Branes in pp-wave backgrounds

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Abstract

We present a detailed discussion of branes with boundary structure in Ramond-Ramond pp-wave backgrounds of type II B superstring theory. Background configurations with nontrivial (Ramond-Ramond) fluxes play a prominent rôle in the context of gauge-gravity dualities, but also in the search for phenomenologically feasible low-energy limits of string theory. Whereas most RR-backgrounds have no presently known tractable worldsheet descriptions, string theory on the particular pp-waves stands out, as it allows for an explicit two dimensional formulation in the light-cone gauge. In particular cases, the resulting gauge fixed field theories can be solved by available methodologies.

After reviewing some background material on the particular type of string solutions we are working with, the Maldacena-Maoz backgrounds, we consider the special spacetime leading to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model. Besides its appearance as string worldsheet theory, this model is interesting in its own right as an example of a (supersymmetric) integrable field theory. We study this two dimensional model in the presence of worldsheet boundaries and derive conditions under which the boundary theory retains the supersymmetry and integrability present in the bulk. We extend previous perturbative approaches and derive exact solutions valid to all orders in the bulk coupling constant. Our findings rely in particular on the introduction of boundary Lagrangians with additional fermionic boundary degrees of freedom.

In the next part we consider the plane wave background, that is, the particular Ramond-Ramond pp-wave solution with maximal spacetime supersymmetry. We construct supersymmetric branes with nontrivial boundary condensates. In contradistinction to flat space, a nonzero boundary condensate necessarily breaks some of the supersymmetries on certain branes in the plane wave background.

We present a detailed worldsheet description of the maximally supersymmetric configurations by using methods from open and closed string theory. As a nontrivial consistency check, cylinder diagrams are determined in both pictures and shown to be identical. The construction involves the definition of new mass and gauge-field dependent modular functions.

In the final part, branes in the plane wave background are constructed under the inclusion of boundary fermions. The relevant boundary fields are similar to those
appearing in the previously discussed $\mathcal{N} = 2$ sine-Gordon theory. We construct new integrable and supersymmetric boundary configurations and find a new maximally supersymmetric brane with only Neumann directions in transverse space. The branes with boundary excitations are again constructed by worldsheet techniques in the open and closed string pictures. By analogous methods as used in the context of boundary condensates, all configurations are furthermore shown to pass the open / closed duality check.


In einem nächsten Abschnitt betrachten wir den 'plane wave' - Hintergrund, d.h. die spezielle Ramond - Ramond 'pp-wave' - Lösung mit maximaler Raumzeitsuperasymmetrie. Wir konstruieren supersymmetrische Branen mit nicht-trivialen Randmagnetfeldern. Im Gegensatz zur Situation in einem flachen Hintergrund, existieren in diesem Fall Branen, deren Supersymmetrie zwangsläufig durch das Eichfeldkondensat gebrochen wird.


Im abschließenden dritten Abschnitt dieser Dissertation betrachten wir Branen im
‘plane-wave’ - Hintergrund mit zusätzlichen fermionischen Randanregungen. Die
dabei definierten Randfelder sind in direkter Analogie zu den Randstrukturen der
zuvor diskutierten sine-Gordon Theorie.
Wir konstruieren neue integrable und supersymmetrische Randkonfigurationen, de¬
er Grenzfall mit ausschließlich Neumann-Randbedingungen im transversalen Raum
eine neue maximal supersymmetrische Brane darstellt. Mit stets dualen Ergebnis¬
sen im offenen und geschlossenen Sektor werden die Branen wie zuvor im Welt¬
flächenzugang diskutiert.
Chapter 1

Introduction

The present day understanding of fundamental physics is based on the Standard Model quantum field theory and the description of gravity in General Relativity. Together, these theories account for all observed physical phenomena ranging from the smallest accessible length scales in high energy physics to the largest structures observed in cosmology.

Despite this impressive success in describing separate energy regimes, the theories have so far resisted with some success attempts to describe them conclusively in a unified manner. There are, however, situations in which both, quantum mechanical and gravitational effects are expected to play an important rôle. As examples we mention the physics of the early universe and the microscopic description of black holes.

String theory is widely considered to be the currently most promising candidate to eventually accomplish the reconciliation of quantum field theory and General Relativity. It is based on the idea that fundamental objects like quarks and leptons derive from extended string-like objects, rather than being point-like particles of conventional quantum field theory.

Internal consistency constraints require (critical) string theory to be defined on an at least ten dimensional spacetime. Any attempt within string theory to make contact with presently known physics therefore necessarily has to include a framework explaining the unobservability of higher dimensions on low energy scales. The most prominent attempts to this problem are known as ‘brane world scenarios’ and string compactifications. Brane worlds are based on the idea that low energy physics resides on a four dimensional extended object (a brane) moving in the ten dimensional background spacetime. The compactification schemes, on the other hand, assume the ten dimensional string spacetimes to be products of four dimensional macroscopically extended spaces with six dimensional compact (‘microscopic’) manifolds.

A framework of recent interest in the second context is known under the name of ‘flux compactifications’. Besides offering a solution to the absence of extra dimensions in low energy physics, it can furthermore be used to obtain a semi-realistic amount of conserved (super)-symmetry and fixed moduli in the resulting low-energy field theory.
Flux compactifications proceed by considering background configurations defined on product manifolds which contain in addition to the standard metric further non-trivial background fields. These additional fluxes are for example described by non-vanishing vacuum expectation values of Ramond-Ramond p-form field-strengths.

Apart from their appearance in string phenomenology, background fluxes also play an important rôle in the construction of gauge-gravity dualities. Starting with the remarkable conjecture of Maldacena in 1998, the AdS/CFT proposal has been one of the main focusses in the string theory community over the last few years. According to this conjecture, string theory defined on a product space containing an anti de Sitter (AdS) spacetime with Ramond-Ramond fluxes is dual to a superconformal field theory defined on a lower dimensional space. The CFT can be interpreted as living on the conformal boundary of the AdS bulk space.

Motivated by the relevance in the previously described contexts, a detailed understanding of string theory with its higher dimensional objects (D-branes) in Ramond-Ramond backgrounds is desirable. By technical obstacles, the knowledge about strings moving in such spacetimes nevertheless has been rather limited for a long time and available data were mostly valid in supergravity approximations, only. The so far most fruitful worldsheet approach to string theory is given by the Neveu-Schwarz-Ramond formalism, employing methods from two dimensional conformal field theory. Up to the present time, it is, however, unknown how to describe Ramond-Ramond backgrounds in this framework. One therefore has to rely on the alternative formulation by Green and Schwarz or on one of the so far less developed new formalisms by Berkovits, the pure spinor or the hybrid approach. We focus on the Green-Schwarz description in this thesis.

While the formulation of string theory in the GS-formalism allows for a non-quantum mechanical discussion of arbitrary RR- and even nontrivial fermionic background fields, it has its own drawbacks upon quantisation. Without fixing a particular gauge, some of the rather involved GS-equations of motion turn into Dirac constraints of first and second class in the quantum theory. It is fair to say that rather limited progress has been made in the covariant quantisation along these lines.

The problems related to the Dirac constraints can be circumvented by choosing a particular gauge, the so called lightcone gauge. It is analogous to the unitary gauge choice in standard quantum field theory and can only be adopted in case of spacetime fields with a particular symmetry structure. The relevant conditions determine the background spacetime to be a manifold of pp-wave structure. We present further details on general pp-wave backgrounds and the particular string solutions we are working with, the Maldacena-Maoz solutions, in the next section.

The worldsheet theories of Maldacena-Maoz solutions in the lightcone gauge are given by $\mathcal{N} = (2, 2)$ supersymmetric Landau-Ginzburg models, parametrised by an otherwise arbitrary holomorphic superpotential. In distinction to flat space, the resulting gauge fixed field theories are therefore generically massive and no longer (super)conformal. Despite having no conformal symmetry at our disposal, it is nevertheless possible to construct theories which display an integrable struc-
ture. This additional symmetry content is expressed by the appearance of infinitely many conserved higher spin currents with pairwise commuting charges. It offers the possibility to derive detailed information about the underlying (interacting) field theory. In particular, one can study branes in the corresponding Maldacena-Maoz backgrounds by using methods developed in treatments of integrable models in the presence of (worldsheet) boundaries.

In this thesis we consider branes in Maldacena-Maoz backgrounds with the $\mathcal{N} = 2$ sine-Gordon or a supersymmetrised Ising model as corresponding worldsheet theory. The first is a supersymmetric extension of the perhaps best understood interacting integrable field theory, the bosonic sine-Gordon model. The supersymmetrised Ising model, on the other hand, appears as worldsheet theory for the particular plane wave background. It is described by suitably combined free massive bosonic and fermionic fields.

We construct a boundary Lagrangian for the $\mathcal{N} = 2$ sine-Gordon model, ensuring the (exact) conservation of integrability and supersymmetry in the boundary theory. Our discussion extends a previous perturbative treatment in which only boundary contributions to first order in the bulk coupling constant were taken into account. The boundary Lagrangian contains additional fermionic boundary degrees of freedom. Structurally identical boundary functions appeared recently in the study of superconformal B-type branes with matrix factorisation techniques.

Subsequently, we present detailed results on maximally supersymmetric branes with boundary structure in the plane wave background. Because of its free worldsheet theory we can explicitly solve the corresponding boundary models in the open and closed sectors.

In a first step we consider branes with nonzero boundary magnetic fields. In contradistinction to flat space, a nonzero boundary condensate necessarily breaks some of the supersymmetries on certain branes. The case of fermionic boundary excitation is thereafter discussed along the lines of the previously studied sine-Gordon model. The fermionic boundary excitations give rise to new supersymmetric branes and as a main result, the limiting case of a brane with only Neumann directions in transverse space is seen to be maximally spacetime supersymmetric.

All boundary configurations in the plane wave background are studied with methods from open and closed string theory. In particular, the results pass in each case the open/closed duality check.

