Master Thesis

Quantum theory of open systems

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Publication Date:
2002

Permanent Link:
https://doi.org/10.3929/ethz-a-005074029

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Quantum Theory of Open Systems

Thesis at ETH Zuerich

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August 15, 2002
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Acknowledgements

This diploma thesis was finished in the summer 2002 at ETH Zürich under the guidance of Prof. Dr. Jürg Fröhlich. I would like to gratefully thank to Prof. Dr. Jürg Fröhlich for his help, support and advice and for the time that he has found to guide me. Many grateful thanks also go to Dr. Marco Merkli for his supporting help and advice. I would like to thank also Dr. Walid Abou Salem for the revision of the thesis.
Abstract

We study a small system with a finite number of energy levels that interchanges energy with a reservoir, consisting of a bosonic field at temperature $T > 0$. We will study the dynamics of this system, which is given by a Liouvillian operator.

It has been proven, that if 0 is a simple eigenvalue of the Liouvillian then the system has the property of return to equilibrium. That means that the system under small perturbations approaches its unique equilibrium state, as time tends to $\infty$.

We will prove that 0 is a simple eigenvalue in leading order perturbation theory and we will apply these results to spin waves in a ferromagnet.

We will see the principal methods and results used in studying the spectrum of the Liouvillian.
Chapter 1

Introduction

In this diploma thesis we will deal with two classes of open systems, namely a finite dimensional open system coupled to a radiation field and an open system under local perturbations. With "Open system" we mean that the system energy is interchanged with the exterior. In our case the exterior consists either of a reservoir of quantized energy, or of a small perturbation of the dynamics of our system (e.g. a local impurity in a lattice or a local perturbation of the temperature in a bosonic field). We are interested in the dynamical behavior of these open systems. In order to study it we will start with a quantum-mechanical model for the open system and the exterior, write down a model Hamiltonian for the interaction between the two, and then investigate the evolution of the system as determined by the complete Hamiltonian. For systems at $T > 0$ we will see in chapter 2 that it will be easier to use Liouvillians instead of Hamiltonians.

We will see that under appropriate conditions on the Hamiltonian/Liouvilian of the open system and on the Hamiltonian/Liouvilian of the interaction, the system possesses the property of return to equilibrium. That is to say the initial excited states of the open system approach a ground state of equilibrium of the coupled system as the time tends to infinite. These equilibrium states of the system are called KMS states. We will study also the uniqueness of these stationary KMS states.
1.1 Description of Physical Systems

Our principal model will be an atom or a molecule coupled to a radiation field. We will study also a model of magnons in a ferromagnet (section 6.4 of chapter 6). As an open system model under local perturbation we will study a model of a free scalar Bose field perturbed by a local impurity (section 3.4 chapter 3).

1.1.1 Finite dimensional quantum system coupled to a radiation field

Under "system" we will understand the system formed by the open system $S$ and by the reservoir $R$:

$$\text{System} = \text{Small Open System} + \text{Infinitely Extended Reservoir}$$

**Fig.1 Diagram of the System**

Small System: It consists of a confined atom or molecule with a finite number of energy levels. The Hilbert space of this small system is:

$$\mathcal{H}_{at} \simeq \mathbb{C}^N \quad N < \infty$$

(1.1)

the Hamiltonian of the small system $H_{at}$ is given by:

$$H_{at} = -\Delta + V(x)$$

(1.2)

We will suppose that $V(x) \geq 0$ and $V(x) \to 0$ as $x \to \infty$. To this Hamiltonian we will associate an atomic Liouvillian $L_{at}$. We will suppose that:

$$\sigma(H_{at}) = \{E_i\}_{i=0}^{N-1} \quad E_0 < E_1 < \cdots < E_{N-1}$$

(1.3)

**Fig.2 Spectrum of $H_{at}$**
We will suppose that the eigenvalues are simple and every eigenvalue \( E_j \) corresponds to an eigenvector \( \varphi_j \):

\[
H_{at}\varphi_j = E_j\varphi_j
\]  

(1.4)

These eigenvectors form a complete orthonormal system in \( H_{at} \), with the scalar product \((\cdot, \cdot)\) on \( H_{at} \):

\[
(\varphi_i, \varphi_j) = \delta_{ij} \quad i, j = 0, 1 \ldots N - 1
\]  

(1.5)

**Reservoir**: It consists of the quantized electromagnetic field. The Hilbert space of a photon is given by:

\[
h_f^{(1)} = L^2(\mathbb{R}^3 \otimes \mathbb{C}^2, d^3k)
\]  

(1.6)

The Hilbert space of the reservoir coincides with the Fock space:

\[
H_f \equiv F := \mathbb{C} \oplus \bigoplus_{n \geq 1} h_f^{(n)} \quad h^{(n)} := h^{\otimes_n}, n \geq 1
\]  

(1.7)

\( h^{\otimes_n} \) denotes a symmetric tensor product since we are dealing with bosons. The Hamiltonian of the reservoir is:

\[
H_f = \int dk \omega(k)a^*(k)a(k)
\]  

(1.8)

To this Hamiltonian we will associate a field Liouvilian \( L_f \).

We will deal with physical photons, that means that the dispersion relation of \( \omega(k) \) is \( \omega(k) := |k| \) and the dimension of the photon momentum space is three. Therefore the last integral is understood as a three dimensional one, but we will write often \( dk \) in order to simplify the notation.

---

**Fig.3 Spectrum of** \( H_f \), \( \sigma_{pp}(H_f) = 0 \) and \( \sigma_{ac}(H_f) = (0, \infty) \)

**System**: Its Hilbert space is:

\[
H = H_{at} \otimes H_f
\]  

(1.9)

On one hand we will consider the uncoupled system (without interaction) with Hamiltonian:

\[
H_0 = H_{at} \otimes 1 + 1 \otimes H_f
\]  

(1.10)
To this Hamiltonian we will associate a Liouvillian $L_0$. 

\[ \{ E_0, \ldots, E_N \} \]

Fig.4 Spectrum of $H_0$, $\sigma_{pp}(H_0) = \{ E_0, \ldots, E_N \}$, $\sigma_{ac}(H_0) = [E_0, \infty)$

As it can be seen in Fig.4 the spectrum of $H_0$ consists of the union of branches $[E_j, \infty)$ starting at the eigenvalues $E_j$ which are thresholds of continuous spectrum.

On the other hand we have the coupled system (taking into account the interaction $I$) with Hamiltonian:

\[ H_g = H_0 + gI \quad g \geq 0 \]

(1.11)

where $g$ is the coupling constant. To this Hamiltonian $H_g$ we will associate the Liouvillian of the coupled system $L_g$.

Our interaction has the form:

\[ I := \int dk \{ G(k) \otimes a^*(k) + G^*(k) \otimes a(k) \} \quad G(k) \in M_{N \times N} \]

(1.12)

where $G(k)$ is a complex $N \times N$ matrix. To the term $I$ we will associate a term $W$ (the Liouvillian of the interaction).

We will suppose that the matrix $G$ fulfil the following conditions, writing $G_{ij}(k) = \langle \varphi_i | G(k) | \varphi_j \rangle$:

- $H_1$: The functions $G_{ij}(e^{-\theta}k)$ for $\theta \in \mathbb{R}$ extend analytically as functions of $\theta$, on a domain in $\mathbb{C}$ containing the strip:

\[ \Sigma_{\vartheta_0} := \{ \theta \mid \text{Im} \theta < \vartheta_0 \} \]

(1.13)

for some $\vartheta_0$, independent of $\vec{k} \in \mathbb{R}^3$ and $i,j$.

- $H_2$: We assume that there exists positive constants $\mu > 0$ and $M < \infty$, such that for all $\theta \in \Sigma_{\vartheta_0}$ and $\vec{k} \in \mathbb{R}^3$:

\[ \sum_{i,j=1}^N |G_{i,j}(e^{-\theta}k)| \leq e^{M|\text{Re} \theta|} \omega(k)\mu \]

(1.14)

where $\omega(k) = |\vec{k}|$.

For theorem 2.2.11. we will need the following condition, which introduces a cutoff in the interaction:
H3: There exists a constant $0 < \Lambda < \infty$ such that, for all $\theta \in \Sigma_{\theta_0}$:

$$\sum_{i,j=1}^{N} \int |G_{i,j}(e^{-\theta k})|^2 [\omega(k) + \omega(k)^{-3}] dk \leq e^{2M|\text{Re}\theta|} \Lambda^2$$  \hspace{1cm} (1.15)

Magnons in a ferromagnet

In this model we study a ferromagnetic material which suffers a perturbation that is shown under the form of a spin wave. Our Hamiltonian is:

$$H = H_S + H_W = -J \sum_{xy} \vec{S}_x \cdot \vec{S}_{x+y} - \sum_{x} j(x) \vec{s}_0 \cdot \vec{S}_x$$ \hspace{1cm} (1.16)

The first term represents the interaction of nearest neighbors between spin sites in the ferromagnetic lattice. The second term represents the interaction between the ferromagnet and the exterior in form of a spin wave. We will see that under an appropriate transformation the Hamiltonian Eq. (1.16) is transformed to the form of Eq. (1.11).

1.1.2 Open System under Local Perturbations

We will analyze the properties of infinite systems in or close to thermal equilibrium and their behavior under small perturbations of their dynamics by coupling them to finite subsystems.

---

Free scalar Bose field perturbed by local impurity

In this case we have a lattice and a local perturbation under the form of phonons. We will use this model to analyze the uniqueness of the equilibrium KMS state.
1.2 Description of Problems to be Solved

Our principal purpose is to analyze the spectral properties of the Liouvillian of the perturbed system ($L_g$).

We know the spectrum of the atomic Liouvillian ($L_{at}$) and the spectrum of the Liouvillian of the reservoir ($L_f$), but we do not know the spectrum of the coupled Liouvillian ($L_g = L_0 + gW$ where $L_0 := L_{at} + L_f$). Therefore, there have been proposed several methods to derive the properties of the Liouvillian of the coupled system from the properties of the known Liouvillians.

We will assume that $L_g$ is a self-adjoint operator which is proved in [7] or [56]. We will see that the spectrum of $L_g$ is absolutely continuous on $\mathbb{R}\setminus0$. At the point 0 the spectrum will be more difficult to analyze. In order to analyze the simplicity of the eigenvalue 0 and the uniqueness of the equilibrium KMS state there are two methods: the renormalization group method and the positive commutator method. From the simplicity of the eigenvalue 0 it follows the property of return to equilibrium. In chapter 6 we will show the simplicity of the eigenvalue 0 in leading order perturbation theory by computing the level shift operator.

We will also calculate the resonance energies in low-order perturbation theory and we will apply all these results to the spin relaxation in a magnon environment.

In chapter 3 we will show that the commutator method is also valid for matrix element functions $G_{ij}(k)$ that behave as $\sqrt{k}$ and $1/\sqrt{k}$ as k tends to zero. We will also study the uniqueness of the KMS state using the positive commutator method.

In chapter 5 we will present the renormalization group method and we will deduce the expressions for the terms $E, T, W$ contributing to our Liouvillian in low-order perturbation theory. We will also show, by using the smooth Feshbach map, that the domain of the spectrum of $L_g(\theta)$ in the proximity of 0 is cuspidal.

1.3 Summary of Methods to be Used

We will present the different methods used to analyze the spectrum of the Liouvillian of the perturbed system ($L_g$).

In chapter 2 we will present the mathematical formalism of the theory of open systems, the results of which will be used in successive chapters (above all in chapter 6). In chapter 4
we will do a review about the complex deformations and the Feshbach Map. The complex
deformation method is based on the equation (see Eq. (4.21)):
\[
\langle \varphi | (L_g - z)^{-1} \psi \rangle = \langle \hat{U}(\bar{\theta}) \varphi | (L_g(\theta) - z)^{-1} \hat{U}(\theta) \psi \rangle
\]  
\hspace{1cm} (1.17)
where \( L_g(\theta) \) is the complex deformed Liouvillian, this equation is the analyticity (in \( \theta \))
property of the complex deformation. If we show that the right hand side is analytic on
\( z \) near the real axis, then the left hand side is also analytic for these values of \( z \). On the
other hand we know that since \( L_g \) is self-adjoint, it has real eigenvalues. Therefore we can
say that in the values of \( z \) on the real axis, where we can make an analytic continuation of
\( L_g(\theta) \) from the upper half-plane \( (z + i\varepsilon) \) to the lower one \( (z - i\varepsilon) \) for \( z \in \mathbb{R} \), for these values
of \( z \) \( (L_g - z)^{-1} \) exists, with the meaning that the spectrum of \( L_g \) is absolutely continuous
for these values of \( z \).
The Feshbach map will permit us to obtain a perturbative analysis for small \( g \) of the
spectrum of \( L_g(\theta) \). We will get an expression of the form:
\[
\mathcal{F}_P(L_g(\theta) - z) = P(L_{at} - z)P + gPW(\theta)P - g^2\Gamma^{(2)}(\theta, z) + \cdots
\]  
\hspace{1cm} (1.18)
where the spectrum of \( L_{at} \) is known, \( P \) is the projection onto the space of the desired
eigenvalue, \( W \) is the Liouvillian of the interaction, and \( \Gamma^{(2)}(\theta, z) \) is the level shift operator,
\( \Gamma^{(2)}(\theta, z) := -\mathcal{F}_P(L_g(\theta, z))^{(2)} \) defined on Eq. (6.26). Using the isospectrality property
(under a Feshbach map transformation the spectrum of our Liouvillian is conserved), we
see from Eq. (1.18) that the spectrum of \( L_g \) is approximated by \( P(L_{at} - z)P \) in order \( g^0 \)
and by \( P(L_{at} - z)P + gPWP - g^2\Gamma^{(2)}(\theta, z) \) in order \( g^2 \).
Near 0 we must iterate this method, in other words we apply Feshbach map transformations successively, which is the essence of the renormalization group method. In order that each step of the iteration looks like the step before, we add two operations after each Feshbach map transformation, namely a dilatation and a rescalation of the spectral parameter. The set of these three operations constitute the renormalization transformation.

The positive commutator method is based on the inequality:

\[ E_\Delta(L_g)[L_g, A]E_\Delta(L_g) \geq \gamma E_\Delta(L_g) \]  

(1.19)

where \( \gamma > 0 \), \([\cdot, \cdot]\) is the commutator and \( E_\Delta(L_g) \) is the spectral projector of \( L_g \) onto the interval \( \Delta \), which contains only one eigenvalue \( \varepsilon_j \neq 0 \) of \( L_0 \). The idea is to construct an antiself-adjoint operator that fulfills this inequality and together with the virial theorem permits us to say that in \( \Delta \) the spectrum of \( L_g \) is absolutely continuous.
Chapter 2

Mathematical Theory of Quantum Open Systems

The purpose of this chapter is to develop the formalism which is necessary in order to work with the spectral properties of our system. We will use this formalism in coming chapters when we discuss under which conditions of the spectrum, the states that are local perturbations of equilibrium states, have the property of return to equilibrium.

2.1 Liouvillian as a Time Evolution Operator

We will work with infinite systems which we will construct as thermodynamic limit of finite systems. Our $C^*$ algebra of observables is given by:

$$A = \bigcup_{i \in \mathbb{N}} A_{\Lambda_i}$$  \hspace{1cm} (2.1)

where $A_{\Lambda_i}$ is an algebra of observables of a system confined to $\Lambda_i$ with $A_{\Lambda_i} \subseteq B(H_{\Lambda})$ ($B(H_{\Lambda})$: bounded operators on $H_{\Lambda}$) and $H_{\Lambda}$ is a separable Hilbert space with $\Lambda$ being a bounded region of physical space. It is obvious that:

$$A_{\Lambda_i} \subseteq A_{\Lambda_j} \subseteq A_{\Lambda} \hspace{1cm} \text{for} \hspace{1cm} \Lambda_i \subseteq \Lambda_j$$  \hspace{1cm} (2.2)

The dynamics of the system is determined by a Hamiltonian $H_{\Lambda}$, which is a semibounded, self-adjoint operator on $H_{\Lambda}$. A mixed state of the system corresponds to a density matrix $\rho$ i.e. to a positive self-adjoint operator on $H_{\Lambda}$ of unit trace, $\rho \in L^1(H_{\Lambda})$ where:
**Definition 2.1.1.** \( \mathcal{L}^1(\mathcal{H}_\Lambda) \) is the two sided-ideal of trace-class operators in \( B(\mathcal{H}_\Lambda) \)

**Definition 2.1.2.** \( \mathcal{K} \equiv \mathcal{L}^2(\mathcal{H}_\Lambda) \) is the two sided-ideal of Hilbert-Schmitt operators in \( B(\mathcal{H}_\Lambda) \).

\[ \kappa := \frac{\rho^{1/2}}{\rho} \in \mathcal{K} \quad \text{since} \quad \text{Tr}[\kappa^2] = \text{Tr}[\kappa^* \kappa] = \text{Tr}[\rho] = 1. \] \( \mathcal{K} \) is a Hilbert algebra (i.e. as a linear space it is a Hilbert space and as a algebra it is a \( * \)-algebra) with scalar product given by:

\[ \langle \kappa_1 | \kappa_2 \rangle := \text{Tr}[\kappa_1^* \kappa_2] \tag{2.3} \]

In chapter 3 we will see that this Hilbert space is isomorphic to \( \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda \)

**Definition 2.1.3.** We define a linear representation \( l \) of the algebra \( \mathcal{A} \) on \( \mathcal{K} \) by:

\[ l[a] \kappa := a \kappa \in \mathcal{K} \quad a \in \mathcal{A}, \kappa \in \mathcal{K} \tag{2.4} \]

**Definition 2.1.4.** We define an antilinear \((r[za] = \bar{z}r[a], z \in \mathbb{C})\) representation \( r \) of the algebra \( \mathcal{A} \) on \( \mathcal{K} \) by:

\[ r[a] \kappa := \kappa a^* \in \mathcal{K} \quad a \in \mathcal{A}, \kappa \in \mathcal{K} \tag{2.5} \]

If we had defined \( r[a] \kappa = \kappa a \) (as a linear representation) then we would not have a representation:

\[ r[a]r[b] \kappa = r[a](\kappa b) = \kappa ba = r[ba] \kappa = r[ab] \kappa \tag{2.6} \]

To every element \( \kappa \in \mathcal{K} \) we can associate a state of the system given by the density matrix:

\[ \rho := \langle \kappa | \kappa \rangle^{-1} \kappa^* \tag{2.7} \]

then the expectation value of an observable \( a \in \mathcal{A}_\Lambda \) in a state \( \rho \) is given by:

\[ \langle a \rangle_{\rho} := \text{Tr}[\rho a] = \langle \kappa | \kappa \rangle^{-1} \text{Tr}[\kappa^* \kappa a] = \langle \kappa | \kappa \rangle^{-1} \text{Tr}[\kappa^* a \kappa] = \langle \kappa | \kappa \rangle^{-1} \langle \kappa | l[a] \kappa \rangle \tag{2.8} \]

We define the time evolution of an observable \( a \in \mathcal{A} \) in the Heisenberg picture as usual by:

\[ \alpha_t(a) := e^{it\mathcal{H}_\Lambda} a e^{-it\mathcal{H}_\Lambda} \tag{2.9} \]

Let us see now how the states \( \kappa \in \mathcal{K} \) evolve in the Schroedinger picture:

\[ \langle \kappa_1 | l[\alpha_t(a)] \kappa_2 \rangle = \text{Tr}[\kappa_1^* \alpha_t(a) \kappa_2] = \text{Tr}[\kappa_1 e^{it\mathcal{H}_\Lambda} a e^{-it\mathcal{H}_\Lambda} \kappa_2] = \text{Tr}[(e^{it\mathcal{H}_\Lambda} \kappa_1 e^{-it\mathcal{H}_\Lambda})^* a (e^{it\mathcal{H}_\Lambda} \kappa_2 e^{-it\mathcal{H}_\Lambda})] = \langle \alpha_{-t}(\kappa_1) | l[a] \alpha_{-t}(\kappa_2) \rangle \tag{2.10} \]

\[ = \text{Tr}[(e^{it\mathcal{H}_\Lambda} \kappa_1 e^{-it\mathcal{H}_\Lambda})^* a (e^{it\mathcal{H}_\Lambda} \kappa_2 e^{-it\mathcal{H}_\Lambda})] = \langle \alpha_{-t}(\kappa_1) | l[a] \alpha_{-t}(\kappa_2) \rangle \tag{2.11} \]
Therefore, we can define the time evolution of an element $\kappa \in K$ as:

$$\kappa_t = \alpha_{-t}(\kappa) := e^{-itH_\Lambda}Ke^{itH_\Lambda}$$  \hspace{1cm} (2.12)

Now we define the Liouvillian operator $L_\Lambda$ on $K$, which for our physical systems is a self-adjoint operator (for a proof see [7], [56]):

$$L_\Lambda \kappa := [H_\Lambda, \kappa]$$  \hspace{1cm} (2.13)

**Proposition 2.1.5.** The time evolution of an element $\kappa \in K$ is also (besides of Eq. (2.12)) given by:

$$\kappa_t = e^{-itL_\Lambda} \kappa$$  \hspace{1cm} (2.14)

The representation on $K$ of the dynamics of the observables is given by a unitary group generated by $L$:

$$l[\alpha_t(a)] = e^{itL_\Lambda}l[a]e^{-itL_\Lambda}$$  \hspace{1cm} (2.15)

*Proof.* We define:

$$F(t) := e^{-itH_\Lambda}Ke^{itH_\Lambda}$$  \hspace{1cm} (2.16)

and it follows:

$$\frac{dF(t)}{dt} = e^{-itH_\Lambda}[\kappa, H_\Lambda]e^{itH_\Lambda} \frac{d^2F(t)}{dt^2} = e^{-itH_\Lambda}[[\kappa, H_\Lambda], H_\Lambda]e^{itH_\Lambda} \ldots$$

$$\frac{d^nF(t)}{dt^n} = e^{-itH_\Lambda}[[ \ldots [\kappa, H_\Lambda], H_\Lambda], H_\Lambda]e^{itH_\Lambda}$$

then from a Taylor series it follows:

$$e^{-itH_\Lambda}Ke^{itH_\Lambda} = \kappa - it[H_\Lambda, \kappa] - \frac{t^2}{2!}[H_\Lambda[H_\Lambda, \kappa] + \ldots = e^{-itL_\Lambda}\kappa$$  \hspace{1cm} (2.17)

the second relation follows from:

$$e^{itL_\Lambda}l[a]e^{-itL_\Lambda}\kappa = e^{itL_\Lambda}l[a]\alpha_{-t}(\kappa) =$$

$$= e^{itL_\Lambda}a\alpha_{-t}(\kappa) = \alpha_t(a\alpha_{-t}(\kappa)) = \alpha_t(a)\kappa = l[\alpha_t(a)]\kappa$$  \hspace{1cm} (2.18)

and from the definition of $l[a]\kappa$ and Eq. (2.12):

$$= e^{itL_\Lambda}a\alpha_{-t}(\kappa) = \alpha_t(a\alpha_{-t}(\kappa)) = \alpha_t(a)\kappa = l[\alpha_t(a)]\kappa$$  \hspace{1cm} (2.19)
2.2 KMS States and GNS Construction

We are interested in knowing what kind of equilibrium states an infinite system possesses. Therefore we try to find a property of the equilibrium states of a bounded system which can help us to characterize the unbounded ones.

From quantum statistics we know that the equilibrium states of a bounded system at inverse temperature $\beta$ are given by a density matrix (big canonical ensemble):

$$\rho_{\beta,Q} := \frac{1}{Z_{\beta,Q}} e^{-\beta(H_\Lambda - Q)}$$  \hspace{1cm} (2.20)

where $Q$ is a conserved charge of the system $Q \in A'_\Lambda$, where $A'_\Lambda$ is the commutant of $A_\Lambda$ (i.e. the von Neumann algebra of all bounded operators on $\mathcal{H}_\Lambda$ that commute with all operators in $A_\Lambda$) and $Z_{\beta,Q} := \text{Tr}[e^{-\beta(H_\Lambda - Q)}] < \infty$.

Taking the square root of $\rho$:

$$\kappa_{\beta,Q} = \frac{1}{2} e^{-\beta(H_\Lambda - Q/2)} U$$  \hspace{1cm} (2.21)

where $U$ is an arbitrary unitary operator on $\mathcal{H}_\Lambda$.

The expectation value of an element $a \in A_\Lambda$ is from quantum statistics:

$$\langle a \rangle_{\beta,Q} = \frac{1}{Z_{\beta,Q}} \text{Tr}[\rho_{\beta,Q} a] = \frac{1}{Z_{\beta,Q}} \text{Tr}[\kappa_{\beta,Q} \kappa_{\beta,Q}^* a] = \text{Tr}[\kappa_{\beta,Q}^* a \kappa_{\beta,Q}] = \langle \kappa_{\beta,Q} | l[a] \kappa_{\beta,Q} \rangle$$  \hspace{1cm} (2.22)

where we have used the cyclicity of the trace and in the last step we have used the definition of scalar product in $K$ Eq. (2.3)). The most important property of these equilibrium states is:

$$\langle a \alpha_t(b) \rangle_{\beta,Q} = \frac{1}{Z_{\beta,Q}} \text{Tr}[e^{-\beta(H_\Lambda - Q)} a e^{itH_\Lambda} b e^{-itH_\Lambda}] = \langle a \alpha_t(b) \rangle_{\beta,Q}$$  \hspace{1cm} (2.23)

using the cyclicity of the trace and the fact that $H_\Lambda$ and $b$ commute with $Q$

$$= Z_{\beta,Q} \text{Tr}[e^{i\beta Q} e^{itH_\Lambda} b e^{-(\beta+it)H_\Lambda} a] = Z_{\beta,Q} \text{Tr}[e^{-\beta(H_\Lambda - Q)} e^{(\beta+it)H_\Lambda} b e^{-(\beta+it)H_\Lambda} a] = \langle a \alpha_{-i\beta+it}(b) \rangle_{\beta,Q}$$  \hspace{1cm} (2.24)

We make now the following definition:

**Definition 2.2.1.** A state $\varphi$ over $B(\mathcal{H}_\Lambda)$ is called normal if it satisfies the following property. For any bounded increasing net of positive operators $A_\nu$ over $B(\mathcal{H}_\Lambda)$:

$$\varphi(\sup A_\nu) = \sup \varphi(A_\nu)$$  \hspace{1cm} (2.25)
In the following proposition we see the equivalence between normal states and states given by a density matrix:

**Proposition 2.2.2.** The functional \( \rho \) over \( B(\mathcal{H}_\Lambda) \) (where \( \rho \) is a positive operator with trace 1 and the sum does not depend on the orthonormal bases \( \{e_\nu\} \)):

\[
\rho(A) = \text{Tr}(\rho A) = \sum_\nu \langle e_\nu | \rho A e_\nu \rangle = \sum_\nu \langle \rho e_\nu | A e_\nu \rangle
\] (2.27)

is a normal state over \( B(\mathcal{H}_\Lambda) \). Conversely, any normal state \( \phi \) over \( B(\mathcal{H}_\Lambda) \) is of this form with \( \rho \) uniquely determined by \( \phi \).

For a proof see [5].

**Definition 2.2.3.** We say that a normal state is a \( (\alpha_t, \beta) \)-KMS (Kubo-Martin-Schwinger) state if it satisfies the KMS condition, namely:

\[
\langle a\alpha_t(b) \rangle_{\beta,Q} = \langle \alpha_{-i\beta+t}^{-1}(b)a \rangle_{\beta,Q}
\] (2.28)

we will denote these KMS states by:

\[
\omega_{\beta,\Lambda}(a) := \langle a \rangle_{\beta,Q}
\] (2.29)

where we use the parameter \( \Lambda \), which is the size of the physical space (we need it in order to take the thermodynamic limit). We recall that this expectation value depends on \( \Lambda \), and we omit the dependence on \( Q \) which is implicit in the definition of \( \rho \). With this notation the KMS condition becomes:

\[
\omega_{\beta,\Lambda}(a\alpha_t(b)) = \omega_{\beta,\Lambda}(\alpha_{-i\beta+t}^{-1}(b)a)
\] (2.30)

**Proposition 2.2.4.** The KMS-states have the following properties:

1. \( \omega_{\beta,\Lambda}(a\alpha_t(b)) = \omega_{\beta,\Lambda}(\alpha_{-t}^{-1}(a)b) \)
2. \( \omega_{\beta,\Lambda}(\alpha_t(a)) = \omega_{\beta,\Lambda}(a) \) the KMS-states are time translation invariant.
3. \( \omega_{\beta,\Lambda}(a^*b) = \omega_{\beta,\Lambda}(\alpha_{-i\beta/2}^{-1}(b)a_{-i\beta/2}^{-1}(a)^*) \)

**Proof.** the first follows immediately from the cyclicity of the trace:

\[
\omega_{\beta,\Lambda}(a\alpha_t(b)) = Z_{\beta,Q}^{-1} \text{Tr}[e^{-\beta(\mathcal{H}_\Lambda - Q)} ae^{it\mathcal{H}_\Lambda} be^{-it\mathcal{H}_\Lambda}] = Z_{\beta,Q}^{-1} \text{Tr}[e^{-it\mathcal{H}_\Lambda} e^{-\beta(\mathcal{H}_\Lambda - Q)} ae^{it\mathcal{H}_\Lambda} b] =
\]

\[
Z_{\beta,Q}^{-1} \text{Tr}[e^{-\beta(\mathcal{H}_\Lambda - Q)} e^{-it\mathcal{H}_\Lambda} ae^{it\mathcal{H}_\Lambda} b] = \omega_{\beta,\Lambda}(\alpha_{-t}^{-1}(a)b)
\]
the second is a consequence of the first for $a = \mathbb{1}$ the unit element of $\mathcal{A}_\Lambda$. The third property follows from the time translation invariance and from the KMS condition:

$$\omega_{\beta,\Lambda}(a^*b) = \omega_{\beta,\Lambda}(\alpha_{i\beta/2}(a^*)\alpha_{i\beta/2}(b)) = \omega_{\beta,\Lambda}(\alpha_{-i\beta/2}(b)\alpha_{i\beta/2}(a^*)) = \omega_{\beta,\Lambda}(\alpha_{-i\beta/2}(b)\alpha_{-i\beta/2}(a)^*)$$

Remark 1. A KMS state has two parameters, namely the one parametric group of automorphisms $\alpha_t$ and the inverse temperature $\beta = 1/kT$ ($k$: Boltzmann constant). For $t = i\beta$ we can rewrite the KMS condition as:

$$\omega_{\beta,\Lambda}(a\alpha_{i\beta}(b)) = \omega_{\beta,\Lambda}(ba) \quad (2.31)$$

where we see that the value $\beta = 0$ is different from the other values of $\beta$. In this case a KMS state is a trace state: $\omega_{\beta,\Lambda}(ab) = \omega_{\beta,\Lambda}(ba)$.

As the size of the system increases, we will obtain a KMS state for an unbounded system as a limit of a KMS state for a bounded system ($\Lambda_i \subseteq \Lambda_j$ for $i \leq j$):

$$\omega^\beta(a) = \lim_{i \to \infty} \omega_{\beta,\Lambda_i}(a) \quad (2.32)$$

using also $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_\Lambda$ we have the following GNS theorem (Gelfand, Naimark and Segal):

**Theorem 2.2.5.** For any time-translation invariant state $\omega^\beta$ over a $C^*$ algebra $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}^\beta$, a representation $l[\cdot]$ of $\mathcal{A}$ on $\mathcal{H}^\beta$, a unit vector $\Omega^\beta$ and a continuous one-parameter group of unitary operators $\{e^{-it\mathcal{L}}\}_{t \in \mathbb{R}}$ where $\mathcal{L}$ is a self-adjoint operator on $\mathcal{H}^\beta$, satisfying the following conditions:

1. For any $a \in \mathcal{A}$

$$\omega^\beta(a) = \langle \Omega^\beta | l[a] \Omega^\beta \rangle \quad (2.33)$$

2. $l[\alpha_t(a)] = e^{it\mathcal{L}}l[a]e^{-it\mathcal{L}}$

3. $\Omega^\beta$ is a cyclic vector of the representation $l[\cdot]$ i.e.

$$l[\mathcal{A}] \Omega^\beta = \{l[a] \Omega^\beta; a \in \mathcal{A}\} \quad (2.34)$$
is dense in $\mathcal{H}^\beta$. If $\omega^\beta$ is an $\alpha_t$-KMS state then there exists a antiunitary map $J$ from $\mathcal{H}^\beta$ to $\mathcal{H}^\beta$ satisfying:

\begin{align}
J e^{it\mathcal{L}} J &= e^{it\mathcal{L}} \quad (2.35) \\
J l[a] J &= r[a] \quad (2.36) \\
J \Omega^\beta &= \Omega^\beta \quad (2.37)
\end{align}

where $r$ is an antilinear representation.

For a proof see [5], [16], [17].

2.2.1 Modular Operator and Modular Conjugation

We define the modular operator on $\mathcal{H}^\beta$ (GNS Hilbert space) as:

\[ S(l[a]|\Omega^\beta\rangle) := l[a]^*|\Omega^\beta\rangle \quad (2.38) \]

for $a \in \mathcal{A}$, $S$ is a bounded antilinear operator. We would be also interested in having an antiunitary operator $J$ on $\mathcal{H}^\beta$ which relates the representation $l$ and $r$. The question of how to define it, is answered in the next proposition:

**Proposition 2.2.6.** Let be an operator $J$ on $\mathcal{H}^\beta$ with

\[ \langle J l[a] \Omega^\beta | J l[b] \Omega^\beta \rangle = \overline{\langle l[a] \Omega^\beta | l[b] \Omega^\beta \rangle} \quad (2.39) \]

then:

\[ J l[a] \Omega^\beta := e^{-\beta \mathcal{L}} l[a]^* |\Omega^\beta\rangle \quad (2.40) \]

The operator $J$ is called modular conjugation.
Proof.

\[ \langle Jl[a]|\Omega^\beta|Jl[b]|\Omega^\beta \rangle = (\langle l[a]|\Omega^\beta|l[b]|\Omega^\beta \rangle) = (\langle l[b]|\Omega^\beta|l[a]|\Omega^\beta \rangle) \] (2.41)

recalling now the GNS construction in particular \( \omega^\beta(a) = \langle \Omega^\beta|l[a]|\Omega^\beta \rangle \):

\[ \langle Jl[a]|\Omega^\beta|Jl[b]|\Omega^\beta \rangle = \langle \Omega^\beta|l[b]^*l[a]|\Omega^\beta \rangle = \langle \Omega^\beta|l[l[b]^*a]|\Omega^\beta \rangle = \omega^\beta(b^*a) \] (2.42)

now we use the KMS condition \( \omega^\beta(aa_\alpha(b)) = \omega^\beta(a_\alpha_{-t+i\beta}(b)a) \quad a, b \in \mathcal{A} \) for \( t = 0 \):

\[ \omega^\beta(b^*a) = \omega^\beta(a_{-i\beta}(a)b^*) \] (2.43)

using now time-translation invariance \( \omega^\beta(\alpha_t(a)) = \omega^\beta(a) \) for \( t = i\beta \):

\[ \omega^\beta(a_{-i\beta}(b)^*) = \omega^\beta(aa_{i\beta}(b)^*) = \omega^\beta(a_{i\beta}(b)^*) \] (2.44)

recalling now the expression of the time translation operator \( \alpha_t(k) = e^{it\mathcal{L}}k \quad k \in \mathcal{K} \) for \( t = i\beta \) and considering that \( \mathcal{L} \) is a self-adjoint operator:

\[ \omega^\beta(aa_{i\beta}(b)^*)) = \langle \Omega^\beta|l[a]|l[a_{i\beta}(b)]^*\Omega^\beta \rangle = \langle \Omega^\beta|l[a]|e^{-i\mathcal{L}}l[b]^*e^{i\mathcal{L}}\Omega^\beta \rangle = \langle \Omega^\beta|l[a]|\Omega^\beta|e^{-\tfrac{\mathcal{L}}{2}}l[b]^*\Omega^\beta \rangle = \langle \Omega^\beta|l[a]|\Omega^\beta|e^{\tfrac{-\mathcal{L}}{2}}l[b]^*\Omega^\beta \rangle \]

where in the first line we have also considered that \( l[\alpha_t(b)] = e^{it\mathcal{L}}l[b]e^{-it\mathcal{L}} \), in our case for \( t = i\beta \) \( l[a_{i\beta}(b)] = e^{-\tfrac{\mathcal{L}}{2}}l[b]e^{\tfrac{\mathcal{L}}{2}} \). From the first to the second line we have noted that \( \mathcal{L}|\Omega^\beta \rangle = 0|\Omega^\beta \rangle \) and we have developed the exponential in a power series.

Our result is:

\[ \langle Jl[a]|\Omega^\beta|Jl[b]|\Omega^\beta \rangle = \langle e^{-\tfrac{\mathcal{L}}{2}}l[a]^*\Omega^\beta|e^{\tfrac{-\mathcal{L}}{2}}l[b]^*\Omega^\beta \rangle \] (2.45)

and from here it follows the assertion. \( \square \)

**Proposition 2.2.7.** The modular operator \( \mathcal{S} \) and the modular conjugation operator \( \mathcal{J} \) are Involution:

\[ \mathcal{J}\mathcal{J} = 1 \quad \mathcal{S}\mathcal{S} = 1 \] (2.46)

**Proof.** From the definition of \( \mathcal{S} \) Eq. (2.38)

\[ \mathcal{S}\mathcal{S}(l[a]|\Omega^\beta) := (S[l[a]^*]|\Omega^\beta) = l[a]|\Omega^\beta \] (2.47)
Because of the ciclycity of $\Omega^\beta$ each vector of $\mathcal{H}^\beta$ can be written as $l[b]\Omega^\beta$ for $b \in \mathcal{A}$. Let $\psi_{n} = l[a_{n}]\Omega^\beta$ be an orthonormal basis, then from the definition of $\mathcal{J}$ Eq. (2.40):

For $\Omega^\beta$:

$$\langle \Omega^\beta | \mathcal{J} \mathcal{J} \Omega^\beta \rangle = \langle \Omega^\beta | \Omega^\beta \rangle = 1 \quad \text{(2.48)}$$

For $\psi_{m}, \psi_{n}, \psi_{m} = l[a_{m}]\Omega^\beta$, $\psi_{n} = l[a_{n}]\Omega^\beta$ :

$$\langle \psi_{m} | \mathcal{J} \mathcal{J} \psi_{n} \rangle = \langle l[a_{m}]\Omega^\beta | \mathcal{J} \mathcal{J} l[a_{n}]\Omega^\beta \rangle = \quad \text{(2.49)}$$

$$= \langle \mathcal{J}^{-1} l[a_{m}]\Omega^\beta | \mathcal{J} l[a_{n}]\Omega^\beta \rangle = \langle e^{-\beta/2\mathcal{L}} l[a_{m}]\Omega^\beta | e^{\beta/2\mathcal{L}} l[a_{n}]\Omega^\beta \rangle = \quad \text{(2.50)}$$

$$= \langle l[a_{m}]\Omega^\beta | l[a_{n}]\Omega^\beta \rangle = \langle \psi_{m} | \psi_{n} \rangle = \delta_{m,n} \quad \text{(2.51)}$$

In particular for $\mathcal{H}_{f}^\beta$ (GNS Hilbert space of the electromagnetic field) we will use in the next chapters the following simple results:

**Definition 2.2.8.** Let $\mathcal{P}$ be the polynomial algebra generated by $\{a(f), a^*(g) | f, g \in S_{0}(\mathbb{R}^{3})\}$, where $a$ and $a^*$ are the creation and annihilation operators and $S_{0}(\mathbb{R}^{3})$ is the Schwartz space of functions vanishing at the origin of $\mathbb{R}^{3}$.

It is easy to see that $\mathcal{P}$ is a * algebra for the operation defined by:

$$(a(f))^* := a^*(\tau f) \quad \text{(2.52)}$$

where $\tau$ is the complex conjugation operator.

