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Non-tidy Spaces and Graph Colorings

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Abstract

One of the starting points of the subject of Topological Combinatorics is Lovász's proof of the Kneser Conjecture, concerning lower bounds for the chromatic number. The pattern to obtain a lower bound of the chromatic number is to associate a topological space to a graph and bound the chromatic number by a topological invariant of this space, e.g. connectivity or $\mathbb{Z}_2$-index.

In this thesis we study lower bounds for the chromatic number obtained by assigning different complexes to graphs. Namely, we are interested in the neighborhood complex, the Lovász complex, various box complexes, and the homomorphism complex. We will see that although these complexes are seemingly quite different their homotopy or $\mathbb{Z}_2$-homotopy types are in fact closely related. This allows us to take a closer look at the hierarchy of the topological methods.

Another part of the thesis is the study of *non-tidy* spaces. This together with our universality statement allows us to construct graphs whose assigned complexes have interesting properties. Moreover with this result we can refine the hierarchy of lower bounds for the chromatic number.

We study homomorphism complexes separately as well since they can be considered as generalizations of the other complexes. We will show that many of them are manifolds, and we will analyze small dimensional examples. One of the most surprising examples comes from mapping the 5-cycle to the complete graph on 4 vertices. This leads to the 3-dimensional projective space.

Amazingly non-tidy spaces arise naturally amongst homomorphism complexes. My conjecture about homomorphism complexes and Stiefel manifolds appears to be the currently best explanation for the difficulty of proving Lovász's conjecture.

Zusammenfassung

Einer der Ausgangspunkte zum Thema Topologische Kombinatorik ist Lovász' Beweis für die Knesersche Vermutung über untere Schranken für die chromatische Zahl. Das sich ergebende Muster zum Nachweis einer unteren Schranke für die chromatische Zahl besteht darin, dem Graphen einen topologischen Raum zuweisen und die chromatische Zahl durch eine topologische Invariante dieses Raumes zu begrenzen, z.B. durch Konnektivität oder durch den $\mathbb{Z}_2$-Index.

In dieser Arbeit untersuchen wir untere Schranken für die chromatische Zahl, die durch Zuweisung unterschiedlicher Komplexe zu Graphen entstehen. Wir interessieren uns für den Nachbarschaftskomplex, den Lovász'schen Komplex, verschiedene Kastenkomplexe, sowie den Homomorphismuskomplex. Es wird gezeigt, dass obwohl diese Komplexe sich voneinander scheinbar ziemlich stark unterscheiden, ihre Homotopie- oder $\mathbb{Z}_2$-Homotopietypen nah verwandt sind. Dies ermöglicht es uns, die Hierarchie der topologischen Methoden näher zu betrachten.

Zusätzlich beschäftigt sich diese Arbeit mit *nichtsauberen* Räumen. In Verknüpfung mit einer Universalitätsaussage, die wir aufstellen, ermöglicht dies uns, Graphen zu konstruieren, deren zugewiesene Komplexe interessante Eigenschaften aufweisen. Mit diesem Ergebnis können wir die Hierarchie der unteren Schranken für die chromatische Zahl weiter verfeinern.

Im Speziellen untersuchen wir die Homomorphismuskomplexe, da diese als Verallgemeinerungen der anderen Komplexen betrachtet werden können. Es wird gezeigt, dass viele von ihnen Mannigfaltigkeiten sind, und wir werden Beispiele in kleinen Dimensionen analysieren. Eines der überraschendsten Beispiele ergibt sich aus der Abbildung des 5-Kreises zum vollständigen Graph auf 4 Knoten. Dies führt zum 3-dimensionalen projektiven Raum.

Interessanterweise treten nichtsaubere Räume auf natürliche Weise unter Homomorphismuskomplexen auf. Meine Vermutung über Homomorphismuskomplexe und Stiefelsche Mannigfaltigkeiten scheint zur Zeit die beste Erklärung dafür zu sein, dass die Lovász'sche Vermutung schwierig zu beweisen ist.

Contents

Chapter 0

Introduction

There are several combinatorial and geometric results whose proofs (the first proofs and often the only known proofs) involve a surprising application of algebraic topology. Lovász's [Lov78] striking proof of Kneser's conjecture [Kne55] from 1978 using the Borsuk–Ulam Theorem was one of the foundation stones of a new discipline: *Topological Combinatorics.*

During the last two decades, topological methods in combinatorics have become more elaborate. Rather advanced parts of algebraic topology have been successfully applied by Babson and Kozlov [BK04]. Perhaps Borsuk–Ulam-type theorems are some of the most often applied tools from topology. Recently Matoušek collected applications of this kind in a book [Mat03] (see the "*User's guide to equivariant methods in combinatorics*" by Živaljević [Živ96, Živ98] and the survey of Bárány [Bár93] as well). Other directions of research in topological methods are surveyed by Björner [Bjö95] and Živaljević [Živ97].

0.1 Topological methods

My research focuses on the classical problem of coloring graphs. It is well known that computing the chromatic number $\chi(G)$ of a graph G

is a very hard problem which means that the worst case performance of any algorithm has most likely exponential running time. Surprisingly even coloring a 3 chromatic graph with 4 colors is NP-hard, which was proven by Khanna, Linial and Safra [KLS00].

For this reason it is very interesting to find non-trivial bounds for the chromatic number. Lovász's original proof of Kneser's conjecture provides a topological lower bound for the chromatic number.

0.1.1 Graph complexes

Matoušek and Ziegler [MZ04] compared various topological lower bounds for the chromatic number. They found that different methods give numerically the same – or nearly the same – bounds. These bounds can be formulated using various complexes assigned to graphs. We will study the neighborhood complex, the Lovász complex, various box complexes and the Hom complex. Note that the Hom complex $\mathrm{Hom}(H, G)$ is a topological space assigned to two graphs. We will compare the $\mathrm{Hom}(K_2, G)$ complexes to other graph complexes. By slightly abusing the notation we will call these $\mathrm{Hom}(K_2, G)$ complexes, when the first graph is K_2, Hom complex as well.

We will show that these graph complexes can be considered as avatars of the same object as far as their homotopy type is concerned.

0.2 Summary and organization

According to the general theme of the results, the rest of the thesis is divided into four chapters. We summarize what are the new results of this thesis.

0.2.1 Preliminaries

In Chapter 1 we summarize what we need from graph theory, about simplicial complexes, posets, and about topology. This chapter is based on books [Bjö95, Die00, Mat03], survey papers [Koz05c, Živ96, Živ98], and research papers [BK03, Koz05b]. We introduce the definitions and the basic properties, which we frequently use later. If we need some notation only once, we might define it only shortly before we actually need it.

0.2.2 Homotopy type results

In Chapter 2 we continue the work initiated by Matoušek and Ziegler [MZ04]. We show that for any graph G the neighborhood complex, the Lovász complex, various box complexes and the Hom complex have the same homotopy[1] or $\mathbb{Z}_2$-homotopy type. We will generalize some of these results into *simple* $\mathbb{Z}_2$-homotopy equivalence in sense of Whitehead [Whi39]. Some of our simple homotopy type results were independently proven by Kozlov [Koz05b], however, in these cases our proofs are simpler.

Matoušek and Ziegler [MZ04] reformulated Sarkaria's and Lovász's bounds for the chromatic number (Theorem 2.1 and 2.2). They proved that the difference between these two lower bounds is at most 1. We strengthen this statement, in Section 2.2 we show that the box complex $\mathrm{B}_0(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $\mathrm{B}(G)$. Matoušek and Ziegler [MZ04] asked the question whether these two lower bounds are the same or there is a graph for which the difference between them is one.

Topology alone is not enough to answer this question. There are topological spaces [Cso04] such that the suspension does not increase the $\mathbb{Z}_2$-index. So the question is whether the box complex of a graph could be such a space (up to homotopy). Surprisingly, box complexes are universal:

Theorem 2.14 (universality theorem). For any finite free simplicial $\mathbb{Z}_2$-complex (K, ν) there is a graph G such that its graph complex is $\mathbb{Z}_2$-homotopy equivalent to (K, ν).

This theorem leads to examples (graphs) such that Lovász's lower

[1] On the neighborhood complex there is no free $\mathbb{Z}_2$-action in general, so for it we can not have $\mathbb{Z}_2$-homotopy type result. Later we will speak only about $\mathbb{Z}_2$-homotopy type which assumed to be understood as homotopy type for the case of the neighborhood complex.

bound is strictly better than Sarkaria's bound, thus answering the question of Matoušek and Ziegler.

The universality theorem will be used to give a purely topological construction for graphs showing that Sarkaria's and Lovász's bound can be arbitrarily far from the chromatic number.

We study how graph manipulation changes the homotopy type of Hom complexes. We show that the *fold* in the second parameter does not change the homotopy type. In a special case this was proven by Čukić and Kozlov [ČK04]. Here the general case is settled. Later Kozlov [Koz05a] proved a stronger result that the simple homotopy type is also preserved. Folds in the first parameter were considered by Babson and Kozlov earlier [BK03].

We show how the so-called Mycielski construction changes the homotopy type of the Hom complex. We generalize the result of Gyárfás, Jensen and Stiebitz [GyJS04] into $\mathbb{Z}_2$-homotopy equivalence. This stronger statement was predicted by Simonyi and Tardos [ST04].

The size of Hom complexes is rather large. Even for (relatively) small graphs the complex is too large to perform any experiment with computer (e.g. computing homology by some software [DHSW03, GJ03]). For this reason we introduce a homotopy equivalent smaller complex. This smaller complex can be used to determine the homotopy type of $\mathrm{Hom}(K_3, K_n)$.

Some of the results of Chapter 2 are joint work with Carsten Lange, Ingo Schurr and Arnold Wassmer [CsLSW04].

0.2.3 Non-tidy space constructions

After understanding the homotopy types of graph complexes, we know what sort of spaces do we need. In Chapter 3 we construct a space (simplicial complex) such that the suspension does not increase its $\mathbb{Z}_2$-index. It will turn out that we have to search among non-tidy spaces. We will also present non-tidy $\mathbb{Z}_p$-spaces. Later we will see that non-tidy spaces arise naturally in topological combinatorics. The constructed space together with our universality theorem allows us to construct graphs showing that Lovász's original topological lower bound

[Lov78, MZ04] is strictly better than Sarkaria's bound [Sar90, MZ04].

0.2.4 Graph coloring manifolds

The recent proliferation of results [BK03, BK04, ČK04, ČK04b, CsL05, Eng05, HL04, Koz05a, Koz05c, Koz05d] regarding homomorphism complexes opens up a multitude of new research directions.

We denote by $\mathcal{H}\text{om}(G,H)$ the set of graph homomorphisms between G and H. To study $\mathcal{H}\text{om}(G,H)$ is more difficult than coloring graphs. This is evidenced by the observation that:

$$\chi(G) = \min\{n\colon \mathcal{H}\text{om}(G,K_n) \neq \emptyset\}.$$

Lovász et al. studied related questions; for example, they characterize which graph parameters can be obtained as the size $|\mathcal{H}\text{om}(G,H)|$ for some graph H. For example, $|\mathcal{H}\text{om}(G,K_n)|$ is the number of colorings of G with n colors. For more details and for references see the presentation of Lovász [Lov05].

The set $\mathcal{H}\text{om}(G,H)$ can be extended to a polyhedral complex $\text{Hom}(G,H)$, whose 0-skeleton is $\mathcal{H}\text{om}(G,H)$. The cells are products of simplices corresponding to homomorphisms 'close' to each other.

Among homomorphism complexes, there is an amazingly large number of manifolds (e.g., $\text{Hom}(K_2,K_n)$ is an $(n-2)$-dimensional sphere). The starting point was my conjecture.

Conjecture 4.8. The Hom complex $\text{Hom}(C_5,K_{n+1})$ is homeomorphic to the Stiefel manifold $V_{n,2}$.

We will show that they are manifolds, moreover we characterize when *graph coloring complexes* will be manifolds:

Theorem 4.2. The homomorphism complex $\text{Hom}(G,K_n)$ of a graph G ($n > \chi(G)$) is a piecewise linear (PL) manifold if and only if G is the complement of the 1-skeleton of a flag simplicial (PL) sphere.

This provides infinitely many examples of graph coloring manifolds. The simplest ones are homomorphism complexes of complements of cycles $\text{Hom}(\overline{C}_m,K_n)$. One of the most interesting examples is the case of the five cycle C_5 which appears in Lovász's conjecture as well (which was proven recently by Babson and Kozlov).

Theorem [BK04]. Let G be a graph, and let $r \in \mathbb{Z}$, $r \geq 1$. Then

$$\chi(G) \geq \text{connectivity}\,(\mathrm{Hom}(C_{2r+1}, G)) + 3.$$

In connection with this we show that $\mathrm{Hom}(C_5, K_4)$, the space corresponding to graph homomorphisms from the 5 cycle to the complete graph on 4 vertices, is homeomorphic to the 3-dimensional projective space $\mathbb{RP}^3$. This already explains some of the difficulties of the proof of Lovász's conjecture since it is a non-tidy space. It supports my conjecture as well. More low dimensional examples of graph coloring manifolds, such as $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ and $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$, are analyzed.

Some of the results of Chapter 4 are joint work with Frank Lutz [CsL05].

0.2.5 Limits of the topological method

In Chapter 5 we examine the strength of these topological lower bounds for the chromatic number. It was proven by Walker [Wal83] that they can be arbitrarily bad. We generalize this result. Moreover we present the smallest graphs for which the topological lower bound does not give its chromatic number. Using the universality theorem we give purely topological constructions to show that the topological lower bounds can be arbitrarily bad.

Some of the results Chapter 5 of are joint work with Carsten Lange, Ingo Schurr and Arnold Wassmer [CsLSW04].

Chapter 1

Preliminaries

In this section we recall some basic facts about graphs, simplicial complexes and topology to fix notation. Interested readers are referred to e.g. [Bjö95, Die00, Hat01, Koz05c, Mat03, Zie95].

1.1 Graphs, homomorphisms and chromatic numbers

Any graph G will be assumed to be finite, simple, connected, and undirected unless stated otherwise, i.e. G is given by a finite set $V(G)$ of *vertices* and a set of *edges* $E(G) \subseteq \binom{V(G)}{2}$.
We will denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

Examples: K_n is the *complete graph* on n vertices, i.e. $V(K_n) = [n]$ and $E(K_n) = \binom{[n]}{2}$.
$K_{n,m}$ is the *complete bipartite graph* on $n + m$ vertices, i.e. $V(K_{n,m}) = [n + m]$ and $\{i, j\} \in E(K_{n,m})$ if and only if $i \leq n < j$.
Let $\mathrm{KG}_{n,k}$ denote the *Kneser graph*. Its vertices are the k-element subsets of the n-element set $[n]$, and two of them are connected by an edge if and only if they are disjoint. For example $\mathrm{KG}_{n,1}$ is K_n, $\mathrm{KG}_{2k-1,k}$ is a graph with no edges, $\mathrm{KG}_{2k,k}$ is a matching, and the smallest interesting example is $\mathrm{KG}_{5,2}$. It turns out to be the ubiquitous

Petersen graph (see Figure 1.1). Kneser [Kne55] conjectured that the chromatic number of $\mathrm{KG}_{n,k}$ is $n - 2k + 2$.

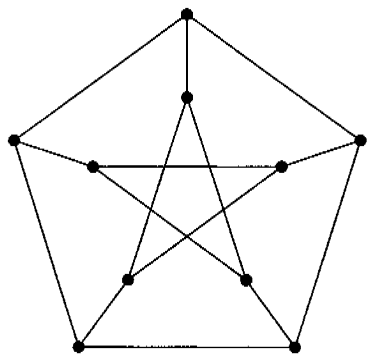

Figure 1.1: The Petersen graph.

The *common neighborhood* of $A \subseteq V(G)$ is

$$\mathrm{CN}(A) \;=\; \{v \in V(G)\colon \{a, v\} \in E(G) \text{ for all } a \in A\}.$$

We define $\mathrm{CN}(\emptyset) := V(G)$. For $A \subseteq B \subseteq V(G)$ the common neighborhood relation satisfies

(a) $A \cap \mathrm{CN}(A) = \emptyset$,

(b) $\mathrm{CN}(B) \subseteq \mathrm{CN}(A)$,

(c) $A \subseteq \mathrm{CN}^2(A)$, and

(d) $\mathrm{CN}(A) = \mathrm{CN}^3(A)$.

For two disjoint sets of vertices $A, B \subseteq V(G)$ we define $G[A, B]$ as the subgraph of G with $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{\{a, b\} \in E(G)\colon a \in A, b \in B\}$.

Definition 1.1. For two graphs H and G, a *graph homomorphism* from H to G is a map $\varphi\colon V(H) \to V(G)$, such that if $\{v, w\} \in E(H)$ then $\{\varphi(v), \varphi(w)\} \in E(G)$. Let the set of all graph homomorphisms from H to G be denoted by $\mathcal{H}om(H, G)$.

Graphs with graph homomorphisms form a category.

Proposition 1.2. *If $\varphi\colon H \to G$ and $\psi\colon G \to T$ are graph homomorphisms, then $\psi \circ \varphi\colon H \to T$ is again a graph homomorphism.*

Definition 1.3. The *chromatic number* of G, $\chi(G)$ is the minimal n such that there exists a graph homomorphism $\varphi\colon G \to K_n$.

Proposition 1.4. *If there exists a graph homomorphism $\varphi\colon H \to G$, then $\chi(H) \leq \chi(G)$.*

In this sense, the problem of vertex-colorings and computing chromatic numbers corresponds to choosing a particular family of graphs, namely the complete graphs K_n, and fixing an evaluation (a function into the real numbers) on this family. Here our evaluation function is given by mapping K_n to n. Now to obtain the chromatic number we have to find a graph homomorphism from a given graph to the chosen family, which would minimize the fixed evaluation. Here we follow the survey on homomorphism complexes written by Kozlov [Koz05c].

There are other families of graphs and evaluations which correspond to other natural and well-studied classes of graph problems.

If we take the larger family, the Kneser graphs, $\mathrm{KG}_{n,k}$, $n \geq 2k$, and choose the evaluation $\mathrm{KG}_{n,k} \to \frac{n}{k}$, we obtain the fractional chromatic number.

Definition 1.5. Let G be a graph. The *fractional chromatic number* of G, $\chi_f(G)$, is defined by

$$\chi_f(G) = \inf_{(n,k)} \frac{n}{k},$$

where the infimum is taken over all pairs (n, k) such that there exists a graph homomorphism from G to $\mathrm{KG}_{n,k}$.

Another possibility is to chose the family to be $U(m, r)$ (see Definition 1.6), and the evaluation is given by $U(m, r) \to r$. Observe that $U(m, m)$ is the complete graph K_m.

Definition 1.6. For positive integers $r \leq m$ we define the graph $U(m, r)$ as follows.

$$\begin{aligned} V(U(m,r)) &= \{(i, A) \colon A \subseteq [m], |A| = r - 1, i \notin A\}, \\ E(U(m,r)) &= \{\{(i, A), (j, B)\} \colon i \in B, j \in A\}. \end{aligned}$$

Let G be a graph. The *local chromatic number* of G, $\psi(G)$, is defined by

$$\psi(G) = \inf_{(m,r)} r,$$

where the infimum is taken over all pairs (m,r) such that there exists a graph homomorphism from G to $U(m,r)$.

One can prove that the local chromatic number is between the fractional chromatic number and the chromatic number:

$$\chi(G) \geq \psi(G) \geq \chi_f(G).$$

Recently Simonyi and Tardos [ST04, ST05] found a topological lower bound for the local chromatic number:

$$\psi(G) \geq \left\lceil \frac{\text{coindex}(\text{Hom}(K_2, G))}{2} \right\rceil + 2.$$

After reading this chapter it will be clear what is on the right hand side of this formula.

1.2 Topological interlude

We assume that the spaces are cell complexes and that the maps between them are continuous unless stated otherwise. We will use standard topological notations. Whoever is familiar with equivariant homotopy theory should skip this section.

A *topological space* is a pair $(X, \mathcal{O})$, where X is a ground set and $\mathcal{O} \subseteq 2^X$ is a set system, whose members are called *open sets*, such that $\emptyset, X \in \mathcal{O}$, the intersection of finitely many open sets is an open set, and so is the union of arbitrary collection of open sets. A map $f\colon X_1 \to X_2$ between the topological spaces $(X_1, \mathcal{O}_1)$ and $(X_2, \mathcal{O}_2)$ is *continuous* if $f^{-1}(V) \in \mathcal{O}_1$ for every $V \in \mathcal{O}_2$. We say that X_1 and X_2 are *homeomorphic* if there is a bijection $\varphi\colon X_1 \to X_2$ such that φ and φ^{-1} are continuous.

A set F in a topological space X is *closed* if and only if its complement $X \setminus F$ is open. The *closure* of a set $Y \subseteq X$, denoted by $\overline{Y}$, is the intersection of all closed sets in X containing Y. The *boundary* of Y is $\partial Y := \{\overline{Y} \cap \overline{X \setminus Y}\}$.

Two maps $f, g\colon X \to Y$ are *homotopic* (written $f \sim g$) if there is a map $F\colon X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

In particular a map $X \to Y$ is *nullhomotopic* if it is homotopic to a constant map that maps all of X to a single point $y \in Y$. It is not hard to verify that "being homotopic" is an equivalence relation on the set of all maps $X \to Y$. $\pi_n(X)$ denotes as the homotopy classes of maps $[0,1]^n \to X$, where homotopies and maps must satisfy that $f(\partial[0,1]^n) = g(\partial[0,1]^n) = F(\partial[0,1]^n, t) = x_0 \in X$. This set becomes a group with the following definition

$$(f+g)(a_1, a_2, \ldots, a_n) = \begin{cases} f(2a_1, a_2, \ldots, a_n) & \text{if} \quad 0 \le a_1 \le \frac{1}{2}, \\ g(2a_1 - 1, a_2, \ldots, a_n) & \text{if} \quad \frac{1}{2} \le a_1 \le 1. \end{cases}$$

For $n \ge 2$ $\pi_n(X)$ is known to be Abelian. Figure 1.2 shows the homotopy groups of spheres [Tod62].

$\pi_i(S^n)$

$n \downarrow$ \ $i \to$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Figure 1.2: Sample of homotopy groups of spheres [Tod62].

A space is *k-connected* ($k \ge -1$) if for every $l = -1, 0, 1, \ldots, k$, each continuous map $f \colon \partial[0,1]^{l+1} \to X$ can be extended to a map $\hat{f} \colon [0,1]^{l+1} \to X$. Here (-1)-connected means that the space is nonempty. We define the *connectivity* of a space X by $\operatorname{connectivity}(X) := \max\{k \colon X \text{ is } k\text{-connected}\}$.

Suppose that X and Y are topological spaces and there is a map $f \colon X \to Y$. If there exist a map $g \colon Y \to X$, such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$, then f is a *homotopy equivalence*. If such an f exists, then the spaces X and Y are called *homotopy equivalent*.
If X is homotopy equivalent to a point we say that X is *contractible*.

Definition 1.7. Let X be a topological space, $A \subseteq X$ a subcomplex, and $i \colon A \hookrightarrow X$ be the inclusion map. A continuous map $f \colon X \to A$ is called a *retraction* if $f|_A = \mathrm{Id}_A$.

Furthermore it is called a *deformation retraction* if $i \circ f\colon X \to X$ is homotopic to the identity map Id_X.
Finally, f is called a *strong deformation retraction* if there exists a homotopy $F\colon X \times [0,1] \to X$ between $i \circ f$ and Id_X, which is a constant on A, i.e., $F(a,t) = a$ for all $t \in [0,1]$ and $a \in A$.

Let us recall that if X and Y are topological spaces, the join $X * Y$ is the quotient of the product space $X \times Y \times [0,1]$ modulo the equivalence $\approx$, where $(x, y, 0) \approx (x', y, 0)$ and $(x, y, 1) \approx (x, y', 1)$ for all $x, x' \in X$ and $y, y' \in Y$. (See Figure 1.3.) It is known that the join of geometric simplicial complexes is homeomorphic to the geometric realization of their simplicial join, which will be defined in the next section. A special case, when Y is the 0-dimensional sphere S^0, is called the *suspension* of X: $\mathrm{susp}(X) := X * S^0$. The *cone* of X is defined by $\mathrm{cone}(X) = X * \text{point}$.

Join can be defined for maps as well. Let $f\colon X_1 \to X_2$ and $g\colon Y_1 \to Y_2$ arbitrary continuous maps. We can define a map $f * g\colon X_1 * Y_1 \to X_2 * Y_2$ by $tx + (1-t)y \to tf(x) + (1-t)g(x)$.

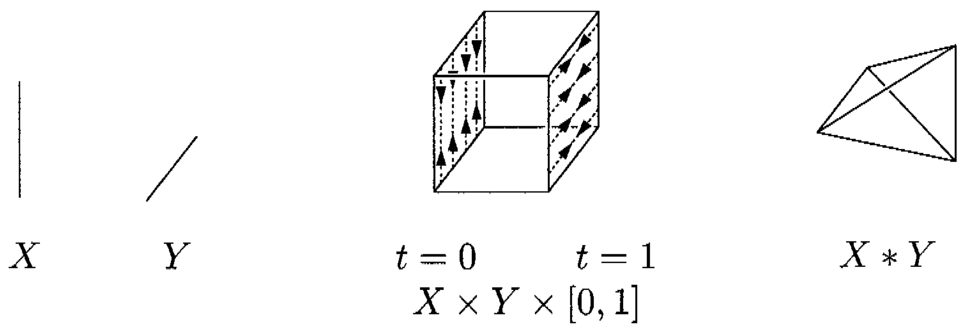

Figure 1.3: Join of spaces.

An *n-cell* is a topological space homeomorphic to $[0,1]^n \subset \mathbb{R}^n$.
A *CW-complex* X is obtained by the following inductive construction.
(1) The 0-skeleton X_0 is a discrete set.
(2) We construct the n-skeleton X_n by the simultaneous attachment of the n-cells to X_{n-1} along their boundaries.
(3) We equip the space $X = \cup_{n-0}^{\infty} X_n$ with the weak topology: $A \subseteq X$ is open if and only if $A \cap X$ is open for any n.

A $\mathbb{Z}_2$-*space* is a pair (X, ν) where X is a topological space and $\nu\colon X \to X$, called the $\mathbb{Z}_2$-action, is a homeomorphism such that $\nu^2 = \nu \circ \nu = \mathrm{Id}_X$. If (X_1, ν_1) and (X_2, ν_2) are $\mathbb{Z}_2$-spaces, a $\mathbb{Z}_2$-

map between them is a continuous mapping $f\colon X_1 \to X_2$ such that $f \circ \nu_1 = \nu_2 \circ f$.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \nu_1}\big\downarrow & & \big\downarrow{\scriptstyle \nu_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

$\mathbb{Z}_2$-spaces with $\mathbb{Z}_2$-maps form a category. We call a $\mathbb{Z}_2$-space *free* if its $\mathbb{Z}_2$-action ν has no fixed point. The sphere S^n is understood as a free $\mathbb{Z}_2$-space with the antipodal homeomorphism $x \to -x$. Sometimes it is denoted by S^n_a to distinguish from other possible actions, e.g., the identity maps makes it a non-free $\mathbb{Z}_2$-space denoted by S^n_t. Since in this thesis we consider only S^n_a we will denote it by simply S^n. We will consider only finite dimensional *free* $\mathbb{Z}_2$-complexes.

One of the central theorems of topology is the Borsuk–Ulam Theorem. In order to state its equivalent forms as well we need the following definitions.
The $\mathbb{Z}_2$*-index* of a $\mathbb{Z}_2$-space (X, ν) is

$$\operatorname{ind}(X) = \min\{n \geq 0\colon \text{ there is a } \mathbb{Z}_2\text{-map } (X, \nu) \to (S^n, -)\}.$$

The $\mathbb{Z}_2$-action ν will be omitted from the notation if it is clear from the context. If $\operatorname{ind}(X_1, \nu_1) > \operatorname{ind}(X_2, \nu_2)$, then there is no $\mathbb{Z}_2$-map $X_1 \to X_2$. One of the equivalent formulation of the Borsuk–Ulam Theorem is $\operatorname{ind}(S^n) = n$.

Another index-like quantity of a $\mathbb{Z}_2$-space, the *coindex* can be defined by

$$\operatorname{coindex}(X) = \max\left\{n \geq 0\colon \text{ there is a } \mathbb{Z}_2\text{-map } S^n \xrightarrow{\mathbb{Z}_2} X\right\}.$$

The notation level and co-level, instead of index and coindex respectively, appears in the literature as well.

Another equivalent formulation of the Borsuk–Ulam Theorem is that $\operatorname{coindex}(X) \leq \operatorname{ind}(X)$. We call a free $\mathbb{Z}_2$-space *tidy* if $\operatorname{coindex}(X) = \operatorname{ind}(X)$.

The Borsuk–Ulam Theorem can be stated in many equivalent forms. Here we state four of them.

Theorem 1.8 (Borsuk–Ulam). *The following are true and equivalent:*

1. *For every continuous map* $f\colon S^k \to \mathbb{R}^k$ *there exists* $x \in S^k$ *for which* $f(x) = f(-x)$.
2. *(Lyusternik-Schnirel'man version) Let* $d \geq 0$ *and let* $\mathcal{A}$ *be a collection of open (or closed) sets covering* S^d *with no* $A \in \mathcal{A}$ *containing a pair of antipodal points. Then* $|\mathcal{A}| \geq d+2$.
3. $S^{d+1} \overset{\mathbb{Z}_2}{\not\longrightarrow} S^d$ *for any* $d \geq 0$.
4. *For a* $\mathbb{Z}_2$*-space* X *we have* $\operatorname{coindex}(X) \leq \operatorname{ind}(X)$.

A $\mathbb{Z}_2$-map $f\colon X \to Y$ is a $\mathbb{Z}_2$*-homotopy equivalence* if there exists a $\mathbb{Z}_2$-map $g\colon Y \to X$ such that $g \circ f$ is $\mathbb{Z}_2$-homotopic to Id_X and $f \circ g$ is $\mathbb{Z}_2$-homotopic to Id_Y. In this case we say that X and Y are $\mathbb{Z}_2$*-homotopy equivalent*. A general reference for group actions, $\mathbb{Z}_2$-spaces and related concepts and facts is the textbook of Bredon [Bre67].

1.3 Simplicial complexes, polytopes and posets

An abstract *simplicial complex* K is a finite hereditary set system. We denote its vertex set by $V(\mathsf{K})$. The formal definition is as follows.

Definition 1.9. An *abstract simplicial complex* is a pair (V, K), where V is a set and $\mathsf{K} \subseteq 2^V$ is a hereditary set system of subsets of V; that is, we require that $F \in \mathsf{K}$ and $G \subseteq F$ imply $G \in \mathsf{K}$. The sets in K are called (abstract) simplices. We define the dimension $\dim(\mathsf{K}) := \max\{|F| - 1\colon F \in \mathsf{K}\}$.

We will now define geometric simplicial complexes. They are obtained by gluing together geometric simplices along their faces.

Definition 1.10. A *simplex* σ is the convex hull of a finite affinely independent set A in $\mathbb{R}^d$. The elements of A are called the *vertices* of σ. The *dimension* of σ is $\dim \sigma := |A| - 1$. The convex hull of an arbitrary subset of vertices of a simplex σ is a *face* of σ.

Definition 1.11. A nonempty family K of simplices is a *simplicial complex* if the following two conditions hold:

1. Each face of any simplex $\sigma \in \mathsf{K}$ is also a simplex of K.
2. The intersection $\sigma_1 \cap \sigma_2$ of any two simplices $\sigma_1, \sigma_2 \in \mathsf{K}$ is a face of both σ_1 and σ_2.

We call $\sigma \in \mathsf{K}$ a d-dimensional simplex if $|\sigma| = d + 1$.

Since an abstract simplicial complex K is a subcomplex of a $d := (|V(\mathsf{K})| - 1)$-dimensional simplex every finite abstract simplicial complex can by realized in $\mathbb{R}^d$. We sometimes denote this geometric realization as $||\mathsf{K}||$ to emphasize that we rather consider it as a topological space, than a set system. The *dimension* of K is $\dim \mathsf{K} := \max\{\dim \sigma \colon \sigma \in \mathsf{K}\}$. The *k-skeleton* of a simplicial complex K is the collection of at most k-dimensional simplices of K. We will denote it by $\mathrm{SK}_{\mathrm{k}}(\mathsf{K})$.

Example: The d-dimensional *crosspolytope* is the unit ball of the l_1-norm: $\left\{x \in \mathbb{R}^d \colon ||x||_1 \leq 1\right\}$. Its boundary is a simplicial complex homeomorphic to the sphere.

A convex *polytope* $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. A *valid* inequality for P is a linear inequality $c^T x \leq c_0$ with $c \in \mathbb{R}^d$, $c_0 \in \mathbb{R}$ satisfied by all $x \in P$. A *face* F of a polytope P is the intersection $F := P \cap \{x \in \mathbb{R}^d \colon c^T x = c_0\}$ where $c^T x \leq c_0$ is a valid inequality.

Definition 1.12. A *polyhedral complex* is a collection $\mathcal{C}$ of polytopes in $\mathbb{R}^d$ satisfying the following conditions:

1. the empty set is in $\mathcal{C}$,
2. for every $P \in \mathcal{C}$ all its faces are in $\mathcal{C}$ and
3. for every $P, Q \in \mathcal{C}$: $P \cap Q$ is a face of both P and Q.

The members of $\mathcal{C}$ are called *cells*.

Definition 1.13. Let K and L be two abstract simplicial complexes. A *simplicial mapping* of K to L is a mapping $f \colon V(\mathsf{K}) \to V(\mathsf{L})$ that maps simplices to simplices, i.e., such that $f(F) \in \mathsf{L}$ whenever $F \in \mathsf{K}$.
A bijective simplicial mapping whose inverse mapping is also simplicial is called an *isomorphism* of abstract simplicial complexes. The existence

of an isomorphism of simplicial complexes K and L will be denoted by $\mathsf{K} \cong \mathsf{L}$.

Clearly the affine extension of a simplicial mapping defines a continuous map $||f||: ||\mathsf{K}|| \to ||\mathsf{L}||$.

We recall that a *partially ordered set*, or *poset* for short, is a pair $(P, \preceq)$, where P is a set and $\preceq$ is a binary relation on P that is reflexive ($x \preceq x$), transitive ($x \preceq y$ and $y \preceq z$ imply that $x \preceq z$), and weakly antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$). When a covering relation $\preceq$ is understood, we say only "a poset P."

Definition 1.14. The *order complex* of a poset P is the simplicial complex $\Delta(P)$, whose vertices are the elements of P and whose simplices are all chains (i.e. $x_1 \prec x_2 \prec \cdots \prec x_k$) in P.
The *face poset* of a simplicial complex K is a poset $\mathcal{F}(\mathsf{K})$, which is the set of all nonempty simplices of K ordered by the inclusion.
For a simplicial complex K, the simplicial complex

$$\mathrm{sd}(\mathsf{K}) := \Delta(\mathcal{F}(\mathsf{K}))$$

is called the (first) *barycentric subdivision* of K.

We will consider maps between two posets $(P_1, \preceq_1)$, $(P_2, \preceq_2)$ which are either monotone or antimonotone, i.e., which satisfy either the condition $x \preceq y \Rightarrow \varphi(x) \preceq \varphi(y)$ or the dual condition $x \preceq y \Rightarrow \varphi(y) \preceq \varphi(x)$. A map between two posets induces a simplicial map between their order complexes.

Sometimes by an abuse of notation we will write K instead of $\mathcal{F}(\mathsf{K})$, and we will speak about the homotopy type of a poset meaning the homotopy type of its order complex.

We will frequently use the following Quillen-type Lemma. Our favorite version which turned out to be especially useful for dealing with Hom complexes was proven by Babson and Kozlov (Proposition 3.2 (page 9) in [BK03]).

Lemma 1.15. *Let $\phi : P \to Q$ be a map of finite posets. If ϕ satisfies*

Condition (A): $\Delta(\phi^{-1}(q))$ *is contractible, for every $q \in Q$,*

and

Condition (B): *For every* $p \in P$ *and* $q \in Q$ *with* $\phi(p) \geq q$ *the poset* $\phi^{-1}(q) \cap P_{\leq p}$ *has a maximal element,*

then ϕ *is a homotopy equivalence.*

Definition 1.16. A poset $(P, \preceq)$ is *involutive* if it is equipped with an involution $\varphi\colon P \to P$ which is either monotone or antimonotone and $\varphi^2 = \mathrm{Id}_P$. Instead of involutive we also say that $(P, \preceq)$ admits a $\mathbb{Z}_2$-*action* or that $(P, \preceq)$ is a $\mathbb{Z}_2$-*poset*. We will call a $\mathbb{Z}_2$-poset $(P, \preceq, \varphi)$ *free* if φ is a free $\mathbb{Z}_2$-action on its order complex.

