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Influence of non monotonic viscosity profiles on the stability of miscible displacement flows in a Hele-Shaw cell

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Influence of non monotonic viscosity profiles on the stability of miscible displacement flows in a Hele-Shaw cell

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Diploma Thesis WS05/06

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Abstract

If a less viscous fluid displaces a more viscous one in a porous medium or Hele-Shaw cell, the unfavorable mobility gradient is known to cause the classical ‘viscous fingering instability’. In the past, this instability has been investigated mostly on the basis of Darcy’s law, which involves certain averaging procedures that are usually not valid for the case of a Hele-Shaw displacement. In this diploma thesis, miscible fingering in a horizontal Hele-Shaw cell is studied by means of Stokes simulations and linear stability analysis. The difference from former studies is that a nonmonotonic concentration viscosity profile is used which can exist in nature for specific fluid pairs. Here, particular emphasis is placed on identifying parameter regimes in terms of the Péclet-number $Pe$, the viscosity contrast $R$, the value and position of the viscosity maximum $\mu_m$ and $c_m$. $\Lambda$ is an additional parameter depending on the end-point derivatives. The existing nonlinear Stokes code for monotonic exponential viscosity profiles developed by N. Goyal was modified to incorporate any general viscosity concentration functional relationship. These two-dimensional simulations lead to a moving finger with a quasisteady state near the tip of the displacement front for sufficiently large Péclet number and high viscosity ratios. The front thickness $d_0$ and the tip velocity $v_{tip}$ of the quasisteady state is primarily affected by the Péclet number $Pe$ and the viscosity contrast $R$. The scale of $d_0$ with $Pe^{-1/2}$ which is found for the monotonic profile is deviating for the nonmonotonic profile. A higher value of the maximum of the nonmonotonic viscosity profile leads to a slower tip and thicker front, while a shift of the maximum from the less viscous fluid to the more viscous fluid leads to a increase of the tip velocity and a decrease of the front thickness.

In a second step the results of the direct numerical Stokes flow simulation are used to examine the stability of the quasisteady front to spanwise perturbations. In the linear stability analysis the viscosity contrast $R$ is the dominant factor that affects the stability of the flow. Manickam&Homsy [10] investigated that for the Darcy flow a negative $\Lambda$ leads to a more stable configuration than a positive $\Lambda$, while we see a reversal of this behavior. Only higher values of $\Lambda$ lead to strong changes for the growth rate $\sigma_{max}$ of the most dangerous mode and the corresponding wavenumber $\beta_{max}$. Because the shape of the profile is also changing a lot with high values of $\Lambda$ it is really hard to figure out what leads to this change. The exact reason could not be identified and would need further investigation.
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Notation

Roman Symbols

\( c \) Concentration
\( c_m \) Location of the maximum of the viscosity profile
\( d_0 \) Front thickness
\( D \) Diffusion coefficient
\( e \) Gap width of the Hele-Shaw cell
\( l \) Length of domain
\( p \) Pressure
\( Pe \) Péclet-number
\( R \) Logarithm of the viscosity ratio
\( t \) Time
\( \vec{u} \) Velocity vector
\( u, v, w \) Velocity components
\( U \) Average gap velocity
\( v_P \) Velocity of the Poiseuille flow
\( v_{ref} \) Reference velocity
\( v_T \) Overall velocity
\( v_{tip} \) Finger tip velocity
\( x, y, z \) Spatial components

Greek Symbols

\( \alpha \) Viscosity ratio
\( \beta \) Wavenumber
\( \delta \) Initial profile thickness
\( \mu \) Viscosity
\( \mu_m \) Maximum of the viscosity profile
\( \sigma \) Growth rate
\( \tau \) Viscous stress tensor
\( \phi \) Eigenvector
\( \Lambda \) Parameter for the end-point derivatives of the viscosity profile
\( \Psi \) Streamfunction

Superscripts

' Deviation/Perturbation
* Dimensionless value
\( \hat{x} \) Eigenvalue
\( \bar{x} \) Mean value/Base state
\( T \) Transpose

Subscripts

\( t \) Time derivation
\( x, y, z \) Spatial derivation
1 Less viscous fluid
2 More viscous fluid
\( \mathbb{X} \) Tensor
1 Introduction

The phenomenon of viscous fingering in porous media is of vital importance in a lot of applications [7]. One specific field of interest is the oil recovery process, where the oil is pumped by displacing it with a less viscous liquid, for example water. The viscous fingering instability reduces the efficiency of the displacement process as the displacing fluid fingers through and bypasses the fluid to be displaced. Those instabilities in miscible displacement processes are already well studied for several cases. Much of the theoretical and computational work on unstable miscible displacements has employed Darcy’s law. Within this framework, [11], [12] investigate the linear stability of the flow as a function of the base flow concentration profile. They find that larger viscosity contrasts and steeper concentration fronts always result in higher growth rates and shorter wavelengths of the most dangerous mode.

In contrast to these Darcy-based stability investigations other researchers employed Hele-Shaw configurations as a substitution for a ‘true’ porous medium. The advantage lies in the ease with which the flow can be experimentally visualized in such apparatuses. It is well known that the analogy between a real porous medium and a Hele-Shaw cell is incomplete. Thus a comparison between Darcy results and Hele-Shaw experiments is not straightforward. The current investigation addresses this deficit by conducting two-dimensional Stokes flow simulations for plane channels, and by subsequently investigating the linear stability of the quasisteady base states thus obtained with respect to spanwise perturbations. The resulting dispersion relations can then be compared with corresponding findings for Darcy flows.

The previous investigation by N. Goyal & E. Meiburg [6], focused on the similarities and discrepancies between Stokes and Darcy results for the viscous fingering instability, using monotonic viscosity concentration profiles. As an extension of their work, this diploma thesis analyzes the problem for non-monotonic viscosity concentration profiles. Such profiles depend on the specific fluid pair employed and can be found in nature in different combinations, for example water and ethanol. Manickam & Homsy [10] used Darcy’s law and found out that the stability of such flows does not depend on the end-point viscosity (as assumed) but on the end-point derivatives of viscosity with respect to concentration. They defined a parameter dependency Λ on the end-point derivatives [Eq. (10)] for which they observed that the flow is unstable for Λ > 0 and is initially stable for Λ < 0, but becomes unstable as the base flow diffuses out. This work is the Stokes counterpart to their study, using a Hele-Shaw cell. We extend the non-linear Stokes code written
by N.Goyal [6] by incorporating non-monotonic concentration viscosity profiles. We run a variety of simulations with different parameters to establish the quasisteady states and compare the results with the earlier monotonic results. Further the two-dimensional, quasisteady states serve as base states for a linear stability analysis with respect to perturbations in the spanwise direction.