This thesis is based on the publications [90, 91, 92]. The discussion of the $\mathcal{N} = 2$ supersymmetric sine-Gordon model is taken from [91] and the results on branes in the plane wave background were originally published in [90, 92].

## 1.1 Maldacena-Maoz backgrounds

In this more technical part of our introduction we briefly review the so called Maldacena-Maoz solutions from [88]. As mentioned beforehand, they are particular
Ramond-Ramond pp-wave backgrounds of type II B superstring theory. Metrics of general pp-wave type are characterised by the presence of a **covariantly constant null Killing vector**. In suitably adapted coordinates, the metric of these spacetimes is of the form

\[
\begin{align*}
\text{d}s^2 &= 2\text{d}X^+\text{d}X^- + g_{++}(X^+,X^k)(\text{d}X^+)^2 + \\
&+ A_{++}(X^+,X^k)\text{d}X^+\text{d}X^i + g_{ij}(X^+,X^k)\text{d}X^i\text{d}X^j
\end{align*}
\]

(1.1)

and the null Killing vector corresponds to the shift symmetry

\[
X^- \rightarrow X^- + X^-_0
\]

(1.2)
in the lightcone coordinate $X^-$. Especially the more restricted **plane wave** metrics with

\[
A_{++} = 0; \quad g_{++}(X^+,X^k) = -\mu_{ij}(X^+)X^iX^j; \quad g_{ij} = \delta_{ij},
\]

(1.3)
in (1.1) have been discussed in the context of four dimensional general relativity for a long time, compare for example with [46, 79, 22]. Together with additional nonzero (Ramond-Ramond) background fields, pp-wave solutions made their first appearance in string theory at the end of the 1980s in the publications [6, 7, 67, 71, 108]. Their prominence in this context nevertheless stems from more recent developments, triggered by the discovery of the maximally supersymmetric plane wave solution in [25]. We will review this particular background in more detail in section 3.1, it nevertheless also reappears as a special case in the subsequent construction.

As mentioned in the last section, the pleasant feature of plane wave physics is the possibility to derive exact string theory data for non-trivial Ramond-Ramond backgrounds. However, in most cases only the situation of **constant** background fluxes is considered. An interesting possibility to go beyond this restrictive setting is offered by the particular solutions discovered by Maldacena and Maoz in [88], see also [93] for an extended discussion. The solutions of [88] are of the form

\[
\begin{align*}
\text{d}s^2 &= 2\text{d}X^+\text{d}X^- + H(X^I)(\text{d}X^+)^2 + \delta_{IJ}\text{d}X^I\text{d}X^J \\
F_5 &= \text{d}X^+ \wedge \varphi_4(X^I)
\end{align*}
\]

(1.4)

with a generically nontrivial four-form $\varphi_4$. As solutions to the pure supergravity equations of motion even more general curved transverse metrics are possible, for conciseness we, however, do not discuss this extension. For the flat transverse metric, the spacetimes with the field content (1.4) are not only supergravity, but also exact superstring solutions without further $\alpha'$-corrections [20].

Introducing complex coordinates

\[
z^j = X^j + iX^{j+4}; \quad \bar{z}^j = X^j - iX^{j+4}; \quad j = 1, \ldots, 4,
\]

(1.5)
the nontrivial fourform components can be written as

\[ \varphi_{mn} = \frac{1}{3!} \varphi_{mijk} \epsilon^{ijkn} \delta_m, \quad \varphi_{nm} = \frac{1}{3!} \varphi_{mijk} \epsilon^{ijkn} \delta_n, \quad \varphi_{im} = \frac{1}{2} \delta^{ss} \varphi_{imss}. \]  

(1.6)

By using this decomposition of \( \varphi_4 \), the \( \mathcal{N} = (2, 2) \) supersymmetric Maldacena-Maoz solutions are given by

\[ ds^2 = 2dX^+dX^- - \left( |\partial_k W|^2 + |\varphi_{jk} z^j|^2 \right) (dz^i) (dz^i) + dz^i d\bar{z}^i \]

\[ \varphi_{mn} = \partial_m \partial_n W, \quad \varphi_{mm} = \partial_m \partial_n \overline{W}, \quad \varphi_{im} = \text{const}, \]

(1.7)

parametrised by a holomorphic function \( W = W(z^1, z^2, z^3, z^4) \) and the additional constants \( \varphi_{im} \), forming the components of a hermitian matrix.

The supersymmetries of the background (1.7) are given by sections in the spinor bundle solving the Killing spinor equation

\[ 0 = D_r \epsilon = \left( \nabla_r + \frac{i}{2} \hat{F} \Gamma_r \right) \epsilon \]

(1.8)

with \( \hat{F} = \frac{1}{8!} (F_5)^{rstuv} \Gamma_{rstuv} \). Solutions of (1.8) split up into two groups, labelled by their respective \( SO(8) \)-chiralities. With

\[ \epsilon = -\frac{1}{2} \Gamma_+ \Gamma_- \epsilon - \frac{1}{2} \Gamma_- \Gamma_+ \epsilon = \epsilon_+ + \epsilon_- \]

(1.9)

from [88], the solutions of type \( \epsilon_+ \) are the so called dynamical, whereas the \( \epsilon_- \) spinors correspond to the kinematical supersymmetries.

The kinematical supersymmetries come from a fermionic shift symmetry in the lightcone-gauge Green-Schwarz formalism as for example described in [40]. As pointed out in [88], these symmetries correspond to free fermionic fields in the respective worldsheet theory. By this, there are in particular always 16 supersymmetries of this kind in pure plane wave backgrounds of type (1.3), but generically none in the more general spacetimes (1.1) (compare with [40]).

The dynamical supersymmetries represented by the sections \( \epsilon_+ \) are generically absent in pp-wave and even plane wave spacetimes. The presence of at least four solutions in case of (1.7) has to be established by a careful study of (1.8). This analysis is to be found in [88].

It is an important observation that worldsheet charges corresponding to dynamical Killing spinors anticommute to the light-cone gauge Hamiltonian, whereas the kinematical charges anticommute to the (constant) gauge parameter \( P^+ \). In particular, the kinematical charges do not commute with the light-cone gauge Hamiltonian and therefore constitute a somewhat non-standard supersymmetry structure. Throughout this thesis, our focus lies on the dynamical supersymmetries.

---

1The field equation / duality constraint on \( F_5 \) in type II B supergravity leads to an anti-selfdual fourform \( \varphi_4 \) in transverse space. By this, \( \varphi_4 \) has only nontrivial components of type (3,1), (1,3), (2,2) in the complex basis (1.5).
Before proceeding to discuss the worldsheet theory corresponding to the background 
(1.7), we want to mention some extensions and related results in the literature. The author of [75] presents solutions of type (1.4) with an additional nontrivial RR-threeform field. More general, but similar backgrounds with mixed NS-NS or RR fields also leading to solvable pp-wave string models can be found in [110, 111, 104, 28].

The worldsheet theories: Integrable models

A particularly attractive feature of the solution (1.7) is its very accessible worldsheet theory in the light-cone gauge Green-Schwarz formulation. As explained in [88, 93], in case of a vanishing matrix $\varphi_{mn}$ it is given by a $\mathcal{N} = (2, 2)$ supersymmetric Landau-Ginzburg model with superpotential $W$. The corresponding explicit (component) Lagrangian is provided in chapter 2 and further details on a superfield formulation can be found in the appendix A. Although we focus on the situation $\varphi_{mn} = 0$, it is worth mentioning that a nontrivial $(2, 2)$-form in (1.7) leads to further fermionic coupling terms in the worldsheet theory, compare with [88], but see also [75] for a more general setting.

Particularly approachable free worldsheet theories are obtained by choosing a quadratic superpotential $W$ in (1.7), that is, by considering string theory on plane wave backgrounds. For the choice

$$W(z) = -im \sum_{j=1}^{4} (z^j)^2; \quad \overline{W}(\overline{z}) = im \sum_{j=1}^{4} (\overline{z}^j)^2$$

one reobtains the maximally supersymmetric background from [25]. We will further discuss this case in chapters 3 and 4.

Only by relying on the preserved $\mathcal{N} = (2, 2)$ supersymmetry structure it is possible to derive a substantial amount of information about Landau-Ginzburg field theories with generic superpotentials. We refer to the book of Vafa and Zaslow [123] for an introduction and a review of the available literature. In particular cases, however, there appear additional symmetries in the two dimensional field theories beyond the standard supersymmetry. This opens up the possibility for a much more detailed understanding of the underlying model and its implications in string theory. The choice of the exponential

$$W(z) = ige^{i\beta z}$$

leads to the superconformal $\mathcal{N} = 2$ Liouville theory and the trigonometric superpotential

$$W(z) = -2ig \cos z$$

gives rise to the $\mathcal{N} = 2$ sine-Gordon theory. As the LG-model with superpotential

$$W(z) = \frac{1}{3!} z^3 - \lambda z$$

leads to the $\mathcal{N} = 2$ Liouville theory and the trigonometric superpotential

$$W(z) = -2ig \cos z$$

gives rise to the $\mathcal{N} = 2$ sine-Gordon theory. As the LG-model with superpotential
which corresponds to a deformation of a superconformal minimal model \cite{48}, the sine-Gordon theory is an example of an \textit{integrable} field theory.

There exists a large literature on various aspects of (supersymmetric) integrable models in two-dimensions. Our focus lies on the construction of symmetry preserving boundary conditions by following the seminal paper \cite{54}.