Chain notation: We denote by $\mathcal{A}$ a chain $A_1 \subset \ldots \subset A_p$ of subsets of the nodes $\mathrm{V}(G)$ of a graph G and by $\mathcal{B}$ a chain $B_1 \subset \ldots \subset B_q$ of subsets of $\mathrm{V}(G)$. For $1 \leq t \leq p$ we denote by $\mathcal{A}_{\leq t}$ the chain $A_1 \subset \ldots \subset A_t$. A similar convention is used for $\mathcal{A}_{\geq t}$. For chains $\mathcal{A}$, $\mathcal{B}$ satisfying $A_p \subseteq B_1$ we define a new chain, the *concatenation of* $\mathcal{A}$ *and* $\mathcal{B}$:

$$\mathcal{A} \sqsubset \mathcal{B} := A_1 \subset \ldots \subset A_p \subseteq B_1 \subset \ldots \subset B_q,$$

where we omit A_p or B_1 in case $A_p = B_1$. If a map f preserves (resp. reverses) inclusions, we write $f(\mathcal{A})$ instead of $f(A_1) \subseteq \ldots \subseteq f(A_p)$ (resp. $f(A_p) \subseteq \ldots \subseteq f(A_1)$). One obtains a chain of proper subsets by omitting multiple copies.

For sets A, B define $A \uplus B := \{(a,0)\colon a \in A\} \cup \{(b,1)\colon b \in B\}$. An important construction in the category of simplicial complexes is the *join operation*. For two simplicial complexes K and L the join $\mathsf{K} * \mathsf{L}$ is defined as

$$\mathsf{K} * \mathsf{L} := \{F \uplus G \mid F \in \mathsf{K} \text{ and } G \in \mathsf{L}\}.$$

Let K be a simplicial complex and $\sigma \in \mathsf{K}$ its simplex. The *star* and the *link* of σ in K is defined by: $\mathrm{star}_{\mathsf{K}}(\sigma) := \{\tau \in \mathsf{K}\colon \tau \cup \sigma \in \mathsf{K}\}$ and $\mathrm{link}_{\mathsf{K}}(\sigma) := \{\tau \in \mathsf{K}\colon \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \mathsf{K}\}$.

1.4 Collapsing, and simple homotopy type

Definition 1.17. Let K be a simplicial complex. Let $\sigma, \tau \in \mathsf{K}$ such that

1. $\tau \subset \sigma$,

2. σ is a maximal simplex, and no other maximal simplex contains τ.

A (simplicial) *collapse* of K is the removal of all simplices γ, such that $\tau \subseteq \gamma \subseteq \sigma$. If in additional $\dim \tau = \dim \sigma - 1$, then this is called an elementary collapse.

When Y is a simplicial subcomplex of X, we say that X *collapses onto* Y if there exists a sequence of elementary collapses leading from X to Y. The reverse of an elementary collapse is called an elementary *expansion*. A sequence of elementary collapses and elementary expansions leading from a complex X to the complex Y is called a *formal deformation*. If such a sequence exists, then the simplicial complexes X and Y are said to have the same *simple homotopy type*, see [Whi39].

The definition of the *$\mathbb{Z}_2$-collapse* and *simple $\mathbb{Z}_2$-homotopy type* is self-evident.

A sequence of collapses yields a strong deformation retraction, in particular, a homotopy equivalence. The converse is not true, there are simplicial complexes which are contractible, but not collapsible.
Although there are homotopy equivalent but not simple homotopy equivalent spaces these two notions coincide for contractible spaces:

Theorem 1.18. *A simplicial complex K contractible if and only if there exists a sequence of collapses and expansions leading from K to a vertex.*

It is well-known, see e.g. [Koz05b], that for a simplicial complex X the subdivision $\mathrm{sd}(\mathsf{X})$ and $\mathrm{ssd}(\mathsf{X})$ have the same simple homotopy type as X, since they can be obtained by repeating stellar subdivision. This clearly extends to simple $\mathbb{Z}_2$-homotopy type for $\mathbb{Z}_2$-complexes.

Definition 1.19. An order preserving map φ from a poset P to itself is called a *descending closure operator* if $\varphi^2 = \varphi$ and $\varphi(x) \preceq x$, for any $x \in P$; analogously, φ is called an *ascending closure operator* if $\varphi^2 = \varphi$ and $\varphi(x) \succeq x$, for any $x \in P$.

A $\mathbb{Z}_2$-descending closure operator is a descending closure operator which is a $\mathbb{Z}_2$-map as well.

That ascending and descending closure operators induce strong deformation is well known in topological combinatorics, see e.g., [Bjö95].

We will use the following generalization due to Kozlov which is suitable to deal with simple homotopy equivalences and prove collapsibility.

Theorem 1.20 ([Koz05a, Theorem 2.1]). *Let P be a poset, and let φ be a descending closure operator, then $\Delta(P)$ collapses onto $\Delta(\varphi(P))$. By symmetry the same is true for an ascending closure operator.*

Actually, one can adapt the proof of the previous theorem to obtain the $\mathbb{Z}_2$-version which we will need.

Theorem 1.21. *Let P be a poset with a free involution ω, and let a $\mathbb{Z}_2$-map φ be a descending closure operator, then $\Delta(P)$ $\mathbb{Z}_2$-collapses onto $\Delta(\varphi(P))$. By symmetry the same is true for an $\mathbb{Z}_2$-ascending closure operator.*

Proof. We use induction on $|P| - |\varphi(P)|$. If $|P| = |\varphi(P)|$, then φ is the identity map and the statement is obvious. Assume that $P \setminus \varphi(P) \neq \emptyset$ and let $x \in P$ be one of the minimal elements of $P \setminus \varphi(P)$.

Since φ fixes each element in $P_{<x}$, $\varphi(x) < x$, and φ is order preserving, we see that $P_{<x}$ has $\varphi(x)$ as a maximal element, see Figure 1.4. Thus the link of x in $\Delta(P)$ is $\Delta(P_{>x}) * \Delta(P_{<x}) = \Delta(P_{>x}) * \Delta(P_{<\varphi(x)}) * \varphi(x)$, in particular, it is a cone with apex $\varphi(x)$.

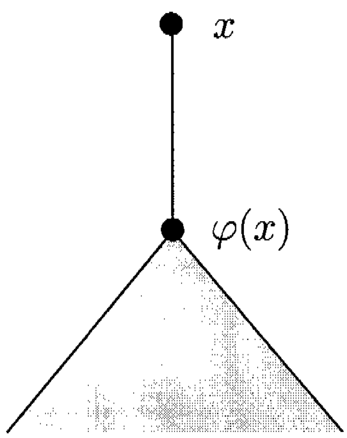

Figure 1.4: $P_{<x} = P_{\leq\varphi(x)}$.

Let $\sigma_1, \ldots, \sigma_t$ be the simplices of $\Delta(P_{>x}) * \Delta(P_{<\varphi(x)})$ ordered so that the dimension is weakly decreasing. Then

$$(\sigma_1 \cup \{x\}, \sigma_1 \cup \{x, \varphi(x)\}), \ldots, (\sigma_t \cup \{x\}, \sigma_t \cup \{x, \varphi(x)\})$$

is a sequence of elementary collapses leading from $\Delta(P)$ to $\Delta(P \setminus \{x\})$. Since φ restricted to $P \setminus \{x\}$ is again a descending closure operator, $\Delta(P \setminus \{x\})$ collapses onto $\Delta(\varphi(P \setminus \{x\})) = \Delta(\varphi(P))$ by the induction assumption. Since ω is a free $\mathbb{Z}_2$-action meanwhile we can do the $\mathbb{Z}_2$-pairs of these collapses as well. □

We introduce the basics of Discrete Morse Theory which was invented by Forman [For98]. It provides a convenient language for describing sequences of elementary collapses.

Definition 1.22. Let P be a poset with the covering relation $\succ$.

- We define a *partial matching* on P to be a pair (Σ, μ) where $\Sigma \subseteq P$ is a set, and $\mu : \Sigma \to P \setminus \Sigma$ is an injective map, such that $\mu(x) \succ x$, for all $x \in \Sigma$.
- The elements of $P \setminus (\Sigma \cup \mu(\Sigma))$ are called *critical*. We let $\mathcal{C}(P, \mu)$ denote the set of critical elements.
- Additionally, such a partial matching μ is called *acyclic* if there exists no sequence of distinct elements $x_1, \ldots, x_t \in \Sigma$, where $t \geq 2$, satisfying $\mu(x_1) \succ x_2$, $\mu(x_2) \succ x_3$, $\ldots$, $\mu(x_t) \succ x_1$.

The partial acyclic matchings and elementary collapses are closely related, as the next proposition shows.

Proposition 1.23 ([Koz02, Proposition 5.4]). *Let Δ be a regular CW complex and Δ' a subcomplex of Δ, then the following are equivalent:*

a) there is a sequence of elementary collapses leading from Δ to Δ';

b) there is a partial acyclic matching on the poset $\mathcal{F}(\Delta)$ with the set of critical cells being exactly $\mathcal{F}(\Delta')$.

1.5 Graph complexes

Now we will define simplicial complexes assigned to graphs.

The *clique complex* Cliq(G) of a graph G is a simplicial complex assigned to G. Its vertex set equals $V(G)$, and the simplices are subsets of vertices forming a clique in G:

$$\mathrm{Cliq(G)} := \{\mathrm{S} \subseteq \mathrm{V(G)} \colon \mathrm{S} \text{ is a clique in } G\}.$$

The *independence complex* Ind(G) of a graph G is a simplicial complex assigned to G. Its vertex set equals $V(G)$, and the simplices are subsets of vertices forming an independent set in G:

$$\mathrm{Ind(G)} := \{\mathrm{S} \subseteq \mathrm{V(G)} \colon \mathrm{S} \text{ is an independent set in } G\}.$$

Clearly Ind(G) is isomorphic to Cliq($\bar{\mathrm{G}}$), where $\bar{G}$ is the complement of G.

The *neighborhood complex* $\mathsf{N}(G)$ was introduced by Lovász [Lov78] in order to solve the Kneser Conjecture [Kne55]. It is a simplicial complex associated with a graph G. Its vertex set equals $V(G)$, and the simplices are subsets of vertices possessing a common neighbor:

$$\mathsf{N}(G) := \{S \subseteq V(G) \colon \mathrm{CN}(S) \neq \emptyset\}.$$

Figure 1.5 shows an example of a graph and its neighborhood complex. The disadvantage of the neighborhood complex is that it does not admit any free $\mathbb{Z}_2$-action.

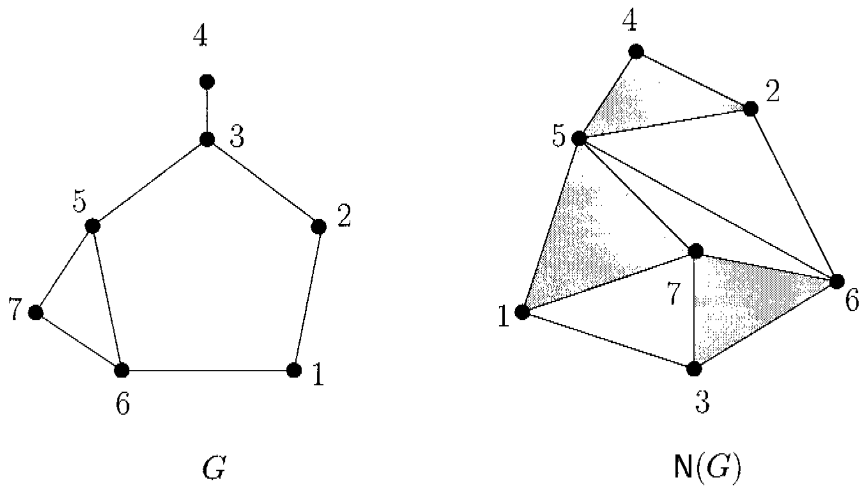

Figure 1.5: The neighborhood complex.

The so-called *Lovász complex* $\mathsf{L}(G)$ is an induced subcomplex of the barycentric subdivision of the neighborhood complex. Its vertex set consists of the sets A on which CN^2 does behave as the identity:

$$V(\mathsf{L}(G)) := \{S \in \mathsf{N}(G) \colon \mathrm{CN}^2(S) = S\}.$$

These sets are called closed. Equivalently it can be defined as $\mathsf{L}(G) = \mathrm{CN}(\mathrm{sd}(\mathsf{N}(G)))$. The map CN, which maps the vertex S to $\mathrm{CN}(S)$, defines a free $\mathbb{Z}_2$-action on $\mathsf{L}(G)$ making it a $\mathbb{Z}_2$-space. Figure 1.6 shows an example of a graph and its Lovász complex.

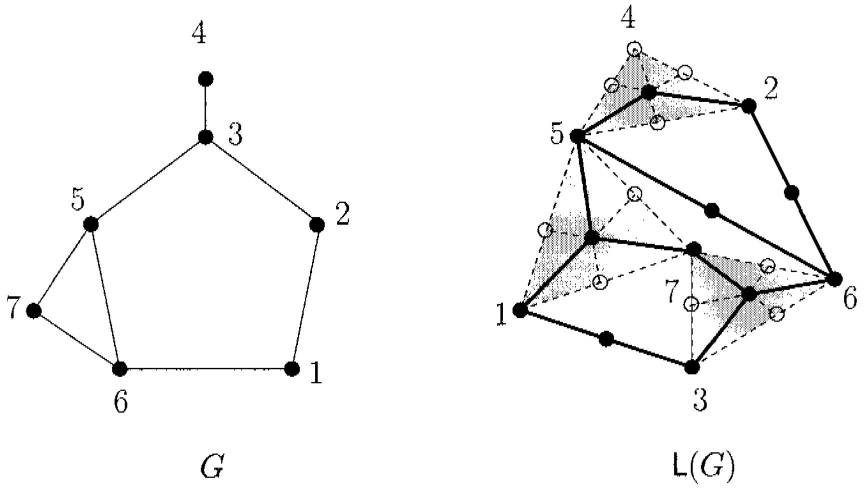

Figure 1.6: The Lovász complex.

It is known that these complexes are simple homotopy equivalent:

Theorem 1.24 ([Koz05b]). *The barycentric subdivision* $\mathrm{sd}(\mathsf{N}(G))$ *of the neighborhood complex collapses into the Lovász complex* $\mathsf{L}(G)$.

Proof. Since $\mathsf{L}(G) = \mathrm{CN}^2(\mathrm{sd}(\mathsf{N}(G)))$ and CN^2 is an ascending closure operator, Theorem 1.20 completes the proof. □

It is worth to note that these complexes provide a surprising and useful topological lower bound for the chromatic number.

Theorem 1.25 ([Lov78]). *For every graph G we have*

$$\chi(G) \geq \mathrm{ind}(\mathsf{L}(G)) + 2 \geq \text{connectivity}\,(\mathsf{N}(G)) + 3.$$

Different versions of a box complex are described by Alon, Frankl, and Lovász [AFL86], Sarkaria [Sar90], Kříž [Kří92], and Matoušek and Ziegler [MZ04]. Among these we will use the one which was introduced by Matoušek and Ziegler.
The *box complex* $\mathsf{B}(G)$ of a graph G can be considered as the subcomplex

of the join $\mathrm{N}(G) * \mathrm{N}(G)$, where we keep only those simplices which correspond to complete bipartite subgraphs of G. More formally,

$$\mathsf{B}(G) := \left\{ \begin{array}{l} A \uplus B\colon A, B \subseteq V(G),\ A \cap B = \emptyset, \\ G[A,B] \text{ is complete bipartite},\ \mathrm{CN}(A) \neq \emptyset \neq \mathrm{CN}(B) \end{array} \right\}.$$

We note that $\mathrm{CN}(\emptyset) = V(G)$. The vertices of the box complex are $V_1 := \{v \uplus \emptyset\colon v \in V(G)\}$ and $V_2 := \{\emptyset \uplus v\colon v \in V(G)\}$. The subcomplexes of $\mathsf{B}(G)$ induced by V_1 and V_2 are disjoint subcomplexes of $\mathsf{B}(G)$ that are both isomorphic to the neighborhood complex $\mathsf{N}(G)$. We refer to these two copies as *shores* of the box complex. The box complex is endowed with a $\mathbb{Z}_2$-action which interchanges the shores.

If the extra condition on "having a common neighbor" is deleted, then we get a *different* box complex [MZ04]

$$\mathsf{B}_0(G) := \left\{ \begin{array}{l} A \uplus B\colon A, B \subseteq V(G),\ A \cap B = \emptyset, \\ G[A,B] \text{ is complete bipartite} \end{array} \right\}.$$

The *cones over the shores* complex $\mathsf{B}_C(G)$ is:

$$\begin{aligned} \mathsf{B}_C(G) \ := \ & \mathsf{B}(G) \cup \{(x, A \uplus \emptyset)\colon A \subseteq V(G), \mathrm{CN}(A) \neq \emptyset\} \\ & \cup \{(\emptyset \uplus B, y)\colon B \subseteq V(G), \mathrm{CN}(B) \neq \emptyset\}, \end{aligned}$$

where we assume that $x, y \notin V(G)$. We need this complex $\mathsf{B}_C(G)$ only for technical reason. We note that $\mathsf{B}(G), \mathsf{B}_0(G), \mathsf{B}_C(G)$ are free $\mathbb{Z}_2$-spaces.

Examples: For the complete graph K_n its neighborhood complex $\mathsf{N}(K_n)$ is the boundary complex of the $(n-1)$-dimensional simplex. Its box complex $\mathsf{B}_0(K_n)$ is the boundary complex of the n-dimensional crosspolytope; while $\mathsf{B}(K_n)$ is the boundary complex of the n-dimensional crosspolytope, with two opposite facets removed. $\mathsf{B}_C(K_n)$ can be obtained from $\mathsf{B}(K_n)$ by attaching cones over its boundary components.

The advantage of the the box complex (compared to the neighborhood or to the Lovász complex) is its natural *functoriality* [Mat03, MZ04]. If $f\colon G \to H$ is a graph homomorphism we get naturally an induced simplicial $\mathbb{Z}_2$-map

$$\mathsf{B}(f)\colon \mathsf{B}(G) \to \mathsf{B}(H).$$

This gives us a possibility for elegant conceptual proofs.

Theorem 2.1. [Mat03, MZ04] For every graph G we have

$$\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)) + 2.$$

Proof. $\chi(G) = n$ means that there is a graph homomorphism $f\colon G \to K_n$. In the above example we have seen that $\mathsf{B}(K_n)$ is the boundary complex of the n-dimensional crosspolytope, with two opposite facets removed. This means that $\mathsf{B}(K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to the sphere S^{n-2} so $\mathrm{ind}(\mathsf{B}(K_n)) = n-2$. The $\mathbb{Z}_2$-map $\mathsf{B}(f)\colon \mathsf{B}(G) \to \mathsf{B}(K_n)$ proves that $\mathrm{ind}(\mathsf{B}(G)) \geq n-2$ what we wanted to show. $\square$

Similarly as before one can obtain a lower bound using $\mathsf{B}_0(G)$.

Theorem 2.2. [MZ04] For every graph G we have

$$\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)) + 2.$$

1.6 Homomorphism complexes

Homomorphism complexes are *generalizations* of box complexes. We will see later that $\mathrm{Hom}(K_2, G)$ is the "middle" of the box complex. Moreover the box complex $B(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the Hom complex $\mathrm{Hom}(K_2, G)$. Lovász's original idea to prove Kneser's conjecture is equivalent to considering 'only' homomorphisms from an edge K_2 to the given graph G.

The *homomorphism complexes* were introduced by Lovász [BK03]. Their 1-skeleton is a well-known graph which was studied earlier by e.g. Brightwell and Winkler [BW99].

Let $\Delta^{V(H)}$ be a simplex whose set of vertices is $V(H)$. Let $C(G,H)$ denote the direct product $\prod_{x\in V(G)} \Delta^{V(H)}$, i.e., the copies of $\Delta^{V(H)}$ are indexed by vertices of G.

Definition 1.26 ([BK03, Koz05c]). For any pair of graphs G and H let the *Hom complex* $\mathrm{Hom}(G,H)$ be a subcomplex of $C(G,H)$ defined by the following condition: $c = \prod_{x\in V(G)} \sigma_x \in \mathrm{Hom}(G,H)$ if and only if for any $x, y \in V(G)$ if $\{x,y\} \in E(G)$, then $H[\sigma_x, \sigma_y]$ is complete bipartite.

The topology of $\mathrm{Hom}(G, H)$ is inherited from the product topology of $C(G, H)$. $\mathrm{Hom}(G, H)$ is a polyhedral complex whose cells are products of simplices and are indexed by functions (multi-homomorphisms) $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $\{\imath, \jmath\} \in E(G)$, then for every $\tilde{\imath} \in \eta(\imath)$ and $\tilde{\jmath} \in \eta(\jmath)$ it follows that $\{\tilde{\imath}, \tilde{\jmath}\} \in E(H)$.

Let $V(G) = \{1, \ldots, m\}$. We encode the functions η by vectors $(\eta(1), \ldots, \eta(m))$ of non-empty sets $\eta(i) \subseteq V(H)$ with the above properties. A cell $(A_1, \ldots, A_m)$ of $\mathrm{Hom}(G, H)$ is a *face* of a cell $(B_1, \ldots, B_m)$ of $\mathrm{Hom}(G, H)$ if $A_i \subseteq B_i$ for all $1 \leq i \leq m$. In particular, $\mathrm{Hom}(G, H)$ has $\mathcal{H}om(G, H)$ as its set of vertices. Moreover, every cell $(A_1, \ldots, A_m)$ of $\mathrm{Hom}(G, H)$ is a product of m simplices of dimension $|A_i| - 1$ for $1 \leq i \leq m$. For brevity, we write sets $A = \{a_1, \ldots, a_k\} \subseteq V(H)$ in compressed form as strings, i.e., $A = a_1 \ldots a_k$.

A cell of $\mathrm{Hom}(G, H)$ is a *maximal face* or *facet* if it is not contained in any higher-dimensional cell of $\mathrm{Hom}(G, H)$. If H is the complete graph K_n on n nodes, then the maximal cells of $\mathrm{Hom}(G, K_n)$ have a particularly simple description: A cell $(A_1, \ldots, A_m)$ of $\mathrm{Hom}(G, K_n)$ is a facet if and only if $\bigcup_{j \in N(i)} A_j = V(K_n)$ for every $1 \leq i \leq m$, where $N(i)$ denotes the set of neighbors of the node i in the graph G, and $A_i \cap A_j = \emptyset$ for $j \in N(i)$.

As remarked above, $\mathrm{Hom}(G, H)$ is a polyhedral complex whose cells are products of simplices. In particular, the barycentric subdivision of $\mathrm{Hom}(G, H)$ is a simplicial complex.

Examples: The cells of the Hom complex $\mathrm{Hom}(K_2, K_3)$ are given by the vectors $(1,2)$, $(1,3)$, $(2,3)$, $(2,1)$, $(3,1)$, $(3,2)$, $(12,3)$, $(13,2)$, $(23,1)$, $(3,12)$, $(2,13)$, and $(1,23)$. Therefore, $\mathrm{Hom}(K_2, K_3)$ is a circle with six edges; see Figure 1.7.

The cells of the Hom complex $\mathrm{Hom}(C_5, K_3)$ is homeomorphic to the disjoint union of two circles. The Hom complex $\mathrm{Hom}(C_7, K_3)$ is homeomorphic to the disjoint union of two Möbius bands. Figure 1.8 shows one component of $\mathrm{Hom}(C_7, K_3)$, a Möbius band. The dashed line corresponds to the embedding $\mathrm{sd}(\mathrm{Hom}(C_5, K_3)) \hookrightarrow \mathrm{sd}(\mathrm{Hom}(C_7, K_3))$ given by $(a, b, c, d, e) \to (a, b, c, d, e, d, e)$.

The Hom complex $\mathrm{Hom}(K_2, K_n)$ is an $(n-2)$-dimensional sphere $(n \geq$

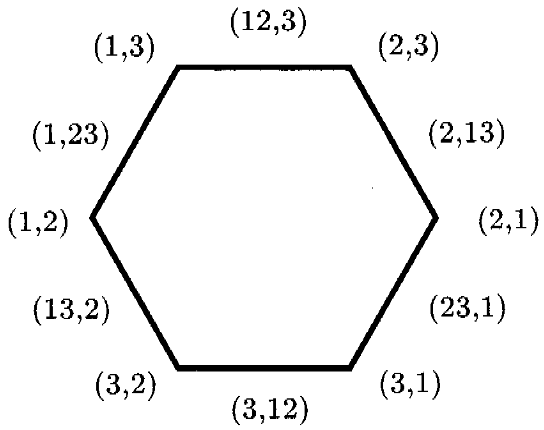

Figure 1.7: The Hom complex $\mathrm{Hom}(K_2, K_3)$.

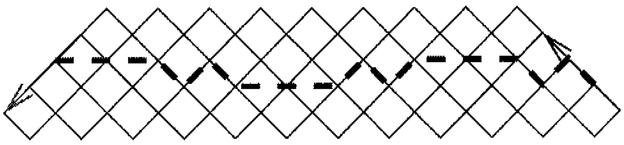

Figure 1.8: One component of the Hom complex $\mathrm{Hom}(C_7, K_3)$ and $\mathrm{Hom}(C_5, K_3)$.

2). In fact, $\mathrm{Hom}(K_2, K_n)$ is the boundary complex of a polytope [BK03, Sect. 4.2]. It can be described as the boundary of the Minkowski sum of an $(n-1)$-dimensional simplex σ_{n-1} and its negative $-\sigma_{n-1}$, as stated in [Mat03, p. 107, Ex. 3 (c)]. We will see a simpler argument later for this fact.

It is useful to note that the *homomorphism complex* $\mathrm{Hom}(K_2, G)$ or actually its barycentric subdivision can be defined as a subcomplex of $\mathrm{sd}(\mathsf{B}(G))$ spanned by the vertices $A \uplus B$ such that $A \neq \emptyset \neq B$. This definition is clearly equivalent to the original definition of homomorphism complex. We will see that actually $\mathrm{sd}(\mathsf{B}(G))$ $\mathbb{Z}_2$-collapses down to this subcomplex (see Section 2.7).

So $\mathrm{Hom}(K_2, K_n)$ can be obtained from intersecting the boundary of the n-dimensional crosspolytope $(\mathsf{B}_0(G))$ with the hyperplane $\sum_{i=1}^{n} x_i = 0$, and therefore $\mathrm{Hom}(K_2, K_n)$ is $\mathbb{Z}_2$-homeomorphic to S_a^{n-2}.

Later we will be interested in subcomplexes of $\mathrm{Hom}(K_2, K_n)$, namely in the cells in the form $(*, 12 \ldots k*)$. Note that $\mathrm{Hom}(K_2, K_{n-1})$ sits inside $\mathrm{Hom}(K_2, K_n)$, we only do not use the color n. Geometrically it corresponds to the intersection of $\mathrm{Hom}(K_2, K_n)$ (as described before) with the hyperplane $x_n = 0$. Now the cells of $(A, B) \in \mathrm{Hom}(K_2, K_n)$ such that $n \in A$ are in the open halfspace $x_n > 0$, while the cells with $n \in B$ are in $x_n < 0$. So returning to our problem the cells in the form $(*, 12 \ldots k*)$ are given by intersections of the sphere $\mathrm{Hom}(K_2, K_n)$ and k orthogonal halfspaces, so it is a disc for $k < n$.

Similarly to the box complexes the Hom complexes are naturally equipped with covariant and contravariant functors [BK03, BK04, Koz05c] as well. If $\varphi\colon G \to H$ is a graph homomorphism, we obtain an induced map $f_\varphi\colon \mathcal{F}(\mathrm{Hom}(T, G)) \to \mathcal{F}(\mathrm{Hom}(T, H))$. This gives functorial continuous maps between the Hom complexes as topological spaces. In the other case we get a contravariant functor. φ induces $f^\varphi\colon \mathcal{F}(\mathrm{Hom}(H, T)) \to \mathcal{F}(\mathrm{Hom}(G, T))$.
It is important for us that if ψ is an automorphism of T such that it flips an edge in T (i.e., two vertices, which are connected by an edge, get interchanged), then the induced map $f^\psi\colon \mathcal{F}(\mathrm{Hom}(T, G)) \to \mathcal{F}(\mathrm{Hom}(T, G))$ is fixed-point free for an arbitrary graph G (without loops).
This makes $\mathrm{Hom}(K_2, G)$ and $\mathrm{Hom}(C_n, G)$ a free $\mathbb{Z}_2$-space. Similarly, as with the box complexes, since $\mathrm{Hom}(K_2, K_n)$ is $\mathbb{Z}_2$-homeomorphic to the $(n-2)$-dimensional sphere, we obtain that

$$\chi(G) \geq \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2.$$

In the next section we will explain that these topological lower bounds are more than closely related.

We will often refer to the neighborhood complex, the Lovász complex, the box complexes and the $\mathrm{Hom}(K_2, G)$ as *graph complexes*.

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Chapter 2

Homotopy types of graph complexes

In this chapter we will study the previously defined graph complexes, the neighborhood complex $\mathsf{N}(G)$, the Lovász complex $\mathsf{L}(G)$, the box complexes $\mathsf{B}(G)$ and $\mathsf{B}_0(G)$, and the Hom complex $\mathrm{Hom}(K_2, G)$. These graph complexes associated to a graph G can be viewed as avatars of the same object, as long as their $\mathbb{Z}_2$-homotopy (or even simple $\mathbb{Z}_2$-homotopy) types are concerned [CsLSW04, Cso04, Koz05b, Živ04]. At first we will see that they are $\mathbb{Z}_2$-homotopy equivalent, and at the end of this section we will show that this can be extended to simple $\mathbb{Z}_2$-homotopy equivalences as well. We will present the universality theorem, which says that up to $\mathbb{Z}_2$-homotopy any $\mathbb{Z}_2$-space can be a graph complex.

In [MZ04] Matoušek and Ziegler compared various topological lower bounds for the chromatic number. They reformulated Lovász's original bound [Lov78] and Sarkaria's bound [Sar90] in terms of the index of various box complexes:

Theorem 2.1 (The Lovász bound [MZ04]). *For any graph G*

$$\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)) + 2.$$

Actually this might be formulated in terms of other graph complexes:

$\chi(G) \geq \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 = \mathrm{ind}(\mathsf{L}(G)) + 2 = \mathrm{ind}(\mathsf{B}(G)) + 2.$

Theorem 2.2 (The Sarkaria bound [MZ04]). *For any graph G*

$$\chi(G) \geq \mathrm{ind}(\mathsf{B}_0(G)) + 1.$$

In Section 2.2 we prove that the box complex $\mathsf{B}_0(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $\mathsf{B}(G)$. This makes the connection between these two bounds explicit. Since $\mathrm{ind}(X) \leq \mathrm{ind}(\mathrm{susp}(X)) \leq \mathrm{ind}(X) + 1$ the difference between the right side of the Lovász and the Sarkaria bound is at most one.

From purely topological point of view it is possible that these two bounds are not the same. In Section 3.5 we construct $\mathbb{Z}_2$-spaces e.g. X_{2h} such that the suspension does not increase their index: $\mathrm{ind}(\mathrm{susp}(X_{2h})) = \mathrm{ind}(X_{2h})$.

However to show that these lower bounds (Theorem 2.1 and 2.2) are not the same for graphs we would need a graph such that its box complex $\mathsf{B}(G)$ has this property (the suspension does not increase the index). In Subsection 2.3.1 we show that graph complexes are universal: their homotopy type can be 'arbitrary'. In Subsection 2.3.2 we extend this result to $\mathbb{Z}_2$-homotopy equivalence. This allows us to construct graphs G such that the gap between these two bounds is 1. This means that the Lovász bound can be strictly better than the Sarkaria bound, which answers a question of Matoušek and Ziegler [MZ04].

2.1 Shore subdivision and useful subcomplexes

In [CsLSW04] we introduced the *shore subdivision* of simplicial complexes and used it to find an upper bound to the topological lower bound (Theorem 2.1) and to construct a strong $\mathbb{Z}_2$-deformation retraction from the box complex to the Lovász complex. In the process, we analyze and clarify the combinatorics of the complexes involved and link their structure via several 'intermediate' complexes.

For a simplicial complex K and any partition of its vertex set V into non-empty sets V_1 and V_2, we call the simplicial subcomplexes K_1

and K_2 induced by V_1 and V_2 its *shores*. In case of the box complex we always consider the canonical partition (see Section 1.5). The *shore subdivision* of K is

$$\mathrm{ssd}(\mathsf{K}) := \{\mathrm{sd}(\sigma \cap \mathsf{K}_1) * \mathrm{sd}(\sigma \cap \mathsf{K}_2) \ \mid \sigma \in \mathsf{K}\}.$$

We apply this definition to the shores of the box complex to obtain the *shore subdivision* $\mathrm{ssd}(\mathsf{B}(G))$ of $\mathsf{B}(G)$. The vertices of $\mathrm{ssd}(\mathsf{B}(G))$ are of type $A \uplus \emptyset$ and $\emptyset \uplus A$ where $\emptyset \neq A \subset V(G)$ with $\mathrm{CN}(A) \neq \emptyset$. A simplex of $\mathrm{ssd}(\mathsf{B}(G))$ is denoted by $\mathcal{A} \uplus \mathcal{B}$ (the simplex spanned by the vertices $A \uplus \emptyset$ and $\emptyset \uplus B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$).

The map $\mathrm{cn}^2 : \mathrm{ssd}(\mathsf{B}(G)) \to \mathrm{ssd}(\mathsf{B}(G))$ defined on the vertices by

$$\mathrm{cn}^2(A \uplus \emptyset) := \mathrm{CN}^2(A) \uplus \emptyset \qquad \text{and} \qquad \mathrm{cn}^2(\emptyset \uplus A) := \emptyset \uplus \mathrm{CN}^2(A).$$

It is simplicial and $\mathbb{Z}_2$-equivariant. We refer to the image of the map cn^2 as *doubled Lovász complex* $\mathsf{DL}(G)$. It is

$$\mathsf{DL}(G) = \left\{ \mathcal{A} \uplus \mathcal{B} \ \middle|\ \begin{array}{c} \mathcal{A}, \mathcal{B} \in \mathsf{L}(G), \text{ for all } A \in \mathcal{A}, B \in \mathcal{B} \\ G[A; B] \text{ is complete bipartite} \end{array} \right\}.$$

A copy of the Lovász complex can be found on each shore of $\mathsf{DL}(G) \subset \mathrm{ssd}(\mathsf{B}(G))$, but these copies do not respect the induced $\mathbb{Z}_2$-action.

We partition the vertex set of the doubled Lovász complex $\mathsf{DL}(G)$ into pairs of type $\{A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A)\}$ to define a simplicial $\mathbb{Z}_2$-map $\jmath : \mathsf{DL}(G) \to \mathsf{DL}(G)$. Our aim is to specify one vertex for every pair and map both vertices of a pair to this chosen "smaller" vertex. To do this we refine the partial order by cardinality to a linear order "$\prec$" on the vertices of the original Lovász complex $\mathsf{L}(G)$ using the lexicographic order:

$$A \prec B \quad :\Longleftrightarrow \quad \begin{cases} |A| < |B| \text{ or} \\ |A| = |B| \text{ and } A <_{\mathrm{lex}} B. \end{cases}$$

In fact any refinement would work in the following. A partial order on the vertices of the doubled Lovász complex $\mathsf{DL}(G)$ is now obtained:

$$A \uplus \emptyset \prec \emptyset \uplus \mathrm{CN}(A) \quad :\Longleftrightarrow \quad A \prec \mathrm{CN}(A).$$

We define the map $\jmath$ on the vertices using this partial order by

$$\jmath(A \uplus \emptyset) := \min_{\prec}\{A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A)\}$$

and

$$j(\emptyset \uplus B) := \min_{\prec}\{\emptyset \uplus B, \mathrm{CN}(B) \uplus \emptyset\}.$$

It is easy to show that j is simplicial. Let $\mathcal{A} \uplus \mathcal{B}$ be a simplex of the doubled Lovász complex $\mathsf{DL}(G)$, that is, the simplex spanned by the vertices $A \uplus \emptyset$ and $\emptyset \uplus B$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$, $A, B \in \mathsf{N}(G)$ are fixed points of CN^2, and $G[A; B]$ is complete bipartite. Suppose that $A_p \prec B_q$ holds for the largest elements A_p and B_q of $\mathcal{A}$ and $\mathcal{B}$. Then $A_p \prec \mathrm{CN}(A_p)$, since $B_q \subseteq \mathrm{CN}(A_p)$. Hence $A \uplus \emptyset$ is a fixed point of j for each $A \in \mathcal{A}$. For some $0 \leq k < q$, the vertices $\emptyset \uplus B$ for $B \in \mathcal{B}_{\leq k}$ are fixed by j, while the vertices $\emptyset \uplus B$ are mapped to $\mathrm{CN}(B) \uplus \emptyset$ for $B \in \mathcal{B}_{>k}$. Since $A_p \prec \mathrm{CN}(B)$ for each $B \in \mathcal{B}_{>k}$, we have

$$j(\mathcal{A} \uplus \mathcal{B}) = (\mathcal{A} \sqsubset \mathrm{CN}(\mathcal{B}_{>k})) \uplus \mathcal{B}_{\leq k},$$

which is a simplex of $\mathsf{DL}(G)$. The argument is the same if $B_q \prec A_p$. Hence j is simplicial.