**Proposition 2.2.9.** Let be $\mathcal{J}_{f}$ the modular conjugation operator and $\mathcal{L}_{f}$ the Liouvillian of the field, then:

$$\mathcal{J}_{f} \mathcal{L}_{f} = -\mathcal{L}_{f} \mathcal{J}_{f} \quad \text{(2.53)}$$

**Proof.** Let be $a \in \mathcal{P}$, then:

$$\mathcal{J}_{f} \mathcal{L}_{f} l[a] |\Omega_{f}\rangle = \mathcal{J}_{f} \left[H_{f}, l[a]\right] |\Omega_{f}\rangle = e^{-\frac{\beta}{2} \mathcal{L}_{f}} l \left[H_{f}, l[a]\right] |\Omega_{f}\rangle = \quad \text{(2.54)}$$

where we have considered the definition of the Liouvillian operator $\mathcal{L}_{f} l[a] = [H_{f}, l[a]]$ and Eq. (2.40). Now we see that $l \left[H_{f}, l[a]\right]^* = [H_{f}, l[a]]^*$:

$$= e^{-\frac{\beta}{2} \mathcal{L}_{f}} [H_{f}, l[a]]^* |\Omega_{f}\rangle = e^{-\frac{\beta}{2} \mathcal{L}_{f}} l[a]^* H_{f} |\Omega_{f}\rangle - e^{-\frac{\beta}{2} \mathcal{L}_{f}} H_{f} l[a]^* |\Omega_{f}\rangle = \quad \text{(2.55)}$$

$$= -H_{f} e^{-\frac{\beta}{2} \mathcal{L}_{f}} l[a]^* |\Omega_{f}\rangle = -H_{f} \mathcal{J}_{f} l[a] |\Omega_{f}\rangle = -[H_{f}, \mathcal{J}_{f} l[a]] |\Omega_{f}\rangle = \mathcal{L}_{f} \mathcal{J}_{f} l[a] |\Omega_{f}\rangle \quad \text{(2.56)}$$
where we have considered $H_f|\Omega_f\rangle = 0$ and again Eq. (2.40)

**Proposition 2.2.10.** Let be $\mathcal{P}$ the $*$-algebra before defined, then:

$$J_f|\Omega_f\rangle = |\Omega_f\rangle$$

(2.57)

**Proof.** the proof is trivial, we use Eq. (2.40) with $a = 1_f$ then:

$$J_f(l[1_f]|\Omega_f^\beta) := e^{-\frac{\beta}{2}\mathcal{L}_f}l[1_f]|\Omega_f^\beta) = |\Omega_f^\beta)$$

(2.58)

where we have considered that $l[1_f]|\Omega_f^\beta) = |\Omega_f^\beta)$ and $\mathcal{L}_f|\Omega_f^\beta) = 0$

2.2.2 KMS States for the System Atom + Reservoir

As an example we will study the KMS state for the system described in section 1.1.1. First we study the KMS states without perturbation and second with perturbation.

Our algebra would be:

$$\mathcal{A} := \mathcal{K}_{at} \otimes \mathcal{P}$$

(2.59)

where $\mathcal{K}_{at}$ is the algebra of complex $N \times N$ matrices and $\mathcal{P}$ was defined in 2.2.8.

Without perturbation the KMS state is the tensor product of KMS states for the atom and for the reservoir:

$$\omega_0^\beta = \omega_{at}^\beta \otimes \omega_f^\beta$$

(2.60)

where:

$$\omega_{at}^\beta = Z_{\beta,0}^{-1}\sum_{j=0}^{N-1} e^{-\beta E_j} |\varphi_j\rangle \langle \varphi_j|$$

(2.61)

where $\{\varphi_j\}^{N-1}_{j=1}$ are the eigenvalues of $H_{at}$ corresponding to the eigenvalues $E_0 < E_1 < \cdots < E_{N-1}$. And where $\omega_f^\beta$ is so defined that the expectation value of the number of photons is given by (from quantum statistics):

$$\omega_f^\beta(a^*(k)a(k')) = \frac{\delta(k - k')}{e^{\beta \omega(k)} - 1}$$

(2.62)

where $a^*(k), a(k)$ are the creation and annihilation operators for the photon field. From this equation it is also easy to derive (using the commutation relations for the creation and annihilation operators):

$$\omega_f^\beta(a(k)a^*(k')) = \delta(k - k') + \omega_f^\beta(a^*(k')a(k)) = \delta(k - k') + \frac{\delta(k - k')}{e^{\beta \omega(k)} - 1} = e^{\beta \omega(k)} \frac{\delta(k - k')}{e^{\beta \omega(k)} - 1}$$

(2.63)
From these considerations it is easy to see that a KMS vector in $\mathcal{H}_{at} \otimes \mathcal{H}_f$ is given by:

$$\Omega_0^\beta = Z_{\beta,0}^{-1/2} \sum_{j=0}^{N-1} e^{-\beta E_j/2} \varphi_j \otimes \Omega_f$$  \hspace{1cm} (2.64)$$

where $\Omega_f$ is the vacuum in $\mathcal{H}_f$ and:

$$\omega_0^\beta(a) = \langle \Omega_0^\beta|a|\Omega_0^\beta \rangle$$  \hspace{1cm} (2.65)$$

with $a \in \mathcal{A}$.

We define:

$$L_{g,l} := L_0 + l[I]$$  \hspace{1cm} (2.66)$$
$$L_{g,r} := L_0 - r[I]$$  \hspace{1cm} (2.67)$$

With perturbation we have the following theorem:

**Theorem 2.2.11.** Assume that $G$ fulfils hypothesis $H3$, Eq. (1.15), then the vector $\Omega_0^\beta$ is in the domain of the two unbounded operators $e^{-\beta L_{g,l}/2}$ and $e^{\beta L_{g,r}/2}$ and the vector:

$$\Omega_g^\beta := Z_{\beta,0}^{-1/2} e^{-\beta L_{g,l}/2} \Omega_0^\beta = Z_{\beta,0}^{-1/2} e^{\beta L_{g,r}/2} \Omega_0^\beta$$  \hspace{1cm} (2.68)$$

defines a KMS state $\omega_g^\beta$ on $\mathcal{A} := \mathcal{K}_{at} \otimes \mathcal{P}$ for the time evolution given by $\alpha_t^\beta$ and we have:

$$L_g \Omega_g^\beta = 0$$  \hspace{1cm} (2.69)$$

that is to say $\Omega_g^\beta$ is an eigenvector of $L_g$ with eigenvalue 0.

For a proof see [17]

### 2.3 Return to Equilibrium

The purpose of this section is to establish under which conditions of the spectrum, local perturbations states (given by normal states) of the coupled system evolve to KMS states in the course of time. The condition will be, that 0 is a simple eigenvalue of the spectrum of $L_g$.

We have already seen that a normal state can be written as:

$$\rho(a) = \sum_{n=1}^{\infty} p_n \langle \psi_n|l[a]|\psi_n \rangle$$  \hspace{1cm} (2.70)$$
where $a \in A$, $\psi_n \in \mathcal{H}_\beta$ an orthonormal system and $p_n \geq 0 \forall n$. We know also that from the GNS construction each of the vectors $\psi_n$ can be approximated in norm by vectors of the form $l[b]\Omega^\beta_g$ (i.e. $\Omega^\beta_g$ is cyclic) for a specific $b \in A$.

**Proposition 2.3.1.** Assume that 0 is a simple eigenvalue of $\mathcal{L}_g$ corresponding to the eigenvector $\Omega^\beta_g$ and that the rest of the spectrum of $\mathcal{L}_g$ is absolute continuous. Let $\rho$ be a normal state (local perturbation) and $a \in A$. Then:

$$\lim_{\tau \to \infty} \int_0^\tau \rho_{\pm t}(a)dt = \omega^\beta(a) \quad \text{return to equilibrium in ergodic mean sense} \quad (2.71)$$

If $\sigma(\mathcal{L}_g) \setminus \{0\}$ is absolutely continuous, then:

$$\lim_{t \to \pm \infty} \rho_t(a) = \omega^\beta(a) \quad \text{return to equilibrium} \quad (2.72)$$

**Proof.** We begin with the equality (where $w$ denotes weak-limit):

$$w-\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{\pm it\mathcal{L}_g}dt = |\Omega^\beta_g\rangle\langle \Omega^\beta_g| \quad (2.73)$$

In order to prove it, suppose first a vector $\psi$ with $P_{\Omega^\beta_g}\psi = 0$ then it is clear that:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{\pm it\mathcal{L}_g}dt|\psi\rangle = 0 \quad (2.74)$$

and the right side is:

$$|\Omega^\beta_g\rangle\langle \Omega^\beta_g|\psi\rangle = 0 \quad (2.75)$$

Doing the same now with the cyclic vector $\Omega^\beta_g$, then:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau e^{\pm it\mathcal{L}_g}dt|\Omega^\beta_g\rangle = |\Omega^\beta_g\rangle \quad (2.76)$$

and the right hand side

$$|\Omega^\beta_g\rangle\langle \Omega^\beta_g|\Omega^\beta_g\rangle = |\Omega^\beta_g\rangle$$

where we have used that $\mathcal{L}_g|\Omega^\beta_g\rangle = 0$.

Now we will prove Eq. (2.71):

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \rho_{\pm t}(a)dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n(\psi_n\langle l[a_{\pm t}(a)]\psi_n) \quad (2.77)$$
and now using that $|\psi_n\rangle = l[a_n]|\Omega^\beta_g)$:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n \langle \psi_n | l[\alpha_{\pm t}(a)] | \psi_n \rangle dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n \langle l[a_n] \Omega^\beta_g | l[\alpha_{\pm t}(a)] | l[a_n] \Omega^\beta_g \rangle dt =$$

$$= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n \langle \Omega^\beta_g | l[a_n]^* l[\alpha_{\pm t}(a)] l[a_n] \Omega^\beta_g \rangle dt$$

In the integrand we have $\langle \Omega^\beta_g | l[a_n]^* l[\alpha_{\pm t}(a)] l[a_n] \Omega^\beta_g \rangle = \langle \Omega^\beta_g | l[a_n]^* \alpha_{\pm t}(a) l[a_n] \Omega^\beta_g \rangle = \omega^\beta(a_n^* \alpha_{\pm t}(a) a_n)$, let us develop this expression using first the KMS condition and then that $L_g |\Omega^\beta_g\rangle = 0$:

$$\omega^\beta(a_n^* \alpha_{\pm t}(a) a_n) = \omega^\beta(\alpha_{-i\beta}(a_n) a_n^* \alpha_{\pm t}(a)) =$$

$$= \langle l[a_n] | l[\alpha_{i\beta}(a_n^*)] \Omega^\beta_g | l[\alpha_{\pm t}(a)] \Omega^\beta_g \rangle = \langle l[a_n] | l[\alpha_{i\beta}(a_n^*)] \Omega^\beta_g | e^{\pm itL} l[(a)] \Omega^\beta_g \rangle$$

and finally we have:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n \omega^\beta(a_n^* \alpha_{\pm t}(a) a_n) dt =$$

(2.78)

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n \langle l[a_n] | l[\alpha_{i\beta}(a_n^*)] \Omega^\beta_g | e^{\pm itL} l[(a)] \Omega^\beta_g \rangle dt$$

(2.79)

and using Eq. (2.73):

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \sum_{n=1}^\infty p_n \langle l[a_n] | l[\alpha_{i\beta}(a_n^*)] \Omega^\beta_g | e^{\pm itL} l[(a)] \Omega^\beta_g \rangle dt =$$

(2.80)

(2.81)

$$\sum_{n=1}^\infty p_n \langle l[a_n] | l[\alpha_{i\beta}(a_n^*)] \Omega^\beta_g | \Omega^\beta_g \rangle \langle \Omega^\beta_g | l[(a)] \Omega^\beta_g \rangle = \omega^\beta(\alpha_{-i\beta}(a_n) a_n^* \omega^\beta(a) =$$

$$\omega^\beta(a_n^* a_n) \omega^\beta(a) = \omega^\beta(a)$$

(2.82)

(2.83)

where we have used $\sum p_n = 1$ and:

$$\omega^\beta(a_n^* a_n) = \langle \Omega^\beta_g | l[a_n^* a_n] \Omega^\beta_g \rangle = \langle \Omega^\beta_g | l[a_n^*] l[a_n] \Omega^\beta_g \rangle =$$

(2.84)

$$\langle l[a_n] \Omega^\beta_g | l[a_n] \Omega^\beta_g \rangle = \langle \psi_n | \psi_n \rangle = 1$$

(2.85)

For the case that $\sigma(L_g) \setminus \{0\}$ is absolutely continuous we have instead of Eq. (2.73):

$$w^{-\lim}_{\tau \to \infty} e^{\pm itL_g} = |\Omega^\beta_g \rangle \langle \Omega^\beta_g |$$

(2.86)

and the proof would be similar. \qed
Chapter 3

Hilbert Spaces. Positive Commutator Method

3.1 Introduction

In the last chapter we have worked with the Liouvillian of the perturbed system $L_g$ on the Hilbert space $\mathcal{K}_{at} \otimes \mathcal{H}_f^\beta$. Now we would like to work on a Hilbert space related with our initials $\mathcal{H}_{at}$ and $\mathcal{F}$ see Eq. (1.1) and Eq. (1.9). This will be possible with the use of two isomorphism $I_C$ and $I_T$ as we will see later. We will see that the Hilbert spaces $\mathcal{K}_{at} \otimes \mathcal{H}_f^\beta$ and $\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$ are isomorph.

We will study also the glued Hilbert space $\mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma))$ which is isomorph to $\mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$. In this glued Hilbert space we will analyze the applicability of the Commutator method to interesting physical cases.

In the last section of this chapter we will study the uniqueness of the KMS states with the help of the positive commutator method.

3.2 The Araki-Woods Hilbert Space

The purpose of this section is to obtain the expression of the Liouvillian $L_g$ on

$$\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$$
where $\mathcal{F}(X)$ is the Fock space over $X$ defined for bosons as:

$$\mathcal{F}(X) = \mathbb{C} \oplus \bigoplus_{n \geq 1} X_j^{\otimes n}$$

where the $s$ in the tensor product denotes a symmetric tensor product (Bose-Einstein statistics).

**Remark 2.** The introduction of the polarization of the bosons would be possible with a Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3, d^3k) \otimes \mathbb{C}^2)^{\otimes n}$ but we will work with a simplified model.

On $\mathcal{F} \otimes \mathcal{F}$ we define the creation and annihilation operators:

$$a^\dagger_l(f) := a^\dagger_{\tau}(f) \otimes \mathbb{1}_f$$

$$a^\dagger_r(f) := \mathbb{1}_f \otimes Ta^\dagger(f)T = \mathbb{1}_f \otimes a^\dagger(\tau f)$$

where $a^\dagger(f)$ are the standard creation and annihilation operators which fulfill the following commutation relations:

$$[a^\dagger(f), a^\dagger(g)] = [a(f), a(g)] = 0 \quad [a(\tau f), a^\dagger(\tau g)] = \langle f|g \rangle \mathbb{1}_f$$

where $a^\dagger(f) := \int d^3k f(k) a^\dagger(k)$

where $\tau$ is the complex conjugation operator $\tau f = \overline{f}$ for $f \in L^2(\mathbb{R}^3, d^3k)$. We define an antiunitary operator $T$ on $\mathcal{F}$ the second quantization of $\tau$:

$$T \Omega = \Omega; \quad Ta^\dagger(f)T := a^\dagger(\tau f); \quad T = T^* = T^{-1}$$

where $\Omega$ is the vacuum vector characterized by the property:

$$a(f)|\Omega \rangle = 0 \quad \text{for all } f \in L^2(\mathbb{R}^3, d^3k)$$

**Theorem 3.2.1.**

The Map: $I_T : \mathcal{H}_\beta^f \longrightarrow \tilde{\mathcal{H}} := \mathcal{F} \otimes \mathcal{F}$

defined by:

$$I_Tl_f[a(f)]I_T^{-1} := a_l(\sqrt{1+\rho f}) + a^*_r(\sqrt{\rho}f)$$

$$I_Tr_f[a(f)]I_T^{-1} := a^*_l(\sqrt{\rho}f) + a_r(\sqrt{1+\rho f})$$

$$I_T\Omega_\beta^f = \Omega \otimes \Omega$$

is an isometric isomorphism.
Proof. From this definition it follows that $I_T[l[a^2(f)]I_T^{-1}$ and $I_T[r[a^2(f)]I_T^{-1}$ satisfy the same canonical commutation relations like Eq. (3.4) which we will check with an example, taking into account Eq. (3.4) and that different representations $r$ and $l$ commute:

$$[I_T l_f[a(f)]I_T^{-1}, I_T l_f[a^*(f)]I_T^{-1}] = (a_l(\sqrt{1 + \rho f}) + a^*_r(\sqrt{\rho f}))(a^*_l(\sqrt{1 + \rho f}) + a_r(\sqrt{\rho f})) -$$

$$(a^*_l(\sqrt{1 + \rho f}) + a_r(\sqrt{\rho f}))(a_l(\sqrt{1 + \rho f}) + a^*_r(\sqrt{\rho f})) = a_l(\sqrt{1 + \rho f})a^*_l(\sqrt{1 + \rho f}) -$$

$$-a^*_l(\sqrt{1 + \rho f})a_l(\sqrt{1 + \rho f}) + a_l(\sqrt{1 + \rho f})a_r(\sqrt{\rho f}) - a_r(\sqrt{\rho f})a_l(\sqrt{1 + \rho f}) +$$

$$+a^*_r(\sqrt{\rho f})a^*_l(\sqrt{1 + \rho f}) - a^*_l(\sqrt{1 + \rho f})a^*_r(\sqrt{\rho f}) + a^*_r(\sqrt{\rho f})a_r(\sqrt{\rho f}) - a_r(\sqrt{\rho f})a^*_r(\sqrt{\rho f}) =$$

$$(1 + \rho)\langle f | f \rangle \cdot 1_f - \rho \langle f | f \rangle \cdot 1_f = \langle f | f \rangle \cdot 1_f$$

It is easy to see the relations:

$$I_T l_f(p_1)I_T^{-1} + I_T l_f(p_2)I_T^{-1} = I_T l_f(p_1 + p_2)I_T^{-1} \quad (3.12)$$

$$I_T l_f(p_1)I_T^{-1} I_T l_f(p_2)I_T^{-1} = I_T l_f(p_1 p_2)I_T^{-1} \quad (3.13)$$

$\forall p_1, p_2 \in \mathcal{P}$ where $\mathcal{P}$ is the polynomial algebra generated by $\{a(f), a^*(g)|f, g \in L^2(\mathbb{R}^3, \delta^3 k)\}$. With these relations we see that $l_f[\mathcal{P}]$ is $*$-homomorphic to $I_T l_f[\mathcal{P}]I_T^{-1}$ and in a similar way $r_f[\mathcal{P}]$ is $*$-homomorphic to $I_T r_f[\mathcal{P}]I_T^{-1}$. We will prove now that $I_t$ is an isometry:

$$\langle \Omega \otimes \Omega | I_T l_f[a^*(k)]f_f^{-1} I_T l_f[a(k')]I_T^{-1} \Omega \otimes \Omega \rangle =$$

$$\langle \Omega \otimes \Omega | (a^*_l(\sqrt{1 + \rho f} + a_r(\sqrt{\rho f}))(a_l(\sqrt{1 + \rho f} + a^*_r(\sqrt{\rho f}))) \Omega \otimes \Omega \rangle =$$

$$\langle \Omega \otimes \Omega | a^*_l(\sqrt{1 + \rho f})a_l(\sqrt{1 + \rho f})(a^*_r(\sqrt{\rho f})) \Omega \otimes \Omega \rangle + \langle \Omega \otimes \Omega | a^*_l(\sqrt{1 + \rho f})a^*_r(\sqrt{\rho f})(\Omega \otimes \Omega) \rangle +$$

$$\langle \Omega \otimes \Omega | a_r(\sqrt{1 + \rho f})a_l(\sqrt{1 + \rho f})(\Omega \otimes \Omega) \rangle + \langle \Omega \otimes \Omega | a_r(\sqrt{\rho f})a^*_l(\sqrt{1 + \rho f})(\Omega \otimes \Omega) \rangle =$$

$$\sqrt{\rho(k)\rho(k')} \delta(k - k') = \frac{\delta(k - k')}{\epsilon^3 \omega(k)} - 1 = \omega_f(a^*(k)a(k')) = \langle \Omega^f_{\beta} | l_f[a^*(k)]l_f[a(k')]\Omega^f_{\beta} \rangle \quad (3.14)$$

likewise we would prove:

$$\langle \Omega \otimes \Omega | I_T r_f[a^*(k)]I_T^{-1} I_T r_f[a(k')]I_T^{-1} \Omega \otimes \Omega \rangle = \langle \Omega^f_{\beta} | r_f[a^*(k)]r_f[a(k')]\Omega^f_{\beta} \rangle \quad (3.15)$$

every vector $\psi \in \mathcal{H}_{\beta}^f$ can be approximated by vectors of the form $l[\mathcal{P}]\Omega^f_{\beta}$ because of the cyclicity of $\Omega^f_{\beta}$. Therefore the isomorphism between operators on $\mathcal{H}_{\beta}^f$ and operators on $\mathcal{F} \otimes \mathcal{F}$ means finally that $I_T$ is an isometry isomorphism between vectors in $\mathcal{H}_{\beta}^f$ and vectors in $\mathcal{F} \otimes \mathcal{F}$. □
Now we will find the form of the Liouvillians $L_f, L_{at}, L_g$ on $\mathcal{F} \otimes \mathcal{F}$, $\mathcal{H}_{at} \otimes \mathcal{H}_{at}$ and $\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F} \otimes \mathcal{F}$ respectively. Afterwards we will obtain the form of the interaction on the gluing Hilbert space $\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma))$.

**Proposition 3.2.2.** The Liouvillian of the field $L_f$ on the GNS Hilbert space of the field $H^f_3$ becomes $L_f$ on the Araki-Woods Hilbert space $\mathcal{F} \otimes \mathcal{F}$ with:

$$L_f = \int dk\omega(k)[a^*_l(k)a_l(k) - a^*_r(k)a_r(k)]$$

(3.16)

Proof. As an operator $L_f$ transforms like:

$$\langle \Omega \otimes \Omega | L_f \Omega \otimes \Omega \rangle = \langle \Omega \otimes \Omega | I_T L_f I_T^{-1} \Omega \otimes \Omega \rangle = \langle \Omega \otimes \Omega | \{ I_T l[H_f] I_T^{-1} - I_T r[H_f] I_T^{-1} \} \Omega \otimes \Omega \rangle$$

$$= \langle \Omega \otimes \Omega | \{ \int dk\omega(k) I_T l[a^*(k)a(k)] I_T^{-1} - \int dk\omega(k) I_T r[a^*(k)a(k)] I_T^{-1} \} \Omega \otimes \Omega \rangle = \int \langle [a_l][a_r] \rangle \int dk\omega(k) \{ I_T l[a^*(k)] I_T^{-1} I_T l[a(k)] I_T^{-1} - I_T r[a^*(k)] I_T^{-1} I_T r[a(k)] I_T^{-1} \} \Omega \otimes \Omega \rangle$$

using now Eq. (3.9) and Eq. (3.10) and the commutation relations (using also that different representations commute):

$$= \langle \Omega \otimes \Omega | \int \omega(k)dk \{ (a^*_l(\sqrt{1+\rho}) + a_r(\sqrt{\rho})) (a_l(\sqrt{1+\rho}) + a^*_r(\sqrt{\rho})) - (a_l(\sqrt{\rho}) + a^*_r(\sqrt{1+\rho})) (a^*_l(\sqrt{\rho}) + a_r(\sqrt{1+\rho})) \} \Omega \otimes \Omega \rangle$$

$$= \langle \Omega \otimes \Omega | \int \omega(k)dk \{ a^*_l(\sqrt{1+\rho})a_l(\sqrt{1+\rho}) - a_l(\sqrt{\rho})a^*_l(\sqrt{\rho}) + a_r(\sqrt{\rho})a^*_r(\sqrt{\rho}) - a_l(\sqrt{1+\rho})a_r(\sqrt{\rho}) + a^*_l(\sqrt{1+\rho})a^*_r(\sqrt{1+\rho}) + a_r(\sqrt{1+\rho})a_l(\sqrt{1+\rho}) - a_l(\sqrt{1+\rho})a_r(\sqrt{1+\rho}) \} \Omega \otimes \Omega \rangle$$

(3.17)

(3.18)
We can construct an isomorphism between $K_{at}$, the two sided-ideal of Hilbert-Schmitt operators in $B(H_{at})$ and $H_{at} \otimes H_{at}$, where $H_{at}$ is the Hilbert space of states of the small open system. Namely:

$$I_C : K_{at} \mapsto H_{at} \otimes H_{at}$$ (3.19)

If $\kappa = |\psi_1\rangle\langle\psi_2| \in K_{at}$ then

$$I_C \kappa := \psi_1 \otimes C\psi_2 = |\psi_1\rangle\langle C\psi_2| \in H_{at} \otimes H_{at}$$ (3.20)

where $C$ is an antiunitary involution on $H_{at}$, that is to say:

$$C^2 = 1 \quad \langle C\phi|C\varphi\rangle = \langle \varphi|\phi\rangle = \langle \overline{\phi}|\overline{\varphi}\rangle$$ (3.21)

**Proposition 3.2.3.** The Liouvillian of the atom $L_{at}$ on the $K_{at}$ Hilbert space becomes $L_{at}$ on the $H_{at} \otimes H_{at}$ Hilbert space with:

$$L_{at} = H_{at} \otimes 1_{at} - 1_{at} \otimes CH_{at}C$$ (3.22)

**Proof.** We recall the definitions of the representations of the observable algebra $A$ on $K$ Eq. (2.4) and Eq. (2.5): $l[a] \kappa := a\kappa$ and $r[a] \kappa := \kappa a^*$ we have:

$$L_{at} = I_C L_{at} I_C^{-1} = I_C (l[H_{at}] - r[H_{at}]) I_C^{-1}$$ (3.23)

$$I_C l[H_{at}] \kappa = I_C H_{at} |\psi_1\rangle\langle\psi_2| = |H_{at}\psi_1\rangle\langle C\psi_2| = (H_{at} \otimes 1_{at})(|\psi_1\rangle\langle C\psi_2|) = (H_{at} \otimes 1_{at}) I_C \kappa$$ (3.24)

$$I_C r[H_{at}] \kappa = I_C |\psi_1\rangle\langle \psi_2| H_{at} = |\psi_1\rangle\langle CH_{at}\psi_2| = (1_{at} \otimes CH_{at}C)(|\psi_1\rangle\langle C\psi_2|) = (1_{at} \otimes CH_{at}C) I_C \kappa$$ (3.25)

$$L_{at} I_C \kappa = I_C L_{at} K_{at}^{-1} Y_{at} K_{at} = I_C (l[H_{at}] - r[H_{at}]) \kappa$$ (3.26)

and substituting now Eq. (3.24) and Eq. (3.25)

$$L_{at} I_C \kappa = (H_{at} \otimes 1_{at}) I_C \kappa - (1_{at} \otimes CH_{at}C) I_C \kappa$$ (3.27)

In particular for $C = T$ because of $H_{at}T = TH_{at}$ and $T^2 = 1_{at}$ we have:

$$L_{at} = H_{at} \otimes 1_{at} - 1_{at} \otimes CH_{at}C = H_{at} \otimes 1_{at} - 1_{at} \otimes H_{at}$$ (3.28)
The same procedure as in the proof shows us that:

\[ I_C(l[G] - r[G])I_C^{-1} = G \otimes \mathbb{1}_{at} - \mathbb{1}_{at} \otimes CGC \]  
(3.29)

where \( l, r \) are defined in Eq. (2.4) and Eq. (2.5).

We will use the following notation:

\[ G_l(k) := G \otimes \mathbb{1}_{at} \]  
(3.30)

\[ G_r(k) := \mathbb{1}_{at} \otimes TG(k)T \]  
(3.31)

We will show now the last step in order to obtain the complete Liouvillian \( L_g \). The Liouvillian of the interacting system is defined on \( \mathcal{H}_{at} \otimes \mathcal{F} \) by:

\[ L_g := L_0 + g(l[I] - r[I]) \]  
(3.32)

where:

\[ I = \int dk [G(k) \otimes a^*(k) + G^*(k) \otimes a(k)] \]  
(3.33)

and:

\[ \begin{pmatrix} l \\ r \end{pmatrix} = \begin{pmatrix} l_{at} \\ r_{at} \end{pmatrix} \otimes \begin{pmatrix} l_f \\ r_f \end{pmatrix} \]  
(3.34)

\[ I_0 := I_C \otimes I_T \]  
(3.35)

**Proposition 3.2.4.** The Liouvillian of the interacting system \( \mathcal{L}_g \) on the \( \mathcal{K}_{at} \otimes \mathcal{H}_\beta^f \) Hilbert space becomes \( L_g \) on the \( \mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F} \otimes \mathcal{F} \) Hilbert space with:

\[ L_g = L_{at} + L_f + gI_0(l[I] - r[I])I_0^{-1} \]  
(3.36)

where:

\[ I_0l[I]I_0^{-1} = \int dk \left\{ G \otimes \mathbb{1} \otimes \left[ a^*_t(k) \sqrt{1 + \rho} + a_r(k) \sqrt{\rho} \right] + G^*_r(k) \otimes \mathbb{1} \otimes \left[ a_l(k) \sqrt{1 + \rho} + a^*_r(k) \sqrt{\rho} \right] \right\} \]  
(3.37)

\[ I_0r[I]I_0^{-1} = \int dk \left\{ \mathbb{1} \otimes TG \otimes \left[ a_l(k) \sqrt{\rho} + a^*_r(k) \sqrt{1 + \rho} \right] + \mathbb{1} \otimes TG^*T \otimes \left[ a^*_t(k) \sqrt{\rho} + a_r(k) \sqrt{1 + \rho} \right] \right\} \]  
(3.38)

**Proof.** Using Eq. (3.35):

\[ I_0L_{at} \otimes \mathbb{1}_f I_0^{-1} = I_C L_{at} I_C^{-1} \otimes \mathbb{1}_f = L_{at} \otimes \mathbb{1}_f \]  
(3.39)

\[ I_0 \mathbb{1}_{at} \otimes L_f I_0^{-1} = \mathbb{1}_{at} \otimes I_T L_f I_T^{-1} = \mathbb{1}_{at} \otimes L_f \]  
(3.40)
Using now that:

\[ l[I] = \int dk [l_{at}[G(k)] \otimes l_f[a^*(k)] + l_{at}[G^*(k)] \otimes l_f[a(k)]] \quad (3.41) \]

and a similar equation for \( r[I] \). Transforming this operators with \( I_0 \):

\[ I_0 l[I] I_0^{-1} = I_T \int dk \left\{ \frac{ICl_{at}[G(k)] I_C^{-1} \otimes l_f[a^*(k)] + ICl_{at}[G^*(k)] I_C^{-1} \otimes l_f[a(k)]}{G_t(k)} \right\} I_T^{-1} \quad (3.42) \]

\[ I_0 r[I] I_0^{-1} = I_T \int dk \left\{ \frac{ICr_{at}[G(k)] I_C^{-1} \otimes r_f[a^*(k)] + ICr_{at}[G^*(k)] I_C^{-1} \otimes r_f[a(k)]}{G_t(k)} \right\} I_T^{-1} \quad (3.43) \]

using now that \( I_T \int dk G_t(k) \otimes l_f[a^*(k)] I_T^{-1} = I_T l_f[a^*(G_t(k))] I_T^{-1} \) and similar equations. We make the following definition (analog to Eq. (3.9), Eq. (3.10)):

\[ I_T l_f[a(G)] I_T^{-1} := a_l(\sqrt{1 + \rho G}) + a^*_r(\sqrt{\rho G}) \quad (3.44) \]

\[ I_T r_f[a(G)] I_T^{-1} := a^*_l(\sqrt{\rho G}) + a_r(\sqrt{1 + \rho G}) \quad (3.45) \]

and substituting into Eq. (3.42)) and Eq. (3.43)):

\[ I_0 l[I] I_0^{-1} = I_T l_f[a^*(G_t(k))] I_T^{-1} + I_T l_f[a(G_t^*(k))] I_T^{-1} \quad (3.46) \]

\[ I_0 r[I] I_0^{-1} = I_T r_f[a^*(G_t(k))] I_T^{-1} + I_T r_f[a(G_t^*(k))] I_T^{-1} \quad (3.47) \]

Substituting this expressions and Eq. (3.30), Eq. (3.31) in the last equations give us the desired results. \( \square \)

### 3.3 Glued Hilbert Space

The purpose of this section is to obtain the expression of the Liouvillian \( L_g \) on \( \mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma)) \).

**Theorem 3.3.1.** There is a isometric isomorphism that identifies:

\[ \mathcal{F}(L^2(\mathbb{R}^3, d^3 k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3 k)) \simeq \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma)) \quad (3.48) \]

where \( S^2 \) is the unit sphere and \( d\Sigma \) is the element of solid angle.
Proof. First we see that it exists a trivial isometric isomorphism between $\mathcal{F}(L^2(\mathbb{R}^3, d^3k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$ and $\mathcal{F}(L^2(\mathbb{R}^3, d^3k) \oplus L^2(\mathbb{R}^3, d^3k))$ by doing the correspondence:

$$a^*(f_1) \cdots a^*(f_m) \otimes a^*(g_1) \cdots a^*(g_n) \mapsto a^*((f_1, 0)) \cdots a^*((f_m, 0))a^*((0, g_1) \cdots a^*((0, g_n))$$

(3.49)

Now we will prove the following isometric isomorphism $\mathcal{F}(L^2(\mathbb{R}^3, d^3k) \oplus L^2(\mathbb{R}^3, d^3k)) \simeq \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma))$ we define the map $j_{\phi, \psi}: \forall (f, g) \in \mathcal{F}(L^2(\mathbb{R}^3, d^3k))$

$$j_{\phi, \psi} : (f, g) \mapsto h$$

where:

$$h(u, \alpha) = \begin{cases}
e^{i\phi} u f(u, \alpha) & \text{if } u \geq 0 \\
e^{i\psi} u g(-u, \alpha) & \text{if } u < 0 
\end{cases}$$

(3.50)

Isometry: Let be

$$\|(f, g)\|^2_{L^2(\mathbb{R}^3, d\Sigma)} := \|f\|^2_{L^2} + \|g\|^2_{L^2}$$

(3.52)

$$\|h\|^2_{L^2(\mathbb{R} \times S^2, du \times d\Sigma)} = \int_{\mathbb{R} \times S^2} |h(u, \alpha)|^2 dud\Sigma =$$

$$= \int_{\mathbb{R}^+ \times S^2} |e^{i\phi} u f(u, \alpha)|^2 dud\Sigma + \int_{\mathbb{R}^- \times S^2} |e^{i\psi} u g(-u, \alpha)|^2 dud\Sigma =$$

$$= \int_{\mathbb{R}^+ \times S^2} |e^{i\phi} u f(u, \alpha)|^2 dud\Sigma + \int_{\mathbb{R}^+ \times S^2} |e^{i\psi} u g(u, \alpha)|^2 dud\Sigma = \|f\|^2_{L^2} + \|g\|^2_{L^2}$$

(3.53)

Isomorphism: Given a h we look for f, g so that $j_{\phi, \psi}^{-1}(h) = (f, g) \in L^2(\mathbb{R}^+ \times S^2) \oplus L^2(\mathbb{R}^+ \times S^2)$, in fact:

$$f(u, \alpha) = e^{-i\phi} \frac{1}{u} h(u, \alpha) \text{ for } u > 0$$

(3.54)

$$g(u, \alpha) = e^{-i\psi} \frac{1}{u} h(-u, \alpha) \text{ for } u < 0$$

(3.55)

and finally we can write the expression of the Liouvillian $L_g$ on $\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma))$:

**Proposition 3.3.2.** The expression of the Liouvillian $L_g$ on $\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma))$ is given by:

$$L_g = L_{at} + L_f + a^*(g_1) + a(g_1^*) + a(g_2) + a^*(g_2^*)$$

(3.56)
where:

\[
g_1(u, \alpha) := \begin{cases} 
\sqrt{1 + \rho(u)} uG(u, \alpha) & : \text{if } u \geq 0 \\
\sqrt{\rho(-u)} uG^*(u, \alpha) & : \text{if } u < 0 
\end{cases}
\] (3.57)

\[
g_2(u, \alpha) := \begin{cases} 
\sqrt{\rho(u)} uG(u, \alpha) & : \text{if } u \geq 0 \\
\sqrt{1 + \rho(-u)} uG^*(u, \alpha) & : \text{if } u < 0 
\end{cases}
\] (3.58)

\[g_1, g_2 \in \mathcal{F}(L^2(\mathbb{R} \times S^2, du \times d\Sigma))\]

**Proof.** Using Eq. (3.51), and Eq. (3.2), Eq. (3.3) we can write our Liouvillian in the following form:

\[
\hat{L} = \hat{L}_{at} + \hat{L}_f + a^*_l(\sqrt{1 + \rho G_l}) + a_r(\sqrt{\rho G_l}) + a^*_l(\sqrt{1 + \rho G^*_l}) + a_r(\sqrt{1 + \rho G^*_l}) - (a^*_l(\sqrt{\rho G^*_r}) + a_r(\sqrt{\rho G^*_r}) - (a^*_l(\sqrt{1 + \rho G^*_r}) + a_r(\sqrt{1 + \rho G^*_r})) \quad (3.59)
\]

where we have used the definition:

\[
a^\sharp(G) := \int d^3 k G(k) \otimes a^\sharp(k) \quad (3.61)
\]

the proof consists simply in transforming each of the eight terms of Eq. (3.60) using Eq. (3.2), Eq. (3.3) and the gluing transformation:

\[
a^*_l(\sqrt{1 + \rho G_l}) \rightarrow (a^*(\sqrt{1 + \rho G_l}), 0) \rightarrow \begin{cases} 
a^*(\sqrt{1 + \rho uG_l}) & : \text{if } u \geq 0 \\
0 & : \text{if } u < 0 
\end{cases} \quad (3.62)
\]

\[
a^*_r(\sqrt{\rho G^*_l}) \rightarrow (0, a^*(\sqrt{\rho G^*_l})) \rightarrow \begin{cases} 
0 & : \text{if } u \geq 0 \\
a^*(\sqrt{\rho uG^*_l}) & : \text{if } u < 0 
\end{cases} \quad (3.63)
\]

From these two terms we obtain \(a^*(g_1)\).

\[
a_r(\sqrt{\rho G_l}) \rightarrow (0, a(\sqrt{\rho G_l}) \rightarrow \begin{cases} 
0 & : \text{if } u \geq 0 \\
a(\sqrt{\rho uG_l}) & : \text{if } u < 0 
\end{cases} \quad (3.64)
\]

\[
a_l(\sqrt{1 + \rho G^*_l}) \rightarrow (a(\sqrt{1 + \rho G^*_l}), 0) \rightarrow \begin{cases} 
a(\sqrt{1 + \rho uG^*_l}) & : \text{if } u \geq 0 \\
0 & : \text{if } u < 0 
\end{cases} \quad (3.65)
\]
From these two terms we obtain \( a(g_1^*) \).

\[
a_l(\sqrt{\rho G_r}) \rightarrow (a(\sqrt{\rho G_r}), 0) \rightarrow \begin{cases} 
  a(\sqrt{\rho G_r}) & : \text{if } u \geq 0 \\
  0 & : \text{if } u < 0
\end{cases}
\]  
\[ (3.66) \]

\[
a_r(\sqrt{1 + \rho G_r^*}) \rightarrow (0, a(\sqrt{1 + \rho G_r^*})) \rightarrow \begin{cases} 
  0 & : \text{if } u \geq 0 \\
  a(\sqrt{1 + \rho G_r^*}) & : \text{if } u < 0
\end{cases}
\]  
\[ (3.67) \]

From these two terms we obtain \( a(g_2^*) \), taking into account that \( G_r = \mathbb{1} \otimes TGT = \mathbb{1} \otimes G \) and \( G_r^* = \mathbb{1} \otimes G^* \).

\[
a_r^*(\sqrt{\rho G_r^*}) \rightarrow (a^*(\sqrt{\rho G_r^*}), 0) \rightarrow \begin{cases} 
  a^*(\sqrt{\rho G_r^*}) & : \text{if } u \geq 0 \\
  0 & : \text{if } u < 0
\end{cases}
\]  
\[ (3.68) \]

\[
a_r^*(\sqrt{1 + \rho G_r}) \rightarrow (0, a^*(\sqrt{1 + \rho G_r})) \rightarrow \begin{cases} 
  0 & : \text{if } u \geq 0 \\
  a^*(\sqrt{1 + \rho G_r}) & : \text{if } u < 0
\end{cases}
\]  
\[ (3.69) \]

From these two terms we obtain \( a^*(g_2^*) \). \( \square \)

As a last remark we see from Eq. (3.51) that we can multiply by phase factors in the expression of \( g_1 \) and \( g_2 \):

\[
g_1(u, \alpha) := \begin{cases} 
  e^{i\varphi} \sqrt{1 + \rho(u)} uG(u, \alpha) & : \text{if } u \geq 0 \\
  e^{i\psi} \sqrt{\rho(-u)} uG^*(-u, \alpha) & : \text{if } u < 0
\end{cases}
\]  
\[ (3.70) \]

\[ g_2(u, \alpha) = -g_1(-u, \alpha) \]  
\[ (3.71) \]

### 3.4 Positive commutator Method

In this method we look for an antiself-adjoint operator \( A \) that fulfils the named positive commutator estimate (PC estimate):

\[
E_{\Delta}(L_g)[L_g, A]E_{\Delta}(L_g) \geq \gamma E_{\Delta}(L_g)
\]  
\[ (3.72) \]

with \( \gamma > 0 \) and \( E_{\Delta} \) is the spectral projector of \( L_g \) onto the interval \( \Delta \), which contains an eigenvalue \( \varepsilon_j \neq 0 \) of \( L_0 \) but no other eigenvalue of \( L_0 \). In Eq. (3.72) \([L_g, A]\) is understood in the sense of quadratic forms on \( \mathcal{D}(N^{1/2}) \) that is \( \forall \varphi, \psi \in \mathcal{D}(L_g) \cap \mathcal{D}(A) \)

\[
\langle \psi | [L_g, A] \varphi \rangle := \langle L_g \psi | A \varphi \rangle - \langle A \psi | L_g \varphi \rangle.
\]
If Eq. (3.72) is satisfied then $L_g$ has not eigenvalues in $\Delta$, because let us suppose that $L_g \psi = \lambda \psi$ with $\lambda \in \Delta$ then $E_\Delta(L_g) \psi = \psi$ and the PC estimate gives $\langle \psi | [L_g, A] | \psi \rangle \geq \gamma$, we have taken $\| \psi \| = 1$. But on the other hand we have:

$$
\langle \psi | [L_g, A] | \psi \rangle = \langle \psi | (L_g - \lambda) | A \psi \rangle = \langle (L_g - \lambda) | A \psi \rangle + \langle A \psi | (L_g - \lambda) | \psi \rangle = 2 \text{Re} \langle (L_g - \lambda) | A \psi \rangle = 0
$$

(3.73)

which contradicts the PC estimate, that is to say in $\Delta$ there is no eigenvalue of $L_g$, the spectrum of $L_g$ is absolutely continuous in $\Delta$. However this proof is not completely valid because we are dealing with unbounded operators ($L_g$ and $A$). The problem is that $\psi$ could be in $\mathcal{D}(L_g)$ but not in $\mathcal{D}(A)$ or viceversa and then $[L_g, A] = L_g A - A L_g$ can not be defined for $\psi$. Therefore we study the $\psi \in \mathcal{D}(L_g) \cap \mathcal{D}(A)$ and we hope that $\mathcal{D}(L_g) \cap \mathcal{D}(A)$ is dense in our Hilbert space. That is to say in order to study $\psi [L_g, A] \psi$ with $\psi \notin \mathcal{D}(L_g) \cap \mathcal{D}(A)$ we take $\psi_n \in \mathcal{D}(L_g) \cap \mathcal{D}(A)$ so that $\psi_n \to \psi$.

