 \\ \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_8^2\,, & \text{else}\,. \end{cases}$$

Remark. Note that in all cases of Chapter 3

$$\Gamma_0 = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^{2r} l^{2s};\ r, s \in \mathbb{N}_0\}\,.$$

Conjecture 3.51. *Let Γ be any $\Gamma_{p,l}$ or Γ_{p,l,e_0} of Chapter 3, and let $k \in \mathbb{N}$. Then*

(1)

$$P_h \cong \begin{cases} \mathrm{PSL}_2(p)\,, & \text{if } \left(\frac{l}{p}\right) = 1 \\ \mathrm{PGL}_2(p)\,, & \text{if } \left(\frac{l}{p}\right) = -1 \end{cases}$$

and

$$P_v \cong \begin{cases} \mathrm{PSL}_2(l)\,, & \text{if } \left(\frac{p}{l}\right) = 1 \\ \mathrm{PGL}_2(l)\,, & \text{if } \left(\frac{p}{l}\right) = -1\,. \end{cases}$$

(2)

$$|P_h^{(k)}| = |P_h| \cdot p^{3(k-1)}$$

and

$$|P_v^{(k)}| = |P_v| \cdot l^{3(k-1)}\,.$$

(3) *As a consequence of part (1) and (2):*

$$|P_h^{(k)}| = \begin{cases} p^{3k-2}(p^2-1)/2\,, & \text{if } \left(\frac{l}{p}\right) = 1 \\ p^{3k-2}(p^2-1)\,, & \text{if } \left(\frac{l}{p}\right) = -1 \end{cases}$$

and

$$|P_v^{(k)}| = \begin{cases} l^{3k-2}(l^2-1)/2\,, & \text{if } \left(\frac{p}{l}\right) = 1 \\ l^{3k-2}(l^2-1)\,, & \text{if } \left(\frac{p}{l}\right) = -1\,. \end{cases}$$

Conjecture 3.52. *Let Γ be any group $\Gamma_{p,l}$ or Γ_{p,l,e_0} of Chapter 3, then*

$$|K_h| = p^2 \text{ and } |K_v| = l^2\,.$$

Remark. We have checked that the Conjectures 3.48(2),(3), 3.49, 3.50, 3.51(1), and Conjecture 3.51(2) for $k = 2$ are true for all 130 examples in Table 3.13. The only uncertainty in Conjecture 3.48(1) among those examples is the case $(p, l) = (73, 97)$, where we are not able to compute $[\Gamma,\Gamma]^{ab}$.

3.6 Construction of anti-tori

Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ be a $(2m, 2n)$–group. Let $a \in \langle a_1, \dots, a_m \rangle$, $b \in \langle b_1, \dots, b_n \rangle$ be two elements. The subgroup $\langle a, b \rangle < \Gamma$ is called an *anti-torus* in Γ, if a and b have no commuting non-trivial powers, i.e. if $a^r b^s \neq b^s a^r$ for all $r, s \in \mathbb{Z} \setminus \{0\}$. If $\langle a, b \rangle \cong F_2$, then $\langle a, b \rangle$ is called a *free anti-torus* in Γ. Obviously, a free anti-torus is an anti-torus.

A definition in a much more general context is given by Bridson-Wise. We quote from [10, Definition 9.1]: "Let X be a compact non-positively curved space with universal cover $p : \tilde{X} \to X$. Suppose that there is an isometrically embedded plane in $\tilde{X}$ which contains an axis for each of $\delta, \delta' \in \pi_1(X, x_0)$ and that $\tilde{x}_0 \in p^{-1}x_0$ lies in the intersection of these axes. If δ and δ' do not have powers that commute, then $\mathrm{gp}\{\delta, \delta'\}$ is called an *anti-torus*. If $\mathrm{gp}\{\delta, \delta'\}$ is free then it is called a *free anti-torus*."

The first example of a (non-free) anti-torus was given by Wise [68] (it is $\langle a_2, b_3 \rangle$ in Example 2.36). It was used to construct interesting non-residually finite groups. An existence theorem for free anti-tori (in a class not including $(2m, 2n)$–groups) appears in [10, Proposition 9.2], but no explicit example of a free anti-torus is given there or elsewhere, as far as we know.

The construction of $\Gamma_{p,l}$ in Chapter 3, based on the non-commutativity of quaternion multiplication, can be used to generate many anti-tori. Before giving examples, we will first state some general criteria for the existence of anti-tori in commutative transitive $(2m, 2n)$–groups.

Proposition 3.53. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ be a commutative transitive $(2m, 2n)$–group and let $a \in \langle a_1, \dots, a_m \rangle$, $b \in \langle b_1, \dots, b_n \rangle$ be two elements. Then $\langle a, b \rangle$ is an anti-torus in Γ if and only if a and b do not commute in Γ.*

Proof. Assume first that $\langle a, b \rangle$ is no anti-torus in Γ, i.e. $a^r b^s = b^s a^r$ for some $r, s \neq 0$. Obviously, a commutes with a^r, and b commutes with b^s. Using the assumption that Γ is commutative transitive, we conclude that a and b commute in Γ. The other direction follows immediately from the definition of an anti-torus. □

Corollary 3.54. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ be a commutative transitive $(2m, 2n)$–group and let $a \in \langle a_1, \dots, a_m \rangle$, $b \in \langle b_1, \dots, b_n \rangle$ be two non-trivial elements. Then either $\langle a, b \rangle \cong \mathbb{Z}^2$ or $\langle a, b \rangle$ is an anti-torus in Γ.*

Proof. If a and b do not commute, then $\langle a, b \rangle$ is an anti-torus in Γ by Proposition 3.53. If $a \neq 1$ and $b \neq 1$ commute, then we apply Lemma 3.14 to show that $\langle a, b \rangle \cong \mathbb{Z}^2$. □

Corollary 3.55. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m \cdot n} \rangle$ be a commutative transitive $(2m, 2n)$–group. Then Γ has an anti-torus if and only if $(m, n) \neq (1, 1)$.*

Proof. Any $(2,2)$–group is virtually abelian, hence has no anti-torus. For the other direction, assume that $(m,n) \neq (1,1)$. There are elements $a \in E_h$ and $b \in E_v$ which do not commute; otherwise the $(2m,2n)$–group Γ would be

$$\langle a_1,\dots,a_m\rangle \times \langle b_1,\dots,b_n\rangle \cong F_m \times F_n\,,$$

which is not commutative transitive if $(m,n) \neq (1,1)$. By Proposition 3.53, $\langle a,b\rangle$ is an anti-torus in Γ. □

Wise ([68]) showed that reducible $(2m,2n)$–groups never have anti-tori:

Proposition 3.56. *(Wise [68, Section II.4]) Let $\Gamma = \langle a_1,\dots,a_m,b_1,\dots,b_n \mid R_{m\cdot n}\rangle$ be a $(2m,2n)$–group. If Γ has an anti-torus, then it is irreducible.*

Proof. Let $\langle a,b\rangle$ be an anti-torus in Γ, where $a \in \langle a_1,\dots,a_m\rangle$, $b \in \langle b_1,\dots,b_n\rangle$. Suppose that Γ is reducible. Then by [17, Proposition 1.2], the subgroup $\Lambda_1 \times \Lambda_2$ has finite index in Γ, in particular $[\langle a_1,\dots,a_m\rangle : \Lambda_1]$ and $[\langle b_1,\dots,b_n\rangle : \Lambda_2]$ are finite. It follows that $a^r \in \Lambda_1$, $b^s \in \Lambda_2$ for some $r,s \in \mathbb{N}$. But then $a^r b^s = b^s a^r$, a contradiction. □

Corollary 3.57. *A commutative transitive $(2m,2n)$–group is irreducible if and only if $(m,n) \neq (1,1)$.*

Proof. Any $(2,2)$–group is reducible. If $(m,n) \neq (1,1)$, then we apply a combination of Corollary 3.55 and Proposition 3.56. □

Corollary 3.58. *Let $\Gamma = \langle a_1,\dots,a_m,b_1,\dots,b_n \mid R_{m\cdot n}\rangle$ be a commutative transitive $(2m,2n)$–group and let $b \in \langle b_1,\dots,b_n\rangle$ be an element such that $Z_\Gamma(b) = \langle b\rangle$. Then $\langle a,b\rangle$ is an anti-torus in Γ for each $a \in \langle a_1,\dots,a_m\rangle \setminus \{1\}$.*

Proof. The assumption $Z_\Gamma(b) = \langle b\rangle$ implies that $b \neq 1$ and that b does not commute with any element $a \in \langle a_1,\dots,a_m\rangle \setminus \{1\}$. Now apply Proposition 3.53. □

The groups $\Gamma_{p,l}$ of Section 3.2 are commutative transitive by Proposition 3.20. Therefore, we can restate the preceding results for $\Gamma_{p,l}$:

Corollary 3.59. *Let $\Gamma = \Gamma_{p,l} = \langle a_1,\dots,a_{\frac{p+1}{2}},b_1,\dots,b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}}\rangle$ be as in Section 3.2 and let $a \in \langle a_1,\dots,a_{\frac{p+1}{2}}\rangle$, $b \in \langle b_1,\dots,b_{\frac{l+1}{2}}\rangle$ be two elements. Then*

(1) *$\langle a,b\rangle$ is an anti-torus in Γ if and only if a and b do not commute in Γ.*

(2) *If $a,b \neq 1$, then either $\langle a,b\rangle \cong \mathbb{Z}^2$ or $\langle a,b\rangle$ is an anti-torus in Γ.*

(3) *The group Γ has an anti-torus and is irreducible.*

(4) *If $Z_\Gamma(b) = \langle b\rangle$ and $a \neq 1$, then $\langle a,b\rangle$ is an anti-torus in Γ.*

We can also restate Proposition 3.53 for $\Gamma_{p,l}$ in terms of quaternions:

Proposition 3.60. *Let ψ and $\Gamma = \Gamma_{p,l}$ be as in Section 3.2. Assume that $x, y \in \mathbb{H}(\mathbb{Z})$ have type o_0, $|x|^2 = p^r$, $|y|^2 = l^s$ for some $r, s \in \mathbb{N}$ and $xy \neq yx$. Then $\langle \psi(x), \psi(y) \rangle$ is an anti-torus in Γ.*

Proof. By Lemma 3.19, $\psi(x)$ and $\psi(y)$ do not commute, hence $\langle \psi(x), \psi(y) \rangle$ is an anti-torus in Γ by Proposition 3.53. □

Proposition 3.60 can be applied for example to $\Gamma_{5,17}$ and $\Gamma_{13,17}$ or to any other group $\Gamma_{p,l}$ of Section 3.2, illustrating Corollary 3.59(3):

Corollary 3.61. *Let ψ be as in Section 3.2. Then*

(1) *The group $\langle \psi(1+2i), \psi(1+4k) \rangle$ is an anti-torus in $\Gamma_{5,17}$.*

(2) *The group $\langle \psi(3+2i), \psi(1+4k) \rangle$ is an anti-torus in $\Gamma_{13,17}$.*

(3) *Fix two distinct prime numbers $p, l \equiv 1 \pmod 4$. Choose by Lemma 3.7(1) two quaternions $x = x_0 + x_1 i$, $y = y_0 + y_3 k \in \mathbb{H}(\mathbb{Z})$ such that x_0, y_0 are odd, x_1, y_3 are non-zero even numbers and $|x|^2 = x_0^2 + x_1^2 = p$, $|y|^2 = y_0^2 + y_3^2 = l$. Then $\langle \psi(x), \psi(y) \rangle$ is an anti-torus in $\Gamma_{p,l}$.*

Proof. (1) We apply Proposition 3.60, taking $x = 1+2i$, $y = 1+4k$, $p = 5$, $l = 17, r = 1, s = 1$.

(2) We apply Proposition 3.60, taking $x = 3+2i$, $y = 1+4k$, $p = 13$, $l = 17$, $r = 1, s = 1$.

(3) We apply Proposition 3.60, taking $r = 1, s = 1$ and using the fact that $x_0 + x_1 i$ and $y_0 + y_3 k$ do not commute.

□

Proposition 3.62. *There are distinct prime numbers $p, l \equiv 1 \pmod 4$, a group*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2} \cdot \frac{l+1}{2}} \rangle$$

as in Section 3.2, and an element $b \in \langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle$, such that $\langle a, b \rangle$ is an anti-torus in Γ for each $a \in \langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle \setminus \{1\}$.

We give two different proofs of Proposition 3.62:

First proof of Proposition 3.62. We choose $p = 5, l = 13$ and

$$b = b_1 = \psi(1 + 2i + 2j + 2k) \in \Gamma_{5,13}\,.$$

By Proposition 3.29(7), we have $Z_\Gamma(b) = \langle b \rangle$ and apply now Corollary 3.58. □

Second proof of Proposition 3.62. We take $p = 5, l = 29$,

$$b = \psi(3 + 2j + 4k) \in \Gamma_{5,29} \text{ and } c = j + 2k \in \mathbb{H}(\mathbb{Z}) .$$

Assume that there is a non-trivial element $a \in \langle a_1, a_2, a_3\rangle < \Gamma_{5,29}$ commuting with some power b^t, $t \in \mathbb{N}$. Note that

$$b^t = \psi((3 + 2j + 4k)^t) = \psi(x_0 + \lambda j + 2\lambda k)$$

for some x_0, $\lambda \neq 0$, depending on t. Then, applying Proposition 3.22 to the power $z = (3 + 2j + 4k)^t$, there are $x, y \in \mathbb{Z}$ such that

$$\gcd(x, y) = \gcd(x, pl) = \gcd(y, pl) = 1$$

and $x^2 + 4 \cdot 5y^2 = 5^r 29^s$ for some $r, s \in \mathbb{N}$. But this implies $x^2 = 5(5^{r-1}29^s - 4y^2)$, contradicting $\gcd(x, 5 \cdot 29) = 1$. (What we use here is that such a decomposition $x^2 + 4 \cdot |c|^2 y^2 = p^r l^s$ implies $\gcd(|c|^2, pl) = 1$, as already noted in [54].) $\square$

Proposition 3.63. *There are distinct prime numbers $p, l \equiv 1 \pmod 4$, a group*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}}\rangle$$

as in Section 3.2, and elements $a \in \langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle \setminus \{1\}$, $b \in \langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle \setminus \{1\}$ such that $\langle a, b_j\rangle$ is an anti-torus in Γ for all $b_j \in \{b_1, \ldots, b_{\frac{l+1}{2}}\}$, but $\langle a, b\rangle$ is no anti-torus in Γ, in particular $Z_\Gamma(a) \neq \langle a\rangle$.

Proof. We take $p = 29, l = 41, a = \psi(3 + 4i + 2j)$ and

$$b = \psi(-31 + 24i + 12j) = \psi(1 + 6j - 2k)\psi(1 + 6j + 2k) ,$$

which implies $ab = ba$. It is easy to check that a does not commute with any generator $b_j \in \{b_1, \ldots, b_{21}\}$, in particular $\langle a, b_j\rangle$ is an anti-torus in Γ by Proposition 3.53. $\square$

Also note the following easy corollary of Proposition 3.13, see Corollary 4.3 for a generalization to all $(2m, 2n)$–groups:

Corollary 3.64. *Let $p, l \equiv 1 \pmod 4$ be distinct prime numbers and*

$$\Gamma = \Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}}\rangle$$

as in Section 3.2. Then there are always non-trivial elements $a \in \langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle$ and $b \in \langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle$ such that $\langle a, b\rangle$ is no anti-torus in Γ.

Free anti-tori

The following proposition gives sufficient conditions to generate *free* anti-tori in the groups $\Gamma_{p,l}$ of Section 3.2:

Proposition 3.65. *Let $p, l \equiv 1 \pmod 4$ be two distinct prime numbers and let ψ and $\Gamma_{p,l}$ be as in Section 3.2. Moreover, let $x, y \in \mathbb{H}(\mathbb{Z})$ be of type o_0, such that $|x|^2 = p^r$, $|y|^2 = l^s$ for some $r, s \in \mathbb{N}$. Suppose that x, y generate a free subgroup F_2 in the multiplicative group $U(\mathbb{H}(\mathbb{Q})) = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$ (or equivalently in the subgroup $U(\mathbb{H}(\mathbb{Z}[1/p, 1/l])) < U(\mathbb{H}(\mathbb{Q}))$). Then $\langle \psi(x), \psi(y) \rangle$ is a free anti-torus in $\Gamma_{p,l}$.*

Proof. Extending ψ from the integer to the rational quaternions, let

$$\tilde{\psi} : U(\mathbb{H}(\mathbb{Q})) \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_l)$$

be the map which sends the quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$ to

$$\left(\left[\begin{pmatrix} x_0 + x_1 i_p & x_2 + x_3 i_p \\ -x_2 + x_3 i_p & x_0 - x_1 i_p \end{pmatrix} \right], \left[\begin{pmatrix} x_0 + x_1 i_l & x_2 + x_3 i_l \\ -x_2 + x_3 i_l & x_0 - x_1 i_l \end{pmatrix} \right] \right),$$

where $x_0, x_1, x_2, x_3 \in \mathbb{Q}$, $x \neq 0$. Recall that $U(\mathbb{H}(\mathbb{Q})) = \mathbb{H}(\mathbb{Q}) \setminus \{0\}$ equipped with quaternion multiplication is a non-abelian group, $\tilde{\psi}$ is a group homomorphism such that

$$\ker(\tilde{\psi}) = ZU(\mathbb{H}(\mathbb{Q})) = \{x \in \mathbb{H}(\mathbb{Q}) \setminus \{0\} : x = \overline{x}\},$$

and $\tilde{\psi}(x) = \psi(x)$, if $x \in \mathbb{H}(\mathbb{Z}) \setminus \{0\}$. Now, fix two integer quaternions x and y satisfying the assumptions made in the proposition. We restrict $\tilde{\psi}$ to the free subgroup $F_2 \cong \langle x, y \rangle < U(\mathbb{H}(\mathbb{Q}))$:

$$\tilde{\psi}|_{\langle x,y \rangle} : \langle x, y \rangle \cong F_2 \twoheadrightarrow \langle \tilde{\psi}(x), \tilde{\psi}(y) \rangle = \langle \psi(x), \psi(y) \rangle < \Gamma_{p,l}\,.$$

We have

$$\ker\bigl(\tilde{\psi}|_{\langle x,y \rangle}\bigr) = \langle x, y \rangle \cap ZU(\mathbb{H}(\mathbb{Q})) < Z(\langle x, y \rangle) \cong ZF_2 = \{1\},$$

in particular $\tilde{\psi}|_{\langle x,y \rangle}$ is an isomorphism, i.e. $\langle \psi(x), \psi(y) \rangle \cong F_2$.

By construction, $\psi(x)$ is an element in

$$\langle a_1, \ldots, a_{\frac{p+1}{2}} \rangle = \{\psi(x) : x \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |x|^2 = p^r;\ r \in \mathbb{N}_0\} < \Gamma_{p,l}\,,$$

and $\psi(y)$ an element in

$$\langle b_1, \ldots, b_{\frac{l+1}{2}} \rangle = \{\psi(y) : y \in \mathbb{H}(\mathbb{Z}) \text{ has type } o_0,\ |y|^2 = l^s;\ s \in \mathbb{N}_0\} < \Gamma_{p,l}\,,$$

where the $(p+1, l+1)$–group $\Gamma_{p,l}$ is generated by $a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}}$ as usual. This shows that $\langle \psi(x), \psi(y) \rangle$ is a free anti-torus in $\Gamma_{p,l}$. $\square$

For example, if $\langle 3+2i, 1+4k\rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$, then Proposition 3.65 would give an explicit free anti-torus $\langle \psi(3+2i), \psi(1+4k)\rangle$ in $\Gamma_{13,17}$. (However, we guess that this group is not free.)

Question 3.66. *Is* $\langle 3+2i, 1+4k\rangle \cong F_2$*?*

More generally:

Problem 3.67. *Let p, l be distinct odd prime numbers. Construct a pair $x, y \in \mathbb{H}(\mathbb{Z})$ such that $\langle x, y\rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$, where $|x|^2 = p^r$, $|y|^2 = l^s$ for some $r, s \in \mathbb{N}$.*

The anti-tori constructed in Corollary 3.61(1) and Proposition 3.47(8) are not free:

Proposition 3.68. (1) *Let ψ be as in Section 3.2, $x = 1+2i$, $y = 1+4k$, $a = \psi(x)$ and $b = \psi(y)$. Then the anti-torus $\langle a, b\rangle$ in $\Gamma_{5,17}$ is not free.*

(2) *Let ψ be as in Section 3.4, $x = 1+j+k$, $y = 1+2i$, $a = \psi(x)$, $b = \psi(y)$. Then the anti-torus $\langle a, b\rangle$ in $\Gamma_{3,5}$ is not free.*

Proof. (1) In $\Gamma_{5,17}$, we have found the relation

$$a^3b^2ab^{-1}a^2b^{-1}a^2b^{-1}a^{-4}b^{-2}a^{-1}ba^{-2}b^{-1}a^{-8}b^{-1}ab^2$$
$$ab^{-1}a^{-2}ba^{-1}b^{-2}a^{-2}b^{-2}a^3ba^{-2}b^2a^2b^2ab^{-1}a^2ba^{-1}b^{-2}$$
$$a^{-1}ba^8ba^2b^{-1}ab^2a^4ba^{-2}ba^{-2}ba^{-1}b^{-2}a^{-5}b^{-1}a = 1\,.$$

To get this relation of length 106, we have used the GAP-command ([29])

```
PresentationSubgroupMtc(G,U);
```

where G and U describe Γ and its subgroup $\langle a, b\rangle$, respectively. This command gives 514 relations of lengths between 106 and 5270 and of total length 536176.

The relation in $U(\mathbb{H}(\mathbb{Q}))$ corresponding to the relation in $\Gamma_{5,17}$ given above is

$$x^3y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}x^{-8}y^{-1}xy^2$$
$$xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^3yx^{-2}y^2x^2y^2xy^{-1}x^2yx^{-1}y^{-2}$$
$$x^{-1}yx^8yx^2y^{-1}xy^2x^4yx^{-2}yx^{-2}yx^{-1}y^{-2}x^{-5}y^{-1}x = 1\,,$$

in particular $\langle x, y\rangle \neq F_2$. Note that GAP ([29]) can also be used to show that

$$[\Gamma_{5,17} : \langle a, b\rangle] = 32 \text{ and } \langle a, b\rangle^{ab} \cong \mathbb{Z}_{16} \times \mathbb{Z}_{64}\,.$$

Moreover, $\langle a, b\rangle \cong \langle x, y\rangle / Z\langle x, y\rangle$, where $Z\langle x, y\rangle \neq 1$, since e.g.

$$
\begin{aligned}
&xy^{-1}xy^2x^8yx^{-3}y^{-1}xyx^4y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}y\\
&x^{-2}y^{-1}x^{-8}y^{-1}xy^2xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^2y^{-1}x^2y^2\\
&xy^{-1}x^2yx^{-1}y^{-2}x^{-1}yx^8yx^2y^{-1}xy^2x^4yx^{-2}yx^{-2}yx^{-1}y^{-2}\\
&x^{-4}y^{-1}x^{-1}y^{-1}x^3y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}\\
&x^{-8}y^{-1}xy^2xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^5y^2xy^{-1}x^2y^{-1}\\
&x^4y^2xy^{-1}x^2y^{-1}x^2y^{-1}x^{-4}y^{-2}x^{-1}yx^{-2}y^{-1}x^{-8}y^{-1}xy^2\\
&xy^{-1}x^{-2}yx^{-1}y^{-2}x^{-2}y^{-2}x^2y^{-1} = \frac{1}{17^8} \in Z\langle x, y\rangle \setminus \{1\}.
\end{aligned}
$$

(2) See Proposition 3.47(9). Recall that the subgroups $\langle a^t, b^t\rangle$, $t \in \mathbb{N}$, are never abelian, and that $[\Gamma_{3,5} : \langle a, b\rangle] = 4$. Also note that $[\Gamma_{3,5} : \langle a^2, b^2\rangle] = 896$ is finite, using GAP ([29]). In particular $\langle a^2, b^2\rangle$ is not free by the following general remark.

□

Remark. If $\langle a, b\rangle$ is a free subgroup in a $(2m, 2n)$–group Γ, then the index $[\Gamma : \langle a, b\rangle]$ is infinite. Otherwise, Γ would be virtually free, but this is impossible since being virtually free is a quasi-isometry invariant (see e.g. [32, IV.50]), and using the facts that $(2m, 2n)$–groups are all quasi-isometric (to $F_2 \times F_2$), if $m, n \geq 2$ (see Proposition 4.25(4)), and that there are $(2m, 2n)$–groups which obviously are not virtually free, e.g. the virtually simple groups constructed in Chapter 2. Anyway, it is known that finitely generated, torsion-free, virtually free groups are free ([65]).

The following interesting general question of Wise appears in Bestvina's problem list "Questions in Geometric Group Theory" ([6]):

Question 3.69. *(Wise [6, Question 2.7]) "Let G act properly discontinuously and cocompactly on a CAT(0) space (or let G be automatic). Consider two elements a, b of G. Does there exist $n > 0$ such that either the subgroup $\langle a^n, b^n\rangle$ is free or $\langle a^n, b^n\rangle$ is abelian?"*

Question 3.70. *Let $\Gamma = \Gamma_{3,5}$ be the group of Example 3.46 and $a_1 = \psi(1 + j + k)$, $b_1 = \psi(1 + 2i)$.*

(1) *Is the index $[\Gamma : \langle a_1^3, b_1^3\rangle]$ infinite?*

(2) *Is $\langle a_1^3, b_1^3\rangle$ free?*

Free subgroups of $U(\mathbb{H}(\mathbb{Q}))$ also induce free subgroups in $\mathrm{SO}_3(\mathbb{Q}) < \mathrm{SO}_3(\mathbb{R})$ via the group homomorphism $\vartheta : U(\mathbb{H}(\mathbb{Q})) \to \mathrm{SO}_3(\mathbb{Q})$, which maps the quaternion $x = x_0 + x_1 i + x_2 j + x_3 k \in U(\mathbb{H}(\mathbb{Q}))$ to the (3×3)–matrix

$$\frac{1}{|x|^2}\begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix},$$

see Section 3.2. The proof is similar to a part of the proof of Proposition 3.65: First remember that

$$\ker(\vartheta) = ZU(\mathbb{H}(\mathbb{Q})) = \{x \in \mathbb{H}(\mathbb{Q}) \setminus \{0\} : x = \overline{x}\}.$$

Assume now that $F_2 \cong \langle x, y\rangle < U(\mathbb{H}(\mathbb{Q}))$. Then

$$\vartheta|_{\langle x,y\rangle} : \langle x, y\rangle \twoheadrightarrow \langle \vartheta(x), \vartheta(y)\rangle < \mathrm{SO}_3(\mathbb{Q})$$

is bijective, since it is surjective and

$$\ker\left(\vartheta|_{\langle x,y\rangle}\right) = \langle x, y\rangle \cap ZU(\mathbb{H}(\mathbb{Q})) < Z(\langle x, y\rangle) \cong ZF_2 = \{1\},$$

in particular $\langle \vartheta(x), \vartheta(y)\rangle \cong F_2$.

Note that if

$$\Gamma_{p,l} = \langle a_1, \ldots, a_{\frac{p+1}{2}}, b_1, \ldots, b_{\frac{l+1}{2}} \mid R_{\frac{p+1}{2}\cdot\frac{l+1}{2}}\rangle$$

is the group of Section 3.2, then both free subgroups $\langle a_1, \ldots, a_{\frac{p+1}{2}}\rangle$ and $\langle b_1, \ldots, b_{\frac{l+1}{2}}\rangle$ of $\Gamma_{p,l}$ induce free subgroups of $\mathrm{SO}_3(\mathbb{Q})$ via the homomorphism ϑ (we can combine Corollary 1.11(1) and Theorem 3.12(2), cf. [45, Corollary 2.1.11]). For example, taking $p = 5$ and any distinct prime number $l \equiv 1 \pmod 4$, the subgroup

$$\begin{aligned} &\langle a_1, a_2, a_3\rangle \cong \langle \vartheta(1 + 2i), \vartheta(1 + 2j), \vartheta(1 + 2k)\rangle \\ &= \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix}, \begin{pmatrix} -3/5 & -4/5 & 0 \\ 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \end{aligned}$$

of $\mathrm{SO}_3(\mathbb{Q})$ is isomorphic to F_3.

However, by Proposition 3.68, the following two subgroups of $\mathrm{SO}_3(\mathbb{Q})$ are not free:

$$\langle \vartheta(1 + 2i), \vartheta(1 + 4k)\rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} -15/17 & -8/17 & 0 \\ 8/17 & -15/17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle,$$

$$\langle \vartheta(1 + j + k), \vartheta(1 + 2i)\rangle = \left\langle \frac{1}{3}\begin{pmatrix} -3 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix} \right\rangle.$$

We can use the explicit amalgam decompositions of $\Gamma_{p,l}$ to construct two integer quaternions x and y generating a non-abelian free group in $U(\mathbb{H}(\mathbb{Q}))$ such that $|x|^2$ and $|y|^2$ are not both powers of the same prime number (cf. Problem 3.67). We illustrate this with an example:

Proposition 3.71. *Let ψ be as in Section 3.4, $x = 1+2i+2j+4k$ of norm $|x|^2 = 5^2$, $y = 3 - 2i + j - k$ of norm $|y|^2 = 3 \cdot 5$. Then $\langle x, y\rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$.*

Proof. We have

$$\psi(x) = \psi(1+2i)\psi(1+2j) = b_1 b_2 \in \Gamma_{3,5}$$

and

$$\psi(y) = \psi(1+j+k)\psi(1-2k) = a_1 b_3^{-1} \in \Gamma_{3,5}\,.$$

By the vertical amalgam decomposition of $\Gamma_{3,5}$ given in Appendix A.12

$$F_2 \cong \langle s_1, s_4\rangle = \langle b_1 b_2, a_1 b_3^{-1}\rangle = \langle \psi(x), \psi(y)\rangle < \Gamma_{3,5}\,,$$

hence $\langle x, y\rangle \cong F_2 < U(\mathbb{H}(\mathbb{Q}))$. $\square$

3.7 A construction for $(p, l) = (2, 5)$

Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}(\mathbb{Z})$. Motivated by the three identities ([24])

$$\begin{aligned}
(1+i)(x_0 + x_1 i + x_2 j + x_3 k) &= (x_0 + x_1 i - x_3 j + x_2 k)(1+i)\\
(1+j)(x_0 + x_1 i + x_2 j + x_3 k) &= (x_0 + x_3 i + x_2 j - x_1 k)(1+j)\\
(1+k)(x_0 + x_1 i + x_2 j + x_3 k) &= (x_0 - x_2 i + x_1 j + x_3 k)(1+k)
\end{aligned}$$

we identify

$$\begin{aligned}
a_1 &\cong 1+i, & a_1^{-1} &\cong 1-i,\\
a_2 &\cong 1+j, & a_2^{-1} &\cong 1-j,\\
a_3 &\cong 1+k, & a_3^{-1} &\cong 1-k,\\
\\
b_1 &\cong 1+2i, & b_1^{-1} &\cong 1-2i,\\
b_2 &\cong 1+2j, & b_2^{-1} &\cong 1-2j,\\
b_3 &\cong 1+2k, & b_3^{-1} &\cong 1-2k,
\end{aligned}$$

and get the following $(6, 6)$–group:

Example 3.72. *Let Γ be the group $\langle a_1, a_2, a_3, b_1, b_2, b_3 \mid R_{3\cdot 3}\rangle$, where*

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2, \\ a_2b_1a_2^{-1}b_3, & a_2b_2a_2^{-1}b_2^{-1}, & a_2b_3a_2^{-1}b_1^{-1}, \\ a_3b_1a_3^{-1}b_2^{-1}, & a_3b_2a_3^{-1}b_1, & a_3b_3a_3^{-1}b_3^{-1} \end{array} \right\}.$$

Note that there is no map ψ involved in this construction, in particular Γ behaves completely differently than the groups $\Gamma_{p,l}$ constructed before, e.g. Γ is reducible, $(1+i)^4 = -4$, but $a_1^4 \neq 1_\Gamma$; $1+i$ and $1+2j$ do not commute, but $\langle a_1, b_2\rangle$ is no anti-torus.

Proposition 3.73. *Let Γ be the $(6,6)$–group defined in Example 3.72. Then*

(1) $P_h = 1$, $P_v \cong S_4 < S_6$.

(2) Γ *is reducible.*

(3) $\Lambda_1 \times \Lambda_2 \cong F_{49} \times F_3$ *and* $[\Gamma : \Lambda_1 \times \Lambda_2] = 24$.

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= \rho_v(b_2) = \rho_v(b_3) = (), \\ \rho_h(a_1) &= (2,4,5,3), \\ \rho_h(a_2) &= (1,3,6,4), \\ \rho_h(a_3) &= (1,5,6,2). \end{aligned}$$

(2) This follows from the subsequent Lemma 3.74(1).

(3) Apply Lemma 3.74(3).

□

Lemma 3.74. *Let $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m\cdot n}\rangle$ be a $(2m, 2n)$–group such that $P_h = 1$. Then*

(1) Γ *is reducible and* $P_h^{(k)} = 1$ *for all* $k \in \mathbb{N}$.

(2) $\Lambda_1 \cong \ker \rho_h$ *and* $\Lambda_2 \cong \ker \rho_v^{(k)} = \langle b_1, \dots, b_n\rangle$ *for all* $k \in \mathbb{N}$.

(3) $\Lambda_1 \times \Lambda_2 \cong F_{(m-1)|P_v|+1} \times F_n$ *has index* $|P_v|$ *in* Γ.

Proof. (1) To prove that Γ is reducible, it is enough by Proposition 1.2(2a) to show that $P_h^{(2)} = 1$. Let $b \in E_v$, $a = \hat{a} \cdot \tilde{a} \in E_h^{(2)}$, where $\hat{a}, \tilde{a} \in E_h$, $\hat{a} \neq \tilde{a}^{-1}$. Then $\rho_v(b)(\hat{a}) = \hat{a}$ and $\rho_v(\rho_h(\hat{a})(b))(\tilde{a}) = \tilde{a}$, i.e. $\rho_v^{(2)}(b)(a) = a$. See Figure 3.1 for an illustration of this fact. The proof of Proposition 1.2(2a) shows that $P_h^{(k)} = 1$ for all $k \in \mathbb{N}$.

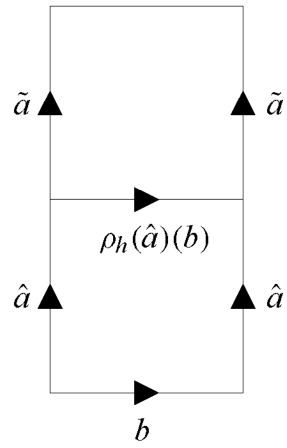


Figure 3.1: Illustrating $P_h^{(2)} = 1$ in Lemma 3.74

(2) Since $\ker \rho_h^{(k+1)} < \ker \rho_h^{(k)}$ for all $k \in \mathbb{N}$, and

$$\Lambda_1 \cong \bigcap_{k \in \mathbb{N}} \ker \rho_h^{(k)},$$

we always have $\Lambda_1 < \ker \rho_h$.
Conversely, $\ker \rho_h < \Lambda_1$ follows from Lemma 1.1(1a) using $P_h = 1$.

To show the second part, observe that $\ker \rho_v^{(k)} = \langle b_1, \ldots, b_n \rangle$ for all $k \in \mathbb{N}$, since $P_h^{(k)} = 1$ for all $k \in \mathbb{N}$. This implies

$$\Lambda_2 \cong \bigcap_{k \in \mathbb{N}} \ker \rho_v^{(k)} = \ker \rho_v^{(k)} = \langle b_1, \ldots, b_n \rangle \text{ for all } k \in \mathbb{N}.$$

(3) This follows from $[\langle a_1, \ldots, a_m \rangle : \Lambda_1] = |P_v|$, which is a direct consequence of part (2) and $P_v \cong \langle a_1, \ldots, a_m \rangle / \ker \rho_h$.

$\square$

Chapter 4

Miscellanea

This chapter consists of six independent sections which we briefly describe now. Given any $(2m, 2n)$–group Γ, we construct in Section 4.1 doubly periodic tilings of the Euclidean plane, where the tiles are the $4mn$ squares corresponding to Γ. It follows that Γ always has free abelian subgroups $\mathbb{Z}^2$. We apply a criterion of Burger-Mozes in Section 4.2 to prove that certain $(2m, 2n)$–groups are not linear. In Section 4.3, we investigate possible relations between reducibility, transitivity properties of the local groups, and finiteness of the abelianization of a $(2m, 2n)$–group. Following Mozes, we associate in Section 4.4 to any $(2m, 2n)$–group two infinite families of finite regular graphs. In Section 4.5, we show that any $(2m, 2n)$–group is quasi-isometric to the group $F_2 \times F_2$, if $m, n \geq 2$, and compute its growth series. We prove in Section 4.6 that $(2m, 2n)$–groups are efficient and compute their deficiency.

4.1 Periodic tilings and $\mathbb{Z}^2$-subgroups

For the moment, let X be a locally compact complete CAT(0)-space and Γ a properly discontinuous and cocompact group of isometries. Then, in this general context, it is an open question if certain free abelian subgroups of Γ exist. We quote from an article of Ballmann [1, Question 2.3]: "Is hyperbolicity equivalent to the non-existence of a subgroup of Γ isomorphic to $\mathbb{Z}^2$? More generally, does Γ contain a subgroup isomorphic to $\mathbb{Z}^k$ if X contains a k-flat? By the work of Bangert and Schroeder [2] the answer is positive in the case of compact, real analytic Riemannian manifolds. Except for this, the answers to these questions are completely open, even in the case where X is a geodesically complete and piecewise Euclidean complex of dimension two!"

We will give in Proposition 4.2(3) an elementary proof that $(2m, 2n)$–groups always contain a $\mathbb{Z}^2$-subgroup. The idea of this proof (and the fact that this result holds) was explained to me by Guyan Robertson.

Let $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$ be a $(2m, 2n)$–group and let $T(\Gamma)$ be the "tile set" consisting of the $4mn$ squares which represent a geometric square in the corresponding $(2m, 2n)$–complex X.

$$T(\Gamma) := \bigcup_{aba'b' \in R_{m\cdot n}} \{aba'b',\ a'b'ab,\ a^{-1}b'^{-1}a'^{-1}b^{-1},\ a'^{-1}b^{-1}a^{-1}b'^{-1}\}\,.$$

It is easy to check that the definition of $T(\Gamma)$ only depends on the group Γ, but not on the choice of the relators in $R_{m\cdot n}$. Recall that the four squares $aba'b'$, $a'b'ab$, $a^{-1}b'^{-1}a'^{-1}b^{-1}$ and $a'^{-1}b^{-1}a^{-1}b'^{-1}$ represent the same geometric square $[aba'b']$. We always visualize them in the Euclidean plane as in Figure 4.1.