Besides this application in the context of branes within string theory, we want to mention a different interesting opportunity offered by the large amount of symmetries in integrable models. Via the well known \textit{thermodynamic Bethe-Ansatz} it is possible to derive detailed information on the spectrum of the underlying interacting field theory. For details on this approach in the context of supersymmetric theories see \cite{81, 97, 17, 45, 44}. Available Bethe-Ansatz data for the particular perturbed minimal model corresponding to (1.13) were for example compared in \cite{122} to the analogous information from a supergravity based calculation.
Chapter 2

Integrability of the $\mathcal{N} = 2$
boundary sine-Gordon model

2.1 Boundary fermions: Introduction

In this chapter we consider boundary Landau-Ginzburg models in the presence of fermionic boundary excitations. In string theory, these field theories appear in the context of brane physics in Maldacena-Maoz backgrounds as introduced in the last chapter. After discussing supersymmetry preserving boundary configurations valid for arbitrary superpotentials, we confine our treatment to the special case of the $\mathcal{N} = 2$ supersymmetric sine-Gordon model. We study in detail the conservation of its integrable structure in the presence of boundaries.

Boundary fermionic fields have been used in the study of boundary phenomena in a number of different field theoretical settings over the last few years. Starting with the work of Ghoshal and Zamolodchikov in [54], they became first of all an indispensible tool for the construction of boundary integrable models. This chapter has its main focus on developments in this context.

Building on the work of Warner in [125], boundary fermions also recently appeared in the study of superconformal models within string theory, see in particular [73, 33] for initial references. In these papers the authors realise a suggestion by Kontsevich to characterise B-type branes in particular Landau-Ginzburg models related to superconformal field theories by using matrix factorisations.

As a final example for an application of boundary fermions we want to mention the papers [21, 68] aiming at constructions of nonabelian extensions of Dirac-Born-Infeld D-brane descriptions. In this context the boundary fields are understood as a physical realisation of Chan-Paton factors, compare for example with the early discussion in [89].

In this introduction we briefly review boundary fermions as appearing in integrable boundary field theories by considering the prominent examples of the Ising model and the supersymmetric extensions of the sine-Gordon theory.

As it is well known, the introduction of spatial boundaries with corresponding
boundary conditions in local quantum field theories generically breaks bulk spacetime symmetries. Correspondingly, most conserved local bulk quantities acquire a time dependence in the boundary theory. It is therefore a natural question to ask, whether there are particular boundary configurations which retain at least a certain fraction of (‘bulk’) symmetries. The corresponding conserved quantities in the boundary theory are then obtained as suitable, mostly linear, combinations of bulk charges.

As a simple example displaying these features, we want to mention the Ising model in two dimensions. It is defined as a free Majorana fermion field theory with bulk Lagrangian

\[ \mathcal{L}_{\text{bulk}} = \psi \partial_x \psi - \overline{\psi} \partial_x \overline{\psi} + m \psi \overline{\psi}. \]  

The Lagrangian (2.1) is presented in the conventions of [54], to be used only in this introduction. As for instance discussed in [127], the Ising model is an example of an integrable field theory. Its integrable structure is expressed by the presence of an infinite set of conserved local currents, leading to time-independent, pairwise commuting higher spin charges. The relevant currents can be found in [127], but see also the appendix E.

Without additional boundary structure there are two sets of boundary conditions compatible with integrability for the Ising model. With the spatial boundary placed at \( z = \overline{z} \), the corresponding boundary conditions are given by

\[ 0 = \left( \psi + \overline{\psi} \right)_{z = \overline{z}}, \]
\[ 0 = \left( \psi - \overline{\psi} \right)_{z = \overline{z}}. \]  

In string theory these conditions are the standard Dirichlet respectively Neumann boundary conditions. More general choices which furthermore interpolate between the previous two cases can be obtained by the introduction of boundary fermions. A suitable boundary Lagrangian reads [54]

\[ \mathcal{L}_{\text{boundary}} = \frac{1}{2} \left( (\psi \overline{\psi})_{z = \overline{z}} + \lambda \tilde{a} \right) + \frac{1}{2} h \left( \psi + \overline{\psi} \right)_{z = \overline{z}} a, \]

where the field \( a = a(t) \) describes the fermionic boundary excitations and the number \( t = \frac{\xi + \bar{\xi}}{2} \) parametrises the boundary at \( z = \overline{z} \). With (2.1) one obtains from (2.3) the boundary conditions

\[ 0 = i \frac{d}{dt} (\psi - \overline{\psi}) - \frac{h^2}{2} (\psi + \overline{\psi}) \bigg|_{z = \overline{z}}, \]

containing \( h \) as a freely adjustable parameter. Likewise (2.2), these new conditions are compatible with integrability, compare with [54]. As the Ising model appears as the fermionic part of the maximally supersymmetric plane wave’s worldsheet theory, the boundary Lagrangian (2.3) will reappear in different conventions in chapter 4 in our construction of branes with nontrivial boundary structure.
As a second example for the use of boundary Lagrangians we consider the sine-Gordon model described by the bulk Lagrangian

$$L_{\text{sine-Gordon}} = \partial_z \varphi \partial_z \varphi - \frac{m^2}{\beta^2} \cos(\beta \varphi)$$

(2.5)

for a single scalar field \( \varphi \). It is probably the best understood interacting integrable model. Following [54], the integrability persists again in the presence of boundaries, this time by using the boundary Lagrangian

$$L_{\text{sg-bound}} = -\alpha \cos \left( \frac{\beta}{2} (\varphi - \varphi_0) \right)$$

(2.6)

Contrary to (2.3), it contains up to additional constants only the bulk field restricted to the boundary and no additional dynamical (boundary) fields. For further details on this boundary Lagrangian see for example [8].

Soon after the introduction of boundary Lagrangians as (2.3) and (2.6) in the context of integrable field theories, Warner initiated in [125] their application in the context of two dimensional supersymmetric theories. The combined situation of supersymmetry and integrability for extensions of the sine-Gordon model was thereafter considered by Nepomechie in [98] for the \( \mathcal{N} = 1 \) case and in [99] for the situation of \( \mathcal{N} = 2 \) supersymmetry. A nice review of this development can be found in [101].

The \( \mathcal{N} = 1 \) supersymmetric extension of the sine-Gordon model is described by the bulk Lagrangian [43, 69]

$$L_{\text{sg,} \mathcal{N}=1} = \partial_z \varphi \partial_z \varphi - \bar{\psi} \partial_z \bar{\psi} + \psi \partial_z \bar{\psi} - 2 \cos \varphi - 2 \psi \bar{\psi} \cos \frac{\varphi}{2}$$

(2.7)

and the conserved currents corresponding to the integrable structure were first presented in [49]. The conservation of supersymmetry and integrability in the presence of boundaries is ensured by a boundary Lagrangian containing suitably combined elements of (2.3) and (2.6). It is given by [98]

$$L_{\mathcal{N}=1 \text{sg-bound}} = \pm \bar{\psi} \psi + i a \bar{\psi} \bar{\psi} - 2 f(\varphi) a \left( \psi \bar{\psi} \right) \right|_{z=\bar{z}}$$

(2.8)

with

$$B(\varphi) = 2 \alpha \cos \left( \frac{1}{2} (\varphi - \varphi_0) \right) ; \quad f(\varphi) = \frac{\sqrt{C}}{2} \sin \left( \frac{1}{4} (\varphi - D) \right)$$

(2.9)

The constants \( C, D \) are known functions of the free parameters \( \alpha, \varphi_0 \). For a discussion of the boundary S-matrix of this theory, see for example [12].

A natural extension of the previous setup is the \( \mathcal{N} = 2 \) supersymmetric version of the sine-Gordon model. The corresponding bulk theory was first discussed in [78] and is for our purposes most conveniently described as a Landau-Ginzburg model with superpotential

$$W(z) = -2ig \cos z$$

(2.10)
as already mentioned in the first chapter. In the presence of boundaries it has been studied by Nepomechie in [99] by defining a boundary Lagrangian analogous to his proposal for the $\mathcal{N} = 1$ case. In contradistinction to (2.8), the boundary Lagrangian for the $\mathcal{N} = 2$ case in [99] is only an approximation to first order in the bulk coupling constant $g$ and leads to exact results in the $g \to 0$ limit, only. In this limit, the boundary Lagrangian contains three real parameters.

The main goal of this chapter is to extend the discussion of [99] and to construct an exact boundary Lagrangian which preserves supersymmetry and integrability to all orders in the bulk coupling constant. The final result will contain up to phases and discrete choices only a single free boundary parameter. Additional parameters appearing in [99] are in our case fixed by constraints of higher order in the bulk coupling constant $g$. We will mainly focus on the (Lorentzian) $\mathcal{N} = 2$ sine-Gordon theory, the supersymmetry considerations in sections 2.2 and 2.3, however, are valid for general Landau-Ginzburg models with arbitrary superpotentials.

Different aspects of the $\mathcal{N} = 2$ supersymmetric sine-Gordon model in the presence of boundaries are discussed for example in [100, 15, 13, 16]. An analogous treatment of the related $\mathcal{N} = 2$ Liouville theory with superpotential as presented in (1.11) can furthermore be found in [2]. The authors of [15] especially propose an explanation for the appearance of fermionic boundary degrees of freedom by using a perturbative CFT approach.

It is worth pointing out that our choice of a boundary Lagrangian to be presented in detail in the next section is identical to the Lagrangians used by [73, 33] in realising Kontsevich's proposal to study topological B-type branes by matrix factorisations. This proposal was further studied in [74, 80, 61, 34, 55, 124, 31, 32, 65, 47], see also [66] for a review and further references. Due to requirements of superconformal symmetry, the treatment of these papers is confined to quasi-homogeneous worldsheet superpotentials.

One of our initial motivations to consider the present problem is the appearance of the sine-Gordon model as the worldsheet theory describing strings in a particular Maldacena-Maoz background from [88], compare with chapter 1. Branes in these general backgrounds without the inclusion of boundary fields have been studied in [62]. Our results can be interpreted as additional supersymmetric brane configurations.