The diploma thesis is structured as follows: The physical problem of the Stokes flow and the non-monotonic profile are defined in section 2. The setup for the two-dimensional direct numerical Stokes simulation is described in section 3, followed by the results in section 4. Section 5 discusses the subsequent linearization of the Stokes equations around the base state, along with the formulation of the numerical eigenvalue problem and the discretization. Section 6 discusses the results of the linear stability analysis.
2 Physical Problem

We are looking at two constant density, miscible fluids of different viscosities in a Hele-Shaw cell, as shown in Fig. 1.

Figure 1: Geometry of the Hele-Shaw cell. The less viscous fluid on the left displaces the more viscous fluid on the right.

The less viscous fluid displaces the more viscous fluid to the right. For narrow gaps the flow velocities will be very small, so that the Reynolds number can be assumed to be zero. Gravity is neglected. In this case the fluid motion can be described by the three-dimensional Stokes equations

\[
\nabla \cdot \vec{u} = 0 , \quad \nabla p = \nabla \cdot \tau , \quad \frac{\partial c}{\partial t} + \vec{u} \cdot \nabla c = D \nabla^2 c .
\]

Here \(\vec{u}\) denotes the flow velocity, and \(c\) indicates the relative concentration of the more viscous fluid. \(\tau\) represents the stress tensor for Newtonian fluids, while \(D\) refers to the diffusion coefficient, which is assumed constant. The cell has a gap of width \(e\), with \(z\) indicating the gapwise or cross-gap direction. The \(x\)-direction will be referred to as the spanwise direction.

The viscosity \(\mu\) is a function of the concentration \(c\) and will depend on the fluid pair used. Several investigators have assumed the viscosity as a monotonic function, but in nature fluid pairs can show nonmonotonic behavior. Manickam&Homsy [10] proposed specific viscosity concentration functional relationships to approximate this behavior.

As opposed to Manickam&Homsy we refer our viscosity to that of the less viscous fluid. The definition of the viscosity ratio \(\alpha\) is
\[ \alpha = \frac{\mu_2}{\mu_1}. \]  

The viscosity profile \( \mu(c) \) is given by

\[ \mu(c) = \mu_1 \cdot \mu_m \sin(\xi), \]  

where

\[ \xi = \xi_0 (1 - \eta) + \xi_1 \eta, \quad \eta = \frac{(1 + a)c}{1 + ac}, \]  

and

\[ a = \frac{(1 - c_m) - \eta_m}{(1 - c_m)(\eta_m - 1)}, \quad \eta_m = \frac{\pi/2 - \xi_0}{\xi_1 - \xi_0}. \]  

The function is a simple sine function modified through a sequence of transformations. The transformations are defined so that the profile has the end-point viscosities \( \mu(0) = 1, \mu(1) = \alpha \) and a maximum value of \( \mu_m \) at \( c = c_m \).

We also introduce the logarithm \( R \) of the viscosity ratio defined as

\[ R = \ln \frac{\mu_2}{\mu_1} = \ln(\alpha) \]  

and a parameter \( \Lambda \) which is a quantity for the relation between the slope of the viscosity at \( c = 0 \) and \( c = 1 \).

\[ \Lambda = \frac{\left. \frac{d\mu}{dc} \right|_{c=0} + \left. \frac{d\mu}{dc} \right|_{c=1}}{\alpha + 1} \]  

In the monotonic case \( \Lambda \) is always positive, in the non monotonic case \( \Lambda \) depends on the magnitude of the gradient at the end points. \( \Lambda < 0 \) means the slope of the viscosity profile at the point \( c = 1 \) is steeper than at the point \( c = 0 \) and \( \Lambda > 0 \) indicates the reverse scenario.

As an example Fig. 2 shows good agreement of our viscosity profiles with experimental data for ethanol.

The governing equations are rendered dimensionless by introducing a characteristic length \( L^* \), velocity \( U^* \), time \( T^* \), pressure \( P^* \) and the viscosity \( \mu^* \) in the form of
Figure 2: Viscosity versus concentration for ethanol [14].

where $U$ refers to the average velocity across the gap of the Hele-Shaw cell. We thus obtain the set of dimensionless equations as

$$\nabla \cdot \vec{u} = 0,$$

$$\nabla p = \nabla \cdot \tau,$$

$$\frac{\partial c}{\partial t} + \vec{u} \cdot \nabla c = \frac{1}{Pe} \nabla^2 c,$$
\[ Pe = \frac{U e}{D}. \]  

(19)

The Péclet number \( Pe \) indicates the relative strength of convective to diffusive transport.


3 Stokes Flow Simulation

3.1 Numerical implementation

To solve the nonlinear Stokes flow and obtain the base state for the subsequent linear stability analysis, we cast the two-dimensional Stokes equations into the fourth order streamfunction formulation.

\[
\mu \nabla^4 \psi = F(y, z), \quad (20)
\]

\[
\frac{\partial c}{\partial t} + \vec{u} \cdot \nabla c = \frac{1}{Pe} \nabla^2 c, \quad (21)
\]

where the streamfunction \( \psi \) is defined by

\[
v = \psi_z, \quad w = -\psi_y \quad (22)
\]

and

\[
\nabla^4 \psi = \psi_{ygyy} + 2\psi_{ygyz} + \psi_{zzzz}, \quad (23)
\]

\[
F(y, z) = (\mu_{yy} - \mu_{zz})(\psi_{zz} - \psi_{yy}) - 4\mu_{yz}\psi_{yz} - 2\mu_{y} (\psi_{yyy} + \psi_{zz}) \quad (24)
\]

The spatial discretization of the concentration derivatives is done by compact finite differences [9] of up to sixth order in the interior of the domain, and fourth order at the boundaries in order to avoid the appearance of numerical instabilities in the convection-diffusion equation. The temporal discretization is accomplished by an explicit, third order Runge-Kutta [15] time integration scheme for updating the concentration field.