In case that $\Delta$ contains the eigenvalue 0 we have to exclude this eigenvalue because we know (see theorem 2.2.11.) that this eigenvalue remains eigenvalue of the perturbed system. Therefore if $\Omega_{\beta \lambda}$ is an eigenvector of the eigenvalue 0 and $P_{\Omega_{\beta \lambda}}$ is the projector onto this eigenvector, the PC estimate would be modified as:

$$
E_\Delta(L_g) P_{\Omega_{\beta \lambda}} [L_g, A] P_{\Omega_{\beta \lambda}} E_\Delta(L_g) \geq \gamma E_\Delta(L_g) P_{\Omega_{\beta \lambda}}
$$

(3.74)

with again $\gamma > 0$.

For further reading see [56], [57]

### 3.5 Positive Commutator Method for physical cases

In [56], [57] it is proven that the commutator method is applicable for cases in which the matrix elements of $G(u)$ behave for $u \to 0$ as $u^p$ with $p > 2$. In this section we will see that we can also apply the commutator method for cases in which our matrix element functions $G$ behave like $\sqrt{u}$ or $1/\sqrt{u}$ as $u$ tends to zero. These two cases are usual in the physical applications.

First we will try to simplify a little the expressions of $g_1$ and $g_2$ of Eq. (3.70) and Eq. (3.71)
in the glued Hilbert space. We will take first $\varphi = \psi = 0$ and then we will vary these values:

$$g_1(u, \alpha) = \begin{cases} \sqrt{1 + \rho(u)} u \phi(u, \alpha) & : \text{if } u \geq 0 \\ \sqrt{\rho(-u)} u \bar{\phi}(-u, \alpha) & : \text{if } u < 0 \end{cases}$$ (3.75)

$$g_2(u, \alpha) = -g_1(-u, \alpha)$$ (3.76)

where $\rho(u) = \frac{1}{e^{\beta u} - 1}$, for $u \geq 0$ we have:

$$g_1(u, \alpha) = \sqrt{1 + \rho(u)} u \phi(u, \alpha) = \sqrt{\frac{e^{\beta u}}{e^{\beta u} - 1}} u \phi(u, \alpha) = e^{\beta u/2} \sqrt{\frac{u}{e^{\beta u} - 1}} u \phi(u, \alpha)$$ (3.77)

and for $u < 0$:

$$g_1(u, \alpha) = \sqrt{\rho(-u)} u \bar{\phi}(-u, \alpha) = \sqrt{\frac{1}{e^{-\beta u} - 1}} u \bar{\phi}(-u, \alpha) = \sqrt{\frac{1}{e^{-\beta u} - 1}} (-1) \sqrt{-u} \sqrt{-u} \bar{\phi}(-u, \alpha) =$$

$$= -\sqrt{\frac{u}{1 - e^{-\beta u}} \sqrt{-u} \bar{\phi}(-u, \alpha)} = -e^{\beta u/2} \sqrt{\frac{u}{e^{-\beta u} - 1}} \sqrt{-u} \bar{\phi}(-u, \alpha)$$

that is to say:

$$g_1(u, \alpha) = e^{\beta u/2} \sqrt{\frac{u}{e^{\beta u} - 1}} \begin{cases} \sqrt{u} \phi(u, \alpha) & : \text{if } u \geq 0 \\ -\sqrt{(-u)} \bar{\phi}(-u, \alpha) & : \text{if } u < 0 \end{cases}$$ (3.78)

by $u \to 0$ we have only problems with $h(u, \alpha)$ because: $e^{\beta u/2} \sqrt{\frac{u}{e^{\beta u} - 1}} \to \frac{1}{\sqrt{\beta}}$. On the other hand $h(u, \alpha)$ is an analytic function of $u$ except maybe at 0 where we must study the behavior. This behavior depends on how $g(u, \alpha)$ behaves at 0.

We will deal now with the two cases mentioned before:

1 Case: $g(u, \alpha) \xrightarrow{u \to 0} \sqrt{u}$

$$h(u, \alpha) = \begin{cases} \sqrt{u} \phi(u, \alpha) & : \text{if } u \geq 0 \xrightarrow{u \to 0} \sqrt{u} \sqrt{u} : \text{if } u \geq 0 \xrightarrow{u \to 0} \\ -\sqrt{(-u)} \bar{\phi}(-u, \alpha) & : \text{if } u < 0 \xrightarrow{u \to 0} -\sqrt{(-u)} \sqrt{(-u)} : \text{if } u < 0 \end{cases}$$

In this case for $g_1(u, \alpha)$ we do not have problems.

2 Case: $g(u, \alpha) \xrightarrow{u \to 0} \frac{1}{\sqrt{u}}$
\[ h(u) = \begin{cases} \sqrt{u} g(u, \alpha) & : \text{if } u \geq 0 \quad \text{as } u \to 0 \\ -\sqrt{(-u)} g(-u, \alpha) & : \text{if } u < 0 \end{cases} \]

and we have a discontinuity at point 0, which implies that the first derivative does not belong to \( L^2 \). Therefore we could not apply the positive commutator method to this case. The way to repair this problem, is to multiply by a phase factor our function \( h(u, \alpha) \) without changing its norm (this is possible because of the fact that the difference in behavior of \( h(u, \alpha) \) for \( u \geq 0 \) and \( u < 0 \) is only a sign).

\[ h(u, \alpha) = \begin{cases} e^{i\varphi} \sqrt{u} g(u, \alpha) & : \text{if } u \geq 0 \\ -e^{i\psi} \sqrt{(-u)} g(-u, \alpha) & : \text{if } u < 0 \end{cases} \]

and we choose \( \varphi = 0 \) and \( \psi = \pi \) then we would have for \( u \to 0 \):

\[ h(u, \alpha) = \begin{cases} e^{0} \sqrt{u} g(u, \alpha) & : \text{if } u \geq 0 \quad \text{as } u \to 0 \\ -e^{i\pi} \sqrt{(-u)} g(-u, \alpha) & : \text{if } u < 0 \end{cases} \]

\[ g_2(u, \alpha) = -g_1(-u, \alpha) = -e^{-\beta u/2} \sqrt{-u} \frac{\sqrt{u} g(-u, \alpha)}{e^{-\beta u} - 1} \]

1 Case: \( g(u, \alpha) \xrightarrow{u \to 0} \sqrt{u} \)

\[ h'(u, \alpha) = \begin{cases} \sqrt{-u} g(-u, \alpha) & : \text{if } u \geq 0 \quad \text{as } u \to 0 \\ (-1)\sqrt{-u} g(u, \alpha) & : \text{if } u < 0 \end{cases} \]
In this case for \( g_2(u, \alpha) \) we do not have problems.

2 Case: \( g(u, \alpha) \xrightarrow{u \to 0} \frac{1}{\sqrt{u}} \)

\[
h'(u) = \begin{cases} \sqrt{-u}g(-u, \alpha) & \text{if } u \geq 0 \\ -\sqrt{u}g(u, \alpha) & \text{if } u < 0 \end{cases} \quad \begin{cases} 1 & \text{if } u \geq 0 \\ -1 & \text{if } u < 0 \end{cases}
\]

(3.86)

Which implies as before that the first derivative does not belong to \( L^2 \). We solve again this problem with phase factors \( \varphi = 0 \) and \( \psi = \pi \):

\[
h'(u, \alpha) = \begin{cases} \sqrt{-u}g(-u, \alpha) & \text{if } u \geq 0 \\ e^{i\pi}(-1)\sqrt{u}g(u, \alpha) & \text{if } u < 0 \end{cases} \quad \begin{cases} 1 & \text{if } u \geq 0 \\ 1 & \text{if } u < 0 \end{cases}
\]

(3.87)

### 3.6 Uniqueness of KMS states

With the positive commutator method we will study now the uniqueness of the KMS state for a model without atomic liouvillian that is to say:

\[
L_g = L_f + g(l[I'] - r[I'])
\]

(3.88)

where \( I' \) is a local perturbation.

From theorem 2.2.11. we know that we can construct (by means of a Dyson-series) an eigenvector of the operator \( L_g \) for the eigenvalue 0:

\[
L_g \Omega_g^\beta = 0
\]

(3.89)

In the same way we construct a vector \( \hat{\Omega}_g^\beta = I_0 \Omega_g^\beta \), with \( I_0 := I_C \otimes I_T \) as in Eq. (3.35), in the Hilbert space \( \mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{H}_f \otimes \mathcal{H}_f \), without interaction we have \((g = 0)\):

\[
\hat{\Omega}_0^\beta = Z_{\beta,0}^{-1/2} \sum_{j=0}^{N-1} e^{-\beta E_j/2} \varphi_j \otimes \varphi_j \otimes \Omega_f \otimes \Omega_f
\]

(3.90)

and with interaction:

\[
\hat{\Omega}_g^\beta := Z_{\beta,0}^{-1/2} e^{-\beta L_g/2} \hat{\Omega}_0^\beta = Z_{\beta,0}^{-1/2} e^{\beta L_g/2} \hat{\Omega}_g^\beta
\]

(3.91)

We have:

\[
0 = L_g \Omega_g^\beta = I_0^{-1} L_g I_0 \Omega_g^\beta = I_0^{-1} L_g \hat{\Omega}_g^\beta
\]

(3.92)
that means:

\[ L_g \hat{\Omega}_g^\beta = 0 \]  

(3.93)

That is to say \( \hat{\Omega}_g^\beta \) is an eigenvector of \( L_g \) for the eigenvalue 0.

Let \( P_\beta \) be the orthogonal projector onto \( \hat{\Omega}_g^\beta \).

Suppose that we can construct an antiself-adjoint operator \( A_\beta \) with:

\[ P_\beta \perp [L_g, A_\beta] P_\beta \geq \gamma P_\beta \]  

(3.94)

for a \( \gamma > 0 \).

From the relation:

\[ \langle \hat{\Omega}_0^\beta | e^{-\beta L_g \beta / 2} \hat{\Omega}_0^\beta \rangle = \sum_{n=0}^{\infty} g^n \int_0^{\beta/2} \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \Omega_0^\beta I(\tau_1) I(\tau_2) \cdots I(\tau_n) \Omega_0^\beta \]  

(3.95)

we see that:

\[ \left| \langle \hat{\Omega}_0^\beta | \hat{\Omega}_0^\beta \rangle \right| \geq 1 - \mathcal{O}(g\beta) \]  

(3.96)

Let us suppose that we have more than one KMS vectors \( \hat{\Omega}_g^{\beta,j} \) for \( j = 1 \ldots k \) with \( \hat{\Omega}_g^{\beta,1} = \hat{\Omega}_g^\beta \) and \( \| \hat{\Omega}_g^{\beta,j} \| = 1 \) i.e.:

\[ L_g \hat{\Omega}_g^{\beta,j} = 0, \forall j \]  

(3.97)

**Proposition 3.6.1.**

\[ |\langle \hat{\Omega}_g^{\beta,j} | \hat{\Omega}_0^\beta \rangle|^2 \geq 1 - const(g\beta)/\gamma, \forall j \]  

(3.98)

**Proof.** It is clear that:

\[ \langle \hat{\Omega}_g^{\beta,j} | [L_g, A_\beta] \hat{\Omega}_g^{\beta,j} \rangle = 0 \]  

(3.99)

Using now that \( P_\beta + P_\beta^\perp = \mathbb{1} \):

\[ 0 = \langle \hat{\Omega}_g^{\beta,j} | [L_g, A_\beta] \hat{\Omega}_g^{\beta,j} \rangle \]  

(3.100)

\[ = \langle P_\beta \hat{\Omega}_g^{\beta,j} | [L_g, A_\beta] P_\beta \hat{\Omega}_g^{\beta,j} \rangle (I) \]  

(3.101)

\[ + 2 Re \langle P_\beta \hat{\Omega}_g^{\beta,j} | [L_g, A_\beta] P_\beta^\perp \hat{\Omega}_g^{\beta,j} \rangle (II) \]  

(3.102)

\[ + \langle P_\beta^\perp \hat{\Omega}_g^{\beta,j} | [L_g, A_\beta] P_\beta^\perp \hat{\Omega}_g^{\beta,j} \rangle (III) \]  

(3.103)

Let us analyze the different terms. We begin with I:

\[ I = (1 - const(g\beta)) \left\{ \frac{\langle \hat{\Omega}_0^\beta | [L_0, A_\beta] \hat{\Omega}_0^\beta \rangle + g\langle \hat{\Omega}_0^\beta | [I^\beta I - r[I, A_\beta] \hat{\Omega}_0^\beta \rangle}{0} \right\} \]  

(3.104)
Now with II:

\[ II = 2 \text{Re} \left\{ (\mathcal{P}_\beta \hat{\Omega}_g^{\beta,j} | L_g A_\beta P_\beta^\perp \hat{\Omega}_g^{\beta,j}) \right\} \]  \hspace{1cm} (IIA) \hspace{1cm} (3.105)

\[ - (\mathcal{P}_\beta \hat{\Omega}_g^{\beta,j} | A_\beta L_g P_\beta^\perp \hat{\Omega}_g^{\beta,j}) \right\} \]  \hspace{1cm} (IIB) \hspace{1cm} (3.106)

noting that \( L_0 P_\beta \hat{\Omega}_g^{\beta(j)} = 0 \), then:

\[ II A = 2 \text{Re} \left\{ (\mathcal{P}_\beta \hat{\Omega}_g^{\beta,j} | L_0 A_\beta P_\beta^\perp \hat{\Omega}_g^{\beta,j}) + \right\} = 0 \]

\[ + 2g \text{Re}(\mathcal{P}_\beta \hat{\Omega}_g^{\beta,j} | ([l[I] - r[I]]) A_\beta P_\beta^\perp \hat{\Omega}_g^{\beta,j}) = O(g) \]  \hspace{1cm} (3.107)

\[ II B = -2 \text{Re} \left\{ (\mathcal{P}_\beta \hat{\Omega}_g^{\beta,j} | A_\beta L_0 \hat{\Omega}_g^{\beta,j}) + \right\} = 0 \]

\[ + 2g \text{Re}(\mathcal{P}_\beta \hat{\Omega}_g^{\beta,j} | A_\beta ([l[I] - r[I]]) P_\beta \hat{\Omega}_g^{\beta,j}) = O(g) \]  \hspace{1cm} (3.109)

and finally III, applying the commutator estimate:

\[ III \geq \gamma |\langle \hat{\Omega}_g^{\beta,j} | P_\beta^\perp \hat{\Omega}_g^{\beta,j} \rangle| \]  \hspace{1cm} (3.111)

Taking all the results together:

\[ 0 = I + II + III \]

\[ = O(g) + O(g) + \gamma [1 - |\langle \hat{\Omega}_g^{\beta,j} | \hat{\Omega}_g^{\beta,j} \rangle|^2] \]  \hspace{1cm} (3.112)

\[ \gamma [1 - |\langle \hat{\Omega}_g^{\beta,j} | \hat{\Omega}_g^{\beta,j} \rangle|^2] \leq \text{const}(g/\beta) \]  \hspace{1cm} (3.113)

and therefore:

\[ |\langle \hat{\Omega}_g^{\beta,j} | \hat{\Omega}_g^{\beta,j} \rangle|^2 \geq 1 - \text{const}(g/\beta)/\gamma \]  \hspace{1cm} (3.114)

and finally:

\[ |\langle \hat{\Omega}_g^{\beta,j} | \hat{\Omega}_g^{\beta,j} \rangle|^2 \geq \frac{3}{4} \]  \hspace{1cm} \forall j \hspace{1cm} (3.115)

From this proposition follows that taking an adequate \( g \) we can have:

\[ |\langle \hat{\Omega}_g^{\beta,j} | \hat{\Omega}_g^{\beta,j} \rangle| \geq \frac{3}{4} \]  \hspace{1cm} \forall j \hspace{1cm} (3.116)
and from this inequality follows the uniqueness of the KMS state with the following reasoning:

We have $\hat{\Omega}^\beta_j = \alpha_j \hat{\Omega}_0^\beta + \chi^{(j)}$ with $\chi^{(j)} \perp \hat{\Omega}_0^\beta$ and $|\alpha_j| \geq \frac{3}{4}$ then, we have:

$$|\langle \hat{\Omega}^\beta_i | \hat{\Omega}^\beta_j \rangle| = |\bar{\alpha}_i \alpha_j + \langle \chi^{(i)} | \chi^{(j)} \rangle| \geq \frac{9}{16} - |\langle \chi^{(i)} | \chi^{(j)} \rangle| \geq \frac{1}{8} \quad (3.117)$$
Chapter 4

Complex Deformations and Feshbach Map

In this chapter we will see the method of complex deformations which consists in deforming the spectrum of our Liouvillian in order to show that some parts of our spectrum are of absolute continuous nature. The idea is that in case that the spectrum is of absolute nature in a interval of the axis, we can make an analytical continuation of the matrix elements of the resolvent of our Liouvillian near the axis in this interval from the upper half plane to the lower half plane. We will see two of such deformations, namely the complex translations and the complex dilatations.

Before we begin with the complex deformations, we will present the spectral information that we know about the Liouvillians. We can deduce the following diagrams from Eq. (3.16) and Eq. (3.28) and from Fig.2 and Fig.3:

\[
\sigma(L_{at}) = \begin{array}{c}
0 \\
\end{array} \quad \text{; } L_{at} = H_{at} \otimes 1 - 1 \otimes H_{at}
\]

\[
\sigma(L_f) = \begin{array}{c}
0 \\
\end{array} \quad \text{; } L_f = \int dk \omega(k) [a^*\!(k)a\!(k) - a^*\!(k)a\!(k)]
\]

\[
\sigma(L_0) = \begin{array}{c}
0 \\
\end{array} \quad \text{; } L_0 = L_{at} + L_f
\]

where the thick lines denote an absolute spectrum of the operator, for example \( L_{at} \) has only pure point spectrum, \( L_f \) has pure point spectrum at 0 and the rest of the real axis...
is absolute continuous. For example if \( \varphi_i \) are eigenvectors of \( H_{at} \) for the eigenvalues \( E_i, \quad i \in \{1 \ldots N\} \) then:

\[
L_{at} \varphi_i \otimes \varphi_j = (H_{at} \otimes \mathbb{1} - \mathbb{1} \otimes H_{at}) \varphi_i \otimes \varphi_j = H_{at} \varphi_i \otimes \varphi_j - \varphi_i \otimes H_{at} \varphi_j = (E_i - E_j) \varphi_i \otimes \varphi_j
\]

(4.1)

which explains why the pure point spectrum of \( L_{at} \) and also for \( L_0 \) are symmetric.

### 4.1 Complex Dilatation

For \( \theta \in \mathbb{R} \) we define the following transformation:

\[
(u(\theta) f)(x) := e^{-3\theta/2} f(e^{-\theta} x) \quad f \in L^2(\mathbb{R}^3, d^3 x)
\]

(4.2)

**Proposition 4.1.1.** For \( \theta \in \mathbb{R} \) is \( u(\theta) \) an unitary transformation

**Proof.**

\[
\|u(\theta)f\|^2_{L^2} = \int_{\mathbb{R}^3} dx |(u(\theta)f)(x)|^2 = \int_{\mathbb{R}^3} dx e^{-3\theta}|f(e^{-\theta} x)|^2
\]

Doing the substitution \( y = e^{-\theta} x \) and Jacobian \( \frac{Dx}{Dy} = e^{3\theta} \)

\[
\|u(\theta)f\|^2_{L^2} = \int_{\mathbb{R}^3} dy \left| \frac{Dx}{Dy} \right| e^{-3\theta}|f(y)|^2 = \int_{\mathbb{R}^3} dy |f(y)|^2 = \|f\|^2_{L^2}
\]

\( \square \)

We define now \( U(\theta) \) as the operator obtained from \( u(\theta) \) by the process of second quantization. Let be:

\[
\psi \in \mathcal{F} \quad \text{i.e.} \quad \psi = \{\psi_n\}_{n \geq 0} \quad \text{with} \quad \psi_n(k_1, k_2, \ldots k_n) \in L^2(\mathbb{R}^{3n}, d^{3n} k)
\]

(4.3)

In the sector \( n \) the dilatation is defined as:

\[
[U(\theta)\psi]|_{n}(k_1, k_2, \ldots k_n) := e^{-(3\theta/2)n}\psi_n(e^{-\theta} k_1 \ldots e^{-\theta} k_n)
\]

(4.4)

We now define the group of dilatations on the Araki-woods Hilbert space:

\[
\hat{U}(\theta) := \mathbb{1}_{at} \otimes \mathbb{1}_{at} \otimes U(\theta) \otimes U(\theta)^{-1}
\]

(4.5)

**Proposition 4.1.2.** Under this unitary transformation the field-Liouvillican transforms like:

\[
L_f(\theta) := \hat{U}(\theta)L_f \hat{U}(\theta)^{-1} = e^{-\theta}(H_f \otimes \mathbb{1}_f) - e^\theta(\mathbb{1}_f \otimes H_f)
\]

(4.6)
Proof. Let us see first how \( H_f \) transforms, we recall:

\[
[H_f \psi]_n(k_1, k_2, \ldots k_n) = \sum_{j=1}^{n} |k_j| \psi_n(k_1, k_2, \ldots k_n) \quad (4.7)
\]

\[
[U(\theta)H_f U(\theta)^{-1}]\psi_n(k_1, k_2, \ldots k_n) = e^{-\theta} [H_f U(\theta)^{-1}]\psi_n(e^{-\theta} k_1, e^{-\theta} k_2, \ldots e^{-\theta} k_n) = \sum_{j=1}^{n} |e^{-\theta} k_j|[U(\theta)^{-1}]\psi_n(e^{-\theta} k_1, e^{-\theta} k_2, \ldots e^{-\theta} k_n) = \]

noting that \( U(\theta)^{-1} = U(-\theta) \) we have:

\[
e^{-\theta} [H_f \psi]_n(k_1, k_2, \ldots k_n) = \sum_{j=1}^{n} |k_j| \psi_n(k_1, k_2, \ldots k_n) = e^{-\theta} [H_f \psi]_n(k_1, k_2, \ldots k_n)
\]

and that means:

\[
U(\theta)H_f U(\theta)^{-1} = e^{-\theta} H_f \quad (4.8)
\]

With this result we have:

\[
\hat{U}(\theta)L_f \hat{U}(\theta)^{-1} = 1_{at} \otimes 1_{at} \otimes U(\theta) \otimes U(\theta)^{-1} (1_{at} \otimes 1_{at} \otimes H_f \otimes 1_{at} \otimes 1_{at} \otimes 1_f \otimes H_f) \times (1_{at} \otimes 1_{at}) \otimes \left(U(\theta)H_f U(\theta)^{-1} \otimes U(\theta)^{-1}U(\theta) - U(\theta)U(\theta)^{-1} \otimes U(\theta)^{-1}H_f U(\theta)\right) = \]

\[
(1_{at} \otimes 1_{at}) \otimes (e^{-\theta} H_f \otimes 1_f - 1_f \otimes e^{\theta} H_f)
\]

\[
\square
\]

With this expression for \( L_f(\theta) \) we can easily deduce the expression for \( L_0(\theta) \):

\[
L_0(\theta) = L_{at} + L_f(\theta) = L_{at} + e^{-\theta}(H_f \otimes 1_f) - e^{\theta}(1_f \otimes H_f) \quad (4.9)
\]

We recall that the unitary transformation \( \hat{U}(\theta) = 1_{at} \otimes 1_{at} \otimes U(\theta) \otimes U(\theta)^{-1} \) will not affect \( L_{at} \) because of the fact that the two first factors are unit operators. We can develop a little more the expression of \( L_f \) using the equation \( e^\theta = \cosh \theta + \sinh \theta \) then:

\[
L_f(\theta) = e^{-\theta}(H_f \otimes 1_f) - e^{\theta}(1_f \otimes H_f) = \quad (4.10)
\]

\[
= (\cosh \theta - \sinh \theta)(H_f \otimes 1_f) - (\cosh \theta + \sinh \theta)(1_f \otimes H_f) = \quad (4.11)
\]

\[
= \cosh \theta (H_f \otimes 1_f - 1_f \otimes H_f) - \sinh \theta (H_f \otimes 1_f + 1_f \otimes H_f) = \quad (4.12)
\]

\[
= L_f \cosh \theta - L_{aux} \sinh \theta \quad (4.13)
\]
where we have defined:

\[ L_{aux} = H_f \otimes 1_f + 1_f \otimes H_f \]  

(4.14)

and finally we have:

\[ L_0(\theta) = L_{at} + L_f \cosh \theta - L_{aux} \sinh \theta \]  

(4.15)

We will use this last equation in the successive chapters. Let us suppose now that \( \theta \notin \mathbb{R} \), we take \( \theta = i\vartheta \) with \( \vartheta > 0 \). We will analyze the spectrum of the different terms of Eq. (4.6):

\[ e^{-i\vartheta}(H_f \otimes 1) : \]

\[ e^{-i\vartheta} \sigma(H_f) \]

\[ \sigma(e^{-i\vartheta}H_f) = e^{-i\vartheta} \sigma(H_f) \]

\[ e^{i\vartheta}(1_f \otimes H_f) : \]

\[ e^{i\vartheta} \sigma(H_f) \]

\[ \sigma(e^{i\vartheta}H_f) = e^{i\vartheta} \sigma(H_f) \]

**Fig. 9 Rotation of the spectrum of** \( H_f \). **Top:** rotated by an angle of \(-\vartheta\). **Bottom:** rotated by an angle of \( \vartheta \)

Superposing both diagrams and taking into account the minus sign of Eq. (4.6) that makes the second diagram go from the first quadrant to the third one we will have:
The shaded area of figure 10 is the spectrum of $L_f(i\vartheta)$. In the same way we would obtain the spectrum of $L_0(i\vartheta)$ by doing the same procedure as before (before it was done for the eigenvalue 0) for each of the embedded eigenvalues of $L_0$.

One can see that there is a dense domain $D \subset \hat{\mathcal{H}}$ with the property that for every $\psi \in D$, $\hat{U}(\theta)\psi$ is an analytic function of $\theta$ for $\theta \in \Sigma_{\frac{\pi}{2}}$, i.e. for $|\theta| < \pi/2$. We can consider for example:

$$D := \text{span}\{\varphi_l \otimes \varphi_r \otimes \psi_l \otimes \psi_r | \varphi_{l/r} \in \mathcal{H}_{\text{at}}, \psi_{l/r} \in \mathcal{D}_{\text{Gauss}}\}$$  \hspace{1cm} (4.17)

where $\mathcal{D}_{\text{Gauss}}$ consists of translations and dilatations of Gaussians:

$$\mathcal{D}_{\text{Gauss}} := \{a^*(f_1) \cdots a^*(f_n)\Omega | f_j(\vec{k}) = \exp[-(\vec{k} - \vec{k}_j)^2/2\sigma_j^2], k_j \in \mathbb{R}^3, \sigma_j > 0\}$$  \hspace{1cm} (4.18)

We consider now the function:

$$f(\theta) = \langle \hat{U}(\theta)\varphi | (L_0(\theta) - z)^{-1}\hat{U}(\theta)\psi \rangle$$  \hspace{1cm} (4.19)
for $\theta \in \mathbb{R}$, we see that:

$$
\langle \hat{U}(\theta)\varphi | (L_0(\theta) - z)^{-1}\hat{U}(\theta)\psi \rangle = \langle \varphi | \hat{U}(-\theta)(L_0(\theta) - z)^{-1}\hat{U}(\theta)\psi \rangle = \\
= \langle \varphi | (\hat{U}(-\theta)L_0(\theta)\hat{U}(\theta) - z)^{-1}\psi \rangle = \langle \varphi | (L_0 - z)^{-1}\psi \rangle
$$

This equality is valid for all $\theta \in \mathbb{R}$. From the analytic continuation principle it follows that it is also valid for $\theta \in \mathbb{C}$. On the other hand from the equation:

$$
\langle \varphi | (L_0 - z)^{-1}\psi \rangle = \langle \hat{U}(\theta)\varphi | (L_0(\theta) - z)^{-1}\hat{U}(\theta)\psi \rangle
$$

we see that the r.s. is an analytic function of $z$ for $z \in \rho(L_0(\theta))$ (resolvent of $L_0(\theta)$). This equation permits us to do an analytic continuation in $z$ of the left hand side of Eq. (4.20) for the values of $z$ in the resolvent set of $L_0(\theta)$. With these considerations we have proved the following proposition:

**Proposition 4.1.3.** Let $\Delta$ be an interval, which does not contain any eigenvalue $\varepsilon_i$ of $L_0$, that is to say:

for $\Delta$ beginning on the positive half axis, there exists a $j$ with $\Delta \subseteq (\varepsilon_j + \delta, \varepsilon_{j+1} - \delta)$

for $\Delta$ beginning on the negative half axis, there exists a $j$ with $\Delta \subseteq (-\varepsilon_{j+1} + \delta, -\varepsilon_j - \delta)$

with $0 < \delta < \min_{i,j} |\varepsilon_i - \varepsilon_j|$; $i, j = -1, 0 \ldots$ and where $\varepsilon_{-1} = -\varepsilon_{-1} = 0$ (see Fig.11), then $L_0$ has an absolutely continuous spectrum on $\Delta$.

We can do a similar analysis for the operator $L_g$ in which case we will have instead of Eq. (4.20):

$$
\langle \varphi | (L_g - z)^{-1}\psi \rangle = \langle \hat{U}(\theta)\varphi | (L_g(\theta) - z)^{-1}\hat{U}(\theta)\psi \rangle
$$

(4.21)

The problem now is that we do not know the spectrum of $L_g(\theta) = L_0(\theta) + gW(\theta)$. If we knew the spectrum of $L_g(\theta)$ we could formulate a similar proposition as proposition 4.1.3. for the intervals $\Delta$ on the real axis, which do not contain any eigenvalue of $L_g(\theta)$. In order to know the spectrum of $L_g(\theta)$ we will use the Feshbach map method. This method will permit us to know this spectrum except near 0 (see Fig.14), where the Feshbach map only gives us an incomplete information. In order to know the spectrum near 0 completely we will use the renormalization group method.

Let us now analyze $W(\theta)$. The expression of $W(\theta)$ is given by the following proposition:
**Proposition 4.1.4.** The expression of $W(\theta)$ on the Hilbert space $\mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes F \otimes F$ is:

$$W(\theta) = e^{-3\theta^2/2} \int dk \left\{ \sqrt{1 + \rho(e^{-\theta}k)G_l(e^{-\theta}k) - \sqrt{\rho(e^{-\theta}k)G_r(e^{-\theta}k)}} a^*_l(k) + 
+ e^{-3\theta^2/2} \int dk \sqrt{1 + \rho(e^{-\theta}k)G^*_l(e^{-\theta}k) - \sqrt{\rho(e^{-\theta}k)G^*_r(e^{-\theta}k)}} a^*_l(k) - 
- e^{3\theta^2/2} \int dk \sqrt{1 + \rho(e^{\theta}k)G^*_r(e^{\theta}k) - \sqrt{\rho(e^{\theta}k)G^*_l(e^{\theta}k)}} a^*_r(k) - 
- e^{3\theta^2/2} \int dk \sqrt{1 + \rho(e^{\theta}k)G_r(e^{\theta}k) - \sqrt{\rho(e^{\theta}k)G_l(e^{\theta}k)}} a_r(k) \right\}$$

(4.22)

**Proof.** The proposition follows immediately from Eq. (3.37), Eq. (3.38) and from $\tilde{U}(\theta) := \mathbb{1}_{at} \otimes \mathbb{1}_{at} \otimes U(\theta) \otimes U(\theta)^{-1}$. 

we try now to write this expression in a easy way, therefore we define:

$$\mu_\theta(k) := \sqrt{\rho(e^{-\theta}k)}$$

(4.23)

$$\nu_\theta(k) := \sqrt{(1 + \rho(e^{-\theta}k)}$$

(4.24)

and defining:

$$G^\theta_{+,l} := e^{-3\theta^2/2} \left\{ \nu_\theta(k)G_l(e^{-\theta}k) - \mu_\theta(k)G^*_r(e^{-\theta}k) \right\}$$

(4.25)

$$G^\theta_{-,l} := e^{-3\theta^2/2} \left\{ \nu_\theta(k)G^*_l(e^{\theta}k) - \mu_\theta(k)G_r(e^{-\theta}k) \right\}$$

(4.26)

$$G^\theta_{+,r} := e^{3\theta^2/2} \left\{ \mu_\theta(k)G^*_l(e^{\theta}k) - \nu_\theta(k)G^*_r(e^{\theta}k) \right\}$$

(4.27)

$$G^\theta_{+,r} := e^{3\theta^2/2} \left\{ \mu_\theta(k)G^*_l(e^{\theta}k) - \nu_\theta(k)G^*_r(e^{\theta}k) \right\}$$

(4.28)

and the expression of $W(\theta)$ becomes:

$$W(\theta) = a^*_l(G^\theta_{+,l}) + a_l(G^\theta_{-,l}) + a^*_r(G^\theta_{+,r}) + a_r(G^\theta_{-,r}) = \sum_{\sigma = \pm} \sum_{\tau = l,r} a^*_\sigma(G^\theta_{\sigma,\tau})$$

(4.29)

We will use this notation in the next chapter with the renormalization group method.

We know the spectrum of $L_0(\theta)$ but we do not know the spectrum of $W(\theta)$. Being $gW(\theta)$ a perturbation, we can hope that a perturbative analysis could help us and so it is. We will see it in the section of the Feshbach map method.
4.2 Complex Translation

In this case the deformation of the spectrum consists in translating the continuous part of the spectrum downwards and the pure point spectrum remains on the real axis. We define the unitary translation group:

\[(u(a)f)(x) := f(x + a) \equiv f^a(s) \quad f \in L^2(\mathbb{R}^3, d^3x)\] (4.30)

It is trivial to see that this defines an unitary transformation. We denote by \(U(a) = \Gamma(u(a))\) the second quantization of \(u(a)\).

**Definition 4.2.1.** Let be \(\Sigma_\delta := \{z \in \mathbb{C} : |\text{Im}z| < \delta\}\), we define \(H^2(\delta, \mathcal{H})\) as the Hilbert space of all functions \(f : \Sigma_\delta \mapsto \mathcal{H}\) which are analytic in \(\Sigma_\delta\) and satisfy:

\[
\|f\|^2_{H^2(\delta, \mathcal{H})} := \sup_{|a| < \delta} \int_{-\infty}^{\infty} \|f(x + ia)\|^2_{\mathcal{H}} dx < \infty
\] (4.31)

We need also the following definition:

**Definition 4.2.2.** Given a function on \(\mathbb{R}^3\) we define a function \(\tilde{f}\) on \(\mathbb{R} \times S^2\) by the formula:

\[
\tilde{f}(u, \alpha) = \begin{cases} 
    u^{1/2}f(u, \alpha) & \text{if } u \geq 0 \\
    -|u|^{1/2}\tilde{f}(-u, \alpha) & \text{if } u < 0
    \end{cases}
\] (4.32)

The principal result with complex translations is:

**Theorem 4.2.3.** Suppose that the form factor \(G\) satisfies:

- **H1:** \((k + k^{-1})G(k) \in L^2(\mathbb{R}^3)\)
- **H2:** There exists \(\delta > 0\) such that \(\tilde{G} \in H^2(\delta, \mathcal{H})\). Then:
  
  There exists a dense subspace \(S \subset \mathcal{H}\) and for each \(\eta \in (0, \delta)\) a constant \(\Lambda(\delta) > 0\) such that for \(g \in (-\Lambda(\delta), \Lambda(\delta))\) and \(\phi, \varphi \in S\), the functions:

\[
z \mapsto \langle \phi | (L_g - z)^{-1} \varphi \rangle
\] (4.33)
have a meromorphic continuation from the upper half-plane onto the region:

\[ \mathcal{O} := \{ z : \text{Im}(z) > -\eta \} \] (4.34)

Moreover, let \( \varepsilon_j \) be a simple eigenvalue of \( L_\Omega \), then the eigenvalues of \( L_g \) satisfy \( \varepsilon_j(g) = \varepsilon_j + g^2 a_j^2 + O(g^4) \), where \( a_j \) is a number related to the level shift operator.

For a proof see [46].

From this theorem we can formulate a result similar to prop. 4.1.3. for an interval \( \Delta \), which does not contain any eigenvalue (see Fig.12).

If we write the Liouvillian in the gluing Hilbert space as explained in chapter 3 in Eq. (3.56) we obtain functions \( g's \) of the form (see also Eq. (3.78)):

\[
g_1(u, \alpha) = e^{\beta u/2} \sqrt{\frac{u}{e^{\beta u} - 1}} \begin{cases} \sqrt{u} g(u, \alpha) & \text{if } u \geq 0 \\ -\sqrt{(-u)} \tilde{g}(-u, \alpha) & \text{if } u < 0 \end{cases}
\] (4.35)

i.e.:

\[
g_1(u, \alpha) = e^{\beta u/2} \left( \frac{u}{e^{\beta u} - 1} \right)^{1/2} \tilde{g}_1(u, \alpha) \] (4.36)

where we have used the definition Eq. (4.32) of \( \tilde{f} \). Therefore we see that if \( \tilde{g}_1 \in H^2(\delta, \mathcal{H}) \) for some \( \delta > 2\pi/\beta \) then \( g_1 \in H^2(\delta - \varepsilon, \mathcal{H}) \) for any \( 0 < \varepsilon < 2\pi/\beta \) but \( g_1 \notin H^2(\delta + \varepsilon, \mathcal{H}) \) for any \( \varepsilon \). Therefore we need \( \delta < 2\pi/\beta \) in order to satisfy the hypothesis of the last theorem. And that means that our gap \( \delta \) (see Fig.12) depends directly proportional on the temperature, which forbids us the use of a limiting argument to analyze the zero temperature case. This is one of the problems of this method (the non-uniformity in the temperature). There is another one, namely the impossibility of applying the last theorem for form factors \( G(k) \) that behave for \( k \to 0 \) like:

\[
|k|^{1/2 + \mu} \quad \mu \text{ small}
\] (4.37)

in which case we have to use complex dilatation followed generally by one Feshbach map or more Feshbach maps (RG). However for form factors which behave when \( k \to 0 \) as \( |k|^\alpha \) with \( \alpha = \pm \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots \) we do not need to change the method and the complex translations with analytic perturbation theory give good results.
4.3 Feshbach Map

The idea of the Feshbach map $F$ is that it permits us to restrict the study of the spectrum of the operator $L_g(\theta)$ on $\mathcal{H}$ to the study of the operator $F_P(\theta)$ on a smaller space $P\mathcal{H}$ without loss of information in the spectrum. In other words, $z \in \sigma(L_g(\theta))$ iff $0 \in \sigma(F_P(L_g(\theta) - z))$. Besides, the form of the Feshbach map is quite appropriate in order to do a perturbative analysis with the perturbation $gW(\theta)$.