Motivated by this possible application in string theory we define the boundary theories on a strip with topology $\mathbb{R} \times [0, \pi]$ instead of the half space as in [99]. This, however, has only notational significance and does not affect any conclusions. A possibly more serious change compared to [99] comes from our choice of a Lorentzian worldsheet signature instead of the Euclidean setting in [99]. The structure of the bosonic fields is almost unaffected by this. Deviating reality properties of the fermions, however, make a direct comparison of the fermionic sectors subtle and we do not attempt to relate them via a Wick rotation.

Although the indefinite worldsheet metric does not directly simplify the calculations concerning the integrability, its consequences in particular in the fermionic sector
make reality requirements more transparent than in [99]. This is especially helpful for studying the structure of the boundary potential $B(z, \bar{z})$ and the conserved supersymmetries.

This chapter is organised as follows. In the next section we will write down a suitable boundary Lagrangian including fermionic boundary excitations and derive the resulting boundary conditions. In the subsequent section 2.3, we deduce constraints under which the boundary theory is $\mathcal{N} = 2$ supersymmetric. The chapter closes in section 2.4 with a discussion of the integrable structure in the boundary theory of the sine-Gordon model. Further details such as the explicit component form of the higher spin conserved currents from [78, 99] are supplied in the appendices A and B.

## 2.2 Landau-Ginzburg models and boundary fermions

In this first part we consider general $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg models with flat target spaces and a vanishing holomorphic Killing vector term. On worldsheets without boundaries these theories are described by the (component) Lagrangian

$$L_{\text{bulk}} = \frac{1}{2} \bar{g}_{\bar{\alpha}\bar{\beta}} \left( \partial_{\bar{+}} \bar{z} \partial_z \bar{z} + \partial_{\bar{+}} \bar{z} \partial_{\bar{-}} \bar{z} + i \bar{\psi}_{\bar{+}} \partial_{\bar{-}} \psi_+ + i \bar{\psi}_{\bar{-}} \partial_{\bar{+}} \psi_- \right) - \frac{1}{2} \partial_{\bar{\alpha}} \partial_{\alpha} W(z) \psi_+ \psi_- - \frac{1}{2} \partial_{\bar{\alpha}} \partial_{\alpha} \bar{W}(\bar{z}) \bar{\psi}_- \bar{\psi}_+ - \frac{1}{4} g^{\beta\gamma} \partial_\beta \bar{W}(\bar{z}) \partial_\gamma \bar{W}(\bar{z}) \quad (2.11)$$

with $\partial_{\bar{\pm}} = \partial_{\bar{\alpha}} \pm \partial_{\alpha}$, compare for example with the extensive publications [64, 63, 123].

When defined on a manifold with boundaries, one might either directly enforce boundary conditions in addition to the equations of motion or add a suitable boundary Lagrangian as discussed in the introduction.

Following the approach of [125, 99], but see also [73, 33], we work with

$$L_{\text{boundary}}^{\alpha = \pi} = \frac{i}{2} \left( b \psi_- \bar{\psi}_+ - b^* \psi_+ \bar{\psi}_- \right) - \frac{i}{2} a \partial_{\tau} \bar{a} + B(z, \bar{z}) + \frac{i}{2} \left( \bar{F}'(\bar{z}) \bar{a} + G'(\bar{z}) a \right) \left( \psi_+ e^{i\beta} \bar{\psi}_- \right) + \frac{i}{2} \left( G'(z) \bar{a} + F'(z) a \right) \left( \psi_+ e^{-i\beta} \bar{\psi}_- \right) \quad (2.12)$$

as boundary Lagrangian in this chapter. An extension containing matrix valued boundary fields is discussed in chapter 4.

As we here restrict attention to superpotentials depending on a single holomorphic coordinate, the boundary Lagrangian (2.12) is written down containing only contributions along the $z = z^1$ direction at $\sigma = \pi$. It is furthermore chosen to be manifestly real and the constant $b$ is determined by consistency of the resulting (fermionic) boundary conditions to $b = e^{-i\beta}$, compare for example with [8].
2.2. THE LANDAU-GINZBURG MODELS AND BOUNDARY FERMIONS

2.2.1 The boundary conditions

Below, we determine the required boundary conditions by using the Lagrangians (2.11) and (2.12). They are obtained by the variation of $L_{\text{boundary}}$ together with boundary terms from partially integrated contributions in $\delta L_{\text{bulk}}$.

The bosonic part of the bulk contributions from (2.11) is given by

$$-g_{ab} \left( \delta z^i \partial_\sigma z^i + \delta \bar{z}^i \partial_\sigma \bar{z}^i \right) \bigg|_{\sigma = 0} = -2 \delta x^I \partial_\sigma x^I \bigg|_{\sigma = 0},$$

whereas the fermionic kinetic parts lead to

$$\frac{1}{2} g_{ab} \left( -i \bar{\psi}_+^i \delta \psi_+^i + i \bar{\psi}_-^i \psi_-^i + i \bar{\psi}_-^i \delta \psi_-^i - i \delta \bar{\psi}_+^i \psi_-^i \right) \bigg|_{\sigma = 0} = i \left( \psi_-^i \delta \psi_-^i - \psi_+^i \delta \psi_+^i \right) \bigg|_{\sigma = 0}. \tag{2.14}$$

Altogether, the boundary conditions for the $z = z^1$ direction at $\sigma = \pi$ are with these terms found to be

\begin{align*}
\partial_\sigma z &= \partial_z B(z, \bar{z}) + \frac{i}{2} \left( \bar{F}'(z) \bar{a} + \bar{G}'(z) a \right) \left( \psi_+ + e^{i\beta} \psi_- \right) \tag{2.15} \\
\partial_\sigma a &= \frac{1}{2} \bar{F}'(z) \left( \psi_+ + e^{i\beta} \psi_- \right) + \frac{1}{2} \bar{G}'(z) \left( \psi_+ + e^{-i\beta} \psi_- \right) \tag{2.16} \\
\psi_+ - e^{-i\beta} \psi_- &= \frac{1}{2} \left( \bar{F}'(z) \bar{a} + \bar{G}'(z) a \right) \tag{2.17}
\end{align*}

together with the complex conjugates

\begin{align*}
\partial_\sigma \bar{z} &= \partial_{\bar{z}} B(z, \bar{z}) + \frac{i}{2} \left( G''(z) \bar{a} + F''(z) a \right) \left( \psi_+ + e^{-i\beta} \psi_- \right) \tag{2.18} \\
\partial_\sigma \bar{a} &= \frac{1}{2} G'(z) \left( \psi_+ + e^{i\beta} \psi_- \right) + \frac{1}{2} F'(z) \left( \psi_+ + e^{-i\beta} \psi_- \right) \tag{2.19} \\
\psi_- - e^{i\beta} \psi_+ &= G'(z) \bar{a} + F'(z) a. \tag{2.20}
\end{align*}

Setting

\begin{align*}
A(z) &= \bar{G}'(z) \bar{a} + F'(z) a \tag{2.21} \\
\bar{A}(z) &= \bar{F}'(z) \bar{a} + \bar{G}'(z) a \tag{2.22}
\end{align*}

and using the suitable fermionic combinations

\begin{align*}
\theta_+ &= \frac{1}{2} \left( \psi_+ + e^{-i\beta} \psi_- \right) & \bar{\theta}_+ &= \frac{1}{2} \left( \psi_+ + e^{i\beta} \psi_- \right) \\
\theta_- &= \frac{1}{2} \left( \psi_+ - e^{-i\beta} \psi_- \right) & \bar{\theta}_- &= \frac{1}{2} \left( \psi_+ - e^{i\beta} \psi_- \right) \tag{2.23}
\end{align*}

\begin{align*}
\psi_+ &= \theta_+ + \theta_- & \bar{\psi}_+ &= \bar{\theta}_+ + \bar{\theta}_- \\
\psi_- &= e^{i\beta} (\theta_+ - \theta_-) & \bar{\psi}_- &= e^{-i\beta} (\bar{\theta}_+ - \bar{\theta}_-) \tag{2.24}
\end{align*}

the boundary conditions finally turn out to be
\[
\begin{align*}
\partial_z z & = \partial_z B(z, \bar{z}) + i \bar{A}'(z) \bar{\theta}_+ \\
\partial_{\bar{z}} \bar{a} & = \bar{F}'(z) \bar{\theta}_+ + G'(z) \theta_+ \\
\theta_- & = \frac{1}{2} \bar{A}(z)
\end{align*}
\] (2.25)

and
\[
\begin{align*}
\partial_{\bar{z}} \bar{z} & = \partial_{\bar{z}} B(z, \bar{z}) + i A'(z) \theta_+ \\
\partial_{\bar{z}} \bar{a} & = G'' \bar{\theta}_+ + F'(z) \theta_+ \\
\bar{\theta}_- & = \frac{1}{2} A(z).
\end{align*}
\] (2.28)

By eliminating the fermionic boundary degrees of freedom in favour of \( \theta_-, \bar{\theta}_- \) in (2.25) and (2.28), one derives bosonic boundary conditions with a quadratic fermionic correction term. Boundary conditions with a comparable structure were for example studied in [83, 3] from a different point of view.

In the next section we discuss how the so far undetermined holomorphic functions \( F, G \) are related to the superpotential \( W \) in case of preserved B-type supersymmetries in the boundary theory.

### 2.3 Matrix factorisation and \( \mathcal{N} = 2 \) supersymmetry

The Landau-Ginzburg bulk theory of (2.11) has the four conserved supercurrents [63, 123]
\[
\begin{align*}
G^0_\pm & = g_{ij} \partial_\pm \bar{z}^i \psi_\pm^j + \frac{i}{2} \bar{\psi}_\pm^j \partial_j W \\
G^1_\pm & = \mp g_{ij} \partial_\pm \bar{z}^i \psi_\pm^j - \frac{i}{2} \bar{\psi}_\pm^j \partial_j W \\
\bar{G}^0_\pm & = g_{ij} \bar{\psi}_\pm^j \partial_\pm z^i + \frac{i}{2} \psi_\pm^j \partial_\mp W \\
\bar{G}^1_\pm & = \mp g_{ij} \bar{\psi}_\pm^j \partial_\pm z^i + \frac{i}{2} \psi_\pm^j \partial_\mp W,
\end{align*}
\] (2.31)

whose corresponding charges
\[
Q_\pm = \int_0^{2\pi} d\sigma \, G^0_\pm, \quad \bar{Q}_\pm = \int_0^{2\pi} d\sigma \, \bar{G}^0_\pm
\] (2.33)

represent the standard \( \mathcal{N} = (2, 2) \) bulk supersymmetry.