Since the image $\operatorname{Im} j$ has half as many vertices as $\mathsf{DL}(G)$, we refer to $\operatorname{Im} j$ as *halved doubled Lovász complex* $\mathsf{HDL}(G)$.

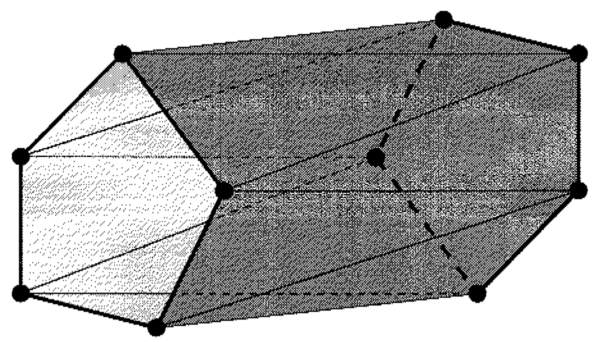


Figure 2.1: The box complex $\mathsf{B}(C_5)$.

A first example: The neighborhood complex $\mathsf{N}(C_5)$ of the 5-cycle C_5 is the 5-cycle; its Lovász complex $\mathsf{L}(C_5)$ is the 10-cycle C_{10}. The box complex $\mathsf{B}(C_5)$, depicted in Figure 2.1, consists of two copies of $\mathsf{N}(C_5)$ (the two shores) such that simplices of different shores are joined if and only if their vertex sets seen as node sets of the graph are common neighbors of each other. The shore subdivision $\mathrm{ssd}(\mathsf{B}(C_5))$ as illustrated in Figure 2.2 is a subdivision of the box complex induced from a barycentric subdivision of the shores. The map cn^2 maps a vertex of $\mathrm{ssd}(\mathsf{B}(C_5))$ to the common neighborhood of its common neighborhood. In our example, every

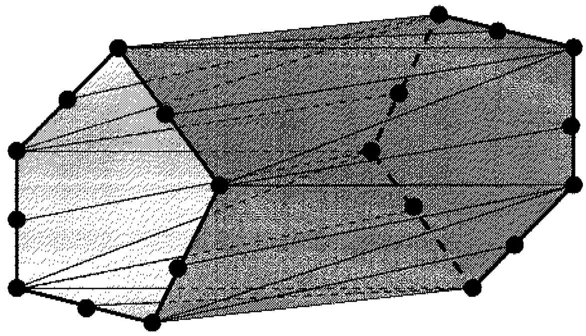


Figure 2.2: The shore subdivision $\mathrm{ssd}(\mathsf{B}(C_5))$ coincides with $\mathsf{DL}(C_5)$.

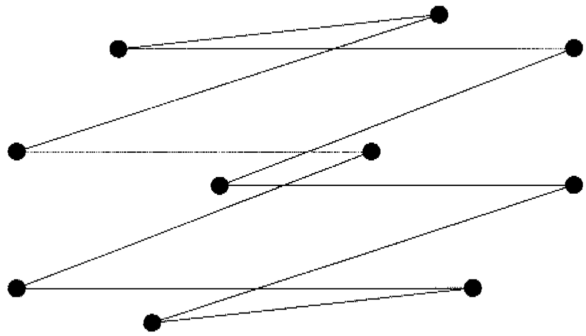


Figure 2.3: The halved doubled Lovász complex of C_5.

vertex is mapped to itself, hence $\mathrm{ssd}(\mathsf{B}(C_5)) = \mathsf{DL}(C_5)$. The partitioning of the vertex set of $\mathsf{DL}(C_5)$ into pairs of type $(A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A))$ can be visualized by edges of $\mathsf{DL}(C_5)$ that connect singletons from one shore with two-element sets from the other. The refined lexicographic order determines the image of such an edge under j: the smaller vertex is a singleton. Hence the map j collapses all edges of type $(A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A))$, which yields the halved doubled Lovász complex $\mathsf{HDL}(G)$ as shown in Figure 2.3.

A second example: Let us first describe the neighborhood complex and the Lovász complex of the complete graph K_n on n nodes. The neighborhood complex of K_n is the boundary of a simplex on n vertices. This follows from the fact that every set of $n-1$ nodes has a common neighbor but the set $[n]$ has empty common neighborhood. The neighborhood complex $\mathsf{N}(K_n)$ is therefore a pure abstract simplicial complex of dimension $n-2$, the set of facets is $\binom{[n]}{n-1}$. The Lovász complex $\mathsf{L}(K_n)$ is its barycentric subdivision, since $\mathrm{CN}(A) = [n] \setminus A$ and therefore $\mathrm{CN}^2(A) = A$ for each $A \subset [n]$. The $\mathbb{Z}_2$-action of $\mathsf{L}(K_n)$ maps a vertex $A \in \mathrm{V}(\mathsf{L}(K_n))$ to its complement in $[n]$. We now describe the box complex $\mathsf{B}(K_n)$. It is the subcomplex of the join $\mathsf{N}(K_n) * \mathsf{N}(K_n)$ that has facets $A \uplus ([n] \setminus A)$ for each non-empty set $A \subset [n]$. The box complex $\mathsf{B}(G)$ can also be interpreted as the boundary of an n-dimensional crosspolytope where a pair of opposite facets is removed. The $\mathbb{Z}_2$-action maps a simplex $A \uplus B$ to the simplex $B \uplus A$. The shore subdivision $\mathrm{ssd}(\mathsf{B}(K_n))$ is a subcomplex of the join $\mathsf{L}(K_n) * \mathsf{L}(K_n)$. Its facets can be described as follows. Consider a non-empty set $A \subset [n]$ and a maximal chain $\mathcal{A}$ of non-empty subsets of A. Such a chain represents a $(|A|-1)$-dimensional face of $\mathsf{L}(K_n)$. Consider a *complementary simplex* $\mathcal{B}$, that is, a maximal chain of non-empty subsets of $[n] \setminus A$. Then $\mathcal{A} \uplus \mathcal{B}$ is a facet of $\mathrm{ssd}(\mathsf{B}(K_n))$. The $\mathbb{Z}_2$-action maps $\mathcal{A} \uplus \mathcal{B}$ to $\mathcal{B} \uplus \mathcal{A}$. Since every vertex of $\mathrm{sd}(\mathsf{N}(K_n))$ is a fixed point of CN^2, the shore subdivision $\mathrm{ssd}(\mathsf{B}(K_n))$ coincides with the doubled Lovász complex $\mathsf{DL}(K_n)$. To define the map j, we consider the following partitioning of the vertices of $\mathsf{DL}(K_n)$ into pairs formed by $A \uplus \emptyset$ and $\emptyset \uplus \mathrm{CN}(A)$. The map j maps both vertices to the smaller one of $A \uplus \emptyset$ and $\emptyset \uplus \mathrm{CN}(A)$ with respect to $\prec$. The image of j is the halved doubled Lovász complex. Its $\mathbb{Z}_2$-action maps $A \uplus \emptyset$ to $\emptyset \uplus A$.

Remark 2.3. *Independently from this work, de Longueville* [dL04] *used shore subdivisions to give a short and elegant proof of the fact that Bier spheres are in fact spheres.*

2.1.1 $\mathsf{L}(G)$ as a $\mathbb{Z}_2$-deformation retract of $\mathsf{B}(G)$

Theorem 2.4. *The Lovász complex* $\mathsf{L}(G)$ *and the halved doubled Lovász complex* $\mathsf{HDL}(G)$ *are* $\mathbb{Z}_2$*-isomorphic.*

The proof makes use of the *chain notation* introduced in Section 1.3.

Proof. Since each shore of $\mathsf{DL}(G)$ is isomorphic (but not $\mathbb{Z}_2$-isomorphic) to $\mathsf{L}(G)$, we have $|\mathrm{V}(\mathsf{L}(G))| = |\mathrm{V}(\mathsf{HDL}(G))|$. To define a simplicial $\mathbb{Z}_2$-map $f : \mathsf{L}(G) \to \mathsf{HDL}(G)$, we partition $\mathrm{V}(\mathsf{L}(G))$ into

$$S := \left\{ A \;\middle|\; \begin{array}{c} A \in \mathrm{V}(\mathsf{L}(G)) \text{ and} \\ j(A \uplus \emptyset) = A \uplus \emptyset \end{array} \right\},$$

and

$$J := \left\{ A \;\middle|\; \begin{array}{c} A \in \mathrm{V}(\mathsf{L}(G)) \text{ and} \\ j(A \uplus \emptyset) = \emptyset \uplus \mathrm{CN}(A) \end{array} \right\},$$

(where "S" and "J" denote the vertices that Stay fixed or Jump to their neighbor), and set

$$f(A) := \begin{cases} A \uplus \emptyset & \text{if } A \in S, \\ \emptyset \uplus \mathrm{CN}(A) & \text{if } A \in J. \end{cases}$$

This map is a bijection between the vertex sets $\mathrm{V}(\mathsf{L}(G))$ and $\mathrm{V}(\mathsf{HDL}(G))$ that commutes on vertex level with the $\mathbb{Z}_2$-actions. We now show that it is also surjective and simplicial. For simpliciality, consider a simplex $\mathcal{A}$ in $\mathsf{L}(G)$. Let t denote the largest index k such that A_k is mapped onto the first shore. The image of $\mathcal{A}$ under f is $\mathcal{A}_{\leq t} \uplus \mathrm{CN}(\mathcal{A}_{\geq t+1})$. This is a simplex since $G\,[A_t; \mathrm{CN}(A_{t+1})]$ is complete bipartite. For surjectivity consider a simplex $\mathcal{A} \uplus \mathcal{B}$ of $\mathsf{HDL}(G)$, i.e. $G\,[A; B]$ is complete bipartite for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. This simplex is the image of the simplex $\mathcal{A} \sqsubset \mathrm{CN}(\mathcal{B})$ of $\mathsf{L}(G)$. □

Theorem 2.5. *The halved doubled Lovász complex* $\mathsf{HDL}(G)$ *is a strong* $\mathbb{Z}_2$*-deformation retract of the box complex* $\mathsf{B}(G)$*.*

Proof. First we observe that $\|\mathsf{DL}(G)\|$ is a strong $\mathbb{Z}_2$-deformation retract of $\|\mathsf{B}(G)\| = \|\mathrm{ssd}(\mathsf{B}(G))\|$. This follows from the fact that a closure operator induces a strong deformation retraction from its domain to its image, [Bjö95, Corollary 10.12 and the following remark]. Explicitly, this map is obtained by sending each point $p \in \|\mathrm{ssd}(\mathsf{B}(G))\|$ towards $\|\mathrm{cn}^2\|(p)$ with uniform speed, which is $\mathbb{Z}_2$-equivariant at any time of the deformation.

To show that $\|\mathsf{HDL}(G)\|$ is a strong $\mathbb{Z}_2$-deformation retract of $\|\mathsf{DL}(G)\|$, we define simplicial complexes and simplicial $\mathbb{Z}_2$-maps

$$\mathsf{DL}(G) =: S_0 \xrightarrow{f_0} S_1 \xrightarrow{f_1} \dots \xrightarrow{f_N} S_{N+1} := \mathsf{HDL}(G),$$

such that S_{i+1} is a $\mathbb{Z}_2$-subcomplex of S_i and S_{i+1} is a strong $\mathbb{Z}_2$-deformation retract of S_i. The composition of the f_i yields the earlier defined map j, i.e.,

$$j = f_N \circ \cdots \circ f_1 \circ f_0.$$

To construct S_{i+1} inductively from S_i, we consider

$$X := \max_{\prec} \{Y \in J \;\mid Y \uplus \emptyset \in S_i\},$$

and obtain S_{i+1} from S_i by deleting each simplex of S_i that contains $X \uplus \emptyset$ or its $\mathbb{Z}_2$-partner $\emptyset \uplus X$, that is,

$$S_{i+1} := \{\sigma \mid \sigma \in S_i \text{ and } X \uplus \emptyset \notin \sigma \text{ and } \emptyset \uplus X \notin \sigma\}.$$

The maximality of X implies that a maximal simplex which contains $X \uplus \emptyset$ (resp. $\emptyset \uplus X$) does also contain $\emptyset \uplus \mathrm{CN}(X)$ (resp. $\mathrm{CN}(X) \uplus \emptyset$). Hence the map f_i defined on the vertices $v \in \mathrm{V}(S_i)$ via

$$f_i(v) := \begin{cases} \emptyset \uplus \mathrm{CN}(X) & \text{if } v = X \uplus \emptyset, \\ \mathrm{CN}(X) \uplus \emptyset & \text{if } v = \emptyset \uplus X, \\ v & \text{otherwise.} \end{cases}$$

It is simplicial and $\mathbb{Z}_2$-equivariant.

Thus $F\colon \|S_i\| \times [0,1] \to \|S_i\|$ given by $F(x,t) := t \cdot x + (1-t) \cdot \|f_i\|(x)$ is a well-defined $\mathbb{Z}_2$-homotopy from $\|f_i\|$ to $\mathrm{Id}_{\|S_i\|}$ that fixes $\|S_{i+1}\|$. $\square$

We end this section with a construction of a $\mathbb{Z}_2$-map $\mathsf{HDL}(f)$ between $\mathsf{HDL}(G)$ and $\mathsf{HDL}(H)$ if we are given a graph homomorphism $f : G \to H$. Once we have chosen the partial orders that define the maps j_G and j_H that give $\mathsf{HDL}(G)$ and $\mathsf{HDL}(H)$, we simply compose the following simplicial $\mathbb{Z}_2$-maps:

- The inclusion $\iota : \mathsf{HDL}(G) \to \mathrm{ssd}(\mathsf{B}(G))$,
- the map $\mathrm{ssd}(\mathsf{B}(f)) : \mathrm{ssd}(\mathsf{B}(G)) \to \mathrm{ssd}(\mathsf{B}(H))$ canonically induced from f,
- the map $\mathrm{cn}^2 : \mathrm{ssd}(\mathsf{B}(H)) \to \mathsf{DL}(H)$, and
- the map $j_H : \mathsf{DL}(H) \to \mathsf{HDL}(H)$.

More precisely, the simplicial $\mathbb{Z}_2$-map $\Psi : \mathsf{HDL}(G) \to \mathsf{HDL}(H)$ is defined by:

$$\Psi := j_H \circ \mathrm{cn}^2 \circ \mathrm{ssd}(\mathsf{B}(f)) \circ \iota.$$

Since the halved doubled Lovász complex $\mathsf{HDL}(G)$ is $\mathbb{Z}_2$-isomorphic to the original Lovász complex $\mathsf{L}(G)$, this map can be interpreted as a simplicial $\mathbb{Z}_2$-map $\mathsf{L}(f)$ between $\mathsf{L}(G)$ and $\mathsf{L}(H)$. This construction is significantly simpler than the construction of the $\mathbb{Z}_2$-map $\mathsf{L}(f) : \mathsf{L}(G) \to \mathsf{L}(H)$ described by Walker, [Wal83].

2.2 The connection between the box complexes $\mathsf{B}_{\mathcal{C}}(G)$, $\mathsf{B}_0(G)$ and $\mathsf{B}(G)$

In this section we will prove that $\mathsf{B}_0(G)$ and $\operatorname{susp}(\mathsf{B}(G))$ are $\mathbb{Z}_2$-homotopy equivalent. The reason is that the box complex is 'nearly' $\operatorname{Hom}(K_2, G) \times [0,1]$.

Theorem 2.6. *$\mathsf{B}_{\mathcal{C}}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to* $\operatorname{susp}(\mathsf{B}(G))$.

Proof. It follows from Lemma 2.8 and 2.9. □

Remark 2.7. *One can use Lovász's bound to prove Kneser's conjecture* [Kne55]. *The box complexes of Kneser graphs (and Schrijver graphs) are tidy spaces* [Lov78] *(spheres up to homotopy* [BL03] *for Schrijver graphs). This means that one can prove Kneser's conjecture by using Sarkaria's bound (or any higher suspension of the box complex) as well.*

Lemma 2.8. *$\mathsf{B}_{\mathcal{C}}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to $\mathsf{B}_0(G)$.*

Proof. $\mathsf{B}_{\mathcal{C}}(G)$ was obtained from $\mathsf{B}(G)$ by attaching two cones C_1, C_2 over the shores, while $\mathsf{B}_0(G)$ is $\mathsf{B}(G)$ plus two simplices Δ_1, Δ_2 covering the shores.
We consider the following two quotient CW-complexes. $(\mathsf{B}_{\mathcal{C}}(G)/C_1)/C_2$ and $(\mathsf{B}_0(G)/\Delta_1)/\Delta_2$ (the order of the factorization does not matter since we collapse disjoint subcomplexes). It is obvious that they are the same CW-complexes and since C_i, Δ_i are contractible subcomplexes $\mathsf{B}_{\mathcal{C}}(G)$ and $\mathsf{B}_0(G)$ are $\mathbb{Z}_2$-homotopy equivalent. □

Lemma 2.9. *$\mathsf{B}_{\mathcal{C}}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to* $\operatorname{susp}(\mathsf{B}(G))$.

Proof. $\mathsf{B}_{\mathcal{C}}(G)$ is a subcomplex of $\operatorname{susp}(\mathsf{B}(G))$. The idea of the proof is to start with $\operatorname{susp}(\mathsf{B}(G))$, and get rid of the extra simplexes one by one (using deformation retraction) such that finally we get $\mathsf{B}_{\mathcal{C}}(G)$. We will work with one cone (half) of the suspension. Since we want a $\mathbb{Z}_2$-retraction, on the other cone we have to do the $\mathbb{Z}_2$-pair of each step.
Let x be the apex of the cone over the first shore in $\operatorname{susp}(\mathsf{B}(G))$ (y is the other apex). We will define (by induction) sequences of simplicial complexes such that

$$\operatorname{susp}(\mathsf{B}(G)) =: X_0 \supset X_1 \supset \cdots \supset X_N = \mathsf{B}_{\mathcal{C}}(G),$$

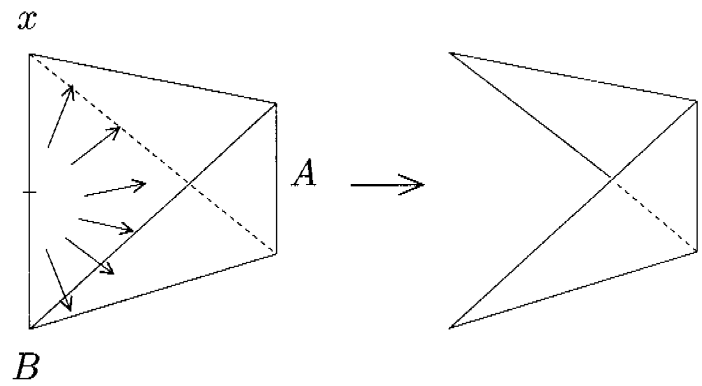
and X_{i+1} is a $\mathbb{Z}_2$-deformation retraction of X_i.

Let assume that we already defined X_n. We choose a simplex $\sigma \in X_n$ such that

1. $x \in \sigma$, and the rest of the vertices of σ are from the second shore,
2. no other simplex in X_n containing x has *more* vertices from the second shore, and it has at least one vertex from the second shore.

The vertex set of σ will be $\{x, \emptyset \uplus b_{j_1}, \ldots, \emptyset \uplus b_{j_{l-1}}\}$ for some $B = \{b_{j_1}, \ldots, b_{j_{l-1}}\} \subseteq V(G)$. Let $A := \mathrm{CN}(B) = \{a_{i_1}, \ldots, a_{i_k}\}$ and $\tilde{\sigma}$ be the $\mathbb{Z}_2$-pair of σ with vertex set $\{y, b_{j_1} \uplus \emptyset, \ldots, b_{j_{l-1}} \uplus \emptyset\}$. We are ready to define X_{n+1}:

$$X_{n+1} := X_n \setminus \{\tau \in X_n : \sigma \in \tau \text{ or } \tilde{\sigma} \in \tau\}.$$



We have to only show that X_{n+1} is the deformation retract of X_n. We know the local structure of our complex X_n around σ. Let assume that it is a face of a bigger simplex Δ with vertex set $\{x, \emptyset \uplus b_{j_1}, \ldots, \emptyset \uplus b_{j_{l-1}}, c\}$. c can not be the other apex. If c were from the second shore, then we would choose Δ instead of σ to define X_{n+1}. So c can be only from the first shore and then $c \in A$. This means that σ is on the boundary of X_n; it is on the boundary of the simplex s with vertex set $\{x, \emptyset \uplus b_{j_1}, \ldots, \emptyset \uplus b_{j_{l-1}}, a_{i_1} \uplus \emptyset, \ldots, a_{i_k} \uplus \emptyset\}$. Moreover every simplex which has σ as face is on the boundary of X_n. So what we delete to get X_{n+1} is on the boundary (except s). The retraction[1]

[1] This deformation retraction of the simplex $\{v_1 = a_{i_1} \uplus \emptyset, \ldots, v_k = a_{i_k} \uplus \emptyset, w_1 = \emptyset \uplus b_{j_1}, \ldots, w_{l-1} = \emptyset \uplus b_{j_{l-1}}, w_l := x\}$ can be explicitly given by:

$$h_t\left(\sum t_i v_i + \sum s_j w_j\right) = \sum \left(\frac{l \cdot t}{k} + t_i\right) v_i + \sum (s_j - t)\, w_j,$$

where $\sum t_i + \sum s_j = 1$. It starts with $h_0 = id$, and ends (for a particular point), just when the first coefficient of w_j become zero. This retraction 'kills' those simplices, which has as a face the simplex $\{w_1, \ldots, w_l\}$, and retracts the 'interior' points to the remaining simplices.

to X_{n+1} can be given as indicated on the picture. □

2.3 Universality of graph complexes

In this section we prove that any $\mathbb{Z}_2$-simplicial complex can be a graph complex up to $\mathbb{Z}_2$-homotopy. First we start with the homotopy version which we extend to $\mathbb{Z}_2$-homotopy in the next subsection (Subsection 2.3.2).

2.3.1 Neighborhood complex

We consider the following natural question about the neighborhood complex. Given a simplicial complex K, is there a graph G such that its neighborhood complex is the given complex, $\mathsf{N}(G) = \mathsf{K}$?

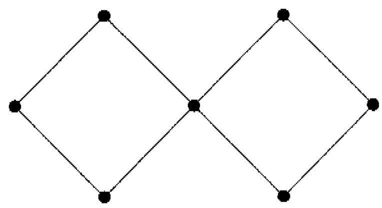


Figure 2.4: K.

For example, if K is the complex on Figure 2.4 then the answer is *no*! The reason is that there is a topological obstruction. The neighborhood complex is homotopy equivalent to the box complex which is a free $\mathbb{Z}_2$-simplicial complex so it has clearly even Euler characteristic. But $\chi(\mathsf{K}) = -1$ is odd.

Another example if K is the complex of Figure 2.5. Now the answer is no again, but there is no topological reason. With the usual antipodal map K becomes a free $\mathbb{Z}_2$-simplicial complex. On the other hand the graph G with $\mathsf{N}(G) = \mathsf{K}$ should have 4 vertices, and by brute force one can check that K is not a neighborhood complex.

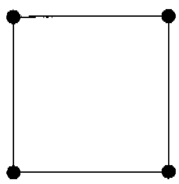


Figure 2.5: K.

Unfortunately we can not answer this question, but we will show that up to homotopy everything is possible.

Theorem 2.10. *Given a free $\mathbb{Z}_2$-simplicial complex (K, ν), there is a graph G such that its neighborhood complex is homotopy equivalent to the given complex,* $\mathsf{N}(G) \simeq \mathsf{K}$.

In order to prove it we will use the following construction of a graph from a $\mathbb{Z}_2$-simplicial complex. Note that the construction does depend only on the 1-skeleton of the $\mathbb{Z}_2$-complex (K, ν). In order to prove Theorem 2.10 we will use the barycentric subdivision of K. The graph complex of $G_{\mathrm{sd}(\mathsf{K})}$ is K up to homotopy.

Construction 2.11 ($\mathsf{K} \to G_{\mathsf{K}}$)**.** *Let* K *be a $\mathbb{Z}_2$-simplicial complex. The vertices of G_{K} are the vertices of* K*, and each vertex is connected to its $\mathbb{Z}_2$-pair and the neighbors*[2] *of the $\mathbb{Z}_2$-pair. Thus if $x, y \in V(G_{\mathsf{K}}) = V(\mathsf{K})$ then there is an edge between them if and only if $\nu(x) = y$ or $\{x, \nu(y)\} \in$* K *(or $\{y, \nu(x)\} \in \mathsf{K}$). An example is in Figure 2.6.*

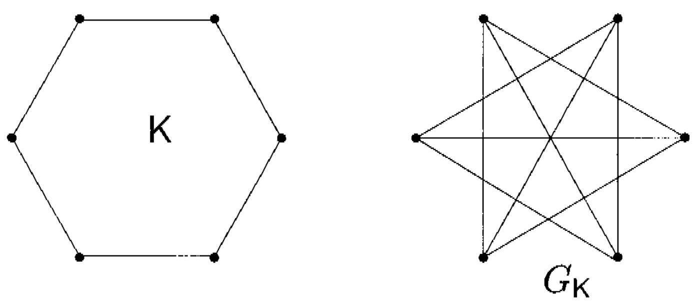


Figure 2.6: Example for the construction.

We need the *nerve theorem* as well.

Definition 2.12 (nerve). Let $\mathcal{F}$ be a set-system. The *nerve* $\mathcal{N}(\mathcal{F})$ of $\mathcal{F}$ is defined as the simplicial complex whose vertices are the sets in $\mathcal{F}$, and $\{X_1, \dots, X_r\} \in \mathcal{N}(\mathcal{F})$ if and only if $X_1, \dots, X_r \in \mathcal{F}$ and $X_1 \cap X_2 \cap \dots \cap X_r \neq \emptyset$.

Theorem 2.13 (nerve theorem). *Let* K *be a simplicial complex and K_i $(i \in I)$ a family of subcomplexes such that $\mathsf{K} = \bigcup_{i \in I} \mathsf{K}_i$. Assume that every nonempty finite intersection $\mathsf{K}_{i_1} \cap \dots \cap \mathsf{K}_{i_r}$ is contractible. Then* K *and the nerve $\mathcal{N}(\bigcup \mathsf{K}_i)$ are homotopy equivalent.*

Proof of Theorem 2.10. For technical reason we need the first barycentric subdivision $\mathrm{sd}(\mathsf{K})$ of K. The free simplicial $\mathbb{Z}_2$-action on $\mathrm{sd}(\mathsf{K})$ will be denoted by ν as well.

[2]in the 1-skeleton of K

We use Construction 2.11 with $\mathrm{sd}(\mathsf{K})$ to obtain $G_{\mathrm{sd}(\mathsf{K})}$. Because of the barycentric subdivision the vertices of $G_{\mathrm{sd}(\mathsf{K})}$ denoted by subsets of $V(\mathsf{K})$. If $A, B \in V(G_{\mathrm{sd}(\mathsf{K})})$ then there is an edge between them if and only if $\nu(A){=}B$ or $\nu(A){\subset}B$ or $\nu(A){\supset}B$.

We denote the vertices of K by $1, 2, \ldots, n$. Let $\mathrm{star}_{\mathrm{sd}(\mathsf{K})}(A)$ be the star[3] of the vertex A in $\mathrm{sd}(\mathsf{K})$. The *nerve* of the set system $\{\mathrm{star}_{\mathrm{sd}(\mathsf{K})}(A) \colon A \in V(G_{\mathrm{sd}(\mathsf{K})})\}$ is clearly the neighborhood complex of $G_{\mathrm{sd}(\mathsf{K})}$. (This is even true without any subdivision: $N(G_{\mathsf{K}}) = \mathcal{N}(\mathcal{S})$ where $\mathcal{S}$ is the set of the vertex stars in K.)

We want to use the *nerve* theorem so we should prove that if $B \in \mathrm{star}_{\mathrm{sd}(\mathsf{K})}(A_1) \cap \cdots \cap \mathrm{star}_{\mathrm{sd}(\mathsf{K})}(A_r) \neq \emptyset$ then this intersection is contractible. We show that this is a cone. We have two cases:

1. If $A_i \subset B$ for all $i = 1, 2, \ldots, r$:
   In this case $\cup A_i$ is a vertex of the barycentric subdivision since it is a subset of B, and it is in the intersection as well. We show that the intersection can be contracted to this point. We construct this deformation retraction by letting each vertex to travel towards $\cup A_i$ with uniform speed. The only thing that we have to check is that whenever $B_1 \subset B_2 \subset \cdots \subset B_q$ is a simplex in the intersection, then with the special vertex $X := \cup A_i$ they form a simplex as well. First observe that there is an edge between X and B_l, $l \in \{1, \ldots, q\}$. If $B_l \subset A_i$ for some i then $B_l \subset X$ as well. Otherwise $X \subset B_l$. For the simplex $B_1 \subset B_2 \subset \cdots \subset B_q$ if $X \subset B_1$ or $X \supset B_q$ then they form a simplex with X. Otherwise there is an index k such that $B_k \subset X \subset B_{k+1}$. This means that $B_1, B_2, \ldots, B_q, X$ form a simplex.
2. If $B \subset A_{i_j}$ for some $j = 1, \ldots, k$ ($k \geq 1$), and $A_i \subset B$ for the rest:
   In this case $B \subset \bigcap_{j=1}^{k} A_{i_j} \neq \emptyset$ [4] is a vertex of the barycentric subdivision and the intersection as well. We show that the intersection can be contracted to this point. We construct this deformation retraction by letting each vertex to travel towards $\cap A_{i_j}$ with uniform speed. We have to show that whenever $B_1 \subset B_2 \subset \cdots \subset B_q$ is a simplex in the intersection, then with the special vertex $X := \cap A_{i_j}$ they form a simplex as well. First observe that there is an edge between X and B_l, $l \in \{1, \ldots, q\}$. If $B_l \supset A_{i_j}$ for some i_j

[3] The star of $\sigma \in \mathsf{K}$: $\mathrm{star}_{\mathsf{K}}(\sigma) = \{\tau \in \mathsf{K} \colon \tau \cup \sigma \in \mathsf{K}\}$

[4] $B \supset \bigcup_{A_i \subset B} A_i$ would be good as well, but it can be the emptyset.

then $B_l \supset X$ as well. Otherwise $X \supset B_l$. For the simplex $B_1 \subset B_2 \subset \cdots \subset B_q$ if $X \subset B_1$ or $X \supset B_q$ then it is true. Otherwise there is an index k such that $B_k \subset X \subset B_{k+1}$ which means that $B_1, B_2, \ldots, B_q, X$ form a simplex.

This completes the proof. $\square$

2.3.2 Box complex: $\mathbb{Z}_2$-universality

In this section we prove the universality theorem. It is the $\mathbb{Z}_2$-extension of Theorem 2.10. These results were already announced by Matoušek and Ziegler [MZ04] (arXiv:math.CO/0208072v2). Later it (and the result from Section 2.2, Theorem 2.6) was proven by Živaljević [Živ04].

Theorem 2.14. *Given a free $\mathbb{Z}_2$-simplicial complex (K, ν), there is a graph G such that its box complex $\mathsf{B}(G)$ is $\mathbb{Z}_2$-homotopy equivalent to the given complex.*

First we need the $\mathbb{Z}_2$-carrier lemma.

Definition 2.15 ($\mathbb{Z}_2$-carrier). Let (K, ν) be a $\mathbb{Z}_2$-simplicial complex and (T, μ) a $\mathbb{Z}_2$-space. A function C taking faces σ of K to subspaces $C(\sigma)$ of T, satisfying $C(\nu(\sigma)) = \mu(C(\sigma))$, is a $\mathbb{Z}_2$-*carrier* if $C(\sigma) \subseteq C(\tau)$ for all $\sigma \subseteq \tau$.

Lemma 2.16 ($\mathbb{Z}_2$-carrier lemma). *Assume that for a $\mathbb{Z}_2$-carrier C for any $\sigma \in \mathsf{K}$ $C(\sigma)$ is contractible. Then any two $\mathbb{Z}_2$-maps $f, g \colon \mathsf{K} \to T$ that are both carried by C are $\mathbb{Z}_2$-homotopic.*

Proof. We proceed similarly as the proof of Theorem II.9.2 in [LW69]. We will construct by cell induction the required homotopy $F \colon \mathsf{K} \times [0,1] \to T$. For a vertex $v \in \mathsf{K}$ since $f(v), g(v) \in C(v)$ and $C(v)$ contractible we can define F on $v \times [0,1]$ such that $F(v \times [0,1]) \subseteq C(v)$. On $\nu(v) \times [0,1]$ we do everything in order to obtain $\mathbb{Z}_2$-homotopy. Assume that $\sigma \in \mathsf{K}$ is a minimal simplex such that F is not defined on $\sigma \times [0,1]$ yet. For a face $\tau \subseteq \sigma$ we have by the induction hypothesis that $F(\tau \times [0,1]) \subseteq C(\tau) \subseteq C(\sigma)$ So on the boundary of $\sigma \times [0,1]$ F is defined, and since $C(\sigma)$ is contractible we can extend F such that

$F(\sigma \times [0,1]) \subseteq C(\sigma)$. On $\nu(\sigma) \times [0,1]$ again we do everything in order to obtain $\mathbb{Z}_2$-homotopy. $\square$

Proof of Theorem 2.14. We will use the same notations as in the proof of Theorem 2.10. Similarly we obtain $G_{\mathrm{sd}(\mathsf{K})}$ by using Construction 2.11 with $\mathrm{sd}(\mathsf{K})$. We need to show that the box complex $\mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$ and (K, ν) are $\mathbb{Z}_2$-homotopy equivalent. In order to prove it we will define $\mathbb{Z}_2$-maps $f\colon \mathrm{sd}(\mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})) \to \mathrm{sd}(\mathsf{K})$ and $g\colon \mathrm{sd}(\mathsf{K}) \to \mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$. To complete the proof we will show that f (and g) is a $\mathbb{Z}_2$-homotopy equivalence.

The definition of g: This is an embedding. We map a vertex $A \in \mathrm{sd}(\mathsf{K})$ to $A \uplus \emptyset \in \mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$ and of course its $\mathbb{Z}_2$-pair $\nu(A) \in \mathrm{sd}(\mathsf{K})$ to $\emptyset \uplus A \in \mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$. Here we had to choose! If we pick $\nu(A)$ first than we mapped $\nu(A)$ to $\nu(A) \uplus \emptyset$ and A to $\emptyset \uplus \nu(A)$. So we have 2 choices for any $\mathbb{Z}_2$-pair $A, \nu(A)$. This defines a $\mathbb{Z}_2$-map g on the vertex level. We have to show that g is simplicial. Let $A_1 \subset \cdots \subset A_l$ be a simplex σ in $\mathrm{sd}(\mathsf{K})$. Since $A_1 \uplus \emptyset, \ldots, A_l \uplus \emptyset$, $\emptyset \uplus \nu(A_1), \ldots, \emptyset \uplus \nu(A_l)$ form a simplex in $\mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$ the image of σ is a simplex. (In $G_{\mathrm{sd}(\mathsf{K})}$ A_i is connected to $\nu(A_i)$ and since $A_i \subset A_j$ or $A_i \supset A_j$ it is connected to $\nu(A_j)$ as well. So $G_{\mathrm{sd}(\mathsf{K})}[\{A_1, \ldots, A_l\}; \{\nu(A_1), \ldots, \nu(A_l)\}]$ is complete bipartite.)