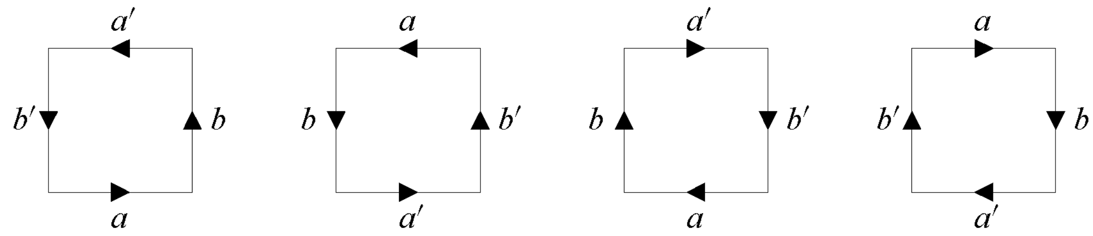


Figure 4.1: Tiles in $T(\Gamma)$ induced by the geometric square $[aba'b']$

Moreover, we assume that each edge of such an element in $T(\Gamma)$ has length 1. Unit squares like this are usually called *Wang tiles* (named after Hao Wang [66]). We define "south-", "east-", "north-" and "west-functions"

$$S, E, N, W : T(\Gamma) \to E_h \sqcup E_v$$

as follows:

$$S(aba'b') := a,\ \ E(aba'b') := b,\ \ N(aba'b') := a'^{-1},\ \ W(aba'b') := b'^{-1}\,.$$

A *tiling* (of the Euclidean plane) is a map $f : \mathbb{Z}^2 \to T(\Gamma)$. We are only interested in *valid* tilings, i.e. tilings where all edges match. To be precise, this means that for each point $(x, y) \in \mathbb{Z}^2$

$$S(f(x,y)) = N(f(x,y-1)) \text{ and } W(f(x,y)) = E(f(x-1,y))\,.$$

A valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ is said to satisfy the adjacency condition (AC) if for each $(x, y) \in \mathbb{Z}^2$

$$\begin{aligned} S(f(x,y)) &\neq N(f(x-1,y-1))^{-1} \\ W(f(x,y)) &\neq E(f(x-1,y-1))^{-1} \end{aligned} \tag{AC}$$

i.e. the two situations illustrated in Figure 4.2 are nowhere allowed in the plane.

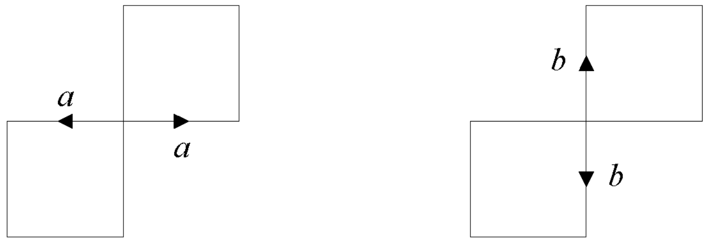


Figure 4.2: Violating (AC)

Note that (AC) is equivalent to the conditions

$$S(f(x-1,y))^{-1} \neq S(f(x,y)) \neq S(f(x+1,y))^{-1}$$
$$N(f(x-1,y))^{-1} \neq N(f(x,y)) \neq N(f(x+1,y))^{-1}$$
$$E(f(x,y-1))^{-1} \neq E(f(x,y)) \neq E(f(x,y+1))^{-1}$$
$$W(f(x,y-1))^{-1} \neq W(f(x,y)) \neq W(f(x,y+1))^{-1}$$

for each $(x,y) \in \mathbb{Z}^2$ and it requires that any word consisting of consecutive horizontal or consecutive vertical edges in the tiling f is freely reduced, where the words of edges are seen as elements in the free groups $\langle a_1,\dots,a_m\rangle < \Gamma$ or $\langle b_1,\dots,b_n\rangle < \Gamma$, respectively.

We say that a valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfies the condition (AC$_j$) for some fixed $j \in \mathbb{Z}$, if for each $i \in \mathbb{Z}$

$$\begin{aligned} S(f(i,i+j)) &\neq N(f(i-1,i-1+j))^{-1} \\ W(f(i,i+j)) &\neq E(f(i-1,i-1+j))^{-1}\,. \end{aligned} \qquad (\mathrm{AC}_j)$$

Note that if (AC$_j$) holds in a valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ for each $j \in \mathbb{Z}$, then also (AC) holds for f.

A valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ is called *periodic* with period $(\tilde{a},\tilde{b}) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, if $f(x,y) = f(x+\tilde{a}, y+\tilde{b})$ for each $(x,y) \in \mathbb{Z}^2$. Observe that if $(\tilde{a},\tilde{b})$ is a period of f then so is $(-\tilde{a},-\tilde{b})$.

The following lemma guarantees the unique extension of any $T(\Gamma)$–valued map f on the main diagonal in $\mathbb{Z}^2$ to a valid tiling of the whole plane satisfying (AC), provided f satisfies the inequalities of condition (AC$_0$).

Lemma 4.1. *Let Γ be a $(2m,2n)$–group and $f : \{(i,i) : i \in \mathbb{Z}\} \to T(\Gamma)$ a map such that for each $i \in \mathbb{Z}$*

$$S(f(i,i)) \neq N(f(i-1,i-1))^{-1} \text{ and } W(f(i,i)) \neq E(f(i-1,i-1))^{-1}\,.$$

Then f uniquely extends to a valid tiling $\mathbb{Z}^2 \to T(\Gamma)$. Moreover, this valid tiling satisfies (AC).

Proof. The existence and uniqueness of a valid tiling $\mathbb{Z}^2 \to T(\Gamma)$ extending the given map f follows directly from the link condition in the $(2m, 2n)$–group Γ. We call this extension again f. By assumption, this f satisfies (AC_0). If $n \in \mathbb{N}_0$, we prove now that condition (AC_n) implies condition (AC_{n+1}). In the same way, one can prove that (AC_{-n}) implies (AC_{-n-1}). By induction, we conclude that $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfies condition (AC).

Fix any $i \in \mathbb{Z}$ and assume that (AC_n) holds. To show (AC_{n+1}), we have to prove that

$$\begin{aligned} S(f(i, i+n+1)) &\neq N(f(i-1, i+n))^{-1} \\ W(f(i, i+n+1)) &\neq E(f(i-1, i+n))^{-1}. \end{aligned}$$

Assume first that

$$N(f(i-1, i+n))^{-1} = S(f(i, i+n+1)) \quad (= N(f(i, i+n))).$$

Since $W(f(i, i+n)) = E(f(i-1, i+n))$, it follows from the link condition in Γ that

$$S(f(i, i+n)) = S(f(i-1, i+n))^{-1} = N(f(i-1, i+n-1))^{-1},$$

contradicting (AC_n). Similarly, assume that

$$W(f(i, i+n+1)) = E(f(i-1, i+n))^{-1} \quad (= W(f(i, i+n))^{-1}).$$

Then $S(f(i, i+n+1)) = N(f(i, i+n))$ implies

$$E(f(i, i+n)) = E(f(i, i+n+1))^{-1} = W(f(i+1, i+n+1))^{-1},$$

again contradicting (AC_n). □

Proposition 4.2. *Fix a $(2m, 2n)$–group $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m \cdot n} \rangle$ and the corresponding tile set $T(\Gamma)$ defined as above. Then*

(1) *There is a periodic valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying* (AC).

(2) *There is a valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying* (AC), *and a number $\tilde{a} \in \mathbb{N}$ such that $f(x, y) = f(x + \tilde{a}, y) = f(x, y + \tilde{a})$ for each $(x, y) \in \mathbb{Z}$, i.e. f has the two periods $(\tilde{a}, 0)$ and $(0, \tilde{a})$ and therefore is* doubly periodic.

(3) *There are commuting elements $a \in \langle a_1, \ldots, a_m \rangle < \Gamma$, $b \in \langle b_1, \ldots, b_n \rangle < \Gamma$ such that*
$$0 < |a| = |b| \leq 64m^2n^2,$$
in particular $\langle a, b \rangle$ is a subgroup of Γ isomorphic to $\mathbb{Z}^2$.

Proof. (1) Given Γ, our goal is to construct a valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$, such that $f(x, y) = f(x + 2, y + 2)$ for each $(x, y) \in \mathbb{Z}^2$. Fix any square

$$t := aba'b' \in T(\Gamma)$$

and define f periodic along the diagonal $\{(i, i) : i \in \mathbb{Z}\}$ as follows. If $a \neq a'$ and $b \neq b'$, then we define $f(i, i) = t$ for each $i \in \mathbb{Z}$. If $a = a'$, then we define

$$f(2i, 2i) = t,\ f(2i + 1, 2i + 1) = a^{-1}b'^{-1}a^{-1}b^{-1} \in T(\Gamma),\ i \in \mathbb{Z}\,.$$

Note that $[a^{-1}b'^{-1}a^{-1}b^{-1}] = [t]$. If $b = b'$, then we define

$$f(2i, 2i) = t,\ f(2i + 1, 2i + 1) = a'^{-1}b^{-1}a^{-1}b^{-1} \in T(\Gamma),\ i \in \mathbb{Z}\,.$$

Also here, $[a'^{-1}b^{-1}a^{-1}b^{-1}] = [t]$. See Figure 4.3 for an illustration of these three cases.

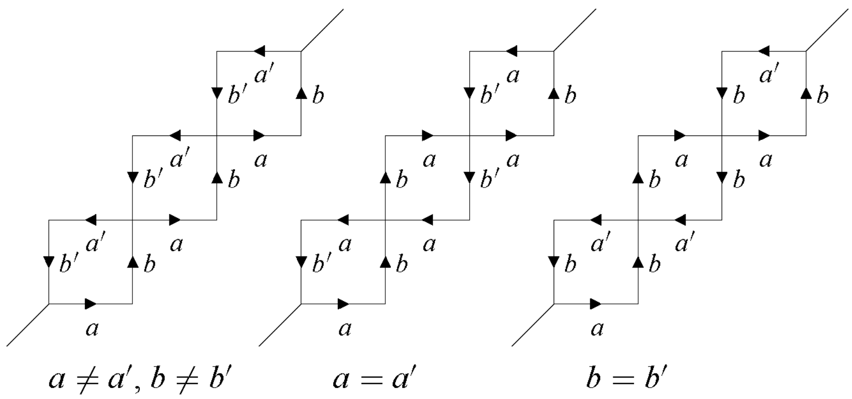


Figure 4.3: Definition of $f(i, i)$ in Proposition 4.2

Now we can apply Lemma 4.1 to the map $f : \{(i, i) : i \in \mathbb{Z}\} \to T(\Gamma)$. The obtained unique extension $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfies (AC) and is obviously periodic with period $(2, 2)$ (in the first case where $a \neq a'$ and $b \neq b'$, there is in fact a smaller period $(1, 1)$).

(2) We use an idea probably going back to Robinson ([60]). It was explained to me by Guyan Robertson. Let $f : \mathbb{Z}^2 \to T(\Gamma)$ be the periodic valid tiling with period $(2, 2)$ satisfying (AC) obtained in part (1). Since $|T(\Gamma)| = 4mn$ is finite, we have

$$|\{(f(i, -i), f(i + 1, -i + 1)) : i \in \mathbb{Z}\}| \leq |T(\Gamma) \times T(\Gamma)| = (4mn)^2 < \infty,$$

in particular there are $i \neq j$, such that $|j - i| \leq (4mn)^2$ and

$$f(i, -i) = f(j, -j) \text{ and } f(i + 1, -i + 1) = f(j + 1, -j + 1)\,.$$

It follows that

$$f(x, y) = f(x + j - i, y + i - j)$$

for each $(x, y) \in \mathbb{Z}^2$. Now, we compute

$$\begin{aligned} f(x, y) &= f(x + j - i, y + i - j) = f(x + 2j - 2i, y + 2i - 2j) \\ &= f(x, y + 4i - 4j) = f(x, y + 4j - 4i) \end{aligned}$$

and

$$\begin{aligned} f(x, y) &= f(x + j - i, y + i - j) = f(x + 2j - 2i, y + 2i - 2j) \\ &= f(x + 4j - 4i, y) = f(x + 4i - 4j, y) . \end{aligned}$$

Note that $0 < |4j - 4i| \le 4(4mn)^2 = 64m^2n^2$.

(3) We use the doubly periodic valid tiling $f : \mathbb{Z}^2 \to T(\Gamma)$ satisfying (AC) of part (2), i.e.

$$f(x, y) = f(x + \tilde{a}, y) = f(x, y + \tilde{a})$$

for each $(x, y) \in \mathbb{Z}$, where $\tilde{a} > 0$. Since any closed edge-path (i.e. any circuit) in this tiling describes a relator in the group Γ, we obviously have two commuting elements $a \in \langle a_1, \dots, a_m \rangle$, $b \in \langle b_1, \dots, b_n \rangle$ corresponding to the periods $(\tilde{a}, 0)$ and $(0, \tilde{a})$. Because of condition (AC), a and b are freely reduced and we therefore have $|a| = |b| = \tilde{a} \in \mathbb{N}$. The upper bound $64m^2n^2$ for the length of $|a|$ and $|b|$ can be obtained by the explicit construction in (2). The statement $\langle a, b \rangle \cong \mathbb{Z}^2$ follows from Lemma 3.14.

□

Remark. The set $T(\Gamma)$ is a *reflection-closed* 4*-way deterministic* tile set (using the terminology of [38]), but $T(\Gamma)$ is never *aperiodic* by Proposition 4.2(1).

We want to illustrate the constructions made in the proof of Proposition 4.2 with a concrete example and take the group $\Gamma = \Gamma_{3,5}$ of Example 3.46 with five generators a_1, a_2, b_1, b_2, b_3 and the six relators in $R_{2\cdot 3}$

$$a_1b_1a_2b_2,\ a_1b_2a_2b_1^{-1},\ a_1b_3a_2^{-1}b_1,\ a_1b_3^{-1}a_1b_2^{-1},\ a_1b_1^{-1}a_2^{-1}b_3,\ a_2b_3a_2b_2^{-1} .$$

This defines the tile set

$$\begin{aligned} T(\Gamma_{3,5}) = &\{a_1b_1a_2b_2,\ a_2b_2a_1b_1,\ a_1^{-1}b_2^{-1}a_2^{-1}b_1^{-1},\ a_2^{-1}b_1^{-1}a_1^{-1}b_2^{-1}\} \\ &\cup \{a_1b_2a_2b_1^{-1},\ a_2b_1^{-1}a_1b_2,\ a_1^{-1}b_1a_2^{-1}b_2^{-1},\ a_2^{-1}b_2^{-1}a_1^{-1}b_1\} \\ &\cup \{a_1b_3a_2^{-1}b_1,\ a_2^{-1}b_1a_1b_3,\ a_1^{-1}b_1^{-1}a_2b_3^{-1},\ a_2b_3^{-1}a_1^{-1}b_1^{-1}\} \\ &\cup \{a_1b_3^{-1}a_1b_2^{-1},\ a_1b_2^{-1}a_1b_3^{-1},\ a_1^{-1}b_2a_1^{-1}b_3,\ a_1^{-1}b_3a_1^{-1}b_2\} \\ &\cup \{a_1b_1^{-1}a_2^{-1}b_3,\ a_2^{-1}b_3a_1b_1^{-1},\ a_1^{-1}b_3^{-1}a_2b_1,\ a_2b_1a_1^{-1}b_3^{-1}\} \\ &\cup \{a_2b_3a_2b_2^{-1},\ a_2b_2^{-1}a_2b_3,\ a_2^{-1}b_2a_2^{-1}b_3^{-1},\ a_2^{-1}b_3^{-1}a_2^{-1}b_2\} . \end{aligned}$$

In Figure 4.4, we recognize a finite part of a periodic valid tiling $f : \mathbb{Z}^2 \to T(\Gamma_{3,5})$ satisfying (AC) induced by $t = a_1 b_1 a_2 b_2 \in T(\Gamma_{3,5})$, with periods

$$(1, 1), (-2, 2), (4, 0), (0, 4) \in \mathbb{Z}(1, 1) + \mathbb{Z}(-2, 2)$$

and commuting elements $a_1 a_2 a_1 a_2^{-1}$, $b_2^{-1} b_1^{-1} b_3^{-1} b_1$, generating the free abelian group

$$\mathbb{Z}^2 \cong \langle a_1 a_2 a_1 a_2^{-1}, b_2^{-1} b_1^{-1} b_3^{-1} b_1 \rangle < \Gamma_{3,5} .$$

Note that the two generators $a_1 a_2 a_1 a_2^{-1}$ and $b_2^{-1} b_1^{-1} b_3^{-1} b_1$ of $\mathbb{Z}^2$ correspond to the two commuting quaternions $5 + 4i + 6j - 2k$ and $-11 - 12i - 18j + 6k$ of norm 3^4 and 5^4, respectively.

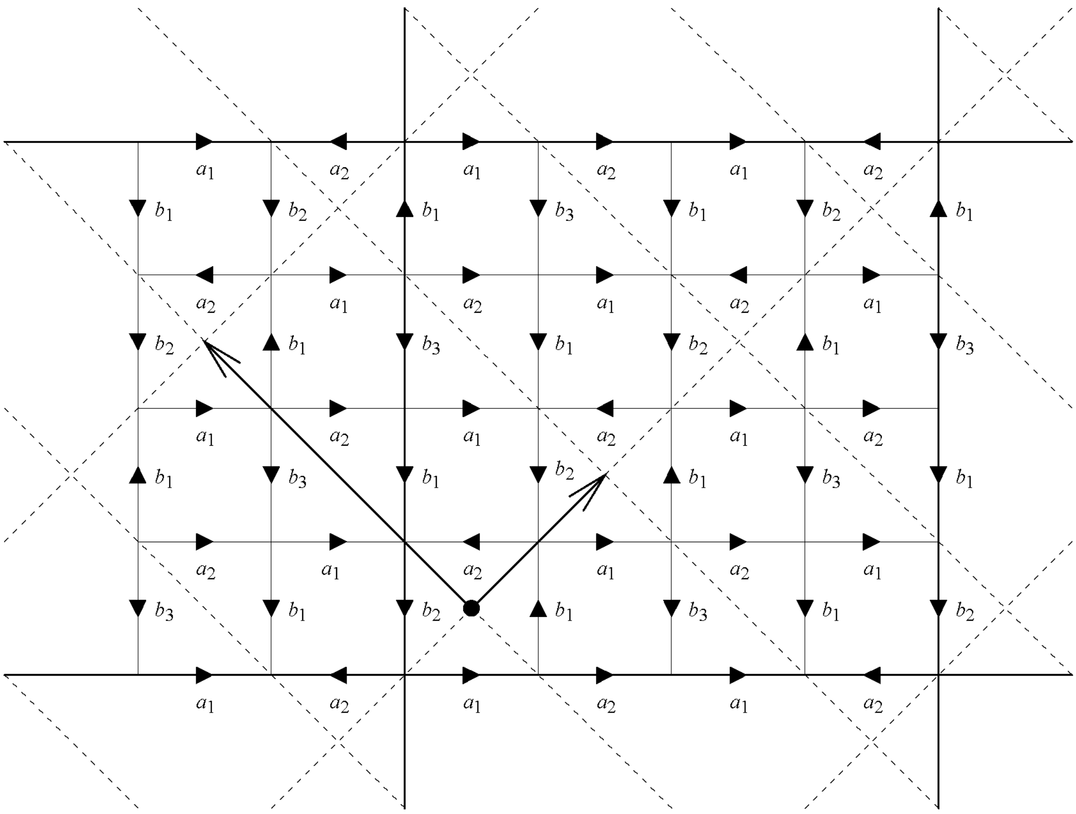


Figure 4.4: Illustration of Proposition 4.2 taking Example 3.46 and $t = a_1 b_1 a_2 b_2$

However, recall that $\langle a_1, b_1 \rangle$ is an anti-torus in $\Gamma_{3,5}$ (see Proposition 3.47(8)), in particular there are also valid non-periodic tilings of the Euclidean plane using the tile set $T(\Gamma_{3,5})$.

See Figure 4.5 for an illustration of a finite part of the non-periodic valid tiling determined by $\langle a_1, b_1\rangle$. Note that all 24 squares of $T(\Gamma_{3,5})$ appear in this picture. To illustrate this, we have equipped the tiles with numbers from 1 to 24.

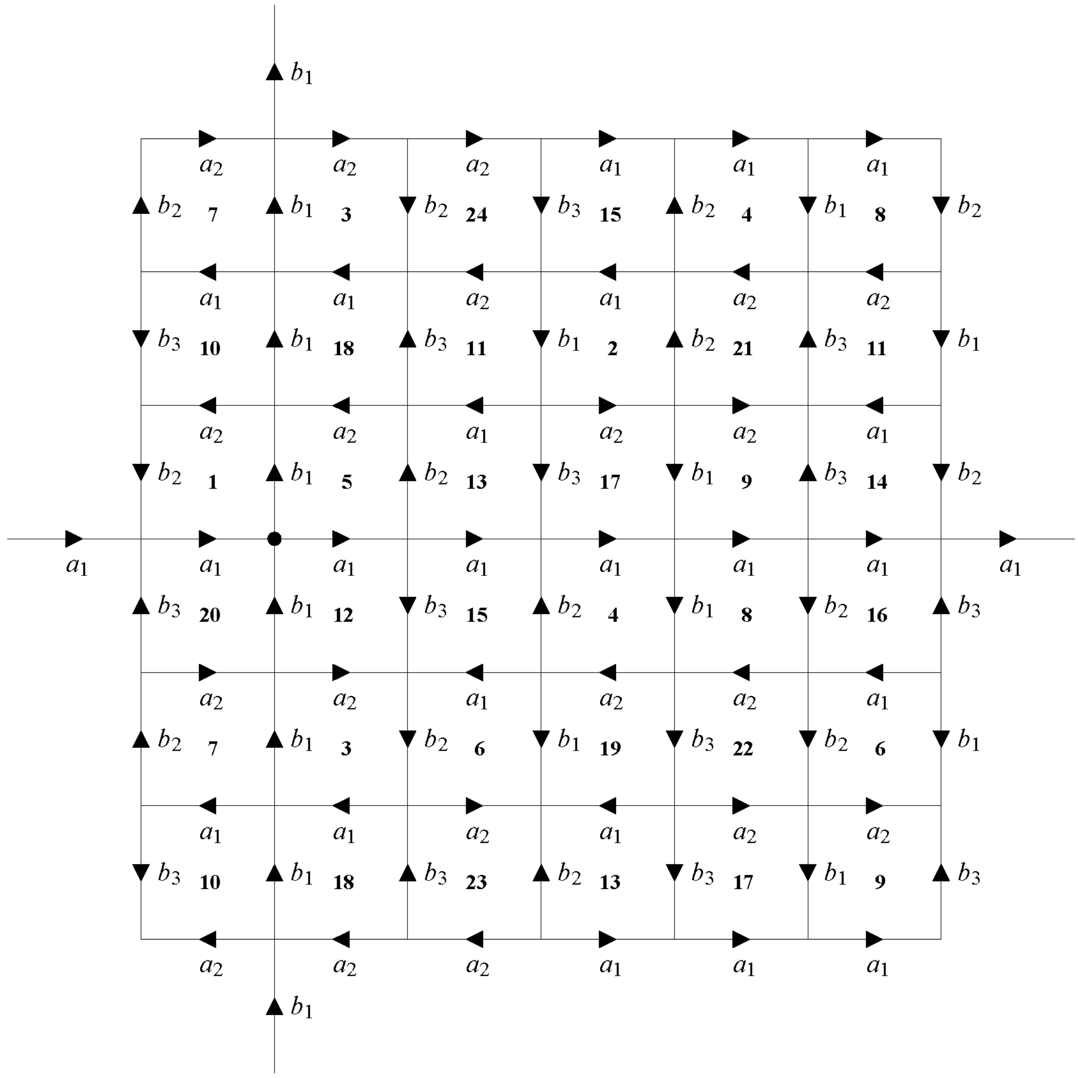


Figure 4.5: A non-periodic tiling in Example 3.46

Corollary 4.3. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m\cdot n}\rangle$ *be a* $(2m, 2n)$*–group. Then there are always non-trivial elements* $a \in \langle a_1, \ldots, a_m\rangle$ *and* $b \in \langle b_1, \ldots, b_n\rangle$ *such that* $\langle a, b\rangle$ *is no anti-torus.*

Proof. This follows directly from Proposition 4.2(3). □

4.2 A criterion for non-linearity

Applying a criterion of Burger-Mozes ([17]), we give here examples of very small irreducible non-linear $(2m, 2n)$–groups Γ, where both P_h and P_v are primitive but not alternating groups.

Proposition 4.4. *(Burger-Mozes, [17, Proposition 1.3, Theorem 1.4]) Let Γ be a $(2m, 2n)$–group such that P_h and P_v are primitive permutation groups. If either K_h or K_v is not a p-group, then Γ is irreducible and not linear over any field.*

Remark. There is no $(2, 2)$–, $(2, 4)$– and $(4, 4)$–group satisfying the assumptions of Proposition 4.4.

Remark. If $m \geq 3$ and Γ is an irreducible (A_{2m}, P_v)–group, i.e. if

$$|P_h^{(2)}| = |A_{2m}| \left(\frac{|A_{2m}|}{2m}\right)^{2m}$$

by Proposition 1.2(1a), then K_h is not a p-group, since $|K_h| = |A_{2m-1}|^{2m-1}$.

We apply now Proposition 4.4 to a $(4, 6)$–group which is moreover a candidate for having a simple subgroup of index 4.

Example 4.5.

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_1^{-1}, \\ a_1 b_3 a_2^{-1} b_1, & a_1 b_3^{-1} a_2 b_3, \\ a_1 b_2^{-1} a_2^{-1} b_3^{-1}, & a_2 b_1 a_2^{-1} b_2 \end{array} \right\}.$$

Proposition 4.6. *Let Γ be the $(4, 6)$–group defined in Example 4.5. Then*

(1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v = S_6$.

(2) $|K_v| = 12441600000 = 2^{14} \cdot 3^5 \cdot 5^5$.

(3) *Γ is irreducible and not linear over any field.*

(4) *$[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 is perfect.*

(5) $Z_\Gamma(b_3) = N_\Gamma(\langle b_3 \rangle) = \langle b_3 \rangle$.

(6) $\mathrm{Aut}(X) \cong \mathbb{Z}_2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,2),\\
\rho_v(b_2) &= (3,4),\\
\rho_v(b_3) &= (1,2,4,3),\\
\rho_h(a_1) &= (1,2)(3,5,6),\\
\rho_h(a_2) &= (1,4,2,6,5).
\end{aligned}$$

(2) GAP ([29]).

(3) Apply Proposition 4.4, using part (1) and (2).

(4) It is an easy computation.

(5) This follows from Proposition 1.12.

(6) Using GAP ([29]), we see that $\mathrm{Aut}(X)$ is generated by

$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_1^{-1}, a_2^{-1}, b_2, b_1, b_3).$$

□

Conjecture 4.7. *The* $(4,6)$*–group* Γ *of Example 4.5 is non-residually finite and*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = \Gamma_0 .$$

Example 4.8.

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, \\ a_1 b_3 a_2^{-1} b_1, & a_1 b_3^{-1} a_2 b_1^{-1}, \\ a_2 b_1 a_2 b_2^{-1}, & a_2 b_2 a_2 b_3 \end{array} \right\}.$$

Proposition 4.9. *Let* Γ *be the* $(4,6)$*–group defined in Example 4.8. Then*

(1) $P_h \cong \mathrm{PGL}_2(3) \cong S_4$, $P_v \cong \mathrm{PGL}_2(5) < S_6$.

(2) $|K_v| = 50000 = 2^4 \cdot 5^5$.

(3) Γ *is irreducible and not linear over any field.*

(4) $[\Gamma, \Gamma] = \Gamma_0$, $\Gamma_0^{ab} \cong \mathbb{Z}_2$, $\Gamma/[\Gamma_0, \Gamma_0] \cong D_4$ *and* $[\Gamma_0, \Gamma_0]$ *is perfect.*

(5) $Z_\Gamma(a_i) = N_\Gamma(\langle a_i \rangle) = \langle a_i \rangle$*, if* $a_i \in \{a_1, a_2\}$.

(6) $\mathrm{Aut}(X) \cong \mathbb{Z}_2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,3,2),\\
\rho_v(b_2) &= (2,3),\\
\rho_v(b_3) &= (2,4,3),\\
\\
\rho_h(a_1) &= (1,4,5,6,3,2),\\
\rho_h(a_2) &= (1,4,2)(3,6,5).
\end{aligned}$$

(2) GAP ([29]).

(3) Apply Proposition 4.4.

(4) This is an easy computation.

(5) This follows from Proposition 1.12.

(6) Using GAP ([29]), we have checked that the group $\mathrm{Aut}(X)$ is generated by the permutation

$$(a_1, a_2, b_1, b_2, b_3) \mapsto (a_1, a_2^{-1}, b_1^{-1}, b_2^{-1}, b_3^{-1})$$

of order 2.

□

Conjecture 4.10. *Let Γ be the $(4,6)$–group defined in Example 4.8. Then Γ is non-residually finite such that*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = [\Gamma_0, \Gamma_0].$$

Question 4.11. *Let Γ be the $(4,6)$–group defined in Example 4.8. Is the index 8 subgroup $[\Gamma_0, \Gamma_0]$ simple?*

We also apply Proposition 4.4 to a $(6,6)$–group:

Example 4.12.

$$R_{3\cdot 3} := \left\{ \begin{array}{lll}
a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_1, \\
\\
a_1b_3^{-1}a_3^{-1}b_3, & a_1b_2^{-1}a_2^{-1}b_1^{-1}, & a_2b_1a_2^{-1}b_2^{-1}, \\
\\
a_2b_3a_3^{-1}b_3^{-1}, & a_3b_1a_3b_2, & a_3b_2^{-1}a_3b_1^{-1}
\end{array} \right\}.$$

Proposition 4.13. *Let Γ be the $(6,6)$–group defined in Example 4.12. Then*

(1) $P_h \cong \mathrm{PSL}_2(5) < S_6$, $P_v \cong \mathrm{PSL}_2(5) < S_6$.

(2) $|K_v| = 100000 = 2^5 \cdot 5^5$.

(3) Γ *is irreducible and not linear over any field.*

(4) $[\Gamma, \Gamma] = \Gamma_0$ *and* Γ_0 *is perfect.*

(5) $Z_\Gamma(b_3) = N_\Gamma(\langle b_3 \rangle) = \langle b_3 \rangle$.

(6) $\mathrm{Aut}(X) \cong \mathbb{Z}_2^2$.

Proof. (1) We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,2)(3,4),\\
\rho_v(b_2) &= (3,4)(5,6),\\
\rho_v(b_3) &= (1,2,3)(4,6,5),\\
\rho_h(a_1) &= (1,5,6,3,2),\\
\rho_h(a_2) &= (1,4,5,6,2),\\
\rho_h(a_3) &= (1,5)(2,6).
\end{aligned}$$

(2) GAP ([29]).

(3) Apply Proposition 4.4.

(4) This is an easy computation.

(5) This follows from Proposition 1.12.

(6) Using GAP ([29]), $\mathrm{Aut}(X)$ is generated by the two automorphisms

$$\begin{aligned}
(a_1, a_2, a_3, b_1, b_2, b_3) &\mapsto (a_2, a_1, a_3, b_1^{-1}, b_2^{-1}, b_3^{-1}),\\
(a_1, a_2, a_3, b_1, b_2, b_3) &\mapsto (a_2^{-1}, a_1^{-1}, a_3^{-1}, b_2, b_1, b_3^{-1}).
\end{aligned}$$

□

Conjecture 4.14. *Let Γ be the $(6,6)$–group defined in Example 4.12. Then Γ is non-residually finite such that*

$$\bigcap_{N \overset{f.i.}{\lhd} \Gamma} N = \Gamma_0 .$$

Question 4.15. *Let Γ be the $(6,6)$–group defined in Example 4.12. Is the subgroup Γ_0 simple?*

4.3 Local groups, irreducibility, abelianization

Two naive attempts to characterize irreducibility for $(2m, 2n)$–groups Γ could be as follows: Γ is irreducible if and only if its abelianization is finite; Γ is irreducible if and only if the local groups P_h and P_v are transitive. Both turn out to be false by small counter-examples given in Proposition 4.16. By [17, Proposition 1.2], any reducible $(2m, 2n)$–group satisfies $\Lambda_1 \neq 1$ and $\Lambda_2 \neq 1$. We give in Proposition 4.16(6) also an irreducible example with this property. Finally, we show that it is not enough to compute for example $P_h^{(2)}$ and P_h, in order to decide by Proposition 1.2(2) that Γ is reducible, even if it *is* reducible.

Proposition 4.16. *There exist examples of $(2m, 2n)$–groups which are*

(1) *reducible such that their local groups P_h and P_v are transitive.*

(2) *irreducible such that P_h and P_v are not transitive.*

(3) *reducible and have finite abelianization.*

(4) *irreducible and have infinite abelianization.*

(5) *irreducible such that P_v is transitive and $\Lambda_2 \neq 1$.*

(6) *irreducible such that $\Lambda_1, \Lambda_2 \neq 1$.*

(7) *reducible but $|P_h| < |P_h^{(2)}|$ and $|P_v| < |P_v^{(2)}|$.*

(8) *reducible but $|P_h^{(3)}| < |P_h^{(4)}|$.*

Proof. (1) Take

$$R_{2\cdot 2} := \left\{ \begin{array}{ll} a_1 b_1 a_2^{-1} b_1, & a_1 b_2 a_2^{-1} b_2, \\ \\ a_1 b_2^{-1} a_1 b_1^{-1}, & a_2 b_1 a_2 b_2 \end{array} \right\}.$$

Then, we have

$$\begin{aligned} \rho_v(b_1) &= (1, 4, 3, 2), \\ \rho_v(b_2) &= (1, 4, 3, 2), \\ \\ \rho_h(a_1) &= (1, 3, 2, 4), \\ \rho_h(a_2) &= (1, 4, 2, 3) \end{aligned}$$

for the corresponding $(4, 4)$–group.

It is reducible, since $|P_h^{(2)}| = |P_h| = 4$.

(2) Embed any irreducible $(2m, 2n)$–complex into a $(2m + 2, 2n + 2)$–complex Y by adding the $m + n + 1$ geometric squares (geometric tori)

$$[a_1 b_{n+1} a_1^{-1} b_{n+1}^{-1}], \ldots, [a_m b_{n+1} a_m^{-1} b_{n+1}^{-1}],$$

$$[a_{m+1} b_1 a_{m+1}^{-1} b_1^{-1}], \ldots, [a_{m+1} b_n a_{m+1}^{-1} b_n^{-1}],$$

$$[a_{m+1} b_{n+1} a_{m+1}^{-1} b_{n+1}^{-1}]$$

and apply Proposition 1.9(3) to show that Y is irreducible. See the example described in part (6) for an explicit realization of this idea.

(3) Taking

$$R_{2\cdot 2} := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_1, & a_1 b_2 a_1 b_2^{-1}, \\ & \\ a_2 b_1 a_2 b_1^{-1}, & a_2 b_2 a_2^{-1} b_2 \end{array} \right\},$$

we have $|P_h| = |P_h^{(2)}| = 4$, which shows that the corresponding $(4, 4)$–group Γ is reducible. A simple computation gives $\Gamma^{ab} \cong \mathbb{Z}_2^4$ of order 16.

(4) Take the subsequent Example 4.18.

Note that if we add to the non-residually finite $(4, 12)$–complex of Example 2.26 the two geometric tori $[a_1 b_7 a_1^{-1} b_7^{-1}]$ and $[a_2 b_7 a_2^{-1} b_7^{-1}]$, then we even get a non-residually finite $(4, 14)$–group Γ having an infinite abelianization $\Gamma^{ab} \cong \mathbb{Z} \times \mathbb{Z}_2^2$.

(5) We take the $(6, 4)$–group Γ given by

$$R_{3\cdot 2} := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_3 b_1^{-1}, & a_1 b_2^{-1} a_3^{-1} b_1, \\ & & \\ a_2 b_1 a_3 b_1, & a_2 b_2 a_2 b_1^{-1}, & a_2 b_2^{-1} a_3 b_2^{-1} \end{array} \right\}.$$

Then

$$\begin{aligned} \rho_v(b_1) &= (1, 4, 2, 5, 3), \\ \rho_v(b_2) &= (2, 4, 6, 3, 5), \\ \rho_h(a_1) &= (1, 2)(3, 4), \\ \rho_h(a_2) &= \rho_h(a_3) = (1, 2, 3, 4), \end{aligned}$$

in particular $P_v \cong D_4 < S_4$ is transitive. Moreover, we compute $P_h \cong A_6$ and $|P_h^{(2)}| = 360 \cdot 60^6$. By Proposition 1.2(1a), Γ is irreducible. Using Lemma 1.1(1b), $B := \{(b_1 b_2)^3, (b_2 b_1)^3, (b_1 b_2)^{-3}, (b_2 b_1)^{-3}\}$ is a subset of Λ_2, since for each $b \in B$ and $a \in E_h$ we have $\rho_v(b)(a) = a$ and $\rho_h(a)(b) \in B$.

(6) Embedding the irreducible (6, 4)–complex just described in the proof of part (5), we construct an irreducible (8, 6)–group such that $\Lambda_1 \neq 1 \neq \Lambda_2$.

$$R_{4\cdot 3} := \left\{ \begin{array}{llll} \underline{a_1 b_1 a_1^{-1} b_2^{-1}}, & \underline{a_1 b_2 a_3 b_1^{-1}}, & a_1 b_3 a_1^{-1} b_3^{-1}, & \underline{a_1 b_2^{-1} a_3^{-1} b_1}, \\ \underline{a_2 b_1 a_3 b_1}, & \underline{a_2 b_2 a_2 b_1^{-1}}, & a_2 b_3 a_2^{-1} b_3^{-1}, & \underline{a_2 b_2^{-1} a_3 b_2^{-1}}, \\ a_3 b_3 a_3^{-1} b_3^{-1}, & a_4 b_1 a_4^{-1} b_1^{-1}, & a_4 b_2 a_4^{-1} b_2^{-1}, & a_4 b_3 a_4^{-1} b_3^{-1} \end{array} \right\}.$$

It is irreducible by Proposition 1.9(3) and we have $a_4 \in \Lambda_1$, $b_3 \in \Lambda_2$, applying Lemma 1.1. Note that P_h and P_v are not transitive, since

$$\begin{aligned} \rho_v(b_1) &= (1, 6, 2, 7, 3), \\ \rho_v(b_2) &= (2, 6, 8, 3, 7), \\ \rho_v(b_3) &= (), \\ \rho_h(a_1) &= (1, 2)(5, 6), \\ \rho_h(a_2) &= \rho_h(a_3) = (1, 2, 5, 6), \\ \rho_h(a_4) &= (). \end{aligned}$$

(7) For the (4, 6)–group given by

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, \\ a_1 b_3 a_1^{-1} b_2, & a_2 b_1 a_2^{-1} b_2^{-1}, \\ a_2 b_2 a_2^{-1} b_1, & a_2 b_3 a_2 b_3^{-1} \end{array} \right\},$$

we compute $|P_h| = 2$, $|P_h^{(2)}| = 4$, $|P_v| = 24$, $|P_v^{(2)}| = 48$. It is reducible by Proposition 1.2(2b), since $|P_v^{(3)}| = 48$. Note that $|P_h^{(3)}| = |P_h^{(4)}| = 8$.