As usual, the introduction of boundaries breaks at least a certain number of bulk symmetries. As explained in [63], there are essentially two possibilities to preserve a \( \mathcal{N} = 2 \) supersymmetry algebra deducing from (2.31) and (2.32). Here we will concentrate on the so called B type case.
2.3. MATRIX FACTORIZATION AND $\mathcal{N} = 2$ SUPERSYMMETRY

Following [54, 63], the (B type) supersymmetries in the boundary theory take on the general form

$$Q = \bar{Q}_+ + e^{i\sigma}Q_- + \Sigma_\pi(\tau) - \Sigma_0(\tau)$$

$$Q^\dagger = Q_+ + e^{-i\sigma}Q_- + \bar{\Sigma}_\pi(\tau) - \bar{\Sigma}_0(\tau),$$

which includes generically nonzero (local) contributions of the boundary fields at $\sigma = \pi$ and $\sigma = 0$. Using the conservation of (2.31) and (2.32), the boundary supersymmetries $Q, Q^\dagger$ are time independent, that is, conserved, if and only if the fluxes fulfill the equations

$$0 = \left[ \bar{G}^1_+ + e^{i\sigma}\bar{G}^1_- \right]_{\sigma=\pi} - \dot{\Sigma}_\pi(\tau)$$

$$0 = \left[ \bar{G}^1_+ + e^{i\sigma}\bar{G}^1_- \right]_{\sigma=0} - \dot{\Sigma}_0(\tau)$$

(2.36) (2.37)

together with their corresponding complex conjugates.

The boundary field $\Sigma_\pi(\tau)$ ($\Sigma_0(\tau)$) is here required to depend only on the boundary degrees of freedom $a(\tau)$ and $\bar{a}(\tau)$ and the bulk fields and their time derivatives at time $\tau$, evaluated at $\sigma = \pi$ ($\sigma = 0$).

2.3.1 $W$ - factorisation

In a next step we evaluate the equation (2.36) to deduce explicit conditions for the boundary fields $F(z), G(z)$ and the boundary potential $B(z, \bar{z})$.

From (2.36) and (2.32) we obtain

$$\frac{d}{dz}\bar{G}(\tau) = \left[ \bar{G}^1_+ + e^{i\sigma}\bar{G}^1_- \right]_{\sigma= \pi} - \dot{\Sigma}_\pi(\tau)$$

(2.38)

evaluated at $\sigma = \pi$. Upon partial integration, (2.38) leads to

$$\partial_\tau \Sigma_\pi(\tau) = -\partial_z (\bar{\psi}_+ - e^{i\sigma}\bar{\psi}_-) - \partial_{\bar{z}} (\psi_+ + e^{i\sigma}\psi_-)$$

$$+ i/2 \partial_\tau W(z) (\psi_+ + e^{i\sigma}\psi_-)$$

(2.39)

from which we deduce

$$\partial_\tau \Sigma_\pi(\tau) = -\partial_\tau (2z\bar{\theta}_- + p(z)a - q(z)\bar{a})$$

$$+ (zG''(z) - q'(z)) \bar{\delta} + (p'(z)) \bar{a} + (zG''(z) - p'(z)) \bar{a}$$

$$+ (zG'(z) - q(z)) \dot{a} + (zF'(z) - p(z)) \dot{a}$$

$$- 2\bar{\theta}_+ \partial_\bar{z} B(z, \bar{z}) + i e^{i\sigma} \theta_+ \partial_\bar{z} W(z)$$

(2.40)
for arbitrary $p$ and $q$. Using
\[ q'(z) = zG''(z) \Rightarrow q(z) = zG'(z) - G(z) \] (2.41)
and
\[ p'(z) = zF''(z) \Rightarrow p(z) = zF'(z) - F(z) \] (2.42)
together with the equations of motion (2.26) and (2.29) for $a$ and $\bar{a}$, we finally arrive at
\[
\partial_\tau \Sigma_\pi(\tau) = -\partial_\tau \left( 2z\bar{\theta}_- - p(z)a - q(z)\bar{a} \right) \\
+ \theta_+ \left[ G(z)F'(z) + F(z)G'(z) + ie^{i\beta} \partial_\tau W(z) \right] \\
+ 2\bar{\theta}_+ \left( \frac{1}{2} G(z)\overline{G'(\bar{z})} + \frac{1}{2} F(z)\overline{F'(\bar{z})} - \partial_\tau B(z, \bar{z}) \right). \tag{2.43}
\]

The conditions for $\mathcal{N} = 2$ supersymmetry therefore read
\begin{align*}
W(z) & = \ i e^{-i\beta} F(z)G(z) + \text{const} \tag{2.44} \\
B(z, \bar{z}) & = \frac{1}{2} \left( F(z)\overline{F(\bar{z})} + G(z)\overline{G(\bar{z})} \right) + \text{const} \tag{2.45}
\end{align*}

and the local boundary field $\Sigma_\pi$ appearing in the ‘boundary adjusted’ supercharges (2.34) and (2.35) is given by
\[
\Sigma_\pi = -2z\bar{\theta}_- + (z\overline{F'} - F')a + (z\overline{G'} - G)a. \tag{2.46}
\]

It explicitly contains contributions from the fermionic boundary degrees of freedom, compare for example with the results in [15].

The requirement (2.44) is of course identical to the matrix factorisation condition from [73, 33], whereas (2.45) so far only determines the structure of the boundary potential $B(z, \bar{z})$. It does not lead to a condition on $F, G$ as the boundary potential remains functionally undetermined by the supersymmetry considerations.

In the context of matrix factorisations in string theory the focus is on quasi-homogeneous superpotentials leading to superconformal field theories in the infrared limit. The latter require a conserved $U(1)$ R-charge which should also be present in the boundary theory, compare for example with [65]. This additional condition requires factorisations of $W$ into quasi-homogeneous functions.

It is worth pointing out that there is no corresponding restriction on $F$ and $G$ in our case. Integrability together with supersymmetry in the context of the boundary Lagrangian (2.12) gives rise to particular trigonometric functions in the case of the sine-Gordon model, but supersymmetry on its own allows for more general choices. Extending our treatment, one might, following [15], add purely bosonic boundary degrees of freedom to (2.12). This opens up the possibility for more general choices of $F$ and $G$ even when enforcing supersymmetry and integrability. We will, however, not pursue this interesting idea in this thesis.
2.4 The $\mathcal{N} = 2$ boundary sine-Gordon model

From now on we specify the superpotential to

$$W(z) = -i\lambda \cos(\omega z)$$

and restrict attention therefore to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model\(^1\). When defined on a manifold without boundaries, this theory is well known to be a supersymmetric and integrable extension of the purely bosonic sine-Gordon theory [78]. Its first nontrivial conserved higher spin currents on whose conservation in the presence of a boundary we will concentrate in the following, were derived in [78, 99]. In our conventions they are given in the appendix A.

With the superpotential (2.47) in (2.11) we can immediately derive the bulk equations of motion. They are found to be

\begin{align*}
\partial_+ \partial_- z &= -ig \sin \bar{z} \bar{\psi}_- \psi_+ - g^2 \sin z \cos \bar{z} \\
\partial_+ \partial_- \bar{z} &= ig \sin z \psi_+ \psi_- - g^2 \cos z \sin \bar{z} \\
\partial_- \psi_+ &= g \cos \bar{z} \bar{\psi}_- \\
\partial_- \bar{\psi}_+ &= g \cos z \psi_- \\
\partial_+ \psi_- &= -g \cos \bar{z} \bar{\psi}_+ \\
\partial_+ \bar{\psi}_- &= -g \cos z \psi_+ , \tag{2.53}
\end{align*}

by setting w.l.o.g. $\omega = 1$. We furthermore redefined the bulk coupling constant to $g = \frac{\lambda}{2}$, resembling the choices in [99].

2.4.1 Integrability in the presence of boundaries

In this section we consider the $\mathcal{N} = 2$ sine-Gordon model in the presence of boundaries and derive conditions for which the following ‘energy-like’ combination

$$I_3 = \int_0^\pi d\sigma \left( T_4 + \bar{T}_4 - \theta_2 - \bar{\theta}_2 \right) - \Sigma^{(3)}_\sigma(\tau) + \Sigma^{(3)}_\sigma(\tau)$$

of the bulk conserved currents from the appendix A is conserved. The inclusion of local boundary currents as $\Sigma^{(3)}_\sigma(\tau)$ and $\Sigma^{(3)}_\sigma(\tau)$ goes back to [54]. Their appearance is by now a well known and frequently used feature in the context of integrable boundary field theories as briefly reviewed in section 2.1. It is in particular independent of the in our case present supersymmetries.

The conservation of a higher spin quantity as $I_3$ is usually regarded as providing strong evidence for the integrability of the underlying two dimensional (boundary) field theory. Our ‘proof’ of integrability is to be understood in this sense.

\(^1\)The phase accompanying the real coupling constant $\lambda$ is chosen for later convenience. Its form does not affect purely bosonic terms in the Lagrangian (2.11). In the fermionic parts of (2.11) it can be absorbed in a redefinition $\psi_\pm \rightarrow e^{i\alpha} \psi_\pm$, $\bar{\psi}_\pm \rightarrow e^{-i\alpha} \bar{\psi}_\pm$. 
As previously done in section 2.3 for the supercurrents (2.31), (2.32) in (2.34) and (2.35), the quantity $I_3$ is conserved iff the condition
\[ \partial_7 \Sigma_{\pi}^{(3)} = T_4 - \bar{T}_4 + \theta_2 - \bar{\theta}_2 \] (2.55)
holds at $\sigma = \pi$. In deriving (2.55) we have used the equations (A.31) and (A.32) from the appendix A. As before, there is an identical condition at $\sigma = 0$.