The streamfunction derivatives are discretized by second order finite differences. This lower order approximation also helps keep the coefficient matrix for the discretized fourth order streamfunction equation sparse, so that it can be solved by means of the sparse direct linear solver implemented in the software package UMFPACK ([2], [3], [4]). Further details can be found in [6].

3.2 Geometry and mesh

Fig. 3 shows the two-dimensional computational domain employed for the simulations. We select the size of the domain in the flow direction sufficiently
Figure 3: Computational domain, with boundary conditions in terms of deviation velocities

large, so that a quasisteady finger can evolve without being affected by the upstream and downstream boundaries. The length of the Hele-Shaw cell in the \( y \)-direction is \( 2L_y \), for most simulations with \( L_y = 6 \). In the \( z \)-direction the computational domain includes only half of the Hele-Shaw cell, because former simulations showed that the finger is symmetrical around the center gap [6].

The grid is a simple equidistant, two-dimensional mesh. We did a couple of test runs with different resolutions to ensure converged results. Starting with a sufficient resolution for the exponential monotonic viscosity problem as a reference case, we increased the gridpoints in \( y \)- and \( z \)-direction with the factors \( 1, 1.25, 1.5, 2, 2.5 \). The test case was a nonmonotonic viscosity profile with a high Péclet number (\( Pe = 2000 \)) and a high viscosity ratio (\( R = 5 \)) and strong gradients in the concentration field. The negligible variation in the quasisteady front thickness \( d_0 \) of the finger confirmed the validity of our simulations. The effect of increasing the resolution in the \( z \)-direction is shown in Figure 4.

Similarly there was no noticeable change in our results, on increasing the value of \( n_z \) higher than 2. The absolute number of gridpoints was \( n_y = 1025 \) in the \( y \)- and \( n_z = 193 \) in the \( z \)-direction.

The simulations are all performed in a reference frame moving with the average gap velocity \( V_{ref} = U \).

### 3.3 Initial and boundary conditions

The initial concentration profile is given by an errorfunction.

\[
c(y,0) = 0.5 + 0.5 \cdot erf \left( \frac{y}{\delta} \right)
\]  

(25)
where $\delta$ indicates the initial profile thickness. Results from simulations done by Goyal & Meiburg [6] showed that the final quasisteady shape of the front is independent of $\delta$. So we employed a value of $\delta = 0.1$.

Further we assume a fully developed Poiseuille flow at time $t = 0$. The velocity profile of the Poiseuille flow is given by

$$v_P = 1.5(1 - 4z^2) \quad (26)$$

To simplify the boundary condition at the upstream and downstream position the overall velocity $v_T$ is split into the 'Poiseuille velocity' $v_P$ plus a deviation from it

$$v_T = v_P + v' - v_{ref}, \quad (27)$$

the term $v_{ref}$ takes care of the velocity of the moving reference frame. Now instead of solving for the overall velocity $v_T$ we can solve for the deviation velocity $v'$. We set $v'(t = 0) = 0$ initially.

The boundary conditions at the downstream and upstream position $\pm L_y$ are now simply given by $v' = 0$ (unaffected Poiseuille flow). The concentration is $c = 0$ at $-L_y$ and $c = 1$ at $+L_y$. On the wall ($z = 0.5$) we assume a
no-slip boundary condition, which leads to \( v = w = 0 \). Further the variation of the concentration \( c \) with \( z \) is set to zero (no diffusion into or out of the wall). At \( z = 0 \) we consider symmetrical boundary conditions. This means the derivatives \( \frac{\partial c}{\partial z} \) and \( \frac{\partial w}{\partial z} \) are set to zero. The velocity \( w \) in \( z \)-direction is also set to zero.

Because the equations are solved in terms of the streamfunction \( \Psi \) instead of the primitive variables, we obtain the following set of boundary and initial conditions for the concentration and deviation component of the streamfunction

\[
\begin{align*}
  y = L_y : & \quad \psi' = 0, \quad \psi'_y = 0, \quad c = 1 \quad (28) \\
  y = -L_y : & \quad \psi' = 0, \quad \psi'_y = 0, \quad c = 0 \quad (29) \\
  z = +0.5 : & \quad \psi' = 0, \quad \psi'_z = 0, \quad c_z = 0 \quad (30) \\
  z = 0 : & \quad \psi' = 0, \quad \psi''_z = 0, \quad c_z = 0. \quad (31)
\end{align*}
\]

### 3.4 Validation and runs

The main modification of the DNS-Code was to change the equation in a way that a general viscosity profile can be used, not only a exponential monotonic one. The modifications required that a few analytical derivatives had to be done numerically instead. To verify the code we compared with the previous exponential profile and found negligible differences.

Because the effect of the Péclet-number \( Pe \) and viscosity ratio \( R \) are already well-investigated [6], we focused more on the two parameters \( \mu_m \) and \( \Lambda \) which characterize the non-monotonic profile. We set up runs for the following set of parameters

\[
\begin{align*}
  Pe &= 500, 1000, 2000 \quad (32) \\
  R &= 2, 3, 5 \quad (33) \\
  \mu_m &= 1.1, 1.5, 2 \quad (34) \\
  \Lambda &= -15, -10, -3, 0, 3, 10, 15 \quad (35)
\end{align*}
\]

The values of the maximum viscosity \( \mu_m \) stands for the multiple of the viscosity of the more viscous fluid. With the resolution mentioned in section 3.2 the simulations took between 20 hours to 3 days on one processor (serial code).
4 DNS Results

4.1 Evolution of the quasisteady displacement front

Upon the start of the flow, the velocity field immediately deforms the concentration distribution in the interior of the cell, while the concentration at the wall changes due to diffusion only. In turn, this change in the concentration distribution modifies the viscosity field, thus leading to the formation of a well-defined finger of the less viscous fluid that propagates along the center of the Hele-Shaw cell (Fig. 5).

Figure 5: Evolution of a quasisteady base state in the area near the tip of the finger for \( Pe = 2000, R = 5, \Lambda = 0 \) and \( \mu_m = 1.1 \).