(8) Take the (4, 6)–group defined by

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_2, \\ a_1 b_3 a_1 b_3^{-1}, & a_2 b_1 a_2 b_2^{-1}, \\ a_2 b_2 a_2 b_3^{-1}, & a_2 b_3 a_2 b_1^{-1} \end{array} \right\}.$$

We compute $|P_h| = 4$, $|P_h^{(2)}| = 8$, $|P_h^{(3)}| = 16$, $|P_h^{(4)}| = 32$. Note that $|P_h^{(5)}| = 32$ and $|P_v| = |P_v^{(2)}| = 24$, in particular the (4, 6)–group is reducible by Proposition 1.2(2).

□

Question 4.17. (1) *Is there a reducible (P_h, P_v)–group Γ such that P_h is transitive and P_v is 2-transitive?*

(2) *Does there exist a reducible (P_h, P_v)–group Γ such that P_h is transitive and P_v is primitive?*

(3) *Is there a reducible (P_h, P_v)–group Γ such that P_h is transitive and P_v is quasi-primitive?*

(4) *Is there a $(2m, 2n)$–group Γ such that P_h and P_v are transitive and Γ^{ab} is infinite?*

The $(6, 6)$–group in the following example not only illustrates Proposition 4.16(4), but has other interesting properties.

Example 4.18.

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2^{-1}, \\ a_1b_3^{-1}a_2b_2, & a_2b_1a_3^{-1}b_1^{-1}, & a_2b_3a_2b_2^{-1}, \\ a_2b_1^{-1}a_3^{-1}b_1, & a_3b_2a_3^{-1}b_2^{-1}, & a_3b_3a_3^{-1}b_3^{-1} \end{array} \right\}.$$

Proposition 4.19. *Let Γ be the $(6, 6)$–group defined in Example 4.18. Then*

(1) *$P_h = A_6$, $P_v \cong \mathbb{Z}_2 < S_6$ and Γ is irreducible.*

(2) *$H_2(x_v)$ is a pro-2 group, where x_v is any vertex of $\mathcal{T}_{2n}$.*

(3) *$\Lambda_2 \neq 1$, in particular $\mathrm{QZ}(H_2) \neq 1$.*

(4) *We have*

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle &\cong \mathrm{pr}_2(\langle a_1, a_2, a_3 \rangle) \\ &\cong \mathrm{pr}_2(\langle a_1, a_2, a_3 \rangle)(x_v) \\ &\cong \mathrm{pr}_2(\Gamma)(x_v) < \mathrm{Aut}(\mathcal{T}_{2n})(x_v) \,. \end{aligned}$$

This group stabilizes pointwise a bi-infinite geodesic in $\mathcal{T}_{2n} = \mathcal{T}_6$ through the vertex x_v.

(5) *$\Gamma^{ab} \cong \mathbb{Z}^2 \times \mathbb{Z}_2$, in particular it is an infinite group.*

Proof. (1) We compute

$$\begin{aligned} \rho_v(b_1) &= (2, 3)(4, 5), \\ \rho_v(b_2) &= (1, 2, 5), \\ \rho_v(b_3) &= (2, 6, 5), \end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (2,3)(4,5),\\
\rho_h(a_2) &= (2,3)(4,5),\\
\rho_h(a_3) &= ().
\end{aligned}$$

To see that Γ is irreducible, compute $|P_h^{(2)}| = 360 \cdot 60^6$.

(2) This follows directly from the subsequent Proposition 4.20.

(3) Using Lemma 1.1(1b), the set $\{b_1^2, b_2^3, b_3^3\}$ is a subset of Λ_2. Note that Λ_2 is a normal subgroup of $\langle b_1, \ldots, b_n\rangle$ of infinite index, since Γ is irreducible. In particular, Λ_2 is a non-finitely generated free normal subgroup of Γ.

(4) The map $\mathrm{pr}_2 : \Gamma \to \mathrm{Aut}(\mathcal{T}_{2n})$ is injective because we know that $\mathrm{QZ}(H_1) = 1$ by [16, Proposition 3.1.2, 1)]. This gives the first claimed isomorphism. The two other isomorphisms are based on the identification

$$\langle a_1, a_2, a_3\rangle \cong \{\gamma \in \Gamma : \mathrm{pr}_2(\gamma)(x_v) = x_v\}$$

proved in [17, Chapter 1]. Since $\rho_h(a)(b_1) = b_1$ for each $a \in E_h$, the bi-infinite geodesic $(b_1^k)_{k\in\mathbb{Z}}$ through x_v is fixed.

(5) This is an easy computation.

□

Proposition 4.20. *Let Γ be a (P_h, P_v)–group such that $|P_v| = 2$. Then $H_2(x_v)$ is a pro-2 group (an infinite group if and only if Γ is irreducible).*

Proof. Consider the following commutative diagram, where p_k, $k \in \mathbb{N}$, is the obvious restriction map.

$$\begin{tikzcd}
\langle a_1, \ldots, a_m\rangle \arrow[r, two heads, "\rho_h^{(k+1)}"] \arrow[dr, two heads, "\rho_h^{(k)}"'] & P_v^{(k+1)} < \mathrm{Sym}(E_v^{(k+1)}) \arrow[d, "p_k"] \\
& P_v^{(k)} < \mathrm{Sym}(E_v^{(k)})
\end{tikzcd}$$

We want to show that $P_v^{(k)}$ is a 2-group for each $k \in \mathbb{N}$. Since $|P_v| = 2$ and $P_v^{(k)} \cong P_v^{(k+1)}/\ker(p_k)$, it is enough to show that $\ker(p_k)$ is a 2-group (or trivial). This follows, if any element $\sigma \in \ker(p_k)$ has order 1 or 2 in $P_v^{(k+1)}$. Given $\sigma \in \ker(p_k)$, write $\sigma = \rho_h^{(k+1)}(a)$ for an appropriate element a in $\langle a_1, \ldots, a_m\rangle$. Let b be any element in $E_v^{(k+1)}$. Decompose $b = b' \cdot b''$, such that $b' \in E_v^{(k)}$, $b'' \in E_v$ and define $\tilde{a} := \rho_v^{(|a|)}(b')(a)$ (see Figure 4.6). Then

$$\sigma^2(b) = \rho_h^{(k+1)}(a^2)(b' \cdot b'') = b' \cdot \rho_h(\tilde{a}^2)(b'') = b' \cdot b'' = b,$$

where the second equation uses the commutativity of the diagram above and the third equation follows from the assumption $|P_v| = 2$. □

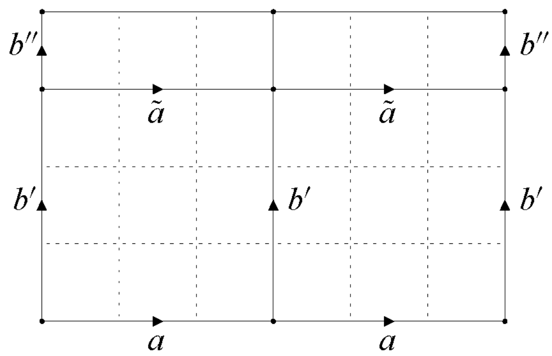


Figure 4.6: Illustration in the proof of Proposition 4.20

The following conjecture is true at least for $k \leq 6$, because we have computed $|P_v^{(2)}| = 4$, $|P_v^{(3)}| = 16$, $|P_v^{(4)}| = 32$, $|P_v^{(5)}| = 128$, $|P_v^{(6)}| = 256$.

Conjecture 4.21. *For Γ defined in Example 4.18 and $l \in \mathbb{N}$*

$$|P_v^{(k)}| = \begin{cases} 2^{3l-1}, & \text{if } k = 2l \\ 2^{3l-2}, & \text{if } k = 2l-1\,. \end{cases}$$

A very natural question is to ask if there is a criterion in terms of properties of the local groups P_h and P_v to decide whether a given $(2m, 2n)$–group is reducible or not. The answer to this question is "no" as shown in the first part of the following proposition.

Proposition 4.22. (1) *In general, it is not possible to determine whether a given $(2m, 2n)$–group is reducible or irreducible only by knowing its local groups P_h and P_v.*

(2) *There exist $(2m, 2n)$–groups Γ_1 and Γ_2 having isomorphic local groups, but different local transitivity properties. More precisely, there are examples such that $P_v(\Gamma_1)$ and $P_h(\Gamma_2)$ are transitive, $P_h(\Gamma_1)$ and $P_v(\Gamma_2)$ are not transitive, although $P_h(\Gamma_1) \cong P_h(\Gamma_2)$ and $P_v(\Gamma_1) \cong P_v(\Gamma_2)$.*

Proof. (1) The idea is to find two $(2m, 2n)$–groups Γ_1 and Γ_2 with permutation isomorphic local groups such that Γ_1 is irreducible but Γ_2 is reducible. We take the $(6, 6)$–group of Example 4.18 as Γ_1, and Γ_2 as $(6, 6)$–group defined as follows:

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2^{-1}, \\ a_1b_3^{-1}a_2b_2, & a_2b_1a_3^{-1}b_1^{-1}, & a_2b_3a_2b_2^{-1}, \\ a_2b_1^{-1}a_3^{-1}b_1, & \underline{a_3b_2a_3^{-1}b_3^{-1}}, & \underline{a_3b_3a_3^{-1}b_2^{-1}} \end{array} \right\}.$$

Note that it has seven (of nine) defining relators in common with those of Example 4.18. The two different relators are underlined. They can be obtained from the corresponding two relators $a_3b_2a_3^{-1}b_2^{-1}$ and $a_3b_3a_3^{-1}b_3^{-1}$ in Example 4.18 by a single "surgery" operation indicated in Figure 4.7. For a more general description of surgery techniques in square complexes, see [17, Section 6.2.2].

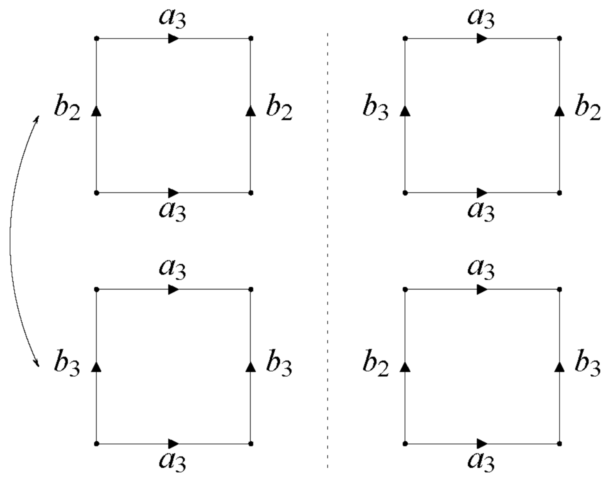


Figure 4.7: "Surgery" on Example 4.18 (on the left)

We compute for Γ_2:

$$\begin{aligned}
\rho_v(b_1) &= (2,3)(4,5),\\
\rho_v(b_2) &= (1,2,5),\\
\rho_v(b_3) &= (2,6,5),\\
\rho_h(a_1) &= \rho_h(a_2) = \rho_h(a_3) = (2,3)(4,5),
\end{aligned}$$

in particular it follows that $P_h = A_6$ and $P_v \cong \mathbb{Z}_2 < S_6$. Moreover, we have $|P_h^{(2)}| = 360 = |P_h|$, hence Γ_2 is reducible by Proposition 1.2(2a). Observe that $|P_v^{(k)}| = 2$ for all $k \in \mathbb{N}$.

(2) The reason for this phenomenon is that the local groups are isomorphic, but not *permutation* isomorphic. Let the $(4,6)$–group Γ_1 be defined by

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, \\ a_1b_3a_2^{-1}b_1^{-1}, & a_1b_3^{-1}a_2^{-1}b_1, \\ a_1b_2^{-1}a_2^{-1}b_3, & a_2b_1a_2^{-1}b_2 \end{array} \right\}.$$

Then

$$\begin{aligned}
\rho_v(b_1) &= (1,2),\\
\rho_v(b_2) &= (3,4),\\
\rho_v(b_3) &= (1,2)(3,4),\\
\rho_h(a_1) &= (1,3,2)(4,5,6),\\
\rho_h(a_2) &= (1,3,2,6,4,5),
\end{aligned}$$

hence $P_h \cong \mathbb{Z}_2^2 < S_4$ is not transitive, $P_v \cong \mathbb{Z}_2 \times A_4 < S_6$ is transitive.

Define the $(4,6)$–group Γ_2 by

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1b_1a_2^{-1}b_2^{-1}, & a_1b_2a_2^{-1}b_2, \\ \\ a_1b_3a_2b_3, & a_1b_3^{-1}a_2b_3^{-1}, \\ \\ a_1b_2^{-1}a_2^{-1}b_1^{-1}, & a_1b_1^{-1}a_2^{-1}b_1 \end{array} \right\}.$$

We compute

$$\begin{aligned}
\rho_v(b_1) &= (1,2)(3,4),\\
\rho_v(b_2) &= (1,2)(3,4),\\
\rho_v(b_3) &= (1,3)(2,4),\\
\rho_h(a_1) &= (1,5,2)(3,4),\\
\rho_h(a_2) &= (2,5,6)(3,4)
\end{aligned}$$

and see that $P_h \cong \mathbb{Z}_2^2 < S_4$ is transitive, but $P_v \cong \mathbb{Z}_2 \times A_4 < S_6$ is not transitive.

□

4.4 Graphs associated to a $(2m, 2n)$–group

Following an idea of Mozes ([52]), we associate to any $(2m, 2n)$–group Γ two infinite families of finite regular graphs $(X_k(\Gamma))_{k\in\mathbb{N}}$ and $(Y_k(\Gamma))_{k\in\mathbb{N}}$. The vertex set of $X_k(\Gamma)$ is identified with the set $E_h^{(k)}$ and the vertex set of $Y_k(\Gamma)$ is identified with $E_v^{(k)}$. Two vertices $a, \tilde{a} \in E_h^{(k)}$ are connected in $X_k(\Gamma)$ by an edge if and only if $\rho_v(b)(a) = \tilde{a}$ holds for some $b \in E_v$. In this case, b and b^{-1} are edges in $X_k(\Gamma)$ such that $o(b) = a$, $t(b) = \tilde{a}$ and $\overline{b} = b^{-1}$. Similarly, two vertices $b, \tilde{b} \in E_v^{(k)}$ are connected in $Y_k(\Gamma)$ by an edge if and only if $\rho_v(a)(b) = \tilde{b}$ for some $a \in E_h$.

See Figure 4.8 and 4.9 for a visualization of $Y_1(\Gamma_{3,5})$ and $X_2(\Gamma_{3,5})$, respectively, where $\Gamma_{3,5}$ is the $(4,6)$–group of Example 3.46.

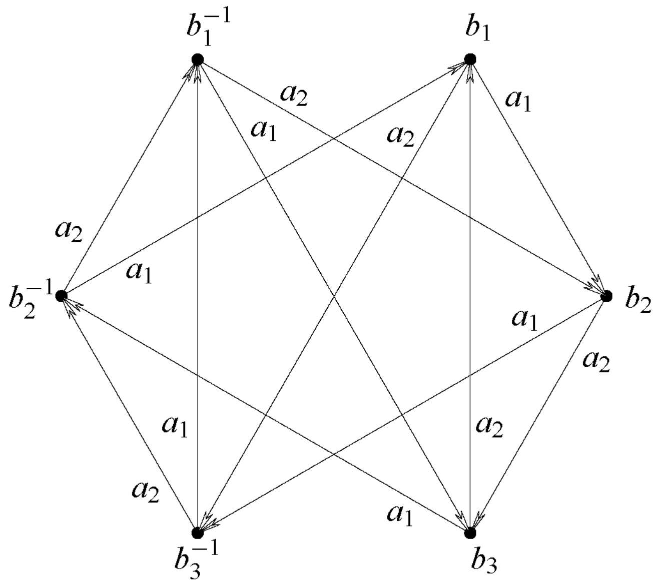


Figure 4.8: The graph $Y_1(\Gamma_{3,5})$

We list now some obvious general properties of the graph $X_k(\Gamma)$ (the properties of $Y_k(\Gamma)$ are analogous):

- $X_k(\Gamma)$ has exactly $2m(2m-1)^{k-1}$ vertices.
- $X_k(\Gamma)$ is $2n$-regular.
- $X_k(\Gamma)$ is connected if and only if $P_h^{(k)}$ is transitive on $E_h^{(k)}$.
- $X_k(\Gamma)$ is connected for each $k \in \mathbb{N}$ if and only if $\mathrm{pr}_1(\Gamma)$ is locally ∞-transitive.
- If $X_k(\Gamma)$ is not connected, then $X_l(\Gamma)$ is not connected for each $l \geq k$.
- If $X_k(\Gamma)$ has no loops, then $X_l(\Gamma)$ has no loops for each $l \geq k$.

Less obvious is the following result of Mozes:

Proposition 4.23. *(Mozes, [52, Theorem, p.323]) If $\Gamma = \Gamma_{p,l}$ is as in Section 3.2, then $(X_k(\Gamma))_{k\in\mathbb{N}}$ and $(Y_k(\Gamma))_{k\in\mathbb{N}}$ are* Ramanujan graphs, *i.e. for every $k \in \mathbb{N}$ and every eigenvalue λ of the adjacency matrix of $X_k(\Gamma)$, either $\lambda = \pm(l+1)$ or $|\lambda| \leq 2\sqrt{l}$, and for every eigenvalue λ of the adjacency matrix of $Y_k(\Gamma)$, either $\lambda = \pm(p+1)$ or $|\lambda| \leq 2\sqrt{p}$.*

Problem 4.24. *Construct other $(2m, 2n)$–groups Γ such that the graphs $(X_k(\Gamma))_{k\in\mathbb{N}}$ and $(Y_k(\Gamma))_{k\in\mathbb{N}}$ are Ramanujan graphs.*

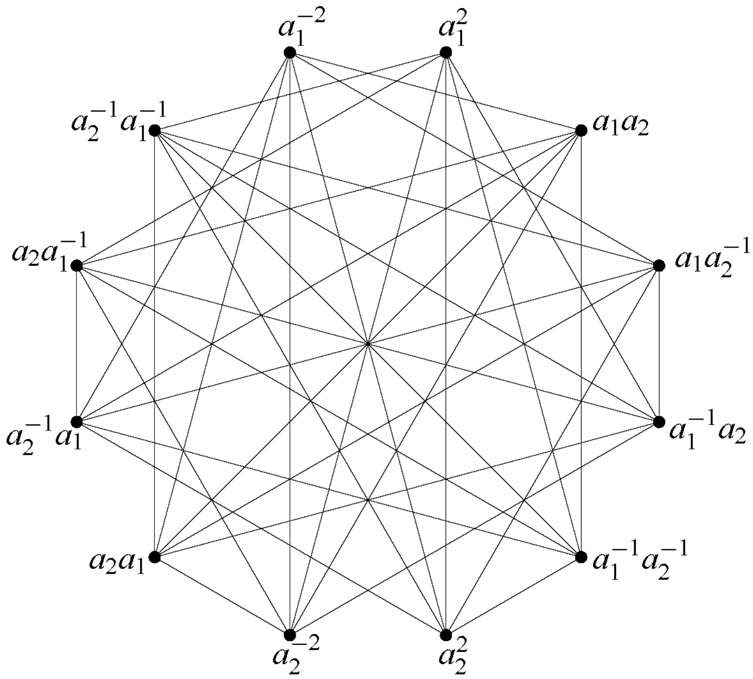


Figure 4.9: Geometric realization of $X_2(\Gamma_{3,5})$

4.5 Growth of $(2m, 2n)$–groups

Let Γ be a finitely generated group and S a finite subset generating Γ. Following [32], we define the *word length* $\ell_S(\gamma)$ of an element $\gamma \in \Gamma \setminus \{1\}$ with respect to S:

$$\ell_S(\gamma) := \min\{i : \gamma = s_1 \dots s_i;\ s_1, \dots, s_i \in S \cup S^{-1}\}, \quad (\text{and } \ell_S(1) := 0),$$

for $k \in \mathbb{N}_0$ the *growth function*

$$k \mapsto \beta(\Gamma, S; k) := |\{\gamma \in \Gamma : \ell_S(\gamma) \leq k\}|,$$

the corresponding *growth series*

$$B(\Gamma, S; z) := \sum_{k=0}^{\infty} \beta(\Gamma, S; k) z^k,$$

the *spherical growth function*

$$k \mapsto \sigma(\Gamma, S; k) := |\{\gamma \in \Gamma : \ell_S(\gamma) = k\}|,$$

and the corresponding *spherical growth series*

$$\Sigma(\Gamma, S; z) := \sum_{k=0}^{\infty} \sigma(\Gamma, S; k) z^k = (1 - z) B(\Gamma, S; z).$$

Observe that $\sigma(\Gamma, S; k) = \beta(\Gamma, S; k) - \beta(\Gamma, S; k-1)$, if $k \in \mathbb{N}$.

Proposition 4.25. *Let* $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_{m\cdot n}\rangle$ *be a* $(2m, 2n)$*–group and* $S := \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ *the set of standard generators of* Γ.

(1) *The Cayley graph of* (Γ, S) *can be identified with the* 1*-skeleton of the product of regular trees* $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$*, in particular the growth functions of* (Γ, S) *only depend on m and n.*

(2) *The spherical growth series is*

$$\begin{aligned}\Sigma(\Gamma, S; z) &= \frac{\left(\frac{1+z}{1-z}\right)^2}{\left(m - (m-1)\frac{1+z}{1-z}\right)\left(n - (n-1)\frac{1+z}{1-z}\right)}\\ &= \frac{1+z}{1-(2m-1)z} \cdot \frac{1+z}{1-(2n-1)z}\\ &= 1 + (2m+2n)z + (4m^2 + 4n^2 + 4mn - 2m - 2n)z^2 + O(z^3)\,.\end{aligned}$$

(3) *If* $(m, n) \neq (1, 1)$*, then* Γ *is of exponential growth. If* $m = n = 1$*, then* Γ *is of polynomial growth.*

(4) *If* $m, n \geq 2$*, then* Γ *is quasi-isometric to* $F_2 \times F_2$.

Proof. (1) See [9, Section I.8A.2] for an explicit identification. Observe that the product $\mathcal{T}_{2m} \times \mathcal{T}_{2n}$ is the universal covering space of the "Cayley complex" of ([9, Section I.8A.2]), which is exactly our $(2m, 2n)$–complex X.

(2) By part (1) we have $\Sigma(\Gamma, S; z) = \Sigma(F_m \times F_n, S; z)$. Note that

$$\Sigma(\mathbb{Z}, \{1\}; z) = \frac{1+z}{1-z}\,.$$

The claim follows now from the behaviour of the spherical growth series with respect to taking free and direct products (see [32, Proposition VI.A.4]). As an intermediate step, we have for example

$$\Sigma(F_m, \{a_1, \ldots, a_m\}; z) = \frac{1+z}{1-(2m-1)z}\,.$$

(3) If $(m, n) \neq (1, 1)$, then the statement follows from the obvious fact that $F_m \times F_n$ contains a non-abelian free subgroup (namely $F_m \times \{1\}$ if $m \geq 2$, or $\{1\} \times F_n$ if $n \geq 2$). If $m = n = 1$, then Γ is virtually abelian, hence is of polynomial growth.

(4) The group $F_m \times F_n$ is isomorphic to a finite index subgroup of $F_2 \times F_2$ (the index is $(m-1)(n-1)$), hence the groups are quasi-isometric by part (1). (Note that for $\ell, \ell' \geq 3$, the tree $\mathcal{T}_\ell$ is quasi-isometric to $\mathcal{T}_{\ell'}$, see [9, Exercise I.8.20(2)]. This is a more general result than (4), since ℓ, ℓ' are allowed to be odd.)

□

Example. Let Γ be a (6, 6)–group. Then

$$\Sigma(\Gamma, S; z) = 1 + 12z + 96z^2 + 660z^3 + 4200z^4 + 25500z^5 + O(z^6)$$
$$B(\Gamma, S; z) = 1 + 13z + 109z^2 + 769z^3 + 4969z^4 + 30469z^5 + O(z^6)\,.$$

4.6 Deficiency of $(2m, 2n)$–groups

Let G be a finitely presented group. The deficiency of a finite presentation P of G is the number of generators minus the number of relations in P. The *deficiency* def(G) of the group G is the maximum of the deficiency of P taken over all possible finite presentations of G. It is well-known (see [27, Lemma 1.2]) that

$$\operatorname{def}(G) \leq \operatorname{rank}(H_1(G; \mathbb{Z})) - d(H_2(G; \mathbb{Z}))\,, \tag{4.1}$$

where $d(H_2(G; \mathbb{Z}))$ denotes the minimal number of generators of the second homology group of G with integer coefficients. The group G is called *efficient* if equality holds in (4.1).

Proposition 4.26. *Let Γ be a $(2m, 2n)$–group. Then Γ is efficient and*

$$\operatorname{def}(\Gamma) = m + n - mn\,.$$

Proof. Since Γ has the finite presentation $\langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m\cdot n}\rangle$, we have

$$\operatorname{def}(\Gamma) \geq m + n - mn\,.$$

On the other hand

$$\begin{aligned}\operatorname{def}(\Gamma) &\leq \operatorname{rank}(H_1(\Gamma; \mathbb{Z})) - d(H_2(\Gamma; \mathbb{Z}))\\ &= \operatorname{rank}(H_1(\Gamma; \mathbb{Z})) - \operatorname{rank}(H_2(\Gamma; \mathbb{Z}))\\ &= 1 - \chi(\Gamma)\\ &= m + n - mn\,.\end{aligned}$$

The inequality is (4.1), and the equalities above are described in [41, Section 6], where $\chi(\Gamma)$ is the Euler characteristic of the $(2m, 2n)$–complex X (or the alternating sums of the ranks of the homology groups of Γ, which is the same here). ☐

Remark. The deficiency def(Γ) for a $(2m, 2n)$–group Γ is attained by its standard presentation

$$\langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_{m\cdot n}\rangle$$

as well as by the natural presentations of their amalgams (provided they exist, see Proposition 1.3)

$$F_n *_{F_{1-2m+2mn}} F_{1-m+mn} \text{ and } F_m *_{F_{1-2n+2mn}} F_{1-n+mn}\,.$$

Similarly as in Proposition 4.26, one can prove that the deficiency of Γ_0 is

$$\mathrm{def}(\Gamma_0) = 4n + 4m - 4mn - 3 .$$

Remark. There are non-efficient torsion-free groups, see [47].

Appendix A

More examples

A.1 Irreducible (A_6, P_v)–groups

In Appendix C.1, we will give a list of all primitive permutation groups in S_{2n}, where $n \leq 7$. There are 33 different such groups (up to isomorphism). Our goal now is to construct for each such primitive group P_v an irreducible (A_6, P_v)–group. We already have constructed an (A_6, A_6)–group in Example 2.2, an (A_6, M_{12})–group in Example 2.18, an $(A_6, \mathrm{ASL}_3(2))$–group in Example 2.21 and an $(A_6, S_5 < S_{10})$–group in Example 2.58. There are no (A_6, S_2)–groups and no (A_6, A_4)–groups, and we have not found an $(A_6, A_5 < S_{10})$–group or an $(A_6, M_{11} < S_{12})$–group. In this section, we construct the 25 remaining (A_6, P_v)–groups and give the generators of the local groups $P_h = A_6$ and P_v. All these examples are irreducible by Proposition 1.2(1a), since we always have $|P_h^{(2)}| = 360 \cdot 60^6$.

Example A.1. *(A_6, S_4)–group:*

$$R_{3\cdot 2} := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_2^{-1}, & a_1 b_2 a_2^{-1} b_1, & a_1 b_2^{-1} a_2 b_1^{-1}, \\ & & \\ a_2 b_1 a_3^{-1} b_1, & a_2 b_2 a_3^{-1} b_2, & a_3 b_1 a_3 b_2 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1, 5, 4, 3, 2), \\ \rho_v(b_2) &= (2, 6, 5, 4, 3), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (1, 3, 4, 2), \\ \rho_h(a_2) &= (1, 3, 2, 4), \\ \rho_h(a_3) &= (1, 4, 2, 3). \end{aligned}$$

Example A.2. $(A_6, \mathrm{PSL}_2(5))$*–group:*

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_1b_2^{-1}, \\ a_1b_3^{-1}a_3^{-1}b_2, & a_2b_1a_3b_2^{-1}, & a_2b_2a_3b_2, \\ a_2b_3a_3b_1^{-1}, & a_2b_3^{-1}a_3b_3^{-1}, & a_2b_1^{-1}a_3b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,4)(3,5), \\ \rho_v(b_2) &= (1,6,5,3)(2,4), \\ \rho_v(b_3) &= (1,2,4,6)(3,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,3)(4,5), \\ \rho_h(a_2) &= (1,3,4,5,2), \\ \rho_h(a_3) &= (2,3,4,6,5). \end{aligned}$$

Example A.3. $(A_6, \mathrm{PGL}_2(5))$*–group:*

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_2, \\ a_1b_3^{-1}a_2b_2^{-1}, & a_2b_1a_3^{-1}b_1^{-1}, & a_2b_2a_3^{-1}b_1, \\ a_2b_3a_3b_3, & a_2b_1^{-1}a_3^{-1}b_2, & a_3b_2a_3b_3^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,3)(4,5), \\ \rho_v(b_2) &= (1,5,4,3,2), \\ \rho_v(b_3) &= (2,6,5,3,4), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,4,5,3), \\ \rho_h(a_2) &= (2,4,3,5,6), \\ \rho_h(a_3) &= (1,5,4,3,2). \end{aligned}$$

Example A.4. (A_6, S_6)*–group:*

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_3^{-1}a_3^{-1}b_3, & a_2b_1a_2^{-1}b_2^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, \\ a_2b_3a_3^{-1}b_1, & a_2b_2^{-1}a_3b_1^{-1}, & a_3b_1a_3b_2 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,4,3), \\ \rho_v(b_2) &= (3,5,4), \\ \rho_v(b_3) &= (1,2,3)(4,6,5), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (), \\ \rho_h(a_2) &= (1,5,6,3,2), \\ \rho_h(a_3) &= (1,4,5)(2,6). \end{aligned}$$

Example A.5. $(A_6, \mathrm{AGL}_1(8))$*–group:*

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_3^{-1}, & a_1b_3a_2^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_1, & a_1b_1^{-1}a_2^{-1}b_3, & a_2b_3a_3^{-1}b_2, \\ a_3b_1a_3^{-1}b_4^{-1}, & a_3b_2a_3b_3, & a_3b_4a_3^{-1}b_1^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,2)(5,6), \\ \rho_v(b_2) &= (1,4,3,2)(5,6), \\ \rho_v(b_3) &= (1,2)(3,6,5,4), \\ \rho_v(b_4) &= (1,2)(5,6), \end{aligned}$$

$$\begin{aligned} \rho_h(a_1) &= (2,6,8,7,5,4,3), \\ \rho_h(a_2) &= (1,2,4,5,6,7,3), \\ \rho_h(a_3) &= (1,4)(2,6)(3,7)(5,8). \end{aligned}$$

Example A.6. $(A_6, \mathrm{A\Gamma L}_1(8))$*–group:*

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_3^{-1}, & a_1b_2a_2b_3, & a_1b_3a_2^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_1, & a_1b_1^{-1}a_2^{-1}b_1^{-1}, & a_2b_2^{-1}a_3^{-1}b_3^{-1}, \\ a_3b_1a_3^{-1}b_1^{-1}, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_2 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,2)(5,6), \\ \rho_v(b_2) &= (1,4,5,6,2), \\ \rho_v(b_3) &= (1,2,3,6,5), \\ \rho_v(b_4) &= (1,2)(5,6), \\ \rho_h(a_1) &= (1,8,7,5,4,3)(2,6), \\ \rho_h(a_2) &= (1,2,4,5,6,8)(3,7), \\ \rho_h(a_3) &= (2,5,6)(3,7,4). \end{aligned}$$

Example A.7. $(A_6, \mathrm{PSL}_2(7))$*–group:*

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_1^{-1}, & a_1b_2a_2^{-1}b_1, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_4^{-1}, & a_1b_1^{-1}a_2^{-1}b_3, & a_2b_3a_3^{-1}b_2, \\ a_3b_1a_3^{-1}b_4, & a_3b_2a_3b_3, & a_3b_4a_3^{-1}b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,2)(5,6), \\ \rho_v(b_2) &= (1,4,3,2)(5,6), \\ \rho_v(b_3) &= (1,2)(3,6,5,4), \\ \rho_v(b_4) &= (1,2)(5,6), \\ \rho_h(a_1) &= (2,6,8)(4,7,5), \\ \rho_h(a_2) &= (1,7,3)(2,4,5), \\ \rho_h(a_3) &= (1,5)(2,6)(3,7)(4,8). \end{aligned}$$

Example A.8. $(A_6, \mathrm{PGL}_2(7))$*–group:*

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_2^{-1}b_3^{-1}, & a_1b_2a_2^{-1}b_1, & a_1b_3a_2^{-1}b_4^{-1}, \\ a_1b_4a_2^{-1}b_4, & a_1b_4^{-1}a_2^{-1}b_2, & a_1b_3^{-1}a_3b_2^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_3, & a_1b_1^{-1}a_3^{-1}b_1^{-1}, & a_2b_1a_3^{-1}b_1, \\ a_2b_3a_3^{-1}b_2, & a_3b_2a_3b_3, & a_3b_4a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (1,3,2)(4,6,5), \\ \rho_v(b_2) &= (1,4,3,2)(5,6), \\ \rho_v(b_3) &= (1,2)(3,6,5,4), \\ \rho_v(b_4) &= (1,2)(5,6), \\ \rho_h(a_1) &= (1,8,2,6,7,5,4,3), \\ \rho_h(a_2) &= (1,7,3,2,4,5,6,8), \\ \rho_h(a_3) &= (1,8)(2,6)(3,7). \end{aligned}$$

Example A.9. (A_6, A_8)*–group:*

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2^{-1}, & a_1b_3a_1^{-1}b_3^{-1}, \\ a_1b_4a_2^{-1}b_4^{-1}, & a_1b_4^{-1}a_2^{-1}b_4, & a_2b_1a_3^{-1}b_2^{-1}, \\ a_2b_2a_3^{-1}b_2, & a_2b_3a_3b_1, & a_2b_3^{-1}a_2b_1^{-1}, \\ a_2b_2^{-1}a_3^{-1}b_3, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (2,3)(4,5), \\ \rho_v(b_3) &= (2,5,3), \\ \rho_v(b_4) &= (1,2)(5,6), \\ \rho_h(a_1) &= (), \\ \rho_h(a_2) &= (1,6,7,2)(3,8), \\ \rho_h(a_3) &= (1,5,6)(2,7,8,4,3). \end{aligned}$$

Example A.10. (A_6, S_8)*–group:*

$$R_{3\cdot 4} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_2, & a_1b_3a_1^{-1}b_3^{-1}, \\ a_1b_4a_2^{-1}b_4^{-1}, & a_1b_4^{-1}a_2^{-1}b_4, & a_2b_1a_3^{-1}b_2^{-1}, \\ a_2b_2a_3^{-1}b_2, & a_2b_3a_3b_1, & a_2b_3^{-1}a_2b_1^{-1}, \\ a_2b_2^{-1}a_3^{-1}b_3, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_4a_3^{-1}b_1 \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5,4), \\ \rho_v(b_2) &= (2,3)(4,5), \\ \rho_v(b_3) &= (2,5,3), \\ \rho_v(b_4) &= (1,2)(5,6), \\ \rho_h(a_1) &= (2,7), \\ \rho_h(a_2) &= (1,6,7,2)(3,8), \\ \rho_h(a_3) &= (1,5,6)(2,7,8,4,3). \end{aligned}$$

Example A.11. $(A_6, \mathrm{PSL}_2(9))$*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_3^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2^{-1}, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_1b_2^{-1}a_3^{-1}b_3, & a_2b_1a_2b_2^{-1}, & a_2b_2a_2b_3, \\ a_2b_5a_2b_1^{-1}, & a_2b_4^{-1}a_2b_3^{-1}, & a_3b_1a_3b_1^{-1}, \\ a_3b_3a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_5^{-1}, & a_3b_5a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= (2,5)(3,4), \\ \rho_v(b_2) &= (2,5)(4,6), \\ \rho_v(b_3) &= (1,3)(2,5), \\ \rho_v(b_4) &= (1,2,5), \\ \rho_v(b_5) &= (2,6,5), \\ \rho_h(a_1) &= (2,3)(4,5)(6,7)(8,9), \\ \rho_h(a_2) &= (1,5,4,8,2)(3,7,6,10,9), \\ \rho_h(a_3) &= (2,3)(4,5)(6,7)(8,9). \end{aligned}$$

Example A.12. $(A_6, S_6 < S_{10})$*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_5, & a_1b_3a_1^{-1}b_3, \\ a_1b_4a_1^{-1}b_1, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_3, & a_2b_2a_2^{-1}b_2^{-1}, & a_2b_3a_3b_1, \\ a_2b_5a_2b_1^{-1}, & a_2b_4^{-1}a_2b_3^{-1}, & a_3b_2a_3^{-1}b_4^{-1}, \\ a_3b_3a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_1, & a_3b_5a_3^{-1}b_5 \end{array} \right\}.$$

$$\begin{aligned} &\rho_v(b_1) = (2, 5, 4),\ \rho_v(b_2) = (), \\ &\rho_v(b_3) = (2, 5, 3), \\ &\rho_v(b_4) = (1, 2, 5), \\ &\rho_v(b_5) = (2, 6, 5), \\ &\rho_h(a_1) = (1, 7, 6, 2)(3, 8)(4, 5, 9, 10), \\ &\rho_h(a_2) = (1, 5, 4, 8)(3, 7, 6, 10), \\ &\rho_h(a_3) = (1, 7, 9, 8)(2, 3, 10, 4)(5, 6). \end{aligned}$$

