Numerical modelling of snow using finite elements

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Numerical Modelling of Snow using Finite Elements

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Foreword

In mountainous regions avalanches are endangering people and man-made structures. During the catastrophic avalanche Winter 1999 in Switzerland 1’200 avalanches were observed resulting in not less than 17 fatalities and damages of over 600 million Swiss francs. To reduce the impact of future catastrophic situations a better understanding of the formation of avalanches is needed.

Based on these considerations, four years ago one of my former Ph.D. students, Dr. Perry Bartelt of the Swiss Federal Institute for Snow and Avalanche Research (SLF) in Davos came to me with the interesting question: “Where and when do avalanches release?” After some discussion we decided to start a Ph.D. project to investigate this challenging question.

In a first step the mechanical material properties of snow were investigated both at the SLF Institute and at the ETH Zurich. Based on these results Martin Stoffel developed numerical models to simulate the visco-elastic, temperature dependent behaviour of snow.

Within his dissertation he designed a framework to simulate a snow cover on an arbitrary mountain slope. This was composed of a Geographic Information System (GIS) interface for selecting the model domain and specifying the snow cover distribution, of a finite element simulation core and of a sophisticated result viewer to display the results in three dimensions.

To address the formation process of avalanche formation he investigated two different methods. The first one used standard linear elastic fracture mechanics using a two dimensional finite element model. The second method employed a three-dimensional model based on the so called “N-Directional” approach developed at our Institute by an other former Ph.D. student, Juan Renau. Both models also included a possibility to model frictionless weak zones within the snow cover, which are assumed to be the main reason for avalanche formation, and to simulate the effects of avalanche defence structures. With these methods it was possible to quantify the minimal weak zones size constraints for an avalanche to occur.

December 2005

Prof. Dr. Edoardo Anderheggen
Abstract

Snow avalanches are a major hazard to people and man-made structures in mountainous regions throughout the world. An analysis of the catastrophic avalanches of the Winter 1999 in the Alps [75] revealed that a quantitative understanding of the processes leading to avalanche formation is still lacking. Research in snow mechanics was subsequently intensified in Switzerland. In a necessary first step, the visco-elastic material behaviour of snow was experimentally investigated over a wide range of densities, strain-rates and temperatures. A constitutive model that could be used within the framework of a numerical simulation program was developed by Scapozza [62].

In this work the stability of the mountain snow cover is investigated using a multi-dimensional, thermo-mechanical finite element model. A visco-elastic constitutive model developed by Scapozza was implemented and amended to include fracture and damage processes.

The two- and three-dimensional finite element model is capable of modelling a layered snowpack, including "super" weak layers, in a complex two- or three-dimensional terrain. The terrain is specified using a digital terrain model that is created using a geographical information system (GIS). A non-equilibrium thermodynamical treatment of the ice and air phases of the snow matrix is proposed that accurately tracks the instationary temperature distribution. The stress and strain-rate distributions in the starting zones of real, well-known avalanche paths can therefore be modelled. Strain-rate concentrations around super weak layers are calculated.

Different methods to determine the "stability" of the snowpack were implemented. The two-dimensional finite element model calculates stress intensity factors using a time-dependent linear fracture mechanics approach. In this case, the measure of material strength is the fracture toughness of snow. Fracture experiments with snow were simulated in order to validate the numerical model. The three-dimensional finite element model uses the so-called $N$-Directional approach which is capable of modelling material damage leading to local or total slope failure. The $N$-Directional approach was devised by Anderheggen and Renau [57]. In this particular application, material damage is defined with respect to the strain-rate in one of the $N$-directions. Thus, this approach is also based on an experimentally based damage criterion.

Both methods show that an imperfection (a so-called "super" weak layer) in the snowpack is a necessary requirement for slab avalanche formation. The failure of a snowpack con-
taining a predefined weak zone could be modelled and the avalanche relevant parameters (size of the weak zone, temperature and density of the snowpack and slope angle) were identified and quantified. We found that the size of the imperfection is independent of the method of investigation. Both the two-dimensional fracture mechanics based approach and the three-dimensional N-Directional approach predict weak layer lengths of approximately 5 m to 10 m are required for an avalanche to occur. The formation process and development over time of such weak layers is not well understood, but this question is not addressed in this work.

Time dependent aspects of avalanche formation were additionally studied using the finite element model. The results revealed that the formation process of a slab avalanche is dominated by a race between the densification of the snow cover due to self-weight — which stabilises the snow cover — and the softening, destabilising influence of a temperature rise. The rate of the temperature rise influences avalanche activity. In comparison, the underlying base topography (slope angle) and snow cover height play only a minor role.

The application of the finite element method in a daily avalanche warning service will never be possible. The spatial and temporal resolution of the weak layer length must be known within $a \pm 0.5 \text{ m}$ over a time period of one hour. This is simply impossible. The uncertainty of the meteorological forecast within this time frame additionally hinders an accurate temporal forecast of an avalanche — even when all other snow properties are known.

Another use of the finite element model is to optimise the layout of defence structures in avalanche starting zones. The influence zone of such structures on the snow cover is likewise investigated with the finite element models.
Zusammenfassung


Beide Methoden haben gezeigt, dass es Störungen ("super-schwache“ Schichten) in der Schneedecke bedarf, um eine Lawine auszulösen. Das Versagen einer Schneedecke mit vordefinierter schwacher Schicht wird simuliert und die Parameter, welche die Lawinenbildung beeinflussen (Grösse der schwachen Schicht, Temperatur und Dichte der Schneedecke
sowie die Hangneigung) werden identifiziert und quantifiziert. Die verwendeten Methoden liefern dabei sehr ähnliche Resultate. Die lineare Bruchmechanik und der "N-Richtungen"-Ansatz sagen voraus, dass schwache Schichten der Größenordnung 5 m x 5 m nötig sind, um eine Lawine zu bilden. Der Bildungsprozess sowie das Verhalten von schwachen Schichten ist noch unklar, dies ist jedoch nicht Teil dieser Arbeit.


Die Anwendung eines finiten Elemente Modells im täglichen Einsatz für die Lawinenwarnung wird wohl nie möglich sein. Die räumliche und zeitliche Auflösung, welche für realistische Aussagen nötig wären, sind schlicht zu groß. Selbst die Ungewissheit in der Wettervorhersage reicht, falls alle anderen Schneeigenschaften bekannt sind, um eine genaue Voraussage über den Zeitpunkt, wann eine Lawine anbricht, zu verunmöglichen.

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Chapter 1

Introduction

The finite element method is now in everyday use in civil, mechanical and electrical engineering [3] [10] [29]. This dissertation explores the possibilities of applying the finite element method to study and understand a dangerous natural phenomenon — snow avalanches. Hence, in a broader sense, this dissertation is concerned with answering the question of whether an advanced numerical method can be effectively applied to study real geophysical problems where the initial and boundary conditions, as well as material properties, are difficult if not impossible to quantify. In our case, these conditions are imposed by the natural environment, outside the control of engineers. This differs dramatically from the usual engineering applications where material properties, geometry and initial conditions are given, can be optimised or changed to fulfil a specific task.

1.1 Modelling snow

Snow is one of the most complex materials appearing in nature [26]. Approximately 14% percent of the world’s surface is covered permanently with snow [71] and 5% of the annual precipitation on earth falls as snow [56]. The physical properties of snow have been studied for many reasons. For example, in many regions of the world, snow runoff is the only available source to replenish drinking water. Therefore, the radiative and energy exchange properties of the snow cover with the atmosphere have been a subject of great interest. Monitoring snow has generated increasing interest because of the debate about the global climate change. In Switzerland the primary reason to study snow has been the threat posed by snow avalanches to mountain communities. The mechanical and dynamical properties of snow have been the subject of research. Interestingly, even Newton in his *Principia* used snow as an example of mass conservation and densification before formulating his famous three laws [51].

The hydrological, ecological and mechanical applications involving snow and the snow cover require models [9] [14] [30], but the modelling of snow remains challenging. These models should be able to treat the following physical properties of snow:
Chapter 1. Introduction

Multicomponent porous media: Snow is a porous material consisting of ice, air and sometimes liquid water. The density of snow in the natural snow cover ranges between $\rho = 50 \text{ kg/m}^3$ to $500 \text{ kg/m}^3$, thus the volumetric content of the solid ice phase ranges from $\theta_i = 0.05$ to 0.55. The porous nature of snow must be addressed when developing numerical models. For example, conservation equations for mass, momentum and energy can be formulated for the bulk continuum [59] or for each component (ice, air and water) [9] [27]. Similarly, constitutive models for mechanical deformation and heat conductivity can be formulated for the ice-air mixture, usually parameterised using the density [63], or in terms of the ice, air and water components [2] [39]. The choice depends on the model application. Bulk models are computational efficient but often cannot simulate the complex behaviour of snow. Multicomponent models can be more accurate [9] [38] [39], however, they require more computation time and the data needed to evaluate model performance is often not available in sufficient detail.

Microstructure: The mechanical and thermophysical properties of snow depend on the size, shape and intergranular bonding of the ice lattice, the so-called microstructure (see Figure 1.1). For example, measurements of Young’s Modulus for snow show a large scatter, especially at low densities. This scatter has been attributed to different microstructure by many authors [62] [67]. The microstructure changes with time, primarily as a function of the thermal gradients in the snow cover [1] [48]. Thus, the mechanical properties of snow change over time as a func-
1.1. Modelling snow

tion of the thermal and load history. Modellers are thus confronted with the task of formulating rate equations describing the microstructural changes as well as formulating constitutive relations based on the microstructural parameters. To date such microstructural formulations have only been developed in one-dimensional snowpack models [9] [38] [39].

- **Temperature:** The mechanical properties of snow are a function of the temperature. For example, measurements [18] [65] show that the apparent thermal activation energy for viscous deformation increase significantly near 0°C. Therefore, the temperature state must always be prescribed when modelling the snow cover since there are large differences in mechanical behaviour for snow at −20°C and at 0°C. Furthermore, snow in alpine regions exists at high homologous temperatures, i.e. very close to its melting temperature. Many applications require including phase changes (melting, freezing and re-freezing) [39]. Finally, as stated earlier, the mechanical properties over time are a function of the thermal history. Models which attempt to track the development of the snow cover, must therefore calculate the thermal state of the snow cover over time, which is given by meteorological forcing — highly non-linear heat and mass exchanges with the atmosphere. Therefore instationary temperature equations must be solved.

- **Large deformations, strength and fracture:** New snow densities can be as low as 50 kg/m³. During its lifetime, snow can compact under self-weight to densities of over 500 kg/m³ (firn densities are even higher). Models which simulate the development of the snow cover must therefore treat large irreversible volumetric deformations on the order of 250%. Although the deformations are large, the material remains isotropic [58] unless material damage occurs. Damage can lead to local brittle (or ductile) failure. Constitutive formulations must therefore address the problem of deformation induced anisotropy leading to loss of material strength and total failure. Few numerical models have addressed the problem of snow failure.

- **Dimensionality:** One-dimensional snowpack models have been developed. It has been shown that these models can track the state of the snow cover over the entire winter, provided the meteorological data is given [9] [38] [39]. The results agree reasonably well with measurements [40] [41]. However, there are many problems which require higher dimensional models. Simple linear elastic states of strain and stress on slopes were calculated with a two-dimensional finite element model by Smith already in 1971 [76]. Bader and Salm investigated the forces on avalanche defence structures and the mechanics of avalanche release using two-dimensional boundary elements [6] [28]. The creep behaviour of the snow cover on idealised three-dimensional slopes has been studied by Kleemayr [34] using a three-dimensional general purpose finite element program. However, all those models used simplistic material properties.

- **Initial and boundary conditions:** When snow is modelled the initial state of the snow cover has to be specified. How the snowpack is built up is very difficult to assess — if not impossible. One would need to account for the microstructure (grain size and shape), density and temperature distribution. All measurement methods known
till now are destructive in the sense that each measurement destroys the measured sample. Additionally only point-measurements [68] or at most measurements of small areas are possible. Due to the apparent variability of the snowpack [36] [37] [69] these measurement give only a rough idea of the state of the snow cover. The surface on which the snow lies is normally well described using digital terrain models (DTM). Other boundary condition such as temperature, radiation, precipitation or wind and the resulting snow drift can also only be measured point-wise and need to be interpolated if they are needed for the whole avalanche slope.

1.2 Avalanche formation

Avalanche formation is the complex interaction between terrain, snowpack and meteorological conditions leading to sudden avalanche release.

There are two general types of dry snow avalanches. The loose snow avalanche and slab avalanche. **Loose snow avalanches** start at or near the surface and they usually involve only surface or near-surface snow. They start from a small area or even a point and spread out as they move down the slope in a triangular pattern. In the following only slab avalanches are considered, since loose snow avalanches seldom entrain enough snow to pose a major threat.

**Slab avalanches** initiate by a failure at depth in the snow cover, normally along a plane between two distinct layers, ultimately resulting in a block of snow, that is entirely cut out by propagating fractures between layers and through the slab. The main question that arises is how natural slab avalanches (i.e. excluding skier or explosive loading) form. As shown by Bader and Salm [6] and McClung [45] slab avalanches within an undisturbed snowpack are highly unlikely. It therefore needs an imperfection in the snow cover or at the bottom soil snow interface. This can be for example a buried depth hoar layer, a wind crust or any other usually thin and unstable layer. Those layers are termed *weak* or even *super weak* layers.

Where and when does what kind of avalanche occur? These are key questions for the back-country traveller, the avalanche forecaster and the snow safety manager alike. “How and why?” are additional questions for the researcher. There is usually no clear answer on the questions of occurrence, and current science is not able to give definite answers concerning the processes involved.

Avalanche formation can be investigated using different approaches. The complex interaction between terrain, snowpack and meteorological conditions are explored by statistical methods such as NXD [16]. Statistical methods are often used to supplement “expert” knowledge and local experience in practice. The second approach is to deterministically model the physical and mechanical processes of avalanche formation. The problem with the latter approach is that due to the stochastic nature of the meteorological processes acting on the snow cover, a purely deterministic approach to the question of ”where and when“ will probably have limited success. The usefulness of the deterministic approach, however, is to formulate general rules governing avalanche formation.
1.2. Avalanche formation

The dry snow slab avalanche is seen as the result of four types of failures, leading to five fracture surfaces: one in tension at the top of the slab (crown), two lateral breaks on the sides of the slab (flanks) mostly in shear, one compressive failure at the lower end (stauchwall) and a failure between the slab and the supporting substratum [21] (see Figure 1.2). The primary failure is between the slab and the stratum — commonly in slope-parallel shear, but occasionally compressive failure leads to the loss of shear support, comparable to adhesive vs. cohesive fracture in layered materials [80]. The compelling argument by Perla and LaChapelle [53] that shear failure at the base of the slab precedes tensile fracture through the slab is based on observations that the tensile fracture surface is perpendicular to the slope.

Different approaches were used in the past to address snow slope failure.

- **Shear strength approach:** The simplest model compares the shear strength of the weak layer to the shear stress due to the load of the overlaying slab and any artificial near surface loading at a specific time and location \( x \). The snow stability \( S \) is calculated as

\[
S = \frac{\tau_f(\sigma, x, t)}{\tau(x, t)}
\]  

where \( \sigma \) is the normal stress, \( \tau \) the shear stress, and \( \tau_f \) the shear strength. Theoretically, unstable conditions will occur when the stability index \( S \) approaches 1. This model is easily introduced into a numerical model because it requires only a
Chapter 1. Introduction

![Diagram of a slab underlain by a shear band with strain softening](image)

**Figure 1.3:** Schematic of a slab underlain by a shear band with strain softening taking place at the tip of the band [45].

The difficulty is that laboratory measurements of $\tau_f$ are so high that failure of the snowpack would never occur. Thus, this method does not capture the physics of failure. The measured values of $\tau_f$ can be empirically reduced to represent predefined “damage”; however, this ad-hoc procedure shifts the problem to defining when, how, and where such pre-damaged zones exist. Such procedures can be found in cellular automaton solutions [81].

- **Strain-rate approach:** Mechanical tests with snow show that snow strength is a function of the strain-rate. Brittle failure occurs at strain-rates near $10^{-4}/s$ [24], [48] [62]. Bader and Salm [6] investigated avalanche formation using a strain-rate based approach. They modelled snow as a purely viscous material and calculated the strain-rates due to self-weight. They stated that snow cannot fail until a critical strain-rate is reached. Interestingly, they found that the critical strain-rate cannot be reached in a homogeneous snowpack without a “super” weak layer. Subsequently, they inserted weak layers into their model and thereby stress and strain-rate concentrations at the upper end of the weak-layer. They found critical weak layer lengths of around 10 meters are needed in order to generate the critical strain-rates needed for failure. This procedure has been criticised since it does not predict how and when the “super” weak layers can form. The method has little predictive value.

- **Fracture mechanics approach:** McClung [44] [45] [48] was the first to apply fracture mechanical principles for snow. His work focused on ductile shear failure at the weak layer, followed by shear fracture and propagation, based on a model for the growth of a shear band (or slip surface) in an over-consolidated clay-mass [52]. A shear band is initiated at a stress concentration in the weak layer. Strain softening at the tip of the band follows. When a critical length $a_c$ is reached, the band propagates rapidly. The approach is similar to a Griffith criterion, which leads to analogous results, with the difference that the stress at the tip of the band is non-singular.
1.2. Avalanche formation

The Mode II propagation criterion is given by

\[
\frac{1 - \nu}{2G} K_{II}^2 = \frac{H(1 - \nu)}{4G} \left( \frac{\tau_g - \tau_r}{H} \right)^2 = (\tau_g - \tau_r) \delta \tag{1.2}
\]

with \(K_{II}\) the stress intensity factor for mode II fracture, \(H\) slab thickness perpendicular to the slope, \(\nu\) Poisson’s ratio, \(G\) shear modulus \(\tau_g = \rho g H \sin(\alpha)\) the shear stress due to the slab, with \(\rho\) average slab density, \(g\) acceleration due to gravity, \(\alpha\) slope angle and with \(\delta\) the displacement in the shear band due to peak stress \(\tau_p\) to residual stress \(\tau_r\) (see Figure 1.3). The two terms on the left are equivalent expressions for the driving term; the rightmost term provides the resistance to shear band extension. Assuming that the end zone length \(\omega\) is small compared to the band extension \(a\), the critical down-slope length \(a_c\) for band extension can be given as

\[
a_c = \frac{H}{\tau_g - \tau_r} \left( \frac{4G}{H(1 - \nu)} (\tau_p - \tau_r) \delta \right) \tag{1.3}
\]

The model can plausibly describe different slab release scenarios [48]. However, the measurement of the critical stress intensity factors \(K_{Ic}\) and \(K_{IIc}\) is not consolidated yet. Additionally it is not clear if linear fracture mechanics is appropriate for avalanche formation. This lies mainly in the fact, that the process zone might be too large for a purely elastic approach [72].

- **Damage mechanics approach**: Another approach to study the stability of the snow slab is to use damage mechanics [42].

The basic idea behind this method can be explained using a simple spring model (see Figure 1.4). For an undamaged ideal elastic spring the force displacement relation is given by Hook’s law

\[
F(x) = -k(x - x_0) \tag{1.4}
\]

where \(k\) is the stiffness of the spring and \(x_0\) the length of the force-free spring. For a given load with mass \(m\) the elongation \(x\) is given as

\[
x = x_0 - \frac{mg}{k} \tag{1.5}
\]

where \(g\) is the gravity constant.

The damage approach states that the spring can loose some of its load bearing capacity due to external and/or internal influences such as material softening, temperature, excessive loading etc. This is described by a damage function \(D(\text{external, internal})\) which can take values in the interval \([0,\infty]\). The stiffness of the spring in the damage model is now multiplied by the damage function which gives the following damage force displacement relation

\[
F(x_d) = -Dk(x_d - x_0) \tag{1.6}
\]

where \(x_d\) denotes the elongation for the damage model. In the undamaged case \((D=1)\) \(x_d\) is equal to \(x\). The elongation \(x_d\) can be computed as

\[
x_d = x_0 - \frac{mg}{Dk} \tag{1.7}
\]
Chapter 1. Introduction

Figure 1.4: Ideal elastic spring in force-free, undamaged and damaged state.

The damage function $D$ is used to change the initial spring model. $D$ can be non-linear and time dependent such that almost any behaviour can be modelled. For $D = 0$ the spring can be viewed as being destroyed. If $D$ is time dependent then $D$ is often a monotonically decreasing function in the interval $[0,1]$. Snow, however, exhibits “self-healing” and thus monotonically decreasing functions cannot be assumed in general.

1.3 Dissertation outline

The idea behind this dissertation is to develop both a two- and a three-dimensional finite element program which can be used to investigate the mechanics of avalanche formation — and other snow-related problems such as the optimisation of avalanche defence structures. Snow is treated as a multi-component porous medium consisting of ice and air. The mechanical response is given by the constitutive properties of the load bearing ice-matrix. Both the temperature in the ice and air are considered; states of thermal non-equilibrium between the ice and air can exist. Such a formulation is necessary to model snow metamorphism. However, the vast problem of modelling snow metamorphism is not the subject of this dissertation. The goal is to develop a numerical model where rate equations governing snow metamorphism can be introduced at a later stage. Instead, the more difficult problem of snow failure is addressed. Three different approaches are applied to study the problem: two-dimensional strain-rate (see Appendix C), two-dimensional fracture mechanics and three-dimensional damage mechanics using the so-called $N$-Directional approach. The creation of three-dimensional layered meshes lying on a mountain slopes is described. The terrain is specified using a digital terrain model.

The dissertation proceeds as follows: In Chapter 2 the formation of the alpine snow cover is described. This introduction is necessary to define the general equations governing mass, momentum and energy conservation. A temperature dependent, visco-elastic constitutive model of snow — based on extensive laboratory testing — is given. The constitutive
model will be used in both the two- and three-dimensional treatments of the alpine snow cover, presented in Chapter 3 and Chapter 5. An interface element is presented which is used to model super weak zones as well as basal gliding. An explicit time-solution method for the visco-elastic model is likewise presented in this chapter. The solution of the instationary temperature model using an implicit integration scheme is also discussed. Results, including extensive fracture mechanics calculations, using the two dimensional model are given in Chapter 4.

The discussion of the three-dimensional finite element formulation begins in Chapter 5. First the mesh generation using DTM based technology for a snowpack on a real slope is outlined. This is then used as a starting point for the finite element discretisation. The model can be used with both the standard tetrahedra element formulation and the so-called N-Directional approach using damage mechanics to study avalanche formation. Simulations and results using the three dimensional model are presented in Chapter 6. These simulations also include an investigation of defence structure placed in staring zones to prevent avalanche release.
Chapter 2

Thermo-Mechanical Properties of the Alpine Snow Cover

2.1 Description of snow

Snow can be described as a porous medium consisting of air and three water phases (ice, vapour and liquid). The ice phase consists of an assemblage of grains which are initially arranged in a random but load bearing skeleton (see Figure 1.1). This assemblage is termed the ice matrix. The ice matrix is surrounded by the ambient air, which can be separated into a dry air component and water vapour.

Figure 2.1 shows a schematic view of a sample of dry snow volume $V$. The volumetric contents of each phase $\theta_i$, $\theta_a$ and $\theta_v$ where the subscript $i$ denotes ice, $a$ stands for air and $v$ for vapour can be defined as follows: Let $I_i(x)$ be an indicator function for ice which returns 1 if we have ice at the point $x$ and 0 otherwise, then

$$\theta_i = \frac{1}{V} \int_V I_i(x)dV$$

(2.1)

defines the volumetric ice content. Analogous indicator functions can be used to define the volumetric air content $\theta_a$ and the volumetric vapour content $\theta_v$. Since $I_i(x) + I_a(x) + I_v(x)=1$ for all $x$ it follows that

$$\theta_i + \theta_a + \theta_v = 1.$$  

(2.2)

Bulk properties of snow are defined using the property $p$ of each phase and weighting it with the volumetric content $\theta$ of each phase as follows

$$p = \theta_ip_i + \theta_ap_a + \thetavp_v.$$  

(2.3)

The density of snow $\rho$ is defined as the mass of the snow sample divided by its volume. Since neither the air nor the water vapour have a significant mass compared to
the ice phase one can express \( \rho \) as a function of \( \theta_i \) only and the specific density of ice \( \rho_i = 917 \text{ kg/m}^3 \)

\[
\rho = \theta_i \rho_i + \theta_a \rho_a + \theta_v \rho_w \approx \theta_i \rho_i. \tag{2.4}
\]

The terms \( \theta_a \rho_a + \theta_v \rho_w \) are approximately 0 since for a dry snowpack \( \theta_v \approx 0 \) and the density of air is much smaller than the density of ice.

For other bulk properties such as temperature, \( T \), this is not the case. As can be seen in Figure 2.1 the volumetric content of air may be more than twice as large as the volumetric ice content. The bulk temperature is given by

\[
T = \theta_i T_i + \theta_a T_a + \theta_v T_v. \tag{2.5}
\]

The pore space of the ice matrix is filled with air and water vapour. This mixture is described using the ideal gas law according to Emil Clapeyron

\[
pV = nRT \tag{2.6}
\]

with pressure \( p \), volume \( V \) and \( n \) the number of moles, \( R \) is the ideal gas constant and \( T \) the temperature in Kelvin. The pressure of non-reacting gases is, according to Dalton’s Law, the sum of the respective partial pressures. For the dry air and water vapour mixture in the pore space of the ice matrix this can be written as

\[
p = p_a + p_v = \frac{n_a RT_a}{V} + \frac{n_v RT_v}{V} = (n_v + n_a) \frac{RT}{V}. \tag{2.7}
\]
2.2 Thermo-mechanical properties of snow

The size and shape of the ice grains constituting the snow can vary significantly. Grain size varies normally between fine (0.2 mm to 0.5 mm) to coarse (1.0 mm to 2.0 mm) and the shape can vary between large and fragile six-fold star-like crystals to quasi spherical grains. For a more complete listing see [23]. The process of transforming the snow crystals from one form to the other is called metamorphism (see [19] for an overview). Metamorphism occurs when the snow is thermodynamically unstable, either due to a large surface area to volume ratio which leads to a high surface free energy, or because of a large temperature gradient. The two processes lead to very different ice crystal morphology.

The surface free energy is minimised by minimising the area to volume ratio. The minimum shape is a sphere. The morphologic change leading to rounded grains is termed equilibrium growth metamorphism.