Let $L_g(\theta)$ be a closed operator on a Hilbert space $\mathcal{H}$ and let $P$ be a closed bounded projection operator whose range is in the domain of $L_g(\theta)$. We define:

$$\mathcal{L}_g(\theta) := PL_g(\theta)P \quad P = 1 - P$$

Let $z$ belong to the resolvent set of $\mathcal{L}_g(\theta)|_{P\mathcal{H}}$. We assume that the operators $PL_g(\theta)P[\mathcal{L}_g(\theta) - z]^{-1/2}$ and $\mathcal{L}_g(\theta) - z|^{-1/2}P\mathcal{L}_g(\theta)P$ are bounded. Then we can define an operator $F_P(L_g(\theta) - z)$ at $L_g(\theta) - z$ associated to the projection $P$ as an operator acting on the Hilbert space $P\mathcal{H}$:

$$F_P(L_g(\theta) - z) := P[(L_g(\theta) - z) - g^2PL_g(\theta)P(\mathcal{L}_g(\theta) - z)^{-1}P\mathcal{L}_g(\theta)]P$$

The following theorem characterizes the important property of the isospectrality of the Feshbach map:

**Theorem 4.3.1.** Assume that $L_g(\theta) = PL_g(\theta)P$ is invertible on $P\mathcal{H}$ and that $(PL_g(\theta)P)^{-1}P$, $PWP(PL_g(\theta)P)^{-1}$, $(PL_g(\theta)P)^{-1}PWP$, $PWP(PL_g(\theta)P)^{-1}PWP$, $PWP$ all extend to bounded operators on $\mathcal{H}_f \otimes (\mathcal{H}_{at} \otimes \mathcal{H}_{at})$. Then:

a) $z$ is an eigenvalue of $L_g(\theta)$ iff $0$ is an eigenvalue of $F_P(L_g(\theta) - z)$ and the multiplicity of $z \in \sigma_{pp}(L_g(\theta))$ is the same as the multiplicity of $0 \in \sigma_{pp}(F_P(L_g(\theta) - z))$.

b) The operator $F_P(L_g(\theta) - z)$ is invertible on $P\mathcal{H}$ iff $L_g(\theta) - z$ is invertible on $\mathcal{H}$. In this case $[F_P(L_g(\theta) - z)]^{-1} = P(L_g - z)^{-1}P$ and for $\varphi, \psi \in P\mathcal{H}$ and $z \notin \sigma(L_g(\theta))$:

$$\langle \varphi | (L_g(\theta) - z)^{-1} \psi \rangle = \langle \varphi | [F_P(L_g(\theta)) - z]^{-1} \psi \rangle.$$

For a proof see [14] or [8].

As projection, we will use:

$$P := P_{\epsilon}^{at} \otimes P_{\rho}^{aux}$$
where \( \varepsilon \) is any eigenvalue of \( L_{at} \) (we recall \( \varepsilon = E_i - E_j \) where \( E_j \) are eigenvalues of \( H_{at} \) for \( j = 0 \ldots N - 1 \)) and \( P_{\rho \text{aux}} \) is defined by:

\[
P_{\rho \text{aux}} := \chi[L_{\text{aux}} < \rho] \quad \rho > 0
\]

where \( \chi \) is a characteristic function:

\[
\chi[L_{\text{aux}} < \rho] = \begin{cases} 
1 & : L_{\text{aux}} < \rho \\
0 & : \text{otherwise}
\end{cases}
\]

We define now a family of subsets which will help us studying the spectrum of \( L_\theta(\theta) \). In the next figure (Fig.13) all the domains are represented.

\[
S := \{ z \in \mathbb{C} | \text{Im}z > -\frac{\sin \theta}{4} \rho_0 \}
\]

we will use:

\[
\rho_0 := g^{2-\epsilon} \quad \rho_1 := g^{2+\epsilon/2}
\]

with \( \epsilon \) small.

We divide this domain in different parts:

\[
S_> := \{ z \in S | \text{dist}(Rez, \sigma[L_{at}]) \geq \frac{\rho_0}{2} \}
\]

\[
S_{ij} := \{ z \in S | |Rez - E_i + E_j| \leq \frac{\rho_0}{2} \}
\]

\[
S_{0>} := \{ z \in S | \frac{\sin \theta}{2} \rho_1 \leq |z| \leq \frac{\rho_0}{2} \}
\]

\[
S_{0<} := \{ z \in S | |z| \leq \frac{\sin \theta}{2} \rho_1 \}
\]

It is evident that:

\[
S \subseteq S_> \cup (\cup_{i \neq j} S_{ij} \cup S_{0>} \cup S_{0<})
\]

The easiest way to visualize these domains is to look at the next figure:
We will now expose the principal results of the analysis of the spectrum of \( L_g(\theta) \) \( (\sigma[L_g(i\vartheta)]) \).

For a proof refer to [7].

We will take:
\[
\theta = i\vartheta \quad 0 < \vartheta_0 < \vartheta < \frac{\pi}{2} \tag{4.51}
\]

**Theorem 4.3.2.** For \( g > 0 \) sufficiently small and for any \( z \in S_\rangle \)
\[
\sigma[L_g(i\vartheta)] \cap S_\rangle = \emptyset \tag{4.52}
\]

**Theorem 4.3.3.** For \( g > 0 \) sufficiently small and for any \( z \in S_{ij} \), it exists a positive constant \( \gamma > 0 \), such that:
\[
S_{ij} \cap \{ z \in \mathbb{C} | \Im z > -g^2\gamma \} \subseteq \rho[L_g(\theta)] \tag{4.53}
\]

**Theorem 4.3.4.** For \( g > 0 \) sufficiently small and for any \( z \in S_{0> \setminus C(\vartheta')} \) where \( C \) is the cone:
\[
C(\vartheta') := \{ |Rez| \leq -\cot \vartheta' Imz \} \quad \vartheta' < \vartheta \tag{4.54}
\]

then:
\[
S_{0> \setminus \{ z \in S_{0> \setminus C(\vartheta')} \}} \subseteq \rho[L_g(\theta)] \tag{4.55}
\]

The mechanism of proving these last three theorems is always the same. We prove first, by means of a Neumann series expansion, that in the desired domain the Feshbach map is well defined. Then we analyze the invertibility of the Feshbach map (i.e. \([F_{P}(L_g(\theta) - z)]^{-1}\) in this domain. The values of \( z \) where the FM is invertible belong to the resolvent set of \( L_g(\theta), \rho[L_g(\theta)] \) (theorem 4.3.1.). The values of \( z \) where the FM is not invertible belong to the spectrum of \( L_g(\theta), \sigma[L_g(\theta)] \) (theorem 4.3.1.).

With these results we infer that the spectrum of \( L_g(\theta) \) in \( S \) is of the following form (where the points indicate that the spectrum continues to the left and to the right in the same form):

![Fig.14 The spectrum of \( L_g(\theta) \) after the application of Theorems 4.3.2., 4.3.3., 4.3.4.](image.png)
From this diagram and similarly to the deduction of proposition 4.1.3. we deduce the following proposition:

**Proposition 4.3.5.** The spectrum of $L_g$ is absolutely continuous in each interval $\Delta$ (see Fig.14) on the real axis, if this interval does not contain the point zero and is sufficiently far away from it (see Fig.14).

In order to analyze the domain $D$ of Fig.14 we need new methods (the positive commutator method and the renormalization group method). The question is to know whether the $N$-fold eigenvalue $0$ of $L_0$ displaces also downwards or remains in the same position under the complex deformation. In the next chapter we will show that the spectrum near $0$ is cuspidal. We will also show in second order perturbation theory (in chapter 6) that $0$ is a simple eigenvalue of $L_g(\theta)$. Therefore $N-1$ of the eigenvalues equal to $0$ of $L_0$ displace downwards by the process of complex dilatation and one remains in the same position as by $L_0(\theta)$ (see next figure).

![Fig.15 The spectrum of $L_g(\theta)$](image-url)
Chapter 5

Renormalization Group Method

5.1 Introduction

We have seen that the values $z \in S_{0}> \mathcal{C}(\vartheta')$ are in the resolvent set of $L_g(\theta)$. Now the question is, what is the spectrum of $L_g(\theta)$ near 0 (for $z \in S_{0}<\) like? To answer this question we use successive Feshbach maps and rescalations of $z$ that permit us to analyze the spectrum of $L_g(\theta)$ in the proximity of 0. This structured set of operations repeated iteratively is known as renormalization group method (RG), which we will present in this Chapter. Mainly we will see it for $T > 0$ and we will see the differences with $T = 0$. Also we will define the smooth Feshbach Method (SFM) and use it to show that the domain of the spectrum of $L_g(\theta)$ is cuspidal near 0.

5.2 General Procedure for $T > 0$

The RG is an iterative process, that is to say, in each step we will repeat the same operations as we did in the step before, over our in each moment effective Liouvillian. In this section we will present this operations for the two first steps and for the case that $T > 0$. The purpose is to enumerate both the operations and the generated terms. In the next section we will make for the effective Liouvillian of step 0 a perturbative analysis in order $g^2$ and we will see how these terms generate.
5.2.1 Step 0

Each step of the renormalization Group consists of the same three operations. In this subsection we will show them for the step 0:

1 Feshbach Map

\[
\mathcal{F}_{P_0}(L_g(\theta) - z) = P_0[(L_g(\theta) - z) - g^2 P_0 W(\theta) P_0 (L_g(\theta) - z)^{-1} P_0 W(\theta)] P_0 \tag{5.1}
\]

where:

\[
P_0 := P_{at}^0 \otimes \chi[L_{aux} < \rho_0] \tag{5.2}
\]

\[
P_{at}^0 = \sum_{i=0}^{N-1} |\phi_i \otimes \phi_i\rangle\langle \phi_i \otimes \phi_i| \tag{5.3}
\]

With this transformation we obtain a new effective Liouvillian \( \tilde{L}_0[z] \) see Eq. (5.5). This is an operator on a smaller vector space \( P_0 \mathcal{H} \). This means that this transformation is a decimation of degrees of freedom of particle and photons with energies \( \geq \rho_0 \), where \( \alpha < \rho_0 < 1 \) was defined in Eq. (4.45).

Because of the isospectrality condition \( z \) is eigenvalue of \( L_g(\theta) \) if and only if 0 is eigenvalue of \( \mathcal{F}_{P_0}(L_g(\theta) - z) \).

This Feshbach Map is only defined where \( (\tilde{L}_g(\theta) - z)^{-1} \) exists. Now we develop Eq. (5.1) using a Neumann series. The details will be presented in the next section:

\[
\mathcal{F}_{P_0}(L_g(\theta) - z) = (L_0(i\theta) - z) P_0 + \sum_{\nu=1}^{\infty} (-1)^{\nu-1} g^\nu P_0 W(\frac{P_0}{L_0(\theta) - z} W P_0)^{\nu-1} \tag{5.4}
\]
defining:

$$\tilde{L}_0[z] - z := \mathcal{F}P_0(L_g(\theta) - z)$$  (5.5)

denoting $L = (L_f, L_{aux})$ and $r = (r_f, r_{aux})$; after doing a Wick ordering (see appendix), we get an expression of the form:

$$\tilde{L}_0[z] = P_0(\tilde{E}_0[z] + \tilde{T}_0[z, L] + \tilde{W}_0[z])P_0$$  (5.6)

with:

$$\tilde{E}_0[z] := \triangle \tilde{E}_0[z] = \tilde{\omega}^0_{0,0}[z, 0]$$  (5.7)

$$\tilde{T}_0[z, L] := L_f(\theta) + \triangle \tilde{T}_0[z, L] = L_f(\theta) + \{\tilde{\omega}^0_{0,0}[z, r] - \tilde{\omega}^0_{0,0}[z, 0]\}$$  (5.8)

$$\tilde{W}_0[z] := gW + \triangle \tilde{W}_0[z] = gW + \sum_{m+n \geq 1} \tilde{W}^{(0)}_{m,n}$$  (5.9)

where $\tilde{E}_0[z] \in \mathbb{C}$ is a number, $\tilde{T}_0[z, L]$ is a function of $L_f$ and $\tilde{W}_0[z]$ is a sum of Wick monomials of the form:

$$\tilde{W}^{(0)}_{m,n}[z] := \iint dk^{(m)}d\tilde{k}^{(n)}a^+(k^{(m)})\tilde{\omega}^0_{m,n}[z; L; k^{(m)}; k^{(n)}]a^-(\tilde{k}^{(n)})$$  (5.10)

where we use the notation:

$$a^+(k) := \tau[a^*(k)]$$  (5.11)

$$a^-(k) := \tau[a(k)]; \quad \tau = l, r$$  (5.12)

and $\tilde{\omega}^0_{m,n}$ is given by:

$$\tilde{\omega}^0_{m,n}[z; L; k^{(M)}; k^{(N)}] = -\sum_{p=0}^{\infty} \sum_{b \in B_{m,n,p}} \sum_{\alpha_1 \ldots \alpha_{\nu-1}} (-1)^{\nu-1}(g)^{\nu} \int dx^{(p)} \mathcal{S}_{m,n}\{F_b[x^{(p)}; x^{(p)}k^{(m)}k^{(n)}]\}d\tilde{x}^{(p)}$$  (5.13)

where $B_{m,n,p}$ denotes the set of partitions $b = (b_k, b_{\tilde{k}}, b_x, b_{\tilde{x}})$ of $\{1, 2, \ldots, m + n + 2p\}$, such that the number of elements of $b_k$ is $m$, $|b_k| = m$, $|b_{\tilde{k}}| = n$ and $|b_x| = |b_{\tilde{x}}| = p$. $b_k, b_{\tilde{k}}, b_x, b_{\tilde{x}}$ are pairwise disjoint subsets of $\{1, 2, \ldots, m + n + 2p\}$ the union of which give $\{1, 2, \ldots, m + n + 2p\}$. Denoting $J := m + n + 2p$ the matrix valued function $F_b$ is defined as:

$$F_b[x^{(p)}; x^{(p)}k^{(m)}; k^{(n)}] := \chi^{(0)}_{at} \chi^{(0)}_{at} \ldots \chi^{(0)}_{at} \chi^{(0)}_{at} \Omega[G^0_b(1, X^{(p)}, K^{(m,n)}) \times \ldots \times G^0_b(1, X^{(p)}, K^{(m,n)})]$$
\[ a^b(1, X^{(p,p)}, K^{(m,n)}) f_{\alpha_1 [L + r]} \cdots f_{\alpha_{\nu-1} [L + r]} G^\theta_b(J, X^{(p,p)}, K^{(m,n)}) a^b(J, X^{(p,p)}, K^{(m,n)}) \Omega \]  
(5.14)

where:

\[ G^\theta_b(j, X^{(p,p)}, K^{(m,n)}) := \begin{cases} 
G^\theta_+(k_l) & \text{if } j \text{ is the } l\text{th member of } b_k \\
G^\theta_-(\tilde{k}_l) & \text{if } j \text{ is the } l\text{th member of } b_{\tilde{k}} \\
G^\theta_+(x_l) & \text{if } j \text{ is the } l\text{th member of } b_x \\
G^\theta_-(\tilde{x}_l) & \text{if } j \text{ is the } l\text{th member of } b_{\tilde{x}} 
\end{cases} \]  
(5.15)

The expressions for \( G^\theta_\pm \) were given in Eq. (4.29) and related equations, where we take:

\[ G^\theta_\pm := \sum_{\tau = l, r} G_{\pm, \tau} \]  
(5.16)

and finally:

\[ a^b(j, X^{(p,p)}, K^{(m,n)}) := \begin{cases} 
1 & \text{if } j \in b_k \cup b_{\tilde{k}} \\
a^+(x_l) & \text{if } j \text{ is the } l\text{th member of } b_x \\
a^-(\tilde{x}_l) & \text{if } j \text{ is the } l\text{th member of } b_{\tilde{x}} 
\end{cases} \]  
(5.17)

and where \( S_{m,n} \) denotes the symmetrization operator:

\[ S_{m,n} := \frac{1}{m! n!} \sum_{\pi \in S_m, \tilde{\pi} \in S_n} F[X^{(p,p)}, k_{\pi(1)} \ldots k_{\pi(m)}; \tilde{k}_{\tilde{\pi}(1)} \ldots \tilde{k}_{\tilde{\pi}(m)}] \]  
(5.18)

In section 5.3, we will deduce these equations in order \( g^2 \) which can help to understand them. 2 Dilatation and Rescalation of energies and momenta

Like in Eq. (4.2) we can define a general dilatation as:

\[ (u_\rho f)(x) := \rho^{3/2} f(\rho x) \quad f \in L^2(\mathbb{R}^3, d^3 x) \]  
(5.19)

like in case of complex dilatation (\( \rho = e^{-\theta} \)) it is easy to see that this is an unitary transformation. We will take \( 0 < \rho < 1 \). We can define now \( U_\rho \) by the process of second quantization as:

\[ [U_\rho \psi]_n(k_1, k_2, \ldots k_n) := \rho^{(3/2)n} \psi_n(\rho k_1 \ldots \rho k_n) \]  
(5.20)

and as in Eq. (4.8) we have:

\[ U_\rho H f U_\rho^{-1} = \rho H f \]  
(5.21)
It is easy to see that the same transformation is defined by:

\[ U_\rho a^\dagger(k) U_\rho^* = \rho^{-\frac{3}{2}} a^\dagger(\rho^{-1}k) \]  
(5.22)

\[ U_\rho a(k) U_\rho^* = \rho^{-\frac{3}{2}} a(\rho^{-1}k) \]  
(5.23)

\[ U_\rho \Omega_f = \Omega_f \]  
(5.24)

The interaction Eq. (1.12) transforms to:

\[ \mathbb{1}_{at} \otimes U_\rho f \mathbb{1}_{at} \otimes U_\rho^* = \int dk \left\{ G(k) \otimes U_\rho a^\dagger(k) U_\rho^* + G(k)^* \otimes U_\rho a(k) U_\rho^* \right\} = \]  
\[ = \int dk \left\{ G(k) \otimes \rho^{-\frac{3}{2}} a^\dagger(\rho^{-1}k) + G(k)^* \otimes \rho^{-\frac{3}{2}} a(\rho^{-1}k) \right\} \]  
(5.25)

making the substitution \( k' = \rho^{-1}k \) and taking into account that the integrals are three-dimensional:

\[ \mathbb{1}_{at} \otimes U_\rho f \mathbb{1}_{at} \otimes U_\rho^* = \int \rho^3 dk' \left\{ G(\rho k') \otimes \rho^{-\frac{3}{2}} a^\dagger(k') + G(\rho k')^* \otimes \rho^{-\frac{3}{2}} a(k') \right\} = \]  
\[ = \int \rho^3 dk \left\{ G(\rho k) \otimes a^\dagger(k) + G(\rho k)^* \otimes a(k) \right\} \]  
(5.25)

and the Hamiltonian \( H_g \) transforms to:

\[ U_\rho H_g U_\rho^* = H_{at} \otimes \mathbb{1}_f + \rho \mathbb{1}_{at} \otimes H_f + \int \rho^2 dk \left\{ G(\rho k) \otimes a^\dagger(k) + G(\rho k)^* \otimes a(k) \right\} \]  
(5.26)

In the same way we can also dilate the Liouvillian \( L_g \) and we would obtain results similar to the ones obtained in chapter 4. With these examples we can see that the effect of this dilatation is to rescale the photon momenta as \( k \to \rho k \).

We apply now these results to \( \tilde{L}_0[z] \) Eq. (5.5), additionally we apply a rescalation by dividing the result of the dilatation by \( \rho_0 \). The new effective Liouvillian after the dilatation and rescalation is:

\[ P_{at} \otimes L'_{ef}[z] := \frac{1}{\rho_0} U_{\rho_0} \left\{ (\mathcal{F}_{P_0}(L_g(\theta)) - z) + z\chi(L_f < \rho_0) \right\} U_{\rho_0}^* \]  
(5.27)

where \( U_{\rho_0} \) is the dilatation operator.

3 Rescalation of the spectral parameter

\( Z : z \to \frac{1}{\rho} z \) and the new effective Liouvillian is:

\[ P_{at} \otimes L_0[z] := \frac{1}{\rho_0} U_{\rho_0} \left\{ (\mathcal{F}_{P_0}(L_g(\theta)) - Z^{-1}(z)) + Z^{-1}(z)\chi(L_f < \rho_0) \right\} U_{\rho_0}^* \]  
(5.28)
The reason of this rescalation is that under a dilatation and a rescalation of 2, the process of transformation of $T$ would be $\text{Dilat}_{\rho_0}(T) = \rho_0 T$ and $\text{Resc}_{\rho_0}(T) = \frac{1}{\rho_0} \text{Dilat}_{\rho_0}(T) = T$ but the way in which $E$ transforms is not adequate since $\text{Resc}_{\rho}(E \cdot \mathbb{1}) = \frac{1}{\rho} \text{Dilat}_{\rho}(E \cdot \mathbb{1}) = \frac{1}{\rho} E \cdot \mathbb{1}$ and with $\rho_0 < 1$ we would have problems. Namely $E/\rho_0^n$ would diverge after a big enough number $n$ of steps of this RG.

Applying these two operations 2 and 3 to Eq. (5.6) we obtain:

$$L_0[z] = P_0(E_0[z] + T_0[z,\mathbb{L}] + W_0[z])P_0$$

where:

$$E_0[z] = \triangle E_0[z] = \rho_0^{-1} \tilde{\omega}_0^0[Z^{-1}(z),0]$$

$$T_0[z,\mathbb{L}] = L_f + \triangle T_0[z,\mathbb{L}] = L_f + \rho_0^{-1}\{\tilde{\omega}_0^0[Z^{-1}(z),\rho_0 E] - \tilde{\omega}_0^0[Z^{-1}(z),0]\}$$

$$W_0[z] = gW + \triangle W_0[z] = gW + \sum_{m+n \geq 1} W^{0}_{m,n} (5.32)$$

where in the same way as in Eq. (5.25) we can calculate the relation between $\omega_{m,n}^{(0)}$ and $\tilde{\omega}_{m,n}^{0}$:

$$\omega_{m,n}^{(0)}[z;\mathbb{L};k^{(m)},\tilde{k}^{(m)}] := \rho_0^{\frac{1}{2}(m+n)-1} \tilde{\omega}_{m,n}^{(0)}[Z^{-1}(z);\rho_0 E;\rho_0 k^{(m)};\rho_0 \tilde{k}^{(m)}]$$

$$W_{m,n}^{(0)} := \int dk^{(m)}dk^{(n)}a^*(k^{(m)})\omega_{m,n}^{(0)}[z;\mathbb{L};k^{(m)},\tilde{k}^{(n)}]a(k^{(n)})$$

The idea behind this RG is to repeat these three operations in a number of steps obtaining after each step a new effective Liouvillian $L_n[z]$, containing a term $W_n[z]$ which is becoming smaller. Because of the isospectrality condition of the renormalization group, the spectrum of our initial Liouvillian $L_g(\theta)$ is the same as the spectrum of $L_n[z]$ which is much easier to analyze with the vanishing term $W_n[z]$. By decimating the degrees of freedom of the system with this RG, the dynamics of the remaining degrees of freedom is approximated by the free photons dynamics.

### 5.2.2 Step 1

In this step we apply the three operations formerly defined to the effective Liouvillian of step 0 $L_0[z]$. The problem to be discussed now is the possibility of doing a new FM. If we realize a new Feshbach Map (FM) onto our new Liouvillian $L_0[z]$, that is to say:

$$\mathcal{F}_{P_1}(L_0[z] - z) = P_1[(L_0[z] - z) - g^2 P_1 W(\theta) \tilde{P}_1 (\tilde{L}_0[z] - z)^{-1} \tilde{P}_1 W(\theta)]P_1$$

(5.35)
with:

\[ P_1 = P_k^{\text{ad}} \otimes \chi[L_{\text{aux}} < \rho_1] \]  

(5.36)

where \( \rho_1 := g^{2+\varepsilon/2} \) (see Eq. (4.45)) and \( k = \sum_{n=0}^{N-1} e^{-\beta E_n/2} \varphi_n \otimes \varphi_n \) and in theorem 6.3.3.

we prove that \( \text{Ker} \Gamma_0 = \mathbb{C}k \), where \( \Gamma_0 \) is the level-shift operator.

This FM would be defined only for the values of z where \( \tilde{L}_0[z] - z \) is invertible.

\[
(\overline{P}_1 L_0[z] \overline{P}_1 - z)^{-1} = \left( \overline{P}_1 P_0 (E_0[z] + T_0[z, L] + W_0[z]) P_0 \overline{P}_1 - z \right)^{-1}
\]

(5.37)

using the polar decomposition, with \( R = |E_0[z] + T_0[z, L] - z|^{-1} \)

\[
E_0[z] + T_0[z, L] - z = U |E_0[z] + T_0[z, L] - z| = UR^{-1}
\]

(5.38)

\[
(\overline{P}_1 L_0[z] \overline{P}_1 - z)^{-1} = R^{1/2} (U + R^{1/2} \overline{P}_1 P_0 W_0[z] \overline{P}_1 P_0 R^{1/2})^{-1} \overline{P}_1 P_0 R^{1/2} =
\]

\[ = R^{1/2} \left[ \sum_{n=0}^{\infty} U^*(-R^{1/2} \overline{P}_1 P_0 W_0[z] \overline{P}_1 P_0 R^{1/2} U^*)^n \right] R^{1/2} \]

(5.39)

and denoting by \( T'[z; \cdot] := \frac{\partial T[z; \cdot]}{\partial r_{\text{aux}}} \) we have:

**Lemma 5.2.1.** Let be \( T_0[z; r] \) with \( \|T'_0[z; \cdot] - 1\|_\infty \leq \delta < 1/6 \) and \( 0 \leq r_{\text{aux}} \leq 1 \) then for any \( z \) so that \( |z - E_0[z]| \leq \rho/2 \) we have:

\[
|T_0[z; \cdot] + E_0[z] - z| \geq \frac{1}{6}(r_{\text{aux}} + \rho)
\]

(5.40)

**Proof.** from \( \|T'_0[z; \cdot] - 1\|_\infty \leq \delta \) we have \(-T'_0 + 1 \leq \delta \) and \( T_0 > r_{\text{aux}}(1 - \delta) \) and therefore:

\[
|T_0[z; \cdot] + E_0[z] - z| \geq |T_0[z; \cdot] - |E_0[z] - z| \geq (1 - \delta)r_{\text{aux}} - \frac{\rho}{2} \geq \left( \frac{1 - \delta}{2} - \frac{1}{4} \right)(r_{\text{aux}} + \rho) \geq \frac{1}{6}(r_{\text{aux}} + \rho)
\]

(5.41)

**Lemma 5.2.2.** Assume that \( |\omega_{M,N}[r; k^{(M)}; k^{(N)}]| \leq \varepsilon \xi^{M+N} \prod_{j=1}^{M} \omega(k_j)^\alpha \prod_{j=1}^{N} \omega(\tilde{k}_j)^\alpha \) then for any \( 0 < \rho \leq 1 \) we have:

\[
\| (r_{\text{aux}} + \rho)^{-1/2} \overline{P}_1 P_0 W_0 \overline{P}_1 P_0 (r_{\text{aux}} + \rho)^{-1/2} \| \leq \frac{2\varepsilon}{\rho^{1/2}} \varepsilon^2
\]

(5.42)

For a proof see [14]

**Theorem 5.2.3.** For any \( z \) so that \( |z - E_0[z]| \leq \rho/2 \) the operator \( (\overline{P}_1 L_0[z] \overline{P}_1 - z) \) is invertible on \( \overline{P}_1 P_0 \).
Proof. With these two lemmas and Eq. (5.39) we have:

\[ \| R^{1/2} \mathcal{P}_1 P_0 W_0 [z] \mathcal{P}_1 P_0 R^{1/2} \| \leq \| (r_{aux} + \rho)^{-1/2} \mathcal{P}_1 P_0 W_0 \mathcal{P}_1 P_0 (r_{aux} + \rho)^{-1/2} \| \times \]
\[ \times \frac{r_{aux} + \rho}{T_0 [z, z] + E_0 [z] - z} \leq \frac{2 \epsilon^2 \epsilon}{\rho^{1/2}} 1/6 \]

\[ \square \]

With this theorem we have the justification for doing a next FM \( \mathcal{F}_P (L_0[z] - z) \) and then writing it in a Wick ordered form:

\[ \mathcal{F}_P (L_0[z] - z) := \tilde{L}_1 [z] + E_0 [z] - z \xrightarrow{\text{Wicks theorem}} \tilde{\omega}_M, N \] (5.43)

where

\[ \tilde{L}_1 [z] = P_1(\tilde{E}_1 [z] + \tilde{T}_1 [z, L] + \tilde{W}_1 [z]) P_1 \]

next we apply a dilatation and rescalation of the spectral parameter and we define the Renormalization Map by:

\[ \mathcal{R}_{\rho_1} (L_0)[z] - z := \rho^{-1} U_{\rho_1} \mathcal{F}_P (L_0[Z^{-1}(z)] - Z^{-1}(z)) U_{\rho_1}^* \] (5.44)

where \( U_{\rho_1} \) is a dilatation operator. We define:

\[ L_1 [z] - z := \mathcal{R}_{\rho_1} (L_0)[z] - z \] (5.45)

and substituting Eq. (5.43):

\[ L_1 [z] - z = \rho^{-1} U_{\rho_1} (\tilde{L}_1 [Z^{-1}(z)] + E_0 [Z^{-1}(z)] - Z^{-1}(z)) U_{\rho_1}^* \] (5.46)

Applying now the Wick’s theorem:

\[ L_1 [z] = P_1 (E_1 [z] + T_1 [z, L] + W_1 [z]) P_1 \] (5.47)

where we obtain again a number \( E_1 [z] \), a function of \( L, T_1 [z, L] \) and a sum of Wick’s monomials:

\[ E_1 [z] = \Delta E_0 [z] = \rho_{1}^{-1} \tilde{\omega}_{0,0}^{1}[Z^{-1}(z), 0] \] (5.48)

\[ T_1 [z, L] = \rho_{1}^{-1} T_0[Z^{-1}(z); \rho_1 L] + \Delta T_1 [z, L] = \rho_{1}^{-1} T_0[Z^{-1}(z); \rho_1 L] + \rho_{1}^{-1} (\tilde{\omega}_{0,0}^{1}[Z^{-1}(z), \rho_1 L] - \tilde{\omega}_{0,0}^{1}[Z^{-1}(z), 0]) \] (5.49)

\[ W_1 [z] = gW + \Delta W_1 [z] = gW + \sum_{M+N \geq 1} W_{M,N}^1 \] (5.50)
5.3 Step 0 of the RG in Second Order in \( g \) for \( T > 0 \)

The purpose of this section is to see the specific form in order \( g^2 \) of the expressions \( \tilde{E}, \tilde{T}, \tilde{W} \) seen in the last section (that is to say step 0 without dilatation and rescalation). In order to do that we will develop the Feshbach Map in powers of \( g \). We will restrict us to order \( g^2 \), afterwards we will see how the terms \( \tilde{E}, \tilde{T}, \tilde{W} \) mentioned in the last sections are generated (until order \( g^2 \)), using Wick’s theorem and lemma A.0.3 of Appendix A. We will use the projector \( P_0 := P_0^{\text{at}} \otimes \chi_{[L_{\text{aux}} < \rho_0]} = P_0^{\text{at}} \otimes \chi_{\rho_0} \) with \( P_0^{\text{at}} = \sum_{i=0}^{N-1} |\varphi_i \otimes \varphi_i \rangle \langle \varphi_i \otimes \varphi_i | \) and \( H_{\text{at}} \varphi_i = \varepsilon_i \varphi_i \).

We proceed now to develop the FM:

\[
\mathcal{F}_{P_0}(L_g(\theta) - z) = P_0[(L_g(\theta) - z) - g^2P_0L_g(\theta)P_0(L_g(\theta) - z)^{-1}P_0L_g(\theta)]P_0 = P_0[(L_g(\theta) - z) - g^2P_0W\bar{P}_0(L_g(\theta) - z)^{-1}\bar{P}_0W]P_0 \tag{5.51}
\]

where we have used \( P_0L_{\text{at}}P_0 = 0 \) and now we use \( \bar{P}_0L_0(\theta)\bar{P}_0 = L_0(\theta)\bar{P}_0 \) to develop the second term of Eq. (5.51) in a Neumann series expansion:

\[
(L_g(\theta) - z)^{-1} = (L_0(\theta) - z + gW(\theta))^{-1} = \left[ (L_0(\theta) - z)(1 + (L_0(\theta) - z)^{-1}gW(\theta)) \right]^{-1}
\]

\[
= (1 + (L_0(\theta) - z)^{-1}gW(\theta))^{-1}(L_0(\theta) - z)^{-1}
\]

using now the Neumann series of \( (1 + (L_0(\theta) - z)^{-1}gW(\theta))^{-1} \):

\[
(L_g(\theta) - z)^{-1} = (L_0(\theta) - z)^{-1} - (L_0(\theta) - z)^{-1}gW(\theta)(L_0(\theta) - z)^{-1} + \nonumber (L_0(\theta) - z)^{-1}gW(\theta)(L_0(\theta) - z)^{-1}gW(\theta)(L_0(\theta) - z)^{-1} + \ldots =
\]

\[
(L_0(\theta) - z)^{-1}\sum_{n=0}^{\infty}(-gW(\theta)(L_0(\theta) - z)^{-1})^n
\]

noting that \( (L_0(\theta) - z)^{-1} = \frac{1}{L_0(\theta) - z} \bar{P}_0 \) and \( \bar{P}_0^2 = \bar{P}_0 \) we have:

\[
(L_g(\theta) - z)^{-1} = \frac{1}{L_0(\theta) - z} \bar{P}_0 \sum_{n=0}^{\infty}(-gW(\theta)\frac{1}{L_0(\theta) - z} \bar{P}_0)^n \tag{5.53}
\]

inserting this result in Eq. (5.51) and using \( P_0L_\theta P_0 = 0 \):

\[
\mathcal{F}_{P_0}(L_g(\theta) - z) = P_0(L_g(\theta) - z) - g^2P_0W(\theta)\bar{P}_0(L_g(\theta) - z)^{-1}\bar{P}_0WP_0 = (L_0(\theta) - z)P_0 + gP_0W(\theta)P_0 - g^2P_0W(\theta)\left[ \frac{1}{L_0(\theta) - z} \bar{P}_0 \sum_{n=0}^{\infty}(-gW(\theta)\frac{1}{L_0(\theta) - z} \bar{P}_0)^n \right]W(\theta)P_0 =
\]
and reordering the terms:

\[
\mathcal{F}_{P_0}(L_0(\theta) - z) = (L_0(\theta) - z)P_0 + gP_0 W(\theta) P_0 -
\]

\[
-g^2 P_0 W(\theta) \left[ \frac{1}{L_0(\theta) - z} \mathcal{T}_0 \right] - \frac{1}{L_0(\theta) - z} \mathcal{T}_0 g W(\theta) \left[ \frac{1}{L_0(\theta) - z} \mathcal{T}_0 \right] + \cdots ] W(\theta) P_0 =
\]

\[
= (L_0(\theta) - z)P_0 + \sum_{\nu=1}^{\infty} (-1)^{\nu-1} g^\nu P_0 W(\theta) \left[ \frac{1}{L_0(\theta) - z} \mathcal{T}_0 \right] W(\theta) P_0
\]

This Neumann series converges if \( \| \mathcal{T}_0 (\theta) \| W(\theta) P_0 < 1 \). We will work with the interaction Liouvillian:

\[
W = l[I] - r[I] = l[a^\ast] \otimes (G \otimes 1) + l[a] \otimes (G^* \otimes 1) - J_f l[a^\ast] J_f \otimes (1 \otimes TGT) - J_f l[a] J_f \otimes (1 \otimes TGT)
\]

and in the same way as by deducing Eq. (5.26), using Eq. (5.22) and Eq. (5.23) (for \( \rho = \phi \theta \)) we can see:

\[
W(\theta) = U(\theta) W U^\ast(\theta) = \int dke^{-\frac{3}{2} \theta} \{ l[a^\ast(k)] \otimes (G(e^{-\theta} k) \otimes 1) + l[a] \otimes (G^*(e^{-\theta} k) \otimes 1) \} -
\]

\[
- \int dke^{-\frac{3}{2} \theta} \{ r[a^\ast(k)] \otimes (1 \otimes \overline{G}(e^{-\theta} k)) + r[a(k)] \otimes (1 \otimes \overline{G}^*(e^{-\theta} k)) \}
\]

and defining:

\[
G_{\theta}(k) := e^{-\frac{3}{2} \theta} G(e^{-\theta} k)
\]

\[
G^*_{\theta}(k) := e^{-\frac{3}{2} \theta} G^*(e^{-\theta} k)
\]

\[
\overline{G}_{\theta}(k) := e^{-\frac{3}{2} \theta} \overline{G}(e^{-\theta} k)
\]

\[
\overline{G}^*_{\theta}(k) := e^{-\frac{3}{2} \theta} \overline{G}^*(e^{-\theta} k)
\]

we can write it as:

\[
W(\theta) = \int dk \{ l[a^\ast(k)] \otimes (G_{\theta}(k) \otimes 1) + l[a] \otimes (G^*_{\theta}(k) \otimes 1) \} -
\]

\[
- \int dk \{ r[a^\ast(k)] \otimes (1 \otimes \overline{G}_{\theta}(k)) + r[a(k)] \otimes (1 \otimes \overline{G}^*_{\theta}(k)) \}
\]

For the convergence of the Neumann series we will need the following lemma:

**Lemma 5.3.1.** With the assumption on \( G \):

\[
\Lambda_1 := \left( \int dk (1 + \omega(k)^{-1})|G(k)|^2 \right)^{1/2} < \infty
\]
we have:

\[
\| (L_{aux} + \rho_0)^{-1/2} W(\theta) (L_{aux} + \rho_0)^{-1/2} \| \leq 4 \Lambda_1 \rho_0^{-1/2}
\]  

(5.63)

Proof. We have that:

\[
\| a(G) \psi \|^2 = \langle a(G) \psi | a(G) \psi \rangle = \int dk dk' |G(k')G(k')\langle a(k) \psi | a(k') \psi \rangle \leq \int dk dk' \frac{|G(k)|}{|k|} \frac{|G(k')|}{|k'|} |\langle \sqrt{|k|} a(k) \psi | \sqrt{|k'|} a(k') \psi \rangle|
\]

applying now the inequality of Cauchy-Schwartz-Buniakowski:

\[
\leq \int dk \frac{|G(k)|}{|k|} \| \sqrt{|k|} a(k) \psi \| \int dk' \frac{|G(k')|}{|k'|} \| \sqrt{|k'|} a(k') \psi \| \leq \left( \int dk \frac{|G(k)|^2}{|k|} \| \sqrt{|k|} a(k) \psi \|^2 \left( \int dk \| \sqrt{|k|} a(k) \psi \|^2 \right) \right) \left( \int dk \frac{|G(k')|^2}{|k'|} \| \sqrt{|k'|} a(k') \psi \|^2 \left( \int dk \| \sqrt{|k'|} a(k') \psi \|^2 \right) \right)^{1/2}
\]

and recalling that \( L_{aux} = H_f \otimes 1_f + 1_f \otimes H_f \), then we have:

\[
\| a(G) \psi \| \leq \left\| \frac{G}{\sqrt{\omega}} \right\|_2 \| L_{aux}^{1/2} \psi \|
\]  

(5.64)

and taking \( \psi := L_{aux}^{-1/2} \chi \) where \( \chi \) is a characteristic function we have:

\[
\| a(G) L_{aux}^{-1/2} \| \leq \left\| \frac{G}{\sqrt{\omega}} \right\|_2
\]  

(5.65)

and therefore:

\[
\| (L_{aux} + \rho_0)^{-1/2} a(G) (L_{aux} + \rho_0)^{-1/2} \| \leq \frac{1}{\sqrt{\rho_0}} \left\| \frac{G}{\sqrt{\omega}} \right\|_2
\]  

(5.66)

for \( \| a^*(G) \psi \| \) we have:

\[
\| a^*(G) \psi \|^2 = \langle \psi | a(G) a^*(G) \psi \rangle = \| G \|_2^2 \| \psi \|^2 + \langle \psi | a^*(G) a(G) \psi \rangle = \| G \|_2^2 \| \psi \|^2 + \| a(G) \psi \|^2
\]  

(5.67)

then for both creation and annihilation operators we have:

\[
\| (L_{aux} + \rho_0)^{-1/2} a^*(G) (L_{aux} + \rho_0)^{-1/2} \| \leq \frac{1}{\sqrt{\rho_0}} \left( \int dk (1 + \frac{1}{\omega(k)}) |G(k)|^2 \right)^{1/2}
\]  

(5.68)

and taking into account the four terms of Eq. (5.61) we obtain Eq. (5.63).
**Proposition 5.3.2.** Let \( z \) be so that \(|\text{Re}z| < \rho_0/2\) and \(|\text{Im}z| \geq -\frac{1}{2}\rho_0 \sin \vartheta\) where \( \rho_0 := g^{2-\epsilon} \) for \( \epsilon \) small and \( 0 < \vartheta < \pi/2\). For these values of \( z \) and \( P_0 = P_{0}^{\text{at}} \otimes \chi_{\rho_0} \), we have:

\[
\left\| \frac{L_{\text{aux}} + \rho_0}{L_0(\theta) - z} P_0 \right\| \leq \frac{4}{\sin \vartheta}
\]  

(5.69)

**Proof.** Using Eq. (4.15) with \( \theta = i\vartheta \):

\[
L_0(\theta) = L_{\text{at}} + L_f \cos \vartheta - iL_{\text{aux}} \sin \vartheta
\]  