Example A.13. $(A_6, \mathrm{PGL}_2(9))$*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_1^{-1}, & a_1b_3a_1^{-1}b_3, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_3, & a_2b_2a_2^{-1}b_2^{-1}, & a_2b_3a_3b_1, \\ a_2b_5a_2b_1^{-1}, & a_2b_4^{-1}a_2b_3^{-1}, & a_3b_2a_3^{-1}b_5, \\ a_3b_3a_3^{-1}b_2^{-1}, & a_3b_4a_3^{-1}b_1, & a_3b_5a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned} &\rho_v(b_1) = (2, 5, 4),\ \rho_v(b_2) = (), \\ &\rho_v(b_3) = (2, 5, 3), \\ &\rho_v(b_4) = (1, 2, 5), \\ &\rho_v(b_5) = (2, 6, 5), \\ &\rho_h(a_1) = (1, 2)(3, 8)(4, 5)(6, 7)(9, 10), \\ &\rho_h(a_2) = (1, 5, 4, 8)(3, 7, 6, 10), \\ &\rho_h(a_3) = (1, 7, 6, 2, 3, 10, 4, 5, 9, 8). \end{aligned}$$

Example A.14. (A_6, M_{10})*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2^{-1}, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_2b_3, & a_2b_5a_2b_1^{-1}, \\ a_2b_4^{-1}a_2b_3^{-1}, & a_2b_2^{-1}a_3b_1, & a_3b_2a_3^{-1}b_5^{-1}, \\ a_3b_3a_3b_3^{-1}, & a_3b_4a_3^{-1}b_1^{-1}, & a_3b_5a_3^{-1}b_4^{-1} \end{array} \right\}.$$

$$\begin{aligned}
&\rho_v(b_1) = (2, 5, 4),\ \rho_v(b_2) = (2, 3, 5),\\
&\rho_v(b_3) = (2, 5)(3, 4),\\
&\rho_v(b_4) = (1, 2, 5),\ \rho_v(b_5) = (2, 6, 5),\\
&\rho_h(a_1) = (2, 3)(4, 5)(6, 7)(8, 9),\\
&\rho_h(a_2) = (1, 5, 4, 8, 2)(3, 7, 6, 10, 9),\\
&\rho_h(a_3) = (1, 4, 5, 2)(6, 9, 10, 7).
\end{aligned}$$

Example A.15. $(A_6, \mathrm{P\Gamma L}_2(9))$*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_1^{-1}b_2^{-1}, \\ a_1b_4a_1^{-1}b_5^{-1}, & a_1b_5a_2^{-1}b_4^{-1}, & a_1b_5^{-1}a_2b_4, \\ a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_2b_3, & a_2b_5a_2b_1^{-1}, \\ a_2b_4^{-1}a_2b_3^{-1}, & a_2b_2^{-1}a_3b_1, & a_3b_2a_3^{-1}b_4, \\ a_3b_3a_3b_3^{-1}, & a_3b_4a_3^{-1}b_5, & a_3b_5a_3^{-1}b_1^{-1} \end{array} \right\}.$$

$$\begin{aligned}
&\rho_v(b_1) = (2, 5, 4),\\
&\rho_v(b_2) = (2, 3, 5),\\
&\rho_v(b_3) = (2, 5)(3, 4),\\
&\rho_v(b_4) = (1, 2, 5),\\
&\rho_v(b_5) = (2, 6, 5),\\
&\rho_h(a_1) = (1, 10)(2, 3)(4, 5)(6, 7)(8, 9),\\
&\rho_h(a_2) = (1, 5, 4, 8, 2)(3, 7, 6, 10, 9),\\
&\rho_h(a_3) = (1, 5, 7, 2)(4, 9, 10, 6).
\end{aligned}$$

Example A.16. (A_6, A_{10})*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_4^{-1}, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_4a_1^{-1}b_2^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_3^{-1}a_2^{-1}b_3, \\ a_2b_1a_3^{-1}b_1^{-1}, & a_2b_2a_3^{-1}b_2, & a_2b_4a_3^{-1}b_5^{-1}, \\ a_2b_5a_2b_4^{-1}, & a_2b_5^{-1}a_3b_4, & a_2b_2^{-1}a_3^{-1}b_1, \\ a_2b_1^{-1}a_3^{-1}b_2^{-1}, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_5a_3^{-1}b_3^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= \rho_v(b_2) = (2,3)(4,5), \\ \rho_v(b_3) &= (1,2)(5,6), \\ \rho_v(b_4) &= (2,5,4), \\ \rho_v(b_5) &= (2,3,5), \\ \rho_h(a_1) &= (2,4)(7,9), \\ \rho_h(a_2) &= (2,10,9)(4,5)(6,7), \\ \rho_h(a_3) &= (1,2,9)(3,5,4)(6,7,8). \end{aligned}$$

Example A.17. (A_6, S_{10})*–group:*

$$R_{3\cdot 5} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_4, & a_1b_3a_2^{-1}b_3^{-1}, \\ a_1b_4a_1^{-1}b_2^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, & a_1b_3^{-1}a_2^{-1}b_3, \\ a_2b_1a_3^{-1}b_1^{-1}, & a_2b_2a_3^{-1}b_2, & a_2b_4a_3^{-1}b_5^{-1}, \\ a_2b_5a_2b_4^{-1}, & a_2b_5^{-1}a_3b_4, & a_2b_2^{-1}a_3^{-1}b_1, \\ a_2b_1^{-1}a_3^{-1}b_2^{-1}, & a_3b_3a_3^{-1}b_4^{-1}, & a_3b_5a_3^{-1}b_3^{-1} \end{array} \right\}.$$

$$\begin{aligned} \rho_v(b_1) &= \rho_v(b_2) = (2,3)(4,5), \\ \rho_v(b_3) &= (1,2)(5,6), \\ \rho_v(b_4) &= (2,5,4), \\ \rho_v(b_5) &= (2,3,5), \\ \rho_h(a_1) &= (2,4,9,7), \\ \rho_h(a_2) &= (2,10,9)(4,5)(6,7), \\ \rho_h(a_3) &= (1,2,9)(3,5,4)(6,7,8). \end{aligned}$$

Example A.18. $(A_6, \mathrm{PSL}_2(11))$*–group:*

$$R_{3\cdot 6} := \left\{ \begin{array}{lll} a_1b_1a_3^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_1^{-1}a_2b_2, & a_2b_1a_2b_3^{-1}, & a_2b_3a_2b_5^{-1}, \\ a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_1^{-1}a_3b_2, & a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_6, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} &\rho_v(b_1) = (2,6,4,3,5),\ \rho_v(b_2) = (1,3,4,2,5), \\ &\rho_v(b_3) = \rho_v(b_5) = \rho_v(b_6) = (2,5)(3,4), \\ &\rho_v(b_4) = (), \\ &\rho_h(a_1) = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12), \\ &\rho_h(a_2) = \rho_h(a_3) = (1,2,7,5,3)(6,11,12,10,8). \end{aligned}$$

Example A.19. $(A_6, \mathrm{PGL}_2(11))$*–group:*

$$R_{3\cdot 6} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_1^{-1}b_4^{-1}, & a_1b_3a_2^{-1}b_1, \\ a_1b_4a_1^{-1}b_6^{-1}, & a_1b_5a_1^{-1}b_3^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_3^{-1}a_3b_1^{-1}, & a_2b_1a_2b_2, & a_2b_3a_2b_5^{-1}, \\ a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_1^{-1}a_3b_3^{-1}, & a_3b_1a_3b_2, & a_3b_3a_3b_5^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_6, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} &\rho_v(b_1) = (1,4,3,5,2), \\ &\rho_v(b_2) = \rho_v(b_5) = \rho_v(b_6) = (2,5)(3,4), \\ &\rho_v(b_3) = (2,4,3,6,5),\ \rho_v(b_4) = (), \\ &\rho_h(a_1) = (1,10,8,7,9,11,12,3,5,6,4,2), \\ &\rho_h(a_2) = \rho_h(a_3) = (1,10,8,6,11)(2,7,5,3,12). \end{aligned}$$

Example A.20. (A_6, A_{12})*–group:*

$$R_{3\cdot 6} := \left\{ \begin{array}{lll} a_1b_1a_3^{-1}b_1^{-1}, & a_1b_2a_3^{-1}b_1, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_2^{-1}a_3^{-1}b_2^{-1}, & a_1b_1^{-1}a_2b_2, & a_2b_1a_2b_3^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3^{-1}b_2, & a_3b_3a_3b_3^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_5^{-1}, & a_3b_6a_3b_6^{-1} \end{array} \right\}.$$

$$\begin{aligned} &\rho_v(b_1) = (1,3)(2,6,4,5),\ \rho_v(b_2) = (1,3,2,5)(4,6), \\ &\rho_v(b_3) = \rho_v(b_5) = \rho_v(b_6) = (2,5)(3,4),\ \rho_v(b_4) = (), \\ &\rho_h(a_1) = (2,11,12)(3,4)(5,6)(7,8)(9,10), \\ &\rho_h(a_2) = (1,2,7,5,3)(6,11,12,10,8), \\ &\rho_h(a_3) = (1,11,2). \end{aligned}$$

Example A.21. (A_6, S_{12})*–group:*

$$R_{3\cdot 6} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_3^{-1}, & a_2b_3a_2b_5^{-1}, \\ a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, & a_2b_6^{-1}a_2b_2^{-1}, \\ a_2b_1^{-1}a_3b_2, & a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, \\ a_3b_4a_3^{-1}b_4^{-1}, & a_3b_5a_3b_6, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned} &\rho_v(b_1) = (1,4,3,5,2),\ \rho_v(b_2) = (2,5,6,3,4), \\ &\rho_v(b_3) = \rho_v(b_5) = \rho_v(b_6) = (2,5)(3,4),\ \rho_v(b_4) = (), \\ &\rho_h(a_1) = (1,2,12,11)(3,4)(5,6)(7,8)(9,10), \\ &\rho_h(a_2) = \rho_h(a_3) = (1,2,7,5,3)(6,11,12,10,8). \end{aligned}$$

Example A.22. $(A_6, \mathrm{PSL}_2(13))$*–group:*

$$R_{3\cdot 7} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_7a_1^{-1}b_7^{-1}, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_7^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_3^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$\rho_v(b_1) = (1, 4, 3, 5, 2),\ \rho_v(b_2) = (2, 5, 6, 3, 4),$
$\rho_v(b_3) = \rho_v(b_5) = \rho_v(b_6) = \rho_v(b_7) = (2, 5)(3, 4),\ \rho_v(b_4) = (),$
$\rho_h(a_1) = (1, 2)(3, 4)(5, 6)(9, 10)(11, 12)(13, 14),$
$\rho_h(a_2) = \rho_h(a_3) = (1, 2, 9, 5, 3, 7)(6, 13, 14, 8, 12, 10).$

Example A.23. $(A_6, \mathrm{PGL}_2(13))$*–group:*

$$R_{3\cdot 7} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_7, \\ a_1b_4a_1^{-1}b_6^{-1}, & a_1b_5a_1^{-1}b_5, & a_1b_6a_1^{-1}b_4^{-1}, \\ a_1b_7a_1^{-1}b_3, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_7^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_3^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$\rho_v(b_1) = (1, 4, 3, 5, 2),\ \rho_v(b_2) = (2, 5, 6, 3, 4),$
$\rho_v(b_3) = \rho_v(b_5) = \rho_v(b_6) = \rho_v(b_7) = (2, 5)(3, 4),\ \rho_v(b_4) = (),$
$\rho_h(a_1) = (1, 2)(3, 8)(4, 6)(5, 10)(7, 12)(9, 11)(13, 14),$
$\rho_h(a_2) = \rho_h(a_3) = (1, 2, 9, 5, 3, 7)(6, 13, 14, 8, 12, 10).$

Example A.24. (A_6, A_{14})*–group:*

$$R_{3\cdot 7} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_7a_1^{-1}b_7^{-1}, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_7^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (1, 4, 3, 5, 2), \\
\rho_v(b_2) &= (2, 5, 6, 3, 4), \\
\rho_v(b_3) &= (2, 5)(3, 4), \\
\rho_v(b_4) &= (), \\
\rho_v(b_5) &= (2, 5)(3, 4), \\
\rho_v(b_6) &= (2, 5)(3, 4), \\
\rho_v(b_7) &= (2, 5)(3, 4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1, 2)(3, 4)(5, 6)(9, 10)(11, 12)(13, 14), \\
\rho_h(a_2) &= (1, 2, 9, 5, 3, 7)(6, 13, 14, 8, 12, 10), \\
\rho_h(a_3) &= (1, 2, 9, 5, 3)(6, 13, 14, 12, 10).
\end{aligned}$$

Example A.25. (A_6, S_{14})*–group:*

$$R_{3\cdot 7} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_2^{-1}, & a_1b_2a_3b_1^{-1}, & a_1b_3a_1^{-1}b_4^{-1}, \\ a_1b_4a_1^{-1}b_3^{-1}, & a_1b_5a_1^{-1}b_6^{-1}, & a_1b_6a_1^{-1}b_5^{-1}, \\ a_1b_7a_1^{-1}b_7, & a_1b_2^{-1}a_2^{-1}b_1, & a_2b_1a_2b_7^{-1}, \\ a_2b_3a_2b_5^{-1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2b_6, \\ a_2b_7a_2b_3^{-1}, & a_2b_6^{-1}a_2b_2^{-1}, & a_2b_1^{-1}a_3b_2, \\ a_3b_1a_3b_3^{-1}, & a_3b_3a_3b_5^{-1}, & a_3b_4a_3^{-1}b_4^{-1}, \\ a_3b_5a_3b_6, & a_3b_7a_3b_7^{-1}, & a_3b_6^{-1}a_3b_2^{-1} \end{array} \right\}.$$

$$\begin{aligned}
\rho_v(b_1) &= (1, 4, 3, 5, 2), \\
\rho_v(b_2) &= (2, 5, 6, 3, 4), \\
\rho_v(b_3) &= (2, 5)(3, 4), \\
\rho_v(b_4) &= (), \\
\rho_v(b_5) &= (2, 5)(3, 4), \\
\rho_v(b_6) &= (2, 5)(3, 4), \\
\rho_v(b_7) &= (2, 5)(3, 4),
\end{aligned}$$

$$\begin{aligned}
\rho_h(a_1) &= (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14), \\
\rho_h(a_2) &= (1, 2, 9, 5, 3, 7)(6, 13, 14, 8, 12, 10), \\
\rho_h(a_3) &= (1, 2, 9, 5, 3)(6, 13, 14, 12, 10).
\end{aligned}$$

A.2 Amalgam decompositions of Example 2.2

Vertical decomposition

We first give the vertical decomposition of the (6, 6)–group Γ of Example 2.2:

$$\Gamma \cong F_3^{(v,b)} *_{F_{13}^{(v,b)} \cong F_{13}^{(v,s)}} F_7^{(v,s)} ,$$

where the factors are defined as follows:

$$F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle, \quad F_7^{(v,s)} = \langle s_1, s_2, s_3, s_4, s_5, s_6, s_7 \rangle .$$

The injective homomorphism $F_{13}^{(v,b)} \hookrightarrow F_3^{(v,b)}$ is given by the description of $F_{13}^{(v,b)}$ as a subgroup of $F_3^{(v,b)}$ of index 6:

$$\begin{aligned} F_{13}^{(v,b)} = \langle & b_1, b_3, b_2 b_3^{-1} b_2,\ b_2^{-1} b_3^{-1} b_2^2,\ b_2^{-1} b_1 b_2^2,\ b_2^{-1} b_1^{-1} b_2^2,\ b_2 b_1^{-2} b_2^{-1},\ b_2 b_3 b_1^{-1} b_2^{-1}, \\ & b_2^2 b_1^{-1} b_2^{-1},\ b_2^{-3} b_1^{-1} b_2^{-1},\ b_2 b_1 b_3^2 b_2^2,\ b_2^{-2} b_3^{-1} b_1 b_3 b_2^2,\ b_2^{-2} b_3^{-1} b_2 b_3 b_2^2 \rangle , \end{aligned}$$

the inclusion $F_{13}^{(v,s)} \hookrightarrow F_7^{(v,s)}$ by

$$\begin{aligned} F_{13}^{(v,s)} = \langle & s_1,\ s_2,\ s_6,\ s_4^{-1} s_3,\ s_5^{-1} s_3,\ s_7^{-1} s_3,\ s_7 s_3^{-1},\ s_5 s_3^{-1}, \\ & s_4 s_3^{-1},\ s_3^{-1} s_6 s_3^{-1},\ s_3^2,\ s_3^{-1} s_1 s_3,\ s_3^{-1} s_2 s_3 \rangle . \end{aligned}$$

The identification

$$\begin{aligned} F_{13}^{(v,b)} &\overset{\cong}{\longleftrightarrow} F_{13}^{(v,s)} \\ b_1 &\longleftrightarrow s_1 \\ b_3 &\longleftrightarrow s_2 \\ b_2 b_3^{-1} b_2 &\longleftrightarrow s_6 \\ b_2^{-1} b_3^{-1} b_2^2 &\longleftrightarrow s_4^{-1} s_3 \\ b_2^{-1} b_1 b_2^2 &\longleftrightarrow s_5^{-1} s_3 \\ b_2^{-1} b_1^{-1} b_2^2 &\longleftrightarrow s_7^{-1} s_3 \\ b_2 b_1^{-2} b_2^{-1} &\longleftrightarrow s_7 s_3^{-1} \\ b_2 b_3 b_1^{-1} b_2^{-1} &\longleftrightarrow s_5 s_3^{-1} \\ b_2^2 b_1^{-1} b_2^{-1} &\longleftrightarrow s_4 s_3^{-1} \\ b_2^{-3} b_1^{-1} b_2^{-1} &\longleftrightarrow s_3^{-1} s_6 s_3^{-1} \\ b_2 b_1 b_3^2 b_2^2 &\longleftrightarrow s_3^2 \\ b_2^{-2} b_3^{-1} b_1 b_3 b_2^2 &\longleftrightarrow s_3^{-1} s_1 s_3 \\ b_2^{-2} b_3^{-1} b_2 b_3 b_2^2 &\longleftrightarrow s_3^{-1} s_2 s_3 \end{aligned}$$

in the amalgam leads to a finite presentation of Γ with 10 generators

$$\{b_1, b_2, b_3, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$$

and 13 relations

$$b_1 = s_1,\ b_3 = s_2,\ b_2b_3^{-1}b_2 = s_6,\ b_2^{-1}b_3^{-1}b_2^2 = s_4^{-1}s_3,\ b_2^{-1}b_1b_2^2 = s_5^{-1}s_3,$$
$$b_2^{-1}b_1^{-1}b_2^2 = s_7^{-1}s_3,\ b_2b_1^{-2}b_2^{-1} = s_7s_3^{-1},\ b_2b_3b_1^{-1}b_2^{-1} = s_5s_3^{-1},$$
$$b_2^2b_1^{-1}b_2^{-1} = s_4s_3^{-1},\ b_2^{-3}b_1^{-1}b_2^{-1} = s_3^{-1}s_6s_3^{-1},\ b_2b_1b_3^2b_2^2 = s_3^2,$$
$$b_2^{-2}b_3^{-1}b_1b_3b_2^2 = s_3^{-1}s_1s_3,\ b_2^{-2}b_3^{-1}b_2b_3b_2^2 = s_3^{-1}s_2s_3\,.$$

Horizontal decomposition

In a similar way, we can describe the horizontal decomposition of

$$\Gamma \cong F_3^{(h,a)} *_{F_{13}^{(h,a)} \cong F_{13}^{(h,u)}} F_7^{(h,u)}$$

by a finite presentation with generators

$$\{a_1, a_2, a_3, u_1, u_2, u_3, u_4, u_5, u_6, u_7\}\,,$$

and relations

$$a_1 = u_1,\ a_3^4 = u_5u_7,\ a_2a_3^{-3} = u_7u_5^{-1},\ a_3^3a_1a_3^{-3} = u_5u_1u_5^{-1},\ a_3a_1a_3^{-2} = u_2u_5^{-1},$$
$$a_3a_2a_3^{-2} = u_3^{-1}u_5^{-1},\ a_3^2a_1a_3^{-1} = u_5u_4,\ a_3^2a_2a_3^{-1} = u_5u_6,\ a_3^3a_2a_1a_2 = u_5u_2,$$
$$a_3^3a_2a_3a_2 = u_5u_6^{-1},\ a_3^3a_2^3 = u_5^2,\ a_2^{-1}a_3a_2^{-1}a_3^{-3} = u_4u_5^{-1},\ a_2^{-1}a_1a_2^{-1}a_3^{-3} = u_3u_5^{-1}\,.$$

Isomorphisms

We recall the set of relators $R_{3\cdot 3}$ of Example 2.2:

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2b_2^{-1}, \\ a_1b_3^{-1}a_3^{-1}b_2, & a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, \\ a_2b_3a_3^{-1}b_1, & a_2b_3^{-1}a_3b_2, & a_2b_1^{-1}a_3^{-1}b_1^{-1} \end{array} \right\}.$$

Explicit isomorphisms between the three given finite presentations of Γ are:

$$\begin{array}{rcccl}
\Gamma^{(v)} & \overset{\cong}{\longleftrightarrow} & \langle a_1, \ldots, b_3 \mid R_{3\cdot 3}\rangle & \overset{\cong}{\longleftrightarrow} & \Gamma^{(h)} \\
 & & & & \\
s_3 b_2^{-2} b_3^{-1} & \longleftrightarrow & a_1 & \longleftrightarrow & a_1 = u_1 \\
b_3 b_2 s_4^{-1} b_2 & \longleftrightarrow & a_2 & \longleftrightarrow & a_2 \\
b_2 s_4^{-1} b_2^2 & \longleftrightarrow & a_3 & \longleftrightarrow & a_3 \\
s_1 = b_1 & \longleftrightarrow & b_1 & \longleftrightarrow & u_7^{-1} a_2 \\
b_2 & \longleftrightarrow & b_2 & \longleftrightarrow & a_2 u_5^{-1} a_3^2 \\
s_2 = b_3 & \longleftrightarrow & b_3 & \longleftrightarrow & a_2^2 u_5^{-1} a_3 \\
s_3 & \longleftrightarrow & a_1 b_3 b_2^2 & & \\
s_4 & \longleftrightarrow & a_1 b_3^2 b_2 & & \\
s_5 & \longleftrightarrow & a_1 b_3 b_1^{-1} b_2 & & \\
s_6 & \longleftrightarrow & b_2 b_3^{-1} b_2 & & \\
s_7 & \longleftrightarrow & a_1 b_3 b_1 b_2 & & \\
 & & a_3 a_1 a_3 b_1^{-1} & \longleftrightarrow & u_2 \\
 & & a_2^{-1} a_1 a_2^{-1} b_1^{-1} & \longleftrightarrow & u_3 \\
 & & a_2^{-1} a_3 a_2^{-1} b_1^{-1} & \longleftrightarrow & u_4 \\
 & & a_3^3 b_1^{-1} & \longleftrightarrow & u_5 \\
 & & (b_1 a_2 a_3 a_2)^{-1} & \longleftrightarrow & u_6 \\
 & & a_2 b_1^{-1} & \longleftrightarrow & u_7\,,
\end{array}$$

where

$$\Gamma^{(v)} = F_3^{(v,b)} *_{F_{13}^{(v,b)} \cong F_{13}^{(v,s)}} F_7^{(v,s)}$$

and

$$\Gamma^{(h)} = F_3^{(h,a)} *_{F_{13}^{(h,a)} \cong F_{13}^{(h,u)}} F_7^{(h,u)}\,.$$

Observe that with this identification, the abelianization map $\Gamma \to \Gamma^{ab} \cong \mathbb{Z}_2^2$ is now given by

$$\begin{aligned}
a_1, a_2, a_3 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\
b_1, b_2, b_3 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}) \\
s_1, s_2, s_6 &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}) \\
s_3, s_4, s_5, s_7 &\mapsto (1 + 2\mathbb{Z}, 1 + 2\mathbb{Z}) \\
u_1 &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}) \\
u_2, u_3, u_4, u_5, u_6, u_7 &\mapsto (1 + 2\mathbb{Z}, 1 + 2\mathbb{Z})\,.
\end{aligned}$$

Local action on trees

The vertical amalgam decomposition of Γ described above gives a natural action of Γ on the first barycentric subdivision $\mathcal{T}_6'$ of $\mathcal{T}_{2m} = \mathcal{T}_6$. See [64, Chapter 4] for the general theory about amalgams and their action on the corresponding tree. Let P be the vertex of $\mathcal{T}_6'$ stabilized by $F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle$. The local action of $\Gamma \cong \mathrm{pr}_1(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2m})$ on $S(x_h, 1)$ in $\mathcal{T}_6$, i.e. the homomorphism $\rho_v : \langle b_1, b_2, b_3 \rangle \twoheadrightarrow P_h < S_{2m}$ determined in the proof of Theorem 2.3(1), can be reconstructed by the action of $F_3^{(v,b)}$ on the set of edges of $\mathcal{T}_6'$ originating at P. These edges are labelled by right cosets $F_{13}^{(v,b)} g_i$, $i = 1, \ldots, 6$, $g_i \in F_3^{(v,b)}$, such that

$$F_3^{(v,b)} = \bigsqcup_{i=1}^{6} F_{13}^{(v,b)} g_i \,.$$

The group $F_3^{(v,b)} = \langle b_1, b_2, b_3 \rangle$ acts by right multiplication on the set of right cosets $\{F_{13}^{(v,b)} g_i\}_{i=1,\ldots,6}$. If we choose $g_1 = 1$, $g_2 = b_2 b_1 b_2$, $g_3 = (b_2 b_1)^2$, $g_4 = b_2 b_1$, $g_5 = b_2$, $g_6 = b_2 b_1 b_3$ and make the identification $F_{13}^{(v,b)} g_i \leftrightarrow i$ for $i = 1, \ldots, 6$, then we exactly get back our homomorphism ρ_v:

$$\begin{aligned}
\rho_v(b_1) &= (2,3)(4,5),\\
\rho_v(b_2) &= (1,5,4,2,3),\\
\rho_v(b_3) &= (2,3,5,4,6),
\end{aligned}$$

generating $P_h = A_6$. In the same way, we compute the action of $F_3^{(h,a)} = \langle a_1, a_2, a_3 \rangle$ by right multiplication on right cosets

$$F_3^{(h,a)} = F_{13}^{(h,a)} \sqcup F_{13}^{(h,a)} a_2^2 a_1 \sqcup F_{13}^{(h,a)} a_2^2 \sqcup F_{13}^{(h,a)} a_3 \sqcup F_{13}^{(h,a)} a_3 a_1 \sqcup F_{13}^{(h,a)} a_2$$

and recover $\rho_h : \langle a_1, a_2, a_3 \rangle \twoheadrightarrow P_v < S_{2n} = S_6$:

$$\begin{aligned}
\rho_h(a_1) &= (2,3)(4,5),\\
\rho_h(a_2) &= (1,6,3,2)(4,5),\\
\rho_h(a_3) &= (1,4,5,6)(2,3),
\end{aligned}$$

generating $P_v = A_6$.

Vertical decompositions of Γ_0

The cell complex X_0 of Example 2.2 corresponding to the subgroup $\Gamma_0 < \Gamma$ is given by the $4 \cdot 9 = 36$ geometric squares illustrated on the next two pages.

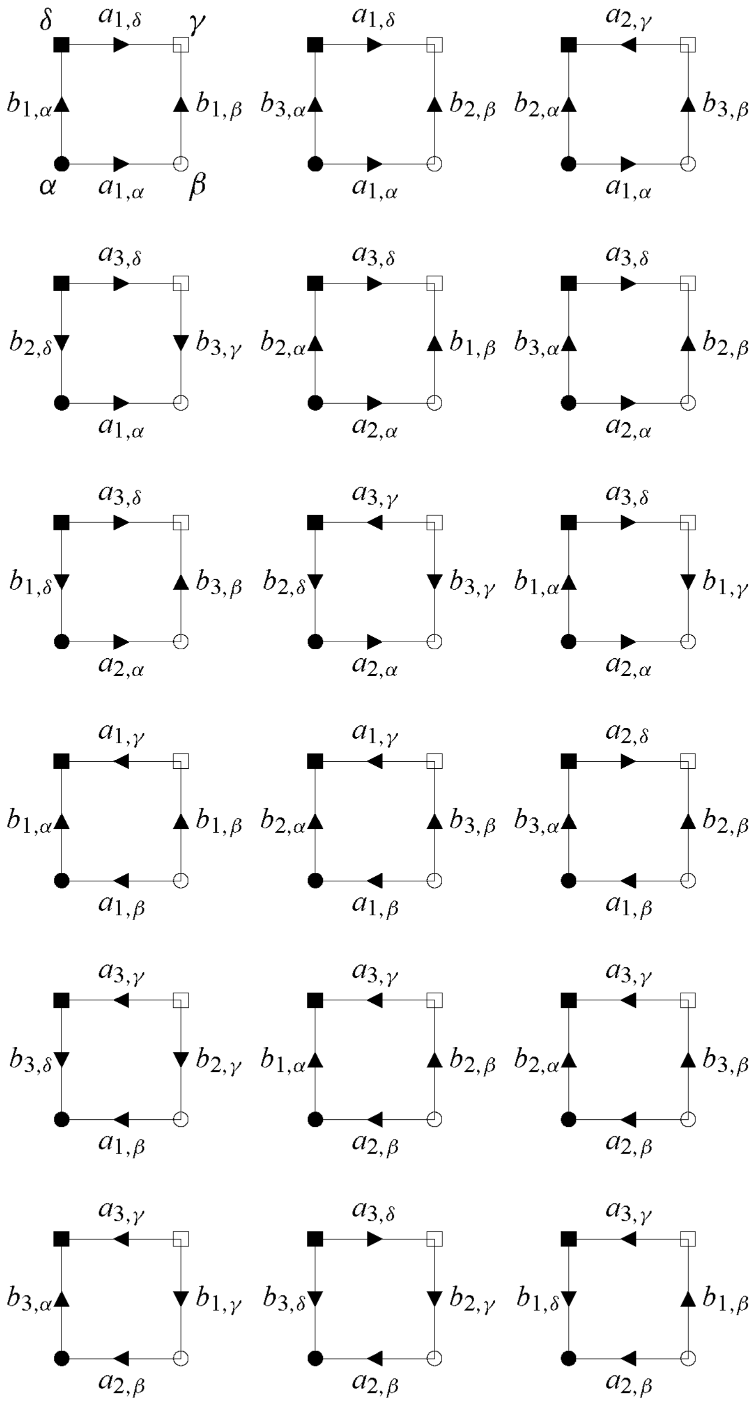


Figure A.1: Complex X_0 of Example 2.2, part I

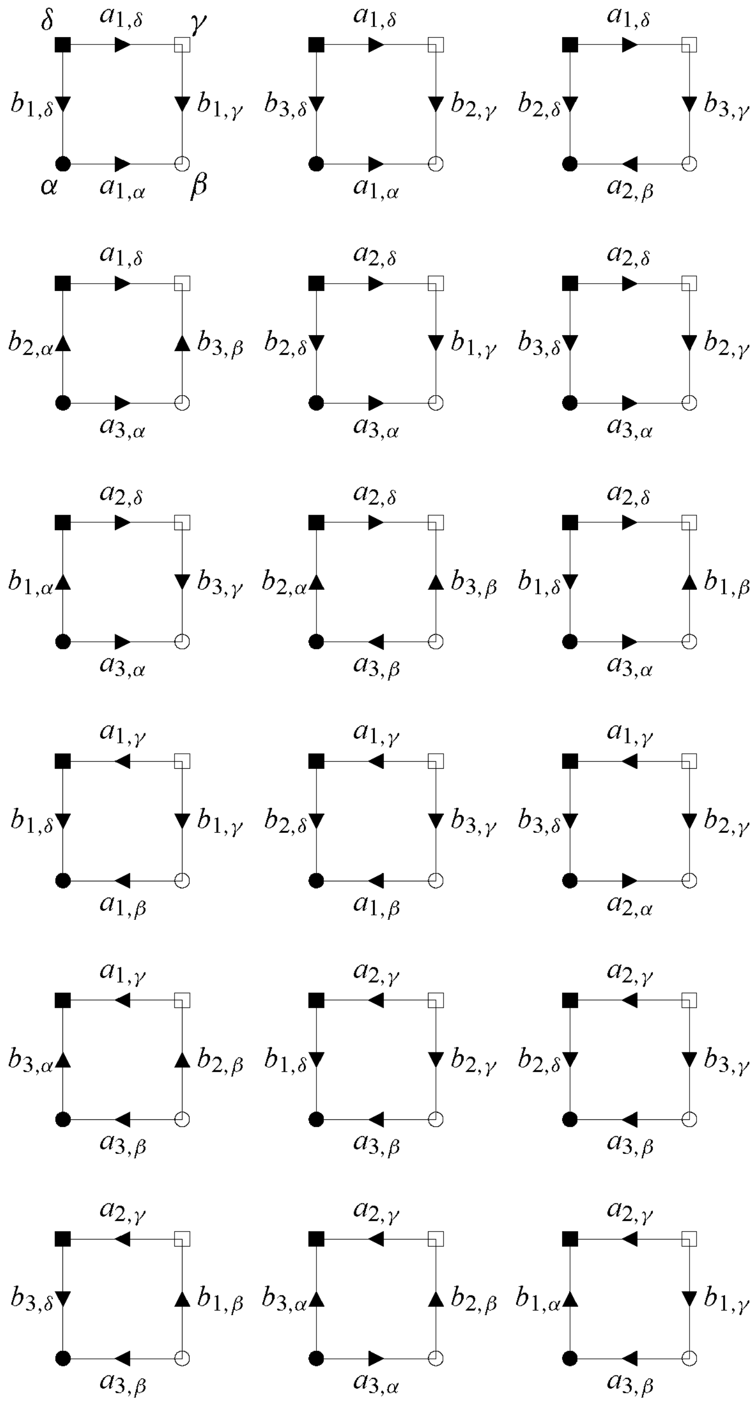


Figure A.2: Complex X_0 of Example 2.2, part II

The amalgam decompositions of Γ_0 are:

$$F_5^{(v,r)} *_{F_{25}^{(v,r)} \cong F_{25}^{(v,q)}} F_5^{(v,q)} \cong \Gamma_0 \cong F_5^{(h,t)} *_{F_{25}^{(h,t)} \cong F_{25}^{(h,w)}} F_5^{(h,w)},$$

where

$$F_5^{(v,r)} = \langle r_1, r_2, r_3, r_4, r_5 \rangle, \quad F_5^{(v,q)} = \langle q_1, q_2, q_3, q_4, q_5 \rangle .$$

The inclusion $F_{25}^{(v,r)} \hookrightarrow F_5^{(v,r)}$ is defined by

$$\begin{aligned} F_{25}^{(v,r)} = \langle & r_2,\ r_5,\ r_3,\ r_1r_5r_3^{-1}r_1^{-1},\ r_1r_4r_3^{-1}r_1^{-1},\ r_1r_3r_1^{-1},\ r_1^{-1}r_5r_1,\ r_1^{-1}r_3r_1, \\ & r_1^{-1}r_4r_1,\ r_1^{-1}r_2r_1^{-1},\ r_4^{-1}r_1^{-1}r_4,\ r_4^{-1}r_5r_1r_4,\ r_4^{-1}r_1^{-1}r_2r_4,\ r_4r_1r_4^{-1}, \\ & r_4r_2r_4^{-1},\ r_4r_5r_4^{-1},\ r_4r_3^{-1}r_4,\ r_4r_3r_2r_1,\ r_4r_3r_4r_3^{-1}r_4^{-1},\ r_4r_3r_5r_3^{-1}r_4^{-1}, \\ & r_4r_3r_1r_3^{-1}r_4^{-1},\ r_4^2r_1r_4,\ r_1r_3r_1^2,\ r_1r_3r_2r_3^{-1}r_4^{-1},\ r_4r_3^2r_1r_4 \rangle \end{aligned}$$

and the other inclusion $F_{25}^{(v,q)} \hookrightarrow F_5^{(v,q)}$ by

$$\begin{aligned} F_{25}^{(v,q)} = \langle & q_1,\ q_5,\ q_4,\ q_2q_4q_2^{-1},\ q_2q_3q_2^{-1},\ q_2q_5^{-1}q_2^{-1},\ q_2^{-1}q_3^{-1}q_2,\ q_2^{-1}q_3^{-1}q_4q_2, \\ & q_2^{-1}q_3^{-1}q_5q_2,\ q_2^{-1}q_1q_2^{-1},\ q_3^{-1}q_5^{-1}q_3,\ q_3^{-1}q_2^{-1}q_3,\ q_3^{-1}q_1q_3, \\ & q_3q_2q_1^{-1}q_3^{-1},\ q_3q_5^{-1}q_1^{-1}q_3^{-1},\ q_3q_1q_3^{-1},\ q_3q_4^{-1}q_3,\ q_3q_1q_4q_1q_3q_2, \\ & q_3q_1q_4q_3q_4^{-1}q_1^{-1}q_3^{-1},\ q_3q_1q_4q_5q_4^{-1}q_1^{-1}q_3^{-1},\ q_3q_1q_4q_2q_4^{-1}q_1^{-1}q_3^{-1}, \\ & q_3q_1q_3^2,\ q_2^2q_3q_2,\ q_2q_1q_4^{-1}q_1^{-1}q_3^{-1},\ q_3q_1q_4^2q_3 \rangle . \end{aligned}$$