If there is a large temperature gradient ($\approx 10^\circ$C/m) the minimisation of the area to volume ratio is not the dominant mechanism any more. The snow builds pillar shaped snow crystals. They are called depth hoar, surface hoar and near surface faceting.

Both processes even take place at temperatures below 0°C by sublimation and condensation of the ice.

Another aspect of the snow cover is the amount of liquid water within the snow. As long as the ambient temperature stays below 0°C the snow can be regarded as dry, which means there is no liquid water within the snow. At temperatures of about 0°C the snow can contain liquid water. In the following, however, only dry snow is assumed.

2.2 Thermo-mechanical properties of snow

2.2.1 Mechanical properties of snow

The mechanical properties of snow have been investigated by many authors. For an overview see [49]. The mechanical response of snow under applied loading depends highly on the loading rate. Therefore, the mechanical properties are usually classified according to the induced strain-rate. High strain-rates, $\dot{\varepsilon} > 10^{-3}$ can be caused by skiers, vehicles or blasting (artificial avalanche release). This work is concerned with problems with low to moderate strain-rates, say $10^{-10} < \dot{\varepsilon} < 10^{-3}$ which are typically found in the natural snow cover. In this regime the loading is usually the self weight of the snow cover.

The mechanical behaviour of snow exhibits a strong similarity to polycrystalline ice which is the constituent material of the load bearing skeleton [64]. Like polycrystalline ice, snow is a visco-elastic material having an elastic, anelastic and viscous part. The anelastic part describes the time dependent reversible response of both polycrystalline ice and snow under loading. Snow does not show a time independent yielding and therefore can not be described as a plastic material, although the irreversible viscous strains can be large.

Even for low to moderate strain-rates the mechanical properties of snow are difficult to assess for the following reasons:
Chapter 2. Thermo-Mechanical Properties of the Alpine Snow Cover

- **Density dependence:** Snow density ranges from 50 kg/m\(^3\) to 500 kg/m\(^3\) in the natural alpine snow cover (not including glaciers). Over this range of densities Young’s modulus and the viscosity vary over several orders of magnitude. Young’s modulus varies between 0.1 MPa to 100 MPa, the compactive viscosity varies between \(10^2\) MPas to \(10^8\) MPas [62]. Therefore, small variations in density can have a large influence on the stress and strain distribution in the snow cover.

- **Temperature dependence:** The temperature in the snow cover is driven by meteorological conditions. Typical temperatures range between \(-40^\circ C\) to \(0^\circ C\). The viscosity of snow is not only density dependent but also highly temperature dependent. An accurate model for snow densification must therefore take this temperature dependent viscosity into account. Similar to polycrystalline ice, Young’s modulus is temperature independent [62].

- **Large strains:** The natural snow cover is in a continual state of deformation. Snow creeps and glides on the ground. For example new snow may have a low density (\(\approx 100\) kg/m\(^3\)) that compacts to 450 kg/m\(^3\). Volumetric strains are in this case on the order of 300%. Thus the mechanical properties must be defined over a wide range of densities.

- **Microstructure:** The mechanical response of snow displays a strong dependence on the microstructure (grain size, grain shape, pore space, grain connectivity) of the constituent ice matrix. The state of micro stress will differ significantly from the state of the snowpack stress when viewed as a continuum.

- **Spatial and temporal variability:** The snow cover consists of layers arising from snowfall events. The mechanical properties of each layer will differ, because the meteorological conditions vary over time. Therefore the snow cover evolves to a layered anisotropic system. The mechanical properties of each layer, however, are usually considered to be isotropic. However, recent high resolution penetrometer tests [68] of the snow cover show that this might be a too simplified model of the snow cover. The layers are often interrupted and there is a considerable variation in density and microstructure.

The mechanical behaviour of snow (i.e. the straining behaviour) can be described in terms of the state of stress in the snow. The stress state of the snow is determined by the forces exerted and the strain describes the resulting deformation. This stress-strain relation for an elastic material is typically written as

\[
\epsilon = f(\sigma)
\]

where \(\epsilon\) is the strain and \(\sigma\) is the stress. The stress \(\sigma\) itself can be dependent on different quantities such as temperature, density etc. Let \(f\) for now be an arbitrary non-linear function, which depends on the snow properties.

Snow, from a mechanical point of view, can be seen as a non-linear visco-elastic material [62]. The total straining of snow resulting from an applied force can be separated in two parts: an elastic and a viscous part. The viscous strain \(\epsilon_v\) is irreversible meaning
2.2. Thermo-mechanical properties of snow

that the straining remains after unloading. The viscous deformation of snow develops over time and is highly temperature dependent. The reversible elastic strain contains two parts, the time and temperature independent elastic strain $\epsilon_e$ and the delayed elastic strain $\epsilon_d$ which is time and temperature dependent. The total strain can therefore be expressed as

$$\epsilon = \epsilon_e + \epsilon_d + \epsilon_v.$$  \hspace{1cm} (2.9)

The change of straining over time, the strain-rate, is then given as

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_d + \dot{\epsilon}_v.$$ \hspace{1cm} (2.10)

Since snow can be viewed as a visco-elastic material it has a time dependent deformation response to loading, termed creep. Creeping is the part of the deformation for a constant load which occurs over time after the initial (elastic) deformation. It can be separated in three phases:

- **Phase I**, primary creep, shows a declining strain-rate over time. Normally phase I creeping stops after some time. For snow this is the phase when the snow is compacted due to the rearrangement of the lose snow grains within the snowpack.

- **Phase II**, secondary creep, shows a constant strain-rate. This occurs if phase I creeping does not stop. Due to the fact, that snow grains can undergo metamorphism for example forming of rounded grains out of pillar shaped crystals, phase II creeping can be observed for snow.

- **Phase III**, tertiary creep, occurs at the end of phase II creeping, when the material starts to lose its capability to deform. It is stamped by an increasing strain-rate and ends often with a failure of the material. For snow this happens when the bonds between the snow grains start to break.

The creeping of snow is described by the viscous and delayed elastic strain. Compared to creep straining, the elastic strains are negligible.

It has to be mentioned that most of the above described mechanisms depend highly on the density and temperature of the snow [46].

### 2.2.2 Thermodynamic properties of snow

Since the mechanical properties of snow depend highly on the temperature it is essential to have a good understanding of the thermodynamical behaviour of snow [4]. The basic question in this regard is the manner in which energy is transported into and through the snowpack?

The thermal energy balance equation written in a general form is given as

$$\int_V \left[ \frac{\partial}{\partial t} (\rho c_v T) + \nabla \cdot \mathbf{q} \right] dV = Q$$ \hspace{1cm} (2.11)
where \( V \) is the sample volume, \( c_v \) the heat capacity of the medium, \( \mathbf{q} \) the energy flux vector and \( Q \) an energy supply like radiation [13].

The four main energy transfer methods are as follows:

- **Conduction** takes place by atom to atom contact, in response to a temperature gradient. The position of the atoms stays the same (apart from Brownian motion) i.e. no mixing takes place. Conduction plays an important role in solids, but can also occur in fluids.

  This is modelled within the same medium using the Fourier Conduction Law

  \[
  \mathbf{q}_c = -k_t \nabla T
  \]  

  where \( k_t \) is the coefficient of thermal conductivity of the material. For the boundary of two medias Newton’s Law of Cooling is used

  \[
  Q = -k_a \frac{dT}{dx} \bigg|_{\text{boundary}} = \dot{h}_c (T_2 - T_1)
  \]  

  where \( \dot{h}_c \) is the heat transfer coefficient specifying the amount of heat interchange between the two materials at the surface.

- **Convection** is transport of energy carried by the molecules in motion. Two forms can be defined: The sensible heat flux in response to a temperature gradient and the latent heat flux through the change of state among solid, liquid and gas phases.

  The sensible energy flux is given as

  \[
  \mathbf{q}_{s} = \rho c_v \mathbf{u}_a T.
  \]  

  Here \( \mathbf{u}_a \) is the velocity vector of the convecting substance.

  The latent heat flux can be written as

  \[
  \mathbf{q}_L = L \dot{m}_{1 \rightarrow 2}
  \]  

  with \( L \) as the latent heat of phase change from phase 1 to 2 and \( \dot{m}_{1 \rightarrow 2} \) mass rate from phase 1 to 2.

- **Advection** is often also called horizontal convection. The energy is transported by mass transfer. Examples are rain, snow and wind drift.

- **Radiation** is the transfer of energy by electromagnetic waves. Each body radiates and absorbs energy. The amount and the peak radiation wavelengths are determined by the temperature of the body.

  The radiative energy flux \( \mathbf{q}_r \) between two bodies can be written as

  \[
  \mathbf{q}_r = \varepsilon \sigma (T_2^4 - T_1^4)
  \]  

  with \( \varepsilon \) the emissivity of the body and \( T_1, T_2 \) the temperatures of the two bodies. It is assumed that the emissivity is equal the absorptivity. \( \sigma \) is the Stefan-Boltzmann constant.
2.2. Thermo-mechanical properties of snow

For a snowpack there are three regimes to be considered where energy transfer can occur; namely the contact surface at the bottom of the snowpack, the snowpack itself and the top surface of the snowpack (see Figure 2.2).

**Figure 2.2: Energy transport mechanisms for the alpine snow cover.**

Energy transfer at the bottom surface

The ground on which the snowpack lies is normally frozen at a temperature of about 0 °C. Exceptions are when it snows onto permafrost or if it snows before the ground is frozen. It will be assumed in the following that the ground stays at a constant temperature of 0 °C during the period it is covered by snow. This is reasonable since soils are generally warmer than the overlying snow which has at most a temperature of 0 °C. The reason is that the ground is a huge energy reservoir which is fed by geothermal heat and the stored heat from the summer. The heat flux from the soil into the snowpack is mainly conduction at the boundary.

Energy transfer at the top surface

At the top surface all the energy transfer mechanisms discussed are involved.
The radiation from the sun gets absorbed by the snow surface, predominately in the visible wavelengths (0.3 \( \mu \)m to 1.2 \( \mu \)m). Short wave radiation can penetrate the snow cover and is partly absorbed at depth. The amount of absorbed energy depends on the grain size and the cleanness of the snow (since dust particles or the like augment the energy absorbed). Long wave radiation (5 \( \mu \)m to 50 \( \mu \)m) from the atmosphere is absorbed and emitted at the surface since ice is close to a black body in the infrared. The net energy transport into the snow is hence the difference between the received radiation and the outgoing radiation from the snowpack (2.16).

Quite contrary to the bottom surface, conduction into the air plays a minor role for the energy transport at the top surface and will not be addressed here.

Convective, latent and sensible heat exchanges play a role at the upper boundary. The sensible heat exchange occurs when there is a temperature gradient in the air just above the snow cover. The temperature gradient drives the exchange of the air within the first section of the snowpack and the snow surface. This exchange, which transports heat from or into the snowpack is accelerated by wind over the snow surface. Latent heat flux occurs if the air is super-saturated or sub-saturated with water vapour with respect to the snow surface. Energy can then be transferred to the snowpack through condensation or removed through sublimation and latent heat exchange. Conversely, if the atmosphere is drier than the snowpack, energy in the snowpack is used to sublime water molecules from the snowpack into the air.

The last form of heat transfer at the surface of the snow cover discussed is advective heat and mass transfer to the snowpack by rain or snowfall. The water vapour in the surrounding air can condensate during cold nights onto the snow and thereby builds so-called surface hoar crystals.

Energy transfer within the snowpack

The energy within the porous snowpack can be transported by conduction and convection. Radiation between ice grains can be neglected and there is no advective heat flux considered (i.e. no liquid water that rinses through the pore space).

Conduction within the snowpack arises from the temperature gradient between the top and bottom surfaces. Conduction is considered relevant in the ice, the air and between these constituents. It is important to note that the ice and the surrounding air need not always be in thermal equilibrium.

As on the top surface convection can take place in the form of latent heat flux. The ice grains sublime water molecules into the air and water molecules can condense onto ice grains. The driving forces are temperature and pressure gradients. Metamorphism results in rounded grains if a pressure gradient is the driving force or in pillar shaped crystals when a temperature gradient dominates the process. The sensible heat flux due to temperature gradients within the snowpack forces air movement through the pore space of the ice matrix. However, since the velocity of the air in the pore space is very small this effect does not play a major role.
2.3. The structure of the alpine snow-cover

The alpine snow-cover is in general horizontally stratified by several more or less distinctive snow layers. There are a number of ways by which these snow layers are formed.

The main mechanism for the snow layers is accumulation from a snow fall event. The density of fresh snow can be quite low (50 kg/m$^3$ to 150 kg/m$^3$) depending on the temperature and humidity during precipitation. Compaction of the fresh snow begins immediately due to the overburden snow, which drives the rearrangement of snow grains and metamorphic processes.

Other, mostly thinner, layers that develop on the top of the snowpack are covered by subsequent snowfalls adding to the stratigraphic variation. There are several ways by which these layers are formed.

Snow can be deposited/eroded due to wind transport. At lee sides a layer of mostly small grained snow is build up. On the windward side the snow often gets eroded until a harder layer forms on the surface, either by compaction of the snow due to wind forces or by excavating a buried hard layer.

Another source for hard layers are ice crusts formed either by surface melting on warm sunny days which is frozen as soon as the temperature drops, or by short rain falls wetting the surface followed by a temperature drop below freezing.

Surface hoar layers are another often seen layer type. They develop when water vapour condenses onto the top surface. Surface hoar crystals can grow into very large and fragile structures (see Figure 2.3).

In addition to the top surface crusts, crusts can also build between two old layers with different densities, and permeability.
Another form of a layer building within the snowpack is depth hoar.

2.4 Physical model and material laws of snow

In this section the physical model of the snow will be introduced. In particular, the material laws used in the finite element model will be presented. If not otherwise stated, all temperatures in this section are in Kelvin.

2.4.1 Mechanical model

The constitutive model for snow presented in the following is based on extensive triaxial testing with snow by von Moos [79] and Scapozza [64] [65].

Under an applied deformation rate Scapozza [63] identified three different strain types. These are:

- $\varepsilon_e$: elastic strain (recoverable),
- $\varepsilon_d$: delayed elastic strain (time dependent, recoverable),
- $\varepsilon_v$: viscous strain (unrecoverable).

The total strain is defined as the sum of the elastic, delayed elastic and viscous strains

$$\varepsilon = \varepsilon_e + \varepsilon_d + \varepsilon_v.$$  \hspace{1cm} (2.17)

The material laws for the three components are formulated in the following. The numerical values are from the dissertation of Scapozza [62].

2.4.1.1 Elastic behaviour

The linear elastic normal strain, $\varepsilon_e$ is related to the axial stress by Young’s modulus $E_0$

$$\sigma = E_0 \varepsilon_e.$$  \hspace{1cm} (2.18)

According to a large number of triaxial tests performed over a wide range of densities (180 kg/m$^3$ to 450 kg/m$^3$) [65] [79] Young’s modulus for snow can be parameterised with the density using an exponential function

$$E_0(\rho) = 0.1873 e^{0.014g\rho} \ (r^2 = 0.928)$$  \hspace{1cm} (2.19)

where $E_0$ is in MPa, $\rho$ the density of the snow in kg/m$^3$ and $r$ the correlation coefficient. This formula is only valid for the fine-grained, well bounded snow investigated and cannot be applied a priori for other snow types (faceted crystals or wet snow). Young’s modulus is found to be independent of the test temperatures, $-20°C < T < -2°C$ [64], which compares well with the ice matrix material.
2.4. Physical model and material laws of snow

2.4.1.2 Delayed elastic behaviour

The triaxial test of Scapozza [63] showed that the delayed elastic strain can amount to 20% of the total elastic strain, depending on the density and temperature of the snow. The polycrystalline ice model of Sinha [74] was modified for snow to predict the delayed elastic or anelastic part of the strain \( \epsilon_d \) as a function of time, stress and temperature

\[
\epsilon_d(T, \rho, t) = \left( \frac{\sigma}{E_0(\rho)} \right)^s \frac{1}{K} [1 - e^{-(a_T t)^b}] = (\epsilon_e)^s \frac{1}{K} [1 - e^{-(a_T t)^b}]
\]

where \( \sigma \) is the stress in kPa, \( E_0 \) the Young’s modulus in kPa, \( s \) and \( b \) are dimensionless exponents (temperature independent), \( \epsilon_e \) is the total elastic strain, \( K \) a dimensionless parameter independent of temperature and stress, \( a_T \) is a temperature dependent parameter in 1/s and \( t \) is the time in seconds. For polycrystalline ice Sinha [73] found \( a_T = 1.94 \times 10^{-4} \) /s for \( T = -10^\circ \text{C} \). Further Sinha assumed \( s = 1 \) and found \( b = 0.37 \). The above formula was first found to be a good qualitative predictor of the delayed elastic strain using Sinha’s values for \( s \) and \( b \). Using somewhat different values \( s \approx 1.4, b \approx 0.12 \) and \( K \) between 0.03 and 0.06 the experimental data could be fitted even better.

It has to be mentioned, that \( a_T \) has to be scaled to the used temperature \( T \) that is used. To do this the Arrhenius law, which describes the effect of temperature on a reaction rate, is used in the following way

\[
a_T(T_1) = \frac{a_T(T_2)}{S_{1,2}}
\]

where \( S_{1,2} \) is given by

\[
S_{1,2} = \frac{Q}{R} \left( \frac{T_1}{T_2} \right).
\]

Here \( R \) is the universal gas constant in kJ/molK and \( Q \) is the apparent activation energy in kJ/mol given by

\[
Q = \begin{cases} 
76 & \text{if } T < 263 \text{ K} \\
10.73T + 2747 & \text{if } T \geq 263 \text{ K}
\end{cases}
\]

where \( Q \) is called apparent activation energy [12] [77] since it is not possible to measure the true activation energy of snow for all temperatures [15] [25]. Other physical properties, that can not be eliminated for temperatures higher than \(-10^\circ \text{C} \), lead to a measured activation energy which is not the true activation energy. The activation energy itself is the energy barrier that needs to be overcome to start a thermodynamically driven process such as creeping [50].

2.4.1.3 Viscous behaviour

The elastic and delayed elastic parts of the total deformation are small compared to the irreversible viscous part. The elastic parts, however, are important since they determine the stress level which is needed to calculate the creep velocity as given in the following. The triaxial tests of Scapozza and Bartelt [63] showed that the viscous strain can be expressed using a hyperbolic potential equation

\[
\dot{\epsilon}_v = A_0 e^{\frac{Q}{RT}} \sinh(\alpha \sigma)^n
\]
where $R$ is the molar gas constant in kJ/molK, $A_0$ is a density and temperature dependent material parameter in 1/s given by
\[
\ln(A_0) = 1.707 \times 10^{-4} \rho^2 + 1.625 \quad \nu^2 = 0.983 \quad \text{for} \quad T_i = -11.3 \degree C \quad (2.25)
\]
where $\rho$ is the snow density in kg/m$^3$. The scaling to different temperatures is done in the following way using the Arrhenius law (see also Equation (2.21))
\[
A_0(T_1) = \frac{A_0(T_2)}{S_{1,2}} \quad (2.26)
\]
where $S_{1,2}$ is given as in Equation (2.22). The exponent $n$ in Equation (2.24) is given by
\[
 n = a(T_i)\rho^2 + b(T_i)\rho + c(T_i) \quad (2.27)
\]
where the coefficients $a$, $b$ and $c$ are as follows
\[
a = 7.2042679 \times 10^{-7} T_i^2 - 3.7731919 \times 10^{-4} T_i + 4.9383224 \times 10^{-2},
\]
\[
b = -4.8868831 \times 10^{-4} T_i + 2.5631217 \times 10^{-1} T_i - 3.3586996 \times 10^{+1},
\]
\[
c = 7.6536355 \times 10^{-2} T_i^2 - 4.0231886 \times 10^{+1} T_i + 5.2827215 \times 10^{+3}. \quad (2.28)
\]
Equation (2.27) is valid for $\rho$ between 180 kg/m$^3$ and 450 kg/m$^3$. For $\rho < 180$ kg/m$^3$ $n$ has a value of $n = 1.4$ (temperature and density independent) and $n = 3.7$ for $\rho > 450$ kg/m$^3$. The remaining two model parameters are the apparent activation energy $Q$ in kJ/mol as given in Equation (2.23) and the fitting parameter $\alpha$ in 1/kPa which is density dependent and given by
\[
\alpha = 2.65489 \times 10^9 \rho^{-1.6497}. \quad (2.29)
\]
Using Equation (2.24) the viscosity of snow can be found according to
\[
\eta_v = \frac{\sigma}{\dot{e}_v} = \sigma \left[ A_0 \rho^2 \sinh(\alpha \sigma) \rho^2 \right]^{-1}. \quad (2.30)
\]

2.4.2 Thermodynamic model

Since the mechanical properties depend highly on temperature it is important to model the temperature of the snow cover. For simplicity, however, not all the details described in Section 2.2.2 are taken into account. It is assumed, that the temperature of the snow (ice and air phase) are given at the atmosphere and soil boundary. So all energy transfers on the top surface (radiation, convection and heat fluxes) and bottom surface (conduction) are determined to yield the temperature.

Even for the energy transfer within the snow some simplifications are made. First of all the snowpack is assumed to be dry, implying that the liquid water content is zero and the temperature stays below 0°C so that there is no melting.

The processes modelled are heat conduction in both the ice and air, convection in the pore space and latent heat. The full energy balances for the ice and air phase are given by the following two equations [31]
\[
\theta_a \rho_a c_a \left( \frac{\partial T_a}{\partial t} + u_a \cdot \nabla T_a \right) - \theta_a k_a \nabla^2 T_a = h_a (T_i - T_a) \quad (2.31)
\]
2.4. Physical model and material laws of snow

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Name</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_a$</td>
<td>heat capacity of air</td>
<td>1000</td>
<td>J/(kg m$^3$)</td>
</tr>
<tr>
<td>$c_i$</td>
<td>heat capacity of ice</td>
<td>2100</td>
<td>J/(kg m$^3$)</td>
</tr>
<tr>
<td>$h_a$</td>
<td>convective heat transfer coefficient</td>
<td>$\approx 0.5$</td>
<td>W/(m$^2$ K)</td>
</tr>
<tr>
<td>$k_a$</td>
<td>conductivity of air</td>
<td>0.026</td>
<td>W/(m$^2$ K)</td>
</tr>
<tr>
<td>$k_i$</td>
<td>conductivity of ice</td>
<td>2.200</td>
<td>W/(m$^2$ K)</td>
</tr>
</tbody>
</table>

Table 2.1: Thermodynamic constants for ice and snow.

for the air temperature, and

$$
\theta_i \rho_i c_i \frac{\partial T_i}{\partial t} - \theta_i k_i \nabla^2 T_i = h_a (T_a - T_i) + L \dot{m}_{i\rightarrow a}.
$$

(2.32)

for ice temperature.

Where $u_a$ is the vector of interstitial air velocities, which are assumed to be in the creeping flow regime ($Re < 10$), $c$ the specific heat capacities, $k_i$ and $k_a$ the microstructural conductivities from the model of Adams and Sato [2], $L$ the latent heat of vapourisation of water, $\dot{m}_{i\rightarrow a}$ mass rate from the ice matrix into the air (negative if from the air onto the ice matrix) and $h_a$ the convective heat transfer coefficient [31] which is an important mode of heat transfer occurring at the inter-facial air–ice boundary. $h_a$ is estimated to be $h_a \approx 0.5$ for snow.

The following simplifications are next made to the equations: First in Equation (2.31) the vector of interstitial air velocities $u_a$ is set to 0, this means, that no energy is transported by the movement of inter-porous air. Second the mass rate $\dot{m}_{i\rightarrow a}$ is set to 0. This implies that no sublimation or re-sublimation occurs. The resulting set of equations then simplifies to

$$
\theta_a \left( \rho_a c_a \frac{\partial T_a}{\partial t} - k_a \nabla^2 T_a \right) = h_a (T_i - T_a),
$$

(2.33)

$$
\theta_i \left( \rho_i c_i \frac{\partial T_i}{\partial t} - k_i \nabla^2 T_i \right) = h_a (T_a - T_i).
$$

(2.34)

The thermodynamic constants are given in Table 2.1.
Chapter 3

2D Finite Element Formulation

The finite element method is firmly accepted as the most powerful general technique for the numerical solution of a variety of problems encountered in engineering. It is established as a general numerical method for the solution of partial differential equation systems subject to known boundary and/or initial conditions.

Since snow is a temperature dependent, highly nonlinear, visco-elastic material an analytical solution for realistic cases is not available.

The finite element method gives an approximate solution of the given differential equation by partitioning the model domain into many small yet finite regions, or "elements" from which the technique derives its name. The physical behaviour of each element is modelled locally. Physical properties of the element nodes are expressed in an algebraic expression, which is then assembled into the global model for the entire structure. By solving the global set of algebraic equations an approximate solution is found.

A two dimensional finite element method for the mechanical and thermo-mechanical model of the snowpack is presented in the following.

3.1 Finite element discretisation

Since the finite element method is an approximate solution method, converging to the true theoretical solution when the element mesh is refined, the first step that needs to be done is to discretise the domain for which the problem is formulated. The model domain is divided into a four-node isoparametric element mesh.

The set of linear element shape functions is defined as $H_i(r, s) = \frac{1}{4}(1 \pm r_i)(1 \pm s_i)$ for $i = 1\ldots4$, where $(r_i, s_i)$ are the corner coordinates of the natural element. The coordinates $(x,y)$ within each element in the global coordinate system can now be expressed in the
natural \((r,s)\) coordinate system as
\[
\begin{align*}
x(r,s) &= \sum_{i=1}^{4} H_i(r,s)x_i, \\
y(r,s) &= \sum_{i=1}^{4} H_i(r,s)y_i
\end{align*}
\tag{3.1}
\]
where \(x_i\) and \(y_i\) are the corner coordinates in the global coordinate system (see Figure 3.1). For notational simplicity the \((r,s)\) arguments are omitted in the following.