(5.70)

and:

\[
P_0 = P(L_{\text{at}} \neq 0) \otimes \chi(L_{\text{aux}} \leq \rho_0) + P(L_{\text{at}} = 0) \otimes \chi(L_{\text{aux}} > \rho_0) + P(L_{\text{at}} \neq 0) \otimes \chi(L_{\text{aux}} > \rho_0)
\]  

(5.71)

defining:

\[Q_1 := P(L_{\text{at}} \neq 0) \otimes \chi(L_{\text{aux}} \leq \rho_0)\]

(5.72)

\[Q_2 := 1_{L_{\text{at}}} \otimes \chi(L_{\text{aux}} > \rho_0)\]

(5.73)

then we have:

\[P_0 = Q_1 + Q_2\]

(5.74)

We will suppose that \([L_0(\theta), Q_{1,2}] = 0\). We have:

\[
\left\| (L_{\text{aux}} + \rho_0)(L_0(\theta) - z)^{-1} P_0 \right\| = \max \left( \left\| (L_{\text{aux}} + \rho_0)(L_0(\theta) - z)^{-1} Q_1 \right\|, \left\| (L_{\text{aux}} + \rho_0)(L_0(\theta) - z)^{-1} Q_2 \right\| \right)
\]  

(5.75)

We use now the inequalities: \(|z| \geq |\text{Im}z|\) and \(|z| \geq |\text{Re}z|\). Let us analyze each of the two expressions \(\left\| (L_{\text{aux}} + \rho_0)(L_0(\theta) - z)^{-1} Q_1 \right\|\) and \(\left\| (L_{\text{aux}} + \rho_0)(L_0(\theta) - z)^{-1} Q_2 \right\|\):

Let \( A \) be the set of values \((r_{\text{at}}, r_f, r_{\text{aux}})\) of \((L_{\text{at}}, L_f, L_{\text{aux}})\) so that:

\[
\begin{cases}
  r_{\text{at}} \in \sigma(L_{\text{at}} \backslash \{0\}) \\
  r_f \in \mathbb{R}, \ |r_f| < r_{\text{aux}} \\
  0 \leq r_{\text{aux}} \leq \rho_0
\end{cases}
\]  

(5.76)

\[
\left\| (L_{\text{aux}} + \rho_0)(L_0(\theta) - z)^{-1} Q_1 \right\| = \sup_{(r_{\text{at}}, r_f, r_{\text{aux}}) \in A} \left| \frac{r_{\text{aux}} + \rho_0}{r_{\text{at}} + r_f \cos \vartheta - i r_{\text{aux}} \sin \vartheta - z} \right| \leq \sup_{(r_{\text{at}}, r_f, r_{\text{aux}}) \in A} \left| \frac{r_{\text{aux}} + \rho_0}{r_{\text{at}} + r_f \cos \vartheta - \text{Re}z} \right|
\]  

(5.77)
Looking at Fig.13 we see that in A:

\[ |r_{at} + r_f \cos \vartheta - Re z| \geq |r_{at} - Re z| - |r_f \cos \vartheta| \geq \frac{3}{2} \rho_0 - \rho_0 = \frac{1}{2} \rho_0 \]  

(5.78)

and:

\[ |r_{aux} + \rho_0| \leq 2 \rho_0 \]  

(5.79)

and then:

\[ \|(L_{aux} + \rho_0)(L_0(\theta) - z)^{-1}Q_1\| \leq \frac{2}{\rho_0} 2 \rho_0 = 4 \]  

(5.80)

Let \( B \) be the set of values \((r_{at}, r_f, r_{aux})\) of \((L_{at}, L_{f}, L_{aux})\) so that:

\[
\begin{cases} 
  r_{at} \in \sigma(L_{at}) \\
  r_f \in \mathbb{R}, |r_f| < r_{aux} \\
  r_{aux} > \rho_0
\end{cases}
\]  

(5.81)

Looking at Fig.13 we see that \( Im z \geq -\frac{1}{2} \rho_0 \sin \vartheta \) and in B:

\[ \frac{r_{aux} + \rho_0}{r_{aux} \sin \vartheta + Im z} \geq \frac{r_{aux} + \rho_0}{r_{aux} \sin \vartheta - \frac{1}{2} \rho_0 \sin \vartheta} = \frac{r_{aux} + \rho_0}{\sin \vartheta (r_{aux} - \rho_0/2)} \]  

(5.82)

and then:

\[ \sup \frac{r_{aux} + \rho_0}{r_{aux} \sin \vartheta + Im z} \leq \sup \frac{r_{aux} + \rho_0}{\sin \vartheta (r_{aux} - \rho_0/2)} \]  

(5.83)

defining now \( y := r_{aux} - \rho_0/2 \) we have:

\[ \sup \frac{r_{aux} + \rho_0}{\sin \vartheta (r_{aux} - \rho_0/2)} = \frac{1}{\sin \vartheta} \sup_{y > \rho_0/2} \frac{y + \rho_0/2 + \rho_0}{y} = \frac{1}{\sin \vartheta} \sup_{y > \rho_0/2} \frac{1 + \rho_0/2 + \rho_0}{y} = \frac{1}{\sin \vartheta} (1 + \frac{\rho_0/2 + \rho_0}{\rho_0/2}) = \frac{4}{\sin \vartheta} \]  

(5.84)

and finally:

\[ \|(L_{aux} + \rho_0)(L_0(\theta) - z)^{-1}P_0\| \leq \max \left(4, \frac{4}{\sin \vartheta}\right) = \frac{4}{\sin \vartheta} \]  

(5.85)
And now we can see the convergence of the Neumann series expansion:

**Theorem 5.3.3.** Let \( z \) be so that \( \text{Re}\,z < \rho_0/2 \) and \( \text{Im}\,z \geq -\frac{1}{2}\rho_0 \sin \vartheta \) where \( \rho_0 := g^{2-\epsilon} \) for \( \epsilon \) small and \( 0 < \vartheta < \pi/2 \). For these conditions and \( g \) sufficiently small \( P_0(L - g(\theta) - z)P_0 \) is invertible on \( \text{Ran}\,P_0 \). That implies that for these conditions and \( g \) sufficiently small the Feshbach map \( \mathcal{F}_{P_0}(L_g(\theta) - z) \) with \( P_0 = P_0^\ast \otimes \chi_{\rho_0} \) is well defined. The series expansion Eq. (5.54)

\[
\mathcal{F}_{P_0}(L_g(\theta) - z) = (L_0(\theta) - z)P_0 + \sum_{\nu=1}^{\infty} (-1)^{\nu-1} g^\nu P_0 W(\theta) (\frac{1}{L_0(\theta) - z} P_0 W(\theta))^\nu P_0
\]

is for these conditions convergent.

**Proof.** The Feshbach map Eq. (5.51):

\[
\mathcal{F}_{P_0}(L_g(\theta) - z) = P_0[(L_g(\theta) - z) - g^2 P_0 L_g(\theta) P_0 (L_g(\theta) - z)^{-1} P_0 L_g(\theta)] P_0
\]

is well defined for all the values of \( z \) where the inverse of \( (L_g(\theta) - z) \) exists.

We will analyze the term Eq. (5.53) \( P_0(\mathcal{L}_g - z)^{-1} P_0 \):

\[
P_0(\mathcal{L}_g(\theta) - z)^{-1} P_0 = \frac{1}{L_0(\theta) - z} P_0 \sum_{n=0}^{\infty} \left( -g W(\theta) \right) \left( \frac{1}{L_0(\theta) - z} P_0 \right)^n = \frac{(L_{\text{aux}} + \rho_0)^{1/2} - \rho_0}{L_0(\theta) - z} P_0 \times
\]

\[
\sum_{n=0}^{\infty} \left\{ (L_{\text{aux}} + \rho_0)^{-1/2} (-g W) (L_{\text{aux}} + \rho_0)^{-1/2} \left( \frac{L_{\text{aux}} + \rho_0}{P_0} \right) \right\}^n \left( \frac{1}{(L_{\text{aux}} + \rho_0)^{1/2}} \right)
\]

(5.89)

this equation can be checked (for \( n=0 \) is trivial),

for \( n = 1 \):

\[
\frac{(L_{\text{aux}} + \rho_0)^{1/2}}{L_0(\theta) - z} P_0 (L_{\text{aux}} + \rho_0)^{-1/2} (-g W)(L_{\text{aux}} + \rho_0)^{-1/2} \left( \frac{L_{\text{aux}} + \rho_0}{P_0} \right) \left( \frac{1}{(L_{\text{aux}} + \rho_0)^{1/2}} \right)
\]

for \( n = 2 \):

\[
\frac{(L_{\text{aux}} + \rho_0)^{1/2}}{L_0(\theta) - z} P_0 (L_{\text{aux}} + \rho_0)^{-1/2} (-g W)(L_{\text{aux}} + \rho_0)^{-1/2} \left( \frac{L_{\text{aux}} + \rho_0}{P_0} \right) \times
\]

\[\times (L_{\text{aux}} + \rho_0)^{-1/2} (-g W)(L_{\text{aux}} + \rho_0)^{-1/2} \left( \frac{L_{\text{aux}} + \rho_0}{P_0} \right) \left( \frac{1}{(L_{\text{aux}} + \rho_0)^{1/2}} \right)\]
and then we have

\[ \|((L_{aux} + \rho_0)^{-1/2}(-gW(\theta))(L_{aux} + \rho_0)^{-1/2})(\frac{L_{aux} + \rho_0}{L_0(\theta) - z} \mathcal{P}_0)\| \leq (4g \Lambda_1)(\frac{4}{\sin \theta}) = \frac{16}{\sin \theta} g \Lambda_1 \ll 1 \]

which is \ll 1 for \( g \) sufficiently small.

We will restrict us to the term in second order in \( g \) of Eq. (5.54):

\[ -g^2 P_0 W(\theta) \frac{\mathcal{P}_0}{L_0(\theta) - z} W(\theta) P_0 \]

we see that:

\[ \mathcal{P}_0 = \mathds{1} - P_0 = \mathcal{P}_0^{at} \otimes \chi_{\rho_0} + \mathcal{P}_0^{at} \otimes \chi_{\rho_0} + \mathcal{P}_0^{at} \otimes \chi_{\rho_0} = \]

\[ = \mathcal{P}_0^{at} \otimes (\chi_{\rho_0} + \chi_{\rho_0}) + \mathcal{P}_0^{at} \otimes \chi_{\rho_0} = \mathcal{P}_0^{at} \otimes \mathds{1} + \mathcal{P}_0^{at} \otimes \chi_{\rho_0} \]

we define:

\[ \mathcal{P}_0 := \sum_{\alpha} \chi_{\alpha}^{at} \otimes \chi_{\alpha}^{0} \]

where \( \chi_{\alpha}^{\sigma}[\omega] := \left\{ \begin{array}{ll} \mathds{1}_f : & \text{for } \alpha = 1, 2, \ldots, M \\ \chi[L_{aux} + \omega > \rho_0] : & \text{for } \alpha = 0 \end{array} \right. \]

and \( \{\varepsilon_0, \ldots, \varepsilon_M\} = \{E_{i,j} | 0 \leq i, j \leq N - 1\} \) with \( \varepsilon_0 := 0 \) and \( M \leq N(N - 1) \), that is we have \( N \) zero eigenvalues (see Eq. (4.1)). These correspond to the differences \( E_{i,j} = E_i - E_j \) when \( E_i = E_j \) and \( N^2 - N = N(N - 1) \) non-zero eigenvalues.

\[ -g^2 P_0 W(\theta) \frac{\mathcal{P}_0}{L_0(\theta) - z} W(\theta) P_0 = -g^2 \chi_{\alpha}^{0} \chi_{\rho_0} W(\theta) \frac{\mathcal{P}_0}{L_0(\theta) - z} W(\theta) \chi_{\alpha}^{0} \]

\[ = -g^2 \chi_{\alpha}^{0} \chi_{\rho_0} W(\theta) \sum_{\omega} \chi_{\alpha}^{0} \otimes \chi_{\omega}^{0} \frac{\mathcal{P}_0}{L_0(\theta) - z} W(\theta) \chi_{\alpha}^{0} \chi_{\rho_0} = -g^2 \sum_{\omega} \chi_{\alpha}^{0} \chi_{\rho_0} \chi_{\alpha}^{0} W(\theta) \frac{\chi_{\omega}^{0}}{L_0(\theta) - z} W(\theta) \chi_{\alpha}^{0} \chi_{\rho_0} \]

Let us analyze the term \( W \frac{\chi_{\omega}^{(0)}}{L_0(\theta) - z} W \) with the interaction Eq. (5.55), considering that \( TGT = G = (G^*)^T \):

\[ W(\theta) \frac{\chi_{\omega}^{(0)}}{L_0(\theta) - z} W(\theta) = \]

\[ = l(I) \frac{\chi_{\omega}^{(0)}}{L_0(\theta) - z} l(I) + r(I) \frac{\chi_{\omega}^{(0)}}{L_0(\theta) - z} r(I) - l(I) \frac{\chi_{\omega}^{(0)}}{L_0(\theta) - z} r(I) - r(I) \frac{\chi_{\omega}^{(0)}}{L_0(\theta) - z} l(I) \]
We will develop now the different terms I, II, III, IV each of which will result in four terms more and each of these will give also four terms. That is to say in total we will have $4 \times 4 \times 4 = 64$ different terms but fortunately many of them are zero.

Taking into account Eq. (5.61), we have:

$$I = [l[a^*] \otimes (G_\theta \otimes 1)] + [l[a] \otimes (G_\theta^* \otimes 1)] = \left[ \frac{\bar{X}_f^\dagger(0)_{L_0(\theta) - z}}{L_0(\theta)} \right] [l[a^*] \otimes (G_\theta \otimes 1)] + [l[a] \otimes (G_\theta^* \otimes 1)] =$$

$$= \int dkdl l[a^*(k)] \otimes (G_\theta(k) \otimes 1) \frac{\bar{X}_f^\dagger(0)_{L_0(\theta) - z}}{L_0(\theta)} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] + (A_1) (5.96)$$

$$+ \int dkdl l[a^*(k)] \otimes (G_\theta(k) \otimes 1) \frac{\bar{X}_f^\dagger(0)_{L_0(\theta) - z}}{L_0(\theta)} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] + (B_1) (5.97)$$

$$+ \int dkdl l[a(k)] \otimes (G_\theta^*(k) \otimes 1) \frac{\bar{X}_f^\dagger(0)_{L_0(\theta) - z}}{L_0(\theta)} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] + (C_1) (5.98)$$

$$+ \int dkdl l[a(k)] \otimes (G_\theta^*(k) \otimes 1) \frac{\bar{X}_f^\dagger(0)_{L_0(\theta) - z}}{L_0(\theta)} [l[a(l)] \otimes (G_\theta^*(l) \otimes 1)] + (D_1) (5.99)$$

In order to become expressions like $\tilde{E}_0, \tilde{T}_0, \tilde{W}_0$ we will need the lemma A.0.3 of Appendix A.

We will use the following notation:

$$\bar{r} = (r_f, r_{aux})$$

$$L = (L_f, L_{aux})$$}

$$L(\theta) + \bar{r}(\theta) := (L_f + r_f) \cosh \theta - (L_{aux} + r_{aux}) \sinh \theta$$

(5.102)

Applying lemma A.0.3 and the Pull-prouch formula (lemma A.0.1) and with the Eq. (4.15) $L_0(\theta) = L_{at} + L_f \cosh \theta - L_{aux} \sinh \theta$:

$$A_1 = \int dkdl l[a^*(k)] \otimes (G_\theta(k) \otimes 1) \frac{\bar{X}_f^\dagger(0)_{L_0(\theta) - z}}{L_0(\theta)} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] =$$

$$= \int dkdl l[a^*(k)] l[a^*(l)] \Omega_f G_\theta(k) \otimes 1 \frac{\bar{X}_f^\dagger(r_{aux} + \omega(l))_{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z}}{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] \bigg|_{r = L} + (5.103)$$

$$+ \int dkdl l[a^*(k)] l[a^*(l)] \Omega_f G_\theta(k) \otimes 1 \frac{\bar{X}_f^\dagger(r_{aux} + \omega(l))_{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z}}{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] \bigg|_{r = L} + (5.104)$$

$$+ \int dl l[a*(l)] \Omega_f [\int dk l[a^*(k)] G_\theta(k) \otimes 1 \frac{\bar{X}_f^\dagger(r_{aux} + \omega(l))_{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z}}{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] \bigg|_{r = L} + (5.105)$$

$$+ \int dl l[a^*(l)] \Omega_f [\int dk l[a^*(k)] G_\theta(k) \otimes 1 \frac{\bar{X}_f^\dagger(r_{aux} + \omega(l))_{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z}}{L_0(\theta) + L_f(\theta) + \bar{r}(\theta) + \omega(l) - z} [l[a^*(l)] \otimes (G_\theta(l) \otimes 1)] \bigg|_{r = L}$$

(5.106)
because of the reasons mentioned before, there are vanishing terms and they are Eq. (5.108)

\[ \int dk dl l[a^*(k)] \otimes (G_\theta(k) \otimes |1\rangle) \frac{\nabla^2 f(0)}{L_0(\theta) - z} l[a(l)] \otimes (G_\theta^*(l) \otimes |1\rangle) = \]

\[ \int dk dl l[a^*(k)] \langle \Omega_f | G_\theta(k) \otimes |1\rangle \frac{\nabla^2 f(0)}{L_0(\theta) + r(\theta) + z} G_\theta^*(l) \otimes |1\rangle \Omega_f \rangle l[a(l)] \bigg|_{r = L} + \]

\[ \int dl \langle \Omega_f | \int dk dl l[a^*(k)] G_\theta(k) \otimes |1\rangle \frac{\nabla^2 f(0)}{L_0(\theta) + r(\theta) + z} G_\theta^*(l) \otimes |1\rangle \Omega_f \rangle l[a(l)] \bigg|_{r = L} + \]

\[ \langle \Omega_f | \int dk dl l[a^*(k)] G_\theta(k) \otimes |1\rangle \frac{\nabla^2 f(0)}{L_0(\theta) + r(\theta) + z} G_\theta^*(l) \otimes |1\rangle \Omega_f \rangle l[a(l)] \bigg|_{r = L} \]

because of the reasons mentioned before, there are vanishing terms and they are Eq. (5.108) and Eq. (5.110), the term Eq. (5.109) do not vanish if \( k = l \).

From now on we will write only the non-vanishing terms:

\[ C_1 = \int dk dl l[a^*(l)] \langle \Omega_f | G_\theta(k) \otimes |1\rangle \frac{\nabla^2 f(0)}{L_0(\theta) + r(\theta) + z} G_\theta^*(l) \otimes |1\rangle \Omega_f \rangle l[a(l)] \bigg|_{r = L} + \]

\[ D_1 = \int dk dl \langle \Omega_f | G_\theta^*(l) \otimes |1\rangle \frac{\nabla^2 f(0)}{L_0(\theta) + r(\theta) + z} G_\theta^*(l) \otimes |1\rangle \Omega_f \rangle l[a(l)] \bigg|_{r = L} \]

Using now the representation \( r(a) = J_f l[a] J_f \) and that \( \overline{G} = T G T \):

\[ II = [r[a^*] \otimes (|1\rangle \otimes \overline{G}_\theta) + r[a] \otimes (|1\rangle \otimes \overline{G}_\theta^*) \frac{\nabla^2 f(0)}{L_0(\theta) - z} [r[a^*] \otimes (|1\rangle \otimes \overline{G}_\theta) + r[a] \otimes (|1\rangle \otimes \overline{G}_\theta^*)] = \]

\[ \int dk dl r[a^*(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a^*(l)] \otimes (|1\rangle \otimes \overline{G}_\theta(l)) + \]

\[ \int dk dl r[a^*(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) + \]

\[ \int dk dl r[a(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a^*(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) + \]

\[ \int dk dl r[a(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) \]

\[ + \int dk dl r[a^*(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a^*(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) + \]

\[ + \int dk dl r[a(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) + \]

\[ + \int dk dl r[a^*(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a^*(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) \]

\[ + \int dk dl r[a(k)] \otimes (|1\rangle \otimes \overline{G}_\theta(k)) \frac{\nabla^2 f(0)}{L_0(\theta) - z} r[a(l)] \otimes (|1\rangle \otimes \overline{G}_\theta^*(l)) \]
\[ A_2 = \int dk dl r[a^*(k)] r[a^*(l)] \langle \Omega_f | 1 \otimes G_\theta(k) \frac{\hat{\mathcal{V}}_f^j(r_{\text{aux}} - \omega(l))}{L_{at} + l \hat{L}(\theta) + r(\theta) - \omega(l) - z} 1 \otimes \overline{G}_\theta(l) \Omega_f \rangle \bigg|_{r=L} \]

\[ B_2 = \int dk dl r[a^*(k)] \langle \Omega_f | 1 \otimes G_\theta(k) \frac{\hat{\mathcal{V}}_f^0(r_{\text{aux}})}{L_{at} + l \hat{L}(\theta) + r(\theta) - z} 1 \otimes \overline{G}_\theta(l) \Omega_f \rangle r[a(l)] \bigg|_{r=L} + \quad (5.119) \]

\[ C_2 = \int dk dl r[a^*(l)] \langle \Omega_f | 1 \otimes G_\theta^*(k) \frac{\hat{\mathcal{V}}_f^j(r_{\text{aux}} - \omega(k) - \omega(l))}{L_{at} + l \hat{L}(\theta) + r(\theta) - \omega(k) - \omega(l) - z} 1 \otimes \overline{G}_\theta(l) \Omega_f \rangle r[a(k)] \bigg|_{r=L} + \quad (5.120) \]

\[ D_2 = \int dk dl \langle \Omega_f | 1 \otimes G_\theta^*(k) \frac{\hat{\mathcal{V}}_f^j(r_{\text{aux}} - \omega(k))}{L_{at} + l \hat{L}(\theta) + r(\theta) - \omega(k) - \omega(l) - z} 1 \otimes \overline{G}_\theta(l) \Omega_f \rangle r[a(k)] r[a(l)] \bigg|_{r=L} \quad (5.121) \]

Doing the same with III and maintaining the non-vanishing terms:

\[ III = [l[a^*] \otimes (G_\theta \otimes 1) + l[a] \otimes (G_\theta^* \otimes 1)] \frac{\hat{\mathcal{V}}_f^j(0)}{L_0(\theta)} - z r[a^*] \otimes (1 \otimes G_\theta) + r[a] \otimes (1 \otimes G_\theta^*) = \]

\[ \int dk dl l[a^*(k)] \otimes (G_\theta(k) \otimes 1) \frac{\hat{\mathcal{V}}_f^j(0)}{L_0(\theta)} - z r[a^*(l)] \otimes (1 \otimes G_\theta(l)) + \quad (A_3) \quad (5.124) \]

\[ + \int dk dl l[a^*(k)] \otimes (G_\theta(k) \otimes 1) \frac{\hat{\mathcal{V}}_f^j(0)}{L_0(\theta)} - z r[a(l)] \otimes (1 \otimes G_\theta^*(l)) + \quad (B_3) \quad (5.125) \]

\[ + \int dk dl l[a(k)] \otimes (G_\theta^*(k) \otimes 1) \frac{\hat{\mathcal{V}}_f^j(0)}{L_0(\theta)} - z r[a^*(l)] \otimes (1 \otimes G_\theta^*(l)) + \quad (C_3) \quad (5.126) \]

\[ + \int dk dl l[a(k)] \otimes (G_\theta^*(k) \otimes 1) \frac{\hat{\mathcal{V}}_f^j(0)}{L_0(\theta)} - z r[a(l)] \otimes (1 \otimes G_\theta^*(l)) \quad (D_3) \quad (5.127) \]

Now we are dealing with different representations l and r and therefore we must be careful because of the relation:

\[ r[a^*(l)] \Omega_f = l[a(l)] e^{\frac{j}{2} \omega(l) \delta \Omega_f} \]

and other similar relations, see Eq. (6.53), Eq. (6.54), Eq. (6.55) and Eq. (6.56).

Taking into account Eq. (6.55):

\[ A_3 = \int dk dl l[a^*(k)] r[a^*(l)] \langle \Omega_f | G_\theta(k) \otimes 1 \frac{\hat{\mathcal{V}}_f^j(r_{\text{aux}} - \omega(l))}{L_{at} + l \hat{L}(\theta) + r(\theta) - \omega(l) - z} 1 \otimes \overline{G}_\theta(l) \Omega_f \rangle \bigg|_{r=L} \quad (5.129) \]
Taking into account Eq. (6.53):

\[ B_3 = \int dkdl l[a^*(k)] \left\langle \Omega_f | G_\theta(k) \otimes \mathbb{1} \frac{\mathcal{X}^f_j(r_{aux})}{L_{at} + \frac{1}{L}(\theta) + r(\theta) - \omega(k) - \omega(l) - \frac{1}{z} \mathcal{G}_\theta(l) \Omega_f} \right| \right| r = L \tag{5.131} \]

Taking into account Eq. (6.55):

\[ C_3 = \int dkdl r[a^*(l)] \left\langle \Omega_f | G_\theta^*(k) \otimes \mathbb{1} \frac{\mathcal{X}^f_j(r_{aux} - \omega(k) + \omega(l))}{L_{at} + \frac{1}{L}(\theta) + r(\theta) - \omega(k) + \omega(l) - \frac{1}{z} \mathcal{G}_\theta(l) \Omega_f} \right| \right| r = L \tag{5.132} \]

Taking into account Eq. (6.53):

\[ D_3 = \int dkdl \left\langle \Omega_f | G_\theta^*(k) \otimes \mathbb{1} \frac{\mathcal{X}^f_j(r_{aux})}{L_{at} + \frac{1}{L}(\theta) + r(\theta) - \omega(k) - \omega(l) - \frac{1}{z} \mathcal{G}_\theta(l) \Omega_f} \right| \right| r = L \tag{5.133} \]

\[ IV = [r[a^*] \otimes (\mathbb{1} \otimes \mathcal{G}_\theta) + r[a] \otimes (\mathbb{1} \otimes \mathcal{G}_\theta^*)] \begin{bmatrix} \mathcal{X}^f_j(\omega(l)) \end{bmatrix} + [l[a^*] \otimes (G_\theta \otimes \mathbb{1}) + l[a] \otimes (G_\theta^* \otimes \mathbb{1})] = \]

\[ \int dkdl \left\langle \Omega_f | G_\theta^*(k) \otimes \mathbb{1} \frac{\mathcal{X}^f_j(\omega(l))}{L_0(\theta) - \omega} \right| \right| \]

\[ + \int dkdl r[a^*(k)] \otimes (\mathbb{1} \otimes \mathcal{G}_\theta(k)) \frac{\mathcal{X}^f_j(\omega(l))}{L_0(\theta) - \omega} \]

\[ + \int dkdl r[a^*(k)] \otimes (\mathbb{1} \otimes \mathcal{G}_\theta^*(k)) \frac{\mathcal{X}^f_j(\omega(l))}{L_0(\theta) - \omega} \]

\[ + \int dkdl r[a^*(k)] \otimes (\mathbb{1} \otimes \mathcal{G}_\theta^*(k)) \frac{\mathcal{X}^f_j(\omega(l))}{L_0(\theta) - \omega} \]

\[ \tag{5.134} \]

\[ \tag{5.135} \]

\[ \tag{5.136} \]

\[ \tag{5.137} \]

\[ \tag{5.138} \]

Once more we deal with different representations r and l therefore we take into account again Eq. (6.53), Eq. (6.54), Eq. (6.55) and Eq. (6.56).

Taking into account Eq. (6.54):

\[ A_4 = \int dkdl r[a^*(k)] l[a^*(l)] \left\langle \Omega_f | [\mathbb{1} \otimes \mathcal{G}_\theta(k)] \frac{\mathcal{X}^f_j(r_{aux} + \omega(l))}{L_{at} + \frac{1}{L}(\theta) + r(\theta) - \omega(k) - \omega(l) - \frac{1}{z} \mathcal{G}_\theta(l) \Omega_f} \right| \right| r = L \tag{5.139} \]

\[ + \left\langle \Omega_f \right| \int \frac{dk}{dl} r[a^*(k)] \mathbb{1} \otimes \mathcal{G}_\theta(k) \frac{\mathcal{X}^f_j(r_{aux})}{L_{at} + \frac{1}{L}(\theta) + r(\theta) - \omega(k) - \omega(l) - \frac{1}{z} \mathcal{G}_\theta(l) \Omega_f} \right| \right| r = L \tag{5.140} \]
Taking into account Eq. (6.54):

\[
B_4 = \int dl \frac{\mathcal{X}_f^\prime(r_{aux})}{L_{at} + L(\theta) + r(\theta) - z} \frac{G^\ast_\theta(l) \otimes \mathbb{1} \Omega_f}{L_{at} + L(\theta) + r(\theta) - z} l[a(l)] \bigg|_{r = L} + \Delta E_0
\]

Taking into account Eq. (6.56):

\[
C_4 = \int dl \frac{\mathcal{X}_f^\prime(r_{aux} - \omega(k) + \omega(l))}{L_{at} + L(\theta) + r(\theta) - z} \frac{G_\theta(l) \otimes \mathbb{1} \Omega_f}{L_{at} + L(\theta) + r(\theta) - z} r[a(k)] \bigg|_{r = L} + \Delta T_0
\]

Taking into account Eq. (6.56):

\[
D_4 = \int dl \frac{\mathcal{X}_f^\prime(r_{aux} - \omega(k))}{L_{at} + L(\theta) + r(\theta) - z} \frac{G^\ast_\theta(l) \otimes \mathbb{1} \Omega_f}{L_{at} + L(\theta) + r(\theta) - z} l[a(l)] \bigg|_{r = L} + \Delta T_0
\]

Recalling the expressions Eq. (5.7), Eq. (5.8), Eq. (5.9) and Eq. (5.10):

\[
\tilde{E}_0[z] = \Delta E_0[z] = \tilde{\omega}_{0,0}[z, 0] \quad (5.145)
\]

\[
\tilde{T}_0[z, L_f] = L_f + \Delta \tilde{T}_0[z, L_f] = L_f + \{\tilde{\omega}_{0,0}[z, r] - \tilde{\omega}_{0,0}[z, 0]\} \quad (5.146)
\]

\[
\tilde{W}_0[z] = gW + \Delta \tilde{W}_0[z] = gW + \sum_{m+n \geq 1} \tilde{W}^{(0)}_{m,n} \quad (5.147)
\]

\[
\tilde{W}^{(0)}_{m,n}[z] := \int dk^{(m)} d\tilde{k}^{(n)} a^+(k^{(m)}) \tilde{\omega}_{m,n}^{(0)}[z; L_f; k^{(m)}; k^{(n)}] a^-(\tilde{k}^{(n)}) \quad (5.148)
\]

In order to identify the different terms of these equations we will transform Eq. (5.109), Eq. (5.112), Eq. (5.120), Eq. (5.130), Eq. (5.134), Eq. (5.140) and Eq. (5.144).

The purpose of this transformation is to fix the point of reference for \( \tilde{T} \) in \( r = 0 \), that is in \( \tilde{T}[z, L] \) we would have \( \tilde{T}[z, 0] = 0 \) and doing this we obtain the terms of \( \tilde{E} \). We will show the procedure with the Eq. (5.112). For the other three the procedure would be the same.

\[
\left\langle \Omega_f \right| \int dl \frac{\mathcal{X}_f^\prime(r_{aux})}{L_{at} + L(\theta) + r(\theta) - z} \frac{G^\ast_\theta(l) \otimes \mathbb{1} \Omega_f}{L_{at} + L(\theta) - z} \bigg|_{r = L} + \Delta E_0
\]

\[
+ \left\langle \Omega_f \right| \int dl \frac{\mathcal{X}_f^\prime(0)}{L_{at} + L(\theta) - z} \bigg|_{r = L} + \Delta T_0
\]

\[
+ \left\langle \Omega_f \right| \int dl \frac{\mathcal{X}_f^\prime(0)}{L_{at} + L(\theta) - z} \bigg|_{r = L} + \Delta T_0
\]
In this way the terms Eq. (5.109), Eq. (5.112), Eq. (5.120), Eq. (5.122), Eq. (5.130), Eq. (5.134), Eq. (5.140) and Eq. (5.144) make contributions to $\Delta \tilde{E}_0$ and $\Delta \tilde{T}_0$. The contributions to $\Delta \tilde{W}[z]$ are the following:

$$\Delta \tilde{W}[z] = W_{11} + W_{20} + W_{02}$$ \hspace{1cm} (5.151)

with:

$$W_{11} = Eq. \ (5.107) + Eq. \ (5.111) + Eq. \ (5.119) + Eq. \ (5.121) + \ldots$$ \hspace{1cm} (5.152)

$$W_{20} = Eq. \ (5.103) + Eq. \ (5.118) + Eq. \ (5.129) + Eq. \ (5.139)$$ \hspace{1cm} (5.154)

$$W_{02} = Eq. \ (5.113) + Eq. \ (5.123) + Eq. \ (5.133) + Eq. \ (5.143)$$ \hspace{1cm} (5.155)

### 5.4 RG for $T=0$

The principal difference with respect to the case $T > 0$ is that we have to work with Hamiltonians instead of Liouvillians and that the spectrum is a little different. That is:

$$L = (L_f, L_{aux}) \rightarrow H_f$$ \hspace{1cm} (5.156)

![Fig.17 Spectrum of $H_0(\theta)$](image)

In Fig.16 we see that instead of triangles like in Fig.10 and Fig.11 we have now lines because of the fact that in Eq. (4.6) we had a difference of two terms but now with the Hamiltonian we have only one, which means only one rotation. We have considered that our model in this section consists of an atom with different levels: the ground state $E_0$, the excited levels $E_1, E_2, \ldots$ which could be degenerated and an ionization energy $\Sigma$. Until now we have avoided degenerated eigenvalues and ionization energies in order not to complicate the problem. We will consider them only in this section. We recall that $\theta = i\vartheta$
In Fig. 17 we see the effect of including the perturbation in the Hamiltonian. As in the case of $T > 0$ (but with other geometry) it produces a displacement of the eigenvalues to the bottom half plane (movement associated with the Fermi-Golden’s rule) and to the sides left or right (movement associated with the Lamb shift). The ground state remains on the axis (but displaced to one side). In the Fig. 17 we have supposed $E_2$ is double and splits into two different domains and that $E_1$ is simple and all displaces to the left.

5.5 Smooth Feshbach Map. Cuspidal Domains of the Spectrum for $T > 0$

In this section we will present the Smooth Feshbach map (SFM) and then we will use it to show that the spectrum of $L_g(\theta)$ is cuspidal near zero.

The SFM is a variant of the FM in which instead of using projections one uses smooth partitions of unity on Hilbert space. We choose a smooth positive operator $\chi$ bounded above by the identity $\mathbb{1}$ on $\mathcal{H}$ and we define:

$$\chi := \sqrt{1 - \chi^2}$$

Taking into account the analysis done before we can write our Liouvillian in each step of the RG as:

$$L = T[L, z] - E[z] \cdot \mathbb{1} + gW[z]$$

where $E[z] \in \mathbb{C}$ is a number which is a purely conventional factor. $T[L, z]$ is a closed operator function of $L$, and $W[z]$ (sum of Wick monomials) is a perturbation defined on the entire domain of $T$. Our purpose is to understand the spectrum of $L_g(\theta)$ near 0. We will use the isospectrality of the renormalization transformation and after a enough number of steps
of the RG the term $W[z]$ tends to zero, simplifying the analysis of the spectrum of $L_g(\theta)$. The domain of the operator $T$ is assumed to be invariant under $\chi$ and $\overline{\chi}$. $T$ is assumed also to commute with $\chi$ and $\overline{\chi}$. We define $L_\chi := T + \chi W \chi$ and $L_{\overline{\chi}} := T + \overline{\chi} W \overline{\chi}$, we assume that both are bounded invertible on the range of $\chi$ and $\overline{\chi}$ respectively. Let $L_{\overline{\chi}} = U |L_{\overline{\chi}}|$ be the polar decomposition of the operator $L_{\overline{\chi}}$. We assume also that $|L_{\overline{\chi}}|^{-1/2} U^{-1} \overline{\chi} W \chi$ and $\chi W \overline{\chi} |L_{\overline{\chi}}|^{-1/2} U^{-1}$ extend to bounded operators on $\mathcal{H}$. With these assumptions we define the smooth Feshbach map as:

$$\mathcal{F}_\chi(L) := T[z;L] - E[z] + g \chi W \chi - g^2 \chi W \overline{\chi} L_{\overline{\chi}}^{-1} \overline{\chi} W \chi$$  \hspace{1cm} (5.159)

For a more complete analysis of the SMF see [6]. The reason why we use SFM is the smoothness of the derivative of these smooth characteristic functions. In contrast, by taking derivatives of projections (case of the FM) we obtain $\delta$-distributions and the calculations are much more involved.

![Fig. 19 Approximative spectrum of $L_g(\theta)$ near 0](image)

Now we will use this SFM to show that the spectrum of $L_g(\theta)$ is cuspidal near 0. In Fig.19 we show it in a graphical form after the application of three steps (step 0,1,2) of the renormalization group method. We begin with the curve 1 which represents an approximation of domain $D$ of Fig.14. In the domain below of this curve we place the center of the new circle $C_1$. This circle correspond to $D_0^n = \{ z \in D_{1/2} | |z| < \rho_0/2 \}$. Now we do the step 0 of the RG for values of $z$ inside of the circle $C_1$. For these values of $z$ we
have proven in Theorem 5.3.3. that the FM is well-defined. We analyze now the values of $z$ inside the circle $C_1$ where $R_0$ is invertible. Because of the isospectrality property of the RG, these values of $z$ are in the spectrum of $L_g(\theta)$. We deduce based on Theorem IV.2 of [14] that above the curve 2 the FM is invertible. The invertibility of $F$ above the curve 2 can also be seen in the following intuitive way, we know that:

$$F_\chi(L) := T[z; L] - E[z] + g\chi W\chi - g^2\chi W\chi L_x^{-1}\chi W\chi$$  \hspace{1cm} (5.160)

$T[z; L] - E[z]$ corresponds in the step 0 to $L_0(\theta) - z$ and we know that this has a spectrum like in Fig.11, i.e. $L_0(\theta) - z$ above the curve 2 is invertible, the other two terms $g\chi W\chi$ and $g^2\chi W\chi L_x^{-1}\chi W\chi$ are small in comparison with the first one (order $O(g\rho^\mu)$ and $O(g^2\rho^\mu)$, $\mu > 0$ where $\rho$ is the parameter of the energy scale). We take a point below of the curve 2 and this will be the center of the circle $C_2$. Using Theorem 5.2.3. the FM is well-defined for the points inside of

$$U_1^{in} = \{ z \in D_{1/2} \mid |z - E_0| < \rho/2 \}.$$  

We do the step 1 of the RG and we analyze where $R_1$ is invertible. Based again on theorem IV.2 of [14] we obtain that the values of $z$ above the curve 3 are in the resolvent set of $L_g(\theta)$. We take a point below of curve 3 and draw the circle $C_3$, etc. In this way we obtain, after an enough number of steps, a cuspidal domain as shown in Fig.19.