The definition of f: Let $A_1 \uplus \emptyset, \ldots, A_l \uplus \emptyset$, $\emptyset \uplus B_1, \ldots, \emptyset \uplus B_k$ be the vertices of a simplex σ in $\mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$. $G_{\mathrm{sd}(\mathsf{K})}[\mathcal{A}; \mathcal{B}]$ is complete bipartite where $\mathcal{A} := \{A_1, \ldots, A_l\}$ and $\mathcal{B} := \{B_1, \ldots, B_k\}$. This means that $\mathcal{A} \subset \mathrm{star}_{\mathrm{sd}(\mathsf{K})}\nu(B_j)$ for any $j \in \{1, \ldots, k\}$ so $\mathcal{A} \subset \bigcap_{j=1}^{k} \mathrm{star}_{\mathrm{sd}(\mathsf{K})}\nu(B_j)$. From the proof of Theorem 2.10 we know that $\bigcap_{j=1}^{k} \mathrm{star}_{\mathrm{sd}(\mathsf{K})}\nu(B_j)$ is a cone with apex X. Since $\mathcal{A}, \nu(\mathcal{B}) \subset \mathrm{star}_{\mathrm{sd}(\mathsf{K})}X$ we have that $Y := \bigcap_{i=1}^{l} \mathrm{star}_{\mathrm{sd}(\mathsf{K})}A_i \bigcap \bigcap_{j=1}^{k} \mathrm{star}_{\mathrm{sd}(\mathsf{K})}\nu(B_j) \neq \emptyset$. From the proof of Theorem 2.10 we know that Y is a cone. We denote its apex by $X_{\mathcal{A}}^{\mathcal{B}}$ which can be chosen to be $\bigcap_{i=1}^{l} A_i \bigcap \bigcap_{j=1}^{k} \nu(B_j)$ if it is not the empty set. Now we are able to define f.

$$f(\mathcal{A} \uplus \mathcal{B}) := \begin{cases} \bigcap_{i=1}^{l} A_i \bigcap \bigcap_{j=1}^{k} \nu(B_j) & \text{if exist,} \\ X_{\mathcal{A}}^{\mathcal{B}} & \text{otherwise.} \end{cases}$$

By the construction it is $\mathbb{Z}_2$ on the vertex level. (We can choose $X_{\mathcal{B}}^{\mathcal{A}} :=$

$\nu(X_{\mathcal{A}}^{\mathcal{B}})$.) It is simplicial. An edge with two vertices $\mathcal{A} \uplus \mathcal{B}$ and $\tilde{\mathcal{A}} \uplus \tilde{\mathcal{B}}$ ($\tilde{\mathcal{A}} \subset \mathcal{A}$, $\tilde{\mathcal{B}} \subset \mathcal{B}$) is mapped to two vertices $S \subset R$ since $X_{\mathcal{A}}^{\mathcal{B}}$ is in the cone of $X_{\tilde{\mathcal{A}}}^{\tilde{\mathcal{B}}}$. Now a simplex is mapped to a chain (since every two vertex is comparable by inclusion).

Next we prove that $f \circ \mathrm{sd}(g)\colon \mathrm{sd}(\mathrm{sd}(\mathsf{K})) \to \mathrm{sd}(\mathsf{K})$ is $\mathbb{Z}_2$-homotopic to Id_{K}. We will use the $\mathbb{Z}_2$-carrier lemma. We have to construct 'only' a contractible $\mathbb{Z}_2$-carrier for $f \circ \mathrm{sd}(g)$ and Id. The image of the vertex $v = \{A_1, \dots, A_l\}$, $A_1 \subset \cdots \subset A_l$ is $\mathrm{sd}(g)(v) = \{A_{i_1}, \dots, A_{i_s}\} \uplus \{\nu(A_{j_1}), \dots, \nu(A_{j_r})\}$. And now $f(\mathrm{sd}(g)(v)) = A_1 \cap \cdots \cap A_l = A_1$ in this case! The image of a simplex with vertex set $\{A_{i_1}\}, \{A_{i_1}, A_{i_2}\}, \dots, \{A_{i_1}, \dots, A_{i_l}\}$ is a face of the simplex $A_1 \subset \cdots \subset A_l$. So for a simplex $\sigma \in \mathrm{sd}(\mathrm{sd}(\mathsf{K}))$ with its maximal vertex $\{A_1, \dots, A_l\}$ we define $C(\sigma) := \{A_1, \dots, A_l\} \in \mathrm{sd}(\mathsf{K})$. This C is a contractible $\mathbb{Z}_2$-carrier what we need. $f \circ \mathrm{sd}(g)$ and Id_{K} are $\mathbb{Z}_2$-homotopic.

Now we show that $g \circ f\colon \mathrm{sd}(\mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})) \to \mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$ is $\mathbb{Z}_2$-homotopic to Id. Again we construct a contractible $\mathbb{Z}_2$-carrier for $g \circ f$ and Id. A vertex $\mathcal{A} \uplus \mathcal{B}$ is mapped to $X_{\mathcal{A}}^{\mathcal{B}}$ by f and to $X_{\mathcal{A}}^{\mathcal{B}} \uplus \emptyset$ or $\emptyset \uplus \nu(X_{\mathcal{A}}^{\mathcal{B}})$ by $g \circ f$. Let $\mathcal{A}_1 \uplus \mathcal{B}_1, \dots, \mathcal{A}_n \uplus \mathcal{B}_n$ the vertex set of a simplex σ in $\mathrm{sd}(\mathsf{B}(G_{\mathrm{sd}(\mathsf{K})}))$. ($\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n$, $\mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n$, $\mathcal{A}_n := \{A_1, \dots, A_l\}$ and $\mathcal{B}_n := \{B_1, \dots, B_k\}$). We consider the subgraph H of $G_{\mathrm{sd}(\mathsf{K})}$ spanned by $A_1, \dots, A_l, B_1, \dots, B_k$, their $\mathbb{Z}_2$-image under ν and $X_{\mathcal{A}_i}^{\mathcal{B}_i}, \nu(X_{\mathcal{A}_i}^{\mathcal{B}_i})$ for any $i \in \{1, \dots, n\}$. We will use H (actually $\mathsf{B}(H)$) to define the desired carrier. First of all $\mathsf{B}(H)$ contains the simplex with vertex set $A_1 \uplus \emptyset, \dots, A_l \uplus \emptyset$, $\emptyset \uplus B_1, \dots, \emptyset \uplus B_k$ which contains σ. Moreover we defined H in such a way that $\mathsf{B}(H)$ contains $(g \circ f)(\sigma)$ as well. Observe that H is bipartite. The neighbors of the vertices $X_{\mathcal{A}_n}^{\mathcal{B}_n}$ and $\nu(X_{\mathcal{A}_n}^{\mathcal{B}_n})$ provides a partition of the vertex set of H. The neighborhood complex $\mathsf{N}(H)$ is the disjoint union of two simplices corresponding to this partition. So the box complex $\mathsf{B}(H) \subset \mathsf{B}(G_{\mathrm{sd}(\mathsf{K})})$ contains two disjoint contractible sets (since it is homotopy equivalent to $\mathsf{N}(H)$). One of these sets covers σ and $(g \circ f)(\sigma)$, so we define our contractible $\mathbb{Z}_2$-carrier $C(\sigma)$ to be this 'half' of $\mathsf{B}(H)$. $\square$

Remark 2.17. *For any free $\mathbb{Z}_2$-simplicial complex* (K, ν) *there is a graph G such that its Hom complex* $\mathrm{Hom}(K_2, G)$ *is $\mathbb{Z}_2$-homotopy equivalent to the given complex, since the box complex* $\mathsf{B}(G)$ *is $\mathbb{Z}_2$-homotopy equivalent to* $\mathrm{Hom}(K_2, G)$. *(The $\mathbb{Z}_2$-maps* $f\colon \mathrm{sd}(\mathsf{B}(G)) \to$

$\mathrm{sd}(\mathrm{Hom}(K_2, G))$ *defined by*

$$A \uplus B \to \begin{cases} (A, \mathrm{CN}(A)) & \textit{if } B = \emptyset, \\ (\mathrm{CN}(B), B) & \textit{if } A = \emptyset, \\ (A, B) & \textit{otherwise}, \end{cases}$$

and $g\colon \mathrm{sd}(\mathrm{Hom}(K_2, G)) \to \mathrm{sd}(\mathsf{B}(G))$ *given by* $(A, B) \to A \uplus B$ *are* $\mathbb{Z}_2$-*homotopy equivalences.* $f \circ g = \mathrm{Id}$ *and* $g \circ f$ *is carried by* Id.*)*

We have seen in Section 2.1.1 that the Lovász complex $\mathsf{L}(G)$ *is* $\mathbb{Z}_2$-*homotopy equivalent to the box complex* $\mathsf{B}(G)$, *so the Lovász complex and the Hom complex* $\mathrm{Hom}(K_2, G)$ *are universal as well.*

It would be interesting to extend this universality result for Hom complexes in general. For example on $\mathrm{Hom}(C_5, G)$ the dihedral group $D_5 = \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ (the symmetry group of the regular pentagon) acts freely. So is it true that every free D_5-space can be $\mathrm{Hom}(C_5, G)$ for some graph G?
Let P_3 the path on 3 vertices. The spaces $\mathrm{Hom}(P_3, G)$ shows some difficulty around the extension of the universality theorem. The space $\mathrm{Hom}(P_3, K_3)$ has only a free $\mathbb{Z}_3$-action, while the complex $\mathrm{Hom}(P_3, P_3)$ is a disjoint union of a square and an interval. So $\mathrm{Hom}(P_3, P_3)$ has no free action at all. One could suspect that $\mathrm{Hom}(P_3, G)$ could be any finite simplicial complex up to homotopy. Using the notation of folding (see Section 2.5) and the result of Babson and Kozlov [BK03], we have that $\mathrm{Hom}(P_3, G)$ is homotopy equivalent to $\mathrm{Hom}(K_2, G)$. So our universality theorem says that $\mathrm{Hom}(P_3, G)$ or more general $\mathrm{Hom}(H, G)$, assuming that H folds to an edge, can be any finite $\mathbb{Z}_2$-simplicial complex up to homotopy.

2.4 The G^+ construction

It is well known (see [Wal83]) that the topological lower bound for the chromatic number can be arbitrarily bad. But now we will be able to give purely topological examples (see Section 5.2).

Definition 2.18. For a graph G let G^+ be the graph obtained from G by adding an extra vertex w and connecting it by edges to all the vertices of G, i.e., $V(G^+) = V(G) \cup \{w\}$ and $E(G^+) = E(G) \cup \{\{v, w\}\colon v \in V(G)\}$.

Remark 2.19. *G^+ is a special case of the Mycielski construction: $G^+ = M_1(G)$. We will reprove the following lemma in Section 2.6.*

Lemma 2.20. *$\mathsf{B}(G^+)$ is $\mathbb{Z}_2$-homotopy equivalent to* $\operatorname{susp}(\mathsf{B}(G))$.

Proof. $\operatorname{susp}(\mathsf{B}(G))$ is a subcomplex of $\mathsf{B}(G^+)$. The difference is only two big simplices (and some of their faces) $V(G) \uplus w$ and $w \uplus V(G)$. We will get rid of the extra simplices one by one using deformation retraction. We will work with one shore, on the other shore we have to do the $\mathbb{Z}_2$-pair of each step.

We will define (by induction) sequences of simplicial complexes such that

$$\mathsf{B}(G^+) =: X_0 \supset X_1 \supset \cdots \supset X_N = \operatorname{susp}(\mathsf{B}(G)),$$

and X_{i+1} is a $\mathbb{Z}_2$-deformation retraction of X_i.

Let assume that we already defined X_n. We choose $A \subseteq V(G)$ such that $A \uplus w \in X_n$, and there is no $A \subset B \subseteq V(G)$ such that $B \uplus w \in X_n$. We define X_{n+1}:

$$X_{n+1} := X_n \setminus \{A \uplus w, w \uplus A, A \uplus \emptyset, \emptyset \uplus A\}.$$

By the definition of X_{n+1} it is clearly a $\mathbb{Z}_2$-deformation retract of X_n since $A \uplus \emptyset$ is on the boundary of X_n. (Map the barycenter of $A \uplus \emptyset$ to $\emptyset \uplus w$.) ☐

2.5 Folding

We will show that folding in the second parameter of the homomorphism complex yields a homotopy equivalence. In the next section we will use this. There we will study how the so called Mycielski construction in the second parameter changes the homotopy type of the homomorphism complex.

Definition 2.21. $G - v$ is called a *fold* of a graph G if there exist $u \in V(G)$, $u \neq v$ such that $\mathrm{N}(u) \supseteq \mathrm{N}(v)$.

So far we only considered special Hom complexes the $\operatorname{Hom}(K_2, G)$ complexes. Now we will work with the general $\operatorname{Hom}(G, H)$ complexes.

It was proven in [BK03, Proposition 5.1] that folds in the first parameter yield homotopy equivalence. It was noticed in [ČK04, Lemma 3.1] that one can fold in the second parameter if the deleted vertex is an identical twin. Now we will show that the fold in the second parameter is a homotopy equivalence in general. This statement was generalized by Kozlov [Koz05a] into a simple homotopy equivalence. Note that now graphs can have loops as well. The proof works in that generality.

Theorem 2.22. *Let G and H be graphs and $u, v \in V(H)$ such that $\mathrm{N}(u) \supseteq \mathrm{N}(v)$. Also, let $i : H - v \hookrightarrow H$ be the inclusion and $\omega : H \to H - v$ the unique graph homomorphism which maps v to u and fixes other vertices. Then, these two maps induce homotopy equivalences $i_H : \mathrm{Hom}(G, H - v) \to \mathrm{Hom}(G, H)$ and $\omega_H : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, H - v)$, respectively.*

Proof. We will show that ω_H satisfies the conditions (A) and (B) of Lemma 1.15. Unfolding definitions, we see that for a cell of $\mathrm{Hom}\,(G, H)$, $\tau : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, we have

$$\omega_H(\tau)(x) = \begin{cases} \tau(x) & \text{if } v \notin \tau(x), \\ (\tau(x) \cup \{u\}) \setminus \{v\} & \text{otherwise.} \end{cases}$$

Let η be a cell of $\mathrm{Hom}(G, H - v)$, $\eta : V(G) \to 2^{V(H)\setminus\{v\}} \setminus \{\emptyset\}$. Then $\omega_H^{-1}(\eta)$ is a set of all η' such that, for all $x \in V(G)$,

1. $\eta'(x) = \eta(x)$, if $u \notin \eta(x)$;
2. or if $u \in \eta(x)$ then (at least theoretically) we have the following possibilities:
   (a) $\eta'(x) = \eta(x)$,
   (b) $\eta'(x) = \eta(x) \setminus \{u\} \cup \{v\}$,
   (c) $\eta'(x) = \eta(x) \cup \{v\}$.

Because of the condition $\mathrm{N}(u) \supseteq \mathrm{N}(v)$, not all η' satisfying $2.(b), 2.(c)$ have to belong to $\mathrm{Hom}\,(G, H)$. Note that if H is simple and $u \in \eta(x)$ and $(x, y) \in E(G)$ then $u \notin \eta(y)$. But this is not true in general. This means that for any x it depends not only on $\mathrm{N}(v)$ that we can use $2.(b), 2.(c)$ to get $\eta' \in \mathrm{Hom}\,(G, H)$. It depends on the choices of $\eta'(y)$ at the neighbors of x.

The map $\varphi\colon \omega_H^{-1}(\eta) \to \omega_H^{-1}(\eta)$ is defined by

$$\varphi(\zeta)(x) = \begin{cases} \zeta(x) & \text{if } u \in \zeta(x), \\ \zeta(x) & \text{if } u, v \notin \zeta(x), \\ \zeta(x) \cup \{u\} & \text{if } u \notin \eta(x) \text{ and } v \in \eta(x), \end{cases}$$

for all $x \in V(G)$. We show that φ is a homotopy equivalence by using Lemma 1.15.

$\varphi^{-1}(\zeta)$ is clearly a cone with apex ζ (it is the maximal element) so it is contractible and condition (A) satisfied for φ.

Take now any $\tau \in \varphi^{-1}\left(\left(\omega_H^{-1}(\eta)\right)_{\geq\zeta}\right)$. The maximal element ξ of the set $\varphi^{-1}(\zeta) \cap (\omega_H^{-1}(\eta))_{\leq\tau}$ is ζ.

Since φ satisfies conditions (A) and (B) it is a homotopy equivalence. The image of φ is a cone with apex η so contractible and condition (A) is satisfied for ω_H.

Take now any $\tau \in \omega_H^{-1}(\mathrm{Hom}(G, H-v)_{\geq\eta})$. The maximal element ξ of the set $\omega_H^{-1}(\eta) \cap (\mathrm{Hom}(G,H))_{\leq\tau}$ is

$$\xi(x) = \begin{cases} \eta(x) & \text{if } u \notin \eta(x), \\ \tau(x) \cap (\eta(x) \cup \{u\}) & \text{otherwise.} \end{cases}$$

Since it satisfies conditions (A) and (B), we conclude that $\mathrm{sd}(\omega_H)$ and hence also ω_H are homotopy equivalences.

It is left to prove that i_H is also a homotopy equivalence. It is clear that $\omega_H \circ i_H = \mathrm{Id}_{\mathrm{Hom}(G,H-v)}$. Let ϑ be the homotopy inverse of ω_H. Then we have $i_H \circ \omega_H \simeq \vartheta \circ \omega_H \circ i_H \circ \omega_H \simeq \vartheta \circ \omega_H \simeq \mathrm{Id}_{\mathrm{Hom}(G,H)}$. $\square$

2.6 Mycielski graphs

Recall (see for example [ST04] page 16) that the Mycielskian $M_r(G)$ of a graph $G = (V, E)$ has vertex set $\{z\} \cup (V \times [r])$, z is connected to all vertices of $V \times \{1\}$, (v, i) is connected to $(u, i+1)$ for all $(u, v) \in E$ and $i = 1, 2, \ldots, r-1$, and a copy of G sits on $V \times \{r\}$.

Our aim is to prove[5] that (as it is predicted in [ST04]):

Theorem 2.23. *For every graph G and every $r \geq 1$, the homomorphism complex* $\mathrm{Hom}(K_2, M_r(G))$ *is $\mathbb{Z}_2$-homotopy equivalent to the suspension* $\mathrm{susp}(\mathrm{Hom}(K_2, G))$.

Our main tools are Bredon's theorem [Bre67] which allows us to use standard topological combinatorics to prove $\mathbb{Z}_2$-homotopy equivalence (see [Živ04] for other applications).

Theorem 2.24 (Bredon). *Suppose that $f\colon X \to Y$ is a (simplicial) $\mathbb{Z}_2$-map of* free *simplicial $\mathbb{Z}_2$-complexes X and Y. The $\mathbb{Z}_2$-map $f\colon X \to Y$ is a $\mathbb{Z}_2$-homotopy equivalence if and only if it is an ordinary homotopy equivalence.*

Proof of Theorem 2.23. We will use induction on r. For $r = 1$ it was proven in Section 2.4. Here we give a new proof.

r=1: We extend the face poset of $\mathrm{Hom}(K_2, G)$ with two non-comparable maximal elements $\max_1$, $\max_2$ to obtain $\mathcal{F}(\mathrm{susp}(\mathrm{Hom}(K_2, G)))$. We define the map

$$f\colon P := \mathcal{F}(\mathrm{Hom}(K_2, M_1(G))) \to \mathcal{F}(\mathrm{susp}(\mathrm{Hom}(K_2, G))) =: Q$$

by (we will denote the cells by (A, B) and $(A \cup \{z\}, B)$ where we assume that $z \notin A, B \subseteq V$ and $A, B \neq \emptyset$)

$$\begin{aligned} f(A,B) &= (A,B), \\ f(A \cup \{z\}, B) &= \max_1, \\ f(z,B) &= \max_1 . \end{aligned}$$

Since we want a $\mathbb{Z}_2$-map f is well defined. (For example $f(B, z) = \max_2$.)

f is clearly monotone (simplicial), as all maps we will introduce later. We will keep using Lemma 1.15.

$f^{-1}(A, B)$ is just (A, B) so in this case (A) and (B) are satisfied. If $f(p) = \max_1$ then $f^{-1}(\max_1) \cap P_{\leq p}$ has a maximal element p. To show that $R := f^{-1}(\max_1)$ is contractible we define $g\colon R \to \mathrm{im}(g)$ by

[5] In [GyJS04] only the homotopy equivalence was proven.

$g(A\cup\{z\}, B) = (z, B)$ and $g(z, B) = (z, B)$. g is a homotopy equivalence since $g^{-1}(z, B)$ is a cone with apex (z, B) and let $q = (z, B)$ and $p = (\tilde{A} \cup \{z\}, \tilde{B})$ such that $g(p) \geq q \quad (B \subseteq \tilde{B})$. Now the maximal element of $g^{-1}(q) \cap R_{\leq p}$ is $(\tilde{A} \cup \{z\}, B)$. Moreover $\mathrm{im}(g)$ is a cone with apex (z, V).

The induction step $r \Rightarrow r + 1$:

The graph homomorphism $\phi\colon M_{r+1}(G) \to M_r(G)$ defined by $\phi(z) = z$ and $\phi(v \times i) = v \times \min\{i, r\}$ gives a $\mathbb{Z}_2$-map

$$f\colon P := \mathcal{F}(\mathrm{Hom}(K_2, M_{r+1}(G))) \to \mathcal{F}(\mathrm{Hom}(K_2, M_r(G))) =: Q.$$

We will show that f is a homotopy equivalence. If $(A \cup B) \cap (\{z\} \cup V \times \{1, 2, \ldots, r-1\}) \neq \emptyset$ then $|f^{-1}(A, B)| = 1$ so in Lemma 1.15 (A) and (B) are satisfied. In the case when $(A \cup B) \subseteq V \times r$, by slightly abuse of notation we will write instead of $(A, B) \quad (A \times r, B \times r)$ showing that in $M_i(G)$ in which copy of V belong to A and B. Let $p = (\tilde{A}_1 \times r \cup \tilde{A}_2 \times (r+1), \tilde{B} \times (r+1))$ such that $f(p) \geq (A \times r, B \times r)$. Now the maximal element of $f^{-1}(A \times r, B \times r) \cap P_{\leq p}$ is $((\tilde{A}_1 \cap A) \times r \cup (\tilde{A}_2 \cap A) \times (r+1), B \times (r+1))$. We should show that $S := f^{-1}(A \times r, B \times r)$ is contractible as well.

We define $g\colon S \to \mathrm{im}(g)$ by $g(A \times r, B \times r) = (A \times r, B \times r)$, $g(A_1 \times r \cup A_2 \times (r+1), B \times (r+1)) = (A_1 \times r \cup A \times (r+1), B \times (r+1))$ $(A_1 \cup A_2 = A)$ and symmetrically $g(A \times (r+1), B_1 \times r \cup B_2 \times (r+1)) = (A \times (r+1), B_1 \times r \cup B \times (r+1))$ $(B_1 \cup B_2 = B)$. $\mathrm{im}(g)$ is a cone with apex $(A \times (r+1), B \times (r+1))$. $g^{-1}(q)$ is a cone with apex q. Let (without loss of generality) $q = (A_1 \times r \cup A \times (r+1), B \times (r+1))$ and $p = (\tilde{A}_1 \times r \cup \tilde{A}_2 \times (r+1), B \times (r+1))$ such that $f(p) \geq q \quad (\tilde{A}_1 \supseteq A_1)$. Now the maximal element of $f^{-1}(q) \cap S_{\leq p}$ is $(A_1 \times r \cup A \times (r+1), B \times (r+1))$. This completes the proof. □

Remark 2.25. *There are interesting consequences of Theorem 2.23. Since* $M_1(K_n) = K_{n+1}$ *and* $\mathrm{Hom}(K_2, K_2)$ *homeomorphic to* S^0 *we get that* $\mathrm{Hom}(K_2, K_n)$ *is* $\mathbb{Z}_2$*-homotopy equivalent*[6] *to* S^{n-2}. *This implies already Lovász's topological lower bound for the chromatic number:*

$$\chi(G) \geq \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2.$$

In general about $\mathrm{Hom}(H, M_r(G))$ *or* $\mathrm{Hom}(M_r(H), G)$ *one can not expect something like Theorem 2.23. Is is shown by the following well known*

[6] It is known [BK03] that $\mathrm{Hom}(K_2, K_n)$ is $\mathbb{Z}_2$-homeomorphic to S^{n-2}.

or easily computable examples: $\mathrm{Hom}(K_3, K_2) = \emptyset$, $\mathrm{Hom}(K_3, M_2(K_2) = C_5) = \emptyset$, $\mathrm{Hom}(K_3, M_1(K_2)) \cong \bigvee^{5} S^0$, $\mathrm{Hom}(K_3, M_1(K_3)) \simeq \bigvee^{13} S^1$, $\mathrm{Hom}(K_3, M_2(K_3)) \simeq \bigvee^{5} S^0$, $\mathrm{Hom}(C_5, K_2) = \emptyset$, $\mathrm{Hom}(C_5, M_2(K_2)) \cong \bigvee^{9} S^0$, $\mathrm{Hom}(C_5, M_1(K_2)) \cong S^1 \cup S^1$, $\mathrm{Hom}(C_5, M_2(C_5)) \simeq \bigvee^{41} S^1$, *and* $\mathrm{Hom}(C_5, M_1(K_3)) \cong \mathbb{RP}^3$. *But still something can be said.*

Theorem 2.26. *If* $n \geq 3$ *and* $r \geq 2$ *then* $\mathrm{Hom}(K_n, M_r(G))$ *is homotopy equivalent to* $\mathrm{Hom}(K_n, G)$.

Proof. Let $\tilde{G}$ be a subgraph of $M_r(G)$ induced by the vertex set $V \times \{r, r-1\}$. Clearly $\mathrm{Hom}(K_n, M_r(G))$ is the same as $\mathrm{Hom}(K_n, \tilde{G})$. It is easy to see that $\tilde{G}$ folds down to G. Now Theorem 2.22 completes the proof. $\square$

Remark 2.27. *Since* $\chi(M_2(G)) = \chi(G) + 1$ *we obtain graphs such that* no *topological lower bound using* $\mathrm{Hom}(K_n, *)$ $(n \geq 3)$ *can give sharp bound on their chromatic number. On the other hand for these graphs* $\mathrm{Hom}(K_2, *)$ *might provide sharp bound.*

It is interesting to mention that $\chi(M_r(G)) > \chi(G)$ *does not hold in general if* $r \geq 3$, *e.g., if* G *is the graph from Figure 5.2 then* $\chi(M_3(G)) = \chi(G) = 4$.

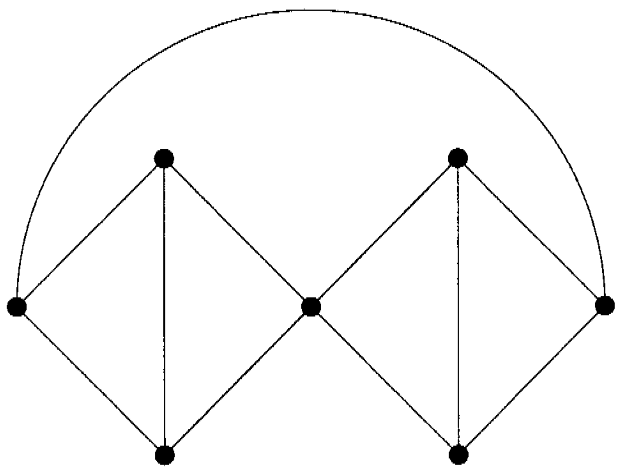


Figure 2.7: G such that $\chi(M_3(G)) = \chi(G) = 4$.

Theorem 2.26 can be stated in more general in the following way.

Proposition 2.28. $\mathrm{Hom}(K_n, G)$ *($n \geq 3$) is homotopy equivalent to* $\mathrm{Hom}(K_n, G - v)$*, if there is no triangle in G containing a vertex $v \in G$.*

Theorem 2.29. *If $2n + 1 \leq 2r$ and $r \geq 2$ then* $\mathrm{Hom}(C_{2n+1}, M_r(G))$ *is homotopy equivalent to* $\mathrm{Hom}(C_{2n+1}, G)$.

Remark 2.30. *The condition $2n + 1 \leq 2r$ in Theorem 2.29 is the best possible since* $\mathrm{Hom}(C_5, K_2) = \emptyset$ *but* $\mathrm{Hom}(C_5, M_2(K_2)) \cong \bigvee^{9} S^0$.

Proof. The same as the proof of Theorem 2.26, just now $\tilde{G}$ should be a subgraph of $M_r(G)$ induced by the vertex set $V(M_r(G)) \setminus \{z\}$. □

Remark 2.31. *As before: using Theorem 2.29 we can construct graphs showing that the topological bound obtained by e.g.* $\mathrm{Hom}(C_5, *)$ *can be arbitrarily bad. On the other hand for these graphs* $\mathrm{Hom}(K_2, *)$ *can provide good bound for their chromatic number.*

2.7 Simple homotopy type of graph complexes

In this section we show that the neighborhood complex $\mathsf{N}(G)$, the box complex $\mathsf{B}(G)$, the homomorphism complex $\mathrm{Hom}(K_2, G)$ and the Lovász complex $\mathsf{L}(G)$ have the same simple $\mathbb{Z}_2$-homotopy type[7], in the sense of Whitehead [Whi39].

The $\mathbb{Z}_2$-homotopy equivalence of the graph complexes were studied in several papers [Cso04, CsLSW04, Mat03, Živ04]. Kozlov independently proved [Koz05b] that the neighborhood complex $\mathsf{N}(G)$, the Lovász complex $\mathsf{L}(G)$ and the homomorphism complex $\mathrm{Hom}(K_2, G)$ has the same *simple* homotopy type. Our proof [Cso05] is a $\mathbb{Z}_2$-version and simpler for the homomorphism complex $\mathrm{Hom}(K_2, G)$ via the box complex $\mathsf{B}(G)$.

[7] Since the neighborhood complex is not a $\mathbb{Z}_2$-space for it we have only simple homotopy type result.

2.7.1 Simple $\mathbb{Z}_2$-homotopy equivalences of graph complexes

In this subsection we will prove that $\mathsf{B}(G)$ collapses onto $\mathsf{N}(G)$, $\mathrm{sd}(\mathsf{B}(G))$ $\mathbb{Z}_2$-collapses onto $\mathrm{sd}(\mathrm{Hom}(K_2, G))$, and $\mathrm{ssd}(\mathsf{B}(G))$ $\mathbb{Z}_2$-collapses onto $\mathsf{L}(G)$.

Theorem 2.32. $\mathsf{B}(G)$ *collapses onto* $\mathsf{N}(G)$.

Proof. We will collapse $\mathsf{B}(G)$ onto its first shore which is isomorphic to $\mathsf{N}(G)$. The idea of the proof is to start with $\mathsf{B}(G)$, and get rid of the extra simplexes one by one (using collapses) such that finally we get $\mathsf{N}(G)$. We will define sequences of simplicial complexes such that

$$\mathsf{B}(G) =: X_0 \supset X_1 \supset \cdots \supset X_N = \mathsf{N}(G),$$

and X_i collapses onto X_{i+1}.

Let assume that we already defined X_n. We choose a maximal simplex $\sigma \in X_n$ such that

1. σ has a vertex from the second shore,
2. no other simplex in X_n has *more* vertices from the second shore.

Let $\tau \subset \sigma$ be the intersection of σ and the second shore. We define X_{i+1} from X_i by collapsing (τ, σ). First of all we have to show that this is a well defined collapse. By contradiction assume that there were a simplex $\tau \cup v \in X_i$ such that $v \notin \sigma$. If v were form the second shore it would rather choose $\tau \cup v$ instead of σ to define X_{i+1}. If v were from the first shore then $\sigma \cup v$ would be a simplex of $\mathsf{B}(G)$. Since σ is maximal, this simplex would be deleted before (together with σ). □

Theorem 2.33. $\mathrm{sd}(\mathsf{B}(G))$ $\mathbb{Z}_2$*-collapses onto* $\mathrm{sd}(\mathrm{Hom}(K_2, G))$.

Proof. $\mathrm{sd}(\mathrm{Hom}(K_2, G))$ is a subcomplex of $\mathrm{sd}(\mathsf{B}(G))$. The extra vertices are vertices on the shores of the box complex $\mathrm{sd}(\mathsf{B}(G))$. (They are in the form $\emptyset \uplus A$ and $B \uplus \emptyset$.) We work only with the first shore; the $B \uplus \emptyset$ part of $\mathrm{sd}(\mathsf{B}(G))$. On the other shore every collapse $\mathbb{Z}_2$-pair is done. We describe an acyclic matching on $P := P(\mathrm{sd}(\mathsf{B}(G)))$. Let $F \in P$. We assume that F has a vertex from the first shore. Its vertices form a chain $A_1 \uplus \emptyset \subset \cdots \subset A_n \uplus \emptyset \subset A_{n+1} \uplus B_1 \subset \cdots \subset A_{n+m} \uplus B_m$. We set $B_0 = \emptyset$ and consider the vertex $\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. Let i be the maximal index such that $A_{n+i} \uplus B_i \subseteq \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. We note that $A_n \uplus B_0 \subseteq \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$ so such an i exists.

If $i = m$ then we can have that $A_{n+m} \uplus B_m = \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. In this case we match F with $F \setminus (\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n))$. Else we match F with $F \cup (\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n))$.

If $i \neq m$ then we consider $X \uplus Y := A_{n+i+1} \uplus B_{i+1} \cap \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. If $(X \uplus Y) \in F$ then we match F with $F \setminus (X \uplus Y)$. If $(X \uplus Y) \notin F$ then we match F with $F \cup (X \uplus Y)$.

Next we show that the obtained matching M acyclic. Assume that there exist a sequence $F_0, \ldots, F_t \in P$ such that all F_i are different, with the exception $F_0 = F_t$, and such that $\mu(F_i) \succ F_{i+1}$ for $0 \leq i \leq t-1$. Assume that $\mu(F_0) = A_1 \uplus \emptyset \subset \cdots \subset A_n \uplus \emptyset \subset A_{n+1} \uplus B_1 \subset \cdots \subset A_{n+m} \uplus B_m$. Observe that $A_{n+m} \uplus B_m = \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. If F_0 were $\mu(F_0) \setminus A_{n+m} \uplus B_m$ then since $F_0 \neq F_1$ it would be not possible to match F_1 upwards unless we delete $A_n \uplus \emptyset$. But matched pairs contains the same many vertices in type $A \uplus \emptyset$, so it can not be a member cycle. Else $F_0 = \mu(F_0) \setminus (A_{n+i} \uplus B_i)$ for some $m > i \geq 1$. Now F_1 should be $\mu(F_0) \setminus (A_{n+i+1} \uplus B_{i+1})$ to be matched up. We see that in F_1 the number of vertices which are subsets of $\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$ is more by 1 than in F_0. Repeating this argument, we see that F_t has t vertices more, therefore $F_0 \neq F_t$. This leads to the conclusion that M is an acyclic matching.

The critical simplices form a subcomplex $\mathrm{sd}(\mathrm{Hom}(K_2, G))$ which completes the proof. □

Theorem 2.34. $\mathrm{ssd}(\mathsf{B}(G))$ *$\mathbb{Z}_2$-collapses onto* $\mathsf{L}(G)$.

Proof. First we show that $\mathrm{ssd}(\mathsf{B}(G))$ $\mathbb{Z}_2$-collapses onto $\mathrm{cn}^2(\mathrm{ssd}(\mathsf{B}(G)))$. This follows from the fact that cn^2 is an ascending closure operator.

Next we show that $\mathrm{cn}^2(\mathrm{ssd}(\mathsf{B}(G)))$ $\mathbb{Z}_2$-collapses onto $\mathsf{L}(G)$.

Unfortunately the previous simple proof does not work directly, we will define simplicial complexes

$$\mathrm{cn}^2(\mathrm{ssd}(\mathsf{B}(G))) =: S_0 \supset S_1 \supset \cdots \supset S_{N+1} = \mathsf{L}(G),$$

such that S_i $\mathbb{Z}_2$-collapses onto S_{i+1} Assume that we already defined S_i. How to define S_{i+1}? We choose a vertex $X \uplus \emptyset \in S_i$ such that

1. $\emptyset \uplus \mathrm{CN}(X) \in S_i$, and
2. $|X| \geq |\mathrm{CN}(X)|$, and
3. there is no Y such that $Y \uplus \emptyset \in S_i$, $\emptyset \uplus \mathrm{CN}(Y) \in S_i$, $|Y| \geq |\mathrm{CN}(Y)|$ and $|Y| > |X|$.