In the isoparametric formulation the element displacements \(u\) and \(v\) and their variations \(\delta u\) and \(\delta v\) are interpolated in the same way as the geometry; i.e. one uses
\[
\begin{align*}
u &= \sum_{i=1}^{4} H_i u_i, \\
v &= \sum_{i=1}^{4} H_i v_i
\end{align*}
\tag{3.2}
\]
where \(u_i\) and \(v_i\) are the nodal displacements and \(\delta u_i\) and \(\delta v_i\) the corresponding variations in the global coordinate system. These relations can also be written in matrix form as
\[
\begin{bmatrix}
u \\ u
\end{bmatrix} =
\begin{bmatrix}
H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4 & 0 \\
0 & H_1 & 0 & H_2 & 0 & H_3 & 0 & H_4
\end{bmatrix}
\begin{bmatrix}
u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4
\end{bmatrix} = [H] \begin{bmatrix}
u \\ u
\end{bmatrix}
\tag{3.3}
\]

### 3.2 Visco-Elastic solution

To solve the visco-elastic equations, first an elastic solution is calculated resulting in the elastic displacements \((u_e, v_e)\) of the nodes. From this the resulting elastic strains \(\epsilon_e\) and stress state \(\sigma_e\) are determined.

Using the elastic stress state \(\sigma_e\) a viscosity \(\eta\) is calculated according to Equation (2.30) which then is used to calculate a creep velocity \(\dot{\epsilon}_{cr}\). This creep velocity is used to evolve the system through time by simply using an explicit Euler scheme.

The governing differential equation to be solved is
\[
-\text{div}(\sigma) = \mathbf{f} = \mathbf{f}_g + \mathbf{f}_e
\tag{3.4}
\]
or written using Einstein’s summation convention
\[
-\sigma_{ij,j} = f_i \quad (i,j = x, y, z)
\tag{3.5}
\]
where \(\sigma\) is the stress tensor and \(\mathbf{f}\) is the force acting on the body. Forces acting on the body are the gravity force \(\mathbf{f}_g\) (weight/body force) and/or an external force \(\mathbf{f}_e\) applied on the boundary of the body.
3.2. Visco-Elastic solution

3.2.1 Elastic element

An elastic body reacts on an exerted force with an instantaneous strain. As soon as the force is removed the body returns to its force-free unstrained configuration. This relation is described by the generalised Hooke’s Law, $\sigma = E \varepsilon$. Where $\sigma$ is the stress tensor, $E$ the elasticity tensor, which is a fourth order tensor representing the linear elastic coefficients, and $\varepsilon$ is the strain vector. The in plane displacements of the body in the $x$ and $y$ direction are denoted by $u$ and $v$. In matrix notation this can be written as

$$
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
E
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
2\varepsilon_{xy}
\end{bmatrix} =
E
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial y}
\end{bmatrix}.
$$

The form presented here is only valid for small strains.

For isotropic materials the matrix of elastic coefficients is reduced to a dependence on only two constants. In the present development Young’s modulus, $E$ (given by 2.19), and Poisson’s ratio, $\nu$, are adopted. In our case the two-dimensional elastic plain strain tensor is used

$$
E = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1 & \frac{\nu}{1-\nu} & 0 \\
\frac{\nu}{1-\nu} & 1 & 0 \\
0 & 0 & \frac{1-2\nu}{2(1-\nu)}
\end{bmatrix}.
$$

Figure 3.1: Natural representation of the isoparametric element and its connection to the actual element in global coordinates.
Applying the principle of virtual work to Equation (3.4) leads to the equation which is solved using a finite element scheme

\[- \int_\Omega [\delta e]^T \sigma d\Omega = \int_\Omega [\delta u]^T f_e d\Omega + \int_{\Gamma_e} [\delta u]^T f_d d\Gamma_e\]  

(3.8)

where \( \Omega \) is the volume of the body and \( \Gamma_e \) the boundary on which \( f_e \) is applied. The term on the left hand side of Equation (3.8) is equal to the virtual work as the result of a virtual displacement \( \delta u \).

Using the constitutive relation \( \sigma = E \varepsilon \) this can be written as

\[- \int_\Omega [\delta e]^T E \varepsilon d\Omega = \int_\Omega [\delta u]^T f_e d\Omega + \int_{\Gamma_e} [\delta u]^T f_d d\Gamma_e.\]  

(3.9)

The left hand side term contains the strain \( \varepsilon \) and the virtual strain \( \delta \varepsilon \). The strains within each element can be expressed in terms of the element displacements \( \varepsilon = Bu \) where

\[ B = [B_1, B_2, B_3, B_4]; \quad B_i = \begin{bmatrix} \frac{\partial H_i}{\partial x} & 0 & \frac{\partial H_i}{\partial y} \\ 0 & \frac{\partial H_i}{\partial y} & \frac{\partial H_i}{\partial x} \end{bmatrix}. \]  

(3.10)

The element strains are obtained in terms of derivatives of element displacements with respect to the global coordinates. Because the element displacements are defined in the natural coordinate system using (3.2) the \( x,y \) derivatives are related to the \( r,s \) derivatives. Using this relation \( \partial / \partial x, \partial / \partial y \) can be calculated using the strain-displacement transformation matrix by applying the chain rule

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x}, \]  

(3.11)

\[ \frac{\partial}{\partial y} = \frac{\partial}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial}{\partial s} \frac{\partial s}{\partial y}. \]  

(3.12)

The inverse relation which is used later can be written in matrix notation as

\[ \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \]  

(3.13)

\[ \frac{\partial}{\partial r} = J \frac{\partial}{\partial x} \]  

(3.14)

where \( J \) is the Jacobian operator.

Using the above relations the left hand side of Equation (3.8) can be expressed for each element as

\[ [k]u = \int_{-1}^{1} \int_{-1}^{1} [B]^T [E][B] |\text{det}[J]| dr ds u. \]  

(3.15)

This is the element stiffness matrix which will be summed into the global stiffness matrix \( K \).
3.2. Visco-Elastic solution

The first integral on the right hand side of Equation (3.8) for each element can be shown to be

\[ f_g = \int_{-1}^{1} \int_{-1}^{1} [H]^T f^e \det[J] dr ds \] (3.16)

which is the area distributed over the nodes times the gravity force where \( f^e_g \) is given for each element by

\[ f^e_g = [0 -\rho g 0 -\rho g 0 -\rho g 0 -\rho g]^T. \] (3.17)

The second integral on the right hand side of Equation (3.8) for each element is

\[ f_e = \int_{-1}^{1} [H] f_i dr \] (3.18)

along an edge with constant \( s \).

Assembling the elements stiffness matrices and force vectors one obtains the final matrix equation which has to be solved for the nodal displacements \( u \)

\[ [K]u = [f] = [f^e_g] + [f^e_r]. \] (3.19)

Using the displacements from the solution to Equation (3.19) one gets the elastic stress \( \sigma_e \) and deformation state.

The numerical integration of the above integrals is accomplished using four point Gauss integration. The integrands are evaluated at the four Gauss integration points \( p_i \) and weighted with the weights \( w_i \) given in Table 3.1. An integral of the form

\[ A = \int_{-1}^{1} \int_{-1}^{1} f(r,s) dr ds \] (3.20)

is evaluated as

\[ A = \sum_{i=1}^{4} w_i f(p_i). \] (3.21)

This integration is exact for the linear shape functions \( H_i \) used in our discretisation model.
Chapter 3. 2D Finite Element Formulation

3.2.2 Viscous element

As stated previously, snow shows a viscous aspect to its behaviour. Analogous to the elastic stress-strain relation a viscous stress strain-rate relation: $\sigma = V \dot{\epsilon}$ exists, where

$$V = \eta \begin{bmatrix} (1 + \frac{1}{1+m}) & \frac{1}{m-2} & 0 \\ \frac{1}{m-2} & (1 + \frac{1}{1+m}) & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$  \quad (3.22)

In this matrix $\eta$ is the viscosity (calculated using Equation (2.30)) and $m$ is the viscous analogous of the Poisson’s number.

As for the elastic solution, this stress strain-rate relation is an idealised constitutive law in the differential Equation (3.4). The discretisation process is the same as for the elastic problem, with the difference that strains and displacements are replaced by strain-rates and displacement velocities resulting in the matrix equation

$$[S]\ddot{u} = [F] \quad (3.23)$$

where $[S]$, the so-called viscous stiffness matrix (this “stiffness” is a material property analogous to the elastic tensor, which is also often referred to as the elastic stiffness matrix. This should not be confused with the element stiffness, which incorporates these), is assembled from the elements viscosity matrices which are given by

$$[s] = \int_{-1}^{1} \int_{-1}^{1} [B]^T[V][B] |\text{det}[J]| dr ds. \quad (3.24)$$

The solution of the matrix equation results in the displacement velocity $\ddot{u}$ which will be used to evolve the system through time.

3.2.3 Boundary conditions

The boundary for the snowpack model domain can normally be divided into four parts. The top, left (downward facing), bottom and right (upward facing) boundaries, for which a boundary conditions can or has to be defined.

For the elastic equation the following constrains are used

$$u_i|_{\text{bottom}} = 0,$$  \quad (3.25)

$$\sigma_{ij}n_j|_{\text{top}} = 0$$

where $n_j$ is the outward pointing normal to the top surface. This means that the elements are fixed at the bottom to the ground (i.e. no displacement occurs) and that the top boundary is force free.

Analogous boundary conditions are used for the viscous equation

$$\dot{u}_i|_{\text{bottom}} = 0,$$  \quad (3.26)

$$\sigma_{ij}n_j|_{\text{top}} = 0.$$
For the left and right boundary one can either use a stress free formulation, periodic boundary conditions (resulting in an infinitely long slope) or fixing the displacements in any direction (for modelling a snow defence structure for example).

### 3.2.4 Weak layer elements

To model a crack in a snowpack a special finite element was introduced. This element will be called weak layer element hereafter.

As Bader and Salm [6] showed, a homogeneous snowpack without any initial inhomogeneities, i.e. weak layer, cracks or the like, is very unlikely to avalanche. Therefore a crack model is needed.

The idea is to change the behaviour of the standard isoparametric element in such a way, that the physical properties in the natural $s$ direction (see Figure 3.1) are transported as if the element did not exist. The thickness (in local $s$ direction) is by definition set to zero. In the other direction (local $r$) the behaviour should be freely definable.

Based on the work of Beer [11], Samtani et al [61] the following modifications to the isoparametric element are applied. In a first step it is assumed, that the two directions $r$ and $s$ are treated independently. There is no interaction between the two directions, which means for the elastic part that the Poisson’s number $\nu = 0$ and for the viscous part $m$ set to $\infty$. Additionally the lower right entry of the elasticity and viscosity matrices is set to 0. This results in the following two matrices for a weak layer element

$$
E_{\text{weak}} = \begin{bmatrix}
E_r & 0 & 0 \\
0 & E_s & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

and

$$
V_{\text{weak}} = \begin{bmatrix}
\eta_r & 0 & 0 \\
0 & \eta_s & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

(3.27)

The resulting stiffness and viscosity matrix for the weak layer elements will then be diagonal as well, and can be assembled into the global stiffness matrices.

Figure 3.2 depicts a weak layer element and its deformation directions. By setting $E_s$ (and $\eta_s$ respectively) to $\infty$ any forces acting in the $s$ direction do not deform the element in the $s$ direction, the forces are simply transmitted through the element. With this, the height of the weak element will stay zero since $v_{s-top} = v_{s-bottom}$. For $E_r$ and $\eta_r$ a value of zero is chosen. This results in no resistance for the nodes in the $r$ direction so that the relative movement of the nodes from the top and the bottom element are free.

#### 3.2.4.1 Using weak layer elements for gliding

Using the above described weak layer element one can model gliding with a simple change. Instead of using 0 for $E_r$ (and $\eta_r$ respectively) one can use different values, coming from experimental data, using a friction coefficient or any other constitutive law.
3.2.5 Summary of the computational procedure

The basic steps in the solution process are described in Algorithm 3.1. It has to be noted, that the stress and strain history is not explicitly taken into account with this method. The strain-rate from the viscous solution is assumed to be constant over a time step $\Delta t$. The size of the time step $\Delta t$ is determined by the maximal strain-rate over all elements. It is chosen such that the maximal straining for any element is small enough to justify the linearisation and the small strain assumption is fulfilled implicitly with this method.

Algorithm 3.1: 2D Visco-Elastic FE-Scheme

Require: Set temperature $T_i, T_a$, final calculation time $t_{\text{final}}$, simulation time $t = 0$, initial densities $\rho(t = 0)$, initial positions $x(t = 0)$

Ensure: At time $t_{\text{final}}$ stress state $\sigma_{ij}$, strain state $\epsilon_{ij}$, strain-rate $\dot{\epsilon}_{ij}$, movement $u$, creep velocities $\dot{u}$, forces $f$, density $\rho$, positions $x$

1: while $t < t_{\text{final}}$ do
2: \hspace{1em} Solve elastic $\rightarrow$ 3.2
3: \hspace{1em} Solve viscous $\rightarrow$ 3.3
4: \hspace{1em} Determine timestep $\rightarrow$ 3.4
5: \hspace{1em} Advance System $\rightarrow$ 3.5
6: end while

Figure 3.2: Definition of the "weak layer" element showing the deformation directions for the case that $E_s = \infty$. 
Algorithm 3.2: Solve elastic

Require: Density $\rho$ and positions $\mathbf{x}$ for each element

Ensure: Elastic stress state $\sigma$, elastic displacements $u_e$

1: for all elements do
2: Calculate Young’s Modulus $E(\rho)$
3: Calculate element stiffness matrix $[k]$
4: Assemble $[k]$ into the global stiffness matrix $[K]$
5: Calculate element forces $[f]$
6: Assemble $[f]$ into the global force vector $[F]$
7: end for

8: Apply boundary conditions
9: Solve $[K]u_e = [F]$ for the unknown displacements $u_e$
10: for all elements do
11: Calculate the elastic stress state
12: end for

Algorithm 3.3: Solve viscous

Require: Density $\rho$, stress state $\sigma$, temperature $T_i$ and positions $\mathbf{x}$ for each element

Ensure: Strain-rate $\dot{\epsilon}$, creep velocities $\dot{u}$

1: for all elements do
2: Calculate viscosity $\mu(\rho, \sigma, T_i)$
3: Calculate element viscosity matrix $[s]$
4: Assemble $[s]$ into the global stiffness matrix $[S]$
5: Calculate element forces $[f]$
6: Assemble $[f]$ into the global force vector $[F]$
7: end for

8: Apply boundary conditions
9: Solve $[S]\dot{u} = [F]$ for the unknown creep velocities $\dot{u}$
10: for all elements do
11: Calculate the strain-rate
12: end for

Algorithm 3.4: Determine timestep

Require: Strain-rate $\dot{\epsilon}$ for all elements, max timestep $\Delta t_{\text{max}}$

Ensure: Time step $\Delta t$

1: for all finite elements do
2: Find largest value of $\dot{\epsilon}_{\text{max}} = \text{MAX}(|\dot{\epsilon}|)$
3: end for
4: Set $\Delta t = \text{MIN}(\Delta t_{\text{max}}, 0.01(1/\dot{\epsilon}_{\text{max}})$
Chapter 3. 2D Finite Element Formulation

Algorithm 3.5: Advance System

**Require:** Time step $\Delta t$, creep velocities $\dot{u}$ density $\rho(t)$ and positions $x(t)$

**Ensure:** positions $x(t + \Delta t)$, density $\rho(t + \Delta t)$ time $t$

1: for all elements do
2:  Calculate element volume $V_{\text{old}}$ using positions $x(t)$
3: end for
4: for all nodes do
5:  Calculate new positions $x(t + \Delta t) = x(t) + \Delta t\dot{u}$
6: end for
7: for all elements do
8:  Calculate element volume $V_{\text{new}}$ using positions $x(t + \Delta t)$
9:  Calculate $\rho(t + \Delta t) = \rho(t)/V_{\text{old}} \cdot V_{\text{new}}$
10: end for
11: Set $t = t + \Delta t$

3.3 Fracture mechanical modelling

3.3.1 Why fracture mechanics?

The snow cover is viewed, from a fracture mechanical point of view, as a component with a crack. The main question to be solved is if this crack is stable or not. In the following a review of the possible methods is discussed and the reason for choosing the linear elastic fracture mechanics model (LEFM) is given.

Both the creep-growing and the mechanics of damaged continuum are based on the softening of the material over time. Results from tensile tests show that snow shows softening only after a large straining of the material [62]. This behaviour is contradicting the field observations, where an avalanche starts as an abrupt event without preceding large deformations. Models driven by an under critical crack growth are therefore not appropriate for describing the formation of an avalanche.

Elastic plastic fracture mechanics (EPFM) can be viewed as the general theory of fracture mechanics. This method, however, is computationally intensive. Therefore the linear elastic fracture mechanics (LEFM) is chosen. The LEFM is consistent with the EPFM in the sense that with shrinking plastic zone around the crack tip the LEFM converges to the EPFM. Therefore the decision criterion between EPFM and LEFM is the size of the plastic zone close to the crack spike. As long as the plastic zone is well embedded within the elastic stress field the LEFM method can be used, otherwise the EPFM has to be applied [66]. Snow shows, under very high tension as at the tip of a crack, a quasi linear elastic to brittle behaviour. Hence there is no possibility for a plastic zone to be developed. Therefore the LEFM method is applicable for snow.
3.3. Fracture mechanical modelling

3.3.2 Definitions

For the modelling of avalanche formation in a natural snowpack, two crack loading types have to be considered: Mode I, the symmetrical crack opening and mode II, the shear plane crack opening (see Figure 3.3).

The threshold value for a dynamic, over-critic stress induced crack propagation is called $K_{Ic}$ and $K_{IIc}$. Hence a crack propagates if the following inequality is fulfilled

$$K_I > K_{Ic}, \quad K_{II} > K_{IIc} \quad \text{resp.} \quad (3.28)$$

where $K_I$ and $K_{II}$ are the so-called stress intensity factors, a local parameter to characterise the crack stressing for mode I and mode II respectively. $K_{Ic}$ and $K_{IIc}$ are called fracture toughness and represent the fracture resistance for mode I and mode II crack propagation, respectively.

The stress intensity factors can be determined either analytically as a local parameter, using the local stress distribution in the elastic stress field [66] or from the global parameters which characterise the crack stressing. The relationship between the local and global stress intensity factor parameters are as follows

$$G = \frac{K_I^2}{E}, \quad G = \frac{K_{II}^2}{E} \quad \text{resp.} \quad (3.29)$$

where $G$ denotes the energy release rate and $E$ is the elasticity modulus. $G$ is defined as the energy that can be released per unit crack-length as follows

$$G = \frac{1}{B} \frac{dW_{el}}{da} \approx \frac{1}{B} \frac{\Delta W_{el}}{\Delta a} \quad (3.30)$$
where $B$ is the width of the component (see Figure 3.3) and $\Delta W_{el}$ the change in elastic energy due to enlarging the crack by $\Delta a$ (see Figure 3.4).

### 3.3.2.1 Visco-elastic considerations

As shown above the energy release rate $G$ is the driving force for crack propagation. It has to be mentioned, that two forms of mechanical energy are appearing in a visco-elastic model: the reversible elastic energy and the dissipative (irreversible) viscous energy. Only the elastic energy can be used for fracture propagation.

The damage mechanical model, however does not lack this feature. Among other reasons it is therefore used in the three dimensional model.

### 3.3.3 Calculation of the stress intensity factors

To calculate the stress intensity factors the change of strain energy due to the enlargement of the crack has to be calculated according to Equation (3.30). Since we have a two dimensional model, $B$ can be set to 1 then the following relation holds

$$G = \frac{K_{I,II}^2}{E} = \frac{dW_{el}}{da}. \quad (3.31)$$

The force stored in an ideal linear elastic material (i.e. a spring) is by definition of the elasticity module given by

$$f = E \Delta x \quad (3.32)$$

where $\Delta x$ is the amplitude from the equilibrium. The energy stored in the material can be calculated as

$$W = \int_{\Delta x}^0 E x dx = \frac{1}{2} E \Delta x^2. \quad (3.33)$$
Along this line the stored elastic energy in the finite element mesh can be calculated. The energy in each element is calculated as

\[ W_e = \frac{1}{2} u^T [k] u \]  

and hence the energy in the whole system is given by

\[ W_{el} = \sum_{i=0}^{n_e} W_e = \frac{1}{2} u^T [K] u \]  

where \([k]\) is the element stiffness matrix, \([K]\) the global stiffness matrix, \(u\) the elastic deformation vector and \(n_e\) the number of Elements.

The energy released by enlarging the weak layer, i.e. the crack by \(\Delta a\) (see Figure 3.4 and 3.5) can then be expressed as the difference between \(W_{el}(a)\) and \(W_{el}(a + \Delta a)\) normed by \(\Delta a\)

\[ G = \frac{W_{el}(a + \Delta a) - W_{el}(a)}{\Delta a} \]  

or

\[ G = \frac{1}{\Delta a} \left\{ \frac{1}{2} u^T [K] u \big|_{a+\Delta a} - \frac{1}{2} u^T [K] u \big|_a \right\} \]  

\[ \approx \frac{1}{2} u^T \frac{d[K]}{da} u. \]  

The stress intensity factors \(K_I, K_{II}\) can then be calculated as

\[ K^2_{I,II} = GE = \frac{1}{\Delta a} \left\{ \frac{E}{2} u^T [K] u \big|_{a+\Delta a} - \frac{1}{2} u^T [K] u \big|_a \right\}. \]
In the finite element calculation this is done on a per element basis, since the elasticity
\( E \) is dependent on the density and is therefore not constant over the whole domain. This
results in the following formula

\[
K_{i,II}^2 = GE = \frac{1}{\Delta a} \left\{ \frac{1}{2} \sum a^T[k]uE \bigg|_{a+\Delta a} - \frac{1}{2} \sum a^T[k]uE \bigg|_a \right\}.
\] (3.40)

Here \( u, [k] \) and \( E \) are element displacement, element stiffness matrix and elasticity module
of the element. It has to be mentioned that a coordinate system is used such that the
direction of fracture is aligned with the \( x \)-coordinate.

### 3.4 Temperature solution

For the temperature solution the same finite element mesh and shape functions are used.
In the following the equations for the ice phase are developed, and therefore the subscript
\( i \) is omitted wherever possible. The process for the air is similar. The ice temperature
field can be described for each element as

\[
T(r, s) = \sum_{j=1}^{4} H_j(r, s) T_j
\] (3.41)

where \((r, s)\) are the natural coordinates of the element and \( T_j \) the ice temperatures at the
nodes. Additionally it is assumed, that the time dependence can be separated from the
spatial variations such that the time dependence can be decoupled in the following form

\[
T(r, s, t) = \sum_{j=1}^{4} H_j(r, s) T_j(t).
\] (3.42)

Taking the same finite element mesh as for the visco-elastic solution process, the weak
form of Equation (2.34) is obtained by multiplying with a weight function \( w(r, s) \) and
integrating over the element area, which leads to

\[
\int_{\Omega} \left[ \theta \left( \frac{\partial T}{\partial t} - k_i \nabla^2 T_i \right) \right. - \left. w_h(T_a - T_i) \right] d\Omega = 0.
\] (3.43)

The second term of Equation (3.43) is investigated further. First rewritten as

\[
\theta \int_{\Omega} w \left[ -k_{11} \frac{\partial T^2}{\partial x^2} - k_{22} \frac{\partial T^2}{\partial y^2} \right] d\Omega
\] (3.44)

and then integrated by parts in the spatial variables results in

\[
\theta \int_{\Omega} \left( k_{11} \frac{\partial w \partial T}{\partial x^2} + k_{22} \frac{\partial w \partial T}{\partial y^2} \right) d\Omega - \theta \int_{\Gamma} w \left( k_{11} \frac{\partial T}{\partial x} n_x + k_{22} \frac{\partial T}{\partial y} n_y \right) d\Gamma.
\] (3.45)
The last term of this equation contains \( n_x \) and \( n_y \) the outward unit normal vectors of the element. Let
\[
g_n := k_{11} \frac{\partial T}{\partial x} n_x + k_{22} \frac{\partial T}{\partial y} n_y
\] (3.46)
denote the heat flux normal to the boundary of the element (into the element). Using all this, Equation (3.43) can be written as
\[
\int \theta_i \left[ w \rho c_i \frac{\partial T_i}{\partial t} + k_{11} \frac{\partial w}{\partial x} \frac{\partial T_i}{\partial x} + k_{22} \frac{\partial w}{\partial y} \frac{\partial T_i}{\partial y} \right] - w h_a (T_a - T_i) d\Omega - \int_{\Gamma} \theta_i w q_n d\Gamma = 0. \tag{3.47}
\]

One now uses the discretisation Equation (3.42) to express the temperature using the shape functions. And since the equation must hold for any weight function one substitutes \( w \) by \( H_j \) for \( j \) from 1 to 4 such that one gets a system of algebraic equations. This results in
\[
0 = \sum_{k=1}^{4} \left( m_{jk} \frac{dT_k}{dt} + k_{jk} T_k \right) - h_{cj} - q_{nj} \tag{3.48}
\]
for each element. This can be written in matrix form as
\[
[\mathbf{m}] \dot{T} + [\mathbf{k}] T = h + q \tag{3.49}
\]
where \( \dot{T} = dT/dt \) and the matrices are given as
\[
\begin{align*}
[\mathbf{m}]_{jk} &= \theta_i \int_{\Omega} \rho c H_j H_k d\Omega, \\
[\mathbf{k}]_{jk} &= \theta_i \int_{\Omega} (k_{11} \frac{\partial H_j}{\partial x} \frac{\partial H_k}{\partial x} + k_{22} \frac{\partial H_j}{\partial y} \frac{\partial H_k}{\partial y}) d\Omega, \\
[\mathbf{h}]_k &= \int_{\Omega} H_k h_a (T_a - T_i) d\Omega \quad \text{and} \\
[\mathbf{q}]_k &= \theta_i \int_{\Gamma} H_k q_n d\Gamma.
\end{align*}
\]

The same calculations can be made for the air temperature, such that the set of Equations (2.33) and (2.34) can be written in matrix notation using finite element discretisation as
\[
[\mathbf{m}]_a \dot{T}_a + [\mathbf{k}]_a T_a = h_a + q_a \tag{3.54}
\]
and
\[
[\mathbf{m}]_i \dot{T}_i + [\mathbf{k}]_i T_i = h_i + q_i. \tag{3.55}
\]

Defining now
\[
T = \begin{cases} T_a \\ T_i \end{cases}
\]
(3.56)
and using the boundary condition that no heat is transported over the boundary (i.e. \( q_i = q_e = 0 \)) the system can be rewritten as
\[
\begin{bmatrix}
[\mathbf{m}]_{jk} & 0 \\
0 & [\mathbf{m}]_{jk}
\end{bmatrix} \dot{T} + \begin{bmatrix}
[\mathbf{k}]_{jk} & 0 \\
0 & [\mathbf{k}]_{jk}
\end{bmatrix} T = \begin{cases} h_a \\ h_i \end{cases}.
\tag{3.57}
\]
Chapter 3. 2D Finite Element Formulation

It is now time to investigate \( h_a \) and \( h_i \) a little further. Once more the derivations are made for the ice part. Starting from Equation (3.52)

\[
h_k = \int_{\Omega} H_k h_a(T_a - T_i) d\Omega = \int_{\Omega} H_k h_a \left( \sum_{j=1}^{4} H_j T_a - \sum_{j=1}^{4} H_j T_i \right) d\Omega \tag{3.58}
\]

\[
= \int_{\Omega} H_k h_a \sum_{j=1}^{4} H_j (T_a - T_i) d\Omega \tag{3.59}
\]

using the definition of \( T \) in (3.56) this can be written as a matrix equation in the following form

\[
h = \begin{bmatrix} \tilde{h}_{jk} & -\tilde{h}_{jk} \end{bmatrix} T \tag{3.60}
\]

with \( \tilde{h} \) defined as

\[
\tilde{h}_{jk} = \int_{\Omega} h_a H_k H_j d\Omega. \tag{3.61}
\]

An analogous calculation can be made for the air part. Taking the two parts together one can rewrite the right hand side of Equation (3.57) as follows

\[
\begin{bmatrix} h_a \\ h_i \end{bmatrix} = \begin{bmatrix} -\tilde{h}_{jk} & \tilde{h}_{jk} \\ \tilde{h}_{jk} & -\tilde{h}_{jk} \end{bmatrix} \begin{bmatrix} T_a \\ T_i \end{bmatrix} = \tilde{H} T. \tag{3.62}
\]

Finally we can write the differential equation as

\[
\begin{bmatrix} [m_{jk}]_a & 0 \\ 0 & [m_{jk}]_i \end{bmatrix} \begin{bmatrix} \dot{T}_a \\ \dot{T}_i \end{bmatrix} + \begin{bmatrix} [k_{jk}]_a & 0 \\ 0 & [k_{jk}]_i \end{bmatrix} \begin{bmatrix} T_a \\ T_i \end{bmatrix} = \begin{bmatrix} -[\tilde{h}_{jk}] & [\tilde{h}_{jk}] \\ [\tilde{h}_{jk}] & -[\tilde{h}_{jk}] \end{bmatrix} \begin{bmatrix} T_a \\ T_i \end{bmatrix} \tag{3.63}
\]

or in a shorter way as

\[
[M] \dot{T} + [K] T = \tilde{H} T. \tag{3.64}
\]

Where \([M], [K]\) and \(\tilde{H}\) are representing the assembled global matrices. The evaluation of the integrals for the matrix entries is done along the same lines described for the elastic solution using a four point Gaussian integration.