We obtain a finite presentation for the vertical decomposition of Γ_0 with generators

$$\{r_1, r_2, r_3, r_4, r_5, q_1, q_2, q_3, q_4, q_5\}$$

and 25 relations

$$\begin{aligned} & r_2 = q_1,\ r_5 = q_5,\ r_3 = q_4,\ r_1r_5r_3^{-1}r_1^{-1} = q_2q_4q_2^{-1},\ r_1r_4r_3^{-1}r_1^{-1} = q_2q_3q_2^{-1}, \\ & r_1r_3r_1^{-1} = q_2q_5^{-1}q_2^{-1},\ r_1^{-1}r_5r_1 = q_2^{-1}q_3^{-1}q_2,\ r_1^{-1}r_3r_1 = q_2^{-1}q_3^{-1}q_4q_2, \\ & r_1^{-1}r_4r_1 = q_2^{-1}q_3^{-1}q_5q_2,\ r_1^{-1}r_2r_1^{-1} = q_2^{-1}q_1q_2^{-1},\ r_4^{-1}r_1^{-1}r_4 = q_3^{-1}q_5^{-1}q_3, \\ & r_4^{-1}r_5r_1r_4 = q_3^{-1}q_2^{-1}q_3,\ r_4^{-1}r_1^{-1}r_2r_4 = q_3^{-1}q_1q_3,\ r_4r_1r_4^{-1} = q_3q_2q_1^{-1}q_3^{-1}, \\ & r_4r_2r_4^{-1} = q_3q_5^{-1}q_1^{-1}q_3^{-1},\ r_4r_5r_4^{-1} = q_3q_1q_3^{-1},\ r_4r_3^{-1}r_4 = q_3q_4^{-1}q_3, \\ & r_4r_3r_2r_1 = q_3q_1q_4q_1q_3q_2,\ r_4r_3r_4r_3^{-1}r_4^{-1} = q_3q_1q_4q_3q_4^{-1}q_1^{-1}q_3^{-1}, \\ & r_4r_3r_5r_3^{-1}r_4^{-1} = q_3q_1q_4q_5q_4^{-1}q_1^{-1}q_3^{-1},\ r_4r_3r_1r_3^{-1}r_4^{-1} = q_3q_1q_4q_2q_4^{-1}q_1^{-1}q_3^{-1}, \\ & r_4^2r_1r_4 = q_3q_1q_3^2,\ r_1r_3r_1^2 = q_2^2q_3q_2,\ r_1r_3r_2r_3^{-1}r_4^{-1} = q_2q_1q_4^{-1}q_1^{-1}q_3^{-1}, \\ & r_4r_3^2r_1r_4 = q_3q_1q_4^2q_3 . \end{aligned}$$

Horizontal decompositions of Γ_0

The horizontal decomposition of Γ_0 is given by the generators $\{w_1, \ldots, w_5, t_1, \ldots, t_5\}$ and 25 relations

$$\begin{aligned}
&w_1w_5 = t_2t_4,\ w_1w_4^2 = t_2t_5^2,\ w_3 = t_3,\ w_1w_3w_1^{-1} = t_2t_3t_2^{-1},\ w_4w_1^{-1} = t_5t_2^{-1},\\
&w_1w_2 = t_2t_1,\ w_4^{-1}w_1w_4 = t_5^{-1}t_2t_5,\ w_4^{-1}w_3^{-1}w_4 = t_5^{-1}t_1t_5,\ w_4^{-1}w_5w_4 = t_5^{-1}t_3t_5,\\
&w_4^{-1}w_2^{-1}w_4 = t_5^{-1}t_4t_5,\ w_1w_2^{-2} = t_2t_1^{-1}t_2t_1^{-1},\ w_2w_1^{-1}w_2w_1^{-1} = t_1t_5t_1t_2^{-1},\\
&w_1w_2^{-1}w_4^{-1}w_2^{-1} = t_2t_1^{-2},\ w_2w_3w_2w_1^{-1} = t_1t_3t_1t_2^{-1},\ w_2w_5w_2w_1^{-1} = t_1t_4t_1t_2^{-1},\\
&w_1w_5^{-1}w_3w_5^{-1} = t_2t_4^{-1}t_5t_4^{-1},\ w_5w_4^{-1}w_5w_1^{-1} = t_4t_2t_4t_2^{-1},\ w_5^2w_1^{-1} = t_4^2t_2^{-1},\\
&w_5w_2w_5w_1^{-1} = t_4t_1t_4t_2^{-1},\ w_1w_5^{-1}w_1^{-1}w_5^{-1} = t_2t_4^{-1}t_3t_4^{-1},\\
&w_1^{-1}w_5^{-1}w_1^{-2} = t_2^{-1}t_1t_2^{-2},\ w_1^{-1}w_2w_1^{-2} = t_2^{-3},\ w_1^2w_4w_1 = t_2^2t_5t_2,\\
&w_1^2w_3w_1 = t_2^2t_3t_2,\ w_1^3 = t_2^2t_4t_2\,.
\end{aligned}$$

Isomorphisms

Explicit isomorphisms between the two amalgams of Γ_0 described above, and Γ_0 as a subgroup of Γ are given as follows

$$\begin{array}{rcccl}
 & & \Gamma & & \\
 & & \vee & & \\
\Gamma_0^{(v)} & \stackrel{\cong}{\longleftrightarrow} & \Gamma_0 & \stackrel{\cong}{\longleftrightarrow} & \Gamma_0^{(h)} \\
 & & & & \\
r_1 & \longleftrightarrow & b_2b_1^{-1} & \longleftrightarrow & w_1t_2^{-1} \\
r_2 = q_1 & \longleftrightarrow & b_3b_1^{-1} & \longleftrightarrow & w_4t_5^{-1} \\
r_3 = q_4 & \longleftrightarrow & b_1b_3 & \longleftrightarrow & t_4^{-1}w_5 \\
r_4 & \longleftrightarrow & b_1b_2 & \longleftrightarrow & t_1^{-1}w_2w_4 \\
r_5 = q_5 & \longleftrightarrow & b_1^2 & \longleftrightarrow & t_5^{-1}w_4 \\
q_2 & \longleftrightarrow & a_1a_3^{-1}b_2b_1^{-1} & \longleftrightarrow & w_2^{-1}w_1t_2^{-1} \\
q_3 & \longleftrightarrow & a_1a_2^{-1}b_2^2 & \longleftrightarrow & t_2^{-1}t_1^{-1}w_2w_4 \\
r_1r_4q_3^{-1} & \longleftrightarrow & a_2a_1^{-1} & \longleftrightarrow & w_1 \\
r_1q_2^{-1} & \longleftrightarrow & a_3a_1^{-1} & \longleftrightarrow & w_2 \\
q_1^{-1}q_2^{-1}r_1r_3r_2 & \longleftrightarrow & a_1^2 & \longleftrightarrow & w_3 = t_3 \\
q_3^{-1}r_4 & \longleftrightarrow & a_1a_2 & \longleftrightarrow & w_4 \\
q_2^{-1}r_1r_3 & \longleftrightarrow & a_1a_3 & \longleftrightarrow & w_5 \\
r_1q_2^{-1}q_3^{-1} & \longleftrightarrow & a_3a_2b_2^{-1}b_1^{-1} & \longleftrightarrow & t_1 \\
r_1r_4q_3^{-1}q_5^{-1} & \longleftrightarrow & a_2a_1^{-1}b_1^{-2} & \longleftrightarrow & t_2 \\
q_2^{-1}r_1 & \longleftrightarrow & a_1a_3b_3^{-1}b_1^{-1} & \longleftrightarrow & t_4 \\
q_3^{-1}r_4r_5^{-1} & \longleftrightarrow & a_1a_2b_1^{-2} & \longleftrightarrow & t_5\,,
\end{array}$$

using the notation

$$\Gamma_0^{(v)} = F_5^{(v,r)} *_{F_{25}^{(v,r)} \cong F_{25}^{(v,q)}} F_5^{(v,q)}, \quad \Gamma_0^{(h)} = F_5^{(h,t)} *_{F_{25}^{(h,t)} \cong F_{25}^{(h,w)}} F_5^{(h,w)}.$$

A.3 An example illustrating Proposition 2.4

In the notation of the proof of [17, Proposition 6.1] we have $n = 0$, ${}^{(0)}X$ is the (A_6, A_6)–complex X of Example 2.2 and $k = \ell = 4$. Let $C_{k,\ell}$ be the $(4,4)$–complex given by

$$\{a_4b_4a_5^{-1}b_5^{-1},\ a_4b_4^{-1}a_5^{-1}b_5,\ a_4b_5a_5^{-1}b_4^{-1},\ a_4b_5^{-1}a_5^{-1}b_4\}$$

and $C_{4,4}$ (a disjoint copy of $C_{k,\ell}$) be given by

$$\{a_6b_6a_7^{-1}b_7^{-1},\ a_6b_6^{-1}a_7^{-1}b_7,\ a_6b_7a_7^{-1}b_6^{-1},\ a_6b_7^{-1}a_7^{-1}b_6\}.$$

We choose ${}^{(0)}a := a_1$, ${}^{(0)}b := b_1$, $\widehat{a}_1 := a_4$, $\widehat{a}_2 := a_5$, $\widehat{b}_1 := b_4$, $\widehat{b}_2 := b_5$, $\widetilde{a}_1 := a_6$, $\widetilde{a}_2 := a_7$, $\widetilde{b}_1 := b_6$ and $\widetilde{b}_2 := b_7$. The surgery operations which are described in the proof of [17, Proposition 6.1] lead to the irreducible (A_{14}, A_{14})–complex given by the following set $R_{7.7}$ (the relators of the embedded Example 2.2 are underlined)

$$\left\{ \begin{array}{lllll}
\underline{a_1b_1a_1^{-1}b_1^{-1}}, & \underline{a_1b_2a_1^{-1}b_3^{-1}}, & \underline{a_1b_3a_2b_2^{-1}}, & a_1b_4a_7^{-1}b_4^{-1}, & a_1b_5a_1^{-1}b_5^{-1}, \\
a_1b_6a_5^{-1}b_6^{-1}, & a_1b_7a_1^{-1}b_7^{-1}, & a_1b_6^{-1}a_5^{-1}b_6, & a_1b_4^{-1}a_7^{-1}b_4, & \underline{a_1b_3^{-1}a_3^{-1}b_2}, \\
\underline{a_2b_1a_3^{-1}b_2^{-1}}, & \underline{a_2b_2a_3^{-1}b_3^{-1}}, & \underline{a_2b_3a_3^{-1}b_1}, & a_2b_4a_2^{-1}b_4^{-1}, & a_2b_5a_2^{-1}b_5^{-1}, \\
a_2b_6a_2^{-1}b_6^{-1}, & a_2b_7a_2^{-1}b_7^{-1}, & \underline{a_2b_3^{-1}a_3b_2}, & \underline{a_2b_1^{-1}a_3^{-1}b_1^{-1}}, & a_3b_4a_3^{-1}b_4^{-1}, \\
a_3b_5a_3^{-1}b_5^{-1}, & a_3b_6a_3^{-1}b_6^{-1}, & a_3b_7a_3^{-1}b_7^{-1}, & a_4b_1a_4^{-1}b_7^{-1}, & a_4b_2a_4^{-1}b_2^{-1}, \\
a_4b_3a_4^{-1}b_3^{-1}, & a_4b_4a_5^{-1}b_5^{-1}, & a_4b_5a_5^{-1}b_4^{-1}, & a_4b_6a_4^{-1}b_6^{-1}, & a_4b_7a_4^{-1}b_1^{-1}, \\
a_4b_5^{-1}a_5^{-1}b_4, & a_4b_4^{-1}a_5^{-1}b_5, & a_5b_1a_5^{-1}b_1^{-1}, & a_5b_2a_5^{-1}b_2^{-1}, & a_5b_3a_5^{-1}b_3^{-1}, \\
a_5b_7a_5^{-1}b_7^{-1}, & a_6b_1a_6^{-1}b_5^{-1}, & a_6b_2a_6^{-1}b_2^{-1}, & a_6b_3a_6^{-1}b_3^{-1}, & a_6b_4a_6^{-1}b_4^{-1}, \\
a_6b_5a_6^{-1}b_1^{-1}, & a_6b_6a_7^{-1}b_7^{-1}, & a_6b_7a_7^{-1}b_6^{-1}, & a_6b_7^{-1}a_7^{-1}b_6, & a_6b_6^{-1}a_7^{-1}b_7, \\
a_7b_1a_7^{-1}b_1^{-1}, & a_7b_2a_7^{-1}b_2^{-1}, & a_7b_3a_7^{-1}b_3^{-1}, & a_7b_5a_7^{-1}b_5^{-1} &
\end{array} \right\}$$

and local groups determined by

$$
\begin{aligned}
\rho_v(b_1) &= \rho_h(a_1) = (2,3)(12,13),\\
\rho_v(b_2) &= (1,13,12,2,3),\\
\rho_v(b_3) &= (2,3,13,12,14),\\
\rho_v(b_4) &= \rho_h(a_4) = (1,7)(4,5)(8,14)(10,11),\\
\rho_v(b_5) &= \rho_h(a_5) = (4,5)(10,11),\\
\rho_v(b_6) &= \rho_h(a_6) = (1,5)(6,7)(8,9)(10,14),\\
\rho_v(b_7) &= \rho_h(a_7) = (6,7)(8,9),\\
\rho_h(a_2) &= (1,14,3,2)(12,13),\\
\rho_h(a_3) &= (1,12,13,14)(2,3).
\end{aligned}
$$

A.4 A virtually simple (A_8, A_{14})–group

Example A.26.

$$
R_{4\cdot 7} := \left\{
\begin{array}{llll}
\underline{a_1b_1a_1^{-1}b_1^{-1},} & \underline{a_1b_2a_2^{-1}b_3^{-1},} & \underline{a_1b_3a_1^{-1}b_4^{-1},} & \underline{a_1b_4a_1^{-1}b_5^{-1},} \\
\underline{a_1b_5a_1^{-1}b_6^{-1},} & \underline{a_1b_6a_1^{-1}b_2^{-1},} & a_1b_7a_2^{-1}b_7^{-1}, & a_1b_7^{-1}a_3b_7, \\
\underline{a_1b_2^{-1}a_2b_3,} & \underline{a_2b_1a_2^{-1}b_5^{-1},} & \underline{a_2b_2a_2b_3^{-1},} & \underline{a_2b_4a_2^{-1}b_4,} \\
\underline{a_2b_5a_2^{-1}b_1^{-1},} & \underline{a_2b_6a_2^{-1}b_6,} & a_2b_7a_4^{-1}b_7^{-1}, & a_3b_1a_4b_3^{-1}, \\
a_3b_2a_4b_1^{-1}, & a_3b_3a_4b_2, & a_3b_4a_3^{-1}b_5, & a_3b_5a_4b_4, \\
a_3b_6a_3^{-1}b_6^{-1}, & a_3b_7^{-1}a_4b_3, & a_3b_5^{-1}a_4^{-1}b_4^{-1}, & a_3b_3^{-1}a_4b_7, \\
a_3b_2^{-1}a_4b_2^{-1}, & a_3b_1^{-1}a_4b_1, & a_4b_6a_4^{-1}b_6^{-1}, & a_4b_5^{-1}a_4b_4^{-1}
\end{array}
\right\}.
$$

$$
\begin{aligned}
\rho_v(b_1) &= (3,5)(4,6),\\
\rho_v(b_2) &= (2,8,7)(3,5)(4,6),\\
\rho_v(b_3) &= (1,2,7)(3,5)(4,6),\\
\rho_v(b_4) &= (3,4,5),\\
\rho_v(b_5) &= (4,5,6),\\
\rho_v(b_6) &= (),\\
\rho_v(b_7) &= (1,2,4,6)(3,8,7,5),
\end{aligned}
$$

$$\begin{aligned}
\rho_h(a_1) &= (2, 6, 5, 4, 3)(9, 10, 11, 12, 13),\\
\rho_h(a_2) &= (1, 5)(2, 3)(4, 11)(6, 9)(10, 14)(12, 13),\\
\rho_h(a_3) &= (1, 2, 13, 3)(4, 10)(5, 11)(8, 12),\\
\rho_h(a_4) &= (2, 13, 14, 12)(3, 7)(4, 10)(5, 11).
\end{aligned}$$

A.5 Supplement to Example 2.58

Let Γ be the $(6, 10)$–group defined in Example 2.58. We first give a finite presentation of the horizontal decomposition $\Gamma_0 \cong F_5 *_{F_{41}} F_5$ in Example 2.58 with generators

$$\{s_1, s_2, s_3, s_4, s_5, u_1, u_2, u_3, u_4, u_5\}$$

and 41 relations

$$\begin{aligned}
s_1^{-1}s_3s_4^{-1}s_3 &= u_4^{-1}u_1u_4u_3\\
s_3^{-1}s_4^2s_3^{-1}s_4s_1^{-1} &= u_3^{-1}u_1u_3u_1u_3u_1\\
s_3^{-1}s_4^2s_3^3s_4^{-2}s_3 &= u_3^{-1}u_1u_3u_4^{-1}u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3s_1s_3s_4^{-2}s_3 &= u_4^{-1}u_1^{-1}u_2u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-2}s_1s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-1}u_2u_3\\
s_3s_2s_3s_4^{-2}s_3 &= u_4^{-1}u_1^{-1}u_5^{-1}u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_2s_3^2s_4^{-2}s_3 &= u_3^{-1}u_1^{-1}u_2u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4s_2s_4s_3^2s_4^{-2}s_3 &= u_3^{-1}u_5^{-1}u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_1^{-1}s_2s_3s_4^{-1}s_3 &= u_4^{-1}u_2u_4u_3\\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_3^{-1}s_4^{-1}s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_4^{-1}u_1^{-1}u_3\\
s_1^{-1}s_3^{-1}s_4s_3^2s_4^{-2}s_3 &= u_4^{-2}u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-1}s_4s_3^{-1}s_1^{-1} &= u_3^{-1}u_1u_3u_1u_3u_4^{-1}\\
s_4^{-1}s_3^3s_4^{-2}s_3 &= u_5^{-1}u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-1}s_4^{-1}s_3^2 &= u_3^{-1}u_1u_3u_1u_5^{-1}u_4u_1u_3\\
s_5^{-1}s_3^3s_4^{-2}s_3 &= u_2u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_5^{-1}s_3^2s_4^{-2}s_3 &= u_3^{-1}u_1^{-1}u_5^{-1}u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4s_5^{-1}s_4s_3^2s_4^{-2}s_3 &= u_3^{-1}u_2u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_5^{-1}s_4^{-1}s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_5^{-1}u_1^{-1}u_3\\
s_3s_4^{-1}s_1^{-1} &= u_4^{-1}u_1^{-1}u_3^{-1}
\end{aligned}$$

$$
\begin{aligned}
s_1^{-1}s_4^{-1}s_3s_4^{-1}s_3 &= u_4^{-1}u_3^{-1}u_4u_3\\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_4^{-1} &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_3^{-1}u_4\\
s_3^{-1}s_4s_1s_3^2 &= u_3^{-1}u_3^{-1}u_4u_1u_3\\
s_3^{-1}s_4^{-1}s_3^2s_4^{-2}s_3 &= u_3^{-1}u_1^{-1}u_3^{-1}u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-1}s_1s_4s_3^2s_4^{-2}s_3 &= u_3^{-1}u_1u_3u_1u_3^{-1}u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-2}s_5^{-1}s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-1}u_4^{-1}u_3\\
s_3s_5^{-1}s_3s_4^{-2}s_3 &= u_4^{-1}u_1^{-1}u_4^{-1}u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-2}s_2s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-1}u_5^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-1}s_4s_5^{-1}s_1^{-1} &= u_3^{-1}u_1u_3u_1u_3u_5^{-1}\\
s_3^{-1}s_4^2s_2 &= u_3^{-1}u_1u_3u_5^{-1}u_4\\
s_1^{-1}s_5^{-1}s_3s_4^{-1}s_3 &= u_4^{-1}u_5^{-1}u_4u_3\\
s_3^3 &= u_4^{-1}u_1^{-1}u_4u_1u_3\\
s_3^{-1}s_4^2s_3^{-1}s_4s_1s_3^3s_4^{-2}s_3 &= u_3^{-1}u_1u_3u_1u_3^2u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_1s_4^{-1}s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_1s_3s_4^{-1}s_3 &= u_3^{-1}u_1u_3u_4u_3\\
s_3^{-1}s_1s_3^2s_4^{-2}s_3 &= u_3^{-1}u_1^{-1}u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-1}s_2s_3^2 &= u_3^{-1}u_1u_3u_1^2u_4u_1u_3\\
s_2s_3^3s_4^{-2}s_3 &= u_1u_3u_1u_4u_1^{-1}u_3^{-1}u_1^{-1}u_3\\
s_3^{-1}s_4^2s_3^{-1}s_5^{-1}s_3^2 &= u_3^{-1}u_1u_3u_1u_2u_4u_1u_3\\
s_3^{-1}s_4^2s_3^{-1}s_4s_2s_1^{-1} &= u_3^{-1}u_1u_3u_1u_3u_2\\
s_3^{-1}s_4^2s_5^{-1} &= u_3^{-1}u_1u_3u_2u_4\\
s_3^{-1}s_4^2s_3^{-2}s_4^{-1}s_2s_4^{-1}s_3 &= u_3^{-1}u_1u_3u_1u_4^{-1}u_1^{-2}u_2u_1^{-1}u_3\,.
\end{aligned}
$$

In the following table, we have computed $\left|\rho_v^{(k)}(w)\right|$, if $|w| = 2$ and $k \leq 5$. Observe that if $b, \tilde{b} \in \{b_1, \ldots, b_5\}^{\pm 1}$, then

$$|\rho_v^{(k)}(b\tilde{b})| = |\rho_v^{(k)}(\tilde{b}b)| = |\rho_v^{(k)}(b\tilde{b})^{-1}| = |\rho_v^{(k)}(\tilde{b}b)^{-1}|\,.$$

If $\left|\rho_v^{(k)}(w)\right| = \left|\rho_v^{(k+1)}(w)\right|$ for some k and w in the table, then we have printed bold the number $\left|\rho_v^{(k+1)}(w)\right|$.

$\lvert\rho_v^{(k)}(w)\rvert$	$k=1$	2	3	4	5
$w=b_1^2$	5	**5**	50	300	1500
b_1b_2	3	15	75	150	2250
b_1b_3	5	10	150	900	9000
b_1b_4	3	15	30	450	4500
b_1b_5	5	30	300	900	5400
$b_1b_5^{-1}$	5	15	450	4500	**4500**
$b_1b_4^{-1}$	5	15	150	900	1800
$b_1b_3^{-1}$	5	25	50	500	3000
$b_1b_2^{-1}$	3	9	54	**54**	1620
b_2^2	5	**5**	50	300	1500
b_2b_3	5	25	50	500	3000
b_2b_4	5	15	150	900	1800
b_2b_5	5	30	300	900	5400
$b_2b_5^{-1}$	5	15	450	4500	**4500**
$b_2b_4^{-1}$	3	15	30	450	4500
$b_2b_3^{-1}$	5	10	150	900	9000
b_3^2	1	5	25	50	500
b_3b_4	2	6	90	180	2700
b_3b_5	1	30	**30**	450	4500
$b_3b_5^{-1}$	1	30	**30**	450	4500
$b_3b_4^{-1}$	2	20	60	600	1800
b_4^2	2	4	20	100	500
b_4b_5	2	10	20	600	6000
$b_4b_5^{-1}$	2	10	20	600	6000
b_5^2	1	2	10	20	600

Table A.1: Orders of some $\rho_v^{(k)}(w)$ in Example 2.58

A.6 Some 4-vertex examples

We give now several examples in a certain class of 4-vertex square complexes. In all examples, the complex will be denoted by Y.

The 1-skeleton of Y is illustrated in Figure A.3, and a typical geometric square of Y is illustrated in Figure A.4, i.e. we always have four vertices α, β, γ, δ, horizontal edges a_1, a_2, a_3 (oriented from α to β), c_1, c_2, c_3 (oriented from δ to γ), and vertical edges $b_1, \dots, b_6$ (oriented from β to γ), $d_1, \dots, d_6$ (oriented from α to δ).

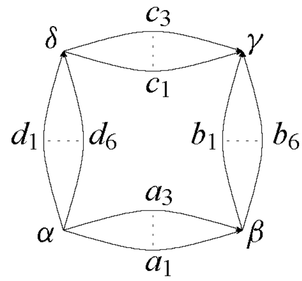


Figure A.3: The 1-skeleton of Y

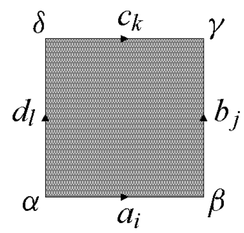


Figure A.4: A typical geometric square of Y

Each of the 18 geometric squares is of the form $a_i b_j = d_l c_k$ (see Figure A.4), and the universal covering space $\tilde{Y}$ is $\mathcal{T}_3 \times \mathcal{T}_6$. By construction of the 1-skeleton and the geometric squares of Y, we have for each $k \in \mathbb{N}$:

$$P_h^{(k)}(\alpha) \cong P_h^{(k)}(\delta)\,,\ P_h^{(k)}(\beta) \cong P_h^{(k)}(\gamma)\,,\ P_v^{(k)}(\alpha) \cong P_v^{(k)}(\beta)\,,\ P_v^{(k)}(\gamma) \cong P_v^{(k)}(\delta)\,.$$

Example A.27. ((1, A_6), *reducible*)
Let Y be given by its geometric squares

$$\begin{array}{lll} a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, \\ a_1b_4 = d_4c_1, & a_1b_5 = d_5c_1, & a_1b_6 = d_6c_1, \\ a_2b_1 = d_1c_2, & a_2b_2 = d_2c_2, & a_2b_3 = d_3c_2, \\ a_2b_4 = d_5c_2, & a_2b_5 = d_6c_2, & a_2b_6 = d_4c_2, \\ a_3b_1 = d_2c_3, & a_3b_2 = d_3c_3, & a_3b_3 = d_4c_3, \\ a_3b_4 = d_1c_3, & a_3b_5 = d_6c_3, & a_3b_6 = d_5c_3. \end{array}$$

Then

$$P_h(\alpha) = 1,\ P_h(\beta) = 1,\ P_v(\alpha) = A_6,\ P_v(\gamma) = A_6,$$
$$P_h^{(2)}(\alpha) = 1,\ P_h^{(2)}(\beta) = 1,\ P_v^{(2)}(\alpha) \cong A_6,\ P_v^{(2)}(\gamma) \cong A_6.$$

Example A.28. *(($\mathbb{Z}_2$, A_6), irreducible)*
Let Y be given by its geometric squares

$$\begin{array}{lll}
a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, \\
a_1b_4 = d_4c_1, & a_1b_5 = d_5c_1, & a_1b_6 = d_6c_1, \\
a_2b_1 = d_1c_2, & a_2b_2 = d_2c_2, & a_2b_3 = d_3c_2, \\
a_2b_4 = d_5c_2, & a_2b_5 = d_6c_2, & a_2b_6 = d_4c_3, \\
a_3b_1 = d_2c_3, & a_3b_2 = d_3c_3, & a_3b_3 = d_5c_3, \\
a_3b_4 = d_6c_3, & a_3b_5 = d_1c_3, & a_3b_6 = d_4c_2.
\end{array}$$

Then

$$P_h(\alpha) \cong \mathbb{Z}_2,\ P_h(\beta) \cong \mathbb{Z}_2,\ P_v(\alpha) = A_6,\ P_v(\gamma) = A_6,$$
$$|P_h^{(2)}(\alpha)| = 4,\ |P_h^{(2)}(\beta)| = 4,\ |P_v^{(2)}(\alpha)| = 360 \cdot 60^6,\ |P_v^{(2)}(\gamma)| = 360 \cdot 60^6.$$

Example A.29. *($P_h(\alpha) \neq P_h(\beta)$, $|P_h^{(2)}(\alpha)| = |P_h(\alpha)|$, irreducible)*
Let Y be given by its geometric squares

$$\begin{array}{lll}
a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, \\
a_1b_4 = d_4c_1, & a_1b_5 = d_5c_2, & a_1b_6 = d_6c_3, \\
a_2b_1 = d_1c_2, & a_2b_2 = d_3c_2, & a_2b_3 = d_4c_2, \\
a_2b_4 = d_6c_2, & a_2b_5 = d_2c_3, & a_2b_6 = d_5c_1, \\
a_3b_1 = d_3c_3, & a_3b_2 = d_1c_3, & a_3b_3 = d_5c_3, \\
a_3b_4 = d_4c_3, & a_3b_5 = d_6c_1, & a_3b_6 = d_2c_2.
\end{array}$$

Then

$$|P_h(\alpha)| = 6,\ |P_h(\beta)| = 3,\ P_v(\alpha) = A_6,\ P_v(\gamma) = A_6,$$
$$|P_h^{(2)}(\alpha)| = 6,\ |P_h^{(2)}(\beta)| = 24,\ |P_v^{(2)}(\alpha)| = 360 \cdot 60^6,\ |P_v^{(2)}(\gamma)| = 360 \cdot 60^6.$$

Example A.30. *($P_h(\alpha) \neq P_h(\beta)$, $P_v(\alpha) \neq P_v(\gamma)$)*
Let Y be given by its geometric squares

$$\begin{array}{lll}
a_1b_1 = d_1c_1, & a_1b_2 = d_2c_1, & a_1b_3 = d_3c_1, \\
a_1b_4 = d_4c_2, & a_1b_5 = d_5c_2, & a_1b_6 = d_6c_3, \\
a_2b_1 = d_1c_2, & a_2b_2 = d_3c_2, & a_2b_3 = d_4c_3, \\
a_2b_4 = d_5c_3, & a_2b_5 = d_6c_1, & a_2b_6 = d_2c_2, \\
a_3b_1 = d_2c_3, & a_3b_2 = d_3c_3, & a_3b_3 = d_6c_2, \\
a_3b_4 = d_4c_1, & a_3b_5 = d_1c_3, & a_3b_6 = d_5c_1.
\end{array}$$

Then $|P_h(\alpha)| = 3$, $|P_h(\beta)| = 6$, $|P_v(\alpha)| = 360$, $|P_v(\gamma)| = 120$.

A.7 Example $\Gamma_{7,23}$

Example A.31.

$$R_{4\cdot 12} := \left\{ \begin{array}{llll}
a_1b_1a_3^{-1}b_4^{-1}, & a_1b_2a_4^{-1}b_5, & a_1b_3a_2b_8, & a_1b_4a_2b_7, \\
a_1b_5a_3^{-1}b_7^{-1}, & a_1b_6a_2^{-1}b_5^{-1}, & a_1b_7a_4^{-1}b_{10}^{-1}, & a_1b_8a_1^{-1}b_{12}, \\
a_1b_9a_4^{-1}b_4, & a_1b_{10}a_3^{-1}b_9^{-1}, & a_1b_{11}a_3b_2, & a_1b_{12}a_3b_3, \\
a_1b_{12}^{-1}a_4^{-1}b_2^{-1}, & a_1b_{11}^{-1}a_2^{-1}b_9, & a_1b_{10}^{-1}a_4b_{11}^{-1}, & a_1b_9^{-1}a_3^{-1}b_{10}, \\
a_1b_7^{-1}a_4b_6^{-1}, & a_1b_6^{-1}a_4^{-1}b_{11}, & a_1b_5^{-1}a_2^{-1}b_6, & a_1b_4^{-1}a_4^{-1}b_8^{-1}, \\
a_1b_3^{-1}a_4b_1^{-1}, & a_1b_2^{-1}a_2^{-1}b_1, & a_1b_1^{-1}a_4b_3^{-1}, & a_2b_1a_4b_9, \\
a_2b_3a_3^{-1}b_{11}, & a_2b_4a_4b_{10}, & a_2b_6a_3^{-1}b_1, & a_2b_9a_3^{-1}b_5^{-1}, \\
a_2b_{10}a_2^{-1}b_7, & a_2b_{12}a_4^{-1}b_{11}^{-1}, & a_2b_{12}^{-1}a_3^{-1}b_8, & a_2b_{11}^{-1}a_4^{-1}b_{12}, \\
a_2b_9^{-1}a_3b_{12}^{-1}, & a_2b_8^{-1}a_4^{-1}b_6, & a_2b_7^{-1}a_3^{-1}b_3^{-1}, & a_2b_5^{-1}a_3b_8^{-1}, \\
a_2b_4^{-1}a_3b_2^{-1}, & a_2b_3^{-1}a_4^{-1}b_2, & a_2b_2^{-1}a_3b_4^{-1}, & a_2b_1^{-1}a_3^{-1}b_{10}^{-1}, \\
a_3b_4a_4^{-1}b_3^{-1}, & a_3b_5a_4b_1, & a_3b_6a_4b_2, & a_3b_8a_4^{-1}b_7^{-1}, \\
a_3b_{10}a_4^{-1}b_{12}^{-1}, & a_3b_{11}a_3^{-1}b_6, & a_3b_7^{-1}a_4^{-1}b_8, & a_4b_5a_4^{-1}b_9
\end{array} \right\}.$$

Generators of $\Gamma_{7,23}$:

$$
\begin{aligned}
a_1 &= \psi(1+2i+j+k), & a_1^{-1} &= \psi(1-2i-j-k),\\
a_2 &= \psi(1+2i+j-k), & a_2^{-1} &= \psi(1-2i-j+k),\\
a_3 &= \psi(1+2i-j+k), & a_3^{-1} &= \psi(1-2i+j-k),\\
a_4 &= \psi(1+2i-j-k), & a_4^{-1} &= \psi(1-2i+j+k),
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \psi(1+2i+3j+3k), & b_1^{-1} &= \psi(1-2i-3j-3k),\\
b_2 &= \psi(1+2i+3j-3k), & b_2^{-1} &= \psi(1-2i-3j+3k),\\
b_3 &= \psi(1+2i-3j-3k), & b_3^{-1} &= \psi(1-2i+3j+3k),\\
b_4 &= \psi(1+2i-3j+3k), & b_4^{-1} &= \psi(1-2i+3j-3k),\\
b_5 &= \psi(3+2i+j+3k), & b_5^{-1} &= \psi(3-2i-j-3k),\\
b_6 &= \psi(3+2i+j-3k), & b_6^{-1} &= \psi(3-2i-j+3k),\\
b_7 &= \psi(3+2i-j+3k), & b_7^{-1} &= \psi(3-2i+j-3k),\\
b_8 &= \psi(3+2i-j-3k), & b_8^{-1} &= \psi(3-2i+j+3k),\\
b_9 &= \psi(3+2i+3j+k), & b_9^{-1} &= \psi(3-2i-3j-k),\\
b_{10} &= \psi(3+2i-3j+k), & b_{10}^{-1} &= \psi(3-2i+3j-k),\\
b_{11} &= \psi(3+2i+3j-k), & b_{11}^{-1} &= \psi(3-2i-3j+k),\\
b_{12} &= \psi(3+2i-3j-k), & b_{12}^{-1} &= \psi(3-2i+3j+k).
\end{aligned}
$$

A.8 Example $\Gamma_{7,31}$

Example A.32.

$$R_{4\cdot 16} := \left\{ \begin{array}{llll}
a_1b_1a_4^{-1}b_8^{-1}, & a_1b_2a_3^{-1}b_{16}^{-1}, & a_1b_3a_1b_{14}^{-1}, & a_1b_4a_4b_1, \\
a_1b_5a_4b_8, & a_1b_6a_1b_{15}^{-1}, & a_1b_7a_4^{-1}b_{10}^{-1}, & a_1b_8a_3^{-1}b_6^{-1}, \\
a_1b_9a_1^{-1}b_9^{-1}, & a_1b_{10}a_4^{-1}b_3^{-1}, & a_1b_{11}a_4b_{14}, & a_1b_{12}a_2^{-1}b_{11}^{-1}, \\
a_1b_{13}a_1b_{12}^{-1}, & a_1b_{14}a_3^{-1}b_4^{-1}, & a_1b_{15}a_4b_{10}, & a_1b_{16}a_4^{-1}b_{13}^{-1}, \\
a_1b_{16}^{-1}a_2^{-1}b_7, & a_1b_{13}^{-1}a_4^{-1}b_{16}, & a_1b_{11}^{-1}a_4^{-1}b_2, & a_1b_{10}^{-1}a_3^{-1}b_{12}, \\
a_1b_8^{-1}a_2^{-1}b_{15}, & a_1b_7^{-1}a_3b_5^{-1}, & a_1b_6^{-1}a_4^{-1}b_{11}, & a_1b_5^{-1}a_3b_7^{-1}, \\
a_1b_4^{-1}a_4^{-1}b_5, & a_1b_3^{-1}a_2^{-1}b_4, & a_1b_2^{-1}a_2b_1^{-1}, & a_1b_1^{-1}a_2b_2^{-1}, \\
a_2b_1a_3^{-1}b_{12}^{-1}, & a_2b_2a_3b_3, & a_2b_4a_2b_{13}^{-1}, & a_2b_5a_2b_{16}^{-1}, \\
a_2b_6a_3^{-1}b_3^{-1}, & a_2b_7a_3b_6, & a_2b_9a_3b_{16}, & a_2b_{10}a_2^{-1}b_{10}^{-1}, \\
a_2b_{11}a_4^{-1}b_9^{-1}, & a_2b_{12}a_3^{-1}b_5^{-1}, & a_2b_{13}a_3b_{12}, & a_2b_{14}a_2b_{11}^{-1}, \\
a_2b_{15}a_3^{-1}b_{14}^{-1}, & a_2b_{15}^{-1}a_4^{-1}b_1, & a_2b_{14}^{-1}a_3^{-1}b_{15}, & a_2b_9^{-1}a_3^{-1}b_8, \\
a_2b_8^{-1}a_4b_6^{-1}, & a_2b_7^{-1}a_3^{-1}b_2, & a_2b_6^{-1}a_4b_8^{-1}, & a_2b_5^{-1}a_4^{-1}b_7, \\
a_2b_4^{-1}a_3^{-1}b_9, & a_2b_3^{-1}a_4^{-1}b_{13}, & a_3b_1a_3b_{16}^{-1}, & a_3b_2a_4^{-1}b_1^{-1}, \\
a_3b_5a_4^{-1}b_{14}^{-1}, & a_3b_8a_3b_{13}^{-1}, & a_3b_{11}a_3^{-1}b_{11}^{-1}, & a_3b_{13}a_4^{-1}b_6^{-1}, \\
a_3b_{15}a_3b_{10}^{-1}, & a_3b_9^{-1}a_4^{-1}b_{10}, & a_3b_4^{-1}a_4b_3^{-1}, & a_3b_3^{-1}a_4b_4^{-1}, \\
a_4b_2a_4b_{15}^{-1}, & a_4b_7a_4b_{14}^{-1}, & a_4b_{12}a_4^{-1}b_{12}^{-1}, & a_4b_{16}a_4b_9^{-1}
\end{array} \right\}.$$