The maximality of X implies that a maximal simplex which contains $X \uplus \emptyset$ (resp. $\emptyset \uplus X$) does also contain $\emptyset \uplus \mathrm{CN}(X)$ (resp. $\mathrm{CN}(X) \uplus \emptyset$).

We define an acyclic matching on $P := P(S_i)$. Let $F \in P$ such that $X \uplus \emptyset$ is its vertex. If $\emptyset \uplus \mathrm{CN}(X)$ is a vertex of F then we match F with the simplex $F \setminus (\emptyset \uplus \mathrm{CN}(X))$. Else we match F with $F \cup (\emptyset \uplus \mathrm{CN}(X))$.

Next we show that the obtained matching M acyclic. Assume that there exist a sequence $F_0, \ldots, F_t \in P$ such that all F_i are different, with the exception $F_0 = F_t$, and such that $\mu(F_i) \succ F_{i+1}$ for $0 \leq i \leq t-1$. $\mu(F_0) = F_0 \cup (\emptyset \uplus \mathrm{CN}(X))$. We must obtain F_1 from $\mu(F_0)$ by deleting 1 vertex in such a way that it matches upwards. It is possible if and only if we delete the vertex $\emptyset \uplus \mathrm{CN}(X)$ therefore $F_0 = F_1$. This leads to the conclusion that M is an acyclic matching.

The critical simplices form a subcomplex as it is proven in Subsection 2.1.1 (Theorem 2.5) as well that the complex what we obtain by the end is $\mathsf{L}(G)$. This completes the proof. $\square$

2.8 Making small Hom complexes

The problem of calculating (by computer) invariants (e.g. homology) of Hom complexes is that they are not simplicial complexes. The natural barycentric subdivision provides a homeomorphic simplicial complex. But in this way one gets simply to large complex even for small graphs!

Since the cells of Hom complexes are products of simplices one can triangulate it without any additional vertex obtaining smaller homeomorphic version. In the following we suggest an only homotopy equivalent, but much smaller complex.

We want to study $\mathrm{Hom}(G, H)$ (definition 1.26). Let I be an independent set in G. We define $\widetilde{\mathrm{Hom}}_I(G, H)$ by deleting this independent set many 'coordinates' form $\mathrm{Hom}(G, H)$ in the following way.

Let $\Delta^{V(H)}$ be a simplex whose set of vertices is $V(H)$. Let $C_I(G, H)$ denote the direct product $\prod_{x \in V(G), x \notin I} \Delta^{V(H)}$, i.e., the copies of $\Delta^{V(H)}$ are indexed by vertices of $G \setminus I$.

Definition 2.35. For any pair of graphs G, H and an independent set I in G we define

$$\widetilde{\mathrm{Hom}}_I(G, H) := \mathrm{Hom}(G, H) \cap C_I(G, H).$$

It is a polyhedral complex whose cells are indexed by functions $\tilde{\eta} : V(G \setminus I) \to 2^{V(H)} \backslash \{\emptyset\}$, such that there exist an extension $\eta : V(G) \to 2^{V(H)} \backslash \{\emptyset\}$ with the following properties:
If $(i, j) \in E(G)$, then for every $\tilde{i} \in \eta(i)$ and $\tilde{j} \in \eta(j)$ it follows that $(\tilde{i}, \tilde{j}) \in E(H)$.

Theorem 2.36. *$\widetilde{\mathrm{Hom}}_I(G, H)$ is homotopy equivalent to* $\mathrm{Hom}(G, H)$.

Proof. We will use Lemma 1.15. We only have to show that the conditions (A) and (B) are satisfied by a map φ: $\mathcal{F}(\mathrm{Hom}(G, H)) \to \mathcal{F}(\widetilde{\mathrm{Hom}}_I(G, H))$ given by restricting $\eta \in \mathcal{F}(\mathrm{Hom}(G, H))$ to $V(G) \setminus I$.
Condition (A): $\Delta(\varphi^{-1}(\eta))$ is clearly the products of (barycentricly subdivided) simplices, so contractible.
Condition (B): For every $\eta \in \mathcal{F}(\mathrm{Hom}(G, H))$ and $\tilde{\nu} \in \mathcal{F}(\widetilde{\mathrm{Hom}}_I(G, H))$ with $\varphi(\eta) \geq \tilde{\nu}$ the poset $\varphi^{-1}(\tilde{\nu}) \cap \mathcal{F}(\mathrm{Hom}(G, H))_{\leq \eta}$ has clearly a maximal element, because I is an independent set. $\square$

Proposition 2.37. $\mathrm{Hom}(K_2, K_n)$ *($n \geq 2$) is homotopy equivalent to S^{n-2}.*

Remark 2.38. *Of course it is well known that* $\mathrm{Hom}(K_2, K_n)$ *($n \geq 2$) is homeomorphic to S^{n-2} (see Section 1.6).*

Proof. Let I be an independent set in K_2 i.e. I is a vertex. $\widetilde{\mathrm{Hom}}_I(K_2, K_n)$ has a very simple structure. It is the boundary of the n-dimensional simplex which shows our claim. $\square$

Remark 2.39. *Observe that $\widetilde{\mathrm{Hom}}_I(K_2, G)$ is the neighborhood complex* $\mathsf{N}(G)$ *of the graph G. So we proved that* $\mathrm{Hom}(K_2, G)$ *is homotopy equivalent to* $\mathsf{N}(G)$.

In [BK04] Babson and Kozlov computes (co)homologies of $\mathrm{Hom}(C_{2r+1}, K_n)$. They only conjecture the $\mathrm{Hom}(C_7, K_4)$ case. Our computer calculation verifies that $H(\mathrm{Hom}(C_7, K_4)) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z})$.

Another application is to determine the homotopy type of the Hom complex $\mathrm{Hom}(K_3, K_m)$. By Theorem 2.36 we have that it is homotopy equivalent to the $(m-3)$-skeleton of $\mathrm{Hom}(K_2, K_m)$. Since $\mathrm{Hom}(K_2, K_m)$ is a boundary of a polytope and it has $2^m - 2$ $(m-2)$-dimensional cells $\mathrm{Hom}(K_3, K_m)$ is homotopy equivalent to the wedges of $2^m - 3$ $(m-3)$-dimensional spheres. It is enough to reprove the theorem of Babson and Kozlov for $n = 3$:

Theorem 2.40 ([BK03]). *If* $\mathrm{Hom}(K_n, G)$ *is* k*-connected, then* $\chi(G) \geq k + n + 1$.

Recently Engström [Eng05] found an interesting application of this idea of removing an independent set (Theorem 2.36).

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Chapter 3

Non-tidy spaces

In the previous chapter we compared the homotopy type of various graph complexes. We studied the topological lower bounds obtained by using them. In order to answer the question of Matoušek and Ziegler [MZ04] we need a space such that the suspension does not increase its index. We will see that we have to search amongst non-tidy spaces. In the next chapter (Subsection 4.2.2) we will observe that non-tidy spaces are around us, for example the Hom complexes $\mathrm{Hom}(C_5, K_{2n})$ are non-tidy!

3.1 The $\mathbb{Z}_2$-index of the join of topological spaces

In combinatorial applications, when we use the appropriate form of the Borsuk–Ulam Theorem [Mat03, ST04, ST05] we usually want to bound the $\mathbb{Z}_2$-index of some space, which we obtain by some construction (e.g. join). The following theorem provides an upper and lower bound for the join of two topological spaces.

Theorem 3.1 ([Mat03]). *For any X and Y (free $\mathbb{Z}_2$-spaces)*

$$\max\Big\{ \mathrm{ind}(X)\,, \mathrm{ind}(Y) \Big\} \le \mathrm{ind}(X * Y) \le \mathrm{ind}(X) + \mathrm{ind}(Y) + 1.$$

The following lemma will provide us plenty of examples showing that the upper bound can be tight. Usually the upper bound of Theorem 3.1 is used in applications, but it is useful to know how big mistake one can make by using it.

Lemma 3.2. *If X, Y are tidy $\mathbb{Z}_2$-spaces, then* $\operatorname{ind}(X * Y) = \operatorname{ind}(X) + \operatorname{ind}(Y) + 1$.

Proof. X, Y are tidy spaces, which means that $\operatorname{coindex}(X) = \operatorname{ind}(X) = n$ and $\operatorname{coindex}(Y) = \operatorname{ind}(Y) = m$ for some $n, m \in \mathbb{N}$. This equivalent to the following chains of $\mathbb{Z}_2$-maps $S^n \xrightarrow{\mathbb{Z}_2} X \xrightarrow{\mathbb{Z}_2} S^n$ and $S^m \xrightarrow{\mathbb{Z}_2} Y \xrightarrow{\mathbb{Z}_2} S^m$. The join of these two chains $S^{n+m+1} \xrightarrow{\mathbb{Z}_2} X * Y \xrightarrow{\mathbb{Z}_2} S^{n+m+1}$ shows that $X * Y$ is tidy as well, and $\operatorname{ind}(X * Y) = n + m + 1 = \operatorname{ind}(X) + \operatorname{ind}(Y) + 1$. □

For example if $X = S^n$ and $Y = S^m$ then, since they are tidy spaces, $\operatorname{ind}(X * Y) = \operatorname{ind}(X) + \operatorname{ind}(Y) + 1$.

Let us remark that in Theorem 3.1 the lower bound strictly smaller then the upper bound

$$\max\left\{ \operatorname{ind}(X), \operatorname{ind}(Y) \right\} < \operatorname{ind}(X) + \operatorname{ind}(Y) + 1.$$

So if we want to find an example (two $\mathbb{Z}_2$-spaces A and B) showing that the lower bound is tight ($\max\left\{ \operatorname{ind}(A), \operatorname{ind}(B) \right\} = \operatorname{ind}(A * B)$), then for this particular example $\operatorname{ind}(A * B) < \operatorname{ind}(A) + \operatorname{ind}(B) + 1$. Lemma 3.2 shows that this is not possible for tidy spaces. Because of that at first we have to search for non-tidy spaces.

3.2 Construction of non-tidy spaces

In this section we will construct non-tidy $\mathbb{Z}_2$-spaces. These examples are based on an earlier construction by Matoušek, Živaljević and the author [Cso04, Mat03, page 100].

We proceed in the following way. We choose a map $f\colon S^{n+k-1} \to S^n$ ($[f] \in \pi_{n+k-1}(S^n)$, $k \geq 1$) and we attach two $(n+k)$-cells (the boundary of the $(n+k)$-cell is S^{n+k-1}) to S^n via f and $-f$. The $\mathbb{Z}_2$-action on $S^n \subset \mathcal{S}_f$ is the standard antipodal map on the sphere and the $\mathbb{Z}_2$-action on $\mathcal{S}_f$ interchanges the two $(n+k)$-cells. We denote this $\mathbb{Z}_2$-space by

$$\mathcal{S}_f := S^n \bigcup_f B^{n+k} \bigcup_{-f} B^{n+k}.$$

For example if $f = \mathrm{id}\colon \mathrm{S^n} \to \mathrm{S^n}$ then $\mathcal{S}_f = S^{n+1}$ which is unfortunately tidy (see Figure 3.1).

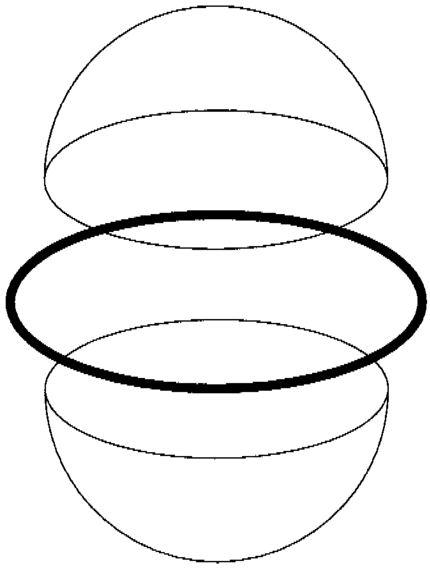

Figure 3.1: $\mathcal{S}_{\mathrm{id}}$ where $\mathrm{id}\colon \mathrm{S^n} \to \mathrm{S^n}$.

Now we compute the $\mathbb{Z}_2$-index and the coindex of $\mathcal{S}_f$. In order to calculate coindex$(\mathcal{S}_f)$ we need an important technical tool, the cellular approximation theorem.

Theorem 3.3 ([Bre67] Proposition II/5.6). *Every $\mathbb{Z}_2$-map $f\colon X \xrightarrow{\mathbb{Z}_2} Y$ between two $\mathbb{Z}_2$-spaces is $\mathbb{Z}_2$-homotopic to a cellular map – which maps the n-skeleton of X into the n-skeleton of Y for every n.*

Using this theorem we are ready to compute coindex$(\mathcal{S}_f)$.

Lemma 3.4. *If $k \geq 2$ then* coindex$(\mathcal{S}_f) = n$.

Proof. The embedding $S^n \hookrightarrow \mathcal{S}_f$ shows that coindex$(\mathcal{S}_f) \geq n$. In order to prove that coindex$(\mathcal{S}_f) = n$ it is enough to show that $S^{n+1} \overset{\mathbb{Z}_2}{\not\longrightarrow} \mathcal{S}_f$.

By contradiction assume that we have a $\mathbb{Z}_2$-map $S^{n+1} \xrightarrow{\mathbb{Z}_2} \mathcal{S}_f$. Using Theorem 3.3 we could assume that S^{n+1} mapped into the n+1 skeleton of $\mathcal{S}_f$ which is S^n. This gives us a $\mathbb{Z}_2$-map $S^{n+1} \xrightarrow{\mathbb{Z}_2} S^n$ contradicting to the Borsuk–Ulam Theorem. $\square$

It is easy to see that $\mathrm{ind}(\mathcal{S}_f) \leq n+1$. Let B^i be the unit ball in $\mathbb{R}^i$ centered at the origin. We assume that $f\colon S^{n+k-1} \to S^n$ maps the n+k−1-dimensional unit sphere, the boundary of the unit ball, into the n-dimensional unit sphere. We define a map $b\colon B^{n+k} \to B^{n+1}$ such that it maps the origin of $\mathbb{R}^{n+k}$ into the origin of $\mathbb{R}^{n+1}$ and if $x \in B^{n+k}$, $\|x\| \neq 0$ then $b(x) := f\left(\frac{x}{\|x\|}\right) \cdot \|x\|$. We know that $S^{n+1} = \mathcal{S}_{\mathrm{id}}$ where $\mathrm{id}\colon \mathrm{S^n} \to \mathrm{S^n}$ so it is enough to construct a map $a\colon \mathcal{S}_f \xrightarrow{\mathbb{Z}_2} \mathcal{S}_{\mathrm{id}}$. a maps $S^n \subset \mathcal{S}_f$ to $S^n \subset \mathcal{S}_{\mathrm{id}}$ identically and these two $(n$+$k)$-cells of $\mathcal{S}_f$ to those two $(n$+1)-cells of $\mathcal{S}_{\mathrm{id}}$ by b and $-b$.

It is slightly more difficult to prove, that for an appropriately chosen f $\mathrm{ind}(\mathcal{S}_f) = n+1$, which shows that $\mathcal{S}_f$ is not tidy. We will use the following tools:

Definition 3.5 ([Hat01] Page 427, Section 4.B). Let $f\colon S^{2n-1} \to S^n$, $(n \geq 2)$, and let $C_f = S^n \bigcup_f B^{2n}$ (we attach a $2n$-cell to S^n via f). The Hopf invariant of f (denoted by $\mathcal{H}(f)$) can be defined such that $\alpha \cup \alpha = \mathcal{H}(f) \cdot \beta$, where $\alpha \in H^n(C_f) = \mathbb{Z}$ and $\beta \in H^{2n}(C_f) = \mathbb{Z}$ are the generators of the corresponding cohomology groups and $\cup$ is the cup product.

We collect some well known facts [Hat01] about the Hopf invariant.

- If n is odd, then $\mathcal{H}(f) = 0$.
- If f is null-homotopic, then $\mathcal{H}(f) = 0$.
- $\mathcal{H} : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism. If n=2, then it is an isomorphism.
- There is an f such that $\mathcal{H}(f) = 2$ if n is even. There exist an f, $\mathcal{H}(f) = 1$ if and only if $n = 2, 4, 8$.
- $\pi_{2n-1}(S^n)$ contains a $\mathbb{Z}$ direct summand when n is even.

Theorem 3.6 ([HW60] Theorem 9.5.9). *Let $f\colon S^{2n-1} \to S^n$ and $g\colon S^n \to S^n$ be continuous maps. Then the following holds: $\mathcal{H}(g \circ f) = \deg(g)^2 \cdot \mathcal{H}(f)$.*

Theorem 3.7 ([Hat01] Proposition 2B.6). *Every $\mathbb{Z}_2$-map $f\colon S^n \xrightarrow{\mathbb{Z}_2} S^n$ must have odd degree.*

Lemma 3.8. *If $f\colon S^{2n-1} \to S^n$ such that $\mathcal{H}(f) \neq 0$ then $\operatorname{ind}(\mathcal{S}_f) = n+1$.*

Proof. By contradiction assume that $\operatorname{ind}(\mathcal{S}_f) \leq n$ which means that there is a $\mathbb{Z}_2$-map $F\colon \mathcal{S}_f \xrightarrow{\mathbb{Z}_2} S^n$. We restrict this map into $S^n \subset \mathcal{S}_f$ obtaining $g\colon S^n \to S^n$. We claim that $g \circ f\colon S^{2n-1} \to S^n$ is null-homotopic. We attached one $2n$-cell to S^n via f. This gives us a map $i\colon B^{2n} \to \mathcal{S}_f$ and $F \circ i\colon B^{2n} \to S^n$. The restriction of $F \circ i$ into $S^{2n-1} = \partial B^{2n}$ is $g \circ f$. So the map $g \circ f$ extends into B^{2n} which proves that $g \circ f$ is null-homotopic.

On the other hand Theorem 3.7 tells us that $\deg(g)$ is odd. (We need now only that it is not zero.) Using Theorem 3.6 we have that $\mathcal{H}(g \circ f) = \deg(g)^2 \cdot \mathcal{H}(f)$. Since we know that $\deg(g) \neq 0$ and $\mathcal{H}(F) \neq 0$ we have that $\mathcal{H}(g \circ f) \neq 0$. This means that $g \circ f$ is not null-homotopic, contradiction. □

Remark 3.9. *It is easy to see that if $f\colon S^{n+k-1} \to S^n$ is null-homotop then $\mathcal{S}_f$ is tidy, i.e., $\operatorname{coindex}(\mathcal{S}_f) = \operatorname{ind}(\mathcal{S}_f) = n$.*

Corollary 3.10. *If $f\colon S^{2n-1} \to S^n$ such that $\mathcal{H}(f) \neq 0$ then $\mathcal{S}_f$ is not tidy.*

Proof. Lemma 3.8 shows that $\operatorname{ind}(\mathcal{S}_f) = n+1$, Lemma 3.4 tells us that $\operatorname{coindex}(\mathcal{S}_f) = n$, hence $\mathcal{S}_f$ is not tidy. □

For example one may obtain easily a cell complex model of a non-tidy $\mathbb{Z}_2$-space in the following way. Let $h\colon S^3 \to S^2$ be the Hopf map[1], then $\mathcal{S}_h$ is not tidy, $\operatorname{ind}(\mathcal{S}_h) = 3$ and $\operatorname{coindex}(\mathcal{S}_h) = 2$.

In fact there are maps $f\colon S^{4n-1} \to S^{2n}$ such that $\mathcal{H}(f) = 2 \neq 0$. Using this f, $\mathcal{S}_f$ is a non-tidy $4n$-dimensional complex.

[1] Considering S^3 as the unit sphere in $\mathbb{C}^2$ and $S^2 = \mathbb{CP}^1$, the Hopf map: $h\colon S^3 \to S^2$ defined by $(z_1, z_2) \to [z_1, z_2] \in \mathbb{CP}^1$. h is the generator of $\pi_3(S^2) = \mathbb{Z}$ and $\mathcal{H}(h) = 1$, (see [Hat01] Example 4.45).

Simonyi and Tardos recently found even simpler examples [ST05]. They show that for example the double torus (the $\mathbb{Z}_2$-action is the reflection through the origin as it is naturally embedded to $\mathbb{R}^3$) has index 2 but coindex 1. Moreover they show that the suspension of this space is tidy with index=coindex=3. However their examples are not good for our purpose.

3.3 Projective spaces

Another example for non-tidy spaces is provided by the odd dimensional projective spaces $\mathbb{RP}^{2n+1}$ $(n \geq 1)$. We present S^{2n+1} as the unit sphere in the complex space $\mathbb{C}^{n+1}$. $\mathbb{RP}^{2n+1}$ is the quotient $S^{2n+1}/(\{\mathbf{x}, -\mathbf{x}\}: \mathbf{x} \in S^{2n+1})$. The free $\mathbb{Z}_2$-action on $\mathbb{RP}^{2n+1}$ is defined by $(v_1, \ldots, v_{n+1}) \to (iv_1, \ldots, iv_{n+1})$, $v_i \in \mathbb{C}$. Stolz [Sto89] showed that the index of $\mathbb{RP}^{2n+1}$ is at least $n+1$, in fact he determined the index exactly:

$$\operatorname{ind}(\mathbb{RP}^{2n-1}) = \begin{cases} n & \text{if } n = 0, 2 \bmod 8, \\ n+1 & \text{if } n = 1, 3, 4, 5, 7 \bmod 8, \\ n+2 & \text{if } n = 6 \bmod 8. \end{cases}$$

On the other hand Živaljević [Živ02] reported that their coindex is 1. In this section we take a closer look into $\mathbb{RP}^3$ since it appears as the Hom-complex $\mathrm{Hom}(C_5, K_4)$ as well!

There is a simplicial $\mathbb{RP}^3$ [Lut] (with 12 vertices, 48 top simplices) with a free simplicial involution presented in Table 3.1. The involution is given by $+6$ modulo 12. For example the $\mathbb{Z}_2$-pair of the vertex 8 is $8+6=14$, which is 2 modulo 12. In Table 3.1 the rows contains the $\mathbb{Z}_2$-pair simplices of $\mathbb{RP}^3$.

{8,4,7,11}	{2,10,1,5}
{4,6,11,8}	{10,12,5,2}
{4,7,5,8}	{10,1,11,2}
{8,3,6,11}	{2,9,12,5}
{8,12,3,11}	{2,6,9,5}
{4,7,12,3}	{10,1,6,9}
{4,7,3,5}	{10,1,9,11}
{2,4,7,12}	{8,10,1,6}
{2,4,7,11}	{8,10,1,5}
{2,4,6,11}	{8,10,12,5}
{2,3,6,11}	{8,9,12,5}
{2,3,5,6}	{8,9,11,12}

{3,5,6,7}	{9,11,12,1}
{9,2,4,6}	{3,8,10,12}
{9,2,4,12}	{3,8,10,6}
{9,5,6,7}	{3,11,12,1}
{9,5,8,7}	{3,11,2,1}
{9,8,7,11}	{3,2,1,5}
{10,3,6,7}	{4,9,12,1}
{10,6,9,7}	{4,12,3,1}
{10,9,7,11}	{4,3,1,5}
{10,2,7,11}	{4,8,1,5}
{10,2,7,12}	{4,8,1,6}
{10,3,7,12}	{4,9,1,6}

Table 3.1: Simplicial $\mathbb{RP}^3$.

It is possible to compute the index and coindex of $\mathbb{RP}^3$ by a direct analysis.

Theorem 3.11. $\mathrm{ind}(\mathbb{RP}^3) = 2$.

Proof. In [CF62] S^3 is considered as the unit sphere in $\mathbb{C}^2$ with a free $\mathbb{Z}_4$-action given by $(z_1, z_2) \to (-\overline{z}_2, \overline{z}_1)$, and S^2 as $\mathbb{CP}^1$ with the free $\mathbb{Z}_2$-action given by $z \to \frac{-1}{\overline{z}}$. Now the Hopf map: $h\colon S^3 \to S^2$ defined by $(z_1, z_2) \to \frac{z_1}{z_2} \in \mathbb{CP}^1$ gives us a $\mathbb{Z}_2$-map $h\colon \mathbb{RP}^3 \to S^2$.

By contradiction assume, that there exists a $\mathbb{Z}_2$-map $m\colon \mathbb{RP}^3 \to S^1$. (Now i is the $\mathbb{Z}_2$-action again!) The map[2]

$$(\cos(\alpha), \sin(\alpha)) \to \left(\cos\left(\frac{\alpha}{2}\right), \sin\left(\frac{\alpha}{2}\right), 0, 0\right), \alpha \in [0, 2\pi]$$

gives us an embedded circle (C) in $\mathbb{RP}^3$, which corresponds to the generator of $\pi_1(\mathbb{RP}^3)$. This embedding is a $\mathbb{Z}_2$-map as well ($S^1 \to \mathbb{RP}^3$). We can choose a CW-structure (not necessary $\mathbb{Z}_2$ invariant), such that this embedded circle is the 1-skeleton. The 2-cell mapped into the 1-skeleton by the map $2\mathrm{id}\colon S^1 \to S^1$. Let $m_{S^1} : S^1 \to S^1$ be the restriction of $m\colon \mathbb{RP}^3 \to S^1$ into $C \subset \mathbb{RP}^3$. We know, that m_{S^1} has odd degree. So $2\mathrm{id} \circ m_{S^1}$ is not null-homotopic. But on the other hand m maps this 2-cell into S^1 so it were null-homotopic. $\square$

Remark 3.12. *Since* $S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^3$, *and the following homology groups of* $\mathbb{RP}^3$ $\mathrm{H}_1(\mathbb{RP}^3)$ *and* $\mathrm{H}_2(\mathbb{RP}^3)$ *are finite, we have that* $\mathrm{coindex}(\mathbb{RP}^3) = 1$ *using Corollary 4.12.*

[2] $\mathbb{RP}^3$ is identified with the set of lines going through the origin of $\mathbb{C}^2 = \mathbb{R}^4$.

3.4 Non-tidy $\mathbb{Z}_p$-spaces

Similarly to the projective space example there are non-tidy $\mathbb{Z}_p$-spaces as well (p is an odd prime). A $\mathbb{Z}_n$*-space* is a pair (X, ν) where X is a topological space and $\nu\colon X \to X$, called the $\mathbb{Z}_n$-action, is a homeomorphism such that $\nu^n = \mathrm{Id}_X$. Their $\mathbb{Z}_n$-index can be defined similarly as the $\mathbb{Z}_2$-index. Now instead of the sphere we will use the $E_n\mathbb{Z}_p$ spaces as yardsticks. They can by defined as iterated joins of p points, $E_n\mathbb{Z}_p = E_{n-1}\mathbb{Z}_p * \mathbb{Z}_p$ $(E_0\mathbb{Z}_p = \mathbb{Z}_p)$.

Definition 3.13. Let X be a $\mathbb{Z}_p$-space. The $\mathbb{Z}_p$-index and $\mathbb{Z}_p$-coindex are defined as follows (see e.g. [Mat03]).

$$\mathrm{ind}_{\mathbb{Z}_p}(X) = \min\{n \in \mathbb{N}\colon \text{ there is a } \mathbb{Z}_p\text{-map } X \to E_n\mathbb{Z}_p = (\mathbb{Z}_p)^{*(n+1)}\},$$

$$\mathrm{coind}_{\mathbb{Z}_p}(X) = \max\{n \in \mathbb{N}\colon \text{ there is a } \mathbb{Z}_p\text{-map } E_n\mathbb{Z}_p \to X\}.$$

Lens spaces are a 'generalization' of the projective spaces. Let $S^{2n-1} \subset \mathbb{C}^n$ be the unit sphere. Now the multiplication by $e^{\frac{2\pi i}{m}}$ defines a free $\mathbb{Z}_m$-action. We choose a $\mathbb{Z}_{p^2}$-action, which gives us a free $\mathbb{Z}_p$-action on $\mathsf{L}^{2n-1}(p) = S^{2n-1}/\mathbb{Z}_p$. Meyer proved in [Mey98] that

$$2\left\lceil \frac{n-2}{p} \right\rceil \leq \mathrm{ind}_{\mathbb{Z}_p}\left(\mathsf{L}^{2n-1}(p)\right) \leq 2\left\lceil \frac{n-2}{p} \right\rceil + 3,$$

if p is an odd prime (here $\lceil x \rceil$ denotes the smallest integer bigger or equal to x, i.e. $\lceil x \rceil = -[-x]$.)

Remark 3.14. *The importance of this result is that*

$$\lim_{n\to\infty} \mathrm{ind}_{\mathbb{Z}_p}\left(\mathsf{L}^{2n-1}(p)\right) = \infty.$$

Lemma 3.15. $\mathrm{coind}_{\mathbb{Z}_p}\left(\mathsf{L}^{2n-1}(p)\right) = 1.$

Proof. Similarly as in [Živ02]. We will use that $S^1 = E_1\mathbb{Z}_p$ ($S^1 := \{z \in \mathbb{C}\colon |z| = 1\}$, the $\mathbb{Z}_p$-action is given by the multiplication with $e^{\frac{2\pi i}{p}}$) and that $(E_1\mathbb{Z}_p = S^1 \subset S^1 * \mathbb{Z}_p =)E_2\mathbb{Z}_p$ is 1-connected. $e^{\alpha i} \to \left(e^{\frac{\alpha i}{p}}, \ldots, e^{\frac{\alpha i}{p}}\right) \in \mathbb{C}^n$, $\alpha \in [0, 2\pi]$ is a $\mathbb{Z}_p$-map $S^1 \to \mathsf{L}^{2n-1}(p)$.

Let $\mu\colon S^1 \to \mathsf{L}^{2n-1}(p)$ be any $\mathbb{Z}_p$-map. If we have a $\mathbb{Z}_p$-map $E_2\mathbb{Z}_p \to \mathsf{L}^{2n-1}(p)$ then the restriction into S^1 ($S^1 \subset E_2\mathbb{Z}_p$) is null-homotopic. So it is enough to show that $[\mu] \in \pi_1\left(\mathsf{L}^{2n-1}(p)\right) \approx \mathbb{Z}_p$ $(n > 1)$ is non-zero. (We will see that $[\mu] = 1$.)

We need the universal p-cover $S^{2n-1} \to \mathsf{L}^{2n-1}(p)$. A point in $\mathsf{L}^{2n-1}(p)$ is considered as a p-fan (p planar half-lines emanating from the origin and the angle between neighbor half-lines is $\frac{2\pi}{p}$). Let $\mu(+1) =: F$ ($F \in \mathsf{L}^{2n-1}(p)$ is a p-fan) and then $\mu(e^{\frac{k2\pi i}{p}}) = e^{\frac{k2\pi i}{p^2}} F$. We lift F into S^{2n-1} which means that we choose a unit vector ($v \in S^{2n-1}$) on the p-fan. Now the lift of $e^{\frac{2\pi i}{p^2}} F$ can be any of $e^{\frac{(pk+i)2\pi}{p^2}} v$. This piece ($S^1$ between $+1$ and $e^{\frac{2\pi i}{p^2}}$) determines μ, and all of the lifts ends in $e^{\frac{2\pi i}{p}} v$ showing that μ is not null-homotopic. □

Corollary 3.16. $\mathsf{L}^{2n-1}(p)$ *is an example where the difference between the* $\mathbb{Z}_p$*-index and coindex can be arbitrarily large.*

Remark 3.17. *Bartsch* [Bar90] *claims, that for free G-spaces Fadell* [Fad80] *was the first to introduce the G-index(−1) as G-genus. But in fact it was introduced already by P. E. Conner and E. E. Floyd in* [CF60]. *Perhaps it was studied even before.*
First $\mathbb{Z}_p$*-actions were used in topological combinatorics by Bárány, Shlosman and Szűcs* [BSSz81].

3.5 The suspension and the index

We turn back to Theorem 3.1. We have already seen that the upper bound of the $\mathbb{Z}_2$-index of join of spaces can be tight. Moreover we know that for tidy spaces we always have equality. In this section we will see how one can construct examples showing that the lower bound can be tight as well, namely spaces such that $\max\left\{\operatorname{ind}(X), \operatorname{ind}(Y)\right\} = \operatorname{ind}(X * Y)$. In our example X will be $\mathcal{S}_f$ for an appropriate $f\colon S^{n+k-1} \to S^n$, and Y will be S^0. $X * S^0$ is called the *suspension* of X, and in this section we will denote it by ΣX. We know that $\mathcal{S}_f * S^0$ is $\mathcal{S}_{\Sigma f}$ where the map $\Sigma f\colon S^{n+k} \to S^{n+1}$ is the join of f and $\mathrm{id}\colon \mathrm{S}^0 \to \mathrm{S}^0$. The suspension gives us a group homomorphism as well $\Sigma\colon \pi_{n+k-1}(S^n) \to \pi_{n+k}(S^{n+1})$. The following theorem will help us to provide examples.

Theorem 3.18. *If* $f\colon S^{2n-1} \to S^n$ *such that* $\mathcal{H}(f) \neq 0$ *and* $\Sigma f\colon S^{2n} \to$

S^{n+1} *is null-homotopic then* $\max\left\{ \operatorname{ind}(\mathcal{S}_f)\,, \operatorname{ind}(S^0) \right\} = \operatorname{ind}(\mathcal{S}_f * S^0)$.

Proof. Lemma 3.8 shows that $\operatorname{ind}(\mathcal{S}_f) = n{+}1$. We know that $\operatorname{ind}(S^0) = 0$ so we have that $\max\left\{ \operatorname{ind}(\mathcal{S}_f)\,, \operatorname{ind}(S^0) \right\} = n+1$. According to the assumption Σf is null-homotopic. Remark 3.9 tells us that $\operatorname{ind}(\mathcal{S}_f * S^0) = \operatorname{ind}(\mathcal{S}_{\Sigma f}) = n{+}1$. So we proved that $\max\left\{ \operatorname{ind}(\mathcal{S}_f)\,, \operatorname{ind}(S^0) \right\} = n+1 = \operatorname{ind}(\mathcal{S}_f * S^0)$. □

It is not clear again how many different dimensional examples can we construct using Theorem 3.18. It is known that $|\pi_{4n-1}(S^{2n})| = \infty$ and $|\pi_{4n}(S^{2n+1})| < \infty$. So there is always a map $f\colon S^{4n-1} \to S^{2n}$ such that $\mathcal{H}(f) \neq 0$ and Σf is null-homotopic. Using this f, $X = \mathcal{S}_f$ is a $4n$-dimensional complex, with $Y = S^0$ they show the tightness of the lower bound.

For example if $h\colon S^3 \to S^2$ is the Hopf map then the Freudenthal Theorem ([Hat01] Corollary 4.24) tells us that $\Sigma\colon \pi_3(S^2) \to \pi_4(S^3)$, which is actually $\mathbb{Z} \to \mathbb{Z}_2$, is surjective. This means that Σh is the generator of $\pi_4(S^3)$. So $\mathcal{S}_h$ is not known to be a good choice for showing equality in the lower bound, but $\mathcal{S}_{2h}$ is since $\Sigma 2h$ is null-homotopic.

3.6 More simplicial complex examples

Now we present more simplicial complex examples. We know that $\mathcal{S}_h$ (h is the Hopf map) is a non-tidy space. If we attach one 4-cell to S^2 via h we get $\mathbb{CP}^2$. In [MS00] there is a simplicial complex $\mathbb{CP}^2$ obtained in this way (Table 3.2), where S^2 is the boundary of the simplex $ABCD$.

After a special barycentric subdivision, when we subdivide only S^2 (the boundary of $ABCD$) (and the neighbor simplices as well) we can define the simplicial $\mathbb{Z}_2$-action on this sphere by $\nu\colon H \to \{A, B, C, D\} \setminus H$ where $H \subseteq \{A, B, C, D\}$. Now we can attach the second 4-ball via $-h$ in order to get $\mathcal{S}_h$. It is the same as gluing two subdivided $\mathbb{CP}^2$ along there embedded spheres in an antipodal way. The result has 24 vertices an 252 top simplexes (Table 3.3). The $\mathbb{Z}_2$-action given by $\imath \to \tilde{\imath}$ for $\imath = 1, 2, 3, 4, 5$ and ν.

{1,2,A,B,C}	{4,5,A,C,D}	{1,2,3,4,5}	{1,3,B,C,D}	{3,4,A,B,D}
{2,3,5,B,D}	{2,3,A,B,C}	{4,5,B,C,D}	{1,2,3,4,D}	{3,5,B,C,D}
{1,3,4,B,D}	{2,3,A,B,D}	{3,1,A,B,C}	{4,5,A,B,D}	{1,2,3,5,D}
{1,4,B,C,D}	{2,3,4,A,D}	{2,3,5,B,C}	{1,3,4,5,A}	{1,2,5,A,B}
{1,5,A,C,D}	{1,3,5,A,C}	{1,4,5,A,B}	{1,2,5,A,D}	{1,2,4,5,B}
{1,2,A,C,D}	{3,4,5,A,C}	{2,5,A,B,D}	{1,3,5,C,D}	{1,3,4,A,B}
{2,3,4,5,C}	{1,2,4,B,C}	{2,4,A,C,D}	{2,4,5,B,C}	{1,2,4,C,D}
{2,3,4,A,C}				

Table 3.2: Simplicial $\mathbb{CP}^2$.