### 3.4.1 Time-integration scheme for the temperature solution

For the solution of the instationary temperature Equation (3.64) the so-called \( \theta \)-method was implemented [55]. This scheme can be used as explicit (Euler forward, \( \theta = 0 \)) or fully implicit (Euler backward, \( \theta = 1 \)) method and belongs to the class of finite difference schemes.

In each solution step given the solution \( T(t^n) = T^n \) at time \( t^n \) (here \( n \) denotes the time step number) the temperature \( T^{n+1} \) at time \( t^{n+1} \) is to be computed.

In a first step the time derivative of the temperature \( \dot{T} \) has to be computed. This is done by the following approximation using a backward difference

\[
T^{n+1} = \frac{1}{\Delta t} (T^{n+1} - T^n) \tag{3.65}
\]
3.4. Temperature solution

where $\Delta t = t^{n+1} - t^n$. With the assumption that the temperature $T$ changes linearly over the time-step $\Delta t$. Therefore the temperature in the interval $[t^n, t^{n+1}]$ can be written as

$$T(t^n + \theta \Delta t) = (1 - \theta)T^n + \theta T^{n+1}$$

(3.66)

for $0 \leq \theta \leq 1$. Equation (3.64) can now be written as

$$[M] \frac{1}{\Delta t} (T^{n+1} - T^n) + [K]((1 - \theta)T^n + \theta T^{n+1}) = [\tilde{H}](1 - \theta)T^n + \theta T^{n+1}).$$

(3.67)

Reordering the terms results in

$$\left\{ \frac{1}{\Delta t}[M] + \theta([K] - [\tilde{H}]) \right\} T^{n+1} = \left\{ \frac{1}{\Delta t}[M] + (1 - \theta)([\tilde{H}] - [K]) \right\} T^n.$$

(3.68)

This matrix equation is now solved for the unknown $T^{n+1}$.

For different values of the parameter $\theta$, one obtains several well-known time approximation schemes:

- $\theta = 0$, the forward difference scheme, also called Euler forward (conditionally stable, explicit)
- $\theta = \frac{1}{2}$, the Crank-Nicholson scheme (unconditionally stable, implicit)
- $\theta = \frac{2}{3}$, the Galerkin scheme (unconditionally stable, implicit)
- $\theta = 1$, the backward difference scheme, also called Euler backward (unconditionally stable, implicit)

Conditionally stable means that the solution is stable only when the time step is smaller than a given maximal value $\Delta t_{cr}$

$$\Delta t < \Delta t_{cr} = \frac{2}{(1 + 2\theta)\lambda_{max}}$$

(3.69)

with $\lambda_{max}$ given as the largest eigenvalue of the eigenvalue problem associated with the matrix equation

$$|[A] - \lambda[1]| = 0.$$ 

(3.70)

Throughout the simulations in the following $\theta = 1$ was used.

The boundary conditions at the top surface can be either specified as a constant temperature (von Neuman)

$$T_i = \text{const},$$

(3.71)

$$T_a = \text{const}$$

(3.72)

or by a function $\Theta(t)$ specifying the temperature history over time (Dirichlet)

$$T_i(t) = \Theta(t),$$

(3.73)

$$T_a(t) = \Theta(t).$$

(3.74)

For the bottom surface a constant temperature, which can be specified is assumed in the model.
3.4.2 Summary of the two dimensional FE computational procedure including the temperature solution

The following algorithm adopts the FE-Scheme as presented in Algorithm 3.1 to take the temperature into account.

**Algorithm 3.6: 2D Visco-Elastic FE-Scheme including temperature solution**

**Require:** Set boundary temperature for time \( t = 0 \), set temperature boundary function \( \Theta(t) \), final calculation time \( t_{\text{final}} \), simulation time \( t = 0 \), initial densities \( \rho(t = 0) \), initial positions \( x(t = 0) \)

**Ensure:** At time \( t_{\text{final}} \) stress state \( \sigma_{ij} \), strain state \( \epsilon_{ij} \), strain-rate \( \dot{\epsilon}_{ij} \), temperature \( T_i, T_a \), movement \( u \), creep velocities \( \dot{u} \), forces \( f \), density \( \rho \), positions \( x \)

1: Solve initial Temperature \( \rightarrow 3.7 \)
2: while \( t < t_{\text{final}} \) do
3: Solve elastic \( \rightarrow 3.2 \)
4: Solve viscous \( \rightarrow 3.3 \)
5: Determine timestep \( \rightarrow 3.4 \)
6: Solve temperature \( \rightarrow 3.8 \)
7: Advance System \( \rightarrow 3.5 \)
8: end while

**Algorithm 3.7: Solve initial Temperature**

**Require:** Temperature at boundary for time \( t = 0 \)

**Ensure:** Stationary temperature distribution for \( T_i(t = 0) \) and \( T_a(t = 0) \)

1: Set \( \Delta t = \infty \)
2: Solve temperature \( \rightarrow 3.8 \)
3: for all nodes do
4: Set \( T_i(0) = T_i(\infty) \)
5: Set \( T_a(0) = T_a(\infty) \)
6: end for

**Algorithm 3.8: Solve temperature**

**Require:** Temperature \( T_i(t), T_a(t) \) and positions \( x \) for all elements, time step \( \Delta t \)

**Ensure:** Temperature \( T_i(t + \Delta t), T_a(t + \Delta t) \)

1: for all elements do
2: Calculate element matrices \([m], [k], [\tilde{h}]\)
3: Assemble the element matrix into the global matrices \([M], [K], [\tilde{H}]\)
4: end for
5: Apply boundary conditions
6: Solve \([M] \dot{T} + [K]T = [\tilde{H}]T\) for Temperature \( T_i(t + \Delta t), T_a(t + \Delta t) \)
In this chapter the two-dimensional finite element model will be applied to investigate two snow engineering problems. The first is to determine the forces exerted by the creeping snowpack on a rigid supporting structure of infinite width and the second is to investigate the mechanics of avalanche formation. To solve these problems, three physical processes — snow creep, snow fracture and snowpack gliding — must be realistically modelled. Hence, in the first part of this chapter laboratory creep and fracture experiments with snow will be simulated using the constitutive values determined from independent triaxial tests [62]. This provides an opportunity to test the constitutive formulations for elasticity and temperature dependent viscosity. In addition, instrumented field tests designed to determine snowpack gliding movements will be back-calculated in order to secure knowledge regarding the basal boundary condition of the snowpack. The accuracy and limits of the finite element model will be determined by comparing the simulation and experimental results. Afterwards the snow force and avalanche formation problems can be effectively treated.

4.1 Snow creep

Different authors have investigated the deformation of the snowpack under self-weight [35] [54]. These have been used to formulate empirical viscosity laws that have been used in one-dimensional snowpack models to determine snowpack settlement [9]. However, the majority of the investigations were performed in the field in conditions that are difficult to reproduce since the meteorological forcing was not documented. In the natural environment wind and temperature variations play an important role in the development of the snowpack. In addition most authors provide little information regarding the initial structure (layering, density and snow microstructure) of the snow cover. For this reason a series of well-documented laboratory settlement experiments performed by M. De Quervain at constant temperature in cold rooms are used to test the model [20].

De Quervain sieved natural snow to a height of 90 cm into 1 m high wooden boxes. The
boxes had a square cross-sectional area of 20 cm x 20 cm (see Figure 4.1). The sides were coated with wax to minimise friction. He selected four cold room temperatures: $T = -2^\circ C$, $-10^\circ C$, $-18.5^\circ C$ and $-32.5^\circ C$. The experiment ran for 100 days and De Quervain measured the density distribution and settlement of the snow at regular intervals. The density of the 90 cm deep sieved snow was initially determined to be $\rho = 115 \text{ kg/m}^3$. Measured densities at the end of the experiments are on the order of $\rho = 300 \text{ kg/m}^3$; the density distribution is far from constant. Because of this range of densities the experiments probably contained both Regime I (near Newtonian) and Regime II (power law) creep. No measurements were performed in order to determine whether the mass of the sample remained constant over the 100 days. Thus sublimation or condensation processes might have occurred to reduce or increase the density of the snow.

A mesh containing square 1 cm by 1 cm elements was used to model the box experiments. In total 1800 elements were employed. Simulation times were approximately 30 minutes on a PC with an Intel Pentium III at 1 GHz. Four simulations were performed for each of the constant temperatures $T = -2^\circ C$, $-10^\circ C$, $-18.5^\circ C$ and $-32.5^\circ C$. Because the temperature remained constant, it was not necessary to solve the transient heat condition equation. Note that the initial density $\rho = 115 \text{ kg/m}^3$ is below the density values used by Scapozza and Bartelt [64] to determine the constitutive law and thus at the beginning of the experiments we are in a region where the constitutive relations may not be valid. Recall that Poisson’s ratio for snow is $\nu = 0$.

Figure 4.3 shows the density distribution and settlement after 100 days at constant temperature. A good qualitative agreement is found with the measurements. The numerical results for $T = -10^\circ C$ are provided in Table 4.1. Note that a good agreement between measurements and simulation is found after 40 days but the settlement is overestimated at the end of 5 days. The density results also differ somewhat from the measurements, depending on where and when the measurement was made (top or bottom). These differences stress the importance of the initial conditions. It is hard to imagine how de Quervain filled the boxes to a constant density of $\rho = 115 \text{ kg/m}^3$. Since the snow columns begin to settle immediately a density distribution is expected. Because the constitutive model is so highly density dependent (see for example the Equations (2.19) and (2.24)) a small difference in initial density results in a significant change in the results. Thus, both the elasticity and viscosity depend on the settlement that takes place before the box is completely filled which results in a higher density at the bottom of the experiment box. Another reason might be that measuring the density is, compared to the measurement of the height, more complicated and error prone.
Figure 4.1: Dimensions of De Quervain’s boxes. The finite element mesh used for the calculations is shown. Sliding boundary conditions without friction were employed in the calculations.
Figure 4.2: Density distribution $\rho$ after 100 days for the temperatures $T = -2^\circ C, -10.0^\circ C, -18.5^\circ C$ and $-32.5^\circ C$ (from left to right, respectively). (Photo by De Quervain).

Figure 4.3: Simulation results. Density distribution $\rho$ after 100 days for the temperatures $T = -2^\circ C, -10.0^\circ C, -18.5^\circ C$ and $-32.5^\circ C$ (from left to right, respectively). The plot depicts the influence of temperature on snow settlement. The scale is the same as for Figure 4.2.
4.1. Temperature influence on snow creep

The model discussed in the preceding Section (see Figure 4.1) was used to study the temperature effect on snow creep and the resulting densification process. Recall from Chapter 3.4, that a non-equilibrium solution procedure for $T_i$ and $T_a$ has been implemented in which the heat transport in the ice and air phases is treated separately. Two simulations were performed. The initial snow density of the snow column was 115 kg/m$^3$ at an initial temperature of $-12^\circ$C for both simulations. In the first simulation the temperature (ice and air) of the snowpack was held constant over the simulated time of 2 days. No temperature gradients therefore existed in the column. The settlement, density and temperature distributions were calculated. In the second simulation a linear temperature rise at the top of the model was applied. Specifically, after a time of 4 hours, the surface temperature was raised from $-12^\circ$C to $-2^\circ$C over a four hour period according to the Dirichlet boundary conditions (see Equations (3.73) and (3.74)). After 8 hours the temperature at the top surface was held constant at $-2^\circ$C. An essential question in this analysis was to find the difference in ice and air temperatures and to find the time needed to reach an equilibrium state. The heat exchange coefficient was taken to be $h = 0.5$ W/(m$^2$ K) according to Bartelt at al. [8]. The latent heat effects between the ice and air phases are not included in the model. The ice temperature $T_i$ is used in the constitutive creep model (see Equation (2.30)).
Figure 4.5: *Comparison of the settlement in the snowpack at constant temperature and with a temperature rise.*

Figure 4.4 depicts the temperature of the ice phase $T_i$ and the surrounding air $T_a$ as well as the difference $T_a - T_i$ over the simulated time. The data point is located at a height of 0.45 m. The air temperature rises earlier and faster than the ice temperature. The maximal difference between the ice and air temperature is larger than 1.8 °C at a simulated time of 10 hours. This suggests that bulk temperature snowpack models [9] can only predict the temperature of the ice-matrix within an accuracy of about 1 °C. The temperature difference then decreases towards 0 °C, which means that the ice and the surrounding air reach an equilibrium state. Of interest is that the temperature of the ice and air phases does not attain an equilibrium temperature after 24 hours of simulation. As shown by Bartelt et al. [8] non-equilibrium temperatures induce local temperature gradients in the ice-matrix which are the driving force for snow metamorphism. The results shown here suggest that the natural snow cover — especially in early Winter — seldom reaches a state of thermal equilibrium due to continually changing meteorological forcing at the surface. Hence, the morphology of the snowpack is continually changing.

A comparison between the settlements of the two simulations is shown in Figure 4.5. The plot shows that the creep process, and therefore the settlement, becomes faster as soon as the temperature starts to rise. The same effect can be seen in Figure 4.6 where a comparison of the density development at the top of the model is shown. The densification of the snow, however, slows down the settlement process, thus working against the faster settlement rates induced by higher temperatures. This can be seen in Figure 4.6 where the difference $\Delta \rho - \rho_{\text{const}}$ curve flattens.
4.2 Snow glide

Snow gliding (at speeds around 50 cm/day or higher) damages man-made structures and forests. Snow gliding was defined as a slow translation of the snow cover on an appropriate substratum by In Der Gand [22]. Snow gliding depends on the vegetation cover (cut or uncut grass) and the meteorological history of a winter. If the ground is not frozen at the time when the first snow falls, snow gliding is often observed [60], indicating a stick or frictionless slip type of behaviour depending on whether the ground temperature is below or above 0 °C, respectively.

Snow gliding was investigated at an experimental test field "Matte"-Frauenkirch, Davos, between the years 1976 and 1981 [7]. The grass field is located at an altitude between 1690 m.a.s.l and 1720 m.a.s.l. The Southern exposed slope is inclined between 30° to 36° (see Figure 4.7). The upper and lower boundaries of the test site are bounded by wooden defence structures. The distance is measured from the upper defence structure. The defence structures were equipped with pressure measuring gauges. To measure snow glide, glide shoes were placed at different locations within the slope at the snow cover-ground interface. These measure the basal movement of the snow cover over time. Precipitation and temperature were measured at regular intervals and snow pits were used to determine snow profiles. In the following one well documented measurement, covering 60 days between 25.12.1979 – 23.02.1980, will be simulated with the finite element model.

The snow cover was defined in the finite element model at the beginning of the simulation.
Figure 4.7: The test site "Matte"-Frauenkirch showing the two rows of wooden defence structures.

<table>
<thead>
<tr>
<th>Density kg/m³</th>
<th>Layer height m</th>
<th>Temperature °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>0.15</td>
<td>-10.0</td>
</tr>
<tr>
<td>200</td>
<td>0.15</td>
<td>-8.0</td>
</tr>
<tr>
<td>220</td>
<td>0.20</td>
<td>-6.0</td>
</tr>
<tr>
<td>240</td>
<td>0.20</td>
<td>-4.0</td>
</tr>
<tr>
<td>260</td>
<td>0.25</td>
<td>-2.0</td>
</tr>
<tr>
<td>280</td>
<td>0.25</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

Table 4.2: Snow cover layer properties used for the simulation of "Matte"-Frauenkirch (from top to bottom).

— no new snow was added during the 60 simulated days. Based on the measured snow profiles, a snow cover consisting of five layers was modelled with initial densities, heights and temperatures as given in Table 4.2. The stated temperature is the mean value defined over the whole winter. Thus, no instationary heat transport solution is needed. The mean slope angle of 32° was used in the simulations. The length of the slope was 35 m.

The weak interface model (see Section 3.2.4) was used to model glide. The glide coefficient at the snow-soil boundary, however, was set to 0 (see Equation (3.27)) in order to model frictionless gliding. The field observations showed that the resistance was small and could be neglected. During this winter, the temperature of the ground was near 0°C.

In Figures 4.8, 4.9 and 4.10 the results from the simulation compared to the measurements are shown at points at the top of the field (far from the wooden defence structures), the
4.3. Snow fracture

Brittle fracture is central to the mechanics of avalanche formation. Slab release is defined to occur when the stress intensity in shear $K_{II}$ is greater than the fracture toughness of snow $K_{IIc}$.

Kirchner et al. [32] [33] determined the fracture toughness of snow in tension $K_{Ic}$ and shear $K_{IIc}$ using notched cantilever beam experiments as depicted in Figures 4.11 and 4.12, respectively. The notch of length $a$ was made with a fishing line having a diameter smaller than the grain size of the snow sample in order to ensure that only bonds were broken.
Chapter 4. 2D Model Calculations

Figure 4.9: Simulated (line) and measured (line with points) movements along the slope in comparison for the middle part of the test site.

Figure 4.10: Simulated (line) and measured (line with points) movements along the slope in comparison for the bottom part of the test site.
and to minimise the disruption of the snow ice lattice. The length $a$ at which the sample fails is used to determine the fracture toughness, assuming linear elastic behaviour [5].

In the first series of experiments, Kirchner et al. [32] found the mixed mode fracture toughness to be $(K_{lc}^2 + K_{IIC}^2)^{1/2} = 430 \text{Pa} \cdot \text{m}^{1/2} \pm 90 \text{Pa} \cdot \text{m}^{1/2}$ from which they deduced $K_{lc}$ and $K_{IIC}$. They found mean values of $K_{lc} = 280 \text{Pa} \cdot \text{m}^{1/2} \pm 95 \text{Pa} \cdot \text{m}^{1/2}$ and $K_{IIC} = 310 \text{Pa} \cdot \text{m}^{1/2} \pm 85 \text{Pa} \cdot \text{m}^{1/2}$. The critical crack length varied between $a_c = 10.25 \text{cm}$ and $15.5 \text{cm}$ (two measurements). The snow used had a density of $\rho = 170 \text{kg/m}^3$. In [33] another series of measurements was made using the geometry shown in Figure 4.12. The snow density used by Kirchner et al. was $247 \text{kg/m}^3 \pm 11 \text{kg/m}^3$. Within the given setup, the crack surfaces can be in three different states: (1) separated, (2) under force-less contact and (3) at contact under pressure. The respective toughness in shear was given as: (1) $K_{IIC} = 430 \text{Pa} \cdot \text{m}^{1/2} \pm 90 \text{Pa} \cdot \text{m}^{1/2}$ if an opening mode I, however small, is present so that the crack surface do not touch; (2) $K_{IIC} = 630 \text{Pa} \cdot \text{m}^{1/2} \pm 90 \text{Pa} \cdot \text{m}^{1/2}$ if no mode I opening is present, the surfaces just touch but without being pressed against each other; and (3) $K_{IIC} = 680 \text{Pa} \cdot \text{m}^{1/2} \pm 90 \text{Pa} \cdot \text{m}^{1/2}$, if the shear crack is closed under normal pressure. The different values are depicted in Figures 4.13 and 4.14. The critical crack length was found to be $a_c \approx 20 \text{cm}$.

The fracture experiments of Kirchner were back-calculated using the finite element model and the procedure described in Section 3.3.3. Both test configurations Figure 4.11 and

Figure 4.11: Experimental setup for measuring $K_{lc}$ fracture toughness. The beam depth is $10 \text{cm}$. The same geometry is used for the calculations.
Figure 4.12: Experimental setup for measuring $K_{IIe}$ fracture toughness, the beam depth is 20 cm. The same geometry is used for the calculations.

Figure 4.13: Calculated stress intensity factor $K_I$ as a function of crack length $a$. Also depicted are the experimental values from Kirchner et al. [32].
4.3. Snow fracture

The calculated stress intensity factors are plotted in Figures 4.13 and 4.14 as a function of the notch length $a$. A good agreement between calculations and experiments is found: the published experimental value for $K_{Ic}$ is $280 \text{ Pa m}^{1/2} \pm 95 \text{ Pa m}^{1/2}$ for a critical crack length $a_c = 10.25 \text{ cm}$ to $15.5 \text{ cm}$ whereas the calculated stress intensity factor for a snow density of $150 \text{ kg/m}^3$ is $125.15 \text{ Pa m}^{1/2}$ for $a = 10.5 \text{ cm}$ and $427.63 \text{ Pa m}^{1/2}$ for $a = 15.5 \text{ cm}$. The mean value over this range $276.39 \text{ Pa m}^{1/2}$ is in excellent agreement with the measured value. A similar result is found for $K_{IIc}$. The comparison between the measured fracture toughness $K_{IIc} 680 \text{ Pa m}^{1/2} \pm 90 \text{ Pa m}^{1/2}$ for $a_c \approx 20 \text{ cm}$ with the calculation at a snow density of $250 \text{ kg/m}^3$ at $a = 20 \text{ cm}$ is $690.1 \text{ Pa m}^{1/2}$.

It should be stressed that only an elastic analysis was performed, no viscous effects were included in the calculations. The good agreement between the measurements and experiments can be attributed to the accuracy of Young’s modulus used in the calculations (see

Figure 4.14: Calculated stress intensity factor $K_{II}$ as a function of crack length $a$. Overlaid on the simulation results are the experimental values from Kirchner et. al [32] and [33].
Chapter 4. 2D Model Calculations

Figure 4.15: A group of defence structures placed in the starting zone of an avalanche.

Equation (2.19)). Finally, because Young’s modulus for snow is temperature independent, there can be no influence of temperature on the calculated stress intensity. Hence, in this linear elastic model, there is no influence of temperature on fracture. This result is not in agreement with our empirical understanding of avalanche formation, which shows obvious temperature dependence.

Additional simulations were performed including viscous straining. The calculations showed viscous stress redistributions only after several hours. Evidently, in Kirchner’s experiments, viscous straining does not induce an immediate redistribution of stress around the crack opening. Therefore, viscous effects do not appear in the results depicted in Figures 4.13 and 4.14.

4.4 Snow forces

Figure 4.15 depicts a group of defence structures placed in the starting zone of an avalanche. These steel structures must withstand the forces exerted on them by the creeping and gliding snowpack, typically on steep slopes with angles between $30^\circ$ and $50^\circ$. In this section, the applicability of the two-dimensional finite element method to calculate the stress field acting on the structure and behind the defence structure using the visco-elastic constitutive model is demonstrated.

In the following the stress distribution is calculated for an infinitely wide slope. A $h = 2\text{ m}$ high snow defence structure was placed at the lower end of an inclined slope of given slope
4.4. Snow forces

Figure 4.16: Model domain of snow force calculations. The dotted line describes the layers and the thick line on the right represents the snow defence structure.

angle $\phi$. Figure 4.16 depicts the model domain. The snowpack is fixed to the ground; there is no basal gliding.

Figure 4.17 depicts the calculated normal stress $\sigma_z$ (negative in compression) acting on the snow defence structure at a height $h = 1.9$ m above the ground for different slope angles $\phi$ and different snowpack lengths $l$. The density of the snowpack was constant at $\rho = 250$ kg/m$^3$. At the height, $h = 1.9$ m the stress reaches its maximum. The stress increases with the snowpack length $l$ until a constant stress state $\sigma_l$ is reached. The constant stress is the normal stress that would exist in the snow cover if no structure was present. Hence, the length $l$ at which $\sigma_l$ is reached is termed the influence length of the structure. Figure 4.17 shows that this length is reached at $l = 10$ m to 12 m. The influence zone of the defence structure is hence around 12 m. Interestingly, the calculations show that the influence length is not a strong function of the slope angle $\phi$.

The stress distribution over the height of the snow defence structure is depicted in Figure 4.18 for three different snow densities $\rho = 200$ kg/m$^3$, 250 kg/m$^3$ and 300 kg/m$^3$. The density is constant over the height of the snow cover. In this particular simulation, the modelled snow cover had a length of $l = 20$ m on a slope with an angle of $\phi = 35^\circ$. Due to the no-slip boundary on the ground, the snow does not exert any force on the defence structure at the ground, hence the stress is near zero. It can be seen that for a homogeneous snowpack the stress increases with the height of the defence structure and is higher for higher density snow.