We will use the following smooth characteristic function:

$$\chi := P_0 \otimes \chi_\rho[L_{aux}]$$  \hspace{1cm} (5.161)

where:

$$\chi_\rho[L_{aux}] = \sin(\frac{\pi}{2} \Theta(L_{aux}/\rho))$$  \hspace{1cm} (5.162)

where $P_0$ is the orthogonal projection onto the space corresponding to the eigenvalue 0 and $\Theta$ is a smooth, positive function on the interval $[0, \infty)$, with $0 \leq \Theta \leq 1$, $\Theta \equiv 1$ on $[0, \frac{3}{4}]$ and $\Theta \equiv 0$ on $[1, \infty)$. We make the following definition:

$$\chi_\rho[L_{aux}] := \sqrt{1 - \chi_\rho[L_{aux}]^2}$$  \hspace{1cm} (5.163)

$\chi_\rho[L_{aux}]$ is smooth as well. Doing the same analysis as with the FM we can make a Neumann series expansion of the SFM, Eq. (5.159):

$$F_\chi(L) := T[z; L] - E[z] + \chi W[z]\chi - \chi W[z]\chi(T[z; L] - E[z])^{-1}\chi W[z]\chi = T[z; L] - E[z] + \chi W[z]\chi - \chi W[z]\chi(T[z; L] - E[z])^{-1}(1 + \frac{\chi W[z]\chi}{T[z; L] - E[z]})^{-1}\chi W[z]\chi =$$
where we have considered \( \chi \) zero eigenvalues. These correspond to the differences \( E \) and \( \{ \}

\[
\left( 1 + \frac{\chi W[z] \chi}{T[z; L] - E[z]} + \frac{\chi W[z] \chi}{T[z; L] - E[z]} + \frac{\chi W[z] \chi}{T[z; L] - E[z]} + \ldots \right) \chi W[z] \chi =
\]

\[
= T[z; L] - E[z] + \chi W[z] \chi - \chi W[z] \chi (T[z; L] - E[z])^{-1} \times
\]

\[
\left( \chi W[z] \chi + \frac{\chi W[z] \chi^2 W \chi}{T[z; L] - E[z]} + \frac{\chi W[z] \chi^2 W \chi}{T[z; L] - E[z]} + \frac{\chi W[z] \chi^2 W \chi}{T[z; L] - E[z]} + \ldots \right)
\]

and the result is:

\[
= T[z; L_{aux}] - E[z] + \sum_{L=1}^{\infty} (-g)^{L-1} \chi W[z] \left( \frac{\chi^2}{T[z; L] - E[z]} \right)^{L-1} \chi \quad (5.164)
\]

Let us analyze the expression of \( \chi^2 \):

\[
\chi^2 = \overline{P}_0 \otimes \chi^2_\rho[L_{aux}] + P_0 \otimes \overline{\chi}^2_\rho[L_{aux}] + \overline{P}_0 \otimes \overline{\chi}^2_\rho[L_{aux}]
\]

where we have considered \( \chi^2 = P^2_0 \otimes \chi^2_\rho[L_{aux}] \) and \( P^2_0 = P_0 \), and we obtain:

\[
\chi^2 = \overline{P}_0 \otimes \chi^2_\rho[L_{aux}] + (\overline{P}_0 + P_0) \chi^2_\rho[L_{aux}]
\]

defining \( \chi_1^2 \) and \( \chi_2^2 \) as:

\[
\overline{\chi}^2 = \overline{P}_0 \otimes \chi^2_\rho[L_{aux}] + \mathbb{1}_{dt} \otimes \overline{\chi}^2_\rho[L_{aux}] = \chi_1^2 + \chi_2^2 \quad (5.165)
\]

We will also use the following expressions already seen (see Eq. (5.93), Eq. (5.94) and Eq. (5.95)) but now \( \chi_\rho \) is a smooth characteristic function:

\[
\overline{\chi}^2 := \sum_{0}^{M} P_\alpha \otimes \overline{\chi}^2_\rho(0)
\]

where \( \overline{\chi}^2_\rho(\omega) := \begin{cases} 
\mathbb{1}_f & : \text{ for } \alpha = 1, 2 \cdots M \\
\chi^2_\rho[L_{aux} + \omega] & : \text{ for } \alpha = 0
\end{cases} \)

where \( P_\eta \) is the projection onto the space generated by the eigenvector of the eigenvalue \( \varepsilon_\alpha \) and \( \{ \varepsilon_0 \cdots \varepsilon_M \} = \{ E_{i,j} | 0 \leq i, j \leq N - 1 \} \) with \( \varepsilon_0 := 0 \) and \( M \leq N(N - 1) \), that is we have \( N \) zero eigenvalues. These correspond to the differences \( E_{ij} = E_i - E_j \) when \( E_i = E_j \) and \( N^2 - N = N(N - 1) \) non-zero eigenvalues. Now we do the same procedure (Wick ordering)
as in Eq. (5.7), Eq. (5.8) and Eq. (5.9) and explained also in order $g^2$ in a past section (see section 4.3). That is to say:

$$\tilde{L}_0[z] - z := \mathcal{F}_\chi(L_g(\theta) - z) = \chi\{\tilde{E}_0[z] - z + \tilde{T}_0[z; L] + \tilde{W}_0[z]\} \chi$$  \hspace{1cm} (5.166)

The terms $W$ are becoming smaller under successive steps of our RG analysis. That means the process tends to $E_n[z] + T_n[z; L]$. We are interested in knowing if each of the terms $E_i + T_i$ generated in this process are bounded, otherwise we could not know the spectrum of $L_i[z]$, which because of the isospectrality condition of the FM coincides with $L_g(\theta)$. Let us analyze whether $\tilde{T}_0$ is bounded. From the past sections Eq. (5.8), Eq. (5.13) for $m = n = 0$ and with the notation of Eq. (5.15) and Eq. (5.17) we know that the expression of $\tilde{T}_0$ has the following form:

$$\tilde{T}_0[z, r] = P_0(\Omega)\left(\mathcal{F}_\chi(L_g(\theta) + r_f(\theta) - z) - \mathcal{F}_\chi(L_g(\theta) - z)\right)\Omega P_0 =$$

$$= r_f(\theta) - \sum (-g)^n \sum_{a_1 \ldots a_n = \pm} \int dk^{(n)} P_0 G^0_{a_1}(k_1) P_{a_1} \ldots P_{a_{\bar{n}}-1} G^0_{a_{\bar{n}}}(k_n) P_0 \times$$

$$\times \langle \Omega | \{ a^{a_1}(k_1) f_{a_1}[L + r] \cdots f_{a_{\bar{n}}-1}[L + r] a^{a_{\bar{n}}}(k_n) - a^{a_1}(k_1) f_{a_1}[L] \cdots f_{a_{\bar{n}}-1}[L] a^{a_{\bar{n}}}(k_n) \} \rangle \bigg|_{r=L}$$  \hspace{1cm} (5.167)

and taking into account the notation of Eq. (5.8):

$$\tilde{T}_0[z, r] = r_f(\theta) + \{ \tilde{\omega}^0_{0,0}[z, r] - \tilde{\omega}^0_{0,0}[z, 0] \} = r_f(\theta) + \Delta \tilde{\omega}^0_{0,0}[z, r]$$  \hspace{1cm} (5.168)

where:

$$L := (L_f, L_{aux})  \hspace{1cm} r := (r_f, r_{aux})$$  \hspace{1cm} (5.169)

$$r_f(\theta) := r_f \cos \vartheta - i r_{aux} \sin \vartheta$$  \hspace{1cm} (5.170)

$$f_{a}[r_f, r_{aux}] := \frac{\chi^f_j(r_{aux})}{\varepsilon_{a} + r_f(\theta) - z} = \frac{\chi^f_j(r_{aux})}{\varepsilon_{a} + r_f \cos \vartheta - i r_{aux} \sin \vartheta - z}$$  \hspace{1cm} (5.171)

$$\frac{\partial f_a[r_f, r_{aux}]}{\partial r_{aux}} = \frac{\partial \chi^f_j(r_{aux})/\partial r_{aux}}{\varepsilon_{a} + r_f \cos \vartheta - i r_{aux} \sin \vartheta - z} + \frac{\chi^f_j(r_{aux})(-1)(-i \sin \vartheta)}{(\varepsilon_{a} + r_f \cos \vartheta - i r_{aux} \sin \vartheta - z)^2}$$  \hspace{1cm} (5.172)

where $\varepsilon_{a}$ are the eigenvalues of $L_{aux}$.

In proposition 5.5.3 we will see that $\| \frac{\partial \tilde{\omega}^0_{0,0}[z, r]}{\partial r_{aux}} \| \leq \mathcal{O}(g^2)$ but before that, we need several lemmas:
Lemma 5.5.1. Let $z$ be so that $|Re z| < \rho_0/2$ and $|Im z| \geq -\frac{1}{2}\rho_0 \sin \vartheta$ (in particular for $z \in S_{0,\zeta}$, see Fig.13) where $\rho_0 := g^{2-\epsilon}$ for $\epsilon$ small and $0 < \vartheta < \pi/2$. For these values of $z$ and $\chi := P_0^{at} \otimes \chi_{\rho_0}[L_{aux}]$, where $\chi_{\rho_0}[L_{aux}]$ is a smooth characteristic function as in Eq. (5.162), then we have:

$$\left\| L_{aux} + \rho_0 \frac{\partial^2}{\partial r_{aux}} \right\| \leq \frac{4}{\sin \vartheta}$$

(5.173)

Proof. we use $\chi^2 = \chi_1^2 + \chi_2^2$ from Eq. (5.165):

$$\chi_1^2 := P_0 \otimes \chi_{\rho}^2[L_{aux}]$$

(5.174)

$$\chi_2^2 := 1_{aux} \otimes \chi_{\rho}^2[L_{aux}]$$

(5.175)

The proof is the same as the proof of Proposition 5.3.2. using $\chi_1^2$ and $\chi_2^2$ instead of $Q_1$ and $Q_2$.

From this lemma and Eq. (5.171) we see that:

$$\left\| L_{aux} + \rho_0 \right\| \leq \frac{4}{\sin \vartheta}$$

(5.176)

We will use this result in the next proposition.

In the same way we would prove:

Lemma 5.5.2. With the same conditions as before, then:

$$\left\| \frac{(L_{aux} + \rho_0)\partial^2}{\partial r_{aux}} \right\| \leq \frac{4}{\sin \vartheta}$$

(5.177)

$$\left\| \frac{(L_{aux} + \rho_0)^2 \chi^2 \sin \vartheta}{(L_0(\theta) - z)^2} \right\| \leq \frac{16}{\sin \vartheta}$$

(5.178)

From this lemma we see using Eq. (5.172) that:

$$\left\| \frac{(L_{aux} + \rho_0)^2 \partial f_0[L]}{\partial r_{aux}} \right\| \leq \frac{20}{\sin \vartheta}$$

(5.179)

which we use in the following proposition:

Proposition 5.5.3. Assume that $z \in S_{0,\zeta}$ (see Fig.13) and $g > 0$ sufficiently small, then:

$$\left\| \frac{\partial \tilde{\omega}_{0,0}[z, r]}{\partial r_{aux}} \right\| = \left\| \frac{\partial \tilde{T}_{0}[z, r]}{\partial r_{aux}} - \frac{\partial r_f(\theta)}{\partial r_{aux}} \right\| \leq O(g^\epsilon)$$

(5.180)
Proof. If we take derivatives in Eq. (5.176)
\[
\frac{\partial T_0[z,r]}{\partial r_{aux}} = \frac{\partial f(\theta)}{\partial r_{aux}} - \frac{\partial}{\partial r_{aux}} \sum_{n=2}^{\infty} (-g)^n \sum_{\alpha_1,\alpha_n=\pm}^{\sigma_1,\sigma_n=\pm} \int dk^{(n)} P_0 G_{\sigma_1}^{\theta}(k_1) P_{\alpha_1} \cdots P_{\alpha_n-1} G_{\sigma_n}^{\theta}(k_n) P_0 \times \]
\[
\times \langle \Omega | a^{\sigma_1}(k_1) f_{\alpha_1} [L + r] \cdots f_{\alpha_n-1} [L + r] a^{\sigma_n}(k_n) \rangle | r=L \rangle,
\]
recalling that \( r_f(\theta) = \rho_f \cos \vartheta - iv_{aux} \sin \vartheta \) and \( \theta = i \vartheta \), we have:
\[
\frac{\partial T_0[z,r]}{\partial r_{aux}} = -i \sin \vartheta - \sum_{n=2}^{\infty} (-g)^n \sum_{\alpha_1,\alpha_n=\pm}^{\sigma_1,\sigma_n=\pm} \int dk^{(n)} P_0 G_{\sigma_1}^{\theta}(k_1) P_{\alpha_1} \cdots P_{\alpha_n-1} G_{\sigma_n}^{\theta}(k_n) P_0 \times \]
\[
\times \sum_{j=1}^{n-1} \langle \Omega | a^{\sigma_1}(k_1) f_{\alpha_1} [L + r] \cdots a^{\sigma_j}(k_1) \frac{\partial f_{\alpha_j} [L + r]}{\partial r_{aux}} \cdots f_{\alpha_n-1} [L + r] a^{\sigma_n}(k_n) \rangle | r=L \rangle.
\]
We analyze now one term (the \( n^{th} \) of the sum \( \sum_{n=2}^{\infty} \sum_{\alpha_1,\alpha_n=\pm}^{\sigma_1,\sigma_n=\pm} \), taking absolute value of it:
\[
\rho_0 \left| (-g)^n \sum_{j=1}^{n-1} \langle \Omega | a^{\sigma_1}(G_{\sigma_1}^{\theta}) f_{\alpha_1} [L + r] \cdots a^{\sigma_j}(G_{\sigma_j}^{\theta}) \frac{\partial f_{\alpha_j} [L + r]}{\partial r_{aux}} \cdots f_{\alpha_n-1} [L + r] a^{\sigma_n}(G_{\sigma_n}^{\theta}) \rangle | r=L \rangle \right|
\]
\[
\rho_0 \left| (-g)^n \sum_{j=1}^{n-1} \langle \Omega | (L_{aux} + \rho_0)^{-1/2} a^{\sigma_1}(G_{\sigma_1}^{\theta})(L_{aux} + \rho_0)^{-1/2} (L_{aux} + \rho_0) f_{\alpha_1} [L + r] (L_{aux} + \rho_0)^{-1/2} \cdots \right.
\]
\[
\left. (L_{aux} + \rho_0)^{-1/2} a^{\sigma_j}(G_{\sigma_j}^{\theta})(L_{aux} + \rho_0)^{-1} (L_{aux} + \rho_0)^{1/2} \frac{\partial f_{\alpha_j} [L + r]}{\partial r_{aux}} (L_{aux} + \rho_0)^{-1} \cdots \right.
\]
\[
\left. (L_{aux} + \rho_0)^{-1/2} (L_{aux} + \rho_0) f_{\alpha_n-1} [L + r] (L_{aux} + \rho_0)^{-1/2} a^{\sigma_n}(G_{\sigma_n}^{\theta})(L_{aux} + \rho_0)^{-1/2} \rangle | r=L \rangle \right|
\]
we can check this equation in the same way as with Eq. (5.89), where the \( \rho_0 \) at the beginning of the last equation comes from the following consideration:
\[
\rho_0 |\langle \Omega | (L_{aux} + \rho_0)^{-1/2} \cdots (L_{aux} + \rho_0)^{-1/2} \rangle | \Omega \rangle \right|
\]
and \( L_{aux} |\Omega \rangle = 0. \) Using now Eq. (5.176) and Eq. (5.179):
\[
(5.181) \leq \rho_0 (n-1) g^n \left( \frac{4}{\sin \vartheta} \right)^{n-2} \left( \frac{20}{\sin \vartheta} \right)^n \| (L_{aux} + \rho_0)^{-1/2} a^{\sigma_j}(G_{\sigma_j}^{\theta})(L_{aux} + \rho_0)^{-1} \| \times \]
\[
\| (L_{aux} + \rho_0)^{-1} a^{\sigma_{j+1}}(G_{\sigma_{j+1}}^{\theta})(L_{aux} + \rho_0)^{-1/2} \| \prod_{i\neq j,j+1} (L_{aux} + \rho_0)^{-1/2} a^{\sigma_i}(G_{\sigma_i}^{\theta})(L_{aux} + \rho_0)^{-1/2} \|
\]
that is to say:

\[
\rho_0(n-1)g^n \frac{4}{\sin \vartheta} \left( \frac{20}{\sin \vartheta} \right) \cdot 20 \sin \vartheta \parallel \left( L_{aux} + \rho_0 \right)^{-1/2} a^{\sigma_i} (G_{\sigma_i}^g) (L_{aux} + \rho_0)^{-1/2} \parallel \parallel (L_{aux} + \rho_0)^{-1/2} a^{\sigma_j+1} (G_{\sigma_j+1}^g) (L_{aux} + \rho_0)^{-1/2} \parallel \times \\
\prod_{i \neq j,j+1} (L_{aux} + \rho_0)^{-1/2} a^{\sigma_i} (G_{\sigma_i}^g) (L_{aux} + \rho_0)^{-1/2} \parallel
\]

taking into account the estimate Eq. (5.63) of lemma 5.3.1.:

\[
\parallel (L_{aux} + \rho_0)^{-1/2} a^{\sigma_i} (G_{\sigma_i}^g) (L_{aux} + \rho_0)^{-1/2} \parallel = O(\rho^{-1/2}) \quad (5.183)
\]

we obtain finally:

\[
\parallel \frac{\partial \tilde{T}_0[z,r]}{\partial r_{aux}} + i \sin \vartheta \parallel \leq \sum_{n=2}^{\infty} K \rho_0 \frac{1}{\rho_0^{1/2}} n
\]

(5.184)

where \( K = (n-1) \frac{4}{\sin \vartheta} \left( \frac{20}{\sin \vartheta} \right) \). Considering that \( \rho_0 \approx g^{2-\varrho} \), then \((\rho_0^{1/2})^2 = g^{2-\varrho} = g^\varrho\) that is to say:

\[
\parallel \frac{\partial \tilde{T}_0[z,r]}{\partial r_{aux}} + i \sin \vartheta \parallel \leq O(g^\varrho) \quad (5.185)
\]

In the same way we would prove (see [7]):

**Proposition 5.5.4.** Assume that \( z \in S_{0,>} \). Then there exists a constant, \( C < \infty \), such that, for \( g > 0 \) sufficiently small:

\[
\parallel E_0[z] + ig^2 \Gamma_0 \parallel \leq C g^{2+2 \varrho} \quad (5.186)
\]

\[
\parallel \nabla^{(0)}_{m,n} \omega \parallel \leq C \rho_0 \left( \frac{C g}{\rho_0} \right)^{m+n} \prod_{j=1}^{m} \frac{\kappa(k_j)}{\omega(k_j)^{1/2-\mu}} \prod_{j=1}^{n} \frac{\kappa(k_j)}{\omega(k_j)^{1/2-\mu}} \quad (5.187)
\]

\[
\int_{B_{r_0}^{m+n}} \parallel \partial_{aux} \omega^{(0)}_{m,n}[z,r, K^{(m,n)}] \parallel \prod_{j=1}^{m} \frac{d^3k_j}{\omega(k_j)^{3/2-\mu}} \prod_{j=1}^{n} \frac{d^3k_j}{\omega(k_j)^{3/2-\mu}} \leq C(Cg)^{m+n} \quad (5.188)
\]

where \( B_r := \{ k | \omega(k) < r \} \)

Similar bounds can be established for the step 1.

After each step of the RG we have expressions of the form:

\[
L_n = \mathcal{J} \left( L_{n-1} \right) := P_n \left( T_n[z; L] - E_n[z] + W_n[z] \right) P_n \quad (5.189)
\]
and then we handle $W_n[z]$ as a perturbation. We develop $L_n$ in a Neumann series and we study the convergence of this Neumann series. The convergence will be secured for sufficiently big values of $T_n[z; L] - E_n[z]$. Taking into account that:

$$T_n[z; L] = T_{n-1}[z; L] + \{\omega^{(n-1)}_0(z, L) - \omega^{(n-1)}_{0,0}(z, 0)\}$$  \hspace{1cm} (5.190)

and that we have seen in theorem 5.5.3.:

$$\frac{\partial}{\partial r_{aux}} \{\omega^{(0)}_{0,0}(z, L) - \omega^0_{0,0}(z, 0)\} \leq O(g^\epsilon) = \Delta_0$$  \hspace{1cm} (5.191)

we could prove inductively that:

$$\frac{\partial}{\partial r_{aux}} \{\omega^{(n-1)}_{0,0}(z, L) - \omega^{n-1}_{0,0}(z, 0)\} \leq \Delta_{n-1}$$  \hspace{1cm} (5.192)

where the bound $\Delta_{n-1}$ depends only on:

$$\sup_{N+M \geq 1} \|W_{M,N}^{(n-1)}\| \leq C g^{2^M} \rho^n$$  \hspace{1cm} (5.193)

From these considerations we can also prove inductively that:

$$|T_n[z; L]| \geq \sin \vartheta (1 - \delta_n) L_{aux} \quad |\delta_n| \leq \frac{1}{2} \quad \forall n$$  \hspace{1cm} (5.194)

From Eq. (5.185), we have:

$$T_0[z, r] \geq \sin \vartheta r_{aux} - \frac{1}{2} L_{aux}$$  \hspace{1cm} (5.195)

Suppose that it is true for $n - 1$:

$$|T_{n-1}[z; L]| \geq \sin \vartheta (1 - \delta_{n-1}) L_{aux}$$  \hspace{1cm} (5.196)

then from Eq. (5.192) and Eq. (5.190):

$$|T_n[z; L]| \geq \sin \vartheta (1 - \delta_{n-1} - \frac{\Delta_{n-1}}{\sin \vartheta}) L_{aux} = \sin \vartheta (1 - \delta_n) L_{aux}$$  \hspace{1cm} (5.197)

that is to say:

$$\delta_n = \delta_{n-1} + \frac{\Delta_{n-1}}{\sin \vartheta}; \quad \Delta_0 = g^\epsilon \quad \delta_0 = 1/2$$  \hspace{1cm} (5.198)

this is the renormalization group flux equation.

Being the meaning of $W_n[z]$ a perturbation and having seen that after the application of an enough number of steps this $W_n[z]$ becomes small (see Eq. (5.187) and Eq. (5.188)), we see that our $L_n$ tends to an unperturbed Liouvillian, the domain of which we have seen in chapter 4 is cuspidal.
Chapter 6

The Level Shift Operator

6.1 Introduction

To uniformize the notation and make it applicable to the magnons in a ferromagnet we will work on the Hilbert space \((\mathcal{H}_{pp} \otimes \mathcal{H}_{pp}) \otimes \mathcal{H}_c^\beta\), where \(\mathcal{H}_c^\beta\) is the Hilbert space formed by the GNS Hilbert space of the Hamiltonian with an absolute continuous spectrum (that would be the electromagnetic field Hamiltonian \(\mathcal{H}_f^\beta\) in the case of the atom coupled to a radiation field) and by \(\mathcal{H}_{pp} \otimes \mathcal{H}_{pp}\), the latter is the Hilbert space obtained from \(\mathcal{K}\) through the isomorphism \(I_C : \mathcal{K} \rightarrow \mathcal{H}_{pp} \otimes \mathcal{H}_{pp}\), where \(\mathcal{H}_{pp}\) is the Hilbert space of the Hamiltonian whose spectrum consists of isolated eigenvalues (that would be the atomic Hilbert space \(\mathcal{H}_{at}\) in the case of the atom coupled to a radiation field). We take a general (minimal coupling and dipole approximation) interaction of the form:

\[
I = \int dk [G(k) \otimes a^*(k) + G^*(k) \otimes a(k)]
\]

where \(G(k)\) is an operator on \(\mathcal{H}_{pp}\) i.e. a \(\mathbb{C}^{N \times N}\)-matrix or in a shorter notation \(I = a^*G + a(G^*)\) our Liouvillian is:

\[
L_g = L_0 + gW
\]

\[
L_0 = L_c \otimes 1_{pp} + 1_c \otimes L_{pp}
\]

\[
L_{pp} = H_{pp} \otimes 1_{pp} - 1_{pp} \otimes H_{pp}
\]

\[
L_c = \int dk \omega(k) [l(a^*(k)l(a(k)) - r(a^*(k)r(a(k)))] \quad \text{where } r(a^2) = J_c l(a^2) J_c
\]
where \( l \) and \( r \) are two representations of \( \mathcal{P} \) on \( \mathcal{H}_c^\beta \). We know (see chapter 2) that these two representations are related by
\[
\begin{align*}
\mathcal{J}_c l[a]|\Omega_c^\beta\rangle &:= e^{-\frac{\beta}{2} \mathcal{L}_c l[a]^*} |\Omega_c^\beta\rangle \\
W &:= l[I] - r[I] \\
l[I] &:= l[a^*] \otimes (G \otimes \mathbb{1}) + l[a] \otimes (G^* \otimes \mathbb{1}) \\
r[I] &:= \mathcal{J}_c l[a^*] \mathcal{J}_c \otimes (\mathbb{1} \otimes TGT) + \mathcal{J}_c l[a] \mathcal{J}_c \otimes (\mathbb{1} \otimes TG^*T)
\end{align*}
\]

We will write \( TGT = \mathcal{C} = (G^*)^t \)

\[
\begin{align*}
\sigma(L_{pp}) &= \begin{array}{cccccccc}
\times & \times & \times & \times & \times & \times & \times & \times \\
& & & & & & & \mathbb{0}
\end{array} \\
\sigma(L_f) &= \begin{array}{cccccccc}
\times & \times & \times & \times \\
& & & \mathbb{0}
\end{array} \\
\sigma(L_0) &= \begin{array}{cccccccc}
\times & \times & \times & \times \\
& & & \mathbb{0} & \times & \times & \times & \times
\end{array}
\end{align*}
\]

**Fig.20**

We will try to understand the stability of these eigenvalues.
- First we do a complex deformation, the equality Eq. (4.21):
\[
\langle \varphi | (L_g - z)^{-1} \psi \rangle = \langle \hat{U}(\bar{\vartheta}) \varphi | (L_g(\vartheta) - z)^{-1} \hat{U}(\vartheta) \psi \rangle
\]
permits us to do an analytic continuation of the left side from the upper half-plane to the lower half-plane for all the values of \( z \) where the right side exists. From now on we will take \( \vartheta = i \vartheta \), with \( \vartheta \in \mathbb{R}^+ \).
- Next we use a Feshbach Map to explore the properties of \( \sigma(L_g(i\vartheta)) \), \( 0 < \vartheta < \vartheta_0 \) with the help of a perturbative method, provided that we know \( \sigma(L_0(i\vartheta)) \).

As Projection we take:
\[
P = P(L_{pp} = E_{ij}) \otimes P_{\Omega_f} \quad (6.10)
\]
where \( E_{ij} = E_i - E_j \); \( i, j = 1 \ldots N - 1 \) are the eigenvalues of \( L_{pp} \) and where \( \Omega_f \) is the KMS state on \( \mathcal{H}_f^\beta \).

We will use:
\[
L_g(\theta, z) := L_g(\theta) - z
\]
also:

\[ L_g(\theta, z) := L_{pp}(z) + L_c(\theta) + gW(\theta) \]  \hspace{1cm} (6.12)

\[ L_0(\theta, z) := L_{pp}(z) + L_c(\theta) \]  \hspace{1cm} (6.13)

\[ L_{pp}(z) := L_{pp} - z \]  \hspace{1cm} (6.14)

With these notations and from Eq. (4.39) the Feshbach Map results in:

\[ \mathcal{F}_P(L_g(\theta, z)) = PL_g(\theta, z)P - g^2PW(\theta)\bar{P}(\bar{L}_g(\theta, z))^{-1}\bar{P}W(\theta)P \]  \hspace{1cm} (6.15)

where we have taken into account that:

\[ PL_g(\theta, z)\bar{P} = gPW(\theta)P \] \hspace{1cm} (P commutes with \( L_0 \)) \hspace{1cm} (6.16)

and \( \bar{P} = 1 - P \), \( \bar{L}_g(\theta, z) = \bar{P}L_g(\theta, z)\bar{P} \) \hspace{1cm} (6.17)

From the form of \( W \), see Eq. (6.7), Eq. (6.8) we see that \( PW\bar{P} = 0 \) also \( PL_c(\theta)P = 0 \) and:

\[ \mathcal{F}_P(L_g(\theta, z)) = PL_{pp}(z)P - g^2PW(\theta)\bar{P}(\bar{L}_g(\theta, z))^{-1}\bar{P}W(\theta)P \]  \hspace{1cm} (6.18)

writing now \( L_g(\theta, z) = L_0(\theta, z) + gW(\theta) \), we have:

\[ L_g(\theta, z)^{-1} = (L_0(\theta, z) + gW(\theta))^{-1} = \left[ L_0(\theta, z)(1 + L_0(\theta, z)^{-1}gW(\theta)) \right]^{-1} \hspace{1cm} (6.19)\]

\[ (1 + L_0(\theta, z)gW(\theta))^{-1}L_0(\theta, z)^{-1} = L_0(\theta, z)^{-1}\sum_{n=0}^{\infty} \left(-gW(\theta)L_0(\theta, z)^{-1}\right)^n \hspace{1cm} (6.20)\]

Substituting this in Eq. (6.18):

\[ \mathcal{F}_P(L_g(\theta, z)) = PL_{pp}(z)P - g^2PW(\theta)\bar{P}\left[ L_0(\theta, z)^{-1}\sum_{n=0}^{\infty} \left(-gW(\theta)L_0(\theta, z)^{-1}\right)^n \right]\bar{P}W(\theta)P \]  \hspace{1cm} (6.21)

One can show that the resolvent on the r.h.s. exists provided \( P \) is replaced by \( P' = P(L_{pp} = E_{ij}) \otimes P_{\text{Laux} \leq \rho} \) for some small \( \rho > 0 \) and \( \text{Im}z \geq -\frac{1}{2}\rho \) and then we use a limit argument as \( \rho \to 0 \). From now on we will set \( z = E_{ij} \). The lowest order of Eq. (6.18) is from Eq. (6.21) \( L_0(\theta, z)^{-1} \) i.e. (recalling Eq. (6.13) and Eq. (4.15)):

\[ L_0(\theta, z) = \bar{P}'(L_{pp} + L_c \cos \vartheta - iL_{\text{aux}} \sin \vartheta - z)\bar{P}' \]  \hspace{1cm} (6.22)
with:
\[ \overline{P} = P(L_{pp} \neq E_{ij}) \otimes P_{L_{aux} \leq \rho} + 1 \otimes P(L_{aux} > \rho) := Q_1 + Q_2 \] (6.23)

with \( Q_1 \) we have:
\[ Q_1 L_0(\theta, z) = Q_1 (L_{pp} + L_c \cos \theta - i L_{aux} \sin \theta - z) = Q_1 (L_{pp} - z) > \frac{d}{2} \] (6.24)

where \( d \) is the minimal distance between eigenvalues of \( L_{pp} \), therefore we have \[ \| Q_1 L_0(\theta, z) \| < \frac{d}{2}. \] For \( Q_2 \) is easy to see that \( \| Q_2 \| \) is bounded.

We consider now only the term in second order in \( g \) of Eq. (6.21):
\[ F_P(L_g(\theta, z))^{(2)} = -PW(\theta)\overline{P}(L_0(\theta, z))^{-1}\overline{PW}(\theta)P \] (6.25)

This term is defined as the Level-shift operator:
\[ \Gamma^{(2)} := -F_P(L_g(\theta, z))^{(2)} = PW(\theta)\overline{P}(L_0(\theta, z))^{-1}\overline{PW}(\theta)P \] (6.26)

We see that since \( PW(\theta)P = 0 \) then \( PW(\theta)\overline{P} = PW(\theta) \) and \( \overline{PW}(\theta)P = W(\theta)P \) and we have:
\[ PW(\theta)(L_0(\theta, z))^{-1}W(\theta)P = \lim_{\varepsilon \to 0^+} PW(\theta)(L_0(\theta) - \varepsilon)^{-1}W(\theta)P \] (6.27)

This last function is analytic in \( \theta \), therefore:
\[ PW(\theta)(L_0(\theta, z))^{-1}W(\theta)P = \lim_{\varepsilon \to 0^+} PW(\overline{L_0} - \varepsilon)^{-1}WP \] (6.28)

We are interested in the imaginary part of \( \Gamma^{(2)} \), which will yield information on life times of resonances:
\[ Im(F_P(L_g(\theta, z)))^{(2)} = \lim_{\varepsilon \to 0^+} \left\{ -g^2 \frac{1}{2i} PW[(\overline{L_0} - z - i\varepsilon)^{-1} - (\overline{L_0} - z + i\varepsilon)^{-1}]WP \right\} = \]
\[ = PW \lim_{\varepsilon \to 0^+} \left( -g^2 \frac{2i\varepsilon}{2i (\overline{L_0} - z)^2 - \varepsilon^2} \right) WP = -g^2 PW \pi \delta(L_0 - z) WP \] (6.29)

where \( z = E_{ij} \). In the last equality we have used:
\[ \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \xrightarrow{\varepsilon \to 0} \delta(x) \] (6.30)

See for example [35].

In the next sections we will calculate:
\[ Im \Gamma^{(2)} := -Im(F_P(L_g(\theta, z)))^{(2)} = \pi PW \delta(L_0(z))WP \] (6.31)
6.2 General Calculation

\( Im \Gamma^{(2)} \) corresponds to an operator \( \Sigma^{(2)} \) on \( \mathcal{H}_{pp} \otimes \mathcal{H}_{pp} \) given by:

\[
\Sigma^{(2)} = \langle \Omega_c | W \delta(\mathcal{L}_c + L_{pp}(z)) W \Omega_c \rangle |_{\mathcal{H}_{pp}} \tag{6.32}
\]

and with Eq. (6.6) we obtain:

\[
\Sigma^{(2)} = \langle \Omega_c | l(I) \delta(\mathcal{L}_c + L_{pp}(z)) l(I) \Omega_c \rangle |_{\mathcal{H}_{pp}} \tag{6.33} \quad I
\]

\[
+ \langle \Omega_c | r(I) \delta(\mathcal{L}_c + L_{pp}(z)) r(I) \Omega_c \rangle |_{\mathcal{H}_{pp}} \tag{6.34} \quad II
\]

\[
- \langle \Omega_c | l(I) \delta(\mathcal{L}_c + L_{pp}(z)) r(I) \Omega_c \rangle |_{\mathcal{H}_{pp}} \tag{6.35} \quad -III
\]

\[
- \langle \Omega_c | r(I) \delta(\mathcal{L}_c + L_{pp}(z)) l(I) \Omega_c \rangle |_{\mathcal{H}_{pp}} \tag{6.36} \quad -IV
\]

Now we develop Eq. (6.33), Eq. (6.34), Eq. (6.35) and Eq. (6.36):

\[
I = \langle \Omega_c | l(I) \delta(\mathcal{L}_c + L_{pp}(z)) l(I) \Omega_c \rangle |_{\mathcal{H}_{pp}} = \langle \Omega_c | \{ (G \otimes \mathbf{1}) \otimes l[a^*] + (G^* \otimes \mathbf{1}) \otimes l[a] \} \delta(\mathcal{L}_c + L_{pp}(z)) \times \\
\times [(G \otimes \mathbf{1}) \otimes l[a^*] + (G^* \otimes \mathbf{1}) \otimes l[a]] \Omega_c \rangle |_{\mathcal{H}_{pp}} \tag{6.37}
\]

From these products we have only to consider the products that contain one \( a \) and one \( a^* \), all the others vanish because of:

**Lemma 6.2.1.**

\[
\langle \Omega_c | l[a(k)] l[a(k')] \Omega_c \rangle = \langle \Omega_c | l[a^*(k)] l[a^*(k')] \Omega_c \rangle = 0 \tag{6.38}
\]

\[
\langle \Omega_c | r[a(k)] r[a(k')] \Omega_c \rangle = \langle \Omega_c | r[a^*(k)] r[a^*(k')] \Omega_c \rangle = 0 \tag{6.39}
\]

\[
\langle \Omega_c | l[a(k)] r[a^*(k')] \Omega_c \rangle = \langle \Omega_c | l[a^*(k)] r[a(k')] \Omega_c \rangle = 0 \tag{6.40}
\]

\[
\langle \Omega_c | r[a(k)] l[a^*(k')] \Omega_c \rangle = \langle \Omega_c | l[a^*(k)] r[a(k')] \Omega_c \rangle = 0 \tag{6.41}
\]

and \( k, k' \) can be equal or different.

**Proof.** Eqs. (6.38) are trivial, for example \( \langle \Omega_c | l[a^*(k)] l[a^*(k')] \Omega_c \rangle = \langle \Omega_c | (k + k') \rangle = 0 \), where we have considered that \( l[a^*(k')] \Omega_c = |k'\rangle \).

Eqs. (6.39) are similar deduced, for example:

\[
\langle \Omega_c | r[a(k)] r[a(k')] \Omega_c \rangle = \langle \Omega_c | \mathcal{J}_c l[a(k)] \mathcal{J}_c \mathcal{J}_c l[a(k')] \mathcal{J}_c \Omega_c \rangle \tag{6.42}
\]
where we have used Eq. (2.40) and now we use Eq. (2.57) \( \mathcal{J}_c\Omega_c = |\Omega_c\rangle \) and Eq. (2.46) \( \mathcal{J}_c\mathcal{J}_c = 1 \):

\[
\langle\Omega_c|l[a(k)]\mathcal{J}_c\mathcal{J}_c[l(a(k'))]|\Omega_c\rangle = \langle\Omega_c|l[a(k)]l[a(k')]|\Omega_c\rangle = \langle(k + k')|\Omega_c\rangle = 0 \tag{6.43}
\]

Eqs. (6.40) and Eqs. (6.41) are similar deduced. We will deduce one of them, for the other the procedure is similar:

\[
\langle\Omega_c|l[a(k)]r[a^*(k')]|\Omega_c\rangle = \langle\Omega_c|l[a(k)]\mathcal{J}_c[l(a^*(k'))|\Omega_c\rangle = \\
\langle\Omega_c|l[a^*(k)]e^{-\frac{\delta}{2L}}l[a^*(k')]|\Omega_c\rangle = \langle\Omega_c|l[a^*(k)]l[a^*(k')]e^{-\frac{\delta}{2}(\mathcal{L}_c + \omega(k'))}|\Omega_c\rangle = 0
\]

where we have used \( \langle\Omega_c|l[a(k)]\mathcal{J}_c = \langle\Omega_c|l[a^*(k)]e^{-\frac{\delta}{2}\mathcal{L}_c} \)

Therefore from Eq. (6.37):

\[
I = \int dk dl \left[ \langle G(k) \otimes 1 \rangle \langle\Omega_c|l[a^*(k)]|\delta(\mathcal{L}_c + L_{pp}(z))l[a(l)]\Omega_c\rangle (G^*(l) \otimes 1) \right]_{\mathcal{H}_{pp}} + \\
+ \langle G^*(k) \otimes 1 \rangle \langle\Omega_c|l[a^*(k)]|\delta(\mathcal{L}_c + L_{pp}(z))l[a^*(k')]|\Omega_c\rangle (G(l) \otimes 1) \right]_{\mathcal{H}_{pp}} \tag{6.44}
\]

Now we apply Lemma Appendix A 3.1.2, \( f(\mathcal{L}_c)|\Omega_c\rangle = f(0)|\Omega_c\rangle \), Eq. (2.62) and Eq. (2.63):

\[
\langle\Omega_c|l[a^*(k)]\delta(\mathcal{L}_c + L_{pp}(z))l[a(l)]\Omega_c\rangle = \langle\Omega_c|\delta(\mathcal{L}_c - \omega(k) + L_{pp}(z))l[a^*(k)]l[a(l)]\Omega_c\rangle = \\
= \delta(L_{pp}(z) - \omega(k))\langle\Omega_c|l[a^*(k)]l[a(l)]\Omega_c\rangle = \delta(L_{pp}(z) - \omega(k))\delta(k - l)\frac{1}{e^{\beta\omega(k)} - 1}
\]

in the same way:

\[
\langle\Omega_c|l[a^*(k)]l[a^*(l)]\Omega_c\rangle = \langle\Omega_c|\delta(\mathcal{L}_c + \omega(k) + L_{pp}(z))l[a(k)]l[a^*(l)]\Omega_c\rangle = \\
= \delta(L_{pp}(z) + \omega(k))\langle\Omega_c|l[a^*(k)]l[a(l)]\Omega_c\rangle = \delta(L_{pp}(z) + \omega(k))\delta(k - l)e^{\beta\omega(k)} \frac{1}{e^{\beta\omega(k)} - 1}
\]

and finally we get:

\[
I = \int dk dl \left[ \langle G(k) \otimes 1 \rangle \delta(L_{pp}(z) - \omega(k))\delta(k - l)\frac{1}{e^{\beta\omega(k)} - 1} (G^*(l) \otimes 1) \right]_{\mathcal{H}_{pp}} + \\
+ \int dk dl \left[ \langle G^*(k) \otimes 1 \rangle \delta(L_{pp}(z) + \omega(k))\delta(k - l)e^{\beta\omega(k)} \frac{1}{e^{\beta\omega(k)} - 1} (G(l) \otimes 1) \right]_{\mathcal{H}_{pp}} \tag{6.45}
\]

and using polar coordinates:

\[
I = \int_{\mathbb{R}^+} d\omega 4\pi \omega^2 (G(k) \otimes 1) \delta(L_{pp}(z) - \omega(k))\frac{1}{e^{\beta\omega(k)} - 1} G^*(k) \otimes 1 \right]_{\mathcal{H}_{pp}} + \\
+ \int_{\mathbb{R}^+} d\omega 4\pi \omega^2 (G^*(k) \otimes 1) \delta(L_{pp}(z) + \omega(k))\frac{e^{\beta\omega(k)}}{e^{\beta\omega(k)} - 1} (G(k) \otimes 1) \right]_{\mathcal{H}_{pp}} \tag{6.46}
\]
Let us make the same procedure for II Eq. (6.35) now with the right representation on $H$

Because of Eq. (6.39) only cross products are different from 0:

\[
\delta(\mathcal{L}_c + L_{pp}(z))(\mathbb{1} \otimes \mathcal{T} G) \otimes \mathcal{J}_c l[a^*] \mathcal{J}_c + (\mathbb{1} \otimes \mathcal{T} G^* T) \otimes \mathcal{J}_c l[a] \mathcal{J}_c
\]

\[
\Omega_c = \langle r(I) \delta(\mathcal{L}_c + L_{pp}(z)) r(I) \Omega_c \rangle_{\mathcal{H}_{pp}} = \langle \mathcal{J}_c l[a^*] \mathcal{J}_c + (\mathbb{1} \otimes \mathcal{T} G^* T) \otimes \mathcal{J}_c l[a] \mathcal{J}_c \rangle_{\mathcal{H}_{pp}} = (6.50)
\]

Because of Eq. (6.39) only cross products are different from 0:

\[
II = \int dkdl \left[ (\mathbb{1} \otimes \mathcal{T} G(k)) \langle \Omega_c | r[a^*(k)] \delta(\mathcal{L}_c + L_{pp}(z)) r[a(l)] \Omega_c \rangle (\mathbb{1} \otimes \mathcal{T} G^*(l)) \right]_{\mathcal{H}_{pp}} + (6.51)
\]

\[
+ \int dkdl \left[ (\mathbb{1} \otimes \mathcal{T} G^*(k)) \langle \Omega_c | r[a(k)] \delta(\mathcal{L}_c + L_{pp}(z)) r[a^*(l)] \Omega_c \rangle (\mathbb{1} \otimes \mathcal{T} G(l)) \right]_{\mathcal{H}_{pp}} (6.52)
\]

We have used that $r[a] = \mathcal{J}_c l[a] \mathcal{J}_c$. Now we use the following calculation:

\[
r[a(l)] \Omega_c = \mathcal{J}_c l[a(l)] \mathcal{J}_c \Omega_c = \mathcal{J}_c l[a(l)] \Omega_c = e^{-\frac{i}{2} \mathcal{L}_c l[a(l)]} \Omega_c = l[a(l)] e^{-\frac{i}{2} (\mathcal{L}_c + \omega(l))} \Omega_c
\]

\[
= l[a(l)]^* e^{-\frac{i}{2} \mathcal{L}_c + \omega(l)} \Omega_c = l[a(l)]^* e^{-\frac{i}{2} \omega(l)} \Omega_c (6.53)
\]

where we have used $\mathcal{J}_c \Omega_c = |\Omega_c\rangle$, $\mathcal{J}_c l[a] \Omega_c = e^{-\frac{i}{2} \mathcal{L}_c l[a^*]} \Omega_c$, the pull-through formula and $\mathcal{L}_c \Omega_c = 0$.