{1,2,A,AB,ABC}	{1,2,A,AC,ABC}	{1,2,B,AB,ABC}	{1,2,B,BC,ABC}
{1,2,C,AC,ABC}	{1,2,C,BC,ABC}	{4,5,A,AC,ACD}	{4,5,A,AD,ACD}
{4,5,C,AC,ACD}	{4,5,C,CD,ACD}	{4,5,D,AD,ACD}	{4,5,D,CD,ACD}
{1,2,3,4,5}	{1,3,B,BC,BCD}	{1,3,B,BD,BCD}	{1,3,C,BC,BCD}
{1,3,C,CD,BCD}	{1,3,D,BD,BCD}	{1,3,D,CD,BCD}	{3,4,A,AB,ABD}
{3,4,A,AD,ABD}	{3,4,B,AB,ABD}	{3,4,B,BD,ABD}	{3,4,D,AD,ABD}
{3,4,D,BD,ABD}	{2,3,5,B,BD}	{2,3,5,BD,D}	{2,3,A,AB,ABC}
{2,3,A,AC,ABC}	{2,3,B,AB,ABC}	{2,3,B,BC,ABC}	{2,3,C,AC,ABC}
{2,3,C,BC,ABC}	{4,5,B,BC,BCD}	{4,5,B,BD,BCD}	{4,5,C,BC,BCD}
{4,5,C,CD,BCD}	{4,5,D,BD,BCD}	{4,5,D,CD,BCD}	{1,2,3,4,D}
{3,5,B,BC,BCD}	{3,5,B,BD,BCD}	{3,5,C,BC,BCD}	{3,5,C,CD,BCD}
{3,5,D,BD,BCD}	{3,5,D,CD,BCD}	{1,3,4,B,BD}	{1,3,4,BD,D}
{2,3,A,AB,ABD}	{2,3,A,AD,ABD}	{2,3,B,AB,ABD}	{2,3,B,BD,ABD}
{2,3,D,AD,ABD}	{2,3,D,BD,ABD}	{3,1,A,AB,ABC}	{3,1,A,AC,ABC}
{3,1,B,AB,ABC}	{3,1,B,BC,ABC}	{3,1,C,AC,ABC}	{3,1,C,BC,ABC}
{4,5,A,AB,ABD}	{4,5,A,AD,ABD}	{4,5,B,AB,ABD}	{4,5,B,BD,ABD}
{4,5,D,AD,ABD}	{4,5,D,BD,ABD}	{1,2,3,5,D}	{1,4,B,BC,BCD}
{1,4,B,BD,BCD}	{1,4,C,BC,BCD}	{1,4,C,CD,BCD}	{1,4,D,BD,BCD}
{1,4,D,CD,BCD}	{2,3,4,A,AD}	{2,3,4,AD,D}	{2,3,5,B,BC}
{2,3,5,BC,C}	{1,3,4,5,A}	{1,2,5,A,AB}	{1,2,5,AB,B}
{1,5,A,AC,ACD}	{1,5,A,AD,ACD}	{1,5,C,AC,ACD}	{1,5,C,CD,ACD}
{1,5,D,AD,ACD}	{1,5,D,CD,ACD}	{1,3,5,A,AC}	{1,3,5,AC,C}
{1,4,5,A,AB}	{1,4,5,AB,B}	{1,2,5,A,AD}	{1,2,5,AD,D}
{1,2,4,5,B}	{1,2,A,AC,ACD}	{1,2,A,AD,ACD}	{1,2,C,AC,ACD}
{1,2,C,CD,ACD}	{1,2,D,AD,ACD}	{1,2,D,CD,ACD}	{3,4,5,A,AC}
{3,4,5,AC,C}	{2,5,A,AB,ABD}	{2,5,A,AD,ABD}	{2,5,B,AB,ABD}
{2,5,B,BD,ABD}	{2,5,D,AD,ABD}	{2,5,D,BD,ABD}	{1,3,5,C,CD}
{1,3,5,CD,D}	{1,3,4,A,AB}	{1,3,4,AB,B}	{2,3,4,5,C}
{1,2,4,B,BC}	{1,2,4,BC,C}	{2,4,A,AC,ACD}	{2,4,A,AD,ACD}
{2,4,C,AC,ACD}	{2,4,C,CD,ACD}	{2,4,D,AD,ACD}	{2,4,D,CD,ACD}
{2,4,5,B,BC}	{2,4,5,BC,C}	{1,2,4,C,CD}	{1,2,4,CD,D}
{2,3,4,A,AC}	{2,3,4,AC,C}	{$\tilde{1}$,$\tilde{2}$,BCD,CD,D}	{$\tilde{1}$,$\tilde{2}$,BCD,BD,D}
{$\tilde{1}$,$\tilde{2}$,ACD,CD,D}	{$\tilde{1}$,$\tilde{2}$,ACD,AD,D}	{$\tilde{1}$,$\tilde{2}$,ABD,BD,D}	{$\tilde{1}$,$\tilde{2}$,ABD,AD,D}
{$\tilde{4}$,$\tilde{5}$,BCD,BD,B}	{$\tilde{4}$,$\tilde{5}$,BCD,BC,B}	{$\tilde{4}$,$\tilde{5}$,ABD,BD,B}	{$\tilde{4}$,$\tilde{5}$,ABD,AB,B}
{$\tilde{4}$,$\tilde{5}$,ABC,BC,B}	{$\tilde{4}$,$\tilde{5}$,ABC,AB,B}	{$\tilde{1}$,$\tilde{2}$,$\tilde{3}$,$\tilde{4}$,$\tilde{5}$}	{$\tilde{1}$,$\tilde{3}$,ACD,AD,A}
{$\tilde{1}$,$\tilde{3}$,ACD,AC,A}	{$\tilde{1}$,$\tilde{3}$,ABD,AD,A}	{$\tilde{1}$,$\tilde{3}$,ABD,AB,A}	{$\tilde{1}$,$\tilde{3}$,ABC,AC,A}
{$\tilde{1}$,$\tilde{3}$,ABC,AB,A}	{$\tilde{3}$,$\tilde{4}$,BCD,CD,C}	{$\tilde{3}$,$\tilde{4}$,BCD,BC,C}	{$\tilde{3}$,$\tilde{4}$,ACD,CD,C}
{$\tilde{3}$,$\tilde{4}$,ACD,AC,C}	{$\tilde{3}$,$\tilde{4}$,ABC,BC,C}	{$\tilde{3}$,$\tilde{4}$,ABC,AC,C}	{$\tilde{2}$,$\tilde{3}$,$\tilde{5}$,ACD,AC}
{$\tilde{2}$,$\tilde{3}$,$\tilde{5}$,AC,ABC}	{$\tilde{2}$,$\tilde{3}$,BCD,CD,D}	{$\tilde{2}$,$\tilde{3}$,BCD,BD,D}	{$\tilde{2}$,$\tilde{3}$,ACD,CD,D}
{$\tilde{2}$,$\tilde{3}$,ACD,AD,D}	{$\tilde{2}$,$\tilde{3}$,ABD,BD,D}	{$\tilde{2}$,$\tilde{3}$,ABD,AD,D}	{$\tilde{4}$,$\tilde{5}$,ACD,AD,A}
{$\tilde{4}$,$\tilde{5}$,ACD,AC,A}	{$\tilde{4}$,$\tilde{5}$,ABD,AD,A}	{$\tilde{4}$,$\tilde{5}$,ABD,AB,A}	{$\tilde{4}$,$\tilde{5}$,ABC,AC,A}
{$\tilde{4}$,$\tilde{5}$,ABC,AB,A}	{$\tilde{1}$,$\tilde{2}$,$\tilde{3}$,$\tilde{4}$,ABC}	{$\tilde{3}$,$\tilde{5}$,ACD,AD,A}	{$\tilde{3}$,$\tilde{5}$,ACD,AC,A}
{$\tilde{3}$,$\tilde{5}$,ABD,AD,A}	{$\tilde{3}$,$\tilde{5}$,ABD,AB,A}	{$\tilde{3}$,$\tilde{5}$,ABC,AC,A}	{$\tilde{3}$,$\tilde{5}$,ABC,AB,A}
{$\tilde{1}$,$\tilde{3}$,$\tilde{4}$,ACD,AC}	{$\tilde{1}$,$\tilde{3}$,$\tilde{4}$,AC,ABC}	{$\tilde{2}$,$\tilde{3}$,BCD,CD,C}	{$\tilde{2}$,$\tilde{3}$,BCD,BC,C}
{$\tilde{2}$,$\tilde{3}$,ACD,CD,C}	{$\tilde{2}$,$\tilde{3}$,ACD,AC,C}	{$\tilde{2}$,$\tilde{3}$,ABC,BC,C}	{$\tilde{2}$,$\tilde{3}$,ABC,AC,C}
{$\tilde{3}$,$\tilde{1}$,BCD,CD,D}	{$\tilde{3}$,$\tilde{1}$,BCD,BD,D}	{$\tilde{3}$,$\tilde{1}$,ACD,CD,D}	{$\tilde{3}$,$\tilde{1}$,ACD,AD,D}
{$\tilde{3}$,$\tilde{1}$,ABD,BD,D}	{$\tilde{3}$,$\tilde{1}$,ABD,AD,D}	{$\tilde{4}$,$\tilde{5}$,BCD,CD,C}	{$\tilde{4}$,$\tilde{5}$,BCD,BC,C}
{$\tilde{4}$,$\tilde{5}$,ACD,CD,C}	{$\tilde{4}$,$\tilde{5}$,ACD,AC,C}	{$\tilde{4}$,$\tilde{5}$,ABC,BC,C}	{$\tilde{4}$,$\tilde{5}$,ABC,AC,C}
{$\tilde{1}$,$\tilde{2}$,$\tilde{3}$,$\tilde{5}$,ABC}	{$\tilde{1}$,$\tilde{4}$,ACD,AD,A}	{$\tilde{1}$,$\tilde{4}$,ACD,AC,A}	{$\tilde{1}$,$\tilde{4}$,ABD,AD,A}
{$\tilde{1}$,$\tilde{4}$,ABD,AB,A}	{$\tilde{1}$,$\tilde{4}$,ABC,AC,A}	{$\tilde{1}$,$\tilde{4}$,ABC,AB,A}	{$\tilde{2}$,$\tilde{3}$,$\tilde{4}$,BCD,BC}
{$\tilde{2}$,$\tilde{3}$,$\tilde{4}$,BC,ABC}	{$\tilde{2}$,$\tilde{3}$,$\tilde{5}$,ACD,AD}	{$\tilde{2}$,$\tilde{3}$,$\tilde{5}$,AD,ABD}	{$\tilde{1}$,$\tilde{3}$,$\tilde{4}$,$\tilde{5}$,BCD}
{$\tilde{1}$,$\tilde{2}$,$\tilde{5}$,BCD,CD}	{$\tilde{1}$,$\tilde{2}$,$\tilde{5}$,CD,ACD}	{$\tilde{1}$,$\tilde{5}$,BCD,BD,B}	{$\tilde{1}$,$\tilde{5}$,BCD,BC,B}
{$\tilde{1}$,$\tilde{5}$,ABD,BD,B}	{$\tilde{1}$,$\tilde{5}$,ABD,AB,B}	{$\tilde{1}$,$\tilde{5}$,ABC,BC,B}	{$\tilde{1}$,$\tilde{5}$,ABC,AB,B}
{$\tilde{1}$,$\tilde{3}$,$\tilde{5}$,BCD,BD}	{$\tilde{1}$,$\tilde{3}$,$\tilde{5}$,BD,ABD}	{$\tilde{1}$,$\tilde{4}$,$\tilde{5}$,BCD,CD}	{$\tilde{1}$,$\tilde{4}$,$\tilde{5}$,CD,ACD}
{$\tilde{1}$,$\tilde{2}$,$\tilde{5}$,BCD,BC}	{$\tilde{1}$,$\tilde{2}$,$\tilde{5}$,BC,ABC}	{$\tilde{1}$,$\tilde{2}$,$\tilde{4}$,$\tilde{5}$,ACD}	{$\tilde{1}$,$\tilde{2}$,BCD,BD,B}
{$\tilde{1}$,$\tilde{2}$,BCD,BC,B}	{$\tilde{1}$,$\tilde{2}$,ABD,BD,B}	{$\tilde{1}$,$\tilde{2}$,ABD,AB,B}	{$\tilde{1}$,$\tilde{2}$,ABC,BC,B}

$\{\bar{1},\bar{2},ABC,AB,B\}$	$\{\bar{3},\bar{4},\bar{5},BCD,BD\}$	$\{\bar{3},\tilde{4},\tilde{5},BD,ABD\}$	$\{\bar{2},\bar{5},BCD,CD,C\}$
$\{\tilde{2},\tilde{5},BCD,BC,C\}$	$\{\tilde{2},\tilde{5},ACD,CD,C\}$	$\{\tilde{2},\bar{5},ACD,AC,C\}$	$\{\tilde{2},\bar{5},ABC,BC,C\}$
$\{\bar{2},\bar{5},ABC,AC,C\}$	$\{\bar{1},\tilde{3},\tilde{5},ABD,AB\}$	$\{\bar{1},\tilde{3},\tilde{5},AB,ABC\}$	$\{\bar{1},\tilde{3},\tilde{4},BCD,CD\}$
$\{\tilde{1},\tilde{3},\tilde{4},CD,ACD\}$	$\{\tilde{2},\tilde{3},\tilde{4},\tilde{5},ABD\}$	$\{\tilde{1},\bar{2},\bar{4},ACD,AD\}$	$\{\tilde{1},\bar{2},\bar{4},AD,ABD\}$
$\{\tilde{2},\tilde{4},BCD,BD,B\}$	$\{\tilde{2},\tilde{4},BCD,BC,B\}$	$\{\bar{2},\bar{4},ABD,BD,B\}$	$\{\bar{2},\tilde{4},ABD,AB,B\}$
$\{\tilde{2},\tilde{4},ABC,BC,B\}$	$\{\tilde{2},\tilde{4},ABC,AB,B\}$	$\{\tilde{2},\bar{4},\bar{5},ACD,AD\}$	$\{\tilde{2},\bar{4},\bar{5},AD,ABD\}$
$\{\bar{1},\bar{2},\bar{4},ABD,AB\}$	$\{\bar{1},\tilde{2},\tilde{4},AB,ABC\}$	$\{\tilde{2},\tilde{3},\tilde{4},BCD,BD\}$	$\{\tilde{2},\tilde{3},\tilde{4},BD,ABD\}$

Table 3.3: Simplicial $\mathcal{S}_h$.

Using a simplicial model for $2h\colon S^3_{12} \to S^2_4$ [Mad02, MS00] one can obtain a simplicial complex model for $\mathcal{S}_{2h}$ as well.

After that one can construct similarly as before a simplicial model of $\mathcal{S}_{2h}$. Unfortunately the triangulation we obtained contains 896 vertices and 14688 top simplices. One can clearly get a much smaller complex.

3.7 Open problems

Unfortunately this construction provides only examples, where $\mathrm{ind}(X) - \mathrm{coindex}(X) \leq 1$. It is known that this defect $\mathrm{ind}(X) - \mathrm{coindex}(X)$ can be arbitrarily large [Živ02] (see Section 3.3). Going back to the original question, we have seen only examples where $(\mathrm{ind}(X) + \mathrm{ind}(Y) + 1) - \mathrm{ind}(X * Y) \leq 1$. Can this defect become arbitrarily large?

Moreover one should answer this question for $\mathbb{Z}_p$-spaces as well.

Chapter 4

Graph coloring manifolds

Our starting point was Lovász's conjecture (see Subsection 4.2.2). Although Babson and Kozlov [BK04] proved this conjecture, the homotopy type of the Hom complex $\mathrm{Hom}(C_5, K_n)$ is not fully understood at present. We tried to prove my conjecture (see Subsection 4.2.2) which says that these are Stiefel manifolds. We studied when $\mathrm{Hom}(G, K_n)$ will be a manifold. The most interesting example is C_5 which means that the Hom complexes $\mathrm{Hom}(C_5, K_n)$ are manifolds indeed. We show that $\mathrm{Hom}(C_5, K_4)$ is homeomorphic to $\mathbb{RP}^3$ and explain some difficulty of the proof of the Lovász's conjecture. We present at the end of this chapter more low dimensional manifold examples.

4.1 Vertex-stars and flag simplicial spheres

Babson and Kozlov asked in [BK03] for what graphs the Hom complex construction provides a connection to polytopes. In this section, we will characterize those graphs G for which $\mathrm{Hom}(G, K_n)$ is a piecewise linear (PL) manifold for all $n \geq \chi(G)$.

A (finite) simplicial or polytopal complex is a *combinatorial* or *PL d-manifold* if and only if every vertex-star, i.e., the collection of facets that contains the respective vertex, is a PL d-ball. A PL ball is a politopal complex whose underlying space is piecewise linearly homeomorphic to the simplex.

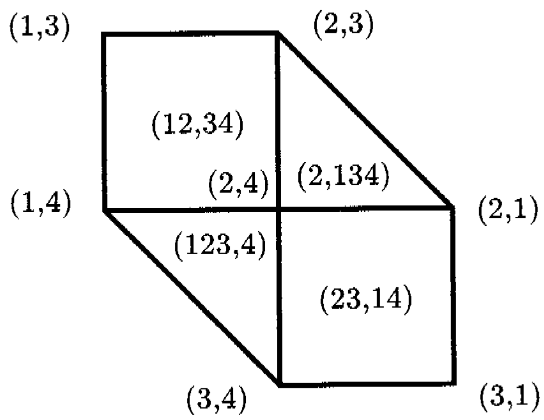

Figure 4.1: The star of the vertex $(2,4)$ in $\mathrm{Hom}(K_2, K_4)$.

Example: In $\mathrm{Hom}(K_2, K_4)$, the vertex $(2,4)$ is contained in the four facets $(12,34)$, $(2,134)$, $(23,14)$, and $(123,4)$, as displayed in Figure 4.1. From the figure we see that the vertex-star has an almost product-like structure: Since the colors 1 and 3 are not present in the vertex $(2,4)$, we can independently add them either to the first or to the second position. If the color 1 is added to the left position, then we move to the left, otherwise we move to the right. On the other hand, if the color 3 is added to the left position, then we move downwards, and upwards otherwise.

Example: We write every cell of $\mathrm{Hom}(C_5, K_n)$ as a vector $(A_1, A_2, A_3, A_4, A_5)$ of non-empty sets $A_i \subseteq V(K_n) = \{1, \ldots, n\}$, $1 \leq i \leq 5$, with the property that $A_i \cap A_{i+1} = \emptyset$, where the indices are taken modulo 5. It is obvious that every $k \in V(K_n)$ can occur in at most two of the sets A_i on the circle C_5. On the other hand, if a number $k \in V(K_n)$ is contained in only one or in none of the sets of a face $(A_1, A_2, A_3, A_4, A_5)$ of $\mathrm{Hom}(C_5, K_n)$, then this face can be complemented by adding one or two copies of k to appropriate sets of the face, respectively. Since this can be done independently for all the elements (color classes) of $V(K_n)$, any facet of $\mathrm{Hom}(C_5, K_n)$ contains exactly two copies of the elements of $V(K_n)$ and therefore has dimension $d = 2n-5$. The star of the vertex $(1,2,1,2,3)$ in $\mathrm{Hom}(C_5, K_4)$ consists of ten facets, as displayed in Figure 4.2. The colors 1 and 2 already occur twice in

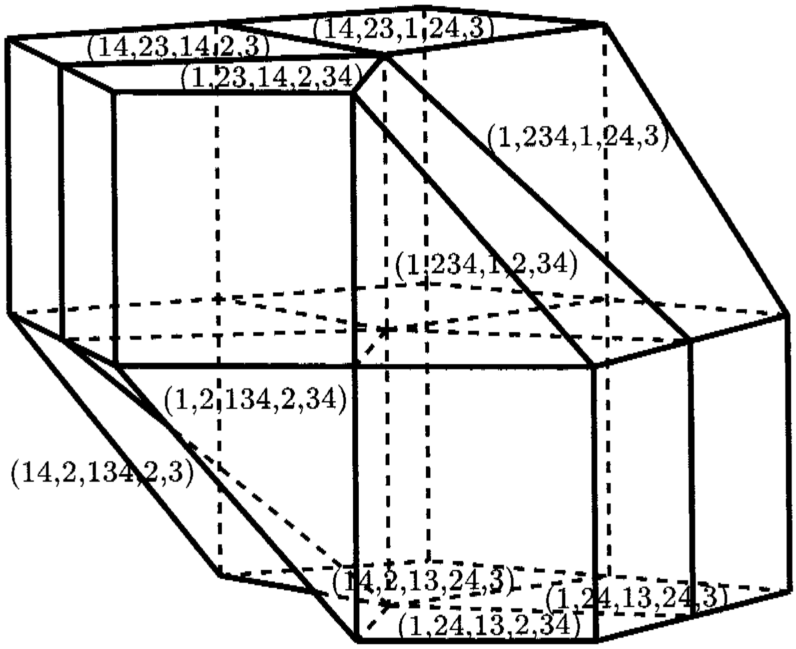

Figure 4.2: The star of the vertex (1,2,1,2,3) in the Hom complex $\mathrm{Hom}(C_5, K_4)$.

(1,2,1,2,3) and thus cannot be used further. The color 3 is used once. If we place its second copy at the second position, then we move upwards. If we place it at the third position, then we move downwards. Altogether, there are five choices to place the two copies of the missing color 4, by which we move around the axis vertical to the central horizontal pentagon. Again, the vertex-star has a product like structure.

Let $\eta \in \mathrm{Hom}(G, K_n)$ be a vertex of the Hom complex $\mathrm{Hom}(G, K_n)$ for which its vertex-star $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ is a PL ball. In the examples above we have seen that the vertex-stars have a product like structure, since colors that are missing in the vertex η can independently be placed to complement η to a facet of $\mathrm{Hom}(G, K_n)$.

We will show in the following that $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ is PL homeomorphic to a cubical complex $X_\eta = X_1 \times \cdots \times X_n$, where each factor X_i, $1 \leq i \leq n$ coresponds to the vertex $i \in K_n$. Since a product $X_1 \times \cdots \times X_n$ is a PL ball if and only if each factor is a PL ball (see [RS82, 2.24 (3)+(5)]), it remains to analyze the factors and to give the PL homeomorphism from $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ to X_η.

We define the factors X_i, $1 \leq i \leq n$, as the collection of cells for which the vertex η is complemented by color i only. Clearly, X_i is a cubical complex, since every of its maximal cells is a product of intervals (and vertices).

In the previous example the maximal cells of the factor X_4 for the vertex $(1,2,1,2,3)$ in $\mathrm{Hom}(C_5, K_4)$ are $(14,2,14,2,3)$, $(1,2,14,2,34)$, $(1,24,1,2,34)$, $(1,24,1,24,3)$, and $(14,2,1,24,3)$. There are three different types of factors for a vertex η of $\mathrm{Hom}(C_5, K_4)$. If the color h is used twice in η, then X_h is a point. If the color i is used once in η, then X_i consists of two intervals. If the color j is not used in η, then X_j consists of five squares. See Figure 4.3 for the factors X_i and X_j (of the vertices $(*,*,*,i,*)$ and $(*,*,*,*,*)$, respectively).

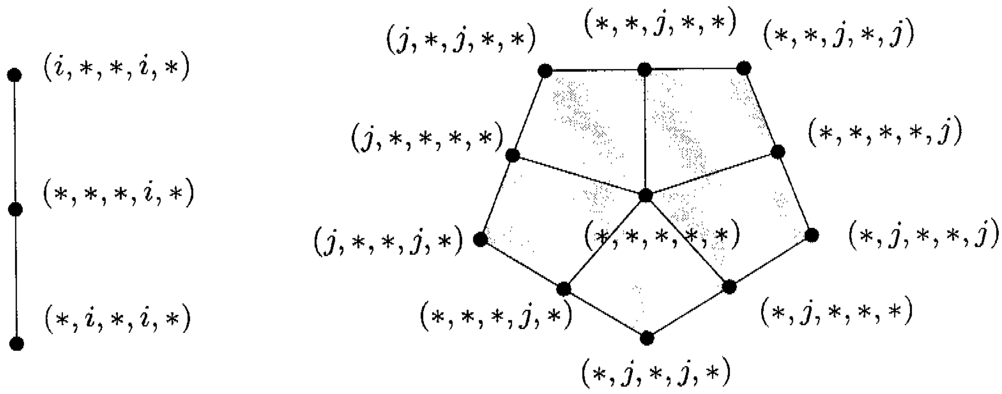

Figure 4.3: Factors of the cubical complex X_η for $G = C_5$.

The PL homeomorphism from $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ to X_η is given by appropriately subdividing facets of the cells of $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$. Every cell ξ of $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ is a product of $|V(G)|$ simplices $\xi(1),\ldots,\xi(|V(G)|)$ with $\eta(v) \in \xi(v)$ for every $v \in V(G)$. We consider each simplex $\xi(v)$ as a cone with apex $\eta(v)$, and we barycentricly subdivide the simplex $\xi(v) \setminus \eta(v)$ as in Figure 4.4. In this way, $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ becomes a cubical complex X_η which is isomorphic to $X_1 \times \cdots \times X_n$.

It remains to discuss, when the factors are indeed PL balls. Suppose that color j is not used in the vertex η, then we can complement η to a facet of the corresponding factor X_j by placing the color j at all positions of a maximal independent set of G. By this correspondence, we see that X_j is isomorphic to the cubical cone over the (simplicial) independence complex $\mathrm{Ind}(G)$ of G. Thus, X_j is a PL ball if and only if $\mathrm{Ind}(G)$ is a PL sphere. In terms of the complement $\overline{G}$ of G, the independence complex $\mathrm{Ind}(G)$ is a PL sphere if and only if the clique complex $\mathrm{Cliq}(\overline{G})$ is a PL sphere.

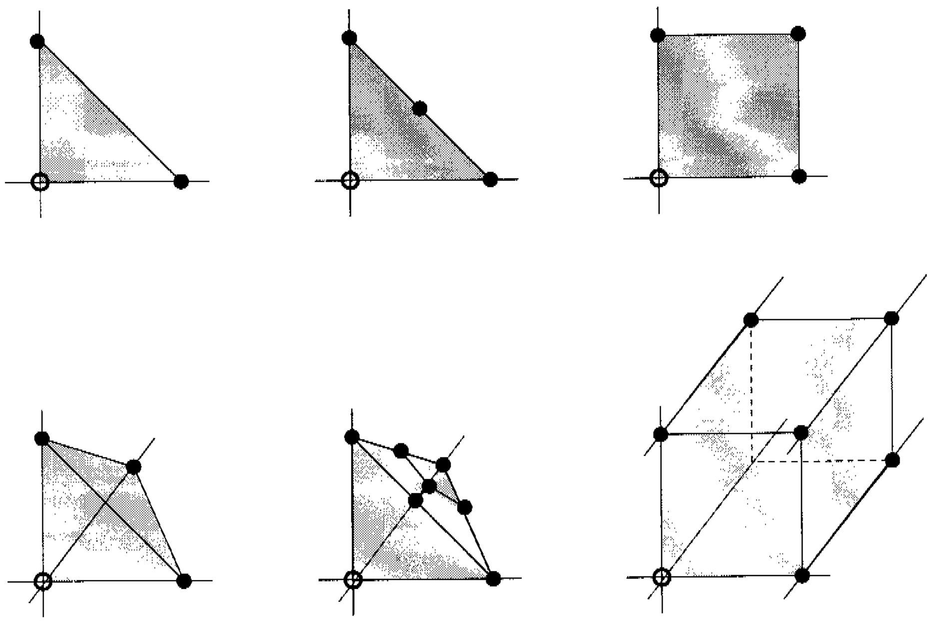

Figure 4.4: Simplex $\xi(v)$ with apex $\eta(v)$ and the barycentric subdivision of $\xi(v) \setminus \eta(v)$ which is identified with a cube.

Definition 4.1. Let K be a (finite) simplicial complex. If K has no "empty simplices", i.e., if every set of vertices of K which form a clique in the 1-skeleton actually spans a simplex, then K is a *flag simplicial complex* (c.f. [CD95]). A *flag simplicial sphere* is a flag simplicial complex which triangulates a sphere.

Theorem 4.2. *Let G be a graph. Then the Hom complex* $\mathrm{Hom}(G, K_n)$ *is a PL manifold for all $n \geq \chi(G)$ if and only if G is the complement of the* 1*-skeleton of a flag simplicial PL sphere.*

Proof. Let $n > \chi(G)$ and let $\mathrm{Hom}(G, K_n)$ be a PL manifold. Since $n > \chi(G)$, there is at least one vertex η of $\mathrm{Hom}(G, K_n)$ that only uses the colors $1, \ldots, \chi(G)$. By the above discussion, $\mathrm{star}_{\mathrm{Hom}(G,K_n)}(\eta)$ is a PL ball if and only if all factors $X_1, \ldots, X_n$ are PL balls. In particular, X_n has to be a PL ball. Since the color n is not used for the vertex η, $\mathrm{Ind}(G)$ is required to be a flag simplicial PL sphere. In other words, G is the complement of the 1-skeleton of a flag simplicial PL sphere. (The factors X_i for $\chi(G) + 1 \leq i \leq n$ are all isomorphic to X_n and therefore are PL balls as well.)

For every $1 \leq i \leq \chi(G)$, the color i is used at least once in the vertex η. In order to complement η to facets of the factor X_i, the color i can be placed at positions of η corresponding to independent sets of G, which contain the positions at which i is already present in η. It follows that X_i is the cubical cone over the link of a face in $\mathrm{Ind}(G)$. Since $\mathrm{Ind}(G)$ is a flag simplicial PL sphere, then also every link of a simplex in $\mathrm{Ind}(G)$ is a flag simplicial PL sphere, and thus X_i is again a PL ball.

Vice versa, let G be the complement of the 1-skeleton of a flag simplicial PL sphere. Then all the factors X_i for all the vertices of $\mathrm{Hom}(G, K_n)$ are PL balls, and thus $\mathrm{Hom}(G, K_n)$ is a PL manifold. □

Remark 4.3. *If* $n < \chi(G)$, *then* $\mathrm{Hom}(G, K_n) = \emptyset$. *If* $n = \chi(G)$, *then every vertex* η *of* $\mathrm{Hom}(G, K_{\chi(G)})$ *uses all colors* $1, \dots, \chi(G)$. *If* $\mathrm{Hom}(G, K_{\chi(G)})$ *is a PL manifold, then it is only required, that the links of vertices (or higher-dimensional faces if every color is used more than once in every vertex of* $\mathrm{Hom}(G, K_{\chi(G)})$*) of* $\mathrm{Ind}(G)$ *are flag simplicial PL spheres. It follows, in particular, that if* G *is the complement of a flag combinatorial manifold, then* $\mathrm{Hom}(G, K_{\chi(G)})$ *is a manifold. As another example, if* G *is a bipartite graph, then* $\mathrm{Hom}(G, K_2) = S^0$ *is a manifold.*

We note that if $\mathrm{Ind}(G)$ *is a pseudo-manifold, then* $\mathrm{Hom}(G, K_n)$ *is clearly a pseudo-manifold.*

Definition 4.4. A Hom complex $\mathrm{Hom}(G, K_n)$ is a *graph coloring manifold* if G is the complement of the 1-skeleton of a flag simplicial PL sphere.

Babson and Kozlov [BK03, Section 2.4] stated as a basic property of Hom complexes that

$$\mathrm{Hom}(G_1 \dot{\cup} G_2, H) = \mathrm{Hom}(G_1, H) \times \mathrm{Hom}(G_2, H), \tag{4.1}$$

from which it follows that if $G = \dot{\bigcup}_{i=1,\dots,k} K_2$ is the complement of the 1-skeleton of the boundary of the k-dimensional crosspolytope, then

$$\mathrm{Hom}\Big(\dot{\bigcup_{i=1,\dots,k}} K_2, K_n\Big) = \prod_{i=1,\dots,k} S^{n-2}. \tag{4.2}$$

Definition 4.5. A flag simplicial PL sphere is *prime* if the complement of its 1-skeleton is connected. A Hom complex $\mathrm{Hom}(G, K_n)$ is a *graph coloring manifold of sphere dimension* d if G is the complement of the 1-skeleton of a prime flag simplicial PL sphere of dimension d.

Since every coloring of a graph G can be regarded as a covering of G by independent sets, the following lower bound holds for the chromatic number $\chi(G)$ of G:

$$\chi(G) \geq \left\lceil \frac{|V|}{\alpha(G)} \right\rceil = \left\lceil \frac{|V|}{\omega(\overline{G})} \right\rceil, \tag{4.3}$$

where $\alpha(G)$ is the *independence number* or *stable set number* of G (i.e., the maximum size of an independent set in G) and $\omega(G)$ is the *clique number* of G (i.e., the maximum size of a clique in G).

If G is the complement of the 1-skeleton of a prime flag simplicial *PL* d-sphere on n vertices, then $\alpha(G) = \omega(\overline{G}) = d+1$. Thus

$$\chi(G) \geq \left\lceil \frac{n}{d+1} \right\rceil \tag{4.4}$$

and

$$\dim(\mathrm{Hom}(G, K_{\chi(G)+k})) = (\chi(G)+k)(d+1) - n \tag{4.5}$$

for all $k \geq 0$.

The lower bound (4.4) can be arbitrarily bad: If G is the complement of the 1-skeleton of the suspension $S^0 * C_{2r+1}$ of an odd cycle C_{2r+1}, $r \geq 2$, then $\chi(G) = 2r+1 > \left\lceil \frac{n}{d+1} \right\rceil = \left\lceil \frac{2r+3}{3} \right\rceil$.

Theorem 4.6. (Čukić and Kozlov [ČK04b]) *Let G be a graph of maximal degree s, then the Hom complex* $\mathrm{Hom}(G, K_n)$ *is at least* $(n-s-2)$*-connected.*

It follows that graph coloring manifolds provide examples of highly connected manifolds.

Conjecture 4.7. *Graph coloring manifolds are orientable.*

Let G be the complement of a flag simplicial PL sphere. If G has maximal degree s, then $\mathrm{Hom}(G, K_n)$ is simply connected and thus orientable for $n \geq s+3$.

4.2 Graph coloring manifolds

Trivially, S^0, consisting of two isolated vertices, it is the only zero-dimensional flag simplicial sphere. The complement of its "1"-skeleton is the complete graph K_2. Hence, $\mathrm{Hom}(K_2, K_n) \cong S^{n-2} \quad (n \geq 2)$ are graph coloring manifolds.

The circles C_m of length $m \geq 4$ are the flag simplicial spheres of dimension one. For $m = 4$ we have that $\mathrm{SK}_1(\overline{C}_4) = \overline{C}_4 = K_2 \dot{\cup} K_2$ and

$$\mathrm{Hom}(K_2 \dot{\cup} K_2, K_n) = \mathrm{Hom}(K_2, K_n) \times \mathrm{Hom}(K_2, K_n) \cong S^{n-2} \times S^{n-2}.$$

If $m \geq 5$, then $\mathrm{SK}_1(\overline{C}_m) = \overline{C}_m$ is connected. In the following, we treat circles of odd and of even length separately.

4.2.1 Hom complexes of complements of odd circles

For $m = 5$, the Hom complexes $\mathrm{Hom}(\overline{C}_5, K_n)$ are perhaps the most interesting examples. It is easy to see that $\mathrm{Hom}(C_5, K_3)$ is homeomorphic to $S^1 \times S^0$.

Now we show that $\mathrm{Hom}(C_5, K_4)$ is homeomorphic to $\mathbb{RP}^3$. The collections of cells in the form $(ijk, *, *, *, *)$ form a solid torus. For example the collection $(123, *, *, *, *)$ is the same as the $(123, 4, *, *, 4)$, where the middle $(*, *)$ part is the six-gon $\mathrm{Hom}(K_2, K_3)$. So $(ijk, *, *, *, *)$ is the product of the triangle and a circle. The cells in the form $(ij, *, *, *, *)$ form a solid torus as well, as it can be seen on Figure 4.5. The cells in the form $(i, *, *, *, *)$ form a solid torus as well, it's boundary torus can be seen on Figure 4.6.