Due to the complexity of the conserved currents as presented in the appendix A, the calculation transforming the right hand side of (2.55) to a total time derivative is rather lengthy and intricate. It nevertheless follows a straightforward strategy which in our case differs slightly from the approach in [99].

In a first step, we use the equations of motion (2.48)-(2.53) and the bosonic boundary conditions (2.25) and (2.28) to remove all $\sigma$-derivatives on the bosonic and fermionic fields appearing in $T_4, \bar{T}_4, \theta_2, \bar{\theta}_2$.

In a second step, we remove (where possible) all time derivatives on the fermionic fields $\theta_+^+$ and $\bar{\theta}_+$ by partial integrations and apply the identities from appendix B to furthermore replace $\theta_-$ and $\bar{\theta}_-$ and their time derivatives by the fermionic boundary fields $\alpha$ and $\bar{\alpha}$.

In doing so, a large number of terms cancel manifestly. There are, however, other contributions as for example those proportional to combinations like $(\theta_+ \partial_+ \theta_+^+)$ or $(\theta_+ \bar{\theta}_+)$ which cannot be reduced further and which cannot be written as a time derivative of a local field. Their prefactors given by expressions containing the boundary potential $B(z, \bar{z})$ and the functions $F(z), G(z)$ and their derivatives therefore necessarily have to vanish.

Together with the conditions (2.45) and (2.44) for the $\mathcal{N} = 2$ supersymmetry, these resulting differential equations are shown to determine the boundary Lagrangian up to two possible choices for the boundary potential including a single free parameter and two additional (discrete) choices.

In the following we present the differential equations determined as explained above and write down their solutions. The explicit form of the boundary field $\Sigma_{\pi}^{(3)}$ appearing in (2.54) is provided in the appendix B.

### 2.4.2 The boundary potential $B(z, \bar{z})$

As explained in [99], the boundary potential $B(z, \bar{z})$ is already determined by the purely bosonic terms in (A.27), (A.29) and (A.28), (A.30). The differential equations for the real field $B$ read
\[ 0 = \partial_z \partial_{\bar{z}} \partial_z B + \frac{1}{4} \partial_{\bar{z}} B \] (2.56)
\[ 0 = \partial_{\bar{z}} \partial_z \partial_{\bar{z}} B + \frac{1}{4} \partial_z B \] (2.57)
together with
\[ \partial_z \partial_{\bar{z}} B = \partial_z \partial_{\bar{z}} B. \] (2.58)
2.4. The \( N = 2 \) Boundary Sine-Gordon Model

This determines \( B \) to

\[
B(z, \overline{z}) = \alpha \cos \frac{z - z_0}{2} \cos \frac{\overline{z} - \overline{z}_0}{2} + b, \quad \alpha, b \in \mathbb{R}, z_0 \in \mathbb{C},
\]  

(2.59)

which is so far exactly the result of [99]. Together with (2.45) we will nevertheless find further conditions on the so far unspecified constant \( z_0 \), arising from contributions of higher order in the bulk coupling constant \( g \) than considered in [99].

2.4.3 The boundary functions \( F, G, \) and \( \overline{F}, \overline{G} \)

From terms quadratic in the fermionic degrees of freedom as for example

\[
16i \left( \partial_z \partial_{\overline{z}} \right) \left( A'''(z) + \frac{1}{4} A'(z) \right) \theta_+ 
\]

(2.60)

and

\[
48i \partial_z \partial_{\overline{z}} \left( A'''(z) + \frac{1}{4} A'(z) \right) \theta_+ 
\]

(2.61)

we obtain the differential equations

\[
0 = A''(z) + \frac{1}{4} A(z) 
\]

(2.62)

\[
0 = \partial_z \left[ F'(z) G'(z) \right] + \frac{1}{2} g \sin z \ e^{i\vartheta} 
\]

(2.63)

and their corresponding complex conjugates. The functions \( F(z) \) and \( G(z) \) are determined from (2.62), (2.21) and (2.22) to

\[
F(z) = A_0 \cos \frac{z - \kappa_1}{2} + C_0
\]

(2.64)

\[
G(z) = B_0 \cos \frac{z - \kappa_2}{2} + D_0,
\]

(2.65)

and the equation (2.63) becomes with the matrix factorisation condition (2.44)

\[
0 = F' \left( G''' + \frac{1}{4} G \right) + G' \left( F''' + \frac{1}{4} F \right)
\]

\[
= \frac{1}{8} \left( A_0 B_0 \sin \frac{z - \kappa_1}{2} + B_0 C_0 \sin \frac{z - \kappa_2}{2} \right).
\]

(2.66)

By combining these results with the expression for \( B(z, \overline{z}) \) found in (2.45), we can in the next step deduce conditions on the so far free parameters in (2.64), (2.65) and (2.59).

Using (2.64) and (2.65) in the differentiated condition (2.44) we obtain

\[
2g e^{i\vartheta} \sin z = -\frac{A_0 B_0}{2} \cos \frac{\kappa_1 + \kappa_2}{2} \sin z + \frac{A_0 B_0}{2} \sin \frac{\kappa_1 + \kappa_2}{2} \cos z
\]

(2.67)
and therefore
\[
\kappa_1 + \kappa_2 = 2\pi n \quad n \in \mathbb{Z} \quad (2.68)
\]
\[
2g \, e^{i\beta} = -\frac{1}{2} A_0 B_0 (-)^n. \quad (2.69)
\]

A constraint on \( z_0 \) appearing in the boundary potential \( B \) arises from equation (2.45). We have
\[
\partial_x \partial_{\bar{z}} B(z, \bar{z}) = \frac{1}{2} \left( F' \bar{F}' + G' \bar{G}' \right). \quad (2.70)
\]

Using (2.59) and (2.64), (2.65) evaluated at \( z = z_0 \), we obtain
\[
0 = A_0 \bar{A}_0 \sin \frac{z_0 - \kappa_1}{2} \sin \frac{\bar{z}_0 - \bar{\kappa}_1}{2} + B_0 \bar{B}_0 \sin \frac{z_0 - \kappa_2}{2} \sin \frac{\bar{z}_0 - \bar{\kappa}_2}{2} \quad (2.71)
\]
and thus
\[
0 = \sin \frac{z_0 - \kappa_1}{2}; \quad 0 = \sin \frac{z_0 - \kappa_2}{2}. \quad (2.72)
\]

Together with (2.68) and the observation that the boundary Lagrangian (2.12) does not depend on the constants \( C_0, D_0 \) in (2.64) and (2.65), we therefore deduce the two following possibilities to ensure integrability in the sense discussed in [54].

Case I:
\[
B(z, \bar{z}) = \alpha \cos \frac{z}{2} \cos \frac{\bar{z}}{2}; \quad F(z) = A_0 \cos \frac{z}{2}; \quad G(z) = B_0 \cos \frac{\bar{z}}{2} \quad (2.73)
\]
with
\[
A_0 B_0 = -4g \, e^{i\beta}; \quad A_0 \bar{A}_0 + B_0 \bar{B}_0 = 2\alpha \quad (2.74)
\]

Case II:
\[
B(z, \bar{z}) = \alpha \sin \frac{z}{2} \sin \frac{\bar{z}}{2}; \quad F(z) = A_0 \sin \frac{z}{2}; \quad G(z) = B_0 \sin \frac{\bar{z}}{2} \quad (2.75)
\]
with
\[
A_0 B_0 = 4g \, e^{i\beta}; \quad A_0 \bar{A}_0 + B_0 \bar{B}_0 = 2\alpha. \quad (2.76)
\]

From (2.74) and (2.76) we have in both cases
\[
A_0 \bar{A}_0 = \alpha \pm \sqrt{\alpha^2 - 16g^2} \quad (2.77)
\]
and therefore
\[
A_0^\pm = e^{i\gamma} \sqrt{\alpha \pm \sqrt{\alpha^2 - 16g^2}}. \quad (2.78)
\]
The undetermined phase $\gamma$ appearing in (2.78) can be absorbed in a redefinition of the fermionic boundary fields $a$ and $\overline{a}$. From (2.77) we furthermore have the condition
\[
\alpha \geq 4g \geq 0 \tag{2.79}
\]
which in particular leads to a positive - semidefinite boundary potential $B(z,\overline{z})$ in (2.12).

With these choices all remaining terms in (2.55) either vanish or can be written as a total time derivative as a rather long calculation shows. Some details are presented in the appendix B. This result ensures the conservation of the higher spin quantity (2.54) in the presence of a boundary to all orders in the bulk coupling constant $g$, providing strong evidence for integrability.

To first order in $g$, the condition (2.70) together with (2.76) and (2.77) does not give rise to a constraint on $F$ and $G$. In this case one reobtains the situation of [99] where the two additional (real) parameters expressed by $z_0$ in the boundary potential (2.59) were found to be compatible with integrability to that order.
Chapter 3

Branes with nontrivial boundary condensates

3.1 The plane wave background: Introduction

We start this chapter by presenting a brief introduction to the maximally supersymmetric plane wave solution of type IIB supergravity from [25]. It is the background spacetime on which all string theories to be discussed in the remainder of this thesis are defined on. As mentioned in chapter 1, the solution of [25] is a particular plane fronted wave with parallel rays (pp-wave) with a metric of the form (1.1) and (1.3). By a slight, but customary abuse of notation, we refer to it as the plane wave solution.