A true snowpack does not have a constant density, rather a density distribution. Usually, the lower density snow is near the top of the snowpack. The stress distribution taking into account a decreasing density distribution is shown in Figure 4.19 for different slope angles $\phi$. The snowpack was modelled using 10 layers with a height of $h_L = 0.2$ cm resulting in a $h = 2$ m thick snow slab. A a from ground to surface decreasing density distribution with $\rho_1 = 300$ kg/m$^3$ for layer 1, $\rho_2 = 290$ kg/m$^3$ for layer 2, $\rho_2 = 280$ kg/m$^3$ for layer
Figure 4.17: Stress acting on the defence structure for different slope lengths $l$ and angles $\phi$ at $h=1.9$ m. The snowpack had a density of $\rho = 250 \text{ kg/m}^3$.

Figure 4.18: Stress acting on defence structure for a slope length of $l = 20$ m, angle $\phi = 35^\circ$ and densities $\rho = 200 \text{ kg/m}^3$, $250 \text{ kg/m}^3$ and $300 \text{ kg/m}^3$. 
4.5 Avalanche formation

Figure 4.19: Stress acting on defence structure for a slope length of \( l = 20 \text{ m} \) and different slope angles \( \phi \). The density is distributed from \( \rho = 300 \text{ kg/m}^3 \) to \( 210 \text{ kg/m}^3 \).

3, ..., \( \rho_{10} = 210 \text{ kg/m}^3 \) for layer 10 was modelled. The length of the snowpack was again chosen to be \( l = 20 \text{ m} \). Comparing these results to the snow cover with constant density (Figure 4.18), the stress distribution is quite different. The maximum stress level is reached in the middle of the snow defence structure and decreases towards the top of the structure.

4.5 Avalanche formation

The plane strain finite element model is now applied to investigate the mechanics of avalanche formation. Previous investigations have relied on either a strength of materials approach [6] [78] or a temperature independent linear elastic fracture mechanics method. In both methods the concept of a weak layer, of unknown length and strength, plays a central role. This layer is assumed to exist at the interface of two homogeneous layers (see Figure 4.20). Bader and Salm [6] determined the shear stress concentrations at the end of the weak layer and compared the induced strain-rates to the failure values of snow. Snow slab release is primarily a function of the relative strength of the homogeneous layers surrounding the weak interface and the weak layer length. They found weak layer lengths of approximately \( a = 10 \text{ m} \) or more were required to reach critical strain-rates. The shear viscosity of the weak layer was assumed to be an order of magnitude smaller than the surrounding layers. This approach was later extended by Stoffel and Bartelt [78]
Chapter 4. 2D Model Calculations

Figure 4.20: Model domain of avalanche formation simulations. The stress intensity factor $K_{II}$ is calculated over time to investigate the influence of the slope angle $\phi$, the height $h_2$, the temperature $T_2$ and density $\rho_2$ on dry snow slab release.

to include a stress, density and temperature dependent viscosity (see Appendix C). The investigated effects (stress, density and temperature) had no significant influence on the critical weak layer lengths, which remained in the same order of magnitude, $a \approx 10$ m.

A linear fracture mechanics approach [44] [70] [82] can also be used to find the critical weak layer lengths $a$. Failure occurs when the calculated stress intensity factor in shear $K_{II}$, which is a function of the material elasticity, is greater than the fracture toughness of snow $K_{IIc}$ [32] [72]. The fracture toughness of snow is density dependent and the measurements display significant scatter. In the following we will calculate $K_{II}$ using the plane strain finite element model presented in the preceding chapter. Creep effects are included such that the evolution of the stress intensity factor over time can be calculated and compared to the measured fracture toughness.

The calculations assume that the length of the weak layer $a$ is known. The calculation does not provide insight into how the weak layer is created, or can be predicted, in real situations.

4.5.1 Description of the model

The model domain for our investigations consists of a two layer snowpack on an inclined slope of constant angle $\phi$. The model domain is depicted in Figure 4.20.

A weak layer of length $a$ is placed between the two layers. The length of the model domain $L$ is chosen such that $L \gg a$. Periodic boundary conditions, simulating an infinitely large slope, at the ends are applied reducing boundary effects. Layer 1 is located at the bottom of the snow cover and represents old, densified snow. It has a height of $h_1 = 1$ m and is modelled using finite elements with a height of 0.5 m and a length of 0.2 m. The density
is given as $\rho_1 = 350 \text{ kg/m}^3$ and the temperature as $T_1 = -2^\circ\text{C}$. These values for the temperature and density were used for all calculations. Layer 2 represents a layer of new snow. It has a height of $h_2$. The initial density is $\rho_2$ and the temperature is $T_2$. The values $h_2$, $\rho_2$ and $T_2$ varied. The length of the slope was $L = 30 \text{ m}$. The initial weak layer length varied from $a = 0 \text{ m}$ to $10 \text{ m}$. This model domain was chosen in order to make direct comparison to previous investigations [6].

Four practical situations important for avalanche prediction were investigated using this model:

- The size of the weak layer length $a$ required to induce failure.
- The influence of slope angle $\phi$, snow density $\rho_2$ and temperature $T_2$ on avalanche formation.
- The influence of new snow accumulation on avalanche formation.
- The influence of temperature changes (warming) on avalanche formation.

Avalanche formation is defined to occur at the weak layer length $a$ at which $K_{II}(t) \geq K_{IIc}$. The calculated $K_{II}$ is a function of time due to material creep and snow densification.

### 4.5.2 Weak layer length $a$

In a first series of simulations we determined the stress intensity factor as a function of the initial weak layer length $a$. The new snow density $\rho_2 = 200 \text{ kg/m}^3$, temperature $T_2 = 6^\circ\text{C}$ and slope angle $\phi = 35^\circ$ remained the same for all calculations. The weak layer was enlarged by $\Delta a = 0.05 \text{ m}$ for the calculation of the stress intensity factor.

Figure 4.21 plots the evolution of the calculated stress intensity factor $K_{II}$ over time. The graph depicts the results for weak layer lengths $a = 2 \text{ m}, 4 \text{ m}, 6 \text{ m}, 8 \text{ m}$ and $10 \text{ m}$. For short weak layers $a < 6 \text{ m}$ only small changes in the calculated stress intensity factor $K_{II}$ can be seen. For larger weak layer lengths, however, an initial rise can be observed which declines after reaching a maximum value between 4 and 8 hours. The calculated $K_{II}$ values are of the same order of magnitude as the measured toughness values $K_{IIc}$ presented by Kirchner [32] [33].

Figure 4.22 shows the calculated stress intensity factor at different simulation times as a function of the weak layer length $a$. The model has a density of $\rho_2 = 200 \text{ kg/m}^3$, temperature $T_2 = -6^\circ\text{C}$ and slope angle $\phi = 35^\circ$. The variation of the calculated stress intensity factor over time is small for small weak layer lengths $a$. The larger the weak layer is, the larger the variations of the calculated stress intensity over time are. The time when a maximum in the calculated stress intensity factor is achieved depends on the weak layer length.
Figure 4.21: Development over time of calculated stress intensity factor $K_{II}$ for different weak layer lengths $a$.

Figure 4.22: Dependence of calculated stress intensity factor $K_{II}$ on the weak layer length $a$ for different times.
4.5. Avalanche formation

4.5.3 Slope angle, density and temperature

In the following the dependence on slope angle, density and temperature for a prescribed weak layer length of \( a = 6 \text{ m} \) is investigated. The above series of calculations used a constant temperature for the upper layer. The simulation investigated the development of \( K_{II} \) over a 12 hour period. The initial density for the upper snow layer \( \rho_2 \) was 200 kg/m\(^3\), the temperature \( T_2 = -6 \text{°C} \) and the slope angle was \( \phi = 35^\circ \).

To investigate the dependence of the stress intensity factor on slope angle, density and temperature each of these three quantities were varied while the other two were held constant. The slope angles were 25°, 30°, 35°, 40° and 45°, the densities \( \rho_2 \) used were 190 kg/m\(^3\), 200 kg/m\(^3\), 210 kg/m\(^3\) and 220 kg/m\(^3\) and the temperatures \( T_2 \) were \(-2 \text{°C}, -4 \text{°C}, -6 \text{°C}, -8 \text{°C}, -10 \text{°C}, -12 \text{°C}\). Where not otherwise stated, the slope angle is 35°, temperature \( T_2 = -6 \text{°C} \) and density \( \rho_2 = 200 \text{ kg/m}^3 \). For all series periodic boundary conditions, simulating an infinitely long slope, at the upper and lower end were applied.

Figure 4.23 depicts the effect of the slope angle on the calculated stress intensity factor. In comparison to the previous results, large changes in the maximum stress intensity factor with increasing slope angle are apparent. Clearly, the forces acting along the weak layer are higher at steeper slope angles. The difference between the initial and maximum calculated stress intensity factor is for all calculated angles around 20 Pa m\(^{1/2}\).

The influence of the upper layer density \( \rho_2 \) on the calculated stress intensity factor is depicted in Figure 4.24. For lower densities the maximum intensity is reached faster than
for higher densities. In addition, the higher the density of the upper layer, the larger the maximum calculated stress intensity factor. For larger densities the load acting on the weak layer is larger and hence the stress concentrations and the change of elastic energy due to the enlargement of the weak layer are higher. The reason why lower densities reach the calculated stress intensity factor maximum earlier than higher densities is due to the fact that the viscosity of lower density snow is higher than that of denser snow. The viscosity law (see Equation (2.30)) is nonlinear in stress, and hence in density. Therefore the creep movement, which is the driving mechanism for the stress redistribution, is faster for low density snow even if the stress itself is smaller.

The results from varying the temperature of the upper layer $T_2$ are shown in Figure 4.25. For all temperatures the maximum calculated stress intensity factor is nearly the same. The time when it is reached, however, depends highly on the snow temperature of the new snow layer. Since snow has a lower viscosity at higher temperatures, the creep velocity is higher for warmer snow. Therefore, the deformations leading to a change of the elastic energy distribution appear earlier than for colder snow. The initial calculated stress intensity factor at the beginning of the simulation is the same for all temperatures, since Young’s modulus for snow is temperature independent.
4.5.4 Influence of a new snow fall event

In the next series of calculations a new snow fall event was simulated. A 12 hour new snowfall event was assumed. During this 12 hour period, snow fell at a constant rate with a density of 150 kg/m$^3$, resulting in a new snow accumulation of 0.25 m, 0.50 m, 0.75 m and 1.00 m by the end of the simulation. The upper layer, on which the snow fell, had a density of $\rho_2 = 200$ kg/m$^3$ and a temperature $T_2$ of $-6^\circ$C. The slope angle is $\phi = 35^\circ$.

The calculated stress intensity factors for the simulation are shown in Figure 4.26. Only a relatively small change arising from the different snowing rates in the maximal calculated stress intensity factors can be seen. After around 6 hours the maximal values are reached. Bear in mind that till then only half of the final load is applied. Nevertheless a raise in the calculated stress intensity factor due to fresh snow loading could be shown.

The timing when the maximum stress intensity factor is reached is determined by the density and temperature of the top layer — and not by the amount of snow precipitated.

4.5.5 Influence of a temperature rise

This series of calculations investigates a temperature rise at the top surface of the snowpack from $-12^\circ$C to $-2^\circ$C in 3 hours, 6 hours, 9 hours and 12 hours beginning at time $t = 0$ hours. Thus, different temperature rise rates $T_r$ are applied to the snow cover.
Chapter 4. 2D Model Calculations

Figure 4.26: Calculated stress intensity factor $K_{II}$ for new snow fall load at different rates using a weak layer length of 6 m.

$(0.3 ^\circ C/h, 0.6 ^\circ C/h, 0.9 ^\circ C/h$ and $1.2 ^\circ C/h)$. Such temperature gradients can be induced from solar radiation, warm winds, etc. The slope had an angle of 35° (a typical avalanche slope) and an upper layer density of $\rho_2 = 200 \text{kg/m}^3$.

The results are shown in Figure 4.27. The temperature rise increases the stress intensity factor by some 10% over the initial value. Depending on the original state, this is clearly enough to initiate failure. Note, that the magnitude of the increase is not dependent on the temperature rise rate. Although, the time when the maximum occurs is a function of the rise rate (the faster the rise, the earlier the maximum value is reached) all the maximum values are reached within two hours of each other.

After reaching the maximum stress intensity values, Figure 4.27 shows that the stress intensity decreases. This corresponds to a relaxation of the snow cover and a diminishing of the avalanche danger. This relaxation occurs although the temperature on the snow cover surface remains constant. Additional simulations showed that if the temperature decreased, which is typical for a winter day, the relaxation is slowed.

### 4.5.6 Conclusions

The following results can be drawn from the simulations:
4.5. Avalanche formation

Figure 4.27: Calculated stress intensity factor $K_{II}$ for a temperature rise at four different rates $T_r$ with a weak layer length of $a = 6\,\text{m}$.

- The slope angle has the largest influence on the calculated stress intensity factor. The steeper the slope, the larger the maximum calculated stress intensity factor for a given weak layer length $a$. Furthermore, the time when the maximum calculated stress intensity factor is reached also depends on the slope angle: the steeper the slope, the earlier the maximum calculated stress intensity factor is reached.

- The upper layer density has only a minor effect on the maximum value of the calculated stress intensity factor. However, the time when the maximum intensity is reached is, among other factors, controlled by the density. For low density snow the maximum intensity is reached earlier than for high density snow.

- A temperature rise induces an increase in the calculated stress intensity factor. The rate of the temperature rise has no effect on the maximum of the calculated stress intensity factor. However, it influences the time when the maximum value is reached. Viscous movements, which are responsible for the stress redistribution, are slower at cold temperatures than at high temperatures. Therefore, the maximum calculated stress intensity factor is reached earlier at higher temperatures than at lower ones. After reaching the maximum values the snowpack relaxes.

- A new snowfall event can influence the maximum calculated stress intensity factor. However, in the calculated example the effect was not very large. Thus, new snowfall can induce a brittle fracture; however, only when the system is already near the critical stress intensity.
These results are not in contradiction to what is known about avalanche formation [47]. However, the simulation model is difficult to apply in practical situations since the initial state of a real snowpack (weak layer length $a$) is unknown. Based on presently available fracture toughness values ($K_{II} \approx 300 \text{ Pa m}^{1/2}$), the simulations show that weak layer lengths of $6 \text{ m} < a < 10 \text{ m}$ are required for avalanche formation. The simulation model does not answer the question how weak layers of this size are formed in the snow cover.
Chapter 5

3D Finite Element Formulation

The mechanical processes leading to avalanche formation are governed by three-dimensional stress and strain fields. The two-dimensional, plane strain finite element model presented in the preceding chapters cannot always include complex terrain features [43], boundary effects, spatially variable snowpack properties [68] or highly nonlinear meteorological forcing (wind, temperature, radiation) [38]. To study the influence of these effects on avalanche formation, a three-dimensional model is required.

This chapter describes the three-dimensional finite element model. The snow cover model consists of a layered finite element mesh resting on a terrain surface. The surface is usually specified using a digital terrain model (DTM), providing the $x$, $y$, and $z$ coordinates of the topography. To study avalanche formation, weak zones within the snow cover, which are located between two adjacent snow layers, must be specified. Boundary conditions describing the domain sides have to be stated. Additionally, avalanche defence structures can be easily specified in the model. Each defence structure introduces two new model boundaries for which boundary conditions have to be given.

As in the preceding chapter, the visco-elastic solution procedure is employed to model the creeping deformation of the snow cover, now in three dimensions. A novel solution method based on the so-called $N$-Directional approach [57] is implemented at the element level. This procedure contains a damage mechanics approach [42] to model material failure. Thus, the linear fracture mechanics approach used in the two-dimensional model is not adopted to study avalanche release. The $N$-Directional approach allows an efficient representation of the time and temperature dependent, non-isotropic material behaviour prior to global failure. A primary difficulty with this solution strategy is separating the visco-elastic creep effects from unsteady local failure within the $N$-Directional elements.

In the following chapter, applications of the three-dimensional model are presented.
Chapter 5. 3D Finite Element Formulation

5.1 Mesh generation

The model domain of a three-dimensional snowpack simulation is a layered snow cover resting on a mountain slope of general topography. This model domain is discretised for the finite element calculation into a tetrahedron mesh. This section describes the mesh generation procedure and how additional features like snow defence structures and weak layers are introduced.

5.1.1 Building of a rectangular base mesh containing avalanche defence structures

In a first step a two-dimensional rectangular base mesh containing no elevation information is generated. The domain is specified using the lower left \((x, y)\) coordinate and the width \(w\) and length \(l\) of the model domain (see Figure 5.1). This domain is then divided into regular rectangles with a given side length specified by the user.

Defence structures are added by specifying the coordinates of the start and endpoint of the structure. This gives a direction to the structure which is interpreted such that the expected snow forces are on the left side of the line when looking from the start to the end point. The rectangular base mesh is adopted such that the sides of the rectangles lie on the line specified by the defence structure. Figure 5.2 depicts a rectangular base mesh with defence structures. The edge points lying on the defence structure are copied and the rectangles on the right side are reconnected to the copied points. The copied points
5.1. Mesh generation

Rectangular base mesh with defence structures

Figure 5.2: Rectangular base mesh with defence structures (bold lines).

Figure 5.3: Adding extra nodes and edges to model a defence structure. (The points are only separated to visualise the difference.)

are marked and separated to visualise the new rectangular connections in Figure 5.3.

In the next step the elevation information is added to the rectangular base mesh. The elevation of the topography is usually specified in a so-called digital terrain model (DTM). A DTM contains the $x$, $y$ and $z$ coordinates (where $z$ is the elevation) on a regularly spaced rectangular mesh. The spacing of the DTM, however, is normally much larger than the spacing in the base mesh. The elevation of the base mesh nodes is calculated by bilinear interpolation of the four points of the rectangle from the DTM within which the node lies. Figure 5.4 depicts the mapped rectangular base mesh with the DTM points used in three dimensions.
5.1.2 Building of the tetrahedron mesh

Starting from the rectangular base mesh including the elevation information the tetrahedron mesh is generated. This is done by first generating a layered cube mesh and then divide each cube into five tetrahedra.

To generate the cube mesh at each node a surface normal has to be calculated. These normals will be the edges of the cubes. Each node of the rectangular base mesh is connected by at least two edges to the neighbouring nodes. For each adjoining pair of edges \( e_i, e_j \) a normal can be calculated by a cross product \( \mathbf{n}_{ij} = e_i \times e_j \) (\( i, j \) in counter clockwise order). The so calculated normals are then added to result in a mean normal \( \mathbf{n} = \mathbf{n}_{12} + \mathbf{n}_{23} + \mathbf{n}_{34} + \mathbf{n}_{41} \) at the node (see Figure 5.5).

A cube is then generated by extruding the rectangular base mesh along the node normals by the specified layer height \( h_l \). This results in a cube mesh where the thickness of the snow cover is constant if measured orthogonal to the ground. If a constant vertical snow height is wanted then the extruding length \( l \) along the normals needs to be calculated for each normal separately as

\[
l = h_l \cdot n_z
\]

with \( n_z \) the z component of the normal vector where the extruding length \( l \) has to be calculated. Figure 5.6 depicts a side view of the two possibilities.

A cube mesh with one layer is depicted in Figure 5.7.
Figure 5.5: Normal at node and adjoining edges.

Figure 5.6: Side view of the cube meshes with vertically constant (left) and normal to ground constant (right) snow height.
Chapter 5. 3D Finite Element Formulation

The last step to the tetrahedron mesh is to divide the cubes into five tetrahedra. The result of this procedure for the cube mesh presented earlier (see Figure 5.7) is depicted in Figure 5.8 where only one layer is modelled. The subdivision of a cube into five tetrahedra is depicted in Figure 5.9.

5.1.3 Adding weak-zone elements to the model

In a final step, the weak-zone elements are added into the mesh structure. The weak zone is specified as a polygon region. It is sandwiched between two snow layers. The interface of the two layers in the tetrahedron mesh consists initially of a triangular mesh, build by the common faces of the tetrahedra of the two layers. Each triangle lying completely within the specified polygon region is replaced by a prism where the two triangular faces represent the faces of the tetrahedra belonging to the upper and lower layer tetrahedra (see Figure 5.10).

Transitional elements are introduced at the boundaries of the weak layer. Figure 5.11 depicts the transition elements from the non-weak layering to the weak zone. For the condensed nodes (3,6 and 2,5) the difference in nodal displacements is set to 0. i.e. \( u_3 = u_6 \) and \( u_2 = u_6 \) respectively.
5.2 Finite element discretisation

5.2.1 Coordinate numbering and generalised coordinates

It is extremely important to use a consistent numbering of the nodes of a tetrahedron for the following considerations. Figure 5.12 shows a general tetrahedron with node numbering. For the numbering of the nodes the following rule is used:

1. Select one node as initial one (numbered 1 in Figure 5.12).
2. Select one face that will contain the first three nodes. The excluded node will be the last one (numbered 4).
3. Number these three nodes in a \textit{counter-clockwise} sense when looking at the face from the last corner (numbered 4).

Using this rule the volume $V$ of the tetrahedron can be calculated as

$$ V = \frac{1}{6} \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} $$

(5.2)
Figure 5.9: Subdivision of a cube into five tetrahedrons with node numbering.
5.2. Finite element discretisation

**Figure 5.10:** Weak element with local node numbers and adjoint tetrahedra (left) and weak element with local coordinate system (right).

**Figure 5.11:** Transition elements from weak to non-weak zone.
where det denotes the determinant of the matrix. Note that the volume $V$ is a signed quantity. If it is negative, the node numbering does not follow the given numbering rule.

Analogous to the formulation for the four-node isoparametric elements used in the two-dimensional formulation, a natural coordinate system for the tetrahedron elements was introduced. The natural tetrahedron coordinates are called $\xi_1$, $\xi_2$, $\xi_3$, $\xi_4$. The value of $\xi_i$ is 1 at the node $i$ and zero at the other three nodes (i.e. on the opposite face) and decays linearly along a line from node $i$ to one of the other three nodes. The sum of the four coordinates for points lying within the tetrahedron is always one. Any function linear in $x, y, z$, say $F(x, y, z)$ that takes the values $F_i$ at the nodes can be interpolated in terms of the tetrahedron coordinates as

$$F(\xi_1, \xi_2, \xi_3, \xi_4) = F_1\xi_1 + F_2\xi_2 + F_3\xi_3 + F_4\xi_4. \quad (5.3)$$

The geometric definition of an element in terms of the natural tetrahedron coordinates is such a linear function in each direction. Remember that $(x_i, y_i, z_i)$ specify the global coordinates of the node $i$

$$x(\xi_1, \xi_2, \xi_3, \xi_4) = x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4, \quad (5.4)$$

$$y(\xi_1, \xi_2, \xi_3, \xi_4) = y_1\xi_1 + y_2\xi_2 + y_3\xi_3 + y_4\xi_4, \quad (5.5)$$

$$z(\xi_1, \xi_2, \xi_3, \xi_4) = z_1\xi_1 + z_2\xi_2 + z_3\xi_3 + z_4\xi_4. \quad (5.6)$$
This can be written in matrix notation as
\[
\begin{bmatrix}
1 \\
x \\
y \\
z
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
z_1 & z_2 & z_3 & z_4
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix},
\]
where the first row describes the fact that \(\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1\) was added to generate a square matrix. Since the determinant of this matrix is six times the volume of the tetrahedron, this matrix is always invertible. The inverted matrix can be written in generic form as
\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix} = 
\frac{1}{6V}
\begin{bmatrix}
6V_1 & a_1 & b_1 & c_1 \\
6V_2 & a_2 & b_2 & c_2 \\
6V_3 & a_3 & b_3 & c_3 \\
6V_4 & a_4 & b_4 & c_4
\end{bmatrix}
\begin{bmatrix}
1 \\
x \\
y \\
z
\end{bmatrix}.
\]

For the description of the displacement field the same formulation as for the coordinates can be used resulting in
\[
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix} = 
\begin{bmatrix}
u_{x1} & u_{x2} & u_{x3} & u_{x4} \\
u_{y1} & u_{y2} & u_{y3} & u_{y4} \\
u_{z1} & u_{z2} & u_{z3} & u_{z4}
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix}.
\]

### 5.2.2 Standard elastic element

The total strain in three dimensions is defined as
\[
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).
\]

Where \(u_{i,j}\) denotes the partial derivative of the displacement \(u\) in i-direction with respect to the j direction. The full strain field is represented in the strain vector defined as
\[
\epsilon = [e_{11}, e_{22}, e_{33}, 2e_{12}, 2e_{23}, 2e_{13}]^T
\]
\[
= [\epsilon_x, \epsilon_y, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}]^T.
\]

The strain-displacement relation can be written in matrix form as
\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 & 0 \\
\frac{\partial}{\partial z} & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y \\
u_z
\end{bmatrix}.
\]

To relate the nodal displacements defined in the canonical coordinate system to the strain components the partial derivatives are needed. From the Equations (5.8) and (5.7) the
following relations can be derived

\[
\frac{\partial x}{\partial \xi_i} = x_i, \quad \frac{\partial y}{\partial \xi_i} = y_i, \quad \frac{\partial z}{\partial \xi_i} = z_i, 
\]

\[
6V \frac{\partial x}{\partial \xi_i} = a_i, \quad 6V \frac{\partial y}{\partial \xi_i} = b_i, \quad 6V \frac{\partial z}{\partial \xi_i} = c_i. 
\]

The derivatives of a function \( F(\xi_1, \xi_2, \xi_3, \xi_4) \) with respect to the Cartesian coordinates follows from Equation (5.14) and the chain rule

\[
\frac{\partial F}{\partial x} = \frac{1}{6V} \left( \frac{\partial F}{\partial \xi_1} a_1 + \frac{\partial F}{\partial \xi_2} a_2 + \frac{\partial F}{\partial \xi_3} a_3 + \frac{\partial F}{\partial \xi_4} a_4 \right),
\]

\[
\frac{\partial F}{\partial y} = \frac{1}{6V} \left( \frac{\partial F}{\partial \xi_1} b_1 + \frac{\partial F}{\partial \xi_2} b_2 + \frac{\partial F}{\partial \xi_3} b_3 + \frac{\partial F}{\partial \xi_4} b_4 \right),
\]

\[
\frac{\partial F}{\partial z} = \frac{1}{6V} \left( \frac{\partial F}{\partial \xi_1} c_1 + \frac{\partial F}{\partial \xi_2} c_2 + \frac{\partial F}{\partial \xi_3} c_3 + \frac{\partial F}{\partial \xi_4} c_4 \right). 
\]

If now the node displacements for an element are written in the following form into a vector

\[
u^{(e)} = [u_{x_1}, u_{y_1}, u_{x_2}, u_{y_2}, u_{x_2}, \ldots u_{x_4}]^T
\]

the relation between strain and node displacement \( \epsilon = \mathbf{B}u^{(e)} \) can be written using Equations (5.9) and (5.14) with \( \mathbf{B} \) as

\[
\mathbf{B} = \frac{1}{6V} \begin{bmatrix}
  a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 \\
  0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 \\
  0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 \\
  b_1 & a_1 & 0 & b_2 & a_2 & 0 & b_3 & a_3 & 0 & b_4 & a_4 & 0 \\
  0 & c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 \\
  c_1 & 0 & a_1 & c_2 & 0 & a_2 & c_3 & 0 & a_3 & c_4 & 0 & a_4
\end{bmatrix}
\]

It has to be mentioned that the \( \mathbf{B} \) matrix is constant for a given tetrahedron.