This finger exhibits a steep concentration front and diffusively spreading concentration layers at its sides. If the Péclet number \( Pe \) and the viscosity contrast \( R \) are sufficiently high the finger tip and the associated concentration field will reach a quasisteady state in the reference frame moving with the finger tip. In the other case where the Péclet number is small the strong diffusive effects prevent the evolution of a quasisteady state. Here the diffusion outweighs convection, so that a steep concentration layer at the tip of the front cannot be maintained. In Fig. 6 these two different behaviors are shown. The simulation with \( Pe = 2000 \) reaches a steady state after about \( t = 4.2 \). The front thickness of the \( Pe = 500 \) simulation reaches a minimum at \( t = 2.3 \), after this point the front thickness starts to rise.
Quasisteady states will be reached for combinations of either high Péclet numbers and low viscosity ratios, high viscosity ratios and low Péclet numbers or both high viscosity ratios and high Péclet numbers. Also there is a strong influence of the two nonmonotonic parameters $\Lambda$ and $\mu_m$. All these behaviors will be discussed in the following sections.

4.2 Influence of the Péclet-number $Pe$ and the viscosity ratio $R$

Fig. 7 displays the tip velocity $v_{tip}$ as a function of the Péclet-number for different values of $R$. The solid lines show the results of the nonmonotonic ($\Lambda = 0, \mu_m = 2$) case while the dashed lines depict the exponential monotonic case. Note that for low $Pe$ and low $R$ the simulations do not really reach a clear steady state, instead there is a nearly steady state only for a short time. In this case the values are taken from the nearly steady state. As expected the velocity will increase with increasing $Pe$ as the ratio of the convection to diffusion gets higher. An increase in the viscosity contrast between the two fluids leads to a higher front velocity.

In the monotonic case at high values of $R$ an increase in $Pe$ leads to a decrease in $v_{tip}$. In the publication of Goyal & Meiburg [6] this effect of
decreasing tip velocity is said to be dependent on the viscosity-concentration relationship. On introducing of a linear monotonic profile

$$\mu = 1 + (e^R - 1) c$$  \hspace{1cm} (36)$$

the tip velocity increases with Péclet number. The nonmonotonic profile, here shows the same behavior as the linear monotonic profile, which underlines their theory about the viscosity-concentration relationship dependency.

The front thickness $d_0$ shows the inverse behavior to the tip velocity. The front thickness decreases with increasing $Pe$ (Fig. 8).

The convective effects dominate the diffusive effects and the front thickness diminishes. Chen&Meiburg [1] did a straightforward one-dimensional balance of strain and diffusion effects on the quasisteady concentration field near the tip of the front according to

$$v \frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}$$  \hspace{1cm} (37)$$
yields that the front thickness $d_0$ should scale inversely with the square root of $Pe$. 

13
Figure 8: Front thickness as a function of $Pe$. The solid lines show a non-monotonic profile with $\Lambda = 0$ and $\mu_m = 2$. The dashed lines depict the exponential monotonic profile.

$$d_0 \sim Pe^{-\frac{1}{2}}$$  \hspace{1cm} (38)

While the monotonic profiles follow the scaling law (38) pretty accurately, nonmonotonic viscosity profiles deviate from this behavior.

### 4.3 Influence of the viscosity profile maximum $\mu_m$

One important parameter which defines the viscosity concentration profile is the maximum $\mu_m$. The influence of this parameter is quite intuitive: the mobility of the displacing fluid will decrease with a higher value, which should lead to a lower tip velocity $v_{\text{tip}}$. This intuition is proved by the simulation and is shown in Fig. 9.

The front thickness $d_0$ shows again the inverse behavior of the tip velocity: The higher the maximum viscosity the thicker the front (Fig. 10). If $R$ is getting higher the influence of the different maxima will decrease.
Figure 9: Top: Tip velocity as a function of $\mu_m$. Bottom: The corresponding viscosity profiles for $R = 5$.
4.4 Influence of the parameter $\Lambda$

In the nonmonotonic viscosity profile the position of the maximum viscosity is very important, because it will denote the place with the lowest mobility. This leads to a kind of damping of the fluid on the left hand side. With an increase in $\Lambda$ the maximum of the viscosity profile will move to a lower concentration, which means that the damping part will move closer to the less viscous fluid. The fluid is moving slower and the finger will start spreading earlier. Figure 11 shows two fingers at the time $t = 3.8$ for two extreme cases $\Lambda = -15$ and $\Lambda = 15$.

If the maximum is at a high concentration, the damping part is close to the tip and seems to be pushed by the more mobile part on the low concentration, which results in a higher tip velocity. The tip velocity $v_{\text{tip}}$ as a function of $\Lambda$ is shown in Fig. 12.

The front thickness $d_0$ increases with $\Lambda$, especially for the case $R = 2$ the influence of the spreading front is clearly observable (Fig. 13). Here we still have to consider that the flow does not really reach a steady state.

To complete the dependencies of the parameter, Fig. 14 shows the tip velocity and front thickness as a function of $\mu_m$ for different $\Lambda$. 

Figure 10: Tip thickness as a function of $\mu_m$
Figure 11: Concentration contour plot for $\Lambda = -15$ and $\Lambda = 15$ at the time $t = 3.8$. The shape of the finger tip changes a lot with increasing $\Lambda$. 

$\text{Pe} = 2000, \ R = 2, \ \mu_m = 1.1$
Figure 12: Top: Tip velocity as a function of $\Lambda$. Bottom: Corresponding viscosity profiles for $R = 5$. 

\[ P_e = 2000, \mu_m = 2 \]
Figure 13: Front thickness $d_0$ as a function of $\Lambda$ for different $R$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13}
\end{figure}
Figure 14: Tip velocity $v_{\text{tip}}$ and front thickness $d_0$ as a function of $\mu_m$ for different $\Lambda$. 
5 Linear stability analysis

5.1 Eigenvalue problem

After the DNS-simulation is done, we will investigate the quasisteady, two-dimensional base state with regard to periodic spanwise perturbations as it is shown in Fig. 15.

![Figure 15: Three-dimensional view of the spanwise instability](image)

The three-dimensional Stokes equations are linearized around the two-dimensional base state

\[
\begin{align*}
    u(x, y, z, t) &= 0 + u'(x, y, z), \\
    v(x, y, z, t) &= \bar{v}(y, z) + v'(x, y, z), \\
    w(x, y, z, t) &= \bar{w}(y, z) + w'(x, y, z), \\
    p(x, y, z, t) &= \bar{p}(y, z) + p'(x, y, z), \\
    c(x, y, z, t) &= \bar{c}(y, z) + c'(x, y, z).
\end{align*}
\]

The variables marked with a 'prime' represent the perturbations, while those marked with a 'bar' denote the base state.