From the results presented in section 1.1, the plane wave can be obtained as a particular Maldacena-Maoz background by choosing a suitable quadratic superpotential. While this is particularly appropriate for discussing boundary fermions in this context (compare with chapter 4), it neither does full justice to this remarkable spacetime, nor does it reflect its original discovery.

The plane wave solution as discovered in [25] is described by the metric

\[ ds^2 = 2dX^+dX^- - \mu^2 X^I X^I dX^+ dX^- + dX^I dX^I \]  

and the nonzero Ramond-Ramond five-form field strength

\[ F_5 = \mu dx^+ \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \].

It is a maximally supersymmetric spacetime, that is, there exist 32 independent sections \( \epsilon \) in the spinor bundle obeying the differential equation

\[ \mathcal{D}_\tau \epsilon = 0. \]

The differential operator \( \mathcal{D}_\tau \) is here the appropriately extended covariant derivative of type II B supergravity with additional contributions of the RR-field (3.2), compare with equation (1.8) in the introduction.
Despite the numerous supergravity or string background solutions with less supersymmetry, there are only 3 cases with maximal supersymmetry in the type IIB case \cite{50}: flat space, the plane wave and the $AdS_5 \times S_5$-solution, famous from the AdS/CFT correspondence \cite{86}.

What is remarkable about the plane wave solution is that it is on the one hand simple enough to give rise to a solvable string theory to be discussed in the next section. On the other hand, it displays sufficiently many nontrivial features to allow for a substantial enlargement of our knowledge about strings in curved backgrounds.

In a certain sense, the plane wave can be understood as lying in between the trivial flat space and the so far still largely intractable $AdS_5 \times S_5$ solution. We present a mathematical justification of this statement below. As discussed in \cite{18}, it has particularly important consequences in the gauge / gravity - duality context from \cite{86}.

The plane wave background (3.1) can be obtained from the $AdS_5 \times S_5$ solution as a ‘Penrose-Güven-limit’, \cite{26}. This limit can be interpreted as a zoom in onto a neighbourhood of a null geodesic in an arbitrary Lorentzian manifold. In the context of general relativity this procedure was introduced by Penrose in \cite{105} and generalised to supergravity theories by Güven in \cite{60}. Following the very accessible introduction \cite{24}, but see also \cite{27}, the limit can be summarised as follows. In suitable coordinates a general Lorentzian metric in a neighbourhood of a null geodesic has the form

$$ds^2 = 2dX^+dX^- + C\left(X^+, X^-, X^\tau\right)\left(dX^+\right)^2 + 2C_i\left(X^+, X^-, X^\tau\right)dX^idX^+ + C_{ij}\left(X^+, X^-, X^\tau\right)dX^idX^j, \quad (3.4)$$

compare with \cite{105}. Following \cite{24}, the Penrose limit follows from (3.4) (‘practically speaking’) by dropping the second and third term and furthermore reducing the coordinate dependency in $C_{ij}$ to

$$C_{ij}(X^+) = C_{ij}(X^+, 0, 0). \quad (3.5)$$

Via a transformation of coordinates, the resulting metric can be brought to the form (1.1) with (1.3). In summary, we have deduced a plane wave metric by starting with a null geodesic in a general Lorentzian spacetime. For a more geometrical account including a discussion of covariance properties, we refer to \cite{24, 27} and references therein.

With the previous procedure, the plane wave solution (3.1) can be derived by starting with the $AdS_5 \times S_5$-metric

$$ds^2 = R^2 \left[- \cosh^2(r)dt^2 + dr^2 + \sinh^2(r)d\Omega_3^2 + \cos^2(\theta)d\psi^2 + d\theta^2 + \sin^2(\theta)d\Omega_3^2\right], \quad (3.6)$$

and performing the Penrose-limit along an appropriate null geodesic with

$$X^+ = t + \psi; \quad X^- = t - \psi, \quad (3.7)$$
compare with [26]. In the extended setting of Güven this also includes the derivation of the nonzero RR-field (3.2).

The particular virtue of this derivation is the possibility to interpret it in the dual field theory described in the AdS/CFT - conjecture. By this conjecture, string theory on $AdS_5 \times S_5$ is dual to a $\mathcal{N} = 4$ superconformal field theory in four spacetime dimensions, [86]. The Penrose limit in this setting amounts to picking special operators in the field theory, compare with the initial paper [18]. This procedure is now referred to as the ‘BMN-limit’.

We do not attempt to discuss the BMN-limit in any further detail and point the interested reader to the extensive reviews [113, 87, 106, 103, 109]. We, however, want to stress that available string theoretical data for the plane wave background and explicit calculations on the field theory side led to tests of Maldacena’s conjecture from [86] far beyond previously available results.

3.1.1 Closed strings in the plane wave background

Below we present a brief review of the closed string theory defined on the maximally supersymmetric plane wave background and fix our notations for future reference. As discussed in chapter 1, the metric (3.1) allows for the choice of the light-cone gauge in the Green-Schwarz description [67, 108]. The resulting gauge-fixed theory was first constructed by Metsaev in [94] and further discussed in [95]. We do not present the derivation of [94, 95] using Cartan forms defined on coset superspaces and just summarise their final result expressed here in the conventions of [58, 19, 51]. The relation to the formulation as a Landau-Ginzburg model will be discussed in detail in chapter 4.

The plane wave worldsheet theory in the light-cone gauge is governed by the following set of equations of motion. For the transverse bosonic fields we have

$$\left(\partial_+ \partial_- + m^2\right) X^r = 0 \quad (3.8)$$

with $r = 1, \ldots, 8$ and the fermionic degrees of freedom are described by the coupled system of differential equations

$$\partial_+ S = m \Pi \tilde{S}; \quad \partial_- \tilde{S} = -m \Pi S, \quad (3.9)$$

using the spinor matrix

$$\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4. \quad (3.10)$$

As explained in [94], the fields $S, \tilde{S}$ reside in the same $SO(8)$-spinor representation as appropriate for the type II B superstring theory of present interest. Our conventions regarding Dirac matrices are taken from chapter 5 of [58], to which we also refer for further details of the corresponding representation theory.

The mass parameter $m$ appearing in (3.8) and (3.9) is proportional to the parameter $\mu$ in the plane wave metric (3.1). Its exact numerical form depends on the light-cone
gauge parameters and will be discussed in section 3.2.

Using the gauge-fixed equations of motion, the closed string theory construction proceeds in the standard fashion. Following [51], the most general solutions to (3.8) and (3.9) are given by

\[X_s(a, t) = A_s \sin(mr) + A_s \cos(mr) + B_s \cosh(ma) + B_s \sinh(ma) + \sum_{n, \omega_n \in \mathbb{C} \setminus \{0\}} \frac{1}{\omega_n} \left( \alpha_n e^{-i(\omega_n \tau - n\sigma)} + \tilde{\alpha}_n e^{-i(\omega_n \tau + n\sigma)} \right)\]

(3.11)

for the bosons and

\[S(a, t) = S_0 \cos(m\tau) + \Pi S_0 \sin(m\tau) + T_0 \cosh(m\sigma) + \Pi T_0 \sinh(m\sigma) + \sum_{n, \omega_n \in \mathbb{C} \setminus \{0\}} c_n \left[ S_n e^{-i(\omega_n \tau - n\sigma)} - \frac{i}{m} (\omega_n - n) \Pi S_n e^{-i(\omega_n \tau + n\sigma)} \right] \]

(3.12)

and

\[\tilde{S}(\tau, \sigma) = -\Pi S_0 \sin(m\tau) + \tilde{S}_0 \cos(m\tau) + \Pi \tilde{T}_0 \sinh(m\sigma) + \sum_{n, \omega_n \in \mathbb{C} \setminus \{0\}} c_n \left[ \tilde{S}_n e^{-i(\omega_n \tau + n\sigma)} - \frac{i}{m} (\omega_n - n) \Pi S_n e^{-i(\omega_n \tau + n\sigma)} \right] \]

(3.13)

for the fermions. To obey the equations of motion we have in both cases

\[\omega_n^2 = n^2 + m^2\]

(3.14)

and the factors \(c_n\) corresponding to the fermionic nonzero modes’ normalisations are chosen to

\[c_n = \frac{m}{\sqrt{2\omega_n (\omega_n - n)}}\]

(3.15)

for later convenience. These general solutions will be of particular interest in the context of open strings with non-standard modings in the field expansions. For the present case of closed string fields additional periodicity requirements as for example

\[X^r(\tau, \sigma) = X^r(\tau, \sigma + 2\pi)\]

(3.16)

lead to significantly simplified results. Explicitly, they are given by

\[X^r(\tau, \sigma) = \cos(m\tau) x_0^r + \frac{1}{m} \sin(m\tau) p_0^r + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{\omega_n} \left( \alpha_n e^{-i(\omega_n \tau - n\sigma)} + \tilde{\alpha}_n e^{-i(\omega_n \tau + n\sigma)} \right)\]

(3.17)

and

\[S(\sigma, \tau) = S_0 \cos(m\tau) + \Pi \tilde{S}_0 \sin(m\tau) + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left[ S_n e^{-i(\omega_n \tau - n\sigma)} + \frac{i}{m} (\omega_n - n) \Pi S_n e^{-i(\omega_n \tau + n\sigma)} \right],\]

(3.18)

\[\tilde{S}(\tau, \sigma) = -\Pi S_0 \sin(m\tau) + \tilde{S}_0 \cos(m\tau) + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left[ \tilde{S}_n e^{-i(\omega_n \tau + n\sigma)} - \frac{i}{m} (\omega_n - n) \Pi S_n e^{-i(\omega_n \tau + n\sigma)} \right].\]

(3.19)
compare with [94, 95]. We furthermore have

\[ \omega_n = \text{sgn}(n) \sqrt{n^2 + m^2} \]  

(3.20)

and the requirements for real fields lead to the well known relations

\[ (\alpha_n^r)^\dagger = \alpha_{-n}^-, \quad (\tilde{\alpha}_n^r)^\dagger = \tilde{\alpha}_{-n}^-, \quad S_n^+ = S_{-n}, \quad \tilde{S}_n^+ = \tilde{S}_{-n}. \]  

(3.21)

For future convenience, we furthermore introduce the following zero-mode combinations from [51]. For the bosons we use

\[ a_0^r = \frac{1}{\sqrt{2m}} (p_0^r + imx_0^r), \quad \bar{a}_0^r = \frac{1}{\sqrt{2m}} (p_0^r - imx_0^r) \]  

(3.22)

and for the fermions

\[ \theta_0 = \frac{1}{\sqrt{2}} \left( S_0 + i\tilde{S}_0 \right), \quad \bar{\theta}_0 = \frac{1}{\sqrt{2}} \left( S_0 - i\tilde{S}_0 \right) \]  

(3.23)

with chiral projections

\[ \theta_R = \frac{1 + \Pi}{2} \theta_0, \quad \bar{\theta}_R = \frac{1 + \Pi}{2} \bar{\theta}_0, \]

\[ \theta_L = \frac{1 - \Pi}{2} \theta_0, \quad \bar{\theta}_L = \frac{1 - \Pi}{2} \bar{\theta}_0. \]  

(3.24)

**Quantisation**

After presenting the mode expansions of the closed string fields in the gauge-fixed worldsheet theory, the canonical quantisation proceeds as in flat space physics, compare for example with the standard reference [58].