A meridian disk of the $(1, *, *, *, *)$ is on Figure 4.7, and its boundary is the red curve on Figure 4.6.

The complement of $(1, *, *, *, *)$ in $\mathrm{Hom}(C_5, K_4)$ is a solid torus as well. Its meridian curve is the blue dashed curve on Figure 4.6, which is a $(2, 1)$ curve. Figure 4.8 explains how can one get this blue meridian circe, since it is enough to understand how the pieces $(2, *, *, *, *)$, $(3, *, *, *, *)$, $(4, *, *, *, *)$ glued together. This $(2, 1)$ curve explains that after gluing them together we obtain the 3-dimensional projective space. $\square$

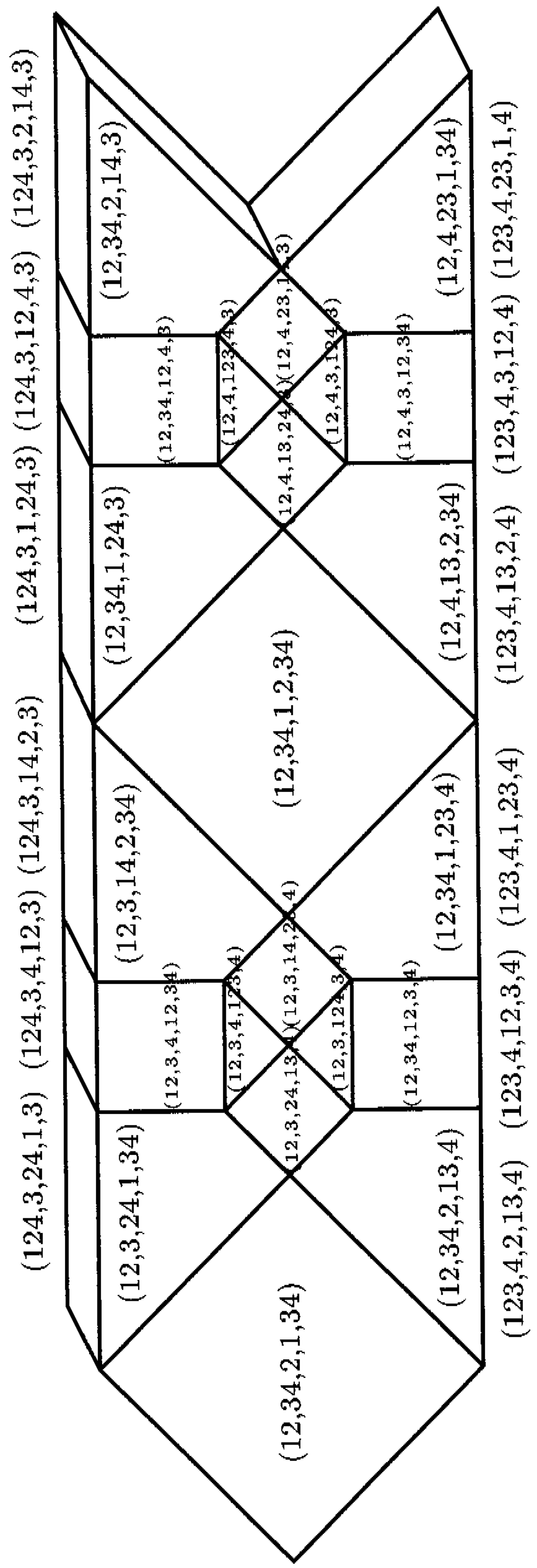

Figure 4.5: The solid torus $(12, *, *, *, *)$ in $\mathrm{Hom}(C_5, K_4)$.

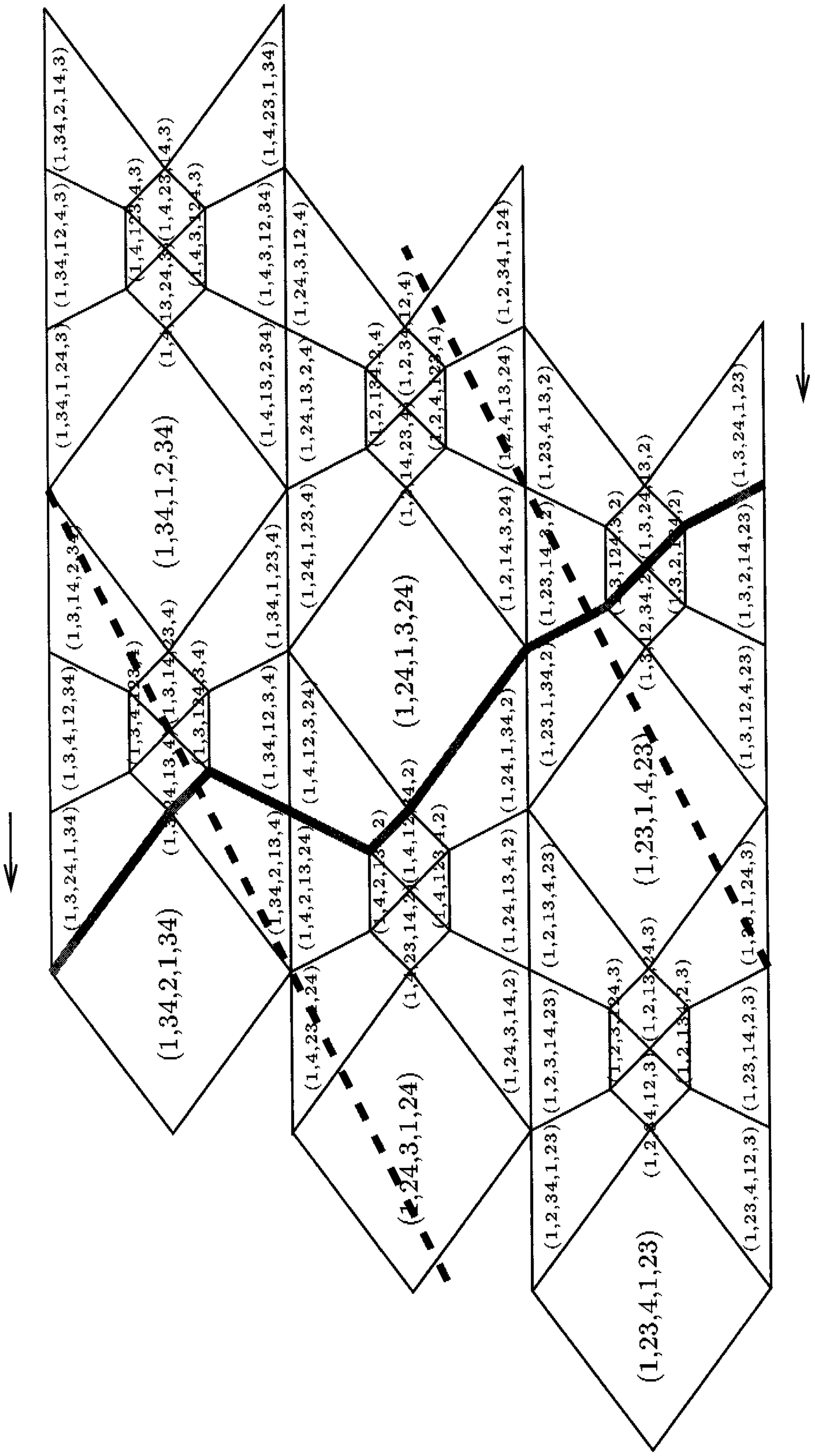

Figure 4.6: The boundary of the solid torus $(1,*,*,*,*)$ in $\mathrm{Hom}(C_5, K_4)$.

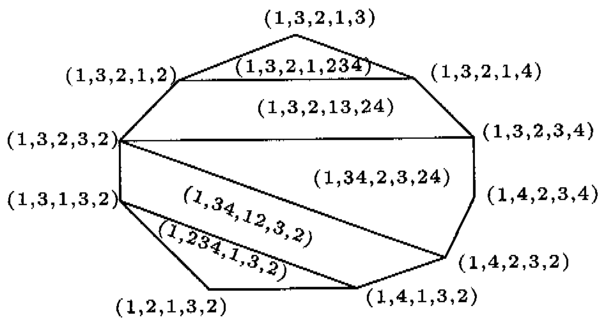

Figure 4.7: The meridian of the solid torus $(1,*,*,*,*)$ in $\mathrm{Hom}(C_5, K_4)$.

Now we turn to the general problem studing the topology of $\mathrm{Hom}(C_5, K_{n+2})$ We will try to prove similarly as above that $\mathrm{Hom}(C_5, K_{n+2})$ looks like a sphere bundle. We prove that the collections of cells $(12\ldots k,*,*,*,*)$ of $\mathrm{Hom}(C_5, K_{n+2})$ form a manifold with boundary. In order to do that we have to show that for every vertex $v \in (12\ldots k,*,*,*,*)$ the star of v is homeomorphic to a ball or to a half ball with v on the boundary. The proof is nearly the same as the proof of Theorem 4.2. Any number $k \in \{1,\ldots,n+2\}$ is either 0,1 or 2 times in v. If the number a is twice in v, then, as before, we define X_a to be a point. If the number b is once in v, for example if $v = (b,\star,\star,\star,\star)$, then we can use b one more time at the third or fourth place. For such a vertex let X_b be the subdivided interval as in Figure 4.3. However, if $v = (\star,\star,\star,b,\star)$, then when b is not in $\{1,\ldots,k\}$ a second copy of b can only be placed at the second position. In this and similar cases, we let X_b to be one single edge as in Figure 4.9. If the number $c \in \{1,\ldots,k\}$ is not in $v = (\star,\star,\star,\star,\star)$, then we can use c twice to complement v to a facet, and X_c can be choosen as in Figure 4.3. But if $c \notin \{1,\ldots,k\}$, then c cannot be placed at the first position. In this case we take for X_c a half disk as in Figure 4.9.

In this way, we define for each vertex $v \in (12\ldots k,*,*,*,*)$ a cubical complex $X_v := X_1 \times \cdots \times X_{n+2}$. As before, we can then identify X_v with the appropriately subdivided vertex-star of v. This shows that $(12\ldots k,*,*,*,*)$ is a manifold with boundary.

We prove that $(12\ldots k,*,*,*,*)$ is homotopy equivalent to S^{n-1}. We will use Lemma 1.15.

The cells $(12\ldots k,*,*,*,*)$ of $\mathrm{Hom}(C_5, K_{n+2})$ (with the incusion) defines a poset P. The map ϕ is defined as $\phi(12\ldots k,A,X,Y,B) :=$

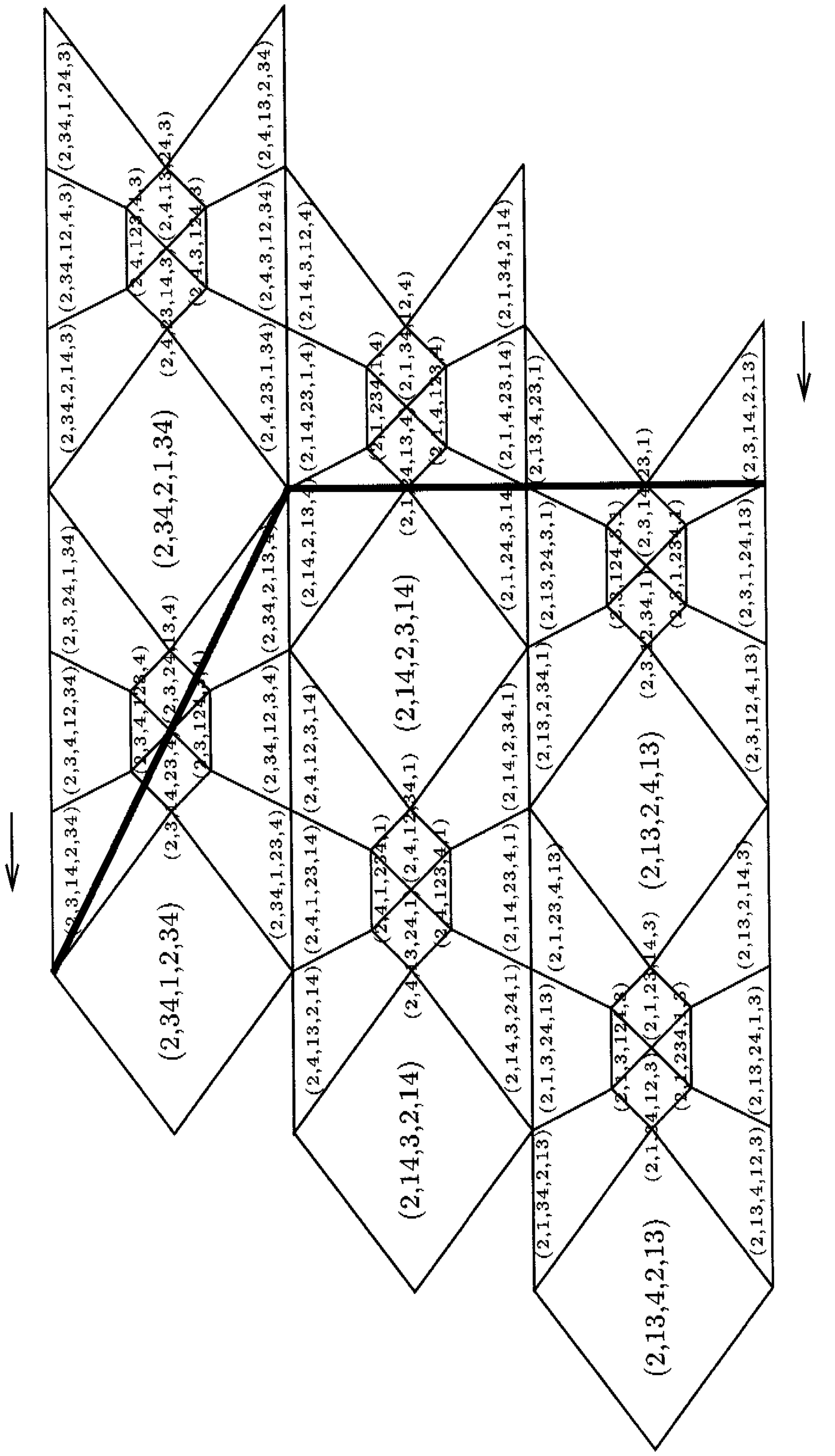

Figure 4.8: The boundary of the solid torus (2,∗,∗,∗,∗) and its meridian curve.

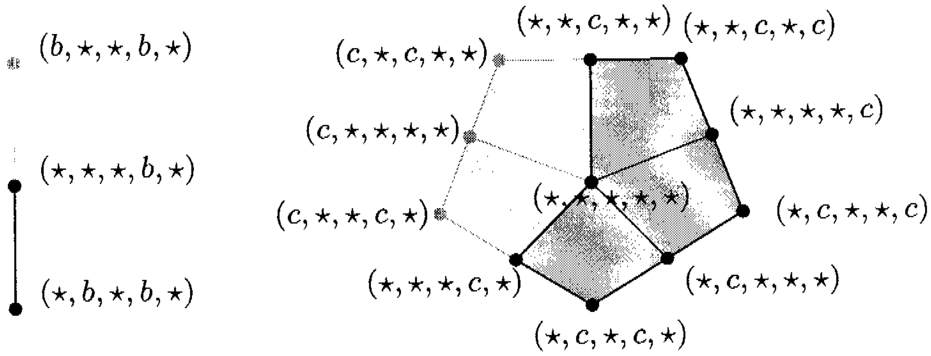

Figure 4.9: Factors of the cubical half ball X_v.

(X, Y), is the projection into the 3rd and 4th coordiante. The image Q is a subcomplex of the sphere $S^n \cong \mathrm{Hom}(K_2, K_{n+2})$. Using Lemma 1.15 we show that the projection is $\phi\colon P \to Q$ is a homotopy equivalence. We have to check that Condition (A) and (B) are fulfilled:

Condition (A): Let $q = (X, Y) \in Q$. $\phi^{-1}(q) = \{(12\ldots k, A, X, Y, B) | A \cap X = \emptyset, B \cap Y = \emptyset, A \cap 12..k = \emptyset, B \cap 12\ldots k = \emptyset\}$. There is a maximal element in $\phi^{-1}(q)$: $p_{\max} = (12\ldots k, A_{\max}, X, Y, B_{\max})$, with $A_{\max} = [n+2] \setminus [k] \setminus X$ and $B_{\max} = [n+2] \setminus [k] \setminus Y$. If $\emptyset \neq A \subseteq A_{\max}$, $\emptyset \neq B \subseteq B_{\max}$ then $(12\ldots k, A, X, Y, B) \in \phi^{-1}(q)$. So $\phi^{-1}(q)$ is the barycentric subdivision of $\Delta_{|A_{\max}|-1} \times \Delta_{|B_{\max}|-1}$ which is the product of two simplices. So it is contractible.

Condition (B): Let $q = (X, Y) \in Q$ and $p = (12\ldots k, A_p, X_p, Y_p, B_p)$ such that $X \subseteq X_p, Y \subseteq Y_p$. Now the maximal element of $\phi^{-1}(q) \cap P_{\leq p}$ is $(12\ldots k, A_p, X, Y, B_p)$.

Since $\phi\colon P \to Q$ is a homotopy equivalence the question is what is the image Q. The image is the subcomplex of the sphere $S^n = \mathrm{Hom}(K_2, K_{n+2})$. The missing elements of the poset $\mathrm{Hom}(\mathrm{K}_2, \mathrm{K}_{n+2})$ are $R := (*, *(k+1)\ldots(n+2))$ and $S := (*(k+1)\ldots(n+2), *)$ since they would be the images of $(12\ldots k, *, *, *(k+1)\ldots(n+2), \emptyset)$ and $(12\ldots k, \emptyset, *(k+1)\ldots(n+2), *, *)$. R and S are clearly n-dimensional disjoint subcomplexes of Q. In Section 1.6 we have already seen that R and S are disks and their complement is clearly homotopy equivalent to S^{n-1}. Here we use Lemma 1.15. We show that R and S are contractible. Since the simmetry of R and S it is enough to deal with R. First consider the following simplicial map: $f\colon R \to R$ defined by $(A, B(k+1)\ldots(n+2)) \to (A, (k+1)\ldots(n+2))$. The image $f(R)$ is a cone with apex $(12\ldots k, (k+1)\ldots(n+2))$ so it is contractible. Finally we have to show that f is a homotopy equivalence. We will use Lemma 1.15. We have to check that Condition (A) and (B) are fulfilled:

Condition (A): Let $q = (A,(k+1)\dots(n+2))$. $f^{-1}(q) = \{(A,B(k+1)\dots(n+2))\,|A\cap B = \emptyset, A\cap(k+1)\dots(n+2) = \emptyset, B\cap(k+1)\dots(n+2) = \emptyset\}$. There is a maximal element in $f^{-1}(q)$: $p_{\max} = (A,[n+2]\setminus A)$, the other elements of $f^{-1}(q)$ are its subsets. So $f^{-1}(q)$ is the cone over the barycentric subdivision of a simplex $\Delta_{n+1-|A|}$ ($B=\emptyset$ can be, which gives the apex of the cone). So $f^{-1}(q)$ is contractible.

Condition (B): Let $p = (A,B(k+1)\dots(n+2))$ and $q = (X,(k+1)\dots(n+2))$ such that $A \supseteq X$. Now the maximal element of $f^{-1}(q)\cap R_{\leq p}$ is $(X,B(k+1)\dots(n+2))$. □

By Theorem 2.36 we know that up to homotopy it is enough to glue together the pieces $(i,*,*,*,*)$ of $\operatorname{Hom}(C_5,K_{n+2})$. Now for a second assume that the pieces $(i,*,*,*,*)$ are $D^n\times S^{n-1}$. Observe that there are only $n+2$ of them for $\operatorname{Hom}(C_5,K_{n+2})$. In order to understand the topology of $\operatorname{Hom}(C_5,K_{n+2})$ it would be useful to find the meridian disk D^n of e.g. $(1,*,*,*,*)$. It would be the following collection of $n+3$ cells: $(1,3,2,1,23\dots(n+2))$, $(1,3,2,13,24\dots(n+2))$, $(1,34,2,3,24\dots(n+2))$, $\dots$, $(1,34\dots(n+2),2,3,2(n+2))$, $(1,34\dots(n+2),12,3,2)$, $(1,23\dots(n+2),1,3,2)$. Here $\dots$ means that the cell $(1,3\dots i,2,3,2i\dots(n+2))$ is followed by $(1,3\dots i(i+1),2,3,2(i+1)\dots(n+2))$.

4.2.2 The Lovász Conjecture

Using my conjecture one could find a new proof of the Lovász Conjecture in a special case. To state it we need to define the Stiefel manifolds $V_{n,k}$. They are the collection of k orthonormal vectors in $\mathbb{R}^n$:

$$V_{n,k} := \left\{(v_1,\dots,v_k)\;\middle|\; v_i\in\mathbb{R}^n; \langle v_i,v_j\rangle = \delta_{i,j}\right\}$$

Examples: $V_{n,1}=S^{n-1}$, $V_{2,2}=S^1\cup S^1$, and $V_{3,2}=\mathbb{RP}^3$.

Conjecture 4.8. *The Hom complex* $\operatorname{Hom}(C_5,K_{n+1})$ *is homeomorphic to the Stiefel manifold* $V_{n,2}$.

We could use the following result of Dai and Lam [DL84].

Theorem 4.9. *Let $\omega(v_1, v_2) = (v_1, -v_2)$ be a $\mathbb{Z}_2$-action on $V_{n,2}$.*

$$\begin{aligned} \operatorname{coindex}(V_{n,2}, \omega) = &\quad n-2 \\ \operatorname{ind}(V_{n,2}, \omega) = &\quad \begin{cases} n-1 \text{ if } n \neq 2,4,8 \\ n-2 \text{ if } n = 2,4,8 \end{cases} \end{aligned}$$

Let us note that ω and $\omega_2(v_1, v_2) = (v_2, v_1)$ are the same $\mathbb{Z}_2$-actions, but $\omega(v_1, v_2) = (-v_1, -v_2)$ can be different. The $\mathbb{Z}_2$-version of the previous conjecture is: The Hom complex $\operatorname{Hom}(C_5, K_{n+1})$ (the $\mathbb{Z}_2$-action given by reflecting an edge in C_5) is $\mathbb{Z}_2$-homeomorphic to the Stiefel manifold $(V_{n,2}, \omega)$.

Now it would be easy to prove Lovász's conjecture for the C_5 case.

Theorem 4.10. [Lovász Conjecture, proven by Babson and Kozlov [BK04].] *Let G be a graph, and let $k \in \mathbb{Z}$ such that $k \geq -1$. If $\operatorname{Hom}(C_5, G)$ is k-connected then $\chi(G) \geq k + 4$.*

Actually we could prove something slightly stronger.

Theorem 4.11 ([BK04]). *Let G be a graph, and let $k \in \mathbb{Z}$ such that $k \geq -1$. If $\operatorname{coindex}(\operatorname{Hom}(C_5, G)) \geq k + 1$ then $\chi(G) \geq k + 4$.*

Proof. It follows from Theorem 4.9 and Conjecture 4.8. □

The (co)homology computation of Babson and Kozlov [BK04] enough to show that $\operatorname{Hom}(C_5, K_{2n})$ are non-tidy spaces. This already explain some difficulty to prove Lovász's conjecture.

Corollary 4.12. *If a $\mathbb{Z}_2$-space X has finite cohomology group $H^n(X, \mathbb{Z})$ then $\operatorname{ind}(X) = \operatorname{coindex}(X) = n$ can not happen.*

Proof. By contradiction if there were $\mathbb{Z}_2$-maps $f\colon S^n \to X$ and $g\colon X \to S^n$ then the composition $H^n(S^n, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to H^n(S^n, \mathbb{Z})$ would be by Theorem 3.7 a multiplication by an odd number. On the other hand since $H^n(X, \mathbb{Z})$ is finite it is not possible, contradiction. □

The graph homomorphism $K_2 \to C_5$ gives a $\mathbb{Z}_2$-map $\phi\colon \operatorname{Hom}(C_5, K_n) \to \operatorname{Hom}(K_2, K_n) \cong S^{n-2}$. Another $\mathbb{Z}_2$-map

$$f : S^{n-3} \cong \operatorname{Hom}(K_2, K_{n-1}) \to \operatorname{Hom}(C_5, K_n)$$

can be defined by $f(x, y) := (n, x, y, n, n-1)$. If n is even than by the cohomology computation of Babson and Kozlov [BK04] we know that $H^{n-2}(\mathrm{Hom}(C_5, K_n), \mathbb{Z})$ and $H^{n-3}(\mathrm{Hom}(C_5, K_n), \mathbb{Z})$ are finite groups. Using Corollary 4.12 we get that $\mathrm{ind}(\mathrm{Hom}(C_5, K_{2k})) = 2k - 2$ and $\mathrm{coindex}(\mathrm{Hom}(C_5, K_{2k})) = 2k - 3$.

4.2.3 More small dimensional examples

Now we will gain some insight into all Hom complexes $\mathrm{Hom}(\overline{C}_{2r+1}, K_n)$ of complements of odd circles $\overline{C}_{2r+1}$, $r \geq 2$. We display the circles C_{2r+1}, $r \geq 2$, in form of a crown; see Figure 4.10 for the *crown representations* of C_5 and of C_7. Clearly, the bottom vertices of a crown representation form a clique, or to be precise, a complete graph K_r, in the complement $\overline{C}_{2r+1}$.

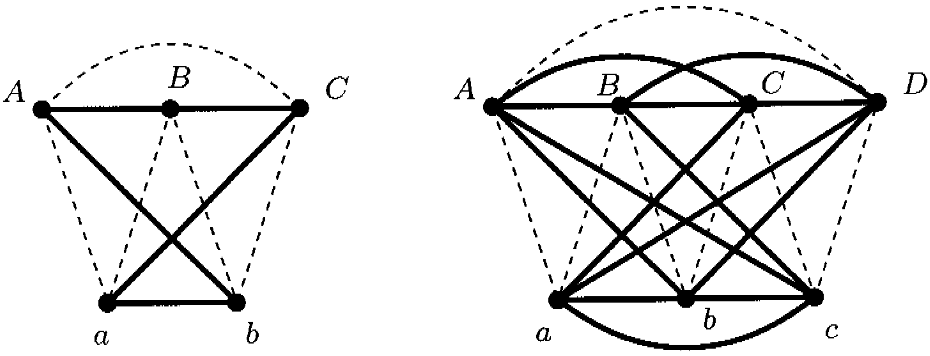

Figure 4.10: The (dashed) circles C_5 and C_7 and their complements.

Let us have a look at the crown representation of C_5. Every cell (a, b, A, B, C) of $\mathrm{Hom}(\overline{C}_5, K_n)$ contains every number $x \in \{1, \ldots, n\}$ at exactly two positions. Since the sets a and b are associated with the bottom vertices that form a clique K_2 in $\overline{C}_5$, the number x can appear in at most one of the sets a and b. If it is contained in, say, a, then the second copy of x can be placed only in the sets A and B that are connected to a by a dashed edge of C_5. The top vertices of $\overline{C}_5$ form a clique minus the (dashed) edge between the leftmost vertex and the rightmost vertex. Hence, if x is contained in neither a nor b, than it is contained in the leftmost top set A and in the rightmost top set C.

If we restrict us further to $n = 3$ colors, then $\mathrm{Hom}(K_2, K_3)$ is a sixgon that we display in solid in Figure 4.11. The cell $(a, b) = (1, 23)$ of

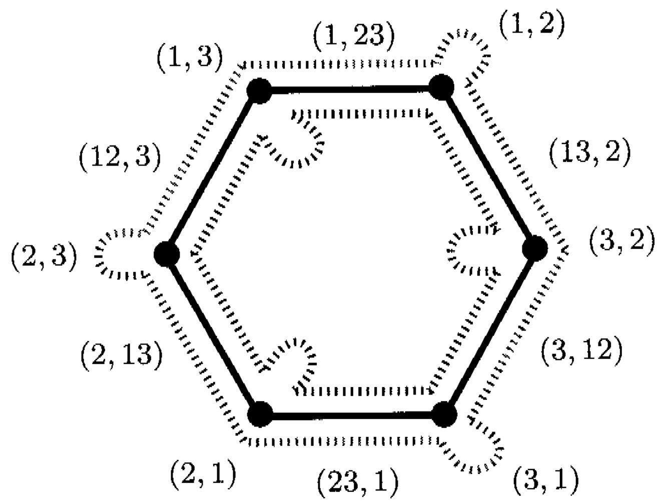

Figure 4.11: $\mathrm{Hom}(K_2, K_3)$ and $\mathrm{Hom}(\overline{C}_5, K_3)$.

$\mathrm{Hom}(K_2, K_3)$ can be extended to a cell $(a, b; A, B, C)$ of $\mathrm{Hom}(\overline{C}_5, K_3)$ in precisely two ways, either to $(1, 23; 1, 2, 3)$ or to $(1, 23; 1, 3, 2)$. We depict these edges of $\mathrm{Hom}(\overline{C}_5, K_3)$ as dashed edges in Figure 4.11, parallel to the edge $(1, 23)$ of $\mathrm{Hom}(K_2, K_3)$. Let $(1, 23; 1, 2, 3)$ be the upper dashed edge. If we move the number 3 from the second to the third position, then we obtain the cell $(1, 2; 13, 2, 3)$ from which we move on to $(1, 2; 3, 12, 3)$, and from there to $(1, 2; 3, 1, 23)$. These three cells of $\mathrm{Hom}(\overline{C}_5, K_3)$ correspond to the vertex $(1, 2)$ of $\mathrm{Hom}(K_2, K_3)$ and are together displayed by a dashed half-circle at the vertex $(1, 2)$ in Figure 4.11. If we move on, then we get to the dashed edge $(13, 2; 3, 1, 2)$, from there to the dashed edge $(3, 12; 3, 1, 2)$, before we again start a half-circle $(3, 1; 23, 1, 2)$, $(3, 1; 2, 13, 2)$, $(3, 1; 2, 3, 12)$, this time at the vertex $(3, 1)$ of $\mathrm{Hom}(K_2, K_3)$. We can then continue on the outer dashed circle until we reach our starting edge $(1, 23; 1, 2, 3)$ of $\mathrm{Hom}(\overline{C}_5, K_3)$. Similarly, we can move around the inner dashed circle when we start with $(1, 23; 1, 3, 2)$.

Proposition 4.13. *The Hom complex* $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ *is the disjoint union of* $r!$ *circles with* $(2r^2 + 3r + 1)$ *vertices each.*

Proof. We first count the number of vertices of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$, i.e., the number of distinct colorings with $r + 1$ colors of $\overline{C}_{2r+1}$. To color the bottom K_r in the crown representation of $\overline{C}_{2r+1}$ we choose r of the $r + 1$ colors and then have $r!$ choices to place these r colors. For one such coloring, say $(1, 2, \ldots, r)$, there are $(2r + 1)$ ways to extend it to a coloring of $\overline{C}_{2r+1}$: If we use the color $r+1$ just once, then we have $r+1$ choices to place it in the top row of the crown; the remaining positions for the colors in the top row are then completely determined by the position of the color $(r+1)$ and by our choice of the colors in the bottom row. If we use the color $r+1$ twice, then we have to put it at the positions 1 and $r+1$ of the top row. We further choose one of the colors $1,\dots,r$ not to be used in the top row; this again determines all the positions for the colors in the top row. Thus we have $(r+1)$ choices if color $r+1$ appears once in the top row and r choices if color $r+1$ appears twice in the top row. Altogether we have

$$\binom{r+1}{r} r!(r+1+r) = (2r^2+3r+1)r!$$

different colorings of $\overline{C}_{2r+1}$ with $r+1$ colors.

Since every number $1,\dots,r+1$ appears exactly twice in a cell of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$, the dimension of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ is $2(r+1) - (2r+1) = 1$. If we move for the edge $(1,2,\dots,r-1,r(r+1);1,2,\dots,r,r+1)$ of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ the number $r+1$ from the last position of the bottom row to the first position of the top row and then continue until we reach the edge $(r+1,1,2,\dots,(r-1)r;r+1,1,2,\dots,r-1,r)$, then this takes $r+1+r = 2r+1$ steps. After $r+1$ such rounds we return to our starting edge $(1,2,\dots,r-1,r(r+1);1,2,\dots,r-1,r,r+1)$. Hence, by symmetry, every circle of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ has length $(2r+1)(r+1) = 2r^2+3r+1$. Since $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ has $(2r^2+3r+1)r!$ vertices, it follows that $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ consists of $r!$ circles with $(2r^2+3r+1)$ vertices each. □

As before in the case of $\mathrm{Hom}(\overline{C}_5, K_3)$, every edge of $\mathrm{Hom}(K_r, K_{r+1})$ can be extended in exactly two ways to an edge of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$. We can interpret this behavior geometrically by thickening every edge of the 1-dimensional manifold $\mathrm{Hom}(K_r, K_{r+1})$ to a 2-dimensional strip and then gluing these strips together at the vertices of $\mathrm{Hom}(K_r, K_{r+1})$. As result, we get a two-dimensional manifold with boundary, where the boundary is homeomorphic to $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$.

In Figure 4.12 we display the Hom complex $\mathrm{Hom}(K_3, K_4)$, consisting of 24 vertices and 36 edges, together with two of the $3! = 6$ (dotted) circles of $\mathrm{Hom}(\overline{C}_7, K_4)$.

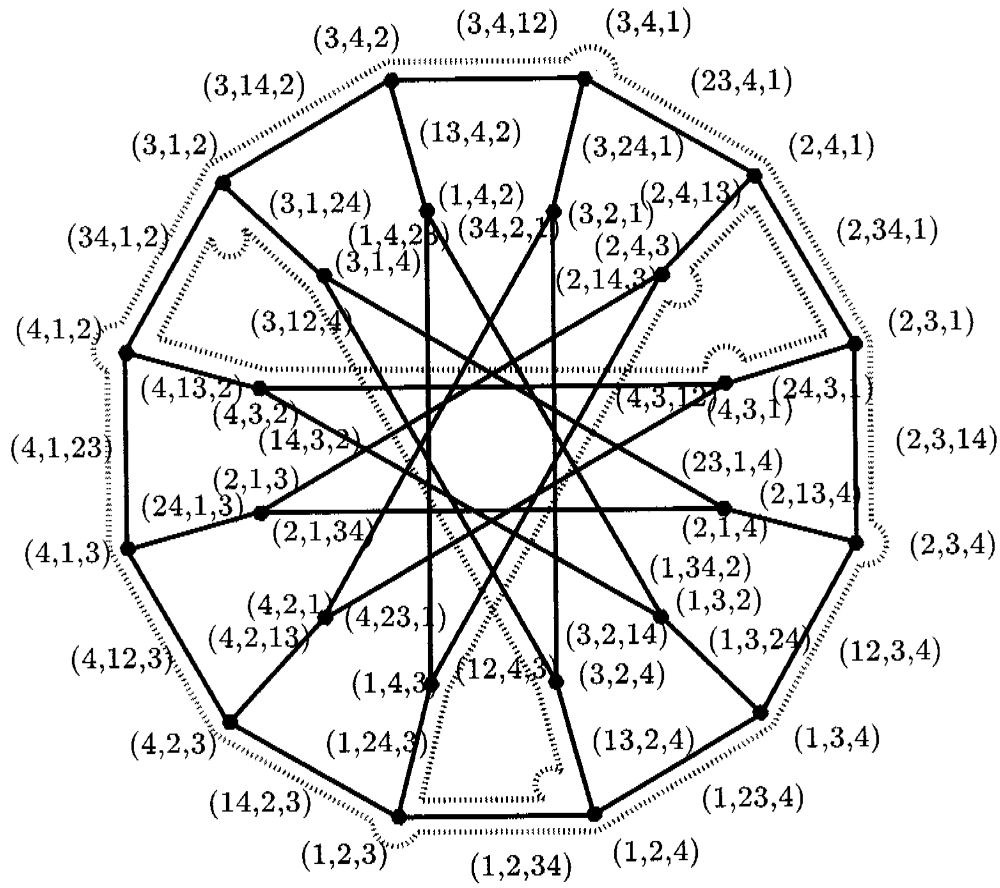

Figure 4.12: The Hom complex $\mathrm{Hom}(K_3, K_4)$.

Remark that every vertex of $\mathrm{Hom}(K_r, K_{r+1})$ can be extended in $r+1$ ways to an edge of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$. These $r+1$ edges form a path that we display as a half-circle in the Figures 4.11 and 4.12.

Conjecture 4.14 (Lutz). *The 3-dimensional graph coloring manifolds* $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+2})$, $r \geq 2$, *have homology* $H_*(\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+2})) = (\mathbb{Z}, \mathbb{Z}^{2(r!-2)} \oplus \mathbb{Z}_{r!}, \mathbb{Z}^{2(r!-2)}, \mathbb{Z})$.