The above expressions are substituted into the governing equations

\[
\begin{align*}
    \nabla \cdot \bar{u} &= 0, \\
    \nabla p &= \nabla (\mu \nabla \bar{u}), \\
    \frac{\partial c}{\partial t} + \bar{u} \cdot \nabla c &= \frac{1}{Pe} \nabla^2 c.
\end{align*}
\]
the base state is subtracted out, and all terms second order and higher in the disturbances are neglected. Because the viscosity $\mu = \mu(c)$ is not a trivial function in $c$ a Taylor expansion for the perturbation is used

$$
\mu(\bar{c} + c') = \mu(\bar{c}) + \frac{\partial \mu}{\partial c} \bigg|_{c=\bar{c}} \cdot c'
$$

(47)

(48)

The perturbations are assumed to be wavelike in the spanwise $x-$direction, so we can write

$$
u'(x, y, z, t) = \hat{\nu}(y, z) \cdot \sin(\beta x) \cdot e^{\sigma t},
$$

(49)

$$v'(x, y, z, t) = \hat{\nu}(y, z) \cdot \cos(\beta x) \cdot e^{\sigma t},
$$

(50)

$$w'(x, y, z, t) = \hat{\nu}(y, z) \cdot \cos(\beta x) \cdot e^{\sigma t},
$$

(51)

$$\rho'(x, y, z, t) = \hat{\rho}(y, z) \cdot \sin(\beta x) \cdot e^{\sigma t},
$$

(52)

$$c'(x, y, z, t) = \hat{c}(y, z) \cdot \cos(\beta x) \cdot e^{\sigma t}.
$$

(53)

where $\sigma$ is the growth rate and $\beta$ the wavenumber which reflects the perturbation wavelengths. The variables with the 'hat' represent the eigenfunctions.

After these steps we end up with an algebraic system of the form

$$A \phi = \sigma B \phi.
$$

(54)

This represents a generalized eigenvalue problem with the growth rate $\sigma$ as the eigenvalue, while the eigenvector $\phi$ reflects the shape of the perturbations. As usual, a positive (negative) eigenvalue indicates unstable (stable) behavior. $A$ and $B$ denote the coefficient matrices and are given by

$$A = \begin{pmatrix}
0 & \beta I & \partial_y & \partial_z & 0 \\
\beta I & M_1 & M_2 & M_3 & 0 \\
-\partial_y & 0 & M_4 & M_5 & M_6 \\
-\partial_z & 0 & M_7 & M_8 & M_9 \\
0 & 0 & -\bar{c}_y & -\bar{c}_z & M_{10}
\end{pmatrix},
$$

(55)

with
\begin{align}
M_1 &= \bar{p} \left( \partial_{yy} + \partial_{zz} - \beta^2 \mathbb{I} \right) + \bar{p}_z \partial_z + \bar{p}_y \partial_y, \\
M_2 &= -\beta \bar{p}_y \mathbb{I}, \\
M_3 &= -\beta \bar{p}_z \mathbb{I}, \\
M_4 &= 2\bar{p}_y \partial_y + \bar{p} \left( \partial_{yy} + \partial_{zz} - \beta^2 \mathbb{I} \right) + \bar{p}_z \partial_z, \\
M_5 &= \bar{p}_z \partial_y, \\
M_6 &= 2\bar{p}_y \left( \frac{\partial \bar{\mu}_c}{\partial y} I + \bar{\mu}_c \partial_y \right) + (\bar{v}_z + \bar{w}_y) \left( \frac{\partial \bar{\mu}_c}{\partial z} I + \bar{\mu}_c \partial_z \right) + (\bar{v}_{yy} + \bar{w}_{zz}) \bar{\mu}_c I, \\
M_7 &= \bar{p}_y \partial_z, \\
M_8 &= 2\bar{p}_z \partial_z + \bar{p} \left( \partial_{yy} + \partial_{zz} - \beta^2 \mathbb{I} \right) + \bar{v}_y \partial_y, \\
M_9 &= 2\bar{w}_z \left( \frac{\partial \bar{\mu}_c}{\partial z} I + \bar{\mu}_c \partial_z \right) + (\bar{v}_z + \bar{w}_y) \left( \frac{\partial \bar{\mu}_c}{\partial y} I + \bar{\mu}_c \partial_y \right) + (\bar{w}_{yy} + \bar{w}_{zz}) \bar{\mu}_c I, \\
M_{10} &= \frac{1}{\bar{P_c}} \left( -\beta^2 \mathbb{I} + \partial_{yy} + \partial_{zz} \right) - \bar{v} \partial_y - \bar{w} \partial_z. 
\end{align}

and

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\quad \text{and} \quad
\phi = \begin{pmatrix}
\hat{p} \\
\hat{u} \\
\hat{v} \\
\hat{w} \\
\hat{\phi}
\end{pmatrix}
\]
derivatives are interpolated from the DNS grid to the stability grid using sixth order Lagrangian interpolation.

Since large gradients in both directions exist near the tip of the finger, we follow the approach outlined by [13] in order to concentrate the grid points around the tip of the finger. In the $z$-direction the nodes are spaced equidistantly, while a stretched grid is employed on the two subdomains $y \geq y_{tip}$ and $y \leq y_{tip}$ that concentrates the grid points near $y_{tip}$. For this purpose a mapping function provided by [5] is employed

\[ y_i = s_i \frac{l}{2}, \quad (67) \]

where

\[ s_i = P\eta_i + (1 - P) \left( 1 - \frac{\tanh [Q (1 - \eta_i)]}{\tanh Q} \right), \quad (68) \]

with

\[ \eta_i = \frac{i - 1}{n - 1}. \quad (69) \]

Here $n$ represents the number of axial points within each subdomain, while $P$ and $Q$ are parameters to be chosen appropriately, in order to obtain a suitable distribution of grid points.

The calculations are performed in a reference frame moving with the fingertip velocity $v_{tip}$.