Generators of $\Gamma_{7,31}$:

$$
\begin{aligned}
a_1 &= \psi(1+2i+j+k), & a_1^{-1} &= \psi(1-2i-j-k),\\
a_2 &= \psi(1+2i+j-k), & a_2^{-1} &= \psi(1-2i-j+k),\\
a_3 &= \psi(1+2i-j+k), & a_3^{-1} &= \psi(1-2i+j-k),\\
a_4 &= \psi(1+2i-j-k), & a_4^{-1} &= \psi(1-2i+j+k),
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \psi(1+2i+j+5k), & b_1^{-1} &= \psi(1-2i-j-5k),\\
b_2 &= \psi(1+2i+j-5k), & b_2^{-1} &= \psi(1-2i-j+5k),\\
b_3 &= \psi(1+2i-j+5k), & b_3^{-1} &= \psi(1-2i+j-5k),\\
b_4 &= \psi(1+2i-j-5k), & b_4^{-1} &= \psi(1-2i+j+5k),\\
b_5 &= \psi(1+2i+5j+k), & b_5^{-1} &= \psi(1-2i-5j-k),\\
b_6 &= \psi(1+2i+5j-k), & b_6^{-1} &= \psi(1-2i-5j+k),\\
b_7 &= \psi(1+2i-5j+k), & b_7^{-1} &= \psi(1-2i+5j-k),\\
b_8 &= \psi(1+2i-5j-k), & b_8^{-1} &= \psi(1-2i+5j+k),\\
b_9 &= \psi(5+2i+j+k), & b_9^{-1} &= \psi(5-2i-j-k),\\
b_{10} &= \psi(5+2i+j-k), & b_{10}^{-1} &= \psi(5-2i-j+k),\\
b_{11} &= \psi(5+2i-j+k), & b_{11}^{-1} &= \psi(5-2i+j-k),\\
b_{12} &= \psi(5+2i-j-k), & b_{12}^{-1} &= \psi(5-2i+j+k),\\
b_{13} &= \psi(3+2i+3j+3k), & b_{13}^{-1} &= \psi(3-2i-3j-3k),\\
b_{14} &= \psi(3+2i+3j-3k), & b_{14}^{-1} &= \psi(3-2i-3j+3k),\\
b_{15} &= \psi(3+2i-3j+3k), & b_{15}^{-1} &= \psi(3-2i+3j-3k),\\
b_{16} &= \psi(3+2i-3j-3k), & b_{16}^{-1} &= \psi(3-2i+3j+3k).
\end{aligned}
$$

A.9 Example $\Gamma_{7,23,e_0}$

Example A.33.

$$R_{4\cdot 12} := \left\{ \begin{array}{llll}
a_1b_1a_3b_9, & a_1b_2a_1^{-1}b_{12}^{-1}, & a_1b_3a_3^{-1}b_2^{-1}, & a_1b_4a_3b_{10}, \\
a_1b_5a_2b_1, & a_1b_6a_2b_2, & a_1b_7a_2^{-1}b_8^{-1}, & a_1b_8a_1^{-1}b_4^{-1}, \\
a_1b_9a_4b_5, & a_1b_{10}a_1^{-1}b_6^{-1}, & a_1b_{11}a_4^{-1}b_{10}^{-1}, & a_1b_{12}a_4b_8, \\
a_1b_{12}^{-1}a_3^{-1}b_{11}, & a_1b_{11}^{-1}a_2^{-1}b_9^{-1}, & a_1b_9^{-1}a_2^{-1}b_{11}^{-1}, & a_1b_7^{-1}a_3^{-1}b_5^{-1}, \\
a_1b_6^{-1}a_4^{-1}b_7, & a_1b_5^{-1}a_3^{-1}b_7^{-1}, & a_1b_4^{-1}a_2^{-1}b_3, & a_1b_3^{-1}a_4^{-1}b_1^{-1}, \\
a_1b_1^{-1}a_4^{-1}b_3^{-1}, & a_2b_3a_2^{-1}b_7^{-1}, & a_2b_5a_2^{-1}b_{12}, & a_2b_6a_3^{-1}b_{11}^{-1}, \\
a_2b_7a_3^{-1}b_{10}^{-1}, & a_2b_8a_3b_5^{-1}, & a_2b_{10}a_2^{-1}b_1, & a_2b_{12}a_3b_9^{-1}, \\
a_2b_{12}^{-1}a_4^{-1}b_3, & a_2b_{11}^{-1}a_4^{-1}b_2, & a_2b_9^{-1}a_4b_{10}, & a_2b_8^{-1}a_4b_6^{-1}, \\
a_2b_6^{-1}a_4b_8^{-1}, & a_2b_4^{-1}a_3b_2^{-1}, & a_2b_2^{-1}a_3b_4^{-1}, & a_2b_1^{-1}a_4b_4, \\
a_3b_1a_3^{-1}b_6, & a_3b_2a_4b_1^{-1}, & a_3b_3a_4^{-1}b_8^{-1}, & a_3b_4a_4^{-1}b_7^{-1}, \\
a_3b_6a_4b_5^{-1}, & a_3b_8a_3^{-1}b_9, & a_3b_{11}a_3^{-1}b_3^{-1}, & a_3b_{12}^{-1}a_4b_{10}^{-1}, \\
a_3b_{10}^{-1}a_4b_{12}^{-1}, & a_4b_2a_4^{-1}b_5, & a_4b_7a_4^{-1}b_{11}^{-1}, & a_4b_9a_4^{-1}b_4
\end{array} \right\}.$$

Generators of $\Gamma_{7,23,e_0}$:

$$
\begin{aligned}
a_1 &= \psi(2+i+j+k), & a_1^{-1} &= \psi(2-i-j-k),\\
a_2 &= \psi(2+i+j-k), & a_2^{-1} &= \psi(2-i-j+k),\\
a_3 &= \psi(2+i-j+k), & a_3^{-1} &= \psi(2-i+j-k),\\
a_4 &= \psi(2-i+j+k), & a_4^{-1} &= \psi(2+i-j-k),
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \psi(2+i+3j+3k), & b_1^{-1} &= \psi(2-i-3j-3k),\\
b_2 &= \psi(2+i+3j-3k), & b_2^{-1} &= \psi(2-i-3j+3k),\\
b_3 &= \psi(2+i-3j-3k), & b_3^{-1} &= \psi(2-i+3j+3k),\\
b_4 &= \psi(2+i-3j+3k), & b_4^{-1} &= \psi(2-i+3j-3k),\\
b_5 &= \psi(2+3i+j+3k), & b_5^{-1} &= \psi(2-3i-j-3k),\\
b_6 &= \psi(2+3i+j-3k), & b_6^{-1} &= \psi(2-3i-j+3k),\\
b_7 &= \psi(2-3i+j-3k), & b_7^{-1} &= \psi(2+3i-j+3k),\\
b_8 &= \psi(2-3i+j+3k), & b_8^{-1} &= \psi(2+3i-j-3k),\\
b_9 &= \psi(2+3i+3j+k), & b_9^{-1} &= \psi(2-3i-3j-k),\\
b_{10} &= \psi(2+3i-3j+k), & b_{10}^{-1} &= \psi(2-3i+3j-k),\\
b_{11} &= \psi(2-3i-3j+k), & b_{11}^{-1} &= \psi(2+3i+3j-k),\\
b_{12} &= \psi(2-3i+3j+k), & b_{12}^{-1} &= \psi(2+3i-3j-k).
\end{aligned}
$$

A.10 Example $\Gamma_{13,17}$

$$
R_{7.9} := \left\{ \begin{array}{lll}
a_1b_1a_3b_3, & a_1b_2a_2b_1, & a_1b_3a_4b_2, \\
a_1b_4a_6b_8, & a_1b_5a_7b_1^{-1}, & a_1b_6a_5b_4, \\
a_1b_7a_2^{-1}b_6^{-1}, & a_1b_8a_7b_6, & a_1b_9a_5b_2^{-1}, \\
a_1b_9^{-1}a_3^{-1}b_8^{-1}, & a_1b_8^{-1}a_2^{-1}b_9, & a_1b_7^{-1}a_6b_3^{-1}, \\
a_1b_6^{-1}a_4^{-1}b_7^{-1}, & a_1b_5^{-1}a_4^{-1}b_4^{-1}, & a_1b_4^{-1}a_3^{-1}b_5, \\
a_1b_3^{-1}a_5b_9^{-1}, & a_1b_2^{-1}a_7b_5^{-1}, & a_1b_1^{-1}a_6b_7, \\
a_2b_2a_3^{-1}b_3^{-1}, & a_2b_3a_6b_6^{-1}, & a_2b_4a_5b_7, \\
a_2b_5a_4b_4^{-1}, & a_2b_6a_6b_1^{-1}, & a_2b_7a_7^{-1}b_9, \\
a_2b_9a_6b_4, & a_2b_9^{-1}a_4b_8^{-1}, & a_2b_8^{-1}a_5b_3, \\
a_2b_6^{-1}a_3b_7^{-1}, & a_2b_5^{-1}a_7^{-1}b_2^{-1}, & a_2b_4^{-1}a_3b_5^{-1}, \\
a_2b_3^{-1}a_4^{-1}b_1, & a_2b_2^{-1}a_5b_8, & a_2b_1^{-1}a_7^{-1}b_5, \\
a_3b_1a_4^{-1}b_2^{-1}, & a_3b_2a_5b_8^{-1}, & a_3b_5a_5b_6, \\
a_3b_6a_7b_9^{-1}, & a_3b_7a_6^{-1}b_1^{-1}, & a_3b_8a_5b_3^{-1}, \\
a_3b_9^{-1}a_6^{-1}b_5, & a_3b_8^{-1}a_4b_9, & a_3b_6^{-1}a_4b_7, \\
a_3b_4^{-1}a_7b_2, & a_3b_3^{-1}a_6^{-1}b_7^{-1}, & a_3b_1^{-1}a_7b_4, \\
a_4b_1a_7b_4^{-1}, & a_4b_4a_7b_2^{-1}, & a_4b_8a_6b_5^{-1}, \\
a_4b_9^{-1}a_5^{-1}b_3^{-1}, & a_4b_7^{-1}a_7b_8, & a_4b_6^{-1}a_6b_1, \\
a_4b_5^{-1}a_5^{-1}b_7^{-1}, & a_4b_3^{-1}a_6b_6, & a_4b_2^{-1}a_5^{-1}b_9, \\
a_5b_1a_5^{-1}b_1^{-1}, & a_5b_7^{-1}a_5b_6^{-1}, & a_5b_5^{-1}a_5b_4^{-1}, \\
a_6b_2a_6^{-1}b_2^{-1}, & a_6b_5a_6b_4^{-1}, & a_6b_9^{-1}a_6b_8^{-1}, \\
a_7b_3a_7^{-1}b_3^{-1}, & a_7b_7a_7b_6^{-1}, & a_7b_9a_7b_8^{-1}
\end{array} \right\}.
$$

A.11 Amalgam decompositions of Example 3.42

We first give the vertical decomposition of the group Γ of Example 3.42:

$$\Gamma \cong F_3^{(b)} *_{F_{17}^{(b)} \cong F_{17}^{(s)}} (\mathbb{Z}_2^{*12} * F_3^{(s)}) ,$$

where

$$\begin{aligned} F_3^{(b)} &= \langle b_1, b_2, b_3 \rangle , \\ \mathbb{Z}_2^{*12} * F_3^{(s)} &= \langle s_1, \ldots, s_{12}, s_{13}, s_{14}, s_{15} \mid s_1^2 = \ldots = s_{12}^2 = 1 \rangle . \end{aligned}$$

The subgroup $F_{17}^{(b)} < F_3^{(b)}$ of index 8 is given by

$$\begin{aligned} F_{17}^{(b)} = \langle & b_1^{-1}b_2,\ b_1^{-1}b_3,\ b_2b_1b_3^{-1},\ b_1^2b_2b_1,\ b_1b_2^2b_1,\ b_1b_3^{-1}b_2b_1, \\ & b_1^{-1}b_2^{-1}b_1b_2b_1^2,\ b_1^{-1}b_2^{-1}b_3^{-1}b_1^2,\ b_3b_1^3,\ b_3^2b_1^2,\ b_3b_2^{-1}b_1^2, \\ & b_3b_1^{-1}b_2^2b_1^2,\ b_3b_1^{-1}b_3b_2b_1^2,\ b_3b_1^{-2}b_2b_1^2,\ b_1^{-1}b_3^{-1}b_2b_1^2, \\ & b_1b_2^{-1}b_3^{-1},\ b_1b_3b_1b_3^{-1} \rangle , \end{aligned}$$

the index 2 subgroup $F_{17}^{(s)} < \mathbb{Z}_2^{*12} * F_3^{(s)}$ by

$$\begin{aligned} F_{17}^{(s)} = \langle & s_1s_2,\ s_1s_3,\ s_{13},\ s_4s_1,\ s_5s_1,\ s_6s_1,\ s_1s_{14}s_1, \\ & s_1s_{15}s_1,\ s_7s_1,\ s_8s_1,\ s_9s_1,\ s_{10}s_1,\ s_{11}s_1, \\ & s_{12}s_1,\ s_1s_{13}s_1,\ s_{15},\ s_{14} \rangle . \end{aligned}$$

The identification in Γ is

$$F_{17}^{(b)} \stackrel{\cong}{\longleftrightarrow} F_{17}^{(s)}$$

$$\begin{aligned} b_1^{-1}b_2 &\longleftrightarrow s_1s_2 \\ b_1^{-1}b_3 &\longleftrightarrow s_1s_3 \\ b_2b_1b_3^{-1} &\longleftrightarrow s_{13} \\ b_1^2b_2b_1 &\longleftrightarrow s_4s_1 \\ b_1b_2^2b_1 &\longleftrightarrow s_5s_1 \\ b_1b_3^{-1}b_2b_1 &\longleftrightarrow s_6s_1 \\ b_1^{-1}b_2^{-1}b_1b_2b_1^2 &\longleftrightarrow s_1s_{14}s_1 \\ b_1^{-1}b_2^{-1}b_3^{-1}b_1^2 &\longleftrightarrow s_1s_{15}s_1 \\ b_3b_1^3 &\longleftrightarrow s_7s_1 \\ b_3^2b_1^2 &\longleftrightarrow s_8s_1 \end{aligned}$$

$$
\begin{aligned}
b_3 b_2^{-1} b_1^2 &\longleftrightarrow s_9 s_1 \\
b_3 b_1^{-1} b_2^2 b_1^2 &\longleftrightarrow s_{10} s_1 \\
b_3 b_1^{-1} b_3 b_2 b_1^2 &\longleftrightarrow s_{11} s_1 \\
b_3 b_1^{-2} b_2 b_1^2 &\longleftrightarrow s_{12} s_1 \\
b_1^{-1} b_3^{-1} b_2 b_1^2 &\longleftrightarrow s_1 s_{13} s_1 \\
b_1 b_2^{-1} b_3^{-1} &\longleftrightarrow s_{15} \\
b_1 b_3 b_1 b_3^{-1} &\longleftrightarrow s_{14}\,.
\end{aligned}
$$

Recall the presentation of Γ given in Section 3.4:

$$\Gamma = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid R \rangle\,,$$

where

$$
R = \left\{
\begin{array}{lll}
\mathbf{a_1 b_1 a_1 b_1}, & \mathbf{a_1 b_2 a_1 b_2}, & \mathbf{a_1 b_3 a_1 b_3}, \\
a_1 b_3^{-1} a_4 b_2^{-1}, & a_1 b_2^{-1} a_2 b_1^{-1}, & a_1 b_1^{-1} a_3 b_3^{-1}, \\
\mathbf{a_2 b_1 a_2 b_1}, & \mathbf{a_2 b_2 a_2 b_2}, & a_2 b_3 a_4^{-1} b_1^{-1}, \\
\mathbf{a_2 b_3^{-1} a_2 b_3^{-1}}, & a_2 b_2^{-1} a_3^{-1} b_3, & \mathbf{a_3 b_1 a_3 b_1}, \\
\mathbf{a_3 b_3 a_3 b_3}, & \mathbf{a_3 b_2^{-1} a_3 b_2^{-1}}, & a_3 b_1^{-1} a_4^{-1} b_2, \\
\mathbf{a_4 b_2 a_4 b_2}, & \mathbf{a_4 b_3 a_4 b_3}, & \mathbf{a_4 b_1^{-1} a_4 b_1^{-1}}
\end{array}
\right\}.
$$

The isomorphism to the amalgam described above is

$$F_3^{(b)} *_{F_{17}^{(b)} \cong F_{17}^{(s)}} (\mathbb{Z}_2^{*12} * F_3^{(s)}) \overset{\cong}{\longleftrightarrow} \Gamma = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid R \rangle$$

$$
\begin{aligned}
s_1 &\longleftrightarrow a_1 b_1 \\
s_2 &\longleftrightarrow a_1 b_2 \\
s_3 &\longleftrightarrow a_1 b_3 \\
s_4 &\longleftrightarrow a_1 b_2^{-1} b_1^{-2} \\
s_5 &\longleftrightarrow a_1 b_2^{-2} b_1^{-1} \\
s_6 &\longleftrightarrow a_1 b_2^{-1} b_3 b_1^{-1} \\
s_7 &\longleftrightarrow a_1 b_1^{-2} b_3^{-1} \\
s_8 &\longleftrightarrow a_1 b_1^{-1} b_3^{-2}
\end{aligned}
$$

$$\begin{aligned}
s_9 &\longleftrightarrow a_1 b_1^{-1} b_2 b_3^{-1} \\
s_{10} &\longleftrightarrow a_1 b_1^{-1} b_2^{-2} b_1 b_3^{-1} \\
s_{11} &\longleftrightarrow a_1 b_1^{-1} b_2^{-1} b_3^{-1} b_1 b_3^{-1} \\
s_{12} &\longleftrightarrow a_1 b_1^{-1} b_2^{-1} b_1^2 b_3^{-1} \\
s_{13} &\longleftrightarrow b_2 b_1 b_3^{-1} \\
s_{14} &\longleftrightarrow b_1 b_3 b_1 b_3^{-1} \\
s_{15} &\longleftrightarrow b_1 b_2^{-1} b_3^{-1} \\
s_1 b_1^{-1} &\longleftrightarrow a_1 \\
b_1^{-2} s_4 b_1 &\longleftrightarrow a_2 \\
b_3^{-2} s_8 b_3 &\longleftrightarrow a_3 \\
b_2^{-1} b_1 b_3^{-1} s_{10} b_3 b_1^{-1} &\longleftrightarrow a_4 \\
b_1 &\longleftrightarrow b_1 \\
b_2 &\longleftrightarrow b_2 \\
b_3 &\longleftrightarrow b_3 \,.
\end{aligned}$$

We describe now the (vertical) amalgam decomposition of the subgroup Γ_0:

$$\Gamma_0 \cong F_5^{(r)} *_{F_{33}^{(r)} \cong F_{33}^{(q)}} F_5^{(q)} \,,$$

where

$$F_5^{(r)} = \langle r_1, r_2, r_3, r_4, r_5 \rangle \,,$$

$$F_5^{(q)} = \langle q_1, q_2, q_3, q_4, q_5 \rangle \,,$$

$$\begin{aligned}
F_{33}^{(r)} = \langle & r_3^{-1} r_5,\ r_4^{-1} r_5,\ r_5 r_1 r_5,\ r_4 r_1 r_5,\ r_2^{-1} r_1 r_5,\ r_1 r_4 r_2 r_5,\ r_1 r_3 r_2 r_5, \\
& r_1 r_2 r_5,\ r_2 r_5^2,\ r_2 r_3 r_5,\ r_2 r_1^{-1} r_5,\ r_5^{-1} r_1^{-2} r_3^{-1},\ r_5^{-1} r_1^{-1} r_2^{-1} r_3^{-1}, \\
& r_5^{-1} r_1^{-1} r_5 r_3^{-1},\ r_1^{-1} r_3 r_1 r_5,\ r_1^{-1} r_2 r_3 r_1 r_5,\ r_1^{-1} r_4^{-1} r_3 r_1 r_5, \\
& r_2 r_4 r_5 r_2 r_5,\ r_2 r_4 r_1 r_5 r_2 r_5,\ r_2 r_4 r_3^{-1} r_5 r_2 r_5,\ r_1^{-1} r_3^{-2} r_1 r_5, \\
& r_1^{-1} r_3^{-1} r_2^{-1} r_3^{-1} r_1 r_5,\ r_1^{-1} r_3^{-1} r_1^{-1} r_3^{-1} r_1 r_5,\ r_2 r_4 r_2 r_1^{-1}, \\
& r_1^{-1} r_3^{-1} r_5 r_4^{-1} r_2^{-1},\ r_5^{-1} r_1^{-1} r_3 r_5^{-1} r_3 r_1 r_5,\ r_1^{-1} r_3^{-1} r_4 r_3^{-1}, \\
& r_1^{-1} r_5^{-1} r_1^{-1},\ r_5^{-1} r_2^{-1} r_1 r_3 r_1 r_5,\ r_5^{-1} r_1^{-1} r_4 r_5,\ r_5^{-1} r_2 r_5 r_2 r_5, \\
& r_3 r_2^{-1},\ r_5^{-1} r_1^{-1} r_3 r_4^{-1} r_5 r_2 r_5 \rangle \,,
\end{aligned}$$

$$\begin{aligned}F_{33}^{(q)} = \langle & q_2,\ q_1,\ q_4^{-1}q_5^{-1},\ q_4^{-1}q_1^{-1}q_5^{-1},\ q_4^{-1}q_3q_5^{-1},\ q_3^{-1}q_1^{-1}q_4^{-1},\\ & q_3^{-1}q_2^{-1}q_4^{-1},\ q_3^{-1}q_5q_4^{-1},\ q_5^{-1}q_3^{-1},\ q_5^{-1}q_2^{-1}q_3^{-1},\ q_5^{-1}q_4q_3^{-1},\\ & q_5q_2q_4q_5^{-1}q_4,\ q_5q_2q_3q_5^{-1}q_4,\ q_5q_2q_5^{-1}q_4,\ q_4^{-1}q_2^{-1}q_5^{-1}q_4q_5^{-1},\\ & q_4^{-1}q_2^{-1}q_3^{-1}q_4q_5^{-1},\ q_4^{-1}q_2^{-1}q_1q_4q_5^{-1},\ q_5^{-1}q_3q_5^{-1}q_3q_4^{-1},\\ & q_5^{-1}q_3q_4^{-1}q_3q_4^{-1},\ q_5^{-1}q_3q_2q_3q_4^{-1},\ q_4^{-1}q_2^{-1}q_5q_1q_2^{-1}q_5^{-1},\\ & q_4^{-1}q_2^{-1}q_3q_1q_2^{-1}q_5^{-1},\ q_4^{-1}q_2^{-1}q_4q_1q_2^{-1}q_5^{-1},\ q_5^{-1}q_3^2,\\ & q_4^{-1}q_2^{-1}q_1^{-1}q_3^{-1}q_5,\ q_5q_2q_1^{-1}q_2q_4q_5^{-1},\ q_4^{-1}q_2^{-2}q_5^{-1}q_4,\\ & q_4^{-1}q_2^{-1}q_4^{-1}q_3,\ q_4^2q_5^{-1},\ q_5q_2q_5q_3^{-1},\ q_3q_1q_3q_4^{-1},\\ & q_4^{-1}q_5q_1q_5,\ q_5q_2q_1^{-1}q_3q_4^{-1}\rangle\,,\end{aligned}$$

$$F_{33}^{(r)} \xleftrightarrow{\cong} F_{33}^{(q)}$$

$$\begin{aligned}
r_3^{-1}r_5 &\longleftrightarrow q_2\\
r_4^{-1}r_5 &\longleftrightarrow q_1\\
r_5r_1r_5 &\longleftrightarrow q_4^{-1}q_5^{-1}\\
r_4r_1r_5 &\longleftrightarrow q_4^{-1}q_1^{-1}q_5^{-1}\\
r_2^{-1}r_1r_5 &\longleftrightarrow q_4^{-1}q_3q_5^{-1}\\
r_1r_4r_2r_5 &\longleftrightarrow q_3^{-1}q_1^{-1}q_4^{-1}\\
r_1r_3r_2r_5 &\longleftrightarrow q_3^{-1}q_2^{-1}q_4^{-1}\\
r_1r_2r_5 &\longleftrightarrow q_3^{-1}q_5q_4^{-1}\\
r_2r_5^2 &\longleftrightarrow q_5^{-1}q_3^{-1}\\
r_2r_3r_5 &\longleftrightarrow q_5^{-1}q_2^{-1}q_3^{-1}\\
r_2r_1^{-1}r_5 &\longleftrightarrow q_5^{-1}q_4q_3^{-1}\\
r_5^{-1}r_1^{-2}r_3^{-1} &\longleftrightarrow q_5q_2q_4q_5^{-1}q_4\\
r_5^{-1}r_1^{-1}r_2^{-1}r_3^{-1} &\longleftrightarrow q_5q_2q_3q_5^{-1}q_4\\
r_5^{-1}r_1^{-1}r_5r_3^{-1} &\longleftrightarrow q_5q_2q_5^{-1}q_4\\
r_1^{-1}r_3r_1r_5 &\longleftrightarrow q_4^{-1}q_2^{-1}q_5^{-1}q_4q_5^{-1}\\
r_1^{-1}r_2r_3r_1r_5 &\longleftrightarrow q_4^{-1}q_2^{-1}q_3^{-1}q_4q_5^{-1}\\
r_1^{-1}r_4^{-1}r_3r_1r_5 &\longleftrightarrow q_4^{-1}q_2^{-1}q_1q_4q_5^{-1}\\
r_2r_4r_5r_2r_5 &\longleftrightarrow q_5^{-1}q_3q_5^{-1}q_3q_4^{-1}\\
r_2r_4r_1r_5r_2r_5 &\longleftrightarrow q_5^{-1}q_3q_4^{-1}q_3q_4^{-1}
\end{aligned}$$

$$
\begin{aligned}
r_2 r_4 r_3^{-1} r_5 r_2 r_5 &\longleftrightarrow q_5^{-1} q_3 q_2 q_3 q_4^{-1}\\
r_1^{-1} r_3^{-2} r_1 r_5 &\longleftrightarrow q_4^{-1} q_2^{-1} q_5 q_1 q_2^{-1} q_5^{-1}\\
r_1^{-1} r_3^{-1} r_2^{-1} r_3^{-1} r_1 r_5 &\longleftrightarrow q_4^{-1} q_2^{-1} q_3 q_1 q_2^{-1} q_5^{-1}\\
r_1^{-1} r_3^{-1} r_1^{-1} r_3^{-1} r_1 r_5 &\longleftrightarrow q_4^{-1} q_2^{-1} q_4 q_1 q_2^{-1} q_5^{-1}\\
r_2 r_4 r_2 r_1^{-1} &\longleftrightarrow q_5^{-1} q_3^2\\
r_1^{-1} r_3^{-1} r_5 r_4^{-1} r_2^{-1} &\longleftrightarrow q_4^{-1} q_2^{-1} q_1^{-1} q_3^{-1} q_5\\
r_5^{-1} r_1^{-1} r_3 r_5^{-1} r_3 r_1 r_5 &\longleftrightarrow q_5 q_2 q_1^{-1} q_2 q_4 q_5^{-1}\\
r_1^{-1} r_3^{-1} r_4 r_3^{-1} &\longleftrightarrow q_4^{-1} q_2^{-2} q_5^{-1} q_4\\
r_1^{-1} r_5^{-1} r_1^{-1} &\longleftrightarrow q_4^{-1} q_2^{-1} q_4^{-1} q_3\\
r_5^{-1} r_2^{-1} r_1 r_3 r_1 r_5 &\longleftrightarrow q_4^2 q_5^{-1}\\
r_5^{-1} r_1^{-1} r_4 r_5 &\longleftrightarrow q_5 q_2 q_5 q_3^{-1}\\
r_5^{-1} r_2 r_5 r_2 r_5 &\longleftrightarrow q_3 q_1 q_3 q_4^{-1}\\
r_3 r_2^{-1} &\longleftrightarrow q_4^{-1} q_5 q_1 q_5\\
r_5^{-1} r_1^{-1} r_3 r_4^{-1} r_5 r_2 r_5 &\longleftrightarrow q_5 q_2 q_1^{-1} q_3 q_4^{-1}\,.
\end{aligned}
$$

The isomorphism is

$$
F_5^{(r)} *_{F_{33}^{(r)} \cong F_{33}^{(q)}} F_5^{(q)} \stackrel{\cong}{\longleftrightarrow} \Gamma_0 \ < \ \Gamma
$$

$$
\begin{aligned}
r_1 &\longleftrightarrow b_2 b_1^{-1}\\
r_2 &\longleftrightarrow b_3 b_1^{-1}\\
r_3 &\longleftrightarrow b_1 b_3\\
r_4 &\longleftrightarrow b_1 b_2\\
r_5 &\longleftrightarrow b_1^2\\
q_1 &\longleftrightarrow b_2^{-1} b_1\\
q_2 &\longleftrightarrow b_3^{-1} b_1\\
q_3 &\longleftrightarrow a_1 a_2 b_3 b_2^{-1}\\
q_4 &\longleftrightarrow a_1 a_2^{-1} b_1^{-2}\\
q_5 &\longleftrightarrow a_1 a_3^{-1} b_1^{-1} b_3^{-1}\,.
\end{aligned}
$$

A.12 Amalgam decompositions of Example 3.46

We describe the amalgam decompositions of the group $\Gamma_{3,5}$.

$$
\begin{array}{rcccl}
\Gamma^{(v)} & \overset{\cong}{\longleftrightarrow} & \langle a_1, \ldots, b_3 \mid R_{2\cdot 3}\rangle & \overset{\cong}{\longleftrightarrow} & \Gamma^{(h)} \\
\\
s_4 b_3 & \longleftrightarrow & a_1 & \longleftrightarrow & a_1 \\
b_1 s_2 b_2^{-1} & \longleftrightarrow & a_2 & \longleftrightarrow & a_2 \\
b_1 & \longleftrightarrow & b_1 & \longleftrightarrow & u_2^{-1} a_2 a_1 \\
b_2 & \longleftrightarrow & b_2 & \longleftrightarrow & a_2^2 u_2^{-1} a_2^2 \\
b_3 & \longleftrightarrow & b_3 & \longleftrightarrow & a_2 u_2^{-1} a_2 \\
s_1 & \longleftrightarrow & b_1 b_2 & & \\
s_2 & \longleftrightarrow & a_1 b_3 b_2 & & \\
s_3 & \longleftrightarrow & a_1 b_1^{-1} b_2 & & \\
s_4 & \longleftrightarrow & a_1 b_3^{-1} & & \\
s_5 & \longleftrightarrow & a_1 b_1 b_2^2 & & \\
 & & a_1 a_2^{-1} b_1^{-1} & \longleftrightarrow & u_1 \\
 & & a_2 a_1 b_1^{-1} & \longleftrightarrow & u_2 \\
 & & a_1^{-2} a_2 & \longleftrightarrow & u_3 \\
 & & a_1^{-1} a_2^{-1} a_1 b_1^{-1} & \longleftrightarrow & u_4\,,
\end{array}
$$

where

$$
\begin{aligned}
\Gamma^{(v)} &= F_3^{(v,b)} *_{F_9^{(v,b)} \cong F_9^{(v,s)}} F_5^{(v,s)}\,, \\
\Gamma^{(h)} &= F_2^{(h,a)} *_{F_7^{(h,a)} \cong F_7^{(h,u)}} F_4^{(h,u)}\,,
\end{aligned}
$$

$$
\begin{aligned}
F_3^{(v,b)} &= \langle b_1, b_2, b_3\rangle\,, \\
F_5^{(v,s)} &= \langle s_1, s_2, s_3, s_4, s_5\rangle\,, \\
F_9^{(v,b)} &= \langle b_3^{-1} b_1,\ b_2 b_1^2,\ b_3 b_1^2,\ b_1 b_2,\ b_2^{-1} b_3 b_1,\ b_1^{-1} b_2^2,\ b_1^{-2} b_3 b_2,\ b_1^{-3} b_2,\ b_1^{-2} b_2 b_1\rangle\,, \\
F_9^{(v,s)} &= \langle s_3 s_2^{-1},\ s_4 s_2^{-1},\ s_4^{-1} s_2^{-1},\ s_1,\ s_5 s_2^{-1},\ s_2 s_5,\ s_2^2,\ s_2 s_3,\ s_2 s_1 s_2^{-1}\rangle\,,
\end{aligned}
$$

$$
\begin{aligned}
F_2^{(h,a)} &= \langle a_1, a_2\rangle\,, \\
F_4^{(h,u)} &= \langle u_1, u_2, u_3, u_4\rangle\,, \\
F_7^{(h,a)} &= \langle a_1^2 a_2^{-1},\ a_1^{-1} a_2^{-2},\ a_2 a_1 a_2 a_1^{-1},\ a_1^{-2} a_2,\ a_1 a_2^{-2} a_1,\ a_1 a_2^2,\ a_1 a_2^{-1} a_1 a_2\rangle\,, \\
F_7^{(h,u)} &= \langle u_1 u_3 u_2^{-1},\ u_4 u_2^{-1},\ u_2 u_1^{-1},\ u_3,\ u_1^2,\ u_1 u_4,\ u_1 u_2\rangle\,,
\end{aligned}
$$

$$\begin{aligned}
F_9^{(v,b)} &\overset{\cong}{\longleftrightarrow} F_9^{(v,s)}\\
b_3^{-1}b_1 &\longleftrightarrow s_3s_2^{-1}\\
b_2b_1^2 &\longleftrightarrow s_4s_2^{-1}\\
b_3b_1^2 &\longleftrightarrow s_4^{-1}s_2^{-1}\\
b_1b_2 &\longleftrightarrow s_1\\
b_2^{-1}b_3b_1 &\longleftrightarrow s_5s_2^{-1}\\
b_1^{-1}b_2^2 &\longleftrightarrow s_2s_5\\
b_1^{-2}b_3b_2 &\longleftrightarrow s_2^2\\
b_1^{-3}b_2 &\longleftrightarrow s_2s_3\\
b_1^{-2}b_2b_1 &\longleftrightarrow s_2s_1s_2^{-1}\,,
\end{aligned}$$

$$\begin{aligned}
F_7^{(h,a)} &\overset{\cong}{\longleftrightarrow} F_7^{(h,u)}\\
a_1^2a_2^{-1} &\longleftrightarrow u_1u_3u_2^{-1}\\
a_1^{-1}a_2^{-2} &\longleftrightarrow u_4u_2^{-1}\\
a_2a_1a_2a_1^{-1} &\longleftrightarrow u_2u_1^{-1}\\
a_1^{-2}a_2 &\longleftrightarrow u_3\\
a_1a_2^{-2}a_1 &\longleftrightarrow u_1^2\\
a_1a_2^2 &\longleftrightarrow u_1u_4\\
a_1a_2^{-1}a_1a_2 &\longleftrightarrow u_1u_2
\end{aligned}$$

and

$$R_{2\cdot 3} := \left\{ \begin{array}{ll} a_1b_1a_2b_2, & a_1b_2a_2b_1^{-1}, \\ a_1b_3a_2^{-1}b_1, & a_1b_3^{-1}a_1b_2^{-1}, \\ a_1b_1^{-1}a_2^{-1}b_3, & a_2b_3a_2b_2^{-1} \end{array} \right\}.$$

Appendix B

GAP-programs

In this appendix, we present and describe the GAP-programs ([29]), which led to the construction of most groups in this work.

B.1 Theory and ideas

Our strategy to generate and analyze $(2m, 2n)$–groups Γ with GAP ([29]) can be resumed as follows:

Step 1: Describe a $(2m, 2n)$–complex X in a way which is manageable for a computer. We write X as a pair of integer valued $(2m \times 2n)$–matrices (lists of lists) A and B.

Step 2: Given "small" m, n, generate *all* pairs of matrices (A, B) corresponding to a $(2m, 2n)$–complex. Given "large" m, n, generate *randomly* many pairs (A, B) corresponding to a $(2m, 2n)$–complex.

Step 3: Starting from a constructed pair (A, B) describing X, provide additional programs which compute the local groups $P_h^{(k)}$, $P_v^{(k)}$ (for $k \in \mathbb{N}$ small) and a finite presentation of $\Gamma = \pi_1(X)$. Then apply the powerful GAP-tools for finite permutation groups to look for examples with interesting local groups and/or use GAP-commands like

```
AbelianInvariants();
```

and

```
LowIndexSubgroupsFpGroup();
```

to get some information on the (normal) subgroup structure of the infinite group Γ.

Following these three steps, we have for instance immediately found an irreducible (A_6, A_6)–group Γ with $[\Gamma, \Gamma] = \Gamma_0$ and Γ_0 perfect (see Example 2.2).

We explain now each of the three steps in detail:

Step 1

We want to define for given $m, n \in \mathbb{N}$ an injective map

$$\begin{aligned} \varphi_{m,n} : \mathcal{X}_{2m,2n} &\to \mathrm{Mat}(2m, 2n, \{1, \dots, 2m\}) \times \mathrm{Mat}(2m, 2n, \{1, \dots, 2n\}) \\ X &\mapsto \varphi_{m,n}(X) = (A, B) \end{aligned}$$

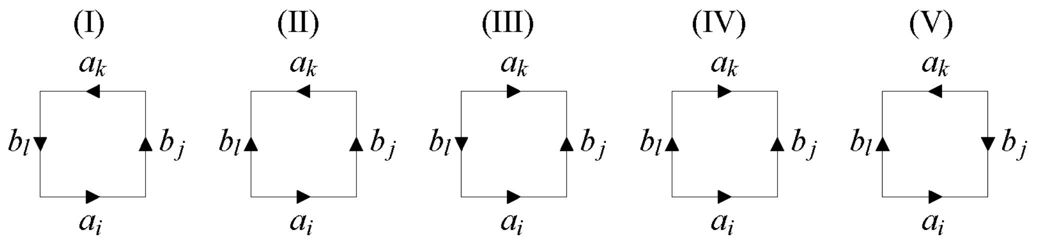
where $\mathcal{X}_{2m,2n}$ denotes the set of $(2m, 2n)$–complexes and $X \in \mathcal{X}_{2m,2n}$ is given as usual by its mn geometric squares, and where $\mathrm{Mat}(2m, 2n, \{1, \dots, 2m\})$ denotes the set of $(2m \times 2n)$–matrices with entries in $\{1, \dots, 2m\}$. Recall that each geometric square $[aba'b']$ of X can be represented by four squares of the form

$$aba'b', \; a'b'ab, \; a^{-1}b'^{-1}a'^{-1}b^{-1}, \; a'^{-1}b^{-1}a^{-1}b'^{-1}\,.$$

To define the map $\varphi_{m,n}$, note that at least one of these four expressions has one of the five types (I)-(V) illustrated in Figure B.1, for suitable

$$i, k \in \{1, \dots, m\} \text{ and } j, l \in \{1, \dots, n\}\,.$$

It is easy to check that each geometric square has a unique type.