The constitutive matrix \( \mathbf{E} \) giving the stress-strain relation in three dimensions is given as

\[
\mathbf{E} = \begin{bmatrix}
  \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
  \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
  \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
  0 & 0 & 0 & \mu & 0 & 0 \\
  0 & 0 & 0 & 0 & \mu & 0 \\
  0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\]

(5.18)

Where \( \lambda \) and \( \mu \) are the Lamé constants defined as

\[
\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}
\]

(5.19)

with \( E \) the elasticity and \( \nu \) the Poisson’s number.
5.2. Finite element discretisation

The element stiffness matrix can now be calculated as

$$[k] = \int_{V(e)} \mathbf{B}^T \mathbf{E} \mathbf{B} dV(e) = V \mathbf{B}^T \mathbf{E} \mathbf{B}.$$  (5.20)

The next step is to calculate the forces acting on the tetrahedron. Assuming a force field \( \mathbf{g} \) given by its three force components

$$\mathbf{g} = [g_x, g_y, g_z]^T.$$  (5.21)

The resulting force for each node \( f \) is given by the integral

$$[f] = \int_{V(e)} \mathbf{N}^T \mathbf{g} dV(e).$$  (5.22)

Here \( \mathbf{N} \) is the 3 by 12 matrix relating the node displacements to the displacement field

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \mathbf{N} \mathbf{u}^{(e)}.$$  (5.23)

with

$$\mathbf{N} = \begin{bmatrix} \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 & 0 \\ 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 & 0 \\ 0 & 0 & \xi_1 & 0 & 0 & \xi_2 & 0 & 0 & \xi_3 & 0 & 0 & \xi_4 \end{bmatrix}.$$  (5.24)

Even if the force field is constant the integral over the element is not. The integral has to be evaluated numerically. For our purposes the calculations are reduced to the following two integrals

$$\int_{V(e)} \xi_i dV(e) = \frac{1}{4} V$$  (5.25)

and

$$\int_{V(e)} \xi_i \xi_j dV(e) = \begin{cases} \frac{1}{12} V & \text{if } i = j, \\ \frac{1}{20} V & \text{if } i \neq j. \end{cases}$$  (5.26)

The general rule for such integrals for tetrahedra with planar faces can be derived from the Beta function as

$$\int_{V(e)} \xi_i \xi_j \xi_k \xi_l dV(e) = \frac{i!j!k!l!!}{(i+j+k+l+3)!} 6V.$$  (5.27)

5.2.3 Viscous element

The viscous element uses the same discretisation as the elastic element. Only for completeness the viscosity matrix \( \mathbf{V} \) in three dimensions is given here

$$\mathbf{V} = \eta \begin{bmatrix} 1 + \frac{1}{1+m} & \frac{1}{m-2} & \frac{1}{m-2} & 0 & 0 & 0 \\ \frac{1}{m-2} & \frac{1}{1+m} & \frac{1}{m-2} & 0 & 0 & 0 \\ \frac{1}{m-2} & \frac{1}{m-2} & \frac{1}{1+m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  (5.28)

where \( \eta \) is the viscosity and \( m \) the inverse analogous to the Poisson’s number.

For more details see Section 3.2.2.
Chapter 5. 3D Finite Element Formulation

5.2.4 Weak layer elements

As for the two-dimensional model a special weak element was introduced to model weak layers. The geometry of this element is shown in Figure 5.10. The local element coordinate system is defined such that \( z_e \) is perpendicular to the triangles and \( x_e, y_e \) lie in the plane spanned by the triangles. The thickness (i.e. \( z_{e,1} - z_{e,4} \)) of a weak layer element is by definition set to zero.

Forces acting along the \( z_e \)-axis are passed on as if the two tetrahedra top and bottom (see Figure 5.10) would be connected directly. In the plane spanned by the two triangles (i.e. coordinates \( x_e, y_e \)), however, the nodes on the bottom can move independently from the nodes of the top tetrahedron. This simulates a friction free interface.

If the node displacement vector of the weak element in local coordinates is defined as

\[
u_{weak}^{(e)} = [u_{x_e,1}, u_{y_e,1}, u_{z_e,1}, u_{x_e,2}, u_{y_e,2}, u_{z_e,2}, \ldots u_{z_e,6}]^T \tag{5.29}
\]

then the element stiffness matrix \([k]_{weak}\) in local coordinates is given by

\[
[k]_{weak} = \begin{bmatrix}
O_z & 0 & 0 & -O_z & 0 & 0 \\
0 & O_z & 0 & 0 & -O_z & 0 \\
-O_z & 0 & 0 & 0 & -O_z & 0 \\
0 & -O_z & 0 & 0 & O_z & 0 \\
0 & 0 & -O_z & 0 & 0 & O_z \\
\end{bmatrix} \tag{5.30}
\]

where

\[
0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \quad \text{and} \quad O_z = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \infty \\
\end{bmatrix}. \tag{5.31}
\]

This local stiffness matrix is rotated into the global coordinate system and assembled into the global stiffness matrix.

5.2.5 Temperature solution

The temperature equations are solved in a manner analogous to the two-dimensional case. The extension to three dimensions, as well as the integration scheme is straightforward and omitted here.

5.2.6 Boundary conditions

The boundary of the three-dimensional model domain can be subdivided into an interface on the ground (bottom), the actual snow surface on the top and the four surfaces on each side (see Figure 5.13). For each avalanche defence structure two additional surfaces are
generated, the front (i.e. in front of the defence structure) and the back side of the defence
structure.

The following boundary conditions are used for the elastic solution

\begin{align*}
    u_i |_{\text{bottom}} &= 0, \\
    \sigma_{ij} n_j |_{\text{top}} &= 0, \\
    \sigma_{ij} n_j |_{\text{side}} &= 0, \\
    u_i n_j |_{\text{front}} &= 0, \\
    \sigma_{ij} n_j |_{\text{back}} &= 0
\end{align*}

(5.32) \quad (5.33) \quad (5.34) \quad (5.35) \quad (5.36)

where \( n_j \) is the outward pointing normal to the given surface and \( u_i \) the movement in
direction \( i \). In words this means that the snow is fixed on the bottom surface, the top
and side surfaces are stress free in the outward pointing direction. Snow on the front side
of the defence structure can freely move in the plane spanned by the structure but not
penetrate it and snow at the back of the defence structure can freely move and is stress
free in the outward pointing direction (towards the defence structure).

For the viscous solution the respective viscous pendant is used

\begin{align*}
    \dot{u}_i |_{\text{bottom}} &= 0, \\
    \sigma_{ij} n_j |_{\text{top}} &= 0, \\
    \sigma_{ij} n_j |_{\text{side}} &= 0, \\
    \dot{u}_i n_j |_{\text{front}} &= 0, \\
    \sigma_{ij} n_j |_{\text{back}} &= 0
\end{align*}

(5.37) \quad (5.38) \quad (5.39) \quad (5.40) \quad (5.41)

where \( \dot{u}_i \) is the creep velocity in \( i \) direction.

The temperature boundary conditions at the top surface can be either specified as a
constant temperature (von Neuman)

\begin{align*}
    T_i &= \text{const}, \\
    T_a &= \text{const}
\end{align*}

(5.42) \quad (5.43)

or by a function \( \Theta(t) \) specifying the temperature history over time (Dirichlet)

\begin{align*}
    T_i(t) &= \Theta(t), \\
    T_a(t) &= \Theta(t)
\end{align*}

(5.44) \quad (5.45)

For the bottom surface a constant temperature that can be specified by the user is as-
sumed.

5.2.7 Numerical procedure

The general solution scheme for the three-dimensional finite element model is the same
as for the two-dimensional finite element which is described in Algorithm 3.6. For the
element matrices, however, their three-dimensional versions as described in this chapter
are used.
5.3 \textit{N}-Directional approach

The \textit{N}-Directional approach [57] is based on the idea of describing the strain state using the normal strain component $\varepsilon_n$ in \textit{N} so-called \textit{evenly distributed} directions instead of the usual tensorial components $\varepsilon_{ij}$. Each direction can be viewed as a uni-axial sub-continuum transmitting only normal stress in its direction, and hence can be modelled by simple spring elements. A one-dimensional material law can be used to describe the complex behaviour of a three-dimensional finite element.

5.3.1 Evenly distributed directions

Let $\vec{d}_n = (d_n^x, d_n^y, d_n^z)^T \ (n = 1 \ldots \text{N})$ be a set of vectors in three-dimensional space with length 1. They are said to be \textit{evenly distributed} if they fulfil the following

\begin{align}
\sum_{n=1}^{\text{N}} d_n^i d_n^j &= \frac{\text{N}}{3} \delta_{ij}, \quad (5.46) \\
\sum_{n=1}^{\text{N}} d_n^i d_n^j d_n^k d_n^l &= \frac{\text{N}}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (5.47)
\end{align}

\forall i, j, k, l \in \{x, y, z\}$ where $\delta_{ij}$ is the Kronecker’s delta. Three sets for $\text{N}=6, 10$ and 15 are given in Appendix B. Due to the properties of the evenly distributed directions $\vec{d}_n$
5.3. N-Directional approach

the tensorial stress components can be split into the so-called N-Directional stresses.

5.3.2 N-Directional stresses

The directional strain $\epsilon_n$ is defined as the projection in direction $\vec{d}_n$ of the strain deformation vector onto a plane perpendicular to $\vec{d}_n$

$$\epsilon_n = \sum_i \sum_j d^i_n d^j_n \epsilon_{ij}. \quad (5.48)$$

In the same way the directional stress $\sigma_n$ is defined as

$$\sigma_n = \sum_i \sum_j d^i_n d^j_n \sigma_{ij}. \quad (5.49)$$

The stress state is described by N-Directional stresses $s_n$ each acting in direction $\vec{d}_n$. They are work-associated to the directional strain $\epsilon_n$ and they can be related to the stress tensor $\sigma_{ij}$ using the principle of virtual work for an initial volume $V$ as follows

$$\int_V \sigma_{ij} \delta \epsilon_{ij} dV = V \sum_{n=1}^N s_n \delta \epsilon_n = V \sum_{n=1}^N s_n d^i_n d^j_n \delta \epsilon_{ij}. \quad (5.50)$$

Rearranging the terms in Equation (5.50) results in the representation of the stress state $\sigma_{ij}$ by a superposition of the N-Directional stresses $s_n$

$$\sigma_{ij} = \sum_{n=1}^N d^i_n d^j_n s_n. \quad (5.51)$$

It has to be noticed, that while $\epsilon_n$ represents the true strain component in the direction $\vec{d}_n$, $s_n$ does not represent the true normal stress in this direction, but only one of the N-Directional components resulting in the stress tensor $\sigma_{ij}$. The $N$ normal strains $\epsilon_n$ are not $N$ independent values since they derive from the 6 independent tensor components $\epsilon_{ij}$ in the three-dimensional case. They are kinematically constrained. The N-Directional stresses $s_n$, however, are independent from each other.

5.3.3 N-Directional constitutive laws

The fundamental idea of the N-Directional approach is to formulate constitutive laws by using the N-Directional stresses $s_n$, assuming each directional stress $s_n$ to be a function of its work-associate directional strain $\epsilon_n$ and the average directional strain $\bar{\epsilon}$ only, and not of the remaining $N-1$ directional strain components

$$\bar{\epsilon} = \frac{1}{N} \sum_{n=1}^N \epsilon_n. \quad (5.52)$$
If the material has isotropic properties, the directional constitutive law is the same for all $N$ directions. Up to the average strain $\bar{\epsilon}$, the directional stress $s_n$ is one-dimensional in $\epsilon_n$. The average strain $\bar{\epsilon}$ is closely related to the volumetric strain and is the same for all directions. This is the only part coupling the $\epsilon_n$'s. In this way, the continuum is modelled by $N$ almost un-coupled uni-directional sub-continua (which can be seen as springs) each with the capacity to transmit the directional stress $s_n$ in its own direction only.

The usually used tensorial constitutive law relating the strain and stress tensor components is replaced by a uni-dimensional constitutive law for each of the $N$ evenly distributed directions $\vec{n}$.

For the linear elastic case the $N$-Directional constitutive law is given by

$$s_n(\epsilon_n, \bar{\epsilon}) = \alpha \epsilon_n + \beta \bar{\epsilon}$$  \hspace{1cm} \text{(5.53)}

with

$$\alpha = \frac{15}{N \frac{E}{2(1+\nu)}} \quad \beta = \frac{9}{N \frac{E}{2(1+\nu)}} \frac{4\nu-1}{1-2\nu}$$  \hspace{1cm} \text{(5.54)}

where $E$ is the elasticity module and $\nu$ the Poisson's number. This constitutive law results in the same stiffness matrix as the usual tensorial formulation. It has to be noticed that for $\nu = \frac{1}{3}$, $\beta = 0$. Therefore the $s_n$ are independent of $\bar{\epsilon}$, hence the $N$-Directions are completely uncoupled.

For the snow model the Poisson's number $\nu$ was set to 0 and the elasticity module $E$ is given by Equation (2.19).

### 5.4 Damage mechanics using the $N$-Directional approach

In the two-dimensional model linear fracture mechanics was used to calculate stress intensity factors. With the help of those factors the stability of a snow slab can be evaluated. Although this would also be possible for the three-dimensional case, we decided not to do so for the following reasons:

- Fracture mechanics was developed using plane stress conditions and was only later extended to three dimensions [5].
- Most real three-dimensional problems are either too complex to model properly or can be reduced to two dimensions.
- The material constants $K_{IC}$ and $K_{IIc}$ for snow are not well known yet.
- The linear fracture mechanics model might not be appropriate for snow due to size effects [83].

The so-called $N$-Directional approach introduced by J. Renau [57] gives a new way to use the damage mechanical model for two- and three-dimensional finite element simulations. In the following, this theory is presented.
Introducing damage mechanics using the N-Directional approach is now straight forward. If each constitutive law $s_n$ is seen as independent from the others a damage function $D_n$ for each direction can be used

$$D_n(\ldots) s_n(\epsilon_n, \bar{\epsilon}) = D_n(\ldots)(\alpha \epsilon_n + \beta \bar{\epsilon}).$$ (5.55)

This allows modelling a different behaviour for each direction, resulting in a non-isotropic material behaviour.

For snow the following damage function was implemented

$$D_n(\dot{\epsilon}_n) = \begin{cases} 0 & \text{if } \dot{\epsilon}_n > 10^{-4} = \dot{\epsilon}_{\text{max}}, \\ 1 & \text{otherwise}. \end{cases}$$ (5.56)

This states that if the normal strain-rate $\dot{\epsilon}_n$ in the $n$-th direction reaches the maximum value $\dot{\epsilon}_{\text{max}}$ then the corresponding uni-axial sub-continuum breaks. The $n$-th normal stress component $s_n$ will vanish in all subsequent straining steps. Since the remaining unbroken sub-continua may still carry a stress in this direction, the continuum as a whole does not lose the capacity to transmit normal stresses $\sigma_n$ in the collapsed direction. However, an irreversible stiffness decrease for strains and stresses in the directions of the broken sub-continua will take place.

5.5 Numerical N-Directional model for snow

This section describes the implementation details of the three-dimensional model for snow using the N-Directional approach. The main task is to build up the constitutive elasticity matrix $E_N$ (see Equation (5.18)) and the viscosity matrix $V_N$ (see Equation (5.28)) for each finite element.

5.5.1 N-Directional constitutive elasticity matrix $E_N$

The normal elasticity matrix $E$ relates the tensorial strain $\epsilon_{ij}$ to the stress state $\sigma_{ij}$

$$\sigma = E\epsilon.$$ (5.57)

For the N-Directional approach, the elasticity matrix $E_N$ needs to give the same relation, such that it can be used in the existing framework. The steps to achieve this are as follows (see also Figure 5.14):

- Calculate the directional strain $\epsilon_n$ from the tensorial strains $\epsilon_{ij}$.
- Apply the N-Directional constitutive equation. $s_n(\epsilon_n, \bar{\epsilon})$ to get the N-Directional stresses $s_n$.
- Calculate the tensorial stress $\sigma_{ij}$ from the N-Directional stresses $s_n$. 
If the strain $\epsilon$ is given as in Equation (5.12) then $\epsilon$ can be calculated as $\epsilon = \text{ND}\epsilon$ where \text{ND} is a $N$ by 6 matrix given as

\[
\text{ND} = \begin{bmatrix}
  d_1^1 d_1^1 & d_1^2 d_1^2 & d_1^3 d_1^3 & d_1^4 d_1^4 & d_1^5 d_1^5 & d_1^6 d_1^6 \\
  d_2^1 d_2^1 & d_2^2 d_2^2 & d_2^3 d_2^3 & d_2^4 d_2^4 & d_2^5 d_2^5 & d_2^6 d_2^6 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  d_N^1 d_N^1 & d_N^2 d_N^2 & d_N^3 d_N^3 & d_N^4 d_N^4 & d_N^5 d_N^5 & d_N^6 d_N^6 
\end{bmatrix}. \tag{5.58}
\]

The $N$-Directional constitutive law $s_n(\epsilon_n, \bar{\epsilon}) = \alpha \epsilon_n + \beta \bar{\epsilon}$ relating the $\epsilon_n$ to the $N$-Directional stresses $s_n$ can also be written in matrix form as a $N$ by $N$ matrix

\[
S_n = \begin{bmatrix}
  \alpha + \frac{1}{N} \beta & \frac{1}{N} \beta & \frac{1}{N} \beta & \cdots & \frac{1}{N} \beta \\
  \frac{1}{N} \beta & \alpha + \frac{1}{N} \beta & \frac{1}{N} \beta & \cdots & \frac{1}{N} \beta \\
  \frac{1}{N} \beta & \frac{1}{N} \beta & \alpha + \frac{1}{N} \beta & \cdots & \frac{1}{N} \beta \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{N} \beta & \frac{1}{N} \beta & \frac{1}{N} \beta & \cdots & \alpha + \frac{1}{N} \beta 
\end{bmatrix}. \tag{5.59}
\]

The tensorial stresses $\sigma$ are calculated from the $N$-Directional stresses $s_n$ as

\[
\sigma = [\text{ND}]^T s_n. \tag{5.60}
\]

Using these matrices the $N$-Directional equivalent to $\sigma = E\epsilon$ can be written as

\[
\sigma = E_N\epsilon = [\text{ND}]^T [S_n][\text{ND}]\epsilon. \tag{5.61}
\]

If the damage mechanics model is also taken into account a damage matrix $D$ is defined
in the following way

\[
D = \text{diag}(D_i) = \begin{bmatrix}
D_1 & 0 & 0 & \cdots & 0 \\
0 & D_2 & 0 & \cdots & 0 \\
0 & 0 & D_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & D_N \\
\end{bmatrix}.
\]  

(5.62)

Using this \(N\)-Directional equivalent to \(\sigma = E\varepsilon\) including damage mechanics is written as

\[
\sigma = E_N\varepsilon = [ND]^T[D]^T[S_n][D][ND]\varepsilon.
\]  

(5.63)

The matrix \(E_N\) can now be used in the usual finite element scheme instead of the usual \(E\) matrix (Equation (5.20)) to calculate the element stiffness matrix.

### 5.5.2 \(N\)-Directional constitutive viscosity matrix \(V_N\)

Analogous to the \(N\)-Directional constitutive viscosity matrix \(E_N\) the \(N\)-Directional constitutive viscosity matrix \(V_N\) has to be build. The basic steps are the same as in the previous section:

- Calculate the directional strain-rates \(\dot{\varepsilon}_n\) from the tensorial strain-rates \(\dot{\varepsilon}_{ij}\).
- Apply the \(N\)-Directional constitutive equation \(s_n^V(\dot{\varepsilon}_n, \ddot{\varepsilon})\) to get the \(N\)-Directional stresses \(s_n\).
- Calculate the tensorial stress \(\sigma_{ij}\) from the \(N\)-Directional stresses \(s_n\).

The viscous analog to the elastic \(N\)-Directional law \(s_n(\varepsilon, \ddot{\varepsilon})\) is given by

\[
s_n^V(\dot{\varepsilon}_n, \ddot{\varepsilon}) = \alpha^V\dot{\varepsilon}_n + \beta^V\ddot{\varepsilon},
\]  

(5.64)

where

\[
\alpha^V = \frac{15\mu}{N^2}, \quad \beta^V = \frac{2}{N} \left(\frac{\mu}{m-2} - \frac{\mu}{4}\right)
\]  

(5.65)

with \(\mu\) the viscosity and \(m\) the inverse analogue to the Poisson's number. The viscosity for snow is given by Equation (2.30). The \(N\)-Directional approach now allows using the directional stress \(\sigma_n\) in the constitutive Equation (2.30), hence to have a different viscosity for each direction.

The \(N\)-Directional equivalent to \(\sigma = V\dot{\varepsilon}\) using damage mechanics and the above definitions is written as

\[
\sigma = V_N\dot{\varepsilon} = [ND]^T[D]^T[S_n^V][D][ND]\dot{\varepsilon}
\]  

(5.66)

where \(S_n^V\) is given by Equation (5.59) with \(\alpha^V, \beta^V\) instead of \(\alpha\) and \(\beta\). For the damage mechanical model Equation (5.56) is used.
## 5.5.3 Numerical procedure using the N-Directional approach

In the following the three-dimensional finite element scheme using the N-Directional approach is presented. The main change is the addition of the damage algorithm. Remember that instead of $E$ and $V$ the N-Directional equivalents $E_N$ and $V_N$ have to be used.

**Algorithm 5.1: 3D FE-Scheme using the N-Directional approach with damage**

Require: Set boundary temperature for time $t = 0$, set final calculation time $t_{\text{final}}$, set simulation time $t = 0$, initial densities $\rho(t = 0)$, initial positions $x(t = 0)$

Ensure: Stress state $\sigma_{ij}$, strain state $\epsilon_{ij}$, strain-rate $\dot{\epsilon}_{ij}$, temperature $T_i, T_a$, movement $u$, creep velocities $\dot{u}$, forces $f$, density $\rho$ and positions $x$

1: for all finite elements do
2: Initialise damage matrix $[D] = \text{diag}(1)$
3: Initialise N-Directions
4: end for
5: Solve initial Temperature $\rightarrow 3.7$
6: while $t < t_{\text{final}}$ do
7: repeat
8: Solve elastic $\rightarrow 3.2$
9: Solve viscous $\rightarrow 3.3$
10: until ! (Check damage $\rightarrow 5.2$)
11: Determine timestep $\rightarrow 3.4$
12: Solve temperature $\rightarrow 3.8$
13: Advance System $\rightarrow 3.5$
14: end while

**Algorithm 5.2: Check damage**

Require: Strain-rate $\dot{\epsilon}$, damage matrix $D$

Ensure: Update damage matrix $D$, return state

1: state=true
2: for all elements do
3: for all N-Directions do
4: if $\dot{\epsilon}_{n,i} \geq \dot{\epsilon}_{\text{max}}$ then
5: $D[i, i] = 0$
6: state=false
7: end if
8: end for
9: end for
10: Return state
In this Chapter the three-dimensional finite element model is applied to investigate the behaviour of the snow cover. In the first section the stress and strain-rate state of a real avalanche slope is investigated. The second section covers the influence of snow defence structures on the snow cover. The final section studies the problem of avalanche formation using the $N$-Directional approach including damage mechanics.

### 6.1 3D calculations of an avalanche slope

The first application of the three-dimensional model involves finding the strain-rate and stress fields in the snow cover on an avalanche slope. For this purpose the starting zone of the Swiss field test site Vallée de la Sionne is modelled (see Figures 6.1 and 6.2) using a DTM with a spacing of 2.5 m. The size of the projected calculation zone is approximately 200 m by 200 m. Over 400,000 elements were used to discretise the 3 m high snow cover. The snow cover modelled consists of three layers with properties given in Table 6.1.

In Figure 6.3 the principal strain-rate $\varepsilon_I$ is shown. Figure 6.4 depicts the principal strain-rate $\varepsilon_{III}$. Both figures show that the values in compression and tension follow the underlying topography; that is, the strain-rates are large on steep slopes. Table 6.2 gives the maximal stress and strain-rate values in compression and the minimal stress and strain-rate values in tension for the simulated snow cover. The strain-rate values are too low for crack initiation and crack propagation, the same holds for the stress concentrations.

<table>
<thead>
<tr>
<th>layer height</th>
<th>density $\text{kg/m}^3$</th>
<th>temperature $\degree C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0 m</td>
<td>300</td>
<td>-4</td>
</tr>
<tr>
<td>1.0 m</td>
<td>250</td>
<td>-6</td>
</tr>
<tr>
<td>1.0 m</td>
<td>200</td>
<td>-8</td>
</tr>
</tbody>
</table>

Table 6.1: Layering used for the simulated snowpack at the Vallée de la Sionne.
Figure 6.1: View of the avalanche starting zone at the Vallée de la Sionne displaying the map. Topographic base map reproduced by permission of swisstopo (BA46644).

Figure 6.2: View of the avalanche starting zone at the Vallée de la Sionne displaying an orthophoto of an avalanche fracture line.
6.1. 3D calculations of an avalanche slope

Figure 6.3: Three dimensional view of the principal strain-rate $\dot{\varepsilon}_I$ in 1/s at the Vallée de la Sionne.