## 5.3 Numerical implementation and resolution

All the first and second order derivatives of the equation (54) are discretized by a second order central finite difference scheme, whereby the matrix $A$ becomes extremely sparse. The entries differing from zero are less than 0.2%. An Arnoldi iteration method based package (ARPACK [8]) is employed to find the largest eigenvalues along with the corresponding eigenvectors. This iterative solver requires an external linear solver, so we avoid establishing the complete matrix by storing only the nonzero elements, and use a sparse direct linear solver implemented in the software package UMFPACK ([2],[3],[4]).

The use of a spectral scheme would need a lower resolution, but on the other hand the matrix $A$ would have many more non-zero entries (about 2%), so it still pays off to use the lower order method, as the overall memory and time requirements are smaller compared to the spectral approach.

The simulations were split in two parts using two different computers: The creation of the sparse matrix needs about 100GB of memory, which forced us
to use the supercomputer “DataStar” at the University of California in San Diego (UCSD). The second part solving the eigenvalue problem was done on the cluster already used for the DNS code.

The effect of the number of gridpoints in the \( z \)-direction is shown in Fig. 17. It shows a similar behavior to the resolution checks for the DNS-simulation. A much higher resolution is necessary for the non monotonic case, as compared to the monotonic profiles. To avoid interpolation errors we decided to use the same resolution in the \( z \)-direction as in the DNS-simulation.

The increase of the numbers in \( z \)-direction by a factor of two caused serious memory problems. The limiting factor was the software package UMFPACK which is not able to allocate more than 2GB of memory, whereby we could not get a sufficient resolution in the \( y \)-direction. Since we could not find a good alternative to this package the problem had to be solved in another way. On the one hand the grid spacing was changed in the way that the resolution around the tip increased and on the other hand we changed the Dirichlet boundary condition in the flow direction to the less restrictive Neumann boundary condition. Due to this change we were able to shorten the computational domain and increase the resolution in the \( y \)-direction without affecting the result. All the resolution checks were done for the case with the steepest concentration gradients at the tip of the front. In Fig. 18 we show the effect of increasing \( n_y \) after the above explained steps. The change from

Figure 16: Illustration of the non equidistant grid with a low resolution.
Figure 17: Change in the growth rate $\sigma$ due to increasing gridpoints in the $z$-direction. In the monotonic case a resolution of $n_z = 97$ was sufficient.

$n_y = 410$ to the absolute maximum of points $n_y = 435$ is less than 0.5% and should be more than sufficient. The resolution used for the final runs was $n_z = 193$ in the $z$- and $n_y = 435$ in the $y$-direction.

5.4 Validation and runs

The changes in the code were again validated by comparing with the result for the monotonic exponential case. As in the DNS-simulations the changes in the code do not affect the results, but it showed that the reference frame velocity affects the result a lot. It is important that the reference velocity and the finger tip velocity agree with each other, otherwise the resolution at the tip front is not high enough. The variation with a slightly different velocity is extremely high.

We were not able to do the linear analysis for all of our DNS runs due to the high utilization of the supercomputer. We decided to do the cases which show a 'real' quasi-steady state. We did the linear stability analysis for all the DNS results with $Pe = 2000$ and the results for $Pe = 500, R = 5$ which show quite a quasi steady state.

Further we decided to simulate only a part of the dispersion relation, which includes the maximum of the growth rate $\sigma_{max}$. In general we calculated the dispersion relation from $\beta = 1.25$ to $\beta = 3$ with an increment of $\Delta \beta = 0.25$.  

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Figure 18: Change in the growth rate $\sigma$ due to increasing gridpoints in the $y$-direction.
6 Linear stability results

6.1 Dispersion relation and eigenfunctions

In the following we discuss the results of the linear stability simulations. As an example for the dispersion relation in Figure 19 the growth rate $\sigma$ is plotted as a function of the spanwise wavenumber $\beta$ for the case $Pe = 2000, R = 3, \Lambda = -3, \mu_m = 2$.

![Dispersion relation plot](image)

Figure 19: Dispersion relation for $Pe = 2000, R = 3, \Lambda = -3, \mu_m = 2$

The circles in the plot show the calculated values of $\sigma$ with increments of $\Delta \beta = 0.25$. In order to extract the maximum of the growth rate $\sigma_{max}$ we fit these points using a polynomial fit curve. Out of that curve we extract $\sigma_{max}$ and the corresponding value $\beta_{max}$. Because of the discrete points of the fitting curve the value $\beta_{max}$ has a deviation of $\pm 0.005$.

Fig. 20 shows the eigenfunction $\hat{u}, \hat{v}, \hat{w}$ and $\hat{c}$ for the wavenumber $\beta = 2$, for the case $Pe = 2000, R = 5, \Lambda = -3, \mu_m = 2$.

If we look closer at the concentration eigenfunction we will see that the maximum of the instability is always centered around the concentration value $c = 0.5$ independent of the viscosity profile. This behavior can be seen in Fig. 21.
Figure 20: Eigenfunctions for (from top to bottom) $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{c}$, for $Pe = 2000, R = 5, \Lambda = -3, \mu_m = 2$. The concentration eigenfunction in the bottom plot is superimposed on the base flow concentration field and shows that the instability develops centered around the tip of the finger. Solid lines denote positive values, while dotted lines indicate negative ones.

Figure 21: Top: The maximum of the concentration eigenfunction is always centered around the concentration $c = 0.5$ independent of the viscosity profile as demonstrated here for $Pe = 2000, R = 3, \Lambda = -3, \mu_m = 1.5$. The bottom plot shows the concentration eigenfunction superimposed with the viscosity $\mu$. 

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6.2 Influence of the viscosity ratio $R$

6.2.1 Growth rate

Fig. 22 shows the growth rate of the most dangerous mode $\sigma_{max}$ as a function of different viscosity ratios $R$ and $Pe = 2000$ for the nonmonotonic and monotonic profile.

The nonmonotonic profile is more stable than the monotonic one. For a low viscosity contrast the change is around 6% and increases to 10% as $R$ increases. For both of the cases an increase in the viscosity ratio is found to be destabilizing. The maximal growth rate increases with a factor of around 3 from $R = 2$ to $R = 5$. The curve starts to flatten out as $R$ increases. Goyal&Meiburg [6] show in their work that for the monotonic profile $\sigma_{max}$ will reach a maximum for a certain value of $R$, e.g. $Pe = 2000, R = 7$. A similar behavior is expected for the nonmonotonic case.