In the same way we have:

\[
\langle \Omega_c | r[a^*(k)] \rangle = \langle \Omega_c | \mathcal{J}_c l[a^*(k)] \mathcal{J}_c = \langle \Omega_c | l[a^*(k)] \mathcal{J}_c =
\]

\[
= \langle \Omega_c | l[a(k)] e^{-\frac{i}{2} \mathcal{L}_c} = \langle \Omega_c | e^{-\frac{i}{2} \mathcal{L}_c + \omega(k)} l[a(k)] = \langle \Omega_c | e^{-\frac{i}{2} \omega(k)} l[a(k)] (6.54)
\]

the other two similar results that we need are:

\[
r[a^*(l)] \Omega_c = l[a(l)] e^{\frac{i}{2} \omega(l)} \Omega_c (6.55)
\]

\[
\langle \Omega_c | r[a(k)] \rangle = \langle \Omega_c | e^{\frac{i}{2} \omega(k)} l[a(k)]^* (6.56)
\]

with these results it follows from Eq. (6.51) and Eq. (6.52):

\[
II = \int dkdl \left[ (\mathbb{1} \otimes \mathcal{T} G(k)) \langle \Omega_c | \delta(\mathcal{L}_c + \omega(k) + L_{pp}(z)) l[a(k)] l[a^*(l)] \Omega_c e^{-\frac{i}{2} (\omega(k) + \omega(l))} (\mathbb{1} \otimes \mathcal{T} G^*(l)) \right]_{\mathcal{H}_{pp}} + (6.57)
\]

\[
+ \int dkdl \left[ (\mathbb{1} \otimes \mathcal{T} G^*(k)) e^{\frac{i}{2} (\omega(k) + \omega(l))} \langle \Omega_c | \delta(\mathcal{L}_c - \omega(k) + L_{pp}(z)) l[a^*(k)] l[a(l)] \Omega_c \rangle (\mathbb{1} \otimes \mathcal{T} G(l)) \right]_{\mathcal{H}_{pp}} (6.58)
\]
and now with Eq. (2.62) and Eq. (2.63):

\[
II = \int dkdl \left[ (1 \otimes \mathcal{G}(k)) \delta(L_{pp}(z) + \omega(k)) e^{-\frac{1}{2} \omega(k)} \frac{e^{\beta\omega(k)}}{e^{\beta\omega(k)} - 1} \delta(k - l)(1 \otimes \mathcal{G}^*(l)) \right]_{\mathcal{H}_{pp}} + (6.59)
\]

\[
+ \int dkdl \left[ (1 \otimes \mathcal{G}^*(k)) \delta(L_{pp}(z) - \omega(k)) e^{\frac{1}{2} \omega(k)} \frac{1}{e^{\beta\omega(k)} - 1} \delta(k - l)(1 \otimes \mathcal{G}(l)) \right]_{\mathcal{H}_{pp}} + (6.60)
\]

using polar coordinates:

\[
II = \int_{R^+} d\omega 4\pi\omega^2 (1 \otimes \mathcal{G}(k)) \delta(L_{pp}(z) + \omega(k)) \frac{1}{e^{\beta\omega(k)} - 1} (1 \otimes \mathcal{G}^*(k)) \right]_{\mathcal{H}_{pp}} (6.61)
\]

\[
+ \int_{R^+} d\omega 4\pi\omega^2 (1 \otimes \mathcal{G}^*(k)) \delta(L_{pp}(z) - \omega(k)) e^{\beta\omega(k)} \frac{1}{e^{\beta\omega(k)} - 1} \mathcal{G}(k) \right]_{\mathcal{H}_{pp}} + (6.62)
\]

For the term III, Eq. (6.35):

\[
III = \langle \Omega_c | l(I) \delta(L_c + L_{pp}(z)) r(I) \Omega_c \rangle_{\mathcal{H}_{pp}} = \langle \Omega_c | \left[ (G \otimes 1) \otimes l[a^*] + (G^* \otimes 1) \otimes l[a] \right] \delta(L_c + L_{pp}(z)) \left(1 \otimes TGT \right) \otimes \mathcal{J}_c l[a^*] \mathcal{J}_c + (1 \otimes TG^* T) \otimes \mathcal{J}_c l[a] \mathcal{J}_c \Omega_c \rangle_{\mathcal{H}_{pp}} (6.63)
\]

We use now Eq. (6.40) and Eq. (6.41), therefore the only surviving terms are the ones that contain both creation operator or both annihilation operators:

\[
III = \int dkdl \left[ (G(k) \otimes 1) \langle \Omega_c | l[a^*(k)] \delta(L_c + L_{pp}(z)) r(a^*(l)) \Omega_c \rangle \mathcal{G}(l) \right]_{\mathcal{H}_{pp}} + (6.64)
\]

\[
+ \int dkdl \left[ (G^*(k) \otimes 1) \langle \Omega_c | l[a(k)] \delta(L_c + L_{pp}(z)) r(a(l)) \Omega_c \rangle \mathcal{G}^*(l) \right]_{\mathcal{H}_{pp}} (6.65)
\]

We take into account now the equations \( r[a(l)] = \mathcal{J}_c l[a(l)] \mathcal{J}_c, \mathcal{J}_c |\Omega_c \rangle = 0, \mathcal{J}_c l[a(l)] \Omega_c \rangle = e^{\frac{1}{2} \mathcal{L}_c l[a^*(l)]} |\Omega_c \rangle \) and similar equations for the creation operators:

\[
III = \int dkdl \left[ (G(k) \otimes 1) \langle \Omega_c | \delta(L_c + L_{pp}(z) - \omega(k)) e^{\frac{1}{2} \omega(l)} l[a^*(k)] l[a(l)] \Omega_c \rangle \mathcal{G}(l) \right]_{\mathcal{H}_{pp}} + (6.66)
\]

\[
+ \int dkdl \left[ (G^*(k) \otimes 1) \langle \Omega_c | \delta(L_c + L_{pp}(z) + \omega(k)) e^{-\frac{1}{2} \omega(l)} l[a(k)] l[a^*(l)] \Omega_c \rangle \mathcal{G}^*(l) \right]_{\mathcal{H}_{pp}} (6.67)
\]

considering now Eq. (2.62) and Eq. (2.63):

\[
III = \int dkdl \left[ (G(k) \otimes 1) \delta(L_{pp}(z) - \omega(k)) e^{\beta\omega(l)} \frac{1}{e^{\beta\omega(k)} - 1} \delta(k - l)(1 \otimes \mathcal{G}(l)) \right]_{\mathcal{H}_{pp}} + (6.68)
\]

\[
+ \int dkdl \left[ (G^*(k) \otimes 1) \delta(L_{pp}(z) + \omega(k)) e^{-\beta\omega(l)} \frac{1}{e^{\beta\omega(k)} - 1} \delta(k - l)(1 \otimes \mathcal{G}^*(l)) \right]_{\mathcal{H}_{pp}} (6.69)
\]
and finally using polar coordinates:

\[
III = \int_{\mathbb{R}^+} d\omega 4\pi \omega^2 (G(k) \otimes \mathbb{1}) \delta(L_{pp}(z) - \omega(k)) \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k)} - 1} (\mathbb{1} \otimes \overline{G}(k)) \bigg|_{\mathcal{H}_{pp}} + \tag{6.70}
\]

\[
+ \int_{\mathbb{R}^+} d\omega 4\pi \omega^2 (G^*(k) \otimes \mathbb{1}) \delta(L_{pp}(z) + \omega(k)) \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k)} - 1} (\mathbb{1} \otimes \overline{G}^*(k)) \bigg|_{\mathcal{H}_{pp}} \tag{6.71}
\]

The last term of the operator \( \Sigma^{(2)} \) is Eq. (6.36):

\[
IV = \langle \Omega_c | r(I) \delta(L_c + L_{pp}(z)) l(l) \Omega_c \rangle \bigg|_{\mathcal{H}_{pp}} = \tag{6.72}
\]

\[
= \langle \Omega_c | (\mathbb{1} \otimes T G^T) \otimes J_c l[a^*] J_c + (\mathbb{1} \otimes T G^* T) \otimes J_c l[a] J_c \rangle \times \delta(L_c + L_{pp}(z)) \bigg[ (G \otimes \mathbb{1}) \otimes l[a^*] + (G^* \otimes \mathbb{1}) \otimes l[a] \bigg] \Omega_c \rangle \bigg|_{\mathcal{H}_{pp}} \tag{6.73}
\]

From Eq. (6.40) and Eq. (6.41) we see that the only surviving terms are the ones that contain both creation operator or both annihilation operators. We also use \( \langle \Omega_c | J_c l[a^*(k)] J_c = \langle \Omega_c | l[a(k)] e^{-\frac{\beta}{2} L_c} = \langle \Omega_c | e^{-\frac{\beta}{2} (L_c + \omega(k))} l[a(k)] = \langle \Omega_c | e^{-\frac{\beta}{2} \omega(k)} l[a(k)] \) and similar equations for the annihilation operator:

\[
IV = \int dk dl \left[ (1 \otimes \overline{G}(k)) \langle \Omega_c | e^{-\frac{\beta}{2} \omega(k)} \delta(L_c + L_{pp}(z) + \omega(k)) l[a(k)] l[a^*(l)] \Omega_c \rangle (G(l) \otimes \mathbb{1}) \right] \bigg|_{\mathcal{H}_{pp}} + \tag{6.74}
\]

\[
+ \int dk dl \left[ (1 \otimes \overline{G}^*(k)) \langle \Omega_c | e^{\frac{\beta}{2} \omega(k)} \delta(L_c + L_{pp}(z) - \omega(k)) l[a^*(k)] l[a(l)] \Omega_c \rangle (G^*(l) \otimes \mathbb{1}) \right] \bigg|_{\mathcal{H}_{pp}} \tag{6.75}
\]

using now Eq. (2.62) and Eq. (2.63)

\[
IV = \int dk dl \left[ (1 \otimes \overline{G}(k)) \delta(L_{pp}(z) + \omega(k)) e^{-\frac{\beta}{2} \omega(k)} \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k)} - 1} \delta(k - l)(G(l) \otimes \mathbb{1}) \right] \bigg|_{\mathcal{H}_{pp}} + \tag{6.76}
\]

\[
+ \int dk dl \left[ (1 \otimes \overline{G}^*(k)) \delta(L_{pp}(z) - \omega(k)) e^{\frac{\beta}{2} \omega(k)} \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k)} - 1} \delta(k - l) \Omega_c \rangle (G^*(l) \otimes \mathbb{1}) \right] \bigg|_{\mathcal{H}_{pp}} \tag{6.77}
\]

and finally expressing it in polar coordinates:

\[
IV = \int_{\mathbb{R}^+} d\omega 4\pi \omega^2 (1 \otimes \overline{G}(k)) \delta(L_{pp}(z) + \omega(k)) \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k)} - 1} (G(k) \otimes \mathbb{1}) \bigg|_{\mathcal{H}_{pp}} + \tag{6.78}
\]

\[
+ \int_{\mathbb{R}^+} d\omega 4\pi \omega^2 (1 \otimes \overline{G}^*(k)) \delta(L_{pp}(z) - \omega(k)) \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k)} - 1} (G^*(k) \otimes \mathbb{1}) \bigg|_{\mathcal{H}_{pp}} \tag{6.79}
\]

So far we have considered projections onto a general \( \mathcal{H}_{pp} \) atomic Hilbert space. Now we will restrict our atomic projectors to smaller spaces in the two next sections. In our model \( \mathcal{H}_{pp} \) is finite dimensional as explained in chapter 1.
6.3 Application to the Eigenvalue $E_i - E_j \neq 0$ of the Atomic Liouvillian

We will analyze in this section the stability of an eigenvalue $E_i - E_j$ of $L_{pp}$ with $i \neq j$. We will suppose that $E_i - E_j$ is non-degenerated.

We will calculate the four terms I, II, III, IV and then we will see that our level-shift operator is positive, which implies the instability of these eigenvalues (for $i, j = 1 \ldots N \ i \neq j$).

We begin with I:

We have to project Eq. (6.46) and Eq. (6.47) onto $\{\varphi_i \otimes \varphi_j\}$ for $i, j$ fixed (the space of eigenvectors of $E_i - E_j$ is one-dimensional because of the non-degeneracy of the eigenvalues different from zero and we have:

$$\langle \varphi_i \otimes \varphi_j | L \varphi_i \otimes \varphi_j \rangle = \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_j | (G(k) \otimes 1) \delta(L_{pp}(z) - \omega(k)) \varphi_m \otimes \varphi_p \rangle \times$$

$$\times \langle \varphi_m \otimes \varphi_p | (G^*(l) \otimes 1) \varphi_i \otimes \varphi_j \rangle \frac{1}{e^{\beta \omega(k)} - 1} +$$

$$+ \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_j | (G^*(k) \otimes 1) \delta(L_{pp}(z) + \omega(k)) \varphi_m \otimes \varphi_p \rangle \times$$

$$\times \langle \varphi_m \otimes \varphi_p | (G(l) \otimes 1) \varphi_i \otimes \varphi_j \rangle \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1}$$

now we use that $(H_{pp} \otimes 1 - 1 \otimes H_{pp})|\varphi_m \otimes \varphi_p\rangle = E_m|\varphi_m\rangle \otimes |\varphi_p\rangle - |\varphi_m\rangle \otimes E_p|\varphi_p\rangle = (E_m - E_p)|\varphi_m \otimes \varphi_p\rangle$, and therefore $f(L_{pp})|\varphi_m \otimes \varphi_p\rangle = f(E_m - E_p)|\varphi_m \otimes \varphi_p\rangle$ writing the matrix elements of $G(k)$, $G_{ij}(k) = \langle \varphi_i | G(k) | \varphi_j \rangle$, we use also that $L_{pp}(z) := L_{pp} - z$ with $z = E_i - E_j$:

$$\langle \varphi_i \otimes \varphi_j | L \varphi_i \otimes \varphi_j \rangle = \sum_{m,p} \int dk G_{im}(k) \delta(E_m - E_p - E_i + E_j - \omega(k)) \frac{1}{e^{\beta \omega(k)} - 1} G_{mi}^*(k) \delta(p) +$$

$$+ \sum_{m,p} \int dk G_{im}^*(k) \delta(E_m - E_p - E_i + E_j + \omega(k)) \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1} G_{mi}(k) \delta(p)$$

taking into account that $(G^*)_{mi}(k) = G_{im}(k)$:

$$\langle \varphi_i \otimes \varphi_j | L \varphi_i \otimes \varphi_j \rangle = \sum_m \int dk |G_{im}(k)|^2 \frac{1}{e^{\beta \omega(k)} - 1} \delta(E_m - E_i - \omega(k))) +$$

$$+ \sum_m \int dk |G_{mi}(k)|^2 \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1} \delta(E_m - E_i + \omega(k)))$$
and finally expressing the result in polar coordinates and using the dispersion relation for phonons, which will be used in the next section $\omega(k) = k^2$, we have:

$$
\langle \varphi_i \otimes \varphi_j | I \varphi_i \otimes \varphi_j \rangle = \sum_{m>i} |G_{im}(E_m - E_i)|^2 \frac{1}{e^{\beta(E_m - E_i)} - 1} 2\pi(E_m - E_i)^{1/2} +
$$

$$
+ \sum_{m<i} |G_{mi}(E_i - E_m)|^2 \frac{e^{\beta(E_i - E_m)}}{e^{\beta(E_i - E_m)} - 1} 2\pi(E_i - E_m)^{1/2}
$$

(6.86)

(6.87)

from $\omega(k) \geq 0$ and if at least one of the elements $G_{im}(E_m - E_i)|_{m>i}$ or at least one of the elements $G_{mi}(E_i - E_m)|_{m<i}$ is different from zero, then:

$$
\langle \varphi_i \otimes \varphi_j | I \varphi_i \otimes \varphi_j \rangle > 0
$$

(6.88)

Now we calculate the second term $II$ of $\sum^{(2)}$, from Eq. (6.59) and Eq. (6.60):

$$
\langle \varphi_i \otimes \varphi_j | II \varphi_i \otimes \varphi_j \rangle = \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_j | (1 \otimes \overline{G}(k))\delta(L_{pp}(z) + \omega(k))\varphi_p \otimes \varphi_m \rangle \times
$$

$$
\times \langle \varphi_p \otimes \varphi_m | (1 \otimes \overline{G}^*(k))\varphi_i \otimes \varphi_j \rangle \frac{1}{e^{\beta\omega(k)} - 1} +
$$

(6.89)

(6.90)

$$
+ \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_j | (\mathbb{1} \otimes \overline{G}^*(k))\delta(L_{pp}(z) - \omega(k))\varphi_p \otimes \varphi_m \rangle \times
$$

$$
\times \langle \varphi_p \otimes \varphi_m | (\mathbb{1} \otimes \overline{G}(k))\varphi_i \otimes \varphi_j \rangle \frac{e^{\beta\omega(k)}}{e^{\beta\omega(k)} - 1}
$$

(6.91)

taking into account $f(L_{pp})|\varphi_m \otimes \varphi_p) = f(E_m - E_p)|\varphi_m \otimes \varphi_p)$ as seen before, $\langle \varphi_i \otimes \varphi_j | (\mathbb{1} \otimes \overline{G}(k))\varphi_p \otimes \varphi_m \rangle = \delta_{ip}(\overline{G}(k))_{jm}$ and $(\overline{G}^*(k))_{mj} = G(k)_{jm}$:

$$
\langle \varphi_i \otimes \varphi_j | II \varphi_i \otimes \varphi_j \rangle = \sum_{m,p} \int dk \overline{G}_{jm}(k)\delta_{ip}\delta(E_p - E_m - E_i + E_j + \omega(k)) \frac{1}{e^{\beta\omega(k)} - 1} G_{jm}(k)\delta_{pi} +
$$

$$
+ \sum_{m,p} \int dk G_{mj}(k)\delta_{ip}\delta(E_p - E_m - E_i + E_j - \omega(k)) \frac{e^{\beta\omega(k)}}{e^{\beta\omega(k)} - 1} \overline{G}_{mj}(k)\delta_{pi}
$$

(6.92)

finally using polar coordinates it becomes:

$$
\langle \varphi_i \otimes \varphi_j | II \varphi_i \otimes \varphi_j \rangle = \sum_{m>j} |G_{jm}(E_m - E_j)|^2 \frac{1}{e^{\beta(E_m - E_j)} - 1} 2\pi(E_m - E_j)^{1/2} +
$$

$$
+ \sum_{m<j} |G_{mj}(E_j - E_m)|^2 \frac{e^{\beta(E_j - E_m)}}{e^{\beta(E_j - E_m)} - 1} 2\pi(E_j - E_m)^{1/2}
$$

(6.93)

(6.94)
from $\omega(k) \geq 0$ and if at least one of the elements $G_{jm}(E_m - E_j)|_{m> j}$ or at least one of the elements $G_{mj}(E_j - E_m)|_{m< j}$ is different from zero, then:

\[ \langle \varphi_i \otimes \varphi_j | II \varphi_i \otimes \varphi_j \rangle > 0 \quad (6.95) \]

If we assume $G_{ii} = 0, \forall i$ then we see that:

\[ \langle \varphi_i \otimes \varphi_j | III \varphi_i \otimes \varphi_j \rangle = \langle \varphi_i \otimes \varphi_j | IV \varphi_i \otimes \varphi_j \rangle = 0 \quad (6.96) \]

We will show it for example for III, from Eq. (6.68) and Eq. (6.69):

\[ \langle \varphi_i \otimes \varphi_j | III \varphi_i \otimes \varphi_j \rangle = \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_j | (G(k) \otimes \mathbb{1}) \delta(L_{pp}(z) - \omega(k)) \varphi_m \otimes \varphi_p \rangle \times \]

\[ \times \langle \varphi_m \otimes \varphi_p | (\mathbb{1} \otimes \overline{G}(k)) \varphi_i \otimes \varphi_j \rangle e^{\frac{\beta}{2} \omega(k)} e^{\beta \omega(k) - 1} + \]

\[ + \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_j | (G^*(k) \otimes \mathbb{1}) \delta(L_{pp}(z) + \omega(k)) \varphi_m \otimes \varphi_p \rangle \times \]

\[ \langle \varphi_m \otimes \varphi_p | (\mathbb{1} \otimes \overline{G}^*(k)) \varphi_j \otimes \varphi_j \rangle e^{\frac{\beta}{2} \omega(k)} e^{\beta \omega(k) - 1} = \quad (6.99) \]

taking into account the same considerations as for I and II:

\[ \sum_{m,p} \int dk G_{im}(k) \delta_{jp} \delta(E_i - E_p - \omega(k)) \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k) - 1}} \overline{G}_{pj}(k) \delta_{mi} + \]

\[ + \sum_{m,p} \int dk \overline{G}_{mi}(k) \delta_{jp} \delta(E_i - E_p + \omega(k)) \frac{e^{\frac{\beta}{2} \omega(k)}}{e^{\beta \omega(k) - 1}} \overline{G}_{jp}(k) \delta_{mj} = 0 \quad (6.101) \]

It vanishes because of the fact that all the matrix elements $G_{ij}$ in this expression are diagonal $(i = j)$.

Taking all the results together we have:

\[ (\Sigma^{(2)})_{ij} > 0 \quad i \neq j \quad (6.102) \]

and that means instability of the eigenvalue $E_i - E_j$. That is to say when we add the interaction to our atomic Liouvillian, the non-degenerated real eigenvalues of $L_{pp}$ different from zero, move to the lower half plane. They become complex numbers with a negative imaginary part which implies instability.
6.4 Application to the Eigenvalue 0 of the Atomic Liouvillian

Let us consider now the eigenvalue 0 of \( L_{pp} \) and the N-dimensional Hilbert space \( \mathcal{H}_{pp} = \{ \varphi_i \otimes \varphi_i \}_{i=0}^{N-1} \).

In this case we have the matrix elements (see Eq. (6.46) and Eq. (6.47)):

\[
I_{ij} = \langle \varphi_i \otimes \varphi_i | I | \varphi_j \otimes \varphi_j \rangle = \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (G(k) \otimes 1)| \varphi_m \otimes \varphi_p \rangle \times \\
\times \langle \varphi_m \otimes \varphi_p | (G^*(l) \otimes 1) | \varphi_j \otimes \varphi_j \rangle \frac{1}{e^{\beta \omega(k)} - 1} + \\
+ \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (G^*(k) \otimes 1) | \varphi_m \otimes \varphi_p \rangle \times \\
\times \langle \varphi_m \otimes \varphi_p | (G(l) \otimes 1) | \varphi_j \otimes \varphi_j \rangle \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1}
\]

(6.103)

(6.104)

(6.105)

taking into account \( f(L_{pp})| \varphi_m \otimes \varphi_p \rangle = f(E_m - E_p) | \varphi_m \otimes \varphi_p \rangle \) and \( \langle \varphi_i \otimes \varphi_i | (G^*(k) \otimes 1) | \varphi_m \otimes \varphi_p \rangle = (G^*(k))_{im} \delta_{ip} \):

\[
= \sum_{m,p} \int dk G_{im}(k) \delta_{ip} \delta(E_m - E_j - \omega(k)) \frac{1}{e^{\beta \omega(k)} - 1} G_{jm}(k) \delta_{pj} + \\
+ \sum_{m,p} \int dk \overline{G}_{mi}(k) \delta_{ip} \delta(E_m - E_j - \omega(k)) \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1} G_{mj}(k) \delta_{pj}
\]

(6.106)

(6.107)

now we use polar coordinates and integrate:

\[
= \delta_{ij} \sum_m G_{im}(E_m - E_j - \frac{1}{e^{\beta (E_m - E_i)} - 1} \overline{G}_{jm}(E_m - E_j) 2\pi (E_m - E_j)^{1/2} + \\
+ \delta_{ij} \sum_m \overline{G}_{mi}(E_j - E_m - \frac{e^{\beta (E_j - E_m)}}{e^{\beta (E_j - E_m)} - 1} G_{mj}(E_j - E_m) 2\pi (E_j - E_m)^{1/2} =
\]

(6.108)

(6.109)

and that becomes:

\[
= \sum_{m>i} |G_{im}(E_m - E_i)|^2 \frac{1}{e^{\beta (E_m - E_i)} - 1} 2\pi (E_m - E_i)^{1/2} + \\
+ \sum_{m<i} |G_{mi}(E_i - E_m)|^2 \frac{e^{\beta (E_i - E_m)}}{e^{\beta (E_i - E_m)} - 1} 2\pi (E_i - E_m)^{1/2}
\]

(6.110)

(6.111)

and our result is:

\[
I_{ij} = \begin{cases} 
Eq. (6.110) + Eq. (6.111) & : i \neq j \\
0 & : otherwise
\end{cases}
\]

(6.112)
In a similar way we calculate \( I_{ij} \) (from Eq. (6.59) and Eq. (6.60)):

\[
I_{ij} = \langle \varphi_i \otimes \varphi_i | I I \varphi_j \otimes \varphi_j \rangle = \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (1 \otimes \overline{G}(k)) \delta(L_{pp}(z) + \omega(k)) \varphi_p \otimes \varphi_m \rangle \times \\
\times \langle \varphi_p \otimes \varphi_m | (1 \otimes \overline{G}(k)) \varphi_j \otimes \varphi_j \rangle \frac{1}{e^{\beta \omega(k)} - 1} + \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1} \tag{6.114}
\]

\[
+ \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (1 \otimes \overline{G}(k)) \delta(L_{pp}(z) - \omega(k)) \varphi_p \otimes \varphi_m \rangle \times \\
\times \langle \varphi_p \otimes \varphi_m | (1 \otimes \overline{G}(k)) \varphi_j \otimes \varphi_j \rangle \frac{1}{e^{\beta \omega(k)} - 1} \tag{6.115}
\]

taking into account \( f(L_{pp}) | \varphi_m \otimes \varphi_p \rangle = f(E_m - E_p) | \varphi_m \otimes \varphi_p \rangle \), \( \langle \varphi_i \otimes \varphi_i | (1 \otimes \overline{G}(k)) \varphi_m \otimes \varphi_p \rangle = \delta_{im}(\overline{G}(k))_{kp} \) and similar relations:

\[
I_{ij} = \sum_{m,p} \int dk \overline{G}_{im}(k) \delta_{ip} \delta(E_i - E_m + \omega(k)) \frac{1}{e^{\beta \omega(k)} - 1} \overline{G}_{jm}(k) \delta_{pj} + \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1} \overline{G}_{mj}(k) \delta_{pj} \tag{6.116}
\]

\[
+ \sum_{m,p} \int dk G_{mi}(k) \delta_{ip} \delta(E_i - E_m - \omega(k)) \frac{e^{\beta \omega(k)}}{e^{\beta \omega(k)} - 1} G_{mj}(k) \delta_{pj} \tag{6.117}
\]

taking polar coordinates and integrating:

\[
I_{ij} = \delta_{ij} \sum_m |G_{im}(E_m - E_i)|^2 \frac{1}{e^{\beta(E_m - E_i)} - 1} G_{jm}(E_m - E_i) 2\pi(E_m - E_i)^{1/2} + \frac{e^{\beta(E_m - E_i)}}{e^{\beta(E_m - E_i)} - 1} \tag{6.118}
\]

\[
+ \delta_{ij} \sum_m G_{mi}(E_i - E_m) \frac{e^{\beta(E_i - E_m)}}{e^{\beta(E_i - E_m)} - 1} \overline{G}_{mj}(E_i - E_m) 2\pi(E_i - E_m)^{1/2} \tag{6.119}
\]

our result is:

\[
I_{ij} = \sum_{m > i} |G_{im}(E_m - E_i)|^2 \frac{1}{e^{\beta(E_m - E_i)} - 1} 2\pi(E_m - E_i)^{1/2} + \frac{e^{\beta(E_m - E_i)}}{e^{\beta(E_m - E_i)} - 1} \tag{6.120}
\]

\[
+ \sum_{m < i} |G_{mi}(E_i - E_m)|^2 \frac{e^{\beta(E_i - E_m)}}{e^{\beta(E_i - E_m)} - 1} 2\pi(E_i - E_m)^{1/2} \tag{6.121}
\]

i.e.:

\[
I_{ij} = \begin{cases} 
Eq. (6.120) + Eq. (6.121) & : i = j \\
0 & : otherwise
\end{cases} \tag{6.122}
\]
We note that Eq. (6.122)=Eq. (6.112).

In the same way we calculate $III_{ij}$ from Eq. (6.68) and Eq. (6.69):

$$III_{ij} = \langle \varphi_i \otimes \varphi_i | III \varphi_j \otimes \varphi_j \rangle =$$

$$= \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (G(k) \otimes 1) \delta(L_{pp}(z) - \omega(k)) \varphi_m \otimes \varphi_p \rangle \times$$

$$\times \langle \varphi_m \otimes \varphi_p | (1 \otimes \overline{G}(k)) \varphi_j \otimes \varphi_j \rangle e^{\frac{\beta \omega(k)}{2}} +$$

$$+ \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (G^\ast(k) \otimes 1) \times \delta(L_{pp}(z) + \omega(k)) \varphi_m \otimes \varphi_p \rangle \times$$

$$\langle \varphi_m \otimes \varphi_p | (1 \otimes \overline{G}^\ast(k)) \varphi_j \otimes \varphi_j \rangle e^{\frac{\beta \omega(k)}{2}}$$

(6.123)

Taking into account $f(L_{pp})|\varphi_m \otimes \varphi_p \rangle = f(E_m - E_p)|\varphi_m \otimes \varphi_p \rangle$, $\langle \varphi_i \otimes \varphi_i | (G(k) \otimes 1) \varphi_m \otimes \varphi_p \rangle = (\overline{G}(k))_{im} \delta_{ip}$ and similar relations:

$$III_{ij} = \sum_{m,p} \int dk G_{im}(k) \delta_{ip} \delta(E_m - E_p - \omega(k)) e^{\frac{\beta \omega(k)}{2}} \overline{G}_{pj}(k) \delta_{mj} +$$

$$+ \sum_{m,p} \int dk G^\ast_{im}(k) \delta_{ip} \delta(E_m - E_p + \omega(k)) e^{\frac{\beta \omega(k)}{2}} \overline{G}^\ast_{pj}(k) \delta_{mj}$$

(6.124)

After some simplifications with the $\delta$ of Kronecker:

$$III_{ij} = \int dk |G_{ij}(k)|^2 e^{\frac{\beta \omega(k)}{2}} \overline{G}_{pj}(k) \delta_{pj}$$

(6.125)

(6.126)

Taking polar coordinates and integrating:

$$III_{ij} = \begin{cases} |G_{ij}(E_j - E_i)|^2 e^{\frac{\beta}{2} (E_j - E_i)} e^{\frac{\beta}{2} (E_j - E_i)} \frac{2\pi (E_j - E_i)^{1/2}}{2\pi (E_j - E_i)^{1/2}} : if \ j > i \\ |G_{ji}(E_i - E_j)|^2 e^{\frac{\beta}{2} (E_i - E_j)} e^{\frac{\beta}{2} (E_i - E_j)} \frac{2\pi (E_i - E_j)^{1/2}}{2\pi (E_i - E_j)^{1/2}} : if \ j < i \end{cases}$$

(6.127)
and finally IV:

\[ IV_{ij} = \langle \varphi_i \otimes \varphi_i | IV \varphi_j \otimes \varphi_j \rangle = \]  
\[ = \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (\mathds{1} \otimes \overline{G}(k)) \delta(L_{pp}(z) + \omega(k)) \varphi_m \otimes \varphi_p \rangle \times \]  
\[ \times \langle \varphi_m \otimes \varphi_p | (G(k) \otimes \mathds{1}) \varphi_j \otimes \varphi_j \rangle \frac{e^{\frac{2}{\beta} \omega(k)}}{e^{\beta \omega(k)} - 1} + \]  
\[ + \sum_{m,p} \int dk \langle \varphi_i \otimes \varphi_i | (\mathds{1} \otimes \overline{G}^*(k)) \delta(L_{pp}(z) - \omega(k)) \varphi_m \otimes \varphi_p \rangle \times \]  
\[ \times \langle \varphi_m \otimes \varphi_p | (G^*(k) \otimes \mathds{1}) \varphi_j \otimes \varphi_j \rangle \frac{e^{\frac{2}{\beta} \omega(k)}}{e^{\beta \omega(k)} - 1} \]  
(6.131)

taking into account \( f(L_{pp})|\varphi_m \otimes \varphi_p \rangle = f(E_m - E_p)|\varphi_m \otimes \varphi_p \rangle \), \( \langle \varphi_i \otimes \varphi_i | (\mathds{1} \otimes \overline{G}(k)) \varphi_m \otimes \varphi_p \rangle = \delta_{im}(\overline{G}(k))_{ip} \) and similar relations:

\[ IV_{ij} = \sum_{m,p} \int dk \overline{G}_{ip}(k) \delta_{im} \delta(E_m - E_p + \omega(k)) \frac{e^{\frac{2}{\beta} \omega(k)}}{e^{\beta \omega(k)} - 1} G_{mj}(k) \delta_{pj} + \]  
(6.132)
\[ + \sum_{m,p} \int dk \overline{G}^*_{ip}(k) \delta_{im} \delta(E_m - E_p - \omega(k)) \frac{e^{\frac{2}{\beta} \omega(k)}}{e^{\beta \omega(k)} - 1} G^*_{mj}(k) \delta_{pj} \]  
(6.133)

simplifying:

\[ IV_{ij} = \int dk |G_{ij}(k)|^2 \frac{e^{\frac{2}{\beta} \omega(k)}}{e^{\beta \omega(k)} - 1} \delta(E_i - E_j - \omega(k)) + \]  
\[ + \int dk |G_{ji}(k)|^2 \frac{e^{\frac{2}{\beta} \omega(k)}}{e^{\beta \omega(k)} - 1} \delta(E_i - E_j + \omega(k)) \]  
(6.134)
\[ = \left\{ \begin{array}{ll}
|G_{ij}(E_j - E_i)|^2 \frac{e^{\frac{2}{\beta} (E_j - E_i)}}{e^{\beta (E_j - E_i)} - 1} 2\pi(E_j - E_i)^{1/2} & : \text{if } j > i \\
|G_{ji}(E_i - E_j)|^2 \frac{e^{\frac{2}{\beta} (E_i - E_j)}}{e^{\beta (E_i - E_j)} - 1} 2\pi(E_i - E_j)^{1/2} & : \text{if } j < i
\end{array} \right. \]  
(6.135)

We introduce a set of functions and another notation for the projection of the level shift operator:

\[ \phi_{im}(E_m - E_i) := 2 |G_{im}(E_m - E_i)|^2 \frac{e^{\beta (E_m - E_i)}}{e^{\beta (E_m - E_i)} - 1} 2\pi(E_m - E_i)^{1/2} \]  
(6.136)
\[ M := \prod_{i=0}^{N-1} |\langle \varphi_i \otimes \varphi_i \rangle|^{\delta_{im}(\overline{G}(k))_{ip}} \]  
(6.137)

\[ M := \prod_{i=0}^{N-1} |\langle \varphi_i \otimes \varphi_i \rangle|^{\delta_{im}(\overline{G}(k))_{ip}} \]  
(6.138)
We can write the matrix elements of $M$ from Eq. (6.112), Eq. (6.122), Eq. (6.130), Eq. (6.138):

$$M_{ii} = I_{ii} + II_{ii} = 2I_{ii}$$

$$M_{ij} = -III_{ij} - IV_{ij} = -2III_{ij}$$

Let us see now an easy example:

$$N = 2, E_1, E_2$$

$$M_{11} = e^{-\beta(E_2-E_1)}\phi_{12}(E_2 - E_1) + 0$$

$$M_{22} = 0 + \phi_{12}(E_2 - E_1)$$

$$M_{12} = -2|G_{12}(E_2 - E_1)|^2 e^{\frac{\beta(E_2-E_1)}{2}} - 1 2\pi(E_2 - E_1)^{1/2}$$

$$M_{21} = -2|G_{12}(E_2 - E_1)|^2 e^{\frac{\beta(E_2-E_1)}{2}} - 1 2\pi(E_2 - E_1)^{1/2}$$

$M$ is symmetric.

**Definition 6.4.1.** A bounded operator $G$ on $L^2$ is called positivity preserving if $Gf$ is positive whenever $f \in L^2(x,d\mu)$ is positive.

**Definition 6.4.2.** A bounded operator $G$ on $L^2$ is called ergodic if and only if it is positivity preserving and for any $u, v \in L^2$ that are both positive there is some $n > 0$ with $\langle u|G^n v \rangle \neq 0$. 
Theorem 6.4.3. Let be $M = \sum_{i=0}^{N-1} (\varphi_{i} \otimes \varphi_{i})$ with $H_{pp} \varphi_{i} = E_{i} \varphi_{i}$ $i = 0 \ldots N - 1$, then:

a) $M$ has an eigenvector $k$ corresponding to the eigenvalue 0 and $k := \sum_{i=0}^{N-1} e^{-\frac{\beta}{2} E_{i}} \varphi_{i} \otimes \varphi_{i}$

b) (Perron-Frobenius) There is a $\gamma \in \mathbb{R}^{+}$ positive number that satisfies the condition $M + \gamma I > 0$ i.e. $\langle \varphi_{i} | M + \gamma I \varphi_{i} \rangle > 0 \forall i = 0 \ldots N - 1$. Assume $G := (M + \gamma I)^{-1}$ is ergodic then 0 is a simple eigenvalue of $M$ and $k$ is unique.

Proof. a)

\[
(Mk)_{i} = \sum_{j} M_{ij} k_{j} = M_{ii} k_{i} + \sum_{j \neq i} M_{ij} k_{j} = \sum_{m>1} e^{-\beta(E_{m} - E_{i})} \phi_{im}(E_{m} - E_{i})e^{-\frac{\beta}{2} E_{i}} + \sum_{m>i} \phi_{mi}(E_{i} - E_{m})e^{-\frac{\beta}{2} E_{i}} + \sum_{j>i} (-1)e^{-\beta(E_{j} - E_{i})} \phi_{ij}(E_{j} - E_{i})e^{-\frac{\beta}{2} E_{j}} + \sum_{j<i} e^{-\beta(E_{i} - E_{j})} \phi_{ji}(E_{i} - E_{j})e^{-\frac{\beta}{2} E_{j}} = 0
\]

b) We have seen in Eq. (6.4) that $M_{ii} > 0$ and $M_{ij} \leq 0 \forall i \neq j$. We can write $M = M^{d} + M^{od}$ with:

\[
(M^{d})_{ij} = \begin{cases} M_{ii} : & i = j \\
0 : & i \neq j \end{cases} \quad (6.147)
\]

\[
(M^{od})_{ij} = \begin{cases} 0 : & i = j \\
M_{ij} : & i \neq j \end{cases} \quad (6.148)
\]

It is easy to see that it exists a $\gamma$ so that $M + \gamma I > 0$. It is enough to take $\gamma = |\min_{i \neq j} \{M_{ij}\}|$.

From the definition $G := (M + \gamma I)^{-1}$, $M_{ij}$ is symmetric consequently $G$ is also symmetric:

\[
G = (M^{d} + \gamma I + M^{od})^{-1} = (M^{d} + \gamma I)^{-1}(I + (M + \gamma I)^{-1}M^{od})^{-1} = \sum_{n=0}^{\infty} (M^{d} + \gamma I)^{-1}[-(M^{od})(M^{d} + \gamma I)^{-1}]^{n} \quad \text{and :} \quad (6.149)
\]

\[
G_{ij} = \sum_{n=0}^{\infty} (M^{d} + \gamma I)_{ik_{1}}(M^{od})_{k_{1}k_{2}}(M^{d} + \gamma I)_{k_{2}k_{3}} \ldots (M^{od})_{k_{2n}k_{2n+1}}(M^{d} + \gamma I)_{k_{2n+1}j} \quad (6.150)
\]

that is $G_{ij} \geq 0 \forall i, j$.