$a_i b_j a_k b_l$ $\qquad$ $a_i b_j a_k b_l^{-1}$ $\qquad$ $a_i b_j a_k^{-1} b_l$ $\qquad$ $a_i b_j a_k^{-1} b_l^{-1}$ $\qquad$ $a_i b_j^{-1} a_k b_l^{-1}$

Figure B.1: Possible types of a geometric square

We now define the map $\varphi_{m,n}$ for each possible type of geometric squares, using the following notation for the "inverses":

$$\bar{i} := 2m + 1 - i, \;\; \bar{k} := 2m + 1 - k, \;\; \bar{j} := 2n + 1 - j, \;\; \bar{l} := 2n + 1 - l\,.$$

Type (I) $(a_i b_j a_k b_l)$

$$\begin{array}{ll} A_{ij} := \bar{k} & B_{ij} := \bar{l} \\ A_{kl} := \bar{i} & B_{kl} := \bar{j} \\ A_{\bar{i}\bar{l}} := k & B_{\bar{i}\bar{l}} := j \\ A_{\bar{k}\bar{j}} := i & B_{\bar{k}\bar{j}} := l\,. \end{array}$$

Type (II) $(a_i b_j a_k b_l^{-1})$
$$\begin{aligned} A_{ij} &:= \bar{k} & B_{ij} &:= l \\ A_{k\bar{l}} &:= \bar{i} & B_{k\bar{l}} &:= \bar{j} \\ A_{\bar{i}l} &:= k & B_{\bar{i}l} &:= j \\ A_{\bar{k}\bar{j}} &:= i & B_{\bar{k}\bar{j}} &:= \bar{l}\,. \end{aligned}$$

Type (III) $(a_i b_j a_k^{-1} b_l)$
$$\begin{aligned} A_{ij} &:= k & B_{ij} &:= \bar{l} \\ A_{k\bar{j}} &:= i & B_{k\bar{j}} &:= l \\ A_{\bar{i}\bar{l}} &:= \bar{k} & B_{\bar{i}\bar{l}} &:= j \\ A_{\bar{k}l} &:= \bar{i} & B_{\bar{k}l} &:= \bar{j}\,. \end{aligned}$$

Type (IV) $(a_i b_j a_k^{-1} b_l^{-1})$
$$\begin{aligned} A_{ij} &:= k & B_{ij} &:= l \\ A_{k\bar{j}} &:= i & B_{k\bar{j}} &:= \bar{l} \\ A_{\bar{i}l} &:= \bar{k} & B_{\bar{i}l} &:= j \\ A_{\bar{k}\bar{l}} &:= \bar{i} & B_{\bar{k}\bar{l}} &:= \bar{j}\,. \end{aligned}$$

Type (V) $(a_i b_j^{-1} a_k b_l^{-1})$
$$\begin{aligned} A_{i\bar{j}} &:= \bar{k} & B_{i\bar{j}} &:= l \\ A_{k\bar{l}} &:= \bar{i} & B_{k\bar{l}} &:= j \\ A_{\bar{i}l} &:= k & B_{\bar{i}l} &:= \bar{j} \\ A_{\bar{k}j} &:= i & B_{\bar{k}j} &:= \bar{l}\,. \end{aligned}$$

Thus, each geometric square of X defines exactly four entries in A and in B which describe the corresponding four geometric edges in the link $Lk(X)$. In case of type (I) and (V), two choices are possible, since we have the equalities for geometric squares $[a_i b_j a_k b_l] = [a_k b_l a_i b_j]$ and $[a_i b_j^{-1} a_k b_l^{-1}] = [a_k b_l^{-1} a_i b_j^{-1}]$ respectively, but the given definition of $\varphi_{m,n}$ is independent of this choice. This proves that $\varphi_{m,n}$ is well-defined.

We illustrate this definition in Table B.1 in the case of Example 2.2 given by its nine relators

$$R_{3\cdot 3} := \left\{ \begin{array}{lll} a_1 b_1 a_1^{-1} b_1^{-1}, & a_1 b_2 a_1^{-1} b_3^{-1}, & a_1 b_3 a_2 b_2^{-1}, \\ a_1 b_3^{-1} a_3^{-1} b_2, & a_2 b_1 a_3^{-1} b_2^{-1}, & a_2 b_2 a_3^{-1} b_3^{-1}, \\ a_2 b_3 a_3^{-1} b_1, & a_2 b_3^{-1} a_3 b_2, & a_2 b_1^{-1} a_3^{-1} b_1^{-1} \end{array} \right\}.$$

geometric square	representative	type	A-entries	B-entries
$[a_1b_1a_1^{-1}b_1^{-1}]$	$a_1b_1a_1^{-1}b_1^{-1}$	(IV)	$A_{11}=1,\ A_{16}=1$	$B_{11}=1,\ B_{16}=6$
			$A_{61}=6,\ A_{66}=6$	$B_{61}=1,\ B_{66}=6$
$[a_1b_2a_1^{-1}b_3^{-1}]$	$a_1b_2a_1^{-1}b_3^{-1}$	(IV)	$A_{12}=1,\ A_{15}=1$	$B_{12}=3,\ B_{15}=4$
			$A_{63}=6,\ A_{64}=6$	$B_{63}=2,\ B_{64}=5$
$[a_1b_3a_2b_2^{-1}]$	$a_1b_3a_2b_2^{-1}$	(II)	$A_{13}=5,\ A_{25}=6$	$B_{13}=2,\ B_{25}=4$
			$A_{62}=2,\ A_{54}=1$	$B_{62}=3,\ B_{54}=5$
$[a_1b_3^{-1}a_3^{-1}b_2]$	$a_3b_3a_1^{-1}b_2^{-1}$	(IV)	$A_{33}=1,\ A_{14}=3$	$B_{33}=2,\ B_{14}=5$
			$A_{42}=6,\ A_{65}=4$	$B_{42}=3,\ B_{65}=4$
$[a_2b_1a_3^{-1}b_2^{-1}]$	$a_2b_1a_3^{-1}b_2^{-1}$	(IV)	$A_{21}=3,\ A_{36}=2$	$B_{21}=2,\ B_{36}=5$
			$A_{52}=4,\ A_{45}=5$	$B_{52}=1,\ B_{45}=6$
$[a_2b_2a_3^{-1}b_3^{-1}]$	$a_2b_2a_3^{-1}b_3^{-1}$	(IV)	$A_{22}=3,\ A_{35}=2$	$B_{22}=3,\ B_{35}=4$
			$A_{53}=4,\ A_{44}=5$	$B_{53}=2,\ B_{44}=5$
$[a_2b_3a_3^{-1}b_1]$	$a_2b_3a_3^{-1}b_1$	(III)	$A_{23}=3,\ A_{34}=2$	$B_{23}=6,\ B_{34}=1$
			$A_{56}=4,\ A_{41}=5$	$B_{56}=3,\ B_{41}=4$
$[a_2b_3^{-1}a_3b_2]$	$a_3b_2a_2b_3^{-1}$	(II)	$A_{32}=5,\ A_{24}=4$	$B_{32}=3,\ B_{24}=5$
			$A_{43}=2,\ A_{55}=3$	$B_{43}=2,\ B_{55}=4$
$[a_2b_1^{-1}a_3^{-1}b_1^{-1}]$	$a_3b_1a_2^{-1}b_1$	(III)	$A_{31}=2,\ A_{26}=3$	$B_{31}=6,\ B_{26}=1$
			$A_{46}=5,\ A_{51}=4$	$B_{46}=1,\ B_{51}=6$

Table B.1: Definition of A and B in Example 2.2

Hence, we get

$$A = \begin{pmatrix} 1 & 1 & 5 & 3 & 1 & 1 \\ 3 & 3 & 3 & 4 & 6 & 3 \\ 2 & 5 & 1 & 2 & 2 & 2 \\ 5 & 6 & 2 & 5 & 5 & 5 \\ 4 & 4 & 4 & 1 & 3 & 4 \\ 6 & 2 & 6 & 6 & 4 & 6 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \\ 6 & 3 & 2 & 1 & 4 & 5 \\ 4 & 3 & 2 & 5 & 6 & 1 \\ 6 & 1 & 2 & 5 & 4 & 3 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix}.$$

See Table B.2 for a more compact notation.

$\varphi_{3,3}(X)$	$1 \approx b_1$	$2 \approx b_2$	$3 \approx b_3$	$4 \approx b_3^{-1}$	$5 \approx b_2^{-1}$	$6 \approx b_1^{-1}$
$1 \approx a_1$	1/1	1/3	5/2	3/5	1/4	1/6
$2 \approx a_2$	3/2	3/3	3/6	4/5	6/4	3/1
$3 \approx a_3$	2/6	5/3	1/2	2/1	2/4	2/5
$4 \approx a_3^{-1}$	5/4	6/3	2/2	5/5	5/6	5/1
$5 \approx a_2^{-1}$	4/6	4/1	4/2	1/5	3/4	4/3
$6 \approx a_1^{-1}$	6/1	2/3	6/2	6/5	4/4	6/6

Table B.2: Compact notation of A and B in Example 2.2

Note that given $(A, B) \in \mathrm{im}(\varphi_{m,n})$, we can uniquely and easily reconstruct the $(2m, 2n)$–complex $X = \varphi_{m,n}^{-1}((A, B))$ (this reflects the injectivity of $\varphi_{m,n}$).

Remark. By construction of $\varphi_{m,n}$, there are bijections between the following sets:

$$\begin{aligned} \{(A_{ij}, B_{ij})\}_{i=1,\ldots,2m,\ j=1,\ldots,2n} &\cong \{1, \ldots, 2m\} \times \{1, \ldots, 2n\}\,, \\ \{1, \ldots, 2m\} &\cong \{A_{ij}\}_{i=1,\ldots,2m} \text{ for any } j \in \{1, \ldots, 2n\}\,, \\ \{1, \ldots, 2n\} &\cong \{B_{ij}\}_{j=1,\ldots,2n} \text{ for any } i \in \{1, \ldots, 2m\}\,, \end{aligned}$$

in particular each column of A is a permutation of $\{1, \ldots, 2m\}$, and each row of B is a permutation of $\{1, \ldots, 2n\}$.

Step 2

The idea of Step 2 for small m, n (for example "small" could mean $mn < 10$) is to start with $(2m \times 2n)$–matrices A and B consisting of 0-entries and "fill" them recursively with one geometric square (four non-zero entries in A and B) in each recursion step. This is done systematically, i.e. going through all potential geometric squares S. Of course, S has to satisfy several conditions, e.g. we want all potential new positions in A (and B) coming from S to be free (i.e. zeroes), and all potential new pairs of entries $(A_{\alpha\beta}, B_{\alpha\beta})$ coming from S are required to be new. If the candidate S does not satisfy these conditions, we try the next one. The conditions guarantee that at the end a "full" (i.e. without zero entries) pair of matrices (A, B) indeed describes a $(2m, 2n)$–complex X, in particular having a complete bipartite link $Lk(X)$ as required in the link condition.

B.2 The main program

Our main GAP-program ([29]) looks as follows: (comments in GAP start with the character #)

```
all := function(x1, x2, y1, y2)
# generates the list
# [[x1,y1],...,[x1,y2],...,[x2,y1],...,[x2,y2]]
local w, k, i, j;
w := [ ];
k := 1;
for i in [x1..x2] do
 for j in [y1..y2] do
   w[k] := [i,j];
   k := k+1;
 od;
od;
return w;
end;

test := function(M, N, q, r, s, t, cM, cN)
# checks candidate a_q b_r a_s^-1 b_t^-1
if (s = cM+1-q and t = cN+1-r) or
   M[s][cN+1-r] <> 0 or
   M[cM-q+1][t] <> 0 or
   M[cM+1-s][cN+1-t] <> 0 or
# M[q][r] <> 0 is tested in test2
   ForAny(all(1,cM,1,cN),
   v -> ([M[v[1]][v[2]],N[v[1]][v[2]]] in
     [[s,t], [q,cN+1-t], [cM+1-s,r], [cM+1-q,cN+1-r]]))
then
 return false;
else
 return true;
fi;
end;

part := function(x, y, z)
# we assume y <= z
# generates [[1,1],...,[1,z],...,[x-1,1],...,[x-1,z],
# [x,1],...,[x,y-1]]
local w, k, i1, i2, j;
```

```
w := [ ];
k := 1;
for i1 in [1..x-1] do
 for i2 in [1..z] do
  w[k] := [i1,i2];
  k := k+1;
 od;
od;
for j in [1..y-1] do
 w[k] := [x,j];
 k := k+1;
od;
return w;
end;

test2 := function(A, x, y, z)
# returns true if (x,y) is
# the first "free" position in A
if A[x][y] = 0 and
 ForAll(part(x,y,z), v -> A[v[1]][v[2]] <> 0)
then
 return true;
else
 return false;
fi;
end;

full := function(A)
# returns true if matrix A contains no 0
if ForAny(A, x -> 0 in x) then
 return false;
else
 return true;
fi;
end;

main := function(A, B)
# main program
local cA, cB, i, j, k, l, AA, BB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2]; # = DimensionsMat(B)[2]
```

```
for i in [1..cA/2] do
 for j in [1..cB] do
  if test2(A,i,j,cB) then
   # (i,j) is first free position in A
   for k in [1..cA] do
     for l in [1..cB] do
      if test(A,B,i,j,k,l,cA,cB) then
       # tests if $a_i b_j a_k^{-1} b_l^{-1}$ is ok
       AA := StructuralCopy(A);
       BB := StructuralCopy(B);
       AA[i][j] := k;
       BB[i][j] := l;
       AA[k][cB-j+1] := i;
       BB[k][cB-j+1] := cB+1-l;
       AA[cA+1-i][l] := cA+1-k;
       BB[cA+1-i][l] := j;
       AA[cA+1-k][cB+1-l] := cA+1-i;
       BB[cA+1-k][cB+1-l] := cB+1-j;
       if full(AA) then
        # (AA,BB) now describes a (cA,cB)-complex
        # now we can check for conditions on AA, BB,
        # e.g.  if conditions(AA,BB) then
        # Print(AA, " ", BB, "\n"); fi;
       else
        main(AA, BB); # recursive step
       fi;
      fi;
     od;
   od;
  fi;
 od;
od;
end;

# can be applied as follows:
# for example main(NullMat(4, 6), NullMat(4, 6));
# generates now all (4,6)-complexes,
# or use main(C,D); for an embedding, where C, D describe
# any partial complex, i.e. some given geometric squares
```

This procedure can a priori also be applied for large integers m, n (for example if $mn \geq 10$), but the time required to finish (that is to generate *all* $(2m, 2n)$–complexes) grows very rapidly with increasing m and n. One reason for this is that the filling process needs mn recursion steps for each $(2m, 2n)$–complex but another reason is that the number of different $(2m, 2n)$–complexes becomes very large soon. This is illustrated in Table B.3. Observe that the number of non-isomorphic corresponding fundamental groups is much smaller, but unknown in general, even for (4, 4)–groups. Kimberley ([40]) has counted the number of "BM relations" for

$$(m, n) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 2), (2, 3)\}.$$

They coincide with those in Table B.3. The number 541 for (4, 4)–complexes also appears in [41, Section 7].

m	n	mn	$\# X$
1	1	1	3
1	2	2	15
1	3	3	105
1	4	4	945
1	5	5	10395
1	6	6	135135
1	7	7	2027025
1	8	8	34459425
2	2	4	541
2	3	6	35235
2	4	8	3690009
2	5	10	570847095
3	3	9	27712191

Table B.3: Number of $(2m, 2n)$–complexes generated by our programs

Therefore, to get a better "distribution" of the examples for large m and n, we also have written a program which *randomly* generates many $(2m, 2n)$–complexes for fixed $m, n \in \mathbb{N}$.

B.3 A random program

```
# the functions full(), all(), test(), part(), test2()
# are defined as before

Ma := function(m, n)
# generates (m x n)-matrix A, A[i][j] = i
local i, j, w;
```

```
w := NullMat(m,n);
for i in [1..m] do
 for j in [1..n] do
  w[i][j] := i;
 od;
od;
return w;
end;

Mb := function(m, n)
# generates (m x n)-matrix A, A[i][j] = j
local i, j, w;
w := NullMat(m,n);
for i in [1..m] do
 for j in [1..n] do
  w[i][j] := j;
 od;
od;
return w;
end;

out := [ ];

rdm := function(A, B, p)
local cA, cB, i, j, k, l, AA, BB, kl, pp, z;
z := 0;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
for i in [1..cA/2] do
 for j in [1..cB] do
  if test2(A,i,j,cB) then
   repeat kl := Random(p); # p:available edges in link
          z := z+1; # z counts number of attempts,
          # here we set the maximal number to 30, but it
          # can be chosen larger or smaller if needed
   until test(A,B,i,j,kl[1],kl[2],cA,cB) or z = 30;
   AA := StructuralCopy(A);
   BB := StructuralCopy(B);
   if z < 30 then # test ok
     AA[i][j] := kl[1];
     BB[i][j] := kl[2];
```

```
        AA[kl[1]][cB-j+1] := i;
        BB[kl[1]][cB-j+1] := cB+1-kl[2];
        AA[cA+1-i][kl[2]] := cA+1-kl[1];
        BB[cA+1-i][kl[2]] := j;
        AA[cA+1-kl[1]][cB+1-kl[2]] := cA+1-i;
        BB[cA+1-kl[1]][cB+1-kl[2]] := cB+1-j;
        pp := StructuralCopy(p);
        RemoveSet(pp,kl);
        RemoveSet(pp,[i,cB+1-kl[2]]);
        RemoveSet(pp,[cA+1-kl[1],j]);
        RemoveSet(pp,[cA+1-i,cB+1-j]);
        # removes used edges in link
        if full(AA) then
         out := StructuralCopy([AA,BB,cA,cB]);
        else
         rdm(AA, BB, pp);
        fi;
      fi;
    fi;
  od;
od;
return out;
end;

slc := function(aa,bb)
local res;
repeat out := [Ma(aa,bb),Mb(aa,bb),aa,bb]; res :=
 rdm(NullMat(aa, bb), NullMat(aa, bb), all(1,aa,1,bb));
until
# conditions(res[1],res[2]); whatever we want to check
Print(res[1],"\n",res[2],"\n");
end;

# e.g. slc(6,6); generates now randomly a (6,6)-complex
# satisfying additional conditions
```

One nice feature of both programs is that we can start with any k given geometric squares (where $0 \leq k < mn$) and generate all (or randomly some, respectively) $(2m, 2n)$–complexes containing these k geometric squares. This was very useful in Chapter 2, where we have embedded for instance non-residually finite examples in virtually simple $(2m, 2n)$–groups.

B.4 Computing the local groups

Step 3

We have written programs which compute the local groups $P_h^{(k)}$ and $P_v^{(k)}$ for k small enough. Here are the programs for $k = 1$ and $k = 2$. The programs for $k \geq 3$ become more complicated with increasing k, but we do not need any new ideas. Moreover, we give the program to compute the group K_h for $m = 3$.

```
PhPerm := function(j, cA, A)
# generates permutation in $P_h$ induced by $b_j$, i.e. $\rho_v(b_j)$
local v, i;
v := [ ];
for i in [1..cA] do
 v[i] := cA+1-A[cA-i+1][j];
od;
return PermList(v);
end;

Ph := function(A)
# generates $P_h$ as a permutation group
local p, j, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
p := [ ];
for j in [1..cB/2] do
 p[j] := PhPerm(j,cA,A);
od;
return Group(p,());
end;

PvPerm := function(i, cA, cB, B)
# generates permutation in $P_v$ induced by $a_i$, i.e. $\rho_h(a_i)$
local w, j;
w := [ ];
for j in [1..cB] do
 w[j] := B[cA-i+1][j];
od;
return PermList(w);
end;
```

```
Pv := function(B)
# generates P_v
local p, i, cA, cB;
cA := DimensionsMat(B)[1];
cB := DimensionsMat(B)[2];
p := [ ];
for i in [1..cA/2] do
 p[i] := PvPerm(i,cA,cB,B);
od;
return Group(p,());
end;

indx := function(v, x)
# returns index of first appearance of x
# in vector v
local i;
i := 1;
while v[i] <> x do
 i := i+1;
od;
return i;
end;

s2 := function(c)
# generates points in 2-sphere
# of c-regular tree
local v, k, i, j;
v := [ ];
k := 1;
for i in [1..c] do
 for j in [1..c] do
  if i+j <> c+1 then
  # exclude reducible paths
   v[k] := [i,j];
   k := k+1;
  fi;
 od;
od;
return v;
end;
```

```
vPerm2i := function(i, cA, cB, A, B)
# generates i-th permutation in P_v^(2)
local w, j;
w := [ ];
for j in [1..cB*(cB-1)] do
  w[j] := indx(s2(cB), [B[cA+1-i][s2(cB)[j][1]],
         B[A[cA+1-i][s2(cB)[j][1]]][s2(cB)[j][2]]]);
od;
return PermList(w);
end;

P2v := function(A, B)
# generates P_v^(2)
local i, p, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
p := [ ];
for i in [1..cA/2] do
  p[i] := vPerm2i(i, cA, cB, A, B);
od;
return Group(p,());
end;

hPerm2j := function(j, cA, cB, A, B)
# generates j-th permutation in P_h^(2)
local w, i;
w := [ ];
for i in [1..cA*(cA-1)] do
  w[i] := indx(s2(cA), [cA+1-A[cA+1-s2(cA)[i][1]][j],
    cA+1-A[cA+1-s2(cA)[i][2]][B[cA+1-s2(cA)[i][1]][j]]]);
od;
return PermList(w);
end;

P2h := function(A, B)
# generates P_h^(2)
local j, p, cA, cB;
cA := DimensionsMat(A)[1]; cB := DimensionsMat(A)[2];
p := [ ];
for j in [1..cB/2] do
  p[j] := hPerm2j(j, cA, cB, A, B);
```

```
od;
return Group(p,());
end;

Kh6 := function(A, B)
# generates K_h for m = 3
return Stabilizer(Stabilizer(Stabilizer(
       Stabilizer(Stabilizer(Stabilizer(P2h(A, B),
        [1, 2, 3, 4, 5], OnTuples),
        [6, 7, 8, 9, 10], OnSets),
        [11, 12, 13, 14, 15], OnSets),
        [16, 17, 18, 19, 20], OnSets),
        [21, 22, 23, 24, 25], OnSets),
        [26, 27, 28, 29, 30], OnSets);
end;
```

B.5 Computing a presentation

A finite presentation for Γ is obtained as follows (illustrated for $m = n = 3$):

```
F := FreeGroup("a1", "a2", "a3", "b1", "b2", "b3");
# free group generated by a_1, a_2, a_3, b_1, b_2, b_3
a1 := F.1;
a2 := F.2;
a3 := F.3;
b1 := F.4;
b2 := F.5;
b3 := F.6;

NL6a := function(i)
# bijection {1,...,2m} -> E_h
local v;
if i=1 then v := a1;
 elif i=2 then v := a2;
  elif i=3 then v := a3;
  elif i=4 then v := a3^-1;
 elif i=5 then v := a2^-1;
elif i=6 then v := a1^-1;
fi;
return v;
end;
```

```
NL6b := function(j)
# bijection {1,...,2n} → E_v
local v;
if j=1 then v := b1;
 elif j=2 then v := b2;
  elif j=3 then v := b3;
  elif j=4 then v := b3^-1;
 elif j=5 then v := b2^-1;
elif j=6 then v := b1^-1;
fi;
return v;
end;

relation6 := function(A, B)
# generates mn relators of Γ
local i, j, rel, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
rel := [ ];
for i in [1..cA/2] do
 for j in [1..cB] do
  if not NL6a(i)*NL6b(j)*
         NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j]) in rel
     and not NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j])*
         NL6a(i)*NL6b(j) in rel
     and not NL6a(cA+1-A[i][j])^-1*NL6b(j)^-1*
         NL6a(i)^-1*NL6b(cB+1-B[i][j])^-1 in rel then
     Add(rel,NL6a(i)*NL6b(j)*
             NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j]));
  fi;
 od;
od;
return rel;
end;

G := F / relation6(A,B); # definition of Γ
# e.g. AbelianInvariants(G); computes now Γ^ab
# LowIndexSubgroupsFpGroup(G, TrivialSubgroup(G), 8);
# computes all subgroups of low index
# (here of index ≤ 8), only reasonable for small index
```

B.6 A normal form program

Very useful for other investigations are programs which bring a word of Γ in ab- and in ba-normal form, see Proposition 1.10 (again illustrated for $m = n = 3$):

```
# F, a1, a2, a3, b1, b2, b3, NL6a(), NL6b()
# as in Appendix B.5

LN6a := function(w)
# bijection E_h -> {1,...,2m},
# inverse of NL6a
local i;
if w=a1 then i := 1;
 elif w=a2 then i := 2;
  elif w=a3 then i := 3;
  elif w=a3^-1 then i := 4;
 elif w=a2^-1 then i := 5;
elif w=a1^-1 then i := 6;
fi;
return i;
end;

LN6b := function(w)
# bijection E_v -> {1,...,2n},
# inverse of NL6b
local j;
if w=b1 then j := 1;
 elif w=b2 then j := 2;
  elif w=b3 then j := 3;
  elif w=b3^-1 then j := 4;
 elif w=b2^-1 then j := 5;
elif w=b1^-1 then j := 6;
fi;
return j;
end;

SetA6 := [a1, a2, a3, a3^-1, a2^-1, a1^-1];
# E_h

SetB6 := [b1, b2, b3, b3^-1, b2^-1, b1^-1];
# E_v
```

```
nfab := function(A,B,w)
# brings word w in ab-normal form
local i;
for i in [1..Length(w)-1] do
  if Subword(w,i,i) in SetB6 and
     Subword(w,i+1,i+1) in SetA6 then
    return nfab(A,B,SubstitutedWord(w,i,i+1,
      (NL6b(B[LN6a(Subword(w,i+1,i+1)^-1)]
             [LN6b(Subword(w,i,i)^-1)])*
       NL6a(A[LN6a(Subword(w,i+1,i+1)^-1)]
             [LN6b(Subword(w,i,i)^-1)]))^-1));
  fi;
od;
return w;
end;

nfba := function(A,B,w)
# brings word w in ba-normal form
local i;
for i in [1..Length(w)-1] do
  if Subword(w,i,i) in SetA6 and
     Subword(w,i+1,i+1) in SetB6 then
    return nfba(A,B,SubstitutedWord(w,i,i+1,
      NL6b(B[LN6a(Subword(w,i,i))]
             [LN6b(Subword(w,i+1,i+1))])*
      NL6a(A[LN6a(Subword(w,i,i))]
             [LN6b(Subword(w,i+1,i+1))])));
  fi;
od;
return w;
end;
```

B.7 Computing Aut(X)

The following program generates all elements of Aut(X), where X is described by the matrices A and B (again illustrated for $m = n = 3$).

```
# F, a1, a2, a3, b1, b2, b3, NL6a(), NL6b()
# as in Appendix B.5
```

```
relation := function(A, B)
local i, j, k, rel, rel2, cA, cB;
cA := DimensionsMat(A)[1];
cB := DimensionsMat(A)[2];
rel := [ ];
rel2 := [ ];
for i in [1..cA] do
 for j in [1..cB] do
  rel[cB*(i-1)+j] := NL6a(i)*NL6b(j)*
   NL6a(cA+1-A[i][j])*NL6b(cB+1-B[i][j]);
 od;
od;
for k in [1..cA*cB] do
 rel2[k] := Subword(rel[k],2,4)*Subword(rel[k],1,1);
od;
return Union(rel,rel2);
end;

LN := function(w,k1,k2,k3,k4,k5,k6,c)
local n;
if w=a1 then n := k1;
 elif w=a2 then n := k2;
  elif w=a3 then n := k3;
   elif w=b1 then n := k4;
     elif w=b2 then n := k5;
      elif w=b3 then n := k6;
      elif w=b3^-1 then n := c-k6;
     elif w=b2^-1 then n := c-k5;
   elif w=b1^-1 then n := c-k4;
  elif w=a3^-1 then n := c-k3;
 elif w=a2^-1 then n := c-k2;
elif w=a1^-1 then n := c-k1;
fi;
return n;
end;

NL := function(z)
local n;
if z=1 then n := a1;
 elif z=2 then n := a2;
  elif z=3 then n := a3;
```

```
     elif z=4 then n := b1;
        elif z=5 then n := b2;
          elif z=6 then n := b3;
          elif z=7 then n := b3^-1;
        elif z=8 then n := b2^-1;
     elif z=9 then n := b1^-1;
   elif z=10 then n := a3^-1;
  elif z=11 then n := a2^-1;
elif z=12 then n := a1^-1;
fi;
return n;
end;

permute := function(A,B)
local i1, i2, i3, j1, j2, j3, k, PL, L, cA, cB, c;
PL := [ ];
L := relation(A,B);
cA := DimensionsMat(A)[1]; cB := DimensionsMat(A)[2];
c := cA + cB;
for i1 in [1..c] do
  for i2 in Difference([1..c], [i1, c+1-i1]) do
    for i3 in Difference([1..c],
         [i1, c+1-i1, i2, c+1-i2]) do
     for j1 in Difference([1..c],
          [i1, c+1-i1, i2, c+1-i2, i3, c+1-i3]) do
        for j2 in Difference([1..c],
             [i1, c+1-i1, i2, c+1-i2,
             i3, c+1-i3, j1, c+1-j1]) do
          for j3 in Difference([1..c],
              [i1, c+1-i1, i2, c+1-i2, i3, c+1-i3,
                         j1, c+1-j1, j2, c+1-j2]) do
           for k in [1..Size(L)] do
            PL[k] :=
     NL(LN(Subword(L[k],1,1),i1,i2,i3,j1,j2,j3,c+1))*
     NL(LN(Subword(L[k],2,2),i1,i2,i3,j1,j2,j3,c+1))*
     NL(LN(Subword(L[k],3,3),i1,i2,i3,j1,j2,j3,c+1))*
     NL(LN(Subword(L[k],4,4),i1,i2,i3,j1,j2,j3,c+1));
           od;
           if Set(PL) = Set(L) then
            Print(NL(i1)," ",NL(i2)," ",NL(i3)," ",
                 NL(j1)," ",NL(j2)," ",NL(j3)," ","\n");
```

```
          fi;
        od;
      od;
    od;
   od;
  od;
od;
end;
```

For X as in Example 2.2, i.e. for

$$A = \begin{pmatrix} 1 & 1 & 5 & 3 & 1 & 1 \\ 3 & 3 & 3 & 4 & 6 & 3 \\ 2 & 5 & 1 & 2 & 2 & 2 \\ 5 & 6 & 2 & 5 & 5 & 5 \\ 4 & 4 & 4 & 1 & 3 & 4 \\ 6 & 2 & 6 & 6 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 2 & 5 & 4 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \\ 6 & 3 & 2 & 1 & 4 & 5 \\ 4 & 3 & 2 & 5 & 6 & 1 \\ 6 & 1 & 2 & 5 & 4 & 3 \\ 1 & 3 & 2 & 5 & 4 & 6 \end{pmatrix}$$

we get (cf. Theorem 2.3(9))

```
permute(A,B);
a1 a2 a3 b1 b2 b3
a1^-1 a2^-1 a3^-1 b1^-1 b3 b2
```

B.8 A quaternion lattice program

We illustrate the construction of the group $\Gamma_{p,l}$ of Chapter 3 for the smallest example $p = 3, l = 5$ (Example 3.46).

```
psi := function(v,x0,x1,x2,x3)
  return[[x0 + v*x1*E(4), v*x2 + v*x3*E(4)],
         [-v*x2 + v*x3*E(4), x0 - v*x1*E(4)]];
end;
# v = -1 gives the conjugate of x
# E(4)^2 = -1

a := [ ]; b := [ ];

a[1] := psi(1,1,0,1,1);    # ψ(1+j+k)
a[2] := psi(1,1,0,1,-1);   # ψ(1+j-k)
a[3] := psi(-1,1,0,1,-1);  # ψ(1-j+k)
a[4] := psi(-1,1,0,1,1);   # ψ(1-j-k)
```

```
b[1] := psi(1,1,2,0,0);    # ψ(1+2i)
b[2] := psi(1,1,0,2,0);    # ψ(1+2j)
b[3] := psi(1,1,0,0,2);    # ψ(1+2k)
b[4] := psi(-1,1,0,0,2);   # ψ(1−2k)
b[5] := psi(-1,1,0,2,0);   # ψ(1−2j)
b[6] := psi(-1,1,2,0,0);   # ψ(1−2i)

qAB := function(p,l)
local i, j, k, m, A, B;
A := NullMat(p+1,l+1);
B := NullMat(p+1,l+1);
for i in [1..p+1] do
 for j in [1..l+1] do
  for k in [1..l+1] do
   for m in [1..p+1] do
     if a[i]*b[j] = b[k]*a[m] or
        a[i]*b[j] = -b[k]*a[m] then
      A[i][j] := m;
      B[i][j] := k;
     fi;
   od;
  od;
 od;
od;
return([A,B]);
end;

A := qAB(3,5)[1];
B := qAB(3,5)[2];
```

gives

$$A = \begin{pmatrix} 3 & 3 & 2 & 4 & 4 & 2 \\ 1 & 4 & 3 & 1 & 3 & 4 \\ 4 & 2 & 4 & 2 & 1 & 1 \\ 2 & 1 & 1 & 3 & 2 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 5 & 1 & 6 & 2 & 3 & 4 \\ 3 & 6 & 2 & 1 & 4 & 5 \\ 4 & 3 & 1 & 5 & 6 & 2 \\ 2 & 4 & 5 & 6 & 1 & 3 \end{pmatrix}.$$

Appendix C

Some lists

C.1 Primitive permutation groups

We give a list of all primitive permutation groups $G < S_{2n}$, where $n \leq 7$, including some information about the groups like its order $|G|$ or its transitivity on $\{1, \ldots, 2n\}$. A comprehensive introduction to permutation groups, including the definitions of the groups in Table C.1, is given in [25]. See also [13] for a list of all finite primitive permutation groups up to degree 50.

Group G	degree $2n$	transitivity(G)	order $\|G\|$	$G < A_{2n}$
S_2	2	2	2	N
A_4	4	2	12	Y
S_4	4	4	24	N
$\mathrm{PSL}_2(5)$	6	2	60	Y
$\mathrm{PGL}_2(5)$	6	3	120	N
A_6	6	4	360	Y
S_6	6	6	720	N
$\mathrm{AGL}_1(8)$	8	2	56	Y
$\mathrm{A\Gamma L}_1(8)$	8	2	168	Y
$\mathrm{PSL}_2(7)$	8	2	168	Y
$\mathrm{PGL}_2(7)$	8	3	336	N
$\mathrm{ASL}_3(2)$	8	3	1344	Y
A_8	8	6	20160	Y
S_8	8	8	40320	N
A_5	10	1	60	Y
S_5	10	1	120	N
$\mathrm{PSL}_2(9)$	10	2	360	Y
S_6	10	2	720	N
$\mathrm{PGL}_2(9)$	10	3	720	N

M_{10}	10	3	720	Y
$\mathrm{P\Gamma L}_2(9)$	10	3	1440	N
A_{10}	10	8	1814400	Y
S_{10}	10	10	3628800	N
$\mathrm{PSL}_2(11)$	12	2	660	Y
$\mathrm{PGL}_2(11)$	12	3	1320	N
M_{11}	12	3	7920	Y
M_{12}	12	5	95040	Y
A_{12}	12	10	239500800	Y
S_{12}	12	12	479001600	N
$\mathrm{PSL}_2(13)$	14	2	1092	Y
$\mathrm{PGL}_2(13)$	14	3	2184	N
A_{14}	14	12	43589145600	Y
S_{14}	14	14	87178291200	N

Table C.1: Primitive permutation groups

C.2 Quasi-primitive permutation groups

See Table C.2 for all quasi-primitive, but not 2-transitive subgroups of S_{2n}, where $n \leq 8$. Only two of them are not primitive. For the primitive groups, we have used the list in [13] and their notations, in particular the symbol ":" to denote a split extension.

Group G	degree $2n$	primitive	order $\lvert G \rvert$	$G < A_{2n}$
A_5	10	Y	60	Y
S_5	10	Y	120	N
$\mathrm{PSL}_2(5)$	12	N	60	Y
$\mathrm{PSL}_2(7)$	14	N	168	Y
$2^4 : 5$	16	Y	80	Y
$2^4 : D_5$	16	Y	160	Y
$(A_4 \times A_4) : 2$	16	Y	288	Y
$(2^4 : 5) : 4$	16	Y	320	Y
$2^4 : 3^2 : 4$	16	Y	576	Y
$2^4 : S_3 \times S_3$	16	Y	576	Y
$2^4 : A_5$	16	Y	960	Y
$(S_4 \times S_4) : 2$	16	Y	1152	Y
$2^4 : S_5$	16	Y	1920	Y

Table C.2: Quasi-primitive permutation groups

C.3 Locally 2-transitive (6, 6)–groups

We study (6, 6)–groups such that P_h, P_v are 2-transitive and give a complete list of the arising 4-tuples $(|P_h|, |P_v|, |P_h^{(2)}|, |P_v^{(2)}|)$. Without loss of generality, we may assume that $|P_h| \leq |P_v|$ and that $|P_h^{(2)}| \leq |P_v^{(2)}|$ if $|P_h| = |P_v|$. By Table C.1, there are only four 2-transitive subgroups of S_6: $\mathrm{PSL}_2(5)$, $\mathrm{PGL}_2(5)$, A_6 and S_6 of order 60, 120, 360 and 720, respectively. Given $P_\bullet \in \{P_h, P_v\}$, the maximal possible value for $|P_\bullet^{(2)}|$ is $|P_\bullet|(|P_\bullet|/6)^6$. If this maximum is attained, the value of $|P_\bullet^{(2)}|$ is marked in the list with the symbol "$*$" on the right hand side. Observe that in the case $P_\bullet = A_6$ the number $|P_\bullet^{(2)}|$ is always maximal (this is not very surprising by [16, Proposition 3.3.1]).