Since every cell of a Hom complex is a product of simplices, triangulations (without additional vertices) can easily be obtained by the product triangulation construction. For the next example we computed the homology with a computer program [DHSW03].

Proposition 4.15. *The graph coloring manifold* $\mathrm{Hom}(\overline{C}_7, K_5)$ *with* 2520 *vertices has homology groups* $H_*(\mathrm{Hom}(\overline{C}_7, K_5)) = (\mathbb{Z}, \mathbb{Z}^8 \oplus \mathbb{Z}_6, \mathbb{Z}^8, \mathbb{Z})$.

4.2.4 Hom complexes of complements of even circles

Similar to the crown representation of (complements) of odd circles, we split the vertices of even circles C_{2r} into a lower and an upper part, corresponding to the bipartition of C_{2r}. The lower and also the upper part form a complete graph K_r in $\overline{C}_{2r}$, i.e., every maximal cell of $\mathrm{Hom}(\overline{C}_{2r}, K_r)$ contains each color $1, \ldots, r$ exactly twice, once in the lower part and once in the upper part. (Figure 4.13 displays C_6 and its complement $\overline{C}_6$ together with a cell $(a_1, a_2, a_3; A_1, A_2, A_3)$ of $\mathrm{Hom}(\overline{C}_6, K_r)$.)

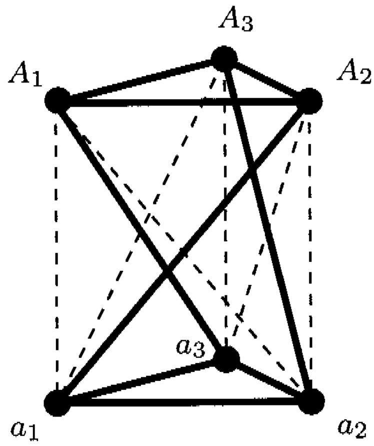

Figure 4.13: The circle C_6 (dashed) and its complement $\overline{C}_6$.

Proposition 4.16. (Babson and Kozlov [BK03]) *The Hom complex* $\mathrm{Hom}(K_s, K_r)$ *is homotopy equivalent to a wedge of* $f(s,r)$ *spheres of dimension* $r-s$*, where the numbers* $f(s,r)$ *satisfy the recurrence relation*

$$f(s,r) = sf(s-1, r-1) + (s-1)f(s, r-1),$$

for $r > s \geq 2$*; with the boundary values* $f(r,r) = r! - 1$*,* $f(1,r) = 0$ *for* $r \geq 1$*, and* $f(s,r) = 0$ *for* $s > r$*.*

We employ this proposition to describe the Hom complexes $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$. Actually we need only that $f(r, r+1) = r!\frac{r^2-r-2}{2} + 1$ (see [ČK04, Proposition 13]).

Proposition 4.17. *The Hom complex* $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$*,* $r \geq 2$*, is an orientable cubical surface of genus* $g(r) = f(r, r+1) = r!\frac{r^2-r-2}{2} + 1$ *with* $n(r) := (2+r^2)\cdot(r+1)!$ *vertices,* $2(n(r)+2g(r)-2) = 2r(r+1)\cdot(r+1)!$ *edges, and* $n(r) + 2g(r) - 2 = r(r+1)\cdot(r+1)!$ *squares.*

Proof. Let $(a_1,\ldots,a_r;A_1,\ldots,A_r)$ be a maximal cell of $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$. Since every color $1,\ldots,r+1$ appears exactly once in $(a_1,\ldots,a_r)$ and once in $(A_1,\ldots,A_r)$ the cell $(a_1,\ldots,a_r;A_1,\ldots,A_r)$ is the product of the edge $(a_1,\ldots,a_r)$ with the edge $(A_1,\ldots,A_r)$. Hence, $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$ is a cubical surface.

We now count the number of vertices of $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$. For every vertex $(v_1,\ldots,v_r,w_1,\ldots,w_r)$ we have to choose r of the $r+1$ colors for the lower part and then have $r!$ choices to place these r colors. Let $(v_1,\ldots,v_r)=(1,\ldots,r)$ be such a placement. If the left-out color $r+1$ does not appear in the upper part, then $(1,\ldots,r)$ can be extended in exactly two ways to a coloring of $\overline{C}_{2r}$, yielding the vertices $(1,\ldots,r;1,\ldots,r)$ and $(1,\ldots,r;2,\ldots,r,1)$ of $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$. If the left-out color $r+1$ *is* used in the top part, then there are r choices to place it, and for each such placement every choice to not use one of the colors $1,\ldots,r$ determines a vertex. Therefore, we have altogether $2+r^2$ choices to extend $(1,\ldots,r)$ to a vertex of $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$; i.e., $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$ has $n(r):=(2+r^2)\cdot(r+1)!$ vertices.

Let M be an orientable cubical surface of genus g with n vertices, e edges, and s squares. Since every square is bounded by four edges and every edge appears in two squares, double counting yields $2e=4s$. By this equation and the Euler relation, $s-e+n=\chi(M)=2-2g$, we get that $s=n+2g-2$ and $e=2(n+2g-2)$.

It therefore remains to show that $\mathrm{Hom}(\overline{C}_{2r},K_{r+1})$ is an orientable surface of genus $g(r)=f(r,r+1)=\binom{r}{2}r!-\sum_{k=1}^{r-1}k(k+2)(k+3)\cdots r$. For this, let us fix an edge, say $(a_1,\ldots,a_r)=(1,2,\ldots,r-1,r(r+1))$, of $\mathrm{Hom}(K_r,K_{r+1})$. Then the sequence of $2r$ squares

$$
\begin{aligned}
&(1,2,\ldots,r-1,r(r+1);1,2,\ldots,r-2,r-1,r(r+1)),\\
&(1,2,\ldots,r-1,r(r+1);1,2,\ldots,r-2,(r-1)r,r+1),\\
&(1,2,\ldots,r-1,r(r+1);1,2,\ldots,(r-2)(r-1),r,r+1),\\
&\ldots\\
&(1,2,\ldots,r-1,r(r+1);1,23,\ldots,r-1,r,r+1),\\
&(1,2,\ldots,r-1,r(r+1);12,3,\ldots,r-1,r,r+1),\\
&(1,2,\ldots,r-1,r(r+1);2,3,\ldots,r-1,r,1(r+1)),\\
&(1,2,\ldots,r-1,r(r+1);2,3,\ldots,r-1,r(r+1),1),\\
&(1,2,\ldots,r-1,r(r+1);2,3,\ldots,r-1,r+1,1r),\\
&(1,2,\ldots,r-1,r(r+1);12,3,\ldots,r-1,r+1,r),
\end{aligned}
$$

$$\begin{aligned}
&(1,2,\ldots,r-1,r(r+1);1,23,\ldots,r-1,r+1,r),\\
&\ldots\\
&(1,2,\ldots,r-1,r(r+1);1,2,\ldots,(r-2)(r-1),r+1,r),\\
&(1,2,\ldots,r-1,r(r+1);1,2,\ldots,r-2,(r-1)(r+1),r),
\end{aligned}$$

forms a cylinder $C_{2r} \times I$. By symmetry, we get such a cylinder for every edge $(a_1, \ldots, a_r)$ of $\mathrm{Hom}(K_r, K_{r+1})$. Since every vertex of the graph $\mathrm{Hom}(K_r, K_{r+1})$ has degree r, we have r cylinders in $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ meeting "at a vertex" of $\mathrm{Hom}(K_r, K_{r+1})$. (In the case of $\mathrm{Hom}(K_3, K_4)$ three cylinders meet at a vertex, which yields a trinoid as depicted in Figure 4.14.) By inspecting the gluing at the vertices, it is easy to

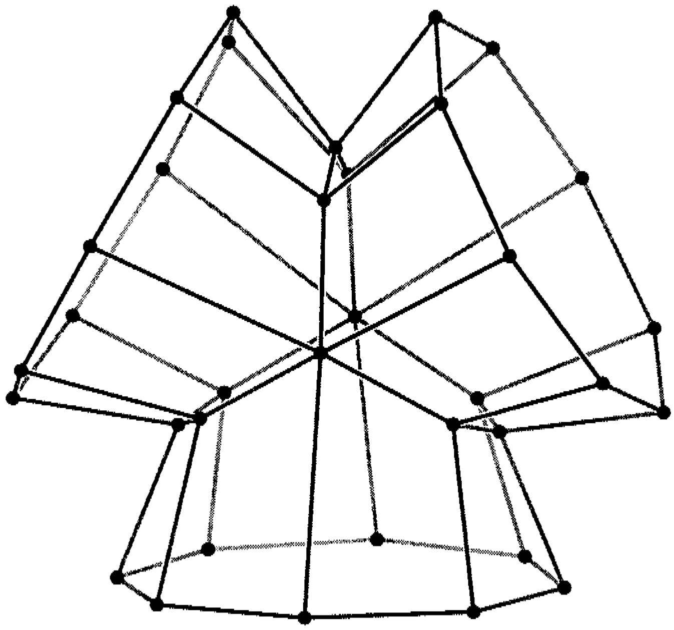

Figure 4.14: Three cylinders forming a trinoid in $\mathrm{Hom}(\overline{C}_6, K_4)$.

deduce that $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ is orientable. It moreover follows that $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ has genus $f(r, r+1)$, which is the number of wedged 1-spheres in the graph $\mathrm{Hom}(K_r, K_{r+1})$. $\square$

As in the case of $\mathrm{Hom}(\overline{C}_{2r+1}, K_{r+1})$ we can interpret $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$ geometrically in the following way. If we thicken the edges of the 1-dimensional manifold $\mathrm{Hom}(K_r, K_{r+1})$ to solid tubes, then for the resulting 3-manifold with boundary the boundary is homeomorphic to $\mathrm{Hom}(\overline{C}_{2r}, K_{r+1})$.

Conjecture 4.18 (Lutz). *The Hom complex* $\mathrm{Hom}(\overline{C}_{2s}, K_r)$ *is homeomorphic to the connected sum of* $2f(s,r)$ *copies of* $S^{r-s} \times S^{r-s}$.

Triangulations of two-dimensional spheres with up to 23 vertices have been enumerated with the program plantri by Brinkmann and McKay [BMcK01].

Table 4.2: Triangulated surfaces with few vertices.

# vertices	6	7	8	9	10
# manifold	3	9	43	655	42426
# spheres	2	5	14	50	233
# flag spheres	1	1	2	4	10

The flag simplicial spheres with up to 8 vertices together with the complements of their 1-skeleton are displayed in Figures 4.15–4.18.

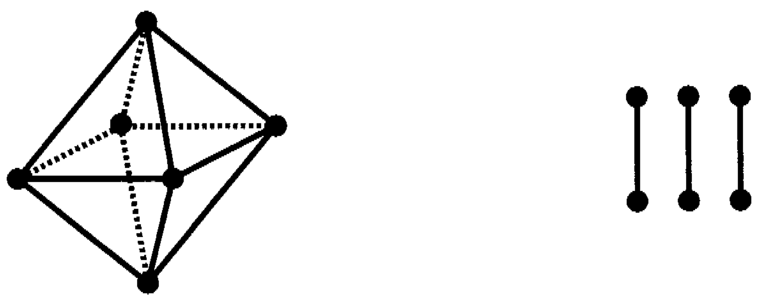

Figure 4.15: The boundary of the octahedron O^2 and the complement of its 1-skeleton.

For the flag 2-spheres O^2, $C_5 * S^0$ and $C_6 * S^0$ the complements of the respective 1-skeleton are not connected, and therefore we have that

$$\begin{aligned}
\mathrm{Hom}(\overline{\mathrm{SK}_1(O^2)}, K_r) &= \times_{i=1}^{3}\mathrm{Hom}(K_2, K_r),\\
\mathrm{Hom}(\overline{\mathrm{SK}_1(C_5 * S^0)}, K_r) &= \mathrm{Hom}(\overline{C}_5, K_r) \times \mathrm{Hom}(K_2, K_r),\\
\mathrm{Hom}(\overline{\mathrm{SK}_1(C_6 * S^0)}, K_r) &= \mathrm{Hom}(\overline{C}_6, K_r) \times \mathrm{Hom}(K_2, K_r), \text{ and}
\end{aligned}$$

For the flag 2-sphere with $n \leq 8$ vertices, for which the complements of their 1-skeleton are connected, we computed the homology of corresponding triangulations of their Hom complexes with few colors.

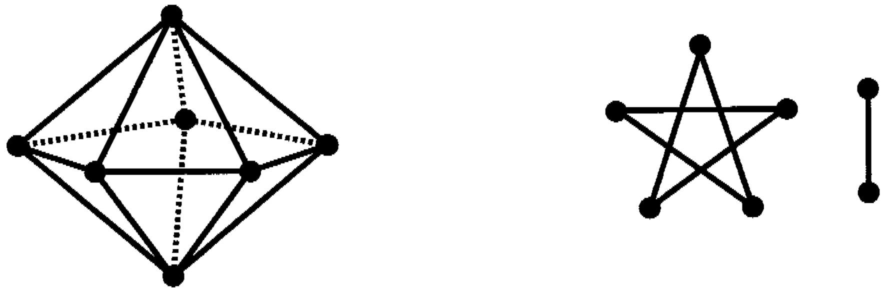

Figure 4.16: The flag sphere $C_5 * S^0$ and the complement of its 1-skeleton.

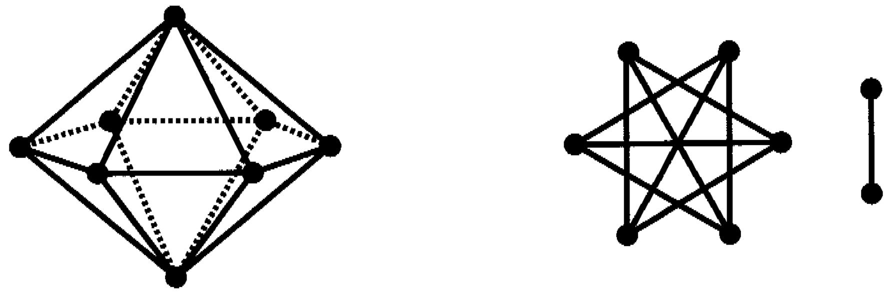

Figure 4.17: The flag sphere $C_6 * S^0$ and the complement of its 1-skeleton.

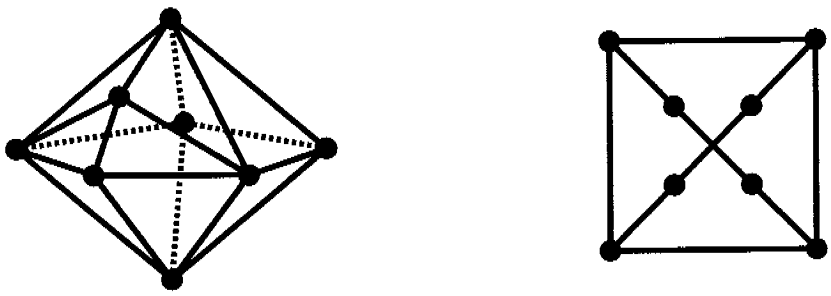

Figure 4.18: A flag 2-sphere X^2 and the complement of its 1-skeleton.

Proposition 4.19. *The Hom complex* $\mathrm{Hom}(\overline{\mathrm{SK}_1(X^2)}, K_3)$ *consists of* 4 *circles with* 24 *vertices each. Moreover,*

$$H_*(\mathrm{Hom}(\overline{\mathrm{SK}_1(X^2)}, K_4)) \quad = \quad (\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}, \mathbb{Z}),$$

where the respective Hom complex have 3624 *vertices.*

Chapter 5

Limit of the topological method

It is known that the topological lower bounds are very useful. Lovász [Lov78] original proof of Kneser's conjecture [Kne55] gives a general lower bound for the chromatic number. His proof was motivating to attack other problems by using topological methods (see [Bjö95, Koz05c, Mat03, Živ98] for references of later developments). There are now short proofs available for the Kneser Conjecture by Bárány [Bár78] and Greene [Gre02]. Moreover Matoušek [Mat04] found a combinatorial proof as well, motivated by discrete version (Tucker's lemma) of the Borsuk–Ulam Theorem.

In this chapter we will show that this topological lower bound can be arbitrarily bad.

5.1 The $\mathrm{K}_{l,m}$-theorem

The following theorem will be useful to show some limit of the topological method.

Theorem 5.1 ($\mathbf{K}_{l,m}$-theorem). *If a graph G does not contain a complete bipartite subgraph $\mathrm{K}_{\ell,m}$ then the index of its box complex is bounded* *by*

$$\operatorname{ind}(\mathsf{B}(G)) \leq \ell + m - 3.$$

We know that $\operatorname{ind}(\mathsf{B}(K_{\ell+m-1})) = \ell+m-3$, since $\mathsf{B}(K_{\ell+m-1})$ is a the boundary of a crosspolytope with an opposite pair of facets removed. Therefore, the statement of the theorem is best possible. On the other hand, we obtain $\operatorname{ind}(\mathsf{B}(K_{k,k})) \leq k-1$ from the theorem, since $K_{1,k+1}$ is not a subgraph of $K_{k,k}$. But $\operatorname{ind}(\mathsf{B}(K_{k,k})) = 0$, since $K_{k,k}$ is bipartite. So the gap in the inequality can be arbitrarily large.

We give two proofs for this theorem. The first one uses the shore subdivision and the halved doubled Lovász complex [CsLSW04], the other is a direct argument on $\mathsf{L}(G)$ along the lines of Walker [Wal83].

Proof.(using shore subdivision) Let $\Phi : \operatorname{ssd}(\mathsf{B}(G)) \to \operatorname{ssd}(\mathsf{B}(G))$ be the simplicial $\mathbb{Z}_2$-map defined by $j \circ \operatorname{cn}^2$ (see Section 2.1). Using that the index is dominated by dimension, it suffices to show the last inequality of

$$\operatorname{ind}(\mathsf{B}(G)) = \operatorname{ind}(\operatorname{ssd}(\mathsf{B}(G))) \leq \operatorname{ind}(\operatorname{Im}\Phi) \leq \dim(\operatorname{Im}\Phi) \leq \ell+m-3.$$

To estimate the dimension of $\operatorname{Im}\Phi = \mathsf{HDL}(G)$, we use that the graph G does not contain a subgraph of type $\mathrm{K}_{\ell,m}$ and assume without loss of generality that $\ell \leq m$. A vertex of $\mathsf{HDL}(G)$ or $\mathsf{DL}(G)$ of the form $A \uplus \emptyset$ or $\emptyset \uplus A$ is called *small* if $|A| < \ell$, *medium* if $\ell \leq |A| < m$, and *large* if $m \leq |A|$. For $\ell = m$ there are no medium vertices. Let $\sigma = \mathcal{A} \uplus \mathcal{B}$ be a simplex of $\mathsf{HDL}(G)$ and consider the set of vertices

$$M_\sigma := \mathrm{V}(j^{-1}(\sigma)) = \bigcup_{A \in \mathcal{A}} \{A \uplus \emptyset, \emptyset \uplus \mathrm{CN}(A)\} \cup \bigcup_{B \in \mathcal{B}} \{\mathrm{CN}(B) \uplus \emptyset, \emptyset \uplus B\}.$$

Clearly, $|M_\sigma|$ is at most twice $|V(\sigma)|$. If σ has a large vertex $A \uplus \emptyset$, then the vertex $\emptyset \uplus \mathrm{CN}(A)$ must be small, otherwise G would contain a $\mathrm{K}_{\ell,m}$. Hence there are at most $2 \cdot 2(\ell - 1)$ many vertices in M_σ that are large or small. Since the number of medium vertices is at most $2(m - \ell)$, we have

$$|M_\sigma| \leq 2 \cdot 2(\ell - 1) + 2(m - \ell) = 2(\ell + m - 2).$$

Hence $|\mathrm{V}(\sigma)| \leq \ell + m - 2$ for all σ, and therefore $\dim(\mathsf{HDL}(G))$ is at most $\ell + m - 3$. $\square$

Proof.(using the Lovász complex) It suffices to prove $\dim(\mathsf{L}(G)) \leq \ell + m - 3$, since

$$\mathrm{ind}(\mathsf{B}(G)) = \mathrm{ind}(\mathsf{L}(G)) \leq \dim(\mathsf{L}(G)).$$

(See e.g. [Mat03, MZ04] or use that $\mathrm{Im}\,\Phi = \mathsf{HDL}(G) \simeq_{\mathbb{Z}_2} \mathsf{L}(G)$ from Section 2.1.1.) Without loss of generality let $\ell \leq m$ and consider a simplex $\mathcal{A} = A_1 \subset \ldots \subset A_p$ of $\mathsf{L}(G)$ of maximal dimension $p-1$. If $p < \ell$ we are done. Suppose that $p \geq \ell$. Then $G\left[A_\ell; \mathrm{CN}(A_\ell)\right]$ is a bipartite subgraph of G and we have $|A_\ell| \geq \ell$ as well as $|\mathrm{CN}(A_\ell)| \geq p-\ell+1$. The assumption that G does not contain a $K_{\ell,m}$ implies that $m > p - \ell + 1$, i.e. $\dim(\mathcal{A}) \leq \ell + m - 3$. $\square$

Already a special case of this theorem, the $\mathsf{K}_{2,2}$-theorem [Wal83], leads to examples showing that the topological lower bound can be arbitrarily bad. Erdős [Erd59] showed that there are graphs with arbitrarily high chromatic number without 4-cycle. The $\mathsf{K}_{2,2}$-theorem provide us that the index of its box complex is at most 1, so the topological lower bound gives us only at most 3.

5.2 Topological construction

It is well known (see the previous sections or [Wal83]) that the topological lower bound for the chromatic number can be arbitrarily bad. But now we are able to give purely topological examples.

We will construct graphs such that $\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)){+}2{+}k$. We will use the universality theorem and the Mycielski construction (Theorem 2.23). First we need a $\mathbb{Z}_2$-space (actually a simplicial complex) X such that $\mathrm{ind}(X) = \mathrm{ind}(\mathrm{susp}^k(X))$. Now let $G := G_{\mathrm{sd}(X)}$. For G we have that $\chi(G) \geq \mathrm{ind}(\mathsf{B}(G)) + 2 = \mathrm{ind}(X) + 2$. We claim that $G^{+k} := \underbrace{M_{i_1}(\ldots(M_{i_k}}_{k}(G)))$ is good for us if $i_j \in \{1,2\}$. (Here we use that the Mycielski construction increases the chromatic number if $i_j \in \{1,2\}$.) Clearly $\chi(G) + k = \chi(G^{+k})$ and $\mathrm{ind}(\mathsf{B}(G^{+k})) = \mathrm{ind}(\mathrm{susp}^k(\mathsf{B}(G))) = \mathrm{ind}(\mathrm{susp}^k(X)) = \mathrm{ind}(X)$. So $\chi(G^{+k}) \geq \mathrm{ind}(\mathsf{B}(G^{+k})) + 2 + k$. In this way we obtain the announced examples.

5.3 The smallest example

The $\mathsf{K}_{\ell,m}$-theorem was invented in order to provide examples such that $\mathrm{ind}(\mathsf{B}(G))+2 < \chi(G)$. Now we will find the smallest example where $\mathrm{ind}(\mathsf{B}(G))+2 \neq \chi(G)$. Let us start with some observations. Their proof is straightforward using the functoriality and that $\mathrm{ind}(\mathsf{B}(K_n)) = n-2$ (see Section 1.5).

Lemma 5.2. *If $K_n \subseteq G$ then $n-2 \leq \mathrm{ind}(\mathsf{B}(G))$.*

Proof. Since $K_n \subseteq G$ we have a $\mathbb{Z}_2$-map $\mathsf{B}(K_n) \xrightarrow{\mathbb{Z}_2} \mathsf{B}(G)$. We have seen that $\mathsf{B}(K_n)$ (the crosspolytope on $2n$ vertices without two antipodal faces) is $\mathbb{Z}_2$-homotopy equivalent to S^{n-2}. So the above map gives a $\mathbb{Z}_2$-map $S^{n-2} \xrightarrow{\mathbb{Z}_2} \mathsf{B}(G)$, which completes the proof. □

Remark 5.3. *We denote by $\omega(G)$ the size of the maximal clique in G. $K_{\omega(G)} \subseteq G$. The previous lemma can be reformulated as*

$$\omega(G) \leq \mathrm{ind}(\mathsf{B}(G)) + 2 \leq \chi(G)\,.$$

Lemma 5.4. $\mathrm{ind}(\mathsf{B}(G)) = 0 \iff$ *graph G bipartite* $\iff \chi(G) = 2$.

Remark 5.5. *If $\chi(G) = 3$ then $\mathrm{ind}(\mathsf{B}(G)) = 1$. (In other words if $\chi(G) = 3$ then $\mathrm{ind}(\mathsf{B}(G)) + 2 = \chi(G) = 3$).*

Definition 5.6. A graph G is called *perfect* if, for each of its induced subgraphs F, the chromatic number of F equals the size of the largest clique.

Going back to our original plan, if we want to find an example such that $\mathrm{ind}(\mathsf{B}(G)) + 2 < \chi(G)$ then Remark 5.5 tells us that $\chi(G) \geq 4$. Moreover if G is perfect then Lemma 5.2 shows that $\mathrm{ind}(\mathsf{B}(G)) + 2 = \chi(G)$. So it is enough to deal with non-perfect graphs. The following theorem will help us to find the non-perfect graphs on few vertices.

Theorem 5.7 (Strong Perfect Graph Theorem). [CRST02] *A graph is perfect if (and only if) it contains no odd hole and no odd antihole, where a hole is a chordless cycle of length at least four; and an antihole is the complement of such a cycle.*

The Strong Perfect Graph Theorem tells us that graphs having at most 4 vertices are prefect, so they can not be an example. On 5 vertices the only non-perfect graph is C_5 (the complement of C_5 is C_5 as well), but since $\chi(C_5) = 3$ this can not be an example.

It is easy to see that the non perfect graphs on 6 vertices are in Figure 5.1. $G_1, \ldots, G_7$ has chromatic number 3, so they are not interesting

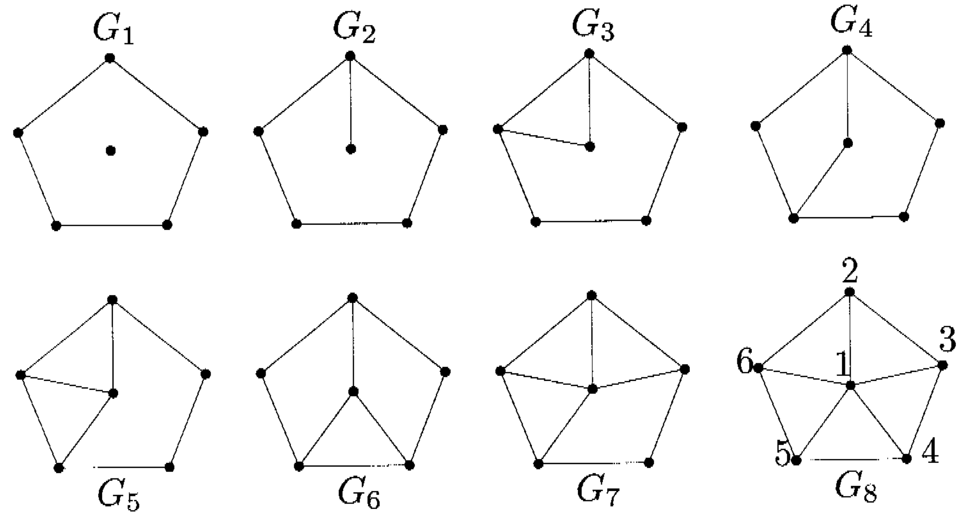

Figure 5.1: The non-perfect graphs on 6 vertices.

any more. The chromatic number of G_8 is 4, and $K_4 \not\subseteq G_8$ so it is a candidate for being the smallest example. We have to compute the $\mathbb{Z}_2$-index of its box complex.

Lemma 5.8. $\operatorname{ind}(\mathsf{B}(G_8)) = 2$ *which means that* $\operatorname{ind}(\mathsf{B}(G_8)) + 2 = \chi(G_8) = 4$.

Proof. We denote the vertex set of G_8 by $V = \{1, 2, 3, 4, 5, 6\}$ as in Figure 5.1. Then the vertex set of the box complex will be $\{1, 2, 3, 4, 5, 6\}$ and $\{\tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, 5, \tilde{6}\}$. (To make the notations simpler here we use A instead of $(A, \emptyset)$ and $\tilde{A}$ instead of $(\emptyset, A)$.) Now one can check that the following triangles forms an embedded sphere (an ikozahedron): $1\tilde{2}\tilde{4}$, $1\tilde{4}\tilde{6}$, $1\tilde{6}\tilde{3}$, $1\tilde{3}\tilde{5}$, $1\tilde{5}\tilde{2}$, $3\tilde{2}\tilde{4}$, $5\tilde{4}\tilde{6}$, $2\tilde{6}\tilde{3}$, $4\tilde{3}\tilde{5}$, $6\tilde{5}\tilde{2}$, $\tilde{3}24$, $\tilde{5}46$, $\tilde{2}63$, $\tilde{4}35$, $\tilde{6}52$, $\tilde{1}24$, $\tilde{1}46$, $\tilde{1}63$, $\tilde{1}35$, $\tilde{1}52$. From this we have that $\operatorname{ind}(\mathsf{B}(G_8)) \geq 2$ and since $\chi(G_8) = 4$ we have that $\operatorname{ind}(\mathsf{B}(G_8)) \leq 2$. □

Remark 5.9. *Another possible way of proving this lemma is using the Mycielski construction (see Theorem 2.23). Since* $G_8 = M_1(C_5)$ *and the box complex* $\mathsf{B}(C_5)$ *is* $\mathbb{Z}_2$*-homotopy equivalent to* S^1 *(see Figure 2.1) we get that* $\mathsf{B}(G_8)$ *is* $\mathbb{Z}_2$*-homotopy equivalent to* S^2.

So the smallest example such that $\mathrm{ind}(\mathsf{B}(G)) + 2 < \chi(G)$ must has at least 7 vertices and the graph on Figure 5.2 has this property.

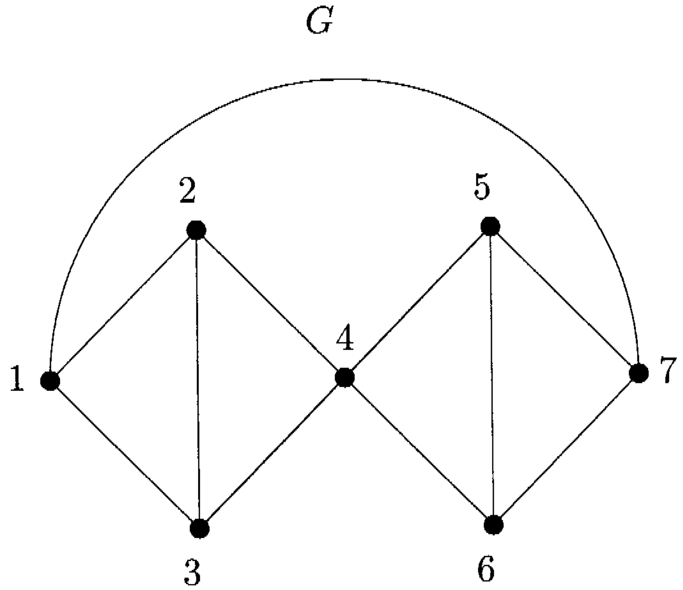

Figure 5.2: The smallest example.

Theorem 5.10. *The $\mathbb{Z}_2$-index of the box complex of the graph on Figure 5.2 is* 1, *while its chromatic number is* 4.

Proof. It is easy to see that the chromatic number is 4.

We denote the vertex set of G by $V = \{1,2,3,4,5,6,7\}$ as in Figure 5.2. Then the vertex set of the box complex will be $\{1,2,3,4,5,6,7\}$ and $\{\tilde{1},\tilde{2},\tilde{3},\tilde{4},\tilde{5},\tilde{6},\tilde{7}\}$. (To make the notations simpler here we use A instead of $(A,\emptyset)$ and $\tilde{A}$ instead of $(\emptyset, A)$.) The simplexes of the box complex are the following simplexes, and its faces as well $\tilde{1}-237$, $\tilde{2}-134$, $\tilde{3}-124$, $\tilde{4}-2356$, $\tilde{5}-467$, $\tilde{6}-457$, $\tilde{7}-156$, $\tilde{1}\tilde{4}-23$, $\tilde{4}\tilde{7}-56$ and its $\mathbb{Z}_2$-images $1-\tilde{2}\tilde{3}\tilde{7},\ldots$. (For example $\tilde{1}\tilde{4}-23$ is a 3-dimensional simplex such that its vertexes are $\tilde{1},\tilde{4},2,3$.) By the $\mathsf{K}_{\ell,m}$-theorem we can reduce the dimension of this complex. (The resulting complex is the Lovász complex what we will obtain in this way.) The stable pairs ($\mathrm{CN}(A)$ and $\mathrm{CN}^2(A)$) are corresponding to the maximal simplexes, so using the method of the $\mathsf{K}_{\ell,m}$-theorem the remaining vertices are $1,2,23,3,4,5,56,6,7$ and their $\mathbb{Z}_2$-pairs $\tilde{1},\tilde{2},\tilde{2}\tilde{3},\tilde{3},\tilde{4},\tilde{5},\tilde{5}\tilde{6},\tilde{6},\tilde{7}$ and the simplices are as in Figure 5.3. We denote this simplicial complex by X. It is easy to see, that it has an embedded circle. The vertices $3,23,2,\tilde{3},\tilde{2}\tilde{3},\tilde{2}$ in this order form a circle, which gives a $\mathbb{Z}_2$-map $S^1 \xrightarrow{\mathbb{Z}_2} X$. This proves that $\mathrm{ind}(X) \leq 1$.

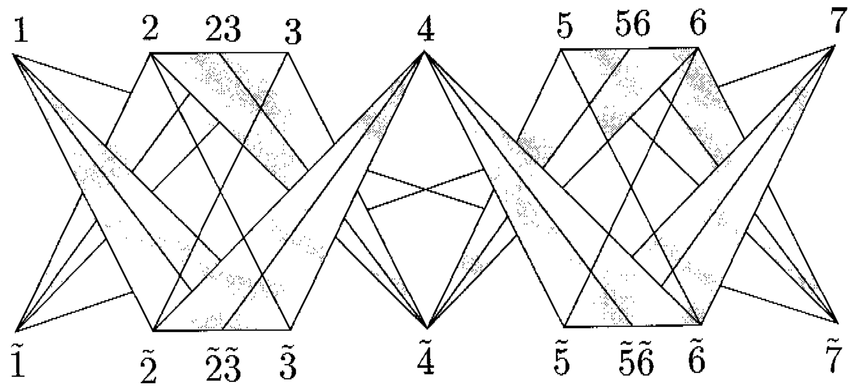

Figure 5.3: The remaining complex.

On the other hand we can map X into S^1, into a 4 cycle as in Figure 5.4. The analysis of this map shows that $\text{ind}(X) \geq 1$, so $\text{ind}(X) = 1$,

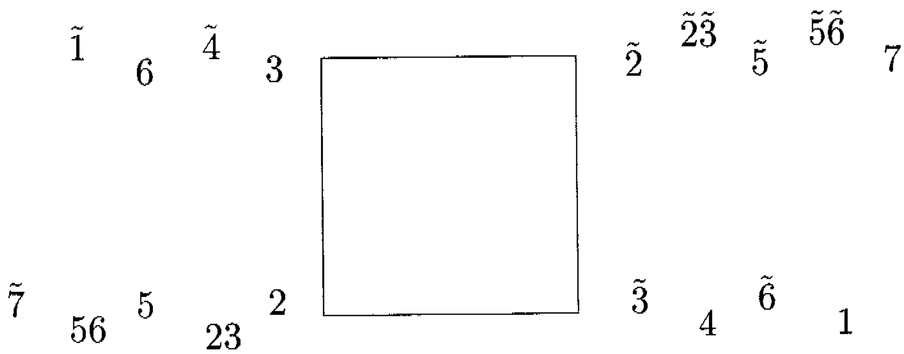

Figure 5.4: The $\mathbb{Z}_2$-map $X \to S^1$.

which is what we wanted to prove. $\square$

Among the graphs with 7 vertices there is another example, the complement of the 7-cycle with this property. Similarly one can see that its graph complex is homotopy equivalent to S^1, so the topological lower bound is 3, while its chromatic number is 4.

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Curriculum Vitae

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