Figure 6.4: Three dimensional view of the principal strain-rate $\dot{\varepsilon}_{II}$ in 1/s at the Vallée de la Sionne.
Table 6.2: Maximal stress and strain-rate values in compression and the minimal stress and strain-rate values in tension for the simulated snow cover.

From the depicted results, we draw the following conclusions:

- Maximum strain-rates are on the order of $10^{-6} / s$. These are far from ($\approx 2$ orders of magnitude) the failure strain-rates ($10^{-4} / s$) needed for snow failure. Snow cover imperfections are still required to induce stress and strain-rate concentrations. This result supports the general conclusions of Bader and Salm [6] that avalanche activity can only exist, if the snow cover contains imperfections or weak zones.

- The highest strain-rates exist on steep flanks of gullies, where the curvature radius is small and the slopes are steep. The distribution of strain-rates and stress follow the local topography closely. This suggests that one dimensional models can be effectively used to predict the state of the snowpack before failure.

- Comparing the strain-rate results of Figures 6.3 and 6.4 with the fracture line depicted in Figure 6.2 shows no obvious reason for the geometry of the fracture line. This supports the necessity of weak zones for avalanche formation, the topography alone seems not to be the most important ingredient. The mechanics of avalanche formation can thus be studied in simplified three-dimensional topographies.

### 6.2 Influence of snow defence structures on the snow cover

The next application of the three-dimensional model is to study the influence of snow defence structures on the strain-rate and stress distribution in an avalanche starting zone. This is of primary interest to engineers who design the layout of the defence structures in complex topography. In Section 4.4 the two-dimensional model was used to determine the length of the influence zone for an infinitely wide slope and the stress distribution at the defence structure. The validity of these numerical studies, as well as the Swiss Guidelines on avalanche defence structures, are now examined using the three dimensional model. Two different series of calculations are performed. The first determines the "neutral" zone of a single defence structure. The second series of calculations is used to find the optimal placement — horizontal and vertical spacing — of several supporting structures. In both series of calculations no weak layer was inserted between the snowpack layers.

An inclined slope with varied length $l$ and width $w$ (see Figure 6.5) is used in the numerical study. The inclination angle $\phi$ varied. A layered snow cover with a total height $h = 2$ m is
6.2. Influence of snow defence structures on the snow cover

Figure 6.5: Model domain of the defence structure calculations showing the model parameters and location of defence structure.

modelled using five layers and a constant snow density $\rho = 250\, \text{kg/m}^3$ at a temperature of $-4^\circ\text{C}$. The model dimensions were width $w = 30\, \text{m}$ and length $l = 30\, \text{m}$. A snow defence structure with length $l_d$ is placed in the centre of the model domain.

In order to define the influence zone surrounding a defence structure calculations are performed with and without the defence structure. The influence zone is defined as the region where the deformation states of the two simulations differ. The calculation without defence structure acts as a control calculation to determine the influence zone. The influence zone of a defence structure is mathematically found by subtracting the results from the simulation without perturbation from the results of the simulation including the perturbation. Figure 6.6 shows a three-dimensional view of the influence zone using the difference in movements.

Figure 6.7 depicts the influence of the defence structure on the snow cover for different defence structure lengths $l_d$. In this particular graphic, the slope angle is $\phi = 40^\circ$. The influence zone for creep deformations is depicted as a top view looking downwards onto the snowpack. It can be seen, that for defence structures longer than $l_d \geq 8\, \text{m}$, the influence zone around the centre becomes stationary. The influence of the defence structure ends is around 3 m to 4 m. The size of the influence in the slope direction $y$ is around 5 m to 10 m, which is in good agreement with the two-dimensional simulations presented in Section 4.4. It has to be mentioned, that Figure 6.7 displays only the initial influence at calculation time $t = 0$. Due to the viscous deformation of the snow cover over time the influence zone grows slowly (less than 1 m for a simulation time of 10 days) over time.
Figure 6.6: Three dimensional view of the influence zone of an avalanche defence structure based on movements, in m. Isolines just before and after the defence structures (top and bottom) and a cut through the centre of the structure (left).
6.2. Influence of snow defence structures on the snow cover

Figure 6.7: Influence zone of avalanche defence structures for different lengths $l_d$ on a slope with angle $\phi = 40^\circ$. The colour scale used is as in Figure 6.6.
Chapter 6. 3D Model Calculations

According to the Swiss Guidelines [17] avalanche defence structures should not only influence the snow cover in a way that no avalanche can start — they should also sustain and stop small avalanches occurring between the rows of an avalanche slope. Therefore, horizontal spacing larger than 10 m, meaning outside of the influence boundary, are appropriate.

The effect of the slope angle $\phi$ on the size of the influence zone is depicted in Figure 6.8. The slope angle determines the magnitude of the creep deformations (they are higher on steeper slopes) but not much the size of the influence zone. This is reasonable since for steeper slopes the forces acting parallel to the defence structure are larger. The decay of this effect is quite fast and therefore the enlargement of the influence zone is small. The size of the influence zone is depicted in Figure 6.8. The shape of the top and bottom boundary changes slightly for different slope angles $\phi$.

Other measures could be used to define the size of the influence zone. Figures 6.7 and 6.8 depicted the influence zone using the difference of creep deformations as a measure. In Figure 6.9 the influence zone is plotted using principal stress and strain instead of the creep deformations. The shape and size of the influence zone in tension ($\sigma_l$ and $\epsilon_l$) are maximum below the defence structure in the slope direction. This is a direct result of the prescribed boundary conditions which state that the snowpack is not attached to the defence structure and is free to displace in the downward direction. The influence at the ends of the structure can be seen in both the tension and compression ($\sigma_{III}$) plots. The maximal values in compression are found at the ends of the structure.

Another question that can be addressed using the three-dimensional simulation tool and the concept of influence zone is the question of the horizontal distance $d$ between snow defence structures. To do so, two $l_d$ 15 m long defence structures were placed next to each other with a varying distance $d$ between them on a slope with angle $\phi = 40^\circ$. The modelled snow cover and model domain have the same values as before. In Figure 6.10 the results of this simulation are shown. The influence zone of the creep deformations is depicted. The Swiss Guidelines [17] give a maximal horizontal distance between two defence structures of 2 m. The simulations show that for a distance $d \leq 2$ m the influence zone above the snow defence structure is only slightly perturbed. The perturbation becomes larger for wider distances $d$. If the distance between the defence structures is larger than $d \geq 10$ m the influence zone is discontinuous, meaning that there is a zone which is not protected by the defence structure. The calculations suggest that the Swiss Guideline specification of 2 m is reasonable, but also rather conservative.
6.2. Influence of snow defence structures on the snow cover

Figure 6.8: Influence of an avalanche defence structure with length 8 m for different slope angles $\phi$. The colour scale used is as in Figure 6.6.
Chapter 6. 3D Model Calculations

principal stress $\sigma_I, \text{N/m}^2$

principal stress $\sigma_{III}, \text{N/m}^2$

principal strain $\epsilon_I, 1$

principal strain $\epsilon_{III}, 1$

Figure 6.9: Principal stress and strain influence for an avalanche defence structure with length $l_d = 8 \text{ m}$ with slope angle $\phi = 40^\circ$.
Figure 6.10: Influence zone for neighbouring avalanche defence structures on a slope with angle $\phi = 40^\circ$. The distance $d$ between the two defence structures is varied. The colour scale used is as in Figure 6.6.
6.3 Avalanche Formation

In Section 4.5 the avalanche formation was discussed using a fracture mechanical approach and a two dimensional model. The main conclusion was that for an avalanche to occur a weak layer \(a\) of length \(6\,\text{m} < a < 10\,\text{m}\) is needed. In this section the \(N\) -Directional approach with the damage mechanics, as described in Section 5.3, is applied to study avalanche formation.

Figure 6.11 depicts the model domain used for the following calculations. It consists of a snowpack on an inclined plane with an inclination angle \(\phi\). In the centre of the snowpack a weak layer surface is embedded with length \(a\) and width \(b\). This weak layer surface is sandwiched between a bottom layer (density \(\rho_b = 300\,\text{kg/m}^3\), height \(h_b = 0.5\,\text{m}\) and temperature \(T_b = -4^\circ\text{C}\)) and the overlaying snow cover. The temperature \(T_o\), density \(\rho_o\) and height \(h_o\) of the upper snow layer was varied. The height is always measured orthogonal to the surface. The length \(l\) and width \(w\) are chosen such that the distance from the side boundary of the snow cover to the weak layer surface is at least 20 m, which is larger than the influence length found in the previous section and Section 4.4. Therefore, boundary effects should have no influence on the results.

Using the \(N\)-Directional approach with a failure criterion of \(\dot{\epsilon} > 10^{-4}/\text{s}\) the avalanche formation process in the simulations proceeds as follows: Initially material damage occurs at the centre of the upper end of the weak zone. The damaged region then grows towards the upper edges of the weak zone and to the surface at the upper boundary. In the next phase the flanks of the weak zone start to fail which leads to the initiation of the avalanche.
6.3. Avalanche Formation

Initial damage occurs at the upper boundary of the weak zone.

Failure starts at the flanks and grows towards the surface.

The lower end of the weak zone starts to fail.

Figure 6.12: Failure initiation and propagation for a generic weak zone. Only the damaged elements are shown where 1 means complete failure.
Figure 6.13: Influence of snow temperature on slope failure angle $\phi_f$ for an infinitely wide weak zone with length $a = 10\,\text{m}$.

This process is depicted in Figure 6.12. The shear failure at the base, preceding the failure described in Figure 6.12, is given by the weak zone in our model, which is equal to the slab size that fails.

The minimal angle for which the first element failed was found by starting the simulation with a flat slope ($\phi = 0^\circ$) and then enlarging the slope angle in steps of $1^\circ$ for each subsequent simulation until a first element failed. The angle found by this method is termed failure angle $\phi_f$. The snowpack is unstable for angles larger than the failure angle ($\phi \geq \phi_f$), and stable for smaller inclinations ($\phi < \phi_f$).

The first series of calculations presented is on the influence of the length $a$ of the weak layer. The width $b$ of the weak layer surface was chosen to be equal to the snowpack width $w$, representing an infinitely wide weak layer surface. The overburden snow had a height $h_o$ of 1 m; the density $\rho_o$ was 180 kg/m$^3$. The temperature of the overburden snow, $T_o$ varied from $-1^\circ\text{C}$ to $-12^\circ\text{C}$. The weak layer length $a$ was varied from 0 m to 20 m.

Figure 6.13 depicts the failure angles for different weak layer lengths $a$ and different temperatures. Since the width of the weak zone is infinite, there is now support form the flanks of the snow cover and therefore the results represent a lower boundary for the failure angle. Both the weak zone length $a$ and the snow temperature have a large influence on the failure angle $\phi_f$. For a failure angle of $\phi_f = 50^\circ$ the weak zone needs to be longer than 5 m at a temperature of $T_o = -4^\circ\text{C}$. For a snow temperature of $T_o = -12^\circ\text{C}$ the weak zone needs a length $a > 10\,\text{m}$, hence a change of $\Delta T_o = 8^\circ\text{C}$ doubled the weak zone length needed for an avalanche to occur. Another conclusion that can be drawn from the
6.3. Avalanche Formation

**Figure 6.14:** Influence of the weak zone width $b$ to the slope failure angle $\phi_f$ for snow at $-1^\circ$C.

Simulation results is, that for the slope angle of interest ($25^\circ < \phi_f < 50^\circ$) a small change in the weak layer length $a$ leads to large differences in $\phi_f$ at high temperatures. Because the values of the weak layer length $a$ will never be known exactly and with sufficient precision ($\pm 2\, \text{m}$), it is unlikely that avalanche formulation will ever be predictable using deterministic methods.

In the next series of calculations the weak layer is no longer infinitely wide ($b \neq w$). The influence of the flanks on the sides of the weak layer boundary is investigated, depending on the weak zone width $w$. The model domain is the same as in the previous series of simulations (see Figure 6.11).

Next to the variation of the weak zone length $a$ (0 m to 20 m) and the snow temperature, the width of the weak zone was varied from 5 m to 20 m. Figures 6.14 and 6.15 show the failure angles for different weak zone widths $b$ for a snow temperature of $-1^\circ$C and $-4^\circ$C respectively. The results suggest that the snow temperature plays a significant role in avalanche formation. This can also be visualised in Figure 6.16 where the failure angle $\phi_f$ is plotted against temperatures for a given weak zone length $a = 10\, \text{m}$. Between $-6^\circ$C and $0^\circ$C the failure angle changes by $5^\circ$ per $1^\circ$C. This means, that since the temperature of the snow cover are seldom known to within an accuracy of $\pm 1^\circ$C the accuracy of a predicted failure slope angle $\phi_f$ is within $\pm 5^\circ$. For the given weak zone length $a = 10\, \text{m}$ and width $w = 10\, \text{m}$ the predicted failure angle $\phi_f$ for a measured snow temperature of $-3^\circ$C lies in the interval of $28^\circ \leq \phi_f \leq 38^\circ$. Hence, if a weak zone of the length $a \geq 10\, \text{m}$ and width $b \geq 10\, \text{m}$ exists for temperatures around $-3^\circ$C almost every relevant slope will
Figure 6.15: Influence of the weak zone width $b$ to the slope failure angle $\phi_f$ for snow at $-4^\circ C$.

Figure 6.16: Temperature influence on slope failure angle $\phi_f$ for weak layer length $a = 10m$. 

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6.3. Avalanche Formation

A summary of the calculations with different weak zone widths $b$ is shown in Figure 6.17. The graphs show that not only the weak zone length $a$, but also the weak zone width $b$ plays a significant role in avalanche formation. Narrow weak zones ($b < 5$ m) are more stable than wider ones.

The influence of the height $h_o$ of the upper snow layer is investigated in the next series of calculations. The model used a weak zone width of $b = 10$ m and a snow density $\rho_o = 180$ kg/m$^3$. The temperature and weak zone length $a$ were varied. The results for three different heights $h_o = 0.5$ m, 0.75 m and 1.0 m are depicted in Figure 6.18. The results suggest, that already a height of $h_o = 0.5$ m is sufficient to start the failure process in the snow cover. The failure angle $\phi_f$ is more dependent on the temperature and the weak zone length $a$ than on the snow height $h_o$.

In the last series of simulations the influence of the density $\rho_o$ on the failure angle $\phi_f$ is investigated. An upper snow layer height $h_o = 1.0$ m and a weak zone width $b = 10$ m was used. The temperature and weak zone length $a$ were varied. For the upper snow layer densities $\rho_o = 180$ kg/m$^3$, 200 kg/m$^3$ and 220 kg/m$^3$ were assumed. Figure 6.19 depicts the failure angle $\phi_f$ for the different densities. The influence of the density shows that

Figure 6.17: Influence of weak zone width $b$ to the failure angle $\phi_f$ for a snow height $h_o = 1.0$ m and density $\rho_o = 180$ kg/m$^3$. 

![Graphs showing the influence of weak zone width on failure angle](image-url)
upper layer snow height $h_o = 0.5 \text{ m}$
density $\rho_o = 180 \text{ kg/m}^3$
weak zone width $b = 10 \text{ m}$

upper layer snow height $h_o = 0.75 \text{ m}$
density $\rho_o = 180 \text{ kg/m}^3$
weak zone width $b = 10 \text{ m}$

upper layer snow height $h_o = 1.0 \text{ m}$
density $\rho_o = 180 \text{ kg/m}^3$
weak zone width $b = 10 \text{ m}$

**Figure 6.18**: Influence of upper snow layer height $h_o$ to the failure angle $\phi_f$ for snow density $\rho_o = 180 \text{ kg/m}^3$. 
6.3. Avalanche Formation

upper layer snow
height $h_o = 1.0$ m
density $\rho_o = 180$ kg/m$^3$
weak zone width $b = 10$ m

upper layer snow
height $h_o = 1.0$ m
density $\rho_o = 200$ kg/m$^3$
weak zone width $b = 10$ m

upper layer snow
height $h_o = 1.0$ m
density $\rho_o = 220$ kg/m$^3$
weak zone width $b = 10$ m

Figure 6.19: *Influence of upper layer snow densities $\rho_o$ to the failure angle $\phi_f$ for a snow height $h_o = 1.0$ m.*
for higher density snow the possibility of avalanches reduces dramatically. For an upper layer density $\rho_o = 220\text{ kg/m}^3$ all simulated slopes remained stable. This does not mean, that avalanches with snow densities larger than $200\text{ kg/m}^3$ are impossible, but according to the model weak zones which are larger than 20 m by 20 m are needed.
Conclusions and Outlook

7.1 Conclusions

Modelling a temperature dependent, highly deformable, non-linear, porous material is a challenge for the finite element method. Using an experimentally-based constitutive law for snow, it was possible to simulate well documented laboratory and field experiments. This served to test the finite element and constitutive formulations. A two- and three-dimensional finite element code capable of modelling a layered snowpack, including super weak layers, was developed. The terrain on which the snow cover lies can be described by a digital terrain model, e.g. from a geographical information system (GIS). Snow is treated as a temperature dependent, visco-elastic material. For both the two- and three-dimensional model the instationary temperature equation for the ice and air phase including the coupling between the two phases was solved.

One of the main problems studied was fracture propagation in the snowpack. This was addressed using two different approaches. The two-dimensional finite element code implements the calculation of linear elastic stress intensity factors which can be compared to the fracture toughness of snow. The simulation results are in excellent agreement with the first measured fracture toughness values for snow found in Kirchner et al. [32] and [33].

The three-dimensional finite element code implements the $N$-Directional approach, capable of modelling material failure resulting in crack initiation and propagation. For the first time the $N$-Directional approach was applied to a geophysical material. It proved to be an appropriate method to simulate snow and its damage mechanisms. Additionally, the strain-rate based failure criterion based on the $N$-Directional approach enabled us to show the influence of the snow temperature on avalanche formation. Linear elastic fracture mechanics does not take the snow temperature into account.

The process of avalanche formation could subsequently be investigated. The model indicates that a perturbation in the form of a weak layer/zone is required for avalanche initiation. This result supports the initial conjecture first stated by Bader and Salm [6].
Chapter 7. Conclusions and Outlook

The weak zone can be located either within the snowpack or at the interface between the soil and the snowpack. Today the formation and life time of weak zones is not fully understood. The three dimensional model was in agreement with the assumed avalanche process; failure of the basal layer (given by the weak zone in the model), initial material damage at the upper end of the weak zone (the crown), followed by failure of the flanks of the slab until the initiation of the avalanche. From the two dimensional model we conclude that a mode II fracture propagation needs weak zone lengths greater than 5 m for propagating the basal crack. The propagating crack will then eventually intersect with some vertical flaw or exceed some normal load in the overlying snow cover and a mode I of mixed mode failure then results in an avalanche.

The calculations revealed that the critical weak zone must be at least 5 m by 5 m in the three dimensional model and longer than 5 m in the two dimensional model. Both the the three-dimensional $N$-Directional material damage approach and the linear elastic fracture mechanical approach used in the two-dimensional analysis yield a similar result.

The critical size of the weak zone was found using the $N$-Directional approach as a function of (1) temperature, (2) density, (3) slope angle and (4) snowpack height (overburden pressure). Failure was defined using the concept of a critical strain-rate. Material damage and total material failure were modelled. For higher density snow ($\rho > 220$ kg/m$^3$) the release of a small avalanche (starting zone less than 20 m by 20 m) is unlikely. This, however, does not mean, that avalanches with snow densities higher than 220 kg/m$^3$ are impossible, rather higher density snow produces larger fracture slabs. For lower density snow ($\rho < 200$ kg/m$^3$) the temperature plays for a constant slope angle, the most important role in avalanche formation. With the finite element model it was possible to quantify when a low density snowpack, with a given perturbation, fails. Furthermore, the stabilising influence of densification, after a new snowfall event, could be simulated. The $N$-Directional approach was successfully applied to simulate the non-linear race between two competing physical processes: temperature rise and densification. Because the slope angle and snow height do not influence the critical time, when material damage/failure occurs, suggests that avalanche activity is clustered in space and time. That is, if an avalanche occurs on a steep slope other avalanche activity in a region with similar snow cover properties can be assumed. This is in good agreement with avalanche practice.

The results based on this computational analysis can be summarised as follows:

- Weak zones are needed for natural slab avalanche formation.
- The size of week need to be at least in the order of 5 m by 5 m in order to propagate the crack.
- Temperature rise and snow densification are competing processes within the mechanics of avalanche formation.
- At low temperatures ($< -10 ^\circ C$) the avalanche activity is small, a temperature increase, if weak zones exist, enlarges the avalanche activity for all avalanche relevant slope angles.
A change in slope angle of 5° has a smaller effect on the avalanche formation process than a temperature change of 1°C of the snow temperature.

We found that small differences in the snow cover properties can have a large influence on calculated failure angle $\phi_f$. For example, a change of weak layer length by $\pm 0.5$ m can lead to a transition from stable to unstable behaviour for all relevant slope angles. Thus, the weak layer length must be known within a range of $\pm 0.5$ m in order to predict avalanche activity with a high temporal resolution. This range defines the spatial resolution of the parameters required to deterministically predict avalanches. Precise measurements of snow temperature, snow cover density and weak zone size at this required spatial and temporal resolution are impossible in practice.

The influence zone of avalanche defence structures was also investigated. The model can be used in future to help design cost effective avalanche defence structures. In this work the first steps were performed by calculating the influence zones of avalanche defence structures. It was shown that the influence zone around defence structures measures about 10 m in slope direction. The horizontal influence, however, is smaller (around 5 m). The Swiss Guidelines suggest a horizontal spacing of at the most 2 m which is conservative but reasonable.

### 7.2 Outlook

Not all questions concerning avalanche formation and the optimal placement of avalanche defence structures were investigated. However, the presented model can be easily extended to explore the additional questions.

Weak zones were modelled with zero shear resistance. This is definitely a worst case scenario, meaning that the weak zone size calculated for an avalanche is a lower boundary. It would be interesting to see how the size of the weak zone needed for an avalanche to occur enlarges if the weak zone has some shear resistance. The shear resistance of weak layers is not known today and a large research effort is needed to quantify it.

The geometry of the weak zone was chosen to be rectangular in the simulations. In the natural snow cover the shape of weak zones are most likely not rectangular and not continuous. The influence of several nearby weak zone patches might be more realistic and could be investigated. However, it can be assumed that the total surface of a patched weak zone needs to be larger than the surface of one continuous weak zone to result in the same avalanche activity. The minimal size and distribution still capable to show avalanche activity would be of great interest, especially in view of the ongoing research of the variability of the snowpack [36].

The damage criterion for each element and each of its directions using the N-Directional approach was chosen to be constant and independent of temperature and density. This can be altered in two ways: First the damage criterion could be made temperature and density dependent. To do this, however, additional laboratory testing would be needed to determine the required material constants. Second the strength could be randomly
distributed for each element and/or direction. With this method similar simulations as used in the models described by cellular automatons [81] could be performed and a comparison of the two methods would be possible. The advantage of this method over cellular automatons is that the true three-dimensional stress state can be used in the simulation.

The influence of a new snowfall was not studied using the three dimensional model. This question could also be addressed in future work by modelling snow accumulation. In general a period of rapid and significant loading is the time in which avalanche forecasters are concerned about the stability of the snowpack.
Appendix A

Nomenclature

\[
\begin{align*}
\text{a} & \quad \text{weak layer length} \\
\text{a}_c & \quad \text{critical weak layer length} \\
\text{B} & \quad \text{natural to global coordinate transformation} \\
\text{b} & \quad \text{weak zone width} \\
\text{c}_a & \quad \text{heat capacity of air} \\
\text{c}_i & \quad \text{heat capacity of ice} \\
\text{D} & \quad \text{damage function} \\
\text{d}_n & \quad \text{evenly distributed direction} \\
\text{E}_0 & \quad \text{Young’s modulus} \\
\text{E} & \quad \text{elasticity matrix} \\
\text{E}_N & \quad \text{N-Directional elasticity matrix} \\
\text{f} & \quad \text{force} \\
\text{G} & \quad \text{energy release rate} \\
\text{g} & \quad \text{gravity} \\
\text{H} & \quad \text{shape function} \\
\text{H} & \quad \text{matrix of base-functions} \\
\text{h}_a & \quad \text{convective heat transfer coefficient} \\
\text{h}_l & \quad \text{layer height} \\
\text{I}_p(\mathbf{x}) & \quad \text{indicator function for phase } p \text{ at } \mathbf{x} \\
\text{J} & \quad \text{Jacobian matrix} \\
\text{K}_{I,II} & \quad \text{(calculated) stress intensity factor} \\
\text{K}_{Ic,IIc} & \quad \text{critical stress intensity factor} \\
\text{K} & \quad \text{global stiffness matrix} \\
\text{k} & \quad \text{element stiffness matrix} \\
\text{k} & \quad \text{stiffness} \\
\text{k}_a & \quad \text{conductivity of air} \\
\text{k}_i & \quad \text{conductivity of ice} \\
\text{L} & \quad \text{latent heat of vapourisation of water} \\
\text{l} & \quad \text{length} \\
\text{l}_d & \quad \text{defence structure length}
\end{align*}
\]
Appendix A. Nomenclature

\[ M \]  global convectivity matrix
\[ m \]  mass
\[ \mathbf{m} \]  element convectivity matrix
\[ n \]  number of moles
\[ n_{x,y,z} \]  normal and its components
\[ \text{ND} \]  relates strain \( \epsilon \) to \( N \)-Directional strain \( \epsilon_n \)
\[ p \]  pressure
\[ Q \]  apparent activation energy or heat energy
\[ q \]  heat flux
\[ R \]  universal gas constant (8314 J/molK)
\[ S \]  stability index
\[ \mathbf{S} \]  global viscosity matrix
\[ s \]  element viscosity matrix
\[ s_n \]  \( N \)-Directional stress
\[ T \]  temperature
\[ t \]  time
\[ U \]  total internal energy
\[ \mathbf{u}_a \]  vector of interstitial air velocities
\[ \mathbf{u} \]  deformation vector
\[ V \]  volume
\[ \mathbf{V} \]  viscosity matrix
\[ \mathbf{V}_N \]  \( N \)-Directional viscosity matrix
\[ v \]  velocity vector
\[ W \]  work
\[ w_i \]  weight
\[ \mathbf{x} \]  coordinates of a point (2D or 3D)

Subscripts

\[ a \]  air
\[ c \]  critical
\[ e \]  element
\[ el \]  elastic
\[ g \]  gravity
\[ i \]  ice
\[ \text{N} \]  \( N \)-Directional
\[ p \]  peak
\[ r \]  residual
\[ v \]  viscous
  or vapour
\[ w \]  water
Greek

\( \delta_{ij} \) Kronecker’s delta
\( \epsilon \) strain
\( \dot{\epsilon} \) strain-rate
\( \epsilon_d \) delayed elastic strain
\( \dot{\epsilon}_d \) delayed elastic strain-rate
\( \epsilon_e \) elastic strain
\( \dot{\epsilon}_e \) elastic strain-rate
\( \epsilon_n \) directional strain
\( \dot{\epsilon}_n \) directional strain-rate
\( \epsilon_v \) viscous strain
\( \dot{\epsilon}_v \) viscous strain-rate
\( \varepsilon \) emissivity
\( \eta \) viscosity
\( \phi \) slope angle
\( \phi_f \) failure angle
\( \Gamma \) boundary of model domain (\( d\Omega \))
\( \nu \) Poisson’s number
\( \Omega \) model domain
\( \omega \) end zone length
\( \rho \) density of snow
\( \rho_i \) specific density of ice
\( \sigma \) stress or
Stefan-Boltzmann constant \( (5.67 \times 10^{-8} \text{W/m}^2\text{K}^4) \)
\( \sigma_{I,II,III} \) principal stresses
\( \tau \) shear strength
\( \tau_f \) failure shear strength
\( \tau_g \) shear stress due to gravity
\( \tau_r \) residual shear strength
\( \theta_a \) volumetric air content
\( \theta_i \) volumetric ice content
\( \theta_w \) volumetric water vapour content
\( \xi \) natural tetrahedron coordinates
Appendix B

N-Directional Approach

B.1 Evenly distributed directions

Three sets of evenly distributed directions for the N-Directional approach are given in the following:

For $N=6$ the evenly distributed directions $\vec{d}_n$ in Cartesian (x,y,z) coordinates are given by

$$\vec{d}_n = \begin{cases} (\cos(\theta) \cos(2n\frac{\pi}{5}), \cos(\theta) \sin(2n\frac{\pi}{5}), \sin(\theta)) & n = 1 \ldots 5 \\ (0, 0, 1) & n = 6 \end{cases}$$  \hspace{1cm} (B.1)

where

$$\theta = \arcsin\left(\frac{1}{2\sin^2\left(\frac{\pi}{5}\right)} - 1\right).$$  \hspace{1cm} (B.2)

They are found by joining two opposite vertices of an icosahedron enclosed by a unit sphere.