$\|P_h\|$	$\|P_v\|$	$\|P_h^{(2)}\|$	$\|P_v^{(2)}\|$
60	60	937500	937500
60	60	937500	60000000 $*$
60	120	7500	15000
60	120	937500	60000000
60	120	937500	120000000
60	120	937500	1920000000
60	120	30000000	1875000
60	120	30000000	60000000
60	120	30000000	1920000000
60	120	60000000 $*$	60000000
60	120	60000000 $*$	120000000
60	120	60000000 $*$	7680000000 $*$
60	360	937500	16796160000000 $*$
60	360	30000000	16796160000000 $*$
60	360	60000000 $*$	16796160000000 $*$
60	720	7500	1074954240000000
60	720	937500	33592320000000
60	720	937500	1074954240000000
60	720	937500	2149908480000000 $*$
60	720	1875000	1074954240000000
60	720	30000000	33592320000000
60	720	30000000	1074954240000000
60	720	30000000	2149908480000000 $*$
60	720	60000000 $*$	33592320000000
60	720	60000000 $*$	67184640000000
60	720	60000000 $*$	1074954240000000
60	720	60000000 $*$	2149908480000000 $*$

120	120	15000		15000	
120	120	1875000		60000000	
120	120	60000000		60000000	
120	120	60000000		1920000000	
120	120	60000000		3840000000	
120	120	1920000000		1920000000	
120	120	1920000000		7680000000	*
120	120	3840000000		7680000000	*
120	360	1875000		16796160000000	*
120	360	60000000		16796160000000	*
120	360	120000000		16796160000000	*
120	360	1920000000		16796160000000	*
120	360	3840000000		16796160000000	*
120	360	7680000000	*	16796160000000	*
120	720	1875000		33592320000000	
120	720	1875000		1074954240000000	
120	720	60000000		33592320000000	
120	720	60000000		67184640000000	
120	720	60000000		1074954240000000	
120	720	60000000		2149908480000000	*
120	720	120000000		33592320000000	
120	720	120000000		1074954240000000	
120	720	120000000		2149908480000000	*
120	720	1920000000		33592320000000	
120	720	1920000000		67184640000000	
120	720	1920000000		1074954240000000	
120	720	1920000000		2149908480000000	*
120	720	3840000000		33592320000000	
120	720	3840000000		67184640000000	
120	720	3840000000		1074954240000000	
120	720	3840000000		2149908480000000	*
120	720	7680000000	*	33592320000000	
120	720	7680000000	*	1074954240000000	
120	720	7680000000	*	2149908480000000	*
360	360	16796160000000	*	16796160000000	*
360	720	16796160000000	*	33592320000000	
360	720	16796160000000	*	67184640000000	
360	720	16796160000000	*	1074954240000000	
360	720	16796160000000	*	2149908480000000	*

720	720	33592320000000	3359232000000
720	720	33592320000000	67184640000000
720	720	33592320000000	1074954240000000
720	720	33592320000000	2149908480000000 *
720	720	67184640000000	1074954240000000
720	720	67184640000000	2149908480000000 *
720	720	1074954240000000	1074954240000000
720	720	1074954240000000	2149908480000000 *
720	720	2149908480000000 *	2149908480000000 *

Table C.3: Local groups in locally 2-transitive (6, 6)–groups

C.4 List of (4, 4)–groups

In the list below, we classify all (4, 4)–groups by the permutation isomorphism types of the local groups P_h and P_v, and by Γ^{ab} (up to interchanging the role of P_h and P_v). In total, we get 32 different types. Note that there are in fact at least 41 and at most 43 non-isomorphic (4, 4)–groups (see [41, Section 7]).

We use the following notation in Table C.4:
2_1: group of order 2, permutation isomorphic to $\langle(1, 2)\rangle < S_4$,
2_2: group of order 2, permutation isomorphic to $\langle(1, 2)(3, 4)\rangle$,
4_1: group of order 4, isomorphic to $\mathbb{Z}_2^2$, permutation isomorphic to $\langle(1, 2), (3, 4)\rangle$,
4_2: as above, but permutation isomorphic to $\langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle$.
trans($P_\bullet$) denotes the transitivity of the group $P_\bullet \in \{P_h, P_v\}$ on the set $\{1, 2, 3, 4\}$.
"N?" means that Γ is possibly irreducible.

P_h	P_v	trans(P_h)	trans(P_v)	reducible	Γ^{ab}
1	1	0	0	Y	$\mathbb{Z}^4$
1	2_1	0	0	Y	$\mathbb{Z}^3 \times \mathbb{Z}_2$
1	2_2	0	0	Y	$\mathbb{Z}^3$
1	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
1	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
1	4_1	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
1	4_2	0	1	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
1	D_4	0	1	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
2_1	2_1	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2^2$
2_1	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
2_1	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_4$

2_1	2_2	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2^3$
2_2	2_2	0	0	Y	$\mathbb{Z}^2 \times \mathbb{Z}_2$
2_2	2_2	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
2_1	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2^2$
2_2	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_8$
2_2	$\mathbb{Z}_4$	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2^2$
2_1	4_1	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2^3$
2_1	4_2	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
2_2	4_1	0	0	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
2_2	4_1	0	0	Y	$\mathbb{Z}^2$
2_2	4_2	0	1	Y	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$
2_1	D_4	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2^2$
2_1	D_4	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4$
2_2	A_4	0	2	Y	$\mathbb{Z} \times \mathbb{Z}_2$
$\mathbb{Z}_4$	$\mathbb{Z}_4$	1	1	Y	$\mathbb{Z}_4 \times \mathbb{Z}_8$
$\mathbb{Z}_4$	4_1	1	0	Y	$\mathbb{Z} \times \mathbb{Z}_4$
4_1	4_1	0	0	Y	$\mathbb{Z}_2^4$
4_1	D_4	0	1	Y	$\mathbb{Z} \times \mathbb{Z}_2$
4_1	D_4	0	1	Y	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
D_4	A_4	1	2	N?	$\mathbb{Z}_2 \times \mathbb{Z}_6$
S_4	S_4	4	4	N?	$\mathbb{Z}_6^2$

Table C.4: Properties of (4, 4)–groups

C.5 List of (4, 6)–groups

Similarly as in Section C.4, we give a certain classification of (4, 6)–groups, but here the groups P_h and P_v are classified only up to isomorphism (*not* up to permutation isomorphism) and up to their transitivity. Notation: "36" denotes the group of order 36 permutation isomorphic to $\langle(1, 2, 3), (1, 4, 2, 5)(3, 6)\rangle$ and "72" denotes the group of order 72 permutation isomorphic to the group $\langle(1, 2, 3), (1, 2), (1, 4)(2, 5)(3, 6)\rangle$. "Y?" means that we do not exclude the existence of a reducible example.

Example	P_h	P_v	trans(P_h)	trans(P_v)	reducible
	1	1	0	0	Y
	1	$\mathbb{Z}_2$	0	0	Y
	1	$\mathbb{Z}_3$	0	0	Y
	1	$\mathbb{Z}_4$	0	0	Y

	1	$\mathbb{Z}_2^2$	0	0	Y
	1	S_3	0	0	Y
	1	S_3	0	1	Y
	1	$\mathbb{Z}_6$	0	1	Y
	1	$\mathbb{Z}_2 \times \mathbb{Z}_4$	0	0	Y
	1	D_4	0	0	Y
	1	A_4	0	1	Y
	1	$\mathbb{Z}_2 \times S_3$	0	1	Y
	1	S_4	0	1	Y
	1	$\mathbb{Z}_2 \times A_4$	0	1	Y
	1	$\mathbb{Z}_2 \times S_4$	0	1	Y
	$\mathbb{Z}_2$	1	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_3$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	0	0	Y
	$\mathbb{Z}_2$	S_3	0	0	Y
	$\mathbb{Z}_2$	S_3	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_6$	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2$	D_4	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_3^2$	0	0	Y
	$\mathbb{Z}_2$	A_4	0	0	Y
	$\mathbb{Z}_2$	A_4	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times S_3$	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_3 \times S_3$	0	1	Y
	$\mathbb{Z}_2$	S_4	0	1	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times A_4$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times A_4$	0	1	Y
	$\mathbb{Z}_2$	36	0	1	Y
	$\mathbb{Z}_2$	$S_3 \times S_3$	0	0	Y
	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times S_4$	0	1	Y
	$\mathbb{Z}_2$	$\mathrm{PSL}_2(5)$	0	2	Y
	$\mathbb{Z}_2$	$\mathrm{PGL}_2(5)$	0	3	Y
	$\mathbb{Z}_2$	A_6	0	4	Y
	$\mathbb{Z}_2$	S_6	0	6	Y
	$\mathbb{Z}_4$	1	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_2$	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_4$	1	0	Y

	$\mathbb{Z}_4$	$\mathbb{Z}_2^2$	1	0	Y
	$\mathbb{Z}_4$	S_3	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	0	Y
	$\mathbb{Z}_4$	D_4	1	0	Y
	$\mathbb{Z}_4$	$\mathbb{Z}_3^2$	1	0	Y
	$\mathbb{Z}_4$	$S_3 \times S_3$	1	0	Y
	$\mathbb{Z}_2^2$	1	0	0	Y
	$\mathbb{Z}_2^2$	1	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_4$	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	1	0	Y
	$\mathbb{Z}_2^2$	S_3	0	0	Y, N?
	$\mathbb{Z}_2^2$	S_3	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	0	0	Y
	$\mathbb{Z}_2^2$	D_4	0	0	Y
	$\mathbb{Z}_2^2$	A_4	0	1	Y
	$\mathbb{Z}_2^2$	A_4	1	0	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times S_3$	0	1	Y, N?
	$\mathbb{Z}_2^2$	S_4	0	1	Y, N?
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times A_4$	0	1	Y
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times A_4$	1	0	Y
	$\mathbb{Z}_2^2$	36	0	1	N?
2.36	$\mathbb{Z}_2^2$	$S_3 \times S_3$	0	0	N?
	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times S_4$	0	1	Y, N?
	$\mathbb{Z}_2^2$	$\mathrm{PSL}_2(5)$	0	2	N?
	$\mathbb{Z}_2^2$	$\mathrm{PGL}_2(5)$	0	3	N?
	$\mathbb{Z}_2^2$	S_6	0	6	N
	D_4	1	1	0	Y
	D_4	$\mathbb{Z}_2$	1	0	Y
	D_4	$\mathbb{Z}_3$	1	0	Y
	D_4	$\mathbb{Z}_4$	1	0	Y
	D_4	$\mathbb{Z}_2^2$	1	0	Y
	D_4	S_3	1	0	Y, N?

	D_4	S_3	1	1	Y
	D_4	$\mathbb{Z}_6$	1	1	Y
	D_4	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	0	Y
	D_4	D_4	1	0	Y
	D_4	$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	0	N?
	D_4	A_4	1	0	Y, N?
	D_4	A_4	1	1	Y
	D_4	S_4	1	1	Y, N?
	D_4	$\mathbb{Z}_2 \times A_4$	1	0	Y, N?
	D_4	$\mathbb{Z}_2 \times A_4$	1	1	Y
	D_4	36	1	1	N?
	D_4	$S_3 \times S_3$	1	0	N?
	D_4	$\mathbb{Z}_2 \times S_4$	1	1	N?
	D_4	$\mathrm{PSL}_2(5)$	1	2	N?
	D_4	$\mathrm{PGL}_2(5)$	1	3	N, Y?
	D_4	A_6	1	4	N
	D_4	S_6	1	6	N
	A_4	$\mathbb{Z}_2$	2	0	Y
	A_4	$\mathbb{Z}_2^2$	2	0	Y
	A_4	S_3	2	0	N?
	A_4	D_4	2	0	N?
	A_4	$\mathbb{Z}_2 \times S_3$	2	1	N?
	A_4	S_4	2	1	N?
	A_4	36	2	1	N?
	A_4	$S_3 \times S_3$	2	0	N?
	A_4	$\mathbb{Z}_2 \times S_4$	2	1	N?
	A_4	S_6	2	6	N
	S_4	$\mathbb{Z}_2$	4	0	Y
	S_4	$\mathbb{Z}_4$	4	0	Y
	S_4	$\mathbb{Z}_2^2$	4	0	Y
	S_4	S_3	4	0	N, Y?
	S_4	$\mathbb{Z}_2 \times \mathbb{Z}_4$	4	0	Y
	S_4	D_4	4	0	Y, N?
	S_4	$\mathbb{Z}_3 \times \mathbb{Z}_3$	4	0	N?
	S_4	S_4	4	0	N?
	S_4	S_4	4	1	N, Y?
	S_4	$S_3 \times S_3$	4	0	N, Y?
	S_4	$\mathbb{Z}_2 \times S_4$	4	0	N?
	S_4	$\mathbb{Z}_2 \times S_4$	4	1	N, Y?

	S_4	$\mathrm{PSL}_2(5)$	4	2	N, Y?
	S_4	72	4	1	N?
3.46	S_4	$\mathrm{PGL}_2(5)$	4	3	N
	S_4	$\mathrm{PGL}_2(5)$	4	3	Y?
	S_4	A_6	4	4	N
	S_4	S_6	4	6	N

Table C.5: Properties of (4, 6)–groups

C.6 Some abelianized (A_{2m}, A_{2n})–groups

We classify some (A_{2m}, A_{2n})–groups Γ by their abelianization Γ^{ab} and by the size of $P_h^{(2)}$ and $P_v^{(2)}$ (we restrict to $2 \leq m \leq n$ and $m + n \leq 8$). If $P_h^{(2)}$ is not maximal (this can only happen if $2m = 4$), then we give the number $12 \cdot 3^4/|P_h^{(2)}|$. The list is complete for $(2m, 2n) = (6, 6)$ and $(2m, 2n) = (4, 8)$. There are no (A_4, A_4)– and (A_4, A_6)–groups.

Example	$2m$	$2n$	$P_h^{(2)}$ max.	$P_v^{(2)}$ max.	$\lvert\Gamma^{ab}\rvert$	Γ^{ab}
	4	8	Y	Y	4	$\mathbb{Z}_2^2$
	4	10	Y	Y	4	$\mathbb{Z}_2^2$
	4	10	3	Y	4	$\mathbb{Z}_2^2$
	4	10	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	10	3	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	10	Y	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	4	10	3	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	4	10	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	4	10	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	10	3	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	10	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$
	4	10	Y	Y	24	$\mathbb{Z}_2^2 \times \mathbb{Z}_6$
	4	10	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	4	12	Y	Y	4	$\mathbb{Z}_2^2$
	4	12	3	Y	4	$\mathbb{Z}_2^2$
	4	12	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	12	3	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	4	12	Y	Y	8	$\mathbb{Z}_2^3$
	4	12	3	Y	8	$\mathbb{Z}_2^3$

	4	12	Y	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	4	12	3	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	4	12	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	12	3	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	4	12	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	4	12	Y	Y	20	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$
	4	12	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$
	4	12	Y	Y	24	$\mathbb{Z}_2^2 \times \mathbb{Z}_6$
	4	12	Y	Y	28	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$
	4	12	Y	Y	32	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
	4	12	3	Y	32	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
	4	12	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	4	12	Y	Y	40	$\mathbb{Z}_2 \times \mathbb{Z}_{20}$
	4	12	Y	Y	40	$\mathbb{Z}_2^2 \times \mathbb{Z}_{10}$
	4	12	Y	Y	48	$\mathbb{Z}_2 \times \mathbb{Z}_{24}$
2.2	6	6	Y	Y	4	$\mathbb{Z}_2^2$
	6	6	Y	Y	8	$\mathbb{Z}_2^3$
	6	6	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	6	6	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	6	6	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$
	6	6	Y	Y	28	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$
2.15	6	6	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	6	8	Y	Y	4	$\mathbb{Z}_2^2$
	6	8	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	6	8	Y	Y	8	$\mathbb{Z}_2^3$
	6	8	Y	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	6	8	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	6	8	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	6	8	Y	Y	16	$\mathbb{Z}_2^4$
	6	8	Y	Y	20	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$
	6	8	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$
	6	8	Y	Y	24	$\mathbb{Z}_2^2 \times \mathbb{Z}_6$
	6	8	Y	Y	28	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$
	6	8	Y	Y	32	$\mathbb{Z}_2 \times \mathbb{Z}_{16}$
	6	8	Y	Y	32	$\mathbb{Z}_2^2 \times \mathbb{Z}_8$
	6	8	Y	Y	36	$\mathbb{Z}_2 \times \mathbb{Z}_{18}$
	6	8	Y	Y	40	$\mathbb{Z}_2 \times \mathbb{Z}_{20}$
	6	8	Y	Y	40	$\mathbb{Z}_2^2 \times \mathbb{Z}_{10}$

	6	8	Y	Y	48	$\mathbb{Z}_2^2 \times \mathbb{Z}_{12}$
	6	8	Y	Y	60	$\mathbb{Z}_2 \times \mathbb{Z}_{30}$
	6	8	Y	Y	80	$\mathbb{Z}_2^2 \times \mathbb{Z}_{20}$
	6	10	Y	Y	4	$\mathbb{Z}_2^2$
	6	10	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	6	10	Y	Y	8	$\mathbb{Z}_2^3$
	6	10	Y	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	6	10	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	6	10	Y	Y	16	$\mathbb{Z}_4^2$
	6	10	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	6	10	Y	Y	20	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$
	6	10	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$
	6	10	Y	Y	24	$\mathbb{Z}_2^2 \times \mathbb{Z}_6$
	6	10	Y	Y	28	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$
	6	10	Y	Y	40	$\mathbb{Z}_2 \times \mathbb{Z}_{20}$
	6	10	Y	Y	40	$\mathbb{Z}_2^2 \times \mathbb{Z}_{10}$
	6	10	Y	Y	108	$\mathbb{Z}_6 \times \mathbb{Z}_{18}$
	8	8	Y	Y	4	$\mathbb{Z}_2^2$
	8	8	Y	Y	8	$\mathbb{Z}_2 \times \mathbb{Z}_4$
	8	8	Y	Y	8	$\mathbb{Z}_2^3$
	8	8	Y	Y	12	$\mathbb{Z}_2 \times \mathbb{Z}_6$
	8	8	Y	Y	16	$\mathbb{Z}_2 \times \mathbb{Z}_8$
	8	8	Y	Y	16	$\mathbb{Z}_4^2$
	8	8	Y	Y	16	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$
	8	8	Y	Y	16	$\mathbb{Z}_2^4$
	8	8	Y	Y	20	$\mathbb{Z}_2 \times \mathbb{Z}_{10}$
	8	8	Y	Y	24	$\mathbb{Z}_2 \times \mathbb{Z}_{12}$
	8	8	Y	Y	24	$\mathbb{Z}_2^2 \times \mathbb{Z}_6$
	8	8	Y	Y	28	$\mathbb{Z}_2 \times \mathbb{Z}_{14}$

Table C.6: Abelianized (A_{2m}, A_{2n})–groups

C.7 More embeddings of Example 2.39

We embed the non-residually finite $(8, 6)$–complex of Example 2.39 into many different $(10, 10)$–complexes X such that P_h and P_v are primitive permutation groups. Let $w := a_2 a_1^{-1} a_3 a_4^{-1}$. In all examples Γ in the subsequent list, the normal subgroup $\langle\langle w \rangle\rangle_\Gamma$ has finite index in Γ, in particular, by Lemma 2.42,

$$\langle\langle w \rangle\rangle_\Gamma = \bigcap_{N \overset{\text{f.i.}}{\lhd} \Gamma} N.$$

If two rows are exactly the same, then the quotients $\Gamma / \langle\langle w \rangle\rangle_\Gamma$ are non-isomorphic non-abelian groups of the same finite order. The (A_{10}, A_{10})–groups are precisely those of Table 2.7.

P_h	P_v	abelianization Γ^{ab}	$\lvert\Gamma^{ab}\rvert$ and $[\Gamma : \langle\langle w \rangle\rangle_\Gamma]$
$S_6 < S_{10}$	A_{10}	[2, 2]	4
$S_6 < S_{10}$	S_{10}	[2, 2]	4
$\mathrm{P\Gamma L}_2(9)$	A_{10}	[2, 2]	4
$\mathrm{P\Gamma L}_2(9)$	S_{10}	[2, 2]	4
$\mathrm{P\Gamma L}_2(9)$	S_{10}	[2, 4]	8
$\mathrm{P\Gamma L}_2(9)$	S_{10}	[2, 2, 2]	8
A_{10}	A_{10}	[2, 2]	4
A_{10}	A_{10}	[2, 4]	8
A_{10}	A_{10}	[2, 2, 2]	8
A_{10}	A_{10}	[2, 6]	12
A_{10}	A_{10}	[2, 2, 4]	16
A_{10}	A_{10}	[2, 8]	16
A_{10}	A_{10}	[2, 10]	20
A_{10}	A_{10}	[2, 12]	24
A_{10}	A_{10}	[2, 2, 6]	24
A_{10}	A_{10}	[2, 2, 8]	32
A_{10}	A_{10}	[2, 20]	40
A_{10}	S_{10}	[2, 2]	4
A_{10}	S_{10}	[2, 4]	8
A_{10}	S_{10}	[2, 2, 2]	8
A_{10}	S_{10}	[2, 2, 2]	8, 16
A_{10}	S_{10}	[2, 6]	12
A_{10}	S_{10}	[2, 8]	16
A_{10}	S_{10}	[4, 4]	16
A_{10}	S_{10}	[2, 2, 4]	16

A_{10}	S_{10}	[2, 10]	20
A_{10}	S_{10}	[2, 12]	24
A_{10}	S_{10}	[2, 2, 6]	24
A_{10}	S_{10}	[2, 14]	28
A_{10}	S_{10}	[2, 2, 8]	32
A_{10}	S_{10}	[2, 16]	32
A_{10}	S_{10}	[2, 20]	40
A_{10}	S_{10}	[2, 2, 10]	40
A_{10}	S_{10}	[2, 24]	48
S_{10}	A_{10}	[2, 2]	4
S_{10}	A_{10}	[2, 4]	8
S_{10}	A_{10}	[2, 2, 2]	8
S_{10}	A_{10}	[2, 2, 2]	8, 16
S_{10}	A_{10}	[2, 2, 2]	8, 16
S_{10}	A_{10}	[2, 6]	12
S_{10}	A_{10}	[2, 2, 4]	16
S_{10}	A_{10}	[2, 8]	16
S_{10}	A_{10}	[4, 4]	16
S_{10}	A_{10}	[2, 10]	20
S_{10}	A_{10}	[2, 12]	24
S_{10}	A_{10}	[2, 2, 6]	24
S_{10}	A_{10}	[2, 14]	28
S_{10}	A_{10}	[2, 2, 8]	32
S_{10}	A_{10}	[2, 18]	36
S_{10}	A_{10}	[6, 6]	36
S_{10}	A_{10}	[2, 20]	40
S_{10}	A_{10}	[2, 22]	44
S_{10}	A_{10}	[2, 28]	56
S_{10}	A_{10}	[2, 32]	64
S_{10}	S_{10}	[2, 2]	4
S_{10}	S_{10}	[2, 4]	8
S_{10}	S_{10}	[2, 2, 2]	8
S_{10}	S_{10}	[2, 2, 2]	8, 16
S_{10}	S_{10}	[2, 2, 2]	8, 16
S_{10}	S_{10}	[2, 6]	12
S_{10}	S_{10}	[2, 8]	16
S_{10}	S_{10}	[2, 2, 4]	16
S_{10}	S_{10}	[2, 2, 4]	16, 32
S_{10}	S_{10}	[2, 2, 4]	16, 32

S_{10}	S_{10}	[2, 2, 4]	16, 32
S_{10}	S_{10}	[4, 4]	16
S_{10}	S_{10}	[2, 10]	20
S_{10}	S_{10}	[2, 12]	24
S_{10}	S_{10}	[2, 2, 6]	24
S_{10}	S_{10}	[2, 2, 6]	24, 48
S_{10}	S_{10}	[2, 14]	28
S_{10}	S_{10}	[2, 16]	32
S_{10}	S_{10}	[2, 2, 8]	32
S_{10}	S_{10}	[2, 4, 4]	32
S_{10}	S_{10}	[4, 8]	32
S_{10}	S_{10}	[2, 18]	36
S_{10}	S_{10}	[6, 6]	36
S_{10}	S_{10}	[2, 20]	40
S_{10}	S_{10}	[2, 2, 10]	40
S_{10}	S_{10}	[2, 22]	44
S_{10}	S_{10}	[2, 24]	48
S_{10}	S_{10}	[2, 2, 12]	48
S_{10}	S_{10}	[2, 26]	52
S_{10}	S_{10}	[2, 28]	56
S_{10}	S_{10}	[2, 30]	60
S_{10}	S_{10}	[2, 32]	64
S_{10}	S_{10}	[2, 36]	72
S_{10}	S_{10}	[2, 38]	76
S_{10}	S_{10}	[2, 40]	80
S_{10}	S_{10}	[2, 44]	88
S_{10}	S_{10}	[2, 50]	100
S_{10}	S_{10}	[10, 10]	100
S_{10}	S_{10}	[2, 52]	104

Table C.7: Example 2.39 embedded into (10, 10)–groups

Appendix D

Miscellanea

D.1 History of simple groups and free amalgams

We give in this section some history of finitely presented (or finitely generated) infinite simple groups and amalgams of finitely generated non-abelian free groups.

- **Aleksandr G. Kuroš 1944** ([42]) He asked for the existence of a finitely generated infinite simple group. (This was positively answered in [34].)

- **Graham Higman 1951** ([34]) He gave the first existence proof of a finitely generated infinite simple group and asked for the existence of a finitely *presented* infinite simple group: "Can an infinite simple group have not only a finite set of generators, but also a finite set of defining relations?" (This was positively answered by Richard J. Thompson in 1965.)

- **Ruth Camm 1953** ([19]) She constructed uncountably many finitely generated infinite simple groups of the form $F_2 *_{F_\infty} F_2$. These groups are torsion-free, 2-generated, but not finitely presentable (by [4]).

- **Richard J. Thompson 1965** (in unpublished notes) He defined two finitely presented infinite simple groups $\widehat{\mathfrak{C}}$ (often called T) and $\widehat{V}$ (often called V). They are not torsion-free. He also defined a third interesting group $\widehat{\mathbb{P}}$ (often called F) which is torsion-free but not simple. For an introduction to these three groups, see [20].

- **Peter M. Neumann 1973** ([56]) "At one time I had hoped that one might construct a finitely presented simple group as a generalised free product of two free groups A, B of finite rank amalgamating finitely generated subgroups H and K. Joan Landman-Dyer and I showed quite easily that if H has infinite index in A or K has infinite index in B then such a group G is not simple." For a proof that G is even SQ-universal under these conditions, see [62, Corollary 2]. For an alternative proof that G is not simple (again provided $[A : H]$ or $[B : K]$ is infinite), see [37, Corollary 2]. Then Neumann posed the following problems (which appeared also in the Kourovka notebook): "Let $G = A *_{H=K} B$ where A, B are non-abelian free groups of finite rank and $|A : H|$, $|B : K|$ are finite. (a) Can it happen that G is simple? (b) Is G always SQ-universal?" ((a) was positively answered in [15]; consequently the answer to (b) is "no".)

- **Graham Higman 1974** ([35]) He generalized Thompson's group V to an infinite family of finitely presented infinite simple groups.

- **Dragomir Ž. Djoković 1981** ([26]) His finitely presented "simple" group with bounded torsion turned out to be *not* simple.

- **Elisabeth A. Scott 1984** ([63]) She constructed another family of finitely presented infinite simple groups, related to the Higman groups.

- **Kenneth S. Brown 1985** ([11]) He generalized the Thompson groups T, V and established some finiteness properties. In 1989 ([12]), he showed that Thompson's group V can be written as a ("positively curved, realizable") triangle of groups with finite vertex groups S_5, S_6, S_7.

- **Meenaxi Bhattacharjee 1994** ([7]) She gave a construction of an amalgam $F_3 *_{F_{13}} F_3$ without non-trivial finite quotients. This group is "nearly simple" in her terminology, but it is not known whether it has proper infinite quotients, or it is simple. More examples like this appear in [7, 8].

- **Geoffrey Mess** (in [57, Problem 5.11 (C)] **1995**) "Let X be a finite aspherical complex. Is there an example of an X with simple fundamental group?" (His question was positively answered in [15].)

- **Daniel T. Wise 1996** ([68]) He constructed a square complex without a non-trivial finite covering and asked: "Does there exist a CSC with (non-trivial) simple π_1? I guess that one does exist." (where CSC stands for complete squared complex; any $(2m, 2n)$–complex is CSC). (Again, this was positively answered in [15].)

- **Marc Burger, Shahar Mozes 1997** ([15]) They constructed an infinite family of finitely presented torsion-free simple groups which are amalgams of finitely generated non-abelian free groups and thereby solved many open problems mentioned above (Neumann, Mess, Wise).

- **Claas E. Röver 1999** ([61]) He gave a construction of finitely presented infinite simple groups that contain Grigorchuk groups.

D.2 Topology of Aut($\mathcal{T}_\ell$)

Throughout this section, let $\mathcal{T}_\ell$ be the ℓ-regular tree and $G = \mathrm{Aut}(\mathcal{T}_\ell)$ its group of automorphisms. We denote by X the countable vertex set of $\mathcal{T}_\ell$ endowed with the discrete topology. Let $X = \{x_1, x_2, \ldots\}$ be a fixed enumeration of X. For subsets $V, W \subseteq X$ and elements $x, v, w \in X$, we define $G_{V,W} := \{g \in G : g(V) \subseteq W\}$, the vertex stabilizer $G_x := G_{\{x\},\{x\}}$, the pointwise stabilizer $G_V := \cap_{x\in V} G_x$ and to simplify the notation we write $G_{v,W} := G_{\{v\},W}$, $G_{v,w} := G_{\{v\},\{w\}}$. We take the product topology on $\prod_{x\in X} X \cong X^X = \{f : X \to X\}$ and let $\mathcal{O}$ be the relative topology for $G \subset X^X$. Let $\pi_i : \prod_{x\in X} X \to X$ be the i-th projection. The product topology guarantees that these maps are continuous. Again, by definition of the product topology, a subbase for $\mathcal{O}$ is given by the sets $G_{v,W}$, where $v \in V \subseteq X$ and $W \subseteq X$. Since $G_{v,W} = \cup_{w\in W} G_{v,w}$, the family of sets $G_{v,w}$, where $v, w \in X$, is another subbase for $\mathcal{O}$. This topology $\mathcal{O}$ is sometimes called *topology of pointwise convergence* (or *topology of simple convergence*), since a sequence $(g_n)_{n\in\mathbb{N}}$ in G converges to $g \in G$ if and only if $(g_n(x))$ converges to $g(x)$ in X for all $x \in X$. Since X carries the discrete topology, this means that for each $x \in X$, there is an integer m such that $g_n(x) = g(x)$ if $n \geq m$. Note that $\mathcal{O}$ is the *compact open topology*, since this has as subbase the sets $G_{V,W}$, where $V \subset X$ is finite, $W \subseteq X$, and since

$$G_{V,W} = \bigcap_{i=1}^{n} \bigcup_{w\in W} G_{v_i,w} \,,$$

where $V = \{v_1, \ldots, v_n\}$.

Proposition D.1. *$(G, \mathcal{O})$ is a locally compact, totally disconnected, second countable, metrizable Hausdorff space. Moreover, it is a topological group, where we take the usual composition of elements in the group G.*

Proof. **Hausdorff:** The space X^X is Hausdorff as a product of Hausdorff spaces (see [39, Theorem III.5]), hence also its subspace G is Hausdorff.

Second countable: This follows immediately since X is countable and the set $\{G_{v,w} : v, w \in X\}$ is a subbase for $\mathcal{O}$.

Metrizable: Let ρ be the discrete metric on X, i.e. $\rho(v, w) := 0$ if $v = w$ and $\rho(v, w) := 1$ if $v \neq w$. We define for $g, h \in G$

$$d(g, h) := \sum_{i=1}^{\infty} \rho(g(x_i), h(x_i)) \,.$$

Then d is a metric on G which induces $\mathcal{O}$ (see [18, Theorem 6.20]).

Locally compact: Let $v, w \in X$. If we can show that $G_{v,w}$ is compact, then any $g \in G$ has a compact neighbourhood. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $G_{v,w}$. By the local finiteness of $\mathcal{T}_\ell$, the set $\{g_n(x_i) : n \in \mathbb{N}\}$ is finite for each $i \in \mathbb{N}$. Therefore, there is an infinite subset $N_1 \subseteq \mathbb{N}$ such that the vertices $g_{n_1}(x_1)$ coincide for all $n_1 \in N_1$. Denote this common vertex by $g(x_1)$. Next, choose an infinite subset $N_2 \subseteq N_1$, such that $g_{n_2}(x_2)$ coincide for all $n_2 \in N_2$ and define $g(x_2) := g_{n_2}(x_2)$ $(n_2 \in N_2)$. Continuing this process $(i = 3, 4, \ldots)$ defines an element $g \in G_{v,w}$. By construction, g is a cluster point of $(g_n)_{n \in \mathbb{N}}$. This shows that $G_{v,w}$ is countably compact. But in a metric space, the notions of countably compactness and compactness are equivalent.

Note that G_x is a profinite group (see [21, Proposition 1.3.5]). Recall that a topological group is profinite if and only if it is compact and totally disconnected.

Observe that X^X is *not* locally compact (this follows from [39, Theorem V.19]).

Separable: A metric space is separable if and only if it has a countable base (see [18, Corollary 7.21]).

Totally disconnected: We show that X^X is totally disconnected. Assume that $K \subset X^X$ is a connected subset such that $k_1, k_2 \in K$. Since the projections π_i are continuous, each image $\pi_i(K)$ is connected in X, i.e. a point. Thus $\pi_i(k_1) = \pi_i(k_2)$ for each i and therefore $k_1 = k_2$. G is totally disconnected as a subspace of X^X.

Topological group: Let $\mathcal{U}$ be the family of sets G_V, where V runs over *finite* subsets of X. Note that $G_V = \cap_{v \in V} G_{v,v}$ is open in G. We first show that

$$\mathcal{B}_1 := \{gU : g \in G,\ U \in \mathcal{U}\}$$

is a base for some topology $\tilde{\mathcal{O}}$ on G such that $(G, \tilde{\mathcal{O}})$ (with the usual composition in the group G) is a topological group and then show that $\tilde{\mathcal{O}} = \mathcal{O}$.
The *sub*base $\mathcal{B}_1 = \{gU : g \in G,\ U \in \mathcal{U}\}$ generates a topology $\tilde{\mathcal{O}}$ on G, in particular, the family $\mathcal{B}_2$ of finite intersections of elements in $\mathcal{B}_1$ is a base for $\tilde{\mathcal{O}}$. Obviously, we have $\mathcal{B}_1 \subseteq \mathcal{B}_2$. If we can prove $\mathcal{B}_2 \subseteq \mathcal{B}_1$, then $\mathcal{B}_1$ is a base for $\tilde{\mathcal{O}}$ as claimed. Let

$$B_2 = \bigcap_{i=1}^{n} g_i U_i \quad (g_i \in G, U_i \in \mathcal{U})$$

be any element in $\mathcal{B}_2$ and let $h \in B_2$. Then $g_i^{-1}h \in U_i$ for each $i = 1, \ldots, n$ and therefore $g_i^{-1}hU_i = U_i$ for each $i = 1, \ldots, n$, using that $U_i = G_{V_i}$ for some finite $V_i \subset X$. Thus,

$$B_2 = \bigcap_{i=1}^{n} hU_i = h\Big(\bigcap_{i=1}^{n} U_i\Big) \in \mathcal{B}_1\,,$$

since $\cap_{i=1}^{n} U_i \in \mathcal{U}$. Recall that the map

$$\begin{aligned} \phi : G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$

is continuous if for each $(g_1, g_2) \in G \times G$ and each open neighbourhood $\hat{U}$ of $g_1 g_2$ in G there is an open neighbourhood $\hat{V}$ of (g_1, g_2) in $G \times G$ such that $\phi(\hat{V}) \subset \hat{U}$.

So let $(g_1, g_2) \in G \times G$ and let $\hat{U} = \cup h_\iota U_\iota$ ($h_\iota \in G$, $U_\iota \in \mathcal{U}$) be an open neighbourhood of $g_1 g_2$ in G, say $g_1 g_2 = h_j u_j \in h_j U_j \subset \hat{U}$ with $U_j = G_{V_j}$. Then $g_2^{-1} G_{g_2(V_j)} g_2 U_j \subset U_j$. It follows that

$$\left(g_1 G_{g_2(V_j)}\right)\left(g_2 U_j\right) \subset g_1 g_2 U_j = h_j u_j U_j = h_j U_j \subset \hat{U}\,.$$

Since $g_1 G_{g_2(V_j)} \times g_2 U_j$ is an open neighbourhood of (g_1, g_2) in $G \times G$, we conclude that ϕ is continuous.

The proof of the continuity of the map $G \to G, g \mapsto g^{-1}$ is similar. We have to show that for each $g \in G$ and each open neighbourhood $\hat{U}$ of g^{-1} there is an open neighbourhood $\hat{V}$ of g such that $\hat{V}^{-1} \subset U$:

Let $g \in G$ and let $\hat{U} = \cup h_\iota U_\iota$ ($h_\iota \in G$, $U_\iota \in \mathcal{U}$) be an open neighbourhood of g^{-1}, say $g^{-1} = h_j u_j \in h_j U_j \subset \hat{U}$ with $U_j = G_{V_j}$ and define $\hat{V} = G_{g^{-1}(V_j)} \in \mathcal{U}$. Then $g\hat{V}^{-1} g^{-1} \subset U_j$ and

$$\left(g\hat{V}\right)^{-1} \subset g^{-1} U_j = h_j u_j U_j = h_j U_j \subset \hat{U}\,.$$

Since $g\hat{V}$ is an open neighbourhood of g, the map $g \mapsto g^{-1}$ is continuous and $(G, \tilde{\mathcal{O}})$ is a topological group.

We know that $\{G_{v,w} : v, w \in X\}$ is a subbase for $\mathcal{O}$ and

$$\{gU : g \in G,\ U = G_V,\ V \subset X \text{ finite}\}$$

is a subbase for $\tilde{\mathcal{O}}$. In fact, $\mathcal{O} = \tilde{\mathcal{O}}$, because on one hand $G_{v,w} = gG_v$ for any $g \in G$ such that $g(v) = w$, and on the other hand

$$gG_V = \bigcap_{v \in V} G_{v,g(v)}\,.$$

$\square$

Proposition D.2. *Let Γ be a subgroup of G and define $\Gamma_x := \Gamma \cap G_x$. Then the following three statements are equivalent:*

i) Γ is discrete.

ii) Γ_x is finite for all $x \in X$.

iii) Γ_x is finite for some $x \in X$.

Proof. i) $\Rightarrow$ ii): A discrete subgroup H of a Hausdorff topological group G is closed in G (see [33, Theorem 5.10]). Applying this theorem, the group Γ is closed in G and $\Gamma_x = \Gamma \cap G_x$ is closed in G_x, hence compact (since G_x is compact). But Γ_x is also discrete (being a subgroup of Γ), thus finite.

ii) $\Rightarrow$ iii): This is obvious.

iii) $\Rightarrow$ i): Write $\Gamma_x = \{\gamma_1, \ldots, \gamma_n\}$. For any $\gamma_i \in \Gamma_x \setminus \{1\}$ there is some (large) integer m_i such that $\gamma_i \notin \Gamma \cap G_{S(x,m_i)}$. Let m be the maximum of the m_i's, then $\Gamma \cap G_{S(x,m)} = \{1\}$. Since $G_{S(x,m)}$ is open in G, $\{1\}$ is open in Γ, and Γ is discrete ($\{\gamma\} = \{\gamma\}\{1\}$ is open in Γ). $\square$

Remark. By Proposition D.2, the full group G is not discrete if $\ell \geq 3$, in particular $\{g\}$ is not open in G. However, $\{g\}$ is closed in G, since

$$\{g\} = G \setminus \bigcup_{i \in \mathbb{N}} G_{x_i, X \setminus \{g(x_i)\}} .$$

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