6.2.2 Wavenumber

The wavenumber of the most dangerous mode $\beta_{max}$ shows the opposite behavior of $\sigma_{max}$ (Fig. 23). The dominant wavenumber rate of the nonmonotonic profile is always higher than the corresponding value for the monotonic
profile. The maximal growth rate is on a high wavenumber for small $R$ and will decrease with a higher viscosity contrast. While the monotonic profile shows for this spectrum a quite linear behavior the nonmonotonic has a much higher wavenumber for $R = 2$. The difference between the monotonic and nonmonotonic profiles gets smaller for higher $R$.

![Figure 23: Comparison of the wavenumber $\beta_{max}$ for the monotonic and non-monotonic ($\Lambda = 0, \mu_m = 1.5$) viscosity profiles for different values of $R$ and $Pe = 2000$.](image)

6.3 Influence of the Péclet-number $Pe$

6.3.1 Growth rate

The influence of the Péclet-number on the maximal growth rate $\sigma_{max}$ is fairly small compared to the influence of the viscosity contrast (Fig. 24). The change from $Pe = 500$ to $Pe = 2000$ is for the non monotonic case ($\Lambda = 0, \mu_m = 2$) for $R = 5$ only around 7%. Even so the dominant growth rate increases with a higher Péclet-number. The influence of $Pe$ seems to be stronger for the non monotonic flow than for the monotonic case. Note that we only display $R = 5$ because we did not get a quasisteady state for lower $R$ and low $Pe$. The difference between the monotonic and the nonmonotonic case is not that high for other $R$. 

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Figure 24: The dominant growth rate $\sigma_{\text{max}}$ as a function of $Pe$ for $R = 5$.

6.3.2 Wavenumber

The wavenumber $\beta_{\text{max}}$ is pretty similar for the monotonic and non monotonic. Both profiles show a shorter wavenumber for a higher $Pe$, which is surprising but already found for the monotonic profile. The change is about 5% (Fig. 25).

6.4 Influence of the viscosity profile maximum $\mu_m$

6.4.1 Growth rate

Fig. 26 displays the dominant growth rate $\sigma_{\text{max}}$ as a function of the maximum viscosity $\mu_m$ for the parameters $Pe = 2000$ and $\Lambda = 0$. The plot shows that the influence of the maximum of the viscosity profile is not that strong compared to the influence of $R$. The change from $\mu_m = 1.1$ to $\mu_m = 2.0$ is at most 22% for $R = 2$ and at least 4% for $R = 3$. The interesting thing is that the growth rate increases for small $R$ and decreases for large $R$.

Additionally we display the growth rate as a function of $\mu_m$ for a constant value of $c_m$, whereby the maximum of the viscosity profile is always at the same concentration $c = 0.5$ (Fig. 27). The corresponding profiles are shown in the plot below.

The figure shows the same behavior as for a constant $\Lambda$. The influence is still pretty small. The growth rate decreases for a higher $\mu_m$ for $R = 5$ as it is already seen for $\Lambda = 0$ in Fig. 26.
Figure 25: $\beta_{\text{max}}$ as a Function of $Pe$ for different $R$.

Figure 26: The dominant growth rate $\sigma_{\text{max}}$ as a function of $\mu_m$.
Figure 27: Top: The growth rate of the most dangerous mode as a function of $\mu_m$ for $c_m = 0.5$. Bottom: The corresponding viscosity profiles.
6.4.2 Wavenumber

Like the growth rate the wavenumber $\beta_{\text{max}}$ does not change a lot with a different maximum $\mu_m$ for the same value of $\Lambda$ (Fig. 28). In general the wavenumber decreases slightly for a higher value of $\mu_m$.

![Figure 28: The wavenumber $\beta_{\text{max}}$ of the dominant mode as a function of the viscosity profile maximum $\mu_m$.](image)

In Fig. 29 the wavenumbers of the dominant mode for the different values of $\mu_m$ are plotted. The change in the wavenumber from $\mu_m = 1.1$ to $\mu_m = 2.0$ is around 4%. To a first approximation we can assume the wavenumber to be quite independent of $\mu_m$.

6.5 Influence of the parameter $\Lambda$

6.5.1 Growth rate

In Fig. 30 the maximal growth rate is plotted as a function of $\Lambda$ for different viscosity contrast $R$. We are focusing on the range of $\Lambda = [-3, 3]$, which Manickam&Homsy [10] used in their on Darcy law based work. We can not find a big difference in the dominant growth rate for these values of $\Lambda$. The viscosity contrast is still the dominant factor that effects the growth rate.

Further Manickam&Homsy investigated in their work that a value of $\Lambda = -3$ is more destabilizing than a value of $\Lambda = 0$ and $\Lambda = 3$. We could not
Figure 29: The wavenumber $\beta_{\text{max}}$ as a function of $\mu_m$ for $c_m = 0.5$.

Figure 30: The growth rate $\sigma_{\text{max}}$ as a function of the parameter $\Lambda$ for different values of $R$. 

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confirm this behavior for the Hele-Shaw cell. In fact the growth rate of the most dangerous mode is higher for \( \Lambda = -3 \) than for \( \Lambda = 0 \) and \( \Lambda = 3 \) as it is shown in Fig. 31.

![Diagram](P e = 2000, R = 5)

Figure 31: The dominant growth rate \( \sigma_{max} \) as a function of the viscosity profile \( \mu_m \) for different values of \( \Lambda \).

Overall it seems that for small values of \( \Lambda \) the relationship between the end-point derivatives of the viscosity profile is not that important. If we extend the range of \( \Lambda \) the effect is much stronger. Fig. 32 shows the maximal growth rate for \( \Lambda = [ -15, 15 ] \) and \( R = 5 \) and the corresponding viscosity profile. For the two extreme values \( \Lambda = \pm 15 \) the growth rate changes a lot. For \( \Lambda = -15 \) the growth rate drops down to \( \sigma_{max} = 0.753 \) which is a change from about 26%. For this case the growth rate is in the range of the values of \( R = 3 \) and is much more stabilizing than the other values. For the other extreme case \( \Lambda = 15 \) the growth rate increases around 8%. Here the results correspond to the investigation for the Darcy flow by Manickam&Homsy, but the question comes up if the change is due to the relationship between the end-point derivatives of the viscosity profile or more because of the strong changes in the shape of the profile. For example is the viscosity at the concentration \( c = 0.5 \), where the peak of the perturbation eigenfunction is located, for the viscosity profile with \( \Lambda = -15 \) is around half as high as for \( \Lambda = 3 \).