A set of $N=10$ evenly distributed directions is found by joining the middle points of two opposite faces. Using the vectors given in B.1 this set can be written as

$$\vec{d}_5 + \vec{d}_1 + \vec{d}_6, \quad \vec{d}_1 + \vec{d}_2 + \vec{d}_6,$$
$$\vec{d}_2 + \vec{d}_3 + \vec{d}_6, \quad \vec{d}_3 + \vec{d}_4 + \vec{d}_6,$$
$$\vec{d}_4 + \vec{d}_5 + \vec{d}_6, \quad \vec{d}_5 + \vec{d}_1 - \vec{d}_3,$$
$$\vec{d}_1 + \vec{d}_2 - \vec{d}_4, \quad \vec{d}_2 + \vec{d}_3 - \vec{d}_5,$$
$$\vec{d}_3 + \vec{d}_4 - \vec{d}_1, \quad \vec{d}_4 + \vec{d}_5 - \vec{d}_2.$$  \hspace{1cm} (B.3)

A set of $N=15$ evenly distributed directions is found by joining the middle points of two
opposite sides. Using the vectors given in B.1 this set can be written as

\[
\begin{align*}
\vec{d}_1 + \vec{d}_6, & \quad \vec{d}_2 + \vec{d}_6, & \quad \vec{d}_3 + \vec{d}_6, \\
\vec{d}_4 + \vec{d}_6, & \quad \vec{d}_5 + \vec{d}_6, & \quad \vec{d}_5 + \vec{d}_1, \\
\vec{d}_1 + \vec{d}_2, & \quad \vec{d}_2 + \vec{d}_3, & \quad \vec{d}_3 + \vec{d}_4, \\
\vec{d}_4 + \vec{d}_5, & \quad \vec{d}_5 - \vec{d}_3, & \quad \vec{d}_1 - \vec{d}_4, \\
\vec{d}_2 - \vec{d}_5, & \quad \vec{d}_3 - \vec{d}_1, & \quad \vec{d}_4 - \vec{d}_2.
\end{align*}
\] (B.4)
MODELLING SNOW SLAB RELEASE USING A
TEMPERATURE-DEPENDENT VISCOELASTIC FINITE ELEMENT
MODEL WITH WEAK LAYERS

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Abstract. A two-dimensional thermo-mechanical plane-strain finite element model for snow is presented. Snow is modeled as a two component porous medium consisting of a solid ice matrix and interstitial pore air. The ice and air phases are not always in thermal equilibrium. Therefore, heat transport is governed by two non-stationary energy conservation equations which are coupled by free convection heat exchanges at the interfacial ice-air boundary. The ice matrix deforms viscoelastically according to an experimentally-based temperature dependent constitutive law. Creep deformation rates are governed by a power law with a density dependent exponent \( n \). The highly nonlinear character of the mechanical model is illustrated by simulating snowcovers with layers of variable height and density. Weak layer interfaces – believed to be the location of initiation of snow slab fracture – are modeled using special finite elements which transfer normal stresses but have little or no shear resistance. Stress and strain-rate concentrations at the boundaries of weak zones are calculated and compared with brittle fracture strain-rates.

Keywords: avalanche formation, porous continua, snow, viscoelasticity, weak layer

1. Introduction

The mechanics of snow slab release have been investigated by Bader et al. (1989) and Bader and Salm (1990). In these works, two-dimensional model calculations were performed to predict stress and strain-rate concentrations along super-weak interfaces, assumed to be located between two homogeneous snow layers. The snow layers were modelled as continua of constant mechanical properties. The steady-state creep response of the snowcover under self-weight was predicted. That is, creep velocities and strain-rates for snowcovers near fracture were determined. Transient effects, from temperature variations, or non-linear material effects, arising from the densification of the snow layers, were not included in the analysis. In addition, the weak interfaces had known length, thickness and mechanical properties.

The primary conclusion derived from the model calculations was that slab avalanches cannot be formed without super-weak layers. The self-weight of the
snowpack alone is not enough to generate the strain or deformation rates required to fracture the homogeneous layers. They concluded that weak layers are a necessary condition for avalanche release. The idea of “super-weak” zones has taken hold in the popular avalanche literature (Munter, 1997). The result is also supported by field observations of avalanche release areas where the basal sliding surface of the slab is usually clearly defined.

The model calculations of Bader and Salm (1990), however, were based on several questionable assumptions. They assumed that the snowcover consisted of two high density snow layers, $\rho > 400 \text{ kgm}^{-3}$. The viscosity of the snow layers was based on the triaxial tests of Salm (1971), $\eta = 5 \times 10^{10} \text{ Pas}$. Both the high values of density and viscosity are particularly advantageous for avalanche release: the high density increases the self-weight of the upper layer, and the high viscosity increases the stiffness of the upper layer. Thus, the applied load causes little deformation and transfers the stress directly to the weak layer. Zones of high strain-rate concentrations $\dot{\varepsilon} \approx 10^{-3} \text{ s}^{-1}$ at the ends of the layer are readily formed. New snow densities and viscosities of the order $\rho \approx 100 \text{ kgm}^{-3}$ and viscosities near $\eta \approx 1 \times 10^9 \text{ Pas}$ would be more appropriate (Bartelt and von Moos, 2000).

The goal of this work is to advance the initial investigations of Bader and Salm (1990). Although this earlier work was incomplete and based on unrealistic assumptions, it was a first attempt to relate the constitutive properties of snow to snowcover stability. It did propose a consistent theory taking into account the viscoelastic properties of snow and the layered structure of the snowpack.

Several recent developments make advances possible. First, the temperature dependent viscoelastic properties of snow have now been quantified over a wide range of densities (von Moos et al., 2003). The newly formulated viscosity laws (Scapozza and Bartelt, 2003a) vary significantly from the constant viscosity assumed by Bader and Salm (1990), but are only valid for fine grained snow of rounded grains. Secondly, numerical methods are now available to treat layer discontinuities. Although the constitutive laws for weak layers are presently unknown, layers of little or no shear strength can be assumed. This approach differs from the earlier investigations, which treated the weak layers as a thin isotropic continuum layer. Finally, the temperature dependence of snow (Scapozza and Bartelt, 2003b), in the form of an apparent activation energy, $Q$, has been determined. The influence of temperature variations on the deformation rates can be predicted.

The paper proceeds as follows. In the next section, the viscoelastic properties of snow are reviewed. Then, the thermal non-equilibrium treatment of snow is discussed. Non-equilibrium refers to the fact that the temperature in the pore space and ice lattice may differ. Heat transfer is governed by the thermal conductivities of each phase and natural convection between the air and ice. Two non-steady energy conservation equations are required. Afterwards the fully implicit thermo-mechanical finite element model with weak interfaces and periodic boundary conditions is formulated. Deformation rates of snowpacks on slopes of constant angle, different weak layer length and variable temperature are then calculated.
The present work attempts to resolve the question whether weak layers (or other perturbations) are a sufficient condition to achieve brittle fracture deformation rates. This does not mean that weak layers are the unique condition for an avalanche to occur. The concomitant breakage of a set of intergranular bonds, for example, can also be a proper mechanism for an avalanche release.

2. The Mechanical Deformation of Snow

It has been shown that, under applied axial loading, snow will deform viscoelastically (Voytkovskiy, 1977). The total axial strain \( \epsilon \) is the sum of the elastic and viscous parts:

\[
\epsilon = \epsilon_e + \epsilon_v. \tag{1}
\]

In general, the elastic strains are small in comparison to the total viscous strains, \( \epsilon_e \ll \epsilon_v \), especially over the course of several days, or weeks. The total volumetric strains of new snow can be over 200% within a few days, depending on temperature. Almost all the strain is viscous and irreversible. However, instantaneous irreversible strains, for example from an external loading like a skier, are not taken into account by this model.

Triaxial tests on snow (von Moos et al., 2003) reveal that the elastic strain can be further divided into two parts, an initial elastic and time-dependent and reversible strain (inelastic strain):

\[
\epsilon_e = \epsilon_0 + \epsilon_a. \tag{2}
\]

Young’s modulus \( E \) relates the state of stress to strain \( \epsilon_0 \); values for \( E \) are provided in Figure 1 as a function of density. It has been found that \( E \) is for the most part strain-rate and temperature independent. Lower density snow, however, exhibits significant inelastic straining in comparison with higher density snow (Scapozza and Bartelt, 2003a).

For the plane-strain problems studied in the following the elasticity matrix, \( E \) is given by:

\[
E = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix}
1 & \frac{\nu}{1-\nu} & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-2\nu}{2(1-\nu)}
\end{bmatrix} \tag{3}
\]

where \( \nu \) is Poisson’s number. As usual, the elasticity matrix relates the plane-strain components \( \epsilon = (\epsilon_x, \epsilon_y, \gamma_{xy})^T \) to the stress components \( \sigma = (\sigma_x, \sigma_y, \tau_{xy})^T \). Poisson’s number is set to zero since triaxial tests show that for snow the deformation
Figure 1. Young’s modulus for snow based on the experiments of von Moos et al. (2003). Values are in good agreement with $E$ values found in Mellor (1974) or Voytkovskiy (1977).

directions are independent of each other until a critical strain is reached (Scapozza and Bartelt, 2003b).

Viscous straining in the granular ice matrix is due to different deformation mechanisms occurring in the ice crystals or ice crystal boundaries (Scapozza and Bartelt, 2003a). For the natural strain-rates and stress levels considered (no skier or other artificial loadings), when extensive bond breakage does not occur, the two rate controlling creep mechanisms are evidently dislocation creep and grain boundary sliding (Petrenko and Whitworth, 1999). Hence, as for polycrystalline ice, the viscous deformation of snow can be modelled with a power law relation:

$$\dot{\varepsilon}_v = A_0 e^{-\frac{Q}{RT}} \sigma_y^n = A \sigma_y^n$$

where $\dot{\varepsilon}_v$ is the viscous strain-rate in s$^{-1}$, $A_0$ is a density dependent material parameter (in kPa·s$^{-1}$), $Q$ is the apparent activation energy (in kJmol$^{-1}$), $R$ is the gas constant (kJmol$^{-1}$·K$^{-1}$), $T$ is the temperature of ice (K), $\sigma_y$ is the yield stress (in Pa) and $n$ is a dimensionless exponent. This relationship is found from extensive triaxial testing for different densities, strain-rates and temperatures (Scapozza and Bartelt, 2003b).

Unlike ice, however, the model parameters $Q$ and $n$ vary with snow density (Figure 2). At densities of $\rho \approx 400$ kgm$^{-3}$ parameters similar to those for polycrystalline ice can be used in the material model, say $n = 3.0$ and $A = 10^{-25}$ (Pa$^{-n}$s$^{-1}$).
MODELLING SNOW SLAB RELEASE

Figure 2. Left: Temperature and density dependence of the exponent $n$ for strain-rates ranging between $\dot{\varepsilon} = 1.1 \times 10^{-6}$ s$^{-1}$ and $\dot{\varepsilon} = 4.4 \times 10^{-5}$ s$^{-1}$. From Scapozza and Bartelt (2003b). Right: Temperature and density dependence of the apparent activation energy $Q$ for different strain-rates.

For the numerical finite element implementation, the viscosity $\eta$ must be defined:

$$\sigma = \eta \dot{\varepsilon}_v.$$  \hspace{1cm} (5)

Thus, for the power law formulation, the viscosity of snow is:

$$\eta = \frac{1}{A \sigma_{\text{II}}^n},$$  \hspace{1cm} (6)

where $\sigma_{\text{II}}$ is the second stress invariant defined as $\sqrt{\frac{1}{2}(\sigma_x^2 + \sigma_y^2) + \tau_{xy}^2}$.

Since the parameters $n$ and $A$ are density, temperature and strain-rate dependent, the viscosity is highly nonlinear, requiring a numerical solution for the problems of interest. Finally, the viscosity matrix $V$ is given by:

$$V = \begin{bmatrix}
\eta \left(1 + \frac{1}{1+m}\right) & \eta \left(1 + \frac{1}{1+m}\right) & 0 \\
\eta \frac{m-2}{m-2} & \eta \frac{m-2}{m-2} & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix}. \hspace{1cm} (7)
$$

This matrix relates viscous strain-rate components $\dot{\varepsilon}_v = (\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\gamma}_{xy})^T$ to the stress components $\sigma = (\sigma_x, \sigma_y, \tau_{xy})^T$. The parameter $m$ is the viscous analogue to Poisson’s number, which we set to a large number, uncoupling the $x$ and $y$ directions. This relation was also used by Bader and Salm (1990) with $m$ values of 5.
3. Non-equilibrium Heat Transfer

The temperature state in the two-dimensional snowcover (coordinates $x$ and $y$) is governed by two non-stationary heat energy conservation equations:

$$
\theta_a \rho_a c_a \left( \frac{\partial T_a}{\partial t} + \mathbf{u}_a \cdot \nabla T_a \right) - \theta_a k_a \nabla^2 T_a = h_a(T_i - T_a)
$$

(8)

$$
\theta_i \rho_i c_i \frac{\partial T_i}{\partial t} - k_i \nabla^2 T_i = h_a(T_a - T_i) + L \dot{m}_{a\rightarrow i}.
$$

(9)

The first equation governs the heat transfer in the interstitial air space (temperature $T_a$, volumetric content $\theta_a$). It includes an advective term, where $\mathbf{u}_a$ is the vector of interstitial air velocities, which is assumed to be in the creeping flow regime, $Re < 10$. The second equation governs the heat transfer in the ice matrix (temperature $T_i$, volumetric content $\theta_i$). The thermal conductivities in the air and ice phases are given by $k_a$ and $k_i$, respectively. For $k_a$ the conductivity of air is used, $k_a = 0.026 \text{ Wm}^{-1} \text{ K}^{-1}$. For $k_i$ the microstructural conductivity model of Adams and Sato (1993) is employed, which relates grain and bond size to conductivity. The specific heats of the air and ice phases are $c_a = 1000 \text{ J kg}^{-1} \text{ K}^{-1}$ and $c_i = 2100 \text{ J kg}^{-1} \text{ K}^{-1}$. Note that the thermal diffusivities of air and ice differ ($\alpha_a = 2.6 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ and $\alpha_i = 1.1 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$) by a factor of 20. Thus, an important mode of heat transfer is the natural convection occurring at the interfacial air-ice boundary, which is a function of the temperature difference between the two phases and the convective heat transfer coefficient, $h_a$ (Kaviany, 1995). Sublimation and deposition of water vapour, (mass rate, $\dot{m}_{a\rightarrow i}$) from or to the ice lattice can also be accounted for; however, the influence of these effects is not included in the present analysis.

Boundary conditions at the lower ($y=0$) and upper surfaces of the snowpack ($y=h$) are specified:

$$
T_i(x, 0, t) = T_a(x, 0, t) = T_0(t)
$$

(10)

and

$$
T_i(x, h, t) = T_a(x, h, t) = T_h(t).
$$

(11)

It is also possible to prescribe only the air temperature $T_a$ and let the convective heat exchange between the ice and air determine the ice temperature $T_i$. 

4. Finite Element Model

A two-dimensional finite element method is employed to solve the two non-stationary heat transfer equations. A fully implicit time-integration scheme is employed; simple four node quadrilateral elements are used for the spatial discretization. This is a standard technique; for more information see Reddy and Garling (1994). This procedure provides the ice and air temperature for all times and points of interest, $T_i(x, y, t)$ and $T_a(x, y, t)$, within the model domain. Note that the thermal conductivities are a function of the volumetric ice and air contents and thus are changing with mechanical deformation. Changes in snow microstructure, which would also influence the thermal conductivity, are not considered since the calculation times are on the order of 24 to 48 hours (not weeks or months).

The viscoelastic solution requires more attention. The governing differential equation for the mechanical deformation is:

$$\nabla \cdot \sigma(T_i, u_i) - f = 0$$  \hspace{1cm} (12)

where $f$ are the applied body forces; in our case, this is the self-weight of the snowpack. The vector $u_i$ represents the deformation of the ice-matrix. The finite element discretization of this equation leads to a system of algebraic equations, expressed in incremental form as:

$$K_t \Delta u = \Delta f_g + \Delta f_v$$  \hspace{1cm} (13)

where $K_t$ is the tangent stiffness matrix, $\Delta u$ is the increment in displacement, $\Delta f_g$ is the increment in body forces (self weight) and $\Delta f_v$ is the increment of creep “forces” from time $t_0$ to $t_1$. The nodal vector $\Delta f_v$ is found by evaluating the element integral:

$$\Delta f_v^e = \int_A B E \dot{\varepsilon}_v \Delta t dA$$  \hspace{1cm} (14)

for all elements. ($B$ is the finite element matrix relating nodal displacements to strains and $A$ is the element area.) The viscous creep strains are found from Equation (4) at each time $t$. Note that since the temperature and density change over time, the vector $\Delta f_v$ must be recalculated at each time step.

An initial static solution is calculated to start the time integration scheme which is a simple explicit Euler scheme. The time step, however, is adaptive using a maximum strain-change criterion in deciding the time step. The same finite element discretization is used for the mechanical solution as for the heat transfer calculation.

At the beginning of the finite element solution, the temperature distribution in the snowcover is known. It is further assumed that the air and ice phases are in thermal equilibrium,

$$T_a(x, y, 0) = T_i(x, y, 0) = T_0(x, y).$$  \hspace{1cm} (15)
The initial stress state \( \sigma(x, y, 0) \) is determined by finding the elastic solution.

After the initialization phase, the time integration scheme is started. At every time \( t \) the temperature distribution in the snowcover is found, for both the ice and air phases. Once the temperature in the ice phase is known, the creep displacements are calculated. This procedure is repeated until the desired calculation time is reached.

5. Weak Interface

Based on the work of Beer (1985) and Day and Potts (1994), the standard isoparametric element formulation was modified to model a zero thickness weak interface. The forces orthogonal to the weak layer interface in the local \( n \)-direction are transported directly to the surrounding layers. The transfer of shear forces in the local \( s \)-direction is governed by different laws such that relative shear motions can occur between the adjacent elements. In Figure 3 a weak interface element is shown. The pseudo elasticity matrix \( E_{\text{weak}} \) for a weak element is given by:

\[
E_{\text{weak}} = \begin{bmatrix}
E_s & 0 \\
0 & E_n
\end{bmatrix}
\]  

(16)

where \( E_s \) is set to 0, such that no resistance in the \( x \)-direction is present \((u_{s-top} \neq u_{s-bottom})\). \( E_n \) is set to a large value, so that no relative displacement in \( n \)-direction can exist \((v_{n-top} - v_{n-bottom} = 0)\), this guarantees, that the element’s height remains zero. Since the weak interface element has no volume, there are no volumetric effects (dilatancy) in the \( E_{\text{weak}} \) matrix. Although non-zero values for \( E_s \) are theoretically possible, there is little experimental data available to justify a particular value; for this reason \( E_s = 0 \) was chosen.

6. Examples

In the first example, the influence of weak layer length \( a \) on the increase of maximum second strain-rate invariants is investigated. The snow density \( \rho_1 \) and temperature \( T_1 \) of the snow layer above the weak layer was also varied. Figure 4 depicts the model domain, a two layer snow pack on a \( \phi = 35^\circ \) slope. The periodic boundary conditions were introduced to model an infinitely long slope. The total length of the snowcover was chosen to be 50 m, much longer than the maximum weak layer length of \( a = 20 \) m.

In the first simulation the density of the top layer varied between \( \rho_1 = 180 \) to 300 kgm\(^{-3}\) in steps of 20 kgm\(^{-3}\). The weak layer length was chosen to be \( a = 12 \) m. The maximum second strain-rate invariants as a function of density are presented in Table I.
Figure 3. Definition of the "weak layer" element showing the deformation directions.

Figure 4. The influence of a temperature rise is investigated on snowcover depicted above. Constant slope at $\phi = 35^\circ$ with a length of 50 m, $d_0 = d_1 = 1$ m, $\rho_0 = 350$ kgm$^{-3}$ and $T_0 = 0^\circ$C.
The results show that the calculated second strain-rate invariants approach the brittle fracture strain-rate \( \dot{\varepsilon}_b > 10^{-4} \text{ s}^{-1} \) as the density decreases.

In the next series of simulations, the weak layer length varied between \( a = 0 \text{ m} \) to 20 m. The density of the top layer remained constant \( \rho_1 = 180 \text{ kgm}^{-3} \) at a temperature of \( T_1 = -6 \text{°C} \). The maximum second strain-rate invariants \( \dot{\varepsilon}_{II} \) along the weak layer are depicted in Figure 5. The results show once more that the calculated strain-rates are greater than brittle fracture strain-rates for weak layer lengths \( a > 10 \text{ m} \).

In the last series of simulations for this example the temperature of the upper layer varied from \( T_1 = -12 \text{°C} \) to \( -2 \text{°C} \). For each temperature a time varying creep calculation was performed. The results of the different simulations are presented in Figure 6. Note that the strain-rates outside the weak layer differ, and the highest strain-rate occurs for \( T \approx -5 \text{°C} \). Thus, contrary to expectation, the strain-rate did not continuously increase with increasing temperature. In general the results are in agreement with the conclusions of Bader and Salm (1990); however, a more realistic snow cover, with experimentally based viscosities, was used in these simulations.

In order to investigate this phenomenon still further, in the next example problem, the influence of a varying temperature rise on the creep strain-rates was investigated. A temperature rise of \( \Delta T = 10 \) is applied to a 2 m high snowpack on a \( \phi = 35^\circ \) slope (to both the air and ice phases). The snowpack consists of two 1 m high homogeneous layers of \( \rho_1 = 350 \text{ kgm}^{-3} \) and \( \rho_2 > 180 \text{ kgm}^{-3} \). A \( a = 10 \text{ m} \) long weak layer exists between the two layers (see Figure 4). The temperature rise is applied over different time intervals, \( \Delta t = 2h, 6h \) and \( 10h \), starting after 6 hours with a constant temperature \( T_1 = -12 \text{°C} \).

Figure 7 shows the second strain-rate invariants at the upper end of the weak layer. The larger the temporal gradient in temperature, \( \dot{T}_1 \), the higher the maximum
strain-rate. Defining the brittle fracture strain-rates to be of the order $\dot{\varepsilon}_b > 10^4 \text{s}^{-1}$, the snow is approaching a brittle fracture regime.

7. Conclusions

Based on the simulation results, we draw the following conclusions concerning snow avalanche formation:

- The initial results of Bader et al. (1988) and Bader and Salm (1990) could be substantiated, however, with realistic snowcovers and mechanical properties. Without perturbations (weak layers, discontinuities, bumps) it is impossible to obtain brittle strain-rate concentrations of $\dot{\varepsilon}_b > 10^4 \text{s}^{-1}$. The larger the weak layer length the higher the deformation rates at the upper and lower ends of the weak layer are. Weak layers of $a > 8 \text{ m}$ are required for all investigated snow densities and snow temperatures in order to achieve the necessary brittle fracture rates.

- The magnitude of the strain-rate concentrations is strongly influenced by temperature and temperature development. For a given temperature rise, slower changes in temperature allow the snowcover to relax and deform, reducing the stress-level and, subsequently, the strain-rates. The highest strain-rates are not
obtained for the highest temperature, but rather for a temperature around $T \approx -5 \, ^{\circ}\text{C}$.

- The brittle tensile strain-rates are concentrated in a small zone at the upper end of the weak layer. The compressive strain-rates at the lower end are spread over a larger region. Thus, the simulation results show that the initial rupture most likely occurs in the tensile zone, as expected.

In future the thermo-mechanical finite element model presented in this work will be applied to investigate snow avalanche formation in greater detail. For example the conditions for snow fracture as well as snow glide at the ground interface will be introduced into the model.

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Figure 7. Second strain rate invariant $\dot{\varepsilon}_{11}$ at the upper end of the weak layer. The shorter the time during which the 10°C temperature change happens, the higher the peak strain-rates are.

References


Bibliography


Bibliography


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All models are wrong some models are useful
by George Box

Martin Stoffel
Zürich, May 2005