$G$ is symmetric, therefore we can diagonalize it and then $\|G\| \in \sigma(G)$ in fact $\|G\| = \ldots$
\[ \max_i |g_i| = \max_i g_i \text{ where } g_i \in \sigma(G). \]

Let be \( \psi^{(0)} \) an eigenvector corresponding to the eigenvalue \( \|G\| \) then taking into account that:

\[
|\langle \psi | G \psi \rangle| = \left| \sum_i \psi_i G_{ij} \psi_j \right| \leq \sum_i |\psi_i| |G| \psi_j | \psi_j | = \langle|\psi| G |\psi| \rangle \quad (6.152)
\]

Assume now that \( \exists \psi_1^{(0)}, \psi_2^{(0)} \) eigenvectors to the eigenvalue \( \|G\| \). \( |\psi_1^{(0)}|, |\psi_2^{(0)}| \) linear independent, \( G^{po} \) symmetric and \( \langle|\psi_1^{(0)}|, |\psi_2^{(0)}| \rangle = 0 \)

\[
0 = \|G\|^p o \langle|\psi_1^{(0)}|, |\psi_2^{(0)}| \rangle = \langle|\psi_1^{(0)}|, G^{po} |\psi_2^{(0)}| \rangle > 0 \quad (6.153)
\]

which is a contradiction, that is to say there is a unique \( \psi_1^{(0)} \) and

\[
\|G\| = \max |\text{eigenvalues of } G| = \min |\text{eigenvalues of } M + \gamma \mathbb{1}| \quad (6.154)
\]

and we already know from part a) that 0 is eigenvalue of M and therefore it is simple. \( \square \)

### 6.5 Magnons in a Ferromagnet

We will begin now with an application of the last sections. Our Hamiltonian is:

\[
H = H_S + H_W = -J \sum_{xy} \hat{S}_x \cdot \hat{S}_{x+y} - \sum_x j(x) \vec{s}_0 \cdot \hat{S}_x \quad (6.155)
\]

where \( S_x \) and \( S_y \) are atomic-spins operators, with \( J > 0 \) and \( j(x) \geq 0 \) which means that the configuration of minimum energy corresponds to all the spins oriented in the same direction, which we will take as z-axis. The first term of Eq. (6.155) represents the interaction of the nearest neighbor between spins of the ferromagnet, \( y \) indicates precisely that the sum runs over the nearest neighbors of each site \( x \). The second term represents the interaction of the external spin wave with the ferromagnet, in which \( j(x) \) is the amplitude of this wave and \( s_0 \) is an unit vector that shows the direction of the spin wave. At first we make a Holstein-Primakoff transformation. We know that the spin-operators satisfy the commutation relations:

\[
[\hat{S}_x^\alpha, \hat{S}_y^\beta] = i\epsilon^{\alpha\beta\gamma} \hat{S}_z^\gamma, \quad \alpha, \beta = 1, 2, 3 \quad [\hat{S}_x^+, \hat{S}_x^-] = 2\hat{S}_x^3 \quad (6.156)
\]
where $\alpha, \beta, \gamma = 1, 2, 3$ the later being the three directions of the space and $\epsilon^{\alpha\beta\gamma}$ represents the antisymmetric tensor. We recall the relations:

$$\hat{S}^+_x = \hat{S}_x^1 + i\hat{S}_x^2$$
$$\hat{S}^-_x = \hat{S}_x^1 - i\hat{S}_x^2$$  \hspace{1cm} (6.157)

On the other hand we know that a spin wave in a spin lattice is a kind of boson called magnon, the creation and annihilation operators of which obey commutation relations $[a, a^*] = 1$. The relation between these last operators and the spin operators is the Holstein-Primakoff transformation (see [3]):

$$\hat{S}^+_x = \hat{S}_x^1 + i\hat{S}_x^2 = (2S)^{1/2}(1 - \frac{a_x^*a_x}{2S})^{1/2}a_x$$ \hspace{1cm} (6.158)
$$\hat{S}^-_x = \hat{S}_x^1 - i\hat{S}_x^2 = (2S)^{1/2}a_x^*(1 - \frac{a_x^*a_x}{2S})^{1/2}$$ \hspace{1cm} (6.159)
$$\hat{S}_3^x = S - a_x^*a_x$$ \hspace{1cm} (6.160)

where $S$ is the spin of our system (typically $1/2$) and where the square roots are to be understood purely formally as infinite series in powers of $\frac{a_x^*a_x}{2S}$. It is only in the subspace of the eigenvectors of the operator $a_x^*a_x$ which belongs to those eigenvalues of this operator which are smaller or equal to unity, where Eq. (6.158), Eq. (6.159) and Eq. (6.160) can be interpreted as the atomic-spin operators. Since it is there where the operators $\hat{S}^+_x, \hat{S}^-_x$ are hermitian conjugates while the operator $S^3$ is self-adjoint. Our system is periodic and we had better characterize the spin waves by a definite momentum, therefore we use the Fourier transformed variables of $a_x, a_x^*$:

$$a_x = \frac{1}{N^{1/2}} \sum_k e^{-i\vec{k} \cdot \vec{x}} b_k$$ \hspace{1cm} (6.161)
$$a_x^* = \frac{1}{N^{1/2}} \sum_k e^{i\vec{k} \cdot \vec{x}} b_k^*$$ \hspace{1cm} (6.162)

where $N$ is the number of unit cells in the body and the wave vector $k$ lies in the first Brillouin zone. It is easy to verify that the operators $b_k, b_k^*$ satisfy the same commutation relations like $a_x$ and $a_x^*$:

$$[b_k, b_k^*] = \delta_{kk'} \hspace{1cm} [b_k, b_k'] = [b_k^*, b_k^*'] = 0$$ \hspace{1cm} (6.163)

The operators $b_k$ and $b_k^*$ create and destroy a magnon or spin-wave excitation of the ferromagnet. These turn out to be excitations where the spins locally deviate only a small
amount from their ground state values as the spin wave passes by. See figure 16:

\[ \hat{S}^+ = (2S)^{1/2}(1 - \frac{a^*_xa_x}{2S})^{1/2}a_x \]
\[ \hat{S}^- = (2S)^{1/2}a_x(1 - \frac{a^*_xa_x}{2S})^{1/2} \]

From \( \hat{S}^+_x + i\hat{S}^1_x \), \( \hat{S}^-_x = \hat{S}^1_x - i\hat{S}^2_x \) we see that:

\[ \hat{S}^+_x \hat{S}^-_{x+y} = \hat{S}^1_x \hat{S}^1_{x+y} + \hat{S}^2_x \hat{S}^2_{x+y} + i\hat{S}^1_x \hat{S}^2_{x+y} - \hat{S}^1_{x+y} \hat{S}^2_x \]
\[ \hat{S}^+_x \hat{S}^-_{x+y} = \hat{S}^1_x \hat{S}^1_{x+y} + \hat{S}^2_x \hat{S}^2_{x+y} + i\hat{S}^1_x \hat{S}^2_{x+y} = \hat{S}^1_x \hat{S}^1_{x+y} + \hat{S}^2_x \hat{S}^2_{x+y} - i\hat{S}^1_{x+y} \hat{S}^2_x \]

That is to say:

\[ \hat{S}^1_x \hat{S}^1_{x+y} + \hat{S}^2_x \hat{S}^2_{x+y} = \frac{1}{2}(\hat{S}^+_x \hat{S}^-_{x+y} + \hat{S}^+_x \hat{S}^-_{x+y}) \]
\[ \hat{S}^1_x \hat{S}^1_{x+y} + \hat{S}^2_x \hat{S}^2_{x+y} + \hat{S}^1_x \hat{S}^1_{x+y} + \frac{1}{2}(\hat{S}^+_x \hat{S}^-_{x+y} + \hat{S}^+_x \hat{S}^-_{x+y}) + \hat{S}^3_x \hat{S}^3_{x+y} \]

Taking into account this result, we will proceed to arrange our Hamiltonian Eq. (6.155) to the typical form:

\[ H = -J \sum_{xy} \hat{S}_x \cdot \hat{S}_{x+y} - \sum_x j(x)s^1_0 \cdot \hat{S}_x = -J \sum_{xy} (\hat{S}^1_x \hat{S}^1_{x+y} + \hat{S}^2_x \hat{S}^2_{x+y} + \hat{S}^3_x \hat{S}^3_{x+y}) - \sum_x j(x)(s^1_0 \hat{S}^1_x + s^2_0 \hat{S}^2_x + s^3_0 \hat{S}^3_x) \]
We note now the result of Eq. (6.169):

\[ H = -J \sum_{xy} \left\{ \frac{1}{2} \left( \hat{S}^+ \hat{S}^- + \hat{S}^+_{x+y} \hat{S}^-_{x+y} + \hat{S}^+ \hat{S}^-_{x+y} \right) - \sum_x j(x) s_0^3 M - \frac{1}{2} \sum_x j(x) (s_0^+ \hat{S}^- + s_0^- \hat{S}^+) \right\} \]

Taking into account now the Holstein-Primakoff transformation for \( S \gg a_x^+ a_x \), Eq. (6.164), Eq. (6.165) and Eq. (6.160):

\[ H = -J \sum_{xy} \left\{ \frac{1}{2} \left[ (2S)^{1/2} a_x (2S)^{1/2} a_{x+y} + (2S)^{1/2} a_{x+y} (2S)^{1/2} a_x \right] + (S - a_x a_x) (S - a_{x+y} a_{x+y}) \right\} - \]

\[ -j_0 s_0^3 M - \frac{1}{2} \sum_x j(x) (s_0^+ (2S)^{1/2} a_x + s_0^- (2S)^{1/2} a_x) \]

and simplifying:

\[ H = -J \sum_{xy} \left\{ S^2 - a_x^+ a_x - a_{x+y} a_{x+y} + a_x a_x^+ a_{x+y} + a_{x+y} a_x^+ a_x + a_x^+ a_x a_{x+y} a_{x+y} \right\} - \]

\[ -j_0 s_0^3 M - \sqrt{\frac{S}{2}} \sum_x j(x) (s_0^+ a_x + s_0^- a_x). \]  

We can neglect the last term of the first line in comparison with the others because of \( S \gg a_x^+ a_x \), and taking into account that \( S a_x a_{x+y} = a_x^+ a_{x+y} a_x \) and \( S a_x a_{x+y}^+ = a_x^+ a_x a_{x+y}^+ \):

\[ H = -J \sum_{xy} \left\{ S^2 - a_x^+ a_x - a_{x+y} a_{x+y} + a_x^+ a_x a_{x+y} + a_{x+y} a_x^+ a_x \right\} - \]

\[ -j_0 s_0^3 M - \sqrt{\frac{S}{2}} \sum_x j(x) (s_0^+ a_x + s_0^- a_x). \]

The last step in the way of obtaining our desired Hamiltonian expression, is to change the creation and annihilation operators following Eq. (6.161) and Eq. (6.162):

\[ H = -J \sum_{xy} \left\{ S^2 - S \frac{1}{N} \sum_k e^{ik \cdot \vec{x}} b_k \sum_{k'} e^{-ik' \cdot \vec{x}} b_{k'} - S \frac{1}{N} \sum_k e^{ik \cdot (\vec{x} + \vec{y})} b_k \sum_{k'} e^{-ik' \cdot (\vec{x} + \vec{y})} b_{k'} + \right. \]

\[ + S \frac{1}{N} \sum_k e^{i k \cdot (\vec{x} + \vec{y})} b_k \sum_{k'} e^{-i k' \cdot \vec{x}} b_{k'} + S \frac{1}{N} \sum_k e^{i k \cdot \vec{x}} b_k \sum_{k'} e^{-i k' \cdot (\vec{x} + \vec{y})} b_{k'} \left\} - \]

\[ -j_0 s_0^3 M - \sqrt{\frac{S}{2}} \sum_x j(x) \left( s_0^+ \frac{1}{\sqrt{N}} \sum_k e^{ik \cdot \vec{x}} b_k + s_0^- \frac{1}{\sqrt{N}} \sum_k e^{-ik \cdot \vec{x}} b_k \right) \]
If \( n \) is the number of nearest neighbors and \( N \) the number of spin places, we have:

\[
H = -\frac{J N n S^2}{2} + J S \frac{1}{N} \sum_{k} \sum_{k'} \sum_{x,y} e^{i(\vec{k} - \vec{k}')} \cdot \vec{x} b_k^* b_{k'} + J S \frac{1}{N} \sum_{k} \sum_{k'} \sum_{x,y} e^{i(\vec{k} - \vec{k}')} \cdot (\vec{x} + \vec{y}) b_k^* b_{k'} - \\
- J S \frac{1}{N} \sum_{k} \sum_{k'} \sum_{x} e^{i(\vec{k} - \vec{k}')} \cdot \vec{x} e^{i\vec{k} \cdot \vec{y}} b_k^* b_{k'} - J S \frac{1}{N} \sum_{k} \sum_{k'} \sum_{x} e^{i(\vec{k} - \vec{k}')} \cdot \vec{x} e^{-i\vec{k} \cdot \vec{y}} b_k^* b_{k'} \}
\]

\[
- j_0 s_0^3 M - \sqrt{\frac{S}{2}} \frac{1}{\sqrt{N}} \sum_{k} \sum_{x} \left( j(x) e^{i\vec{k} \cdot \vec{x}} s_0^+ b_k^* + j(x) e^{-i\vec{k} \cdot \vec{x}} s_0^- b_k \right) \tag{6.174}
\]

We define now the magnon dispersion function, which in this approximation depends only on the positions of the nearest neighbor spins:

\[
\Upsilon_k := \frac{1}{n} \sum_{y} e^{i\vec{k} \cdot \vec{y}}
\tag{6.175}
\]

We assume that \( \Upsilon_k = \Upsilon_{-k} \), which is true for lattices with inversion symmetry. For example for the simple cubic lattice \( (n = 6) \) in three dimensions with lattice constant \( a \):

\[
\Upsilon_k := \frac{1}{n} \sum_{y} e^{i\vec{k} \cdot \vec{y}} = \frac{1}{6}(e^{i k_1 a} + e^{-i k_1 a} + e^{i k_2 a} + e^{-i k_2 a} + e^{i k_3 a} + e^{-i k_3 a}) = \frac{1}{3}(\cos k_1 a + \cos k_2 a + \cos k_3 a)
\tag{6.176}
\]

which is evidently an even function of \( \vec{k} \).

We insert now this function in the expression of the Hamiltonian:

\[
H = -\frac{J N n S^2}{2} + n J S \sum_{k} b_k^* b_k + n J S \sum_{k} b_k^* b_{k'} - n J S \sum_{k} \Upsilon_k b_k^* b_k - n J S \sum_{k} \Upsilon_{-k} b_k^* b_k \}
\]

\[
- j_0 s_0^3 M - \sqrt{\frac{S}{2N}} \sum_{k} \sum_{x} \left( j(x) e^{i\vec{k} \cdot \vec{x}} s_0^+ b_k^* + j(x) e^{-i\vec{k} \cdot \vec{x}} s_0^- b_k \right) \tag{6.177}
\]

which can be simplified as:

\[
H = -\frac{J N n S^2}{2} + 2 n J S \sum_{k} [(1 - \Upsilon_k) b_k^* b_k] - \\
- j_0 s_0^3 M - \frac{S}{2N} \sum_{k} \sum_{x} \left( j(x) e^{i\vec{k} \cdot \vec{x}} s_0^+ b_k^* + j(x) e^{-i\vec{k} \cdot \vec{x}} s_0^- b_k \right) \tag{6.178}
\]
The most important magnons will be those with momenta close to the center of the Brillouin zone, \( k \approx 0 \) so we need to examine the small \( k \)-dispersion function. For a Bravais lattice, like simple cubic, we obtain from Eq. (6.176):

\[
\Upsilon_k = \frac{1}{3} (\cos k_1 a + \cos k_2 a + \cos k_3 a) \quad \tilde{k} \rightarrow 0 \quad \tilde{k} = \frac{1}{3} \cos ka = 1 - \frac{(ka)^2}{2}
\]

and the expression of the Hamiltonian becomes:

\[
H = -\frac{J N n S^2}{2} + 2 n J S \sum_k \left[ \frac{(ka)^2}{2} b_k^* b_k \right] - j_0 s_0^3 M - \sqrt{\frac{S}{2N}} \sum_k \sum_x \left( j(x) e^{i k \cdot \bar{x}} s_0^+ b_k^* + j(x) e^{-i k \cdot \bar{x}} s_0^- b_k \right)
\]

That is to say:

\[
H = -\frac{J N n S^2}{2} + n a^2 J S \sum_k k^2 b_k^* b_k - j_0 s_0^3 M - \sqrt{\frac{S}{2N}} \sum_k \sum_x \left( j(x) e^{i k \cdot \bar{x}} s_0^+ b_k^* + j(x) e^{-i k \cdot \bar{x}} s_0^- b_k \right)
\]

Doing the substitutions \( \sum_k \rightarrow \frac{V}{(2\pi)^3} \int d^3 k \) and \( \sum_x \rightarrow \frac{1}{V} \int d^3 x \) we have finally:

\[
H = -\frac{J N n S^2}{2} - j_0 s_0^3 M + \frac{n a^2 J S V}{(2\pi)^3} \int d k k^2 b^* (k) b(k) - \frac{1}{(2\pi)^3} \sqrt{\frac{S}{2N}} \int d k \left( \hat{j}(k) s_0^+ b^* (k) + \overline{j}(k) s_0^- b(k) \right)
\]

where:

\[
\hat{j}(k) = \int d x j(x) e^{i k \cdot \bar{x}}
\]

\[
\overline{j}(k) = \int d x j(x) e^{-i k \cdot \bar{x}}
\]

since \( j(x) \in \mathbb{R} \).

We can split our Hamiltonian into the following terms:

\[
H = H_{pp} + H_c + I
\]

\[
H_{pp} = -\frac{J N n S^2}{2} \mathbb{1} - j_0 M s_0^3
\]

\[
H_c = \frac{n a^2 J S V}{(2\pi)^3} \int d k k^2 b^* (k) b(k)
\]

\[
I = -\frac{1}{(2\pi)^3} \sqrt{\frac{S}{2N}} \int d k \left( \hat{j}(k) s_0^+ b^* (k) + \overline{j}(k) s_0^- b(k) \right)
\]
we recall:

$$s_0^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_0^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_0^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$  \hspace{1cm} (6.189)

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_0^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$  \hspace{1cm} (6.190)

and then:

$$H_{pp} = \begin{pmatrix} -JNnS^2 \frac{1}{2} - j_0M & 0 \\ 0 & -JNnS^2 \frac{1}{2} + j_0M \end{pmatrix}$$  \hspace{1cm} (6.191)

that is to say:

$$\sigma(H_{pp}) = \{-j_0M - JNnS^2 \frac{1}{2}, j_0M - JNnS^2 \frac{1}{2}\} = \{E_0, E_1\}$$  \hspace{1cm} (6.192)

and our "atomic" Liouvillian is $L_{pp} = H_{pp} \otimes 1 - 1 \otimes H_{pp}$, operator on $\mathbb{C}^2$ with spectrum:

$$\sigma(L_{pp}) = \{E_0 - E_1, 0, E_1 - E_0\} = \{-2j_0M, 0, 2j_0M\}$$  \hspace{1cm} (6.193)

We recall our general form of the interaction Eq. (6.1):

$$I = \int dk [G(k) \otimes b^*(k) + G^*(k) \otimes b(k)]$$

and comparing with Eq. (6.188):

$$G(k) = -\frac{1}{(2\pi)^3} \sqrt{\frac{S}{2N}} j(k)s_0^+ \quad G^*(k) = -\frac{1}{(2\pi)^3} \sqrt{\frac{S}{2N}} j(k)s_0^-$$  \hspace{1cm} (6.194)

Eigenvalue $2j_0M$

In this case $i = 1, j = 0$, considering Eq. (6.86) and Eq. (6.87):

$$\langle \varphi_1 \otimes \varphi_0 | I \varphi_1 \otimes \varphi_0 \rangle = 0 + |G_{01}(E_1 - E_0)|^2 \frac{e^{\beta(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi(E_1 - E_0)^{1/2} =$$

$$= \frac{1}{(2\pi)^6} \frac{1}{4N} |\hat{j}(2j_0M)|^2 \frac{e^{\beta(2j_0M)}}{e^{\beta(2j_0M)} - 1} 2\pi(2j_0M)^{1/2} > 0$$  \hspace{1cm} (6.195)

considering now Eq. (6.93) and Eq. (6.94):

$$\langle \varphi_1 \otimes \varphi_0 | II \varphi_1 \otimes \varphi_0 \rangle = |G_{01}(E_1 - E_0)|^2 \frac{1}{e^{\beta(E_1 - E_0)} - 1} 2\pi(E_1 - E_0)^{1/2} + 0 =$$

$$= \frac{1}{(2\pi)^6} \frac{1}{4N} |\hat{j}(2j_0M)|^2 \frac{e^{\beta(2j_0M)}}{e^{\beta(2j_0M)} - 1} 2\pi(2j_0M)^{1/2} > 0$$  \hspace{1cm} (6.196)
Using Eq. (6.122):

\[ \langle \varphi_0 \otimes \varphi_1 | I | \varphi_0 \otimes \varphi_1 \rangle = |G_{01}(E_1 - E_0)|^2 \frac{1}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} > 0 \]  
(6.199)

Both I and II are positive. This means that when we introduce the perturbation this eigenvalue of \( L_{pp} \) becomes a complex eigenvalue of \( L_g \) with a negative imaginary part which implies the instability of this new eigenvalue.

Eigenvalue \(-2j_0M\)

In this case \( i = 0, j = 1 \), considering Eq. (6.86) and Eq. (6.87):

\[ \langle \varphi_0 \otimes \varphi_1 | I | \varphi_0 \otimes \varphi_1 \rangle = |G_{01}(E_1 - E_0)|^2 \frac{1}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} > 0 \]  
(6.200)

considering now Eq. (6.93) and Eq. (6.94):

\[ \langle \varphi_0 \otimes \varphi_1 | II | \varphi_0 \otimes \varphi_1 \rangle = 0 + |G_{01}(E_1 - E_0)|^2 \frac{e^{\beta(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} > 0 \]  
(6.202)

Both I and II are positive. As for the eigenvalue \( 2j_0M \) this means that when we introduce the perturbation this eigenvalue of \( L_{pp} \) becomes a complex eigenvalue of \( L_g \) with a negative imaginary part which implies the instability of this new eigenvalue.

Eigenvalue 0

Using Eq. (6.112):

\[ I_{00} = \text{Eq. (6.110)} + \text{Eq. (6.111)} \bigg|_{i=j=0} = |G_{01}(E_1 - E_0)|^2 \frac{1}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} > 0 \]  
(6.203)

\[ I_{11} = \text{Eq. (6.110)} + \text{Eq. (6.111)} \bigg|_{i=j=1} = 0 + |G_{01}(E_1 - E_0)|^2 \frac{e^{\beta(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} > 0 \]  
(6.204)

\[ I_{10} = I_{01} = 0 \]  
(6.205)

Using Eq. (6.122):

\[ II_{00} = \text{Eq. (6.120)} + \text{Eq. (6.121)} \bigg|_{i=j=0} = |G_{01}(E_1 - E_0)|^2 \frac{1}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} > 0 \]  
(6.206)

\[ II_{11} = \text{Eq. (6.120)} + \text{Eq. (6.121)} \bigg|_{i=j=1} = 0 + |G_{01}(E_1 - E_0)|^2 \frac{e^{\beta(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi(2j_0M)^{1/2} \]  
(6.207)
Using Eq. (6.130):

\[
III_{01} = \left| G_0 (E_1 - E_0) \right|^2 \frac{e^{\beta/2(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi (E_1 - E_0)^{1/2} \quad (6.209)
\]

\[
III_{10} = \left| G_0 (E_1 - E_0) \right|^2 \frac{e^{\beta/2(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi (E_1 - E_0)^{1/2} \quad (6.210)
\]

\[
III_{00} = III_{11} = 0 \quad (6.211)
\]

Using Eq. (6.138):

\[
IV_{01} = \left| G_0 (E_1 - E_0) \right|^2 \frac{e^{\beta/2(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi (E_1 - E_0)^{1/2} \quad (6.212)
\]

\[
IV_{10} = \left| G_0 (E_1 - E_0) \right|^2 \frac{e^{\beta/2(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 2\pi (E_1 - E_0)^{1/2} \quad (6.213)
\]

\[
IV_{00} = IV_{11} = 0 \quad (6.214)
\]

and then we have the imaginary part of the level-shift operator Eq. (6.31),

\[
Im \Gamma^{(2)} := -Im(\mathcal{F}_P(L_g(\theta, z))^{(2)}) = \pi PW\delta(L_0(z))WP:
\]

\[
Im \Gamma^{(2)} = \pi \left( \begin{array}{cc} I_{00} + II_{00} & -III_{01} - IV_{01} \\ -III_{10} - IV_{10} & I_{11} + II_{11} \end{array} \right) = \quad (6.215)
\]

\[
\left( \begin{array}{cc} 2|G_0 (E_1 - E_0)|^2 \frac{4\pi (E_1 - E_0)^{1/2}}{e^{\beta(E_1 - E_0)} - 1} & -2|G_0 (E_1 - E_0)|^2 \frac{e^{\beta/2(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 4\pi (E_1 - E_0)^{1/2} \\ -2|G_0 (E_1 - E_0)|^2 \frac{e^{\beta/2(E_1 - E_0)}}{e^{\beta(E_1 - E_0)} - 1} 4\pi (E_1 - E_0)^{1/2} & 2|G_0 (E_1 - E_0)|^2 \frac{e^{\beta(E_1 - E_0)} - 1}{e^{\beta(E_1 - E_0)} - 1} 4\pi (E_1 - E_0)^{1/2} \end{array} \right)
\]

\[
= \frac{2|G_0 (E_1 - E_0)|^2}{e^{\beta(E_1 - E_0)} - 1} 4\pi (E_1 - E_0)^{1/2} \left( \begin{array}{cc} 1 & -e^{\beta/2(E_1 - E_0)} \\ -e^{\beta/2(E_1 - E_0)} & e^{\beta(E_1 - E_0)} \end{array} \right) \quad (6.216)
\]

writing \( \alpha = e^{\beta/2(E_1 - E_0)} \) and \( \alpha^2 = e^{\beta(E_1 - E_0)} \) we can calculate the eigenvalues of this last matrix as:

\[
\left| \begin{array}{cc} 1 - \lambda & -\alpha \\ -\alpha & \alpha^2 - \lambda \end{array} \right| = 0 \Rightarrow (1 - \lambda)(\alpha^2 - \lambda) - \alpha^2 = 0 \quad (6.217)
\]

\[
\alpha^2 - \lambda - \lambda\alpha^2 + \lambda^2 - \alpha^2 = 0 \Rightarrow \lambda(\lambda - (1 + \alpha^2)) = 0 \quad (6.218)
\]
with solutions $\lambda_1 = 0$ and $\lambda_2 = 1 + \alpha^2$ and the time of decay of these excitations is:

$$
\tau = \left( \frac{2|G_{01}(E_1 - E_0)|^2}{\epsilon_0(E_1 - E_0) - 1} \right)^{-1} \frac{\lambda_2}{4 \pi (E_1 - E_0)^{1/2}} = \frac{e^{\beta(E_1 - E_0)} - 1}{8 \pi |G_{01}(E_1 - E_0)|^2 (E_1 - E_0)^{1/2}} \frac{1}{1 + \alpha^2} =
$$

$$
= \frac{e^{\beta(E_1 - E_0)} - 1}{8 \pi |G_{01}(E_1 - E_0)|^2 (E_1 - E_0)^{1/2}} \frac{1}{1 + \alpha^2}
$$

(6.219)

Substituting now the values of $\epsilon_0, \epsilon_1$ and $G_{01}(E_1 - E_0)$ we obtain finally:

$$
\tau = \frac{e^{2\beta\lambda_0 M} - 1}{8\pi \left( -\frac{1}{(2\pi)^2} \sqrt{\frac{v}{2N}} \right)^2 [\dot{\varphi}(E_1 - E_0)]^2 (2j_0 M)^{1/2}} \frac{1}{1 + e^{2\beta\lambda_0 M}}
$$

(6.220)

$$
\tau = \frac{2(\sqrt{2}\pi)^5 N}{S^2 \dot{\varphi}(2j_0 M)^{1/2}} \frac{e^{2\beta\lambda_0 M} - 1}{1 + e^{2\beta\lambda_0 M}}
$$

(6.221)

We could also write the KMS states for free dynamic and for the coupled dynamic on the Hilbert space $\mathcal{H}_{pp} \otimes \mathcal{H}_{pp} \otimes \mathcal{H}_{c}^2 \otimes \mathcal{H}_{c}^\beta$. For the free dynamic, see Eq. (3.90):

$$
\hat{\Omega}_0^\beta = Z_{\beta,0}^{-1/2} \sum_{j=0}^{1} e^{-\beta E_j/2} \varphi_j \otimes \varphi_j \otimes \Omega_c \otimes \Omega_f
$$

(6.222)

where $E_0$ and $E_1$ are given in Eq. (6.192) and $\varphi_0$ and $\varphi_1$ are the corresponding eigenvectors.

With interaction we have Eq. (3.91):

$$
\hat{\Omega}_g^\beta := Z_{\beta,0}^{-1/2} e^{-\beta L_{g,l}/2} \hat{\Omega}_0^\beta = Z_{\beta,0}^{-1/2} e^{-\beta L_{g,r}/2} \hat{\Omega}_0^\beta
$$

(6.223)

where:

$$
L_{g,l} = L_{pp} + L_c + I_0[I] I_0^{-1}
$$

(6.224)

$$
L_{g,r} = L_{pp} + L_c - I_0[I] I_0^{-1}
$$

(6.225)

$$
L_{pp} = H_{pp} \otimes \mathbb{1} - \mathbb{1} \otimes H_{pp}
$$

(6.226)

$$
L_c = \int dk k^2 [a_l^*(k)a_l(k) - a_r^*(k)a_r(k)]
$$

(6.227)

where now the dispersion relation is:

$$
\omega(k) = k^2
$$

(6.228)
Appendix A

Algebraic Techniques

This appendix is an adaptation of some results of the appendix of [14] for the case $T > 0$. We will use the following notation: suppose that $\sigma_j \in \{+1, -1\}$ and that $\tau_j \in \{l, r\}$

$$a^+ = \tau[a^*], \quad a^{-1} = \tau[a],$$

where $a^*, a$ are the creation and annihilation operators and $l, r$ are the left and right representation given by the GNS construction. We denote with $\alpha_j = (\sigma_j, \tau_j)$, so that $a^{\alpha_j} = a^{(\sigma_j, \tau_j)} = \tau_j[a^\sigma]$. We define:

$$\pi(\tau) = \begin{cases} 0 : & \text{for } \tau = l \\ 1 : & \text{for } \tau = r \end{cases} \quad (A.1)$$

**Lemma A.0.1. (Pull-through Formula):** Let $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ be a measurable function, obeying $f[r] = O(r)$. Then $f[\mathcal{L}_f]$ is defined on $\mathcal{D}(\mathcal{L}_f)$ and for all $k \in \mathbb{R}^d$:

$$f[\mathcal{L}_f][a^*(k)] = \tau[a^*(k)]f[\mathcal{L}_f] + (-1)^{\pi(\tau)}\omega(k)] \quad (A.2)$$

$$\tau[a(k)]f[\mathcal{L}_f] = f[\mathcal{L}_f + (-1)^{\pi(\tau)}\omega(k)]\tau[a(k)] \quad (A.3)$$

in the sense of operator-valued distributions. Moreover, extending $f$ to the whole real line by setting $f \equiv 0$ on $\mathbb{R}_0^-$ Eq. (A.2) and Eq. (A.3) extend to:

$$\tau[a^*(k)]f[\mathcal{L}_f] = f[\mathcal{L}_f - (-1)^{\pi(\tau)}\omega(k)]\tau[a^*(k)] \quad (A.4)$$

$$f[\mathcal{L}_f][a(k)] = \tau[a(k)]f[\mathcal{L}_f - (-1)^{\pi(\tau)}\omega(k)] \quad (A.5)$$

or in an even more compact notation we can write Eq. (A.2), Eq. (A.3), Eq. (A.4), Eq. (A.5)
Lemma representations commute, we have the following theorem:

\[ f[L_f]a^{\alpha_j}(k_j) = a^{\alpha_j}(k_j)f[L_f + (-1)^{\sigma_j}(\sigma_j \omega(k_j))] \]

\[ a^{\alpha_j}(k_j)f[L_f] = f[L_f - (-1)^{\sigma_j}(\sigma_j \omega(k_j))]a^{\alpha_j}(k_j) \]

Proof. We know that:

\[ L_f = l[H_f] - r[H_f] = \int dk \omega(k)\{l[a^*(k)a(k)] - r[a^*(k)a(k)]\} \]

For the second equation Eq. (A.3) the proof is:

\[ l[a(k')]L_f = \int dk \omega(k)\{l[a^*(k)a(k)] - l[a(k')]r[a^*(k)a(k)]\} \]

using that the \( l \) and \( r \) representations commute and using the commutation relations, i.e.

\[ l[a(k')]l[a^*(k)a(k)] = l[a(k')a^*(k)a(k)] = l[a^*(k)a(k)a(k') + \delta(k' - k)a(k)] = \{l[a^*(k)a(k)] + \delta(k' - k)l[a(k')], \]

\[ l[a(k')]L_f = \int dk \omega(k)\{l[a^*(k)a(k)] + \delta(k' - k) - r[a^*(k)a(k)]\}l[a(k')] = (L_f + \omega(k))l[a(k')] \]

(A.9)

We can develop the function \( f(L_f) \) in a power series of \( L_f \). We prove the equation for one term of this series:

\[ l[a(k')]L_f^p = (L_f + \omega(k))l[a(k')]L_f^{p-1} = \ldots = (L_f + \omega(k))^n l[a(k')] \]

(A.10)

We Denote by a double point the normal ordering, that is to say the creation operators are placed to the left of the annihilation operators. Taking also into account that different representations commute, we have the following theorem:

**Lemma A.0.2. (Wick’s theorem)** Denote \( N := \{1, 2, \ldots, N\} \) and \( \Pi_{j \in A} \equiv \Pi_{j=1}^N \chi[j \in \mathcal{A}] \) for any \( \mathcal{A} \subseteq N \). Then for any \( (\sigma_1, \sigma_2 \ldots \sigma_N) \in \{+1, -1\}^N \) and \( (\tau_1, \tau_2 \ldots \tau_N) \in \{l, r\}^N \), \( \alpha_j = (\sigma_j, \tau_j) \):

\[ \prod_{j \in N} a_{\alpha_j}(k_j) = \sum_{Q \subseteq N} \left( \prod_{j \in N \setminus Q} a_{\alpha_j}(k_j) \right) \prod_{j \in Q} a_{\alpha_j}(k_j) \]

(A.11)

Proof. For a proof we refer to [14], considering in this case that different representations \( l, r \) commute.
Lemma A.0.3. Denote $N := \{1, 2, \ldots, N\}$ and let be $f_j[r] = O(r+1)$ a measurable function on $\mathbb{R}^+$ for any $j = 1, 2, \ldots, N$. Let $\Omega_f$ be the KMS state of the GNS field Hilbert space, then:

$$
\prod_{j=1}^{N} \{a_{j}^{\alpha_j}(k_j)f_j[\mathcal{L}_f]\} = \sum_{\mathcal{Q} \subseteq N} \prod_{j \in \mathcal{Q}} a_{j}^{+}(k_j) \times \\
\times \langle \Omega_f | \prod_{j=1}^{N} \{a_{j}^{\alpha_j}(k_j)\}^{j \notin \mathcal{Q}} f_j[\mathcal{L}_f + r + \sum_{i=1}^{j} (-1)^{\pi(i)} \omega(k_i) + \sum_{i=j+1}^{N} (-1)^{\pi(i)} \omega(k_i)] \Omega_f \rangle \big|_{r=\mathcal{L}_f} \times \\
\times \prod_{j \in \mathcal{Q}} a_{j}^{-}(k_j)
$$

(A.12)

where $[a_{j}^{\alpha_j}(k_j)]^{j \notin \mathcal{Q}} = a_{j}^{\alpha_j}(k_j)$ for $j \notin \mathcal{Q}$ and $[a_{j}^{\alpha_j}(k_j)]^{j \notin \mathcal{Q}} = 1$ for $j \in \mathcal{Q}$

Proof. Using the Pull-through formula (lemma 6.1.1.)Eq. (A.6):

$$
\prod_{j=1}^{N} \{a_{j}^{\alpha_j}(k_j)f_j[\mathcal{L}_f]\} = \prod_{j=1}^{N} a_{j}^{\alpha_j}(k_j) \prod_{j=1}^{N} f_j[\mathcal{L}_f] + \sum_{i=j+1}^{N} (-1)^{\pi_i} \sigma_i \omega_i = \\
\text{(A.13)}
$$

applying now the Wick’s theorem (double point denotes normal ordering):

$$
= \sum_{\mathcal{Q} \subseteq N} \left\langle \prod_{j \in \mathcal{N} \setminus \mathcal{Q}} a_{j}^{\alpha_j}(k_j) \right\rangle \prod_{j \in \mathcal{Q}} a_{j}^{\alpha_j}(k_j) \prod_{j=1}^{N} f_j[\mathcal{L}_f] + \sum_{i=j+1}^{N} (-1)^{\pi_i} \sigma_i \omega_i = \\
\text{(A.14)}
$$

$$
= \sum_{\mathcal{Q} \subseteq N} \prod_{j \in \mathcal{N} \setminus \mathcal{Q}} a_{j}^{+}(k_j) \left\langle \prod_{j \in \mathcal{Q}} a_{j}^{\alpha_j}(k_j) \right\rangle \prod_{j=1}^{N} f_j[\mathcal{L}_f] + \sum_{i=j+1}^{N} (-1)^{\pi_i} \sigma_i \omega_i + \sum_{i \in \mathcal{Q}^-} (-1)^{\pi_i} \omega_i \prod_{j \in \mathcal{Q}^-} a_{j}^{-}(k_j) \\
\text{(A.15)}
$$

now we move the $f_j$'s into the vacuum expectation value using that $f[\mathcal{L}_f] = \langle f[\mathcal{L}_f + r] \rangle |_{r=\mathcal{L}_f}$ and finally we note that:

$$
\sum_{i=j+1}^{N} (-1)^{\pi_i} \omega_i + \sum_{i \in \mathcal{Q}^-} (-1)^{\pi_i} \omega_i = \sum_{i=j+1}^{N} (-1)^{\pi_i} \omega_i - \sum_{i=j+1}^{N} (-1)^{\pi_i} \omega_i + \sum_{i \in \mathcal{Q}^-} (-1)^{\pi_i} \omega_i = \\
\sum_{i=j+1}^{N} (-1)^{\pi_i} \omega_i + \sum_{i \in \mathcal{Q}^-} (-1)^{\pi_i} \omega_i
$$

(A.16)
Appendix B

Operator Algebras

**Definition B.0.4.** A set $\mathcal{A}$ is called a $C^*$-algebra if it has the following properties:

1. $\mathcal{A}$ is an algebra (Two intern operations, with the first it is a commutative group, and with the second it is a semigroup. One extern operation so that with the first intern operation it is a vector space.).

2. One can define on $\mathcal{A}$ a bijection $a \in \mathcal{A} \mapsto a^*$ satisfying the following properties, where $a \in \mathcal{A}, c \in \mathbb{C}$:

   \[
   (c_1a_1 + c_2a_2)^* = c_1a_1^* + c_2a_2^* \\
   (a_1a_2)^* = a_2^*a_1^* \\
   (a^*)^* = a
   \]  

   (a algebra with such an map is called a *-algebra)

3. We have a norm $\|a\|$, and $\mathcal{A}$ is complete with respect to this norm (a complete normed algebra is a Banach algebra).

4. The norm has the following property (the property of a $C^*$ norm):

   \[
   \|a^*a\| = \|a\|^2
   \]  

**Definition B.0.5.** A set of bounded linear operators $\mathcal{N}$ on a Hilbert space $\mathcal{H}$ satisfying the following conditions is called a von Neumann algebra:

1. $\mathcal{N}$ is a *-algebra.

2. $\mathcal{N}$ is closed in the weak operator topology. Namely, if a net of operators $a_n \in \mathcal{N}$ has a weak limit $a$, satisfying:

   \[
   \lim (\psi, a_n\phi) = (\psi, a\phi)
   \]  

   (B.4)
for any vectors $\psi, \phi \in \mathcal{H}$, then $a \in \mathcal{N}$. (We write $a = w - \lim a_n$)

3. $1 \in \mathcal{N}$

$B(\mathcal{H})$: Algebra of all bounded operators on $\mathcal{H}$

Positive operator: A with $\langle \psi | A \psi \rangle \geq 0$ for each $\psi \in \mathcal{H}$. $AA^*$ is a positive operator and also $|A| = (AA^*)^{1/2}$

Trace class operator: A with $\|A\|_1 := \text{Tr}|A| < \infty$, we write $A \in \mathcal{L}^1(\mathcal{H})$
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