Fig. 33 shows \( \sigma_{max} \) as a function of different \( c_m \) for \( R = 5 \). The graph shows the same behavior as we already saw in Fig. 31. The values of \( c_m \) correspond to the following values of \( \Lambda \).
Figure 32: Top: Growth rate $\sigma_{max}$ for a range of $\Lambda = [-15, 15]$ and $R = 3$. Bottom: The corresponding viscosity profiles.
The growth rate is high if the maximum of the viscosity profile is closer to the less viscous profile, decreases as $\Lambda$ is getting close to 0 and increases again for low negative values of $\Lambda$.

$$c_m = 0.3 : \quad \Lambda = 16.8 \quad (70)$$
$$c_m = 0.5 : \quad \Lambda = 4.76 \quad (71)$$
$$c_m = 0.7 : \quad \Lambda = -3.68 \quad (72)$$

Figure 33: The growth rate $\sigma_{max}$ as a function of $c_m$ for $Pe = 2000, R = 5, \mu = 2$

6.5.2 Wavenumber

Fig. 34 displays the wavenumber $\beta_{max}$ as a function of $\Lambda$ for different values of $R$. It looks like for small viscosity ratios the wavenumber is slightly going down for higher values of $\Lambda$, while it is rising for the other two values of $R$. The changes are in general really small, only the change from $\Lambda = 0$ to $\Lambda = 3$ for $R = 2$ is significant here the value changes around 11% from a wavenumber of 2.49 to 2.77.

In Fig. 35 we display the dependency of the wavenumber of $\mu_m$ for different values of $\Lambda$. Note that we again took a high value of $R$ to be sure that all the simulations have reached a quasisteady state. The results show that the wavenumber is not really affected by the change of $\mu_m$ for the same
Figure 34: Wavenumber $\beta_{\text{max}}$ as a function of $\Lambda$ for different values of $R$.

The wavenumber is slightly decreasing for higher values but only for a few percent. The exact match of the values for $\mu_m = 1.1, \Lambda = 3$ and $\mu_m = 1.1, \Lambda = 0$ is a result of the evaluation process.

As for the growth rate we also plot the wavenumber as a function of higher values of $\Lambda$. At first glance it looks like there are big changes for the different values of $\Lambda$, but if we look at the scale of the $y$-axis we recognize that the changes are in the third order so we can assume that the wavenumber does not really change with different values of $\Lambda$ for $R = 5$. 


Figure 35: Wavenumber $\beta_{max}$ as a function of the viscosity maximum $\mu_m$ for different values of $\Lambda$

Figure 36: Growth rate $\sigma_{max}$ for a range of $\Lambda = [-15, 15]$ and $R = 3$
Summary

Viscous fingering for miscible fluids was studied for a horizontal Hele-Shaw cell by means of Stokes simulations and linear stability analysis. The focus of the study was to look at the influence of a nonmonotonic concentration viscosity relationship between the two fluids as it can be found in fluid pairs like water and alcohols. Herby, the nonmonotonic profile is defined through three parameters, the end-point viscosity ratio $R$, the value of the viscosity maximum $\mu_m$ and the position of the viscosity maximum $c_m$. Further a parameter $\Lambda$ is introduced, which is a quantity for the relation between the slope of the viscosity at $c = 0$ and $c = 1$. For these parameters and the flow describing non-dimensional parameter $Pe$ a parameter study was accomplished.

In a first step we extended the nonlinear Stokes code developed by N.Goyal for a monotonic exponential viscosity relationship in the way that a general viscosity profile can be used. Similar to the monotonic profile for large Péclet numbers and viscosity ratios a quasisteady state near the tip of the displacement front $d_0$ is found to develop. Goyal&Meiburg and Chen&Meiburg investigated that the front thickness of this quasisteady state is supposed to scale with $Pe^{-1/2}$. While the monotonic profiles follow the scaling law pretty accurately, nonmonotonic viscosity profiles deviate from this behavior, but the front thickness is still decreasing with a higher Péclet-number. The nonmonotonic profiles further show a thicker front than the monotonic while the tip velocity $v_{tip}$ is lower. As a difference to the monotonic profile the velocity of the nonmonotonic case is rising for all viscosity contrasts $R$ with increasing $Pe$.

Increasing of the maximum of the viscosity profile $\mu_m$ leads to a decreasing of the tip velocity while the front becomes thicker. For the parameter $\Lambda$ the tip velocity decreases as $\Lambda$ grows and the maximum of the viscosity profile moves to the less viscous fluid. The front thickness starts increasing especially for low viscosity ratios.

In the second part of the work we performed a linear stability analysis with respect to perturbations in the spanwise direction. Accordingly to the DNS we modified the existing stability code that the analysis is independent on the viscosity profile.

In the linear stability analysis the viscosity contrast $R$ is the dominant factor and affects the stability primarily. The influence of the Péclet number $Pe$ is really low and quiet negligible. The nonmonotonic profile shows a slightly more stable configuration than the monotonic profile (depending on the parameter deployed). The characteristic that Manickam&Homsy [10] investigated for the Darcy flow that a negative $\Lambda$ leads to a more stable con-
figuration than a positive Λ could not be found for the Hele-Shaw cell. In fact for the values they used in their work Λ = −3, 0, 3 we observed a opposite behavior, even if the differences were really small. It seems that for low values of Λ the effect is small. Only higher values of Λ lead to strong changes for the growth rate $\sigma_{max}$ of the most dangerous mode and the corresponding wavenumber $\beta_{max}$. For Λ = −15 we investigated a strong drop of $\sigma_{max}$ which leads to a more stable flow, while $\sigma_{max}$ rises for Λ = 15. This would fit to the investigation of Manickam&Homsy, but because this leads also to a strong change of the shape of the viscosity profile it is hard to say if the end-point derivative relationship is the determining factor. The exact reason could not be identified and would need further investigations.
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