With the canonical conjugated bosonic momenta

\[ P^r(\tau, \sigma) = \partial_\tau X^r(\tau, \sigma) \]  

(3.25)

the required canonical quantum relations are given by the customary expressions

\[ [X^r(\tau, \sigma), X^s(\tau, \sigma')] = 0, \]  

(3.26)

\[ [X^r(\tau, \sigma), P^s(\tau, \sigma')] = i \delta^{rs} \delta(\sigma - \sigma'), \]  

(3.27)

\[ [P^r(\tau, \sigma), P^s(\tau, \sigma')] = 0 \]  

(3.28)

and

\[ \{S^a(\tau, \sigma), S^b(\tau, \sigma')\} = 2\pi \delta(\sigma - \sigma') \delta^{ab}, \]  

(3.29)

\[ \{\tilde{S}^a(\tau, \sigma), \tilde{S}^b(\tau, \sigma')\} = 2\pi \delta(\sigma - \sigma') \delta^{ab}, \]  

(3.30)

\[ \{S^a(\tau, \sigma), \tilde{S}^b(\tau, \sigma')\} = 0. \]  

(3.31)
As first determined in [94, 95], the field relations (3.26)-(3.31) are fulfilled iff the modes appearing in (3.17), (3.18) and (3.19) obey
\[ \omega_k \delta^{rs} \delta_{kl} \]
\[ \alpha_k^a, \alpha_l^a = 0; \quad \tilde{\alpha}_k^a = 0 \]
for \( k, l \neq 0 \) and
\[ \alpha_0^a, \alpha_0^a = \delta^{rs}; \quad \alpha_0^a, \alpha_0^a = 0 \]
for the zero modes.

### The Superalgebra

Below we present the conserved supercurrents representing the 32 Killing spinors of the maximally supersymmetric background (3.1) in the worldsheet theory. As discussed in chapter 1, the resulting supersymmetries split up into two groups, the dynamical and kinematical, or synonymously linearly and nonlinearly realised charges. For reasons explained in section 1.1, but see also [51], our treatment has its main focus on the dynamical supersymmetries.

As derived in [94, 95], the 16 dynamical Killing spinors correspond to the conserved worldsheet fluxes
\[ Q_a^\gamma = \partial_+ X^a \gamma^\sigma \tilde{S}_a - m X^a (\gamma^\Pi)_{aa} \tilde{S}_a; \quad \tilde{Q}_a^\sigma = \partial_+ X^a \gamma^\sigma \tilde{S}_a + m X^a (\gamma^\Pi)_{aa} \tilde{S}_a \]
\[ \tilde{Q}_a^\gamma = \partial_+ X^a \gamma^\sigma \tilde{S}_a + m X^a (\gamma^\Pi)_{aa} \tilde{S}_a; \quad \tilde{Q}_a^\sigma = -\partial_+ X^a \gamma^\sigma \tilde{S}_a + m X^a (\gamma^\Pi)_{aa} \tilde{S}_a \]
with
\[ 0 = \partial_\mu Q^\mu \quad \text{and} \quad 0 = \partial_\mu \tilde{Q}^\mu. \]
By standard reasoning they give rise to the conserved charges
\[ \sqrt{2} p^+ Q_a^\gamma = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left[ \partial_+ X^a \gamma^\sigma \tilde{S}_a - m X^a \gamma^\Pi \tilde{S}_a \right]_a \]
\[ \sqrt{2} p^+ \tilde{Q}_a^\gamma = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left[ \partial_+ X^a \gamma^\sigma \tilde{S}_a + m X^a \gamma^\Pi \tilde{S}_a \right]_a, \]
which are expressible as
\[ \sqrt{2} p^+ Q_a^\gamma = \sum_r \left[ p_0^a \gamma^\gamma S_0^a - m x_0^a \gamma^\Pi \tilde{S}_0^a + \sum_{n \neq 0} c_n \left( \gamma^\gamma \alpha_{-n}^a S_n^a + i \frac{\omega_n}{m} \gamma^\Pi \tilde{\alpha}_{-n}^a \tilde{S}_n^a \right) \right]_a \]
\[ \sqrt{2} p^+ \tilde{Q}_a^\gamma = \sum_r \left[ p_0^a \gamma^\gamma \tilde{S}_0^a + m x_0^a \gamma^\Pi \tilde{S}_0^a + \sum_{n \neq 0} c_n \left( \gamma^\gamma \tilde{\alpha}_{-n}^a \tilde{S}_n^a - i \frac{\omega_n}{m} \gamma^\Pi \tilde{\alpha}_{-n}^a \tilde{S}_n^a \right) \right]_a \]
by using (3.17)-(3.19), compare for example with [94, 51].

The kinematical Killing spinors are represented by the currents

\[ Q^a = \exp \left[ i m \tau \Pi \right]_{ab} \left( S^b - i S^b \right) \]
\[ Q^a = \exp \left[ -i m \tau \Pi \right]_{ab} \left( S^b + i S^b \right) \]

and the corresponding charges are given by the fermionic zero modes

\[ \sqrt{2p^+} Q^a = S^a_0; \quad \sqrt{2p^+} Q^a = \tilde{S}^a_0. \]

We close this section by noting the component form of the closed string light-cone gauge Hamiltonian for future reference. As discussed in [94, 51], it is given by

\[ 2p^+ H = m \left( a^0 \tilde{a}^0 + i S^a_0 \Pi_{ab} \tilde{S}^a_0 + 4 \right) + \sum_{k=1}^{\infty} \left[ \alpha^r_{-k} \alpha^r_{k} + \tilde{\alpha}^r_{-k} \tilde{\alpha}^r_{k} + \omega_k \left( S^a_0 S^a_k + \tilde{S}^a_0 \tilde{S}^a_k \right) \right] \]

\[ = m \left( a^0 \tilde{a}^0 + \theta^a_L \bar{\theta}^a_L + \theta^a_R \bar{\theta}^a_R \right) + \sum_{k=1}^{\infty} \left[ \alpha^r_{-k} \alpha^r_{k} + \tilde{\alpha}^r_{-k} \tilde{\alpha}^r_{k} + \omega_k \left( S^a_0 S^a_k + \tilde{S}^a_0 \tilde{S}^a_k \right) \right], \]

presented already in its normal ordered form.

### 3.2 Branes in the plane wave background

This section contains a brief introduction to the classification of supersymmetric branes in the maximally supersymmetric plane wave background and explains the context of our subsequent constructions.

Soon after the treatment of closed strings in the spacetime (3.1) in [94, 95] and mainly within the emerging wake of activity following the previously mentioned ‘BMN-proposal’ from [18], branes in the plane wave background have been studied in a number of papers from many different points of view. Starting with the initial papers [23, 41], geometric methods like the probe brane approach were applied in [116, 119, 10, 11, 121, 96, 102], boundary states were used in [19, 51, 52, 42] and discussions using open string constructions can be found in [41, 117, 118, 11, 38, 35, 76]. Closely related setups were considered for example in [4, 112, 29, 5, 70, 77, 9].

Following in particular the flat space treatment of Green and Gutperle in [56, 57], the (maximally) supersymmetric branes in the plane wave background without boundary fields have been classified in [117, 51] by using the spinor matrix

\[ M = \prod_{j \in \mathbb{N}} \gamma^j. \]
The matrix $M$ appears in the standard fermionic boundary conditions and the product (3.47) is understood to span over the Neumann directions of the brane under consideration. From section 3.4 onwards this will be discussed in greater detail in a generalised context.

Following [51, 117], there are two different classes of branes in the plane wave background. Those of class I are characterised by a gluing matrix $M$ fulfilling

$$M_{II} M_{II} = -1,$$

using the matrix $II$ as defined in (3.10). The maximally supersymmetric branes in this class are of structure $(r, r + 2), (r + 2, r)$ with $r = 0, 1, 2$. The notation $(r, s)$ has been introduced in [116] and labels the orientation of the different branes with respect to the $SO(4) \times SO(4)$-background symmetry.

The second class or Class II type branes are described by

$$M_{II} M_{II} = 1$$

and the standard maximally supersymmetric members of this family are the $(0, 0)$ instanton and the $(4, 0), (0, 4)$ branes from [117, 51].

One of our main results is the construction of a maximally supersymmetric class II brane of type $(4, 4)$ to be presented in chapter 4. Its construction uses deformed boundary conditions, originating from the inclusion of fermionic boundary fields as discussed in chapter 2. We provide constructions for branes of type $(n, n)$ for all $n = 0, \ldots, 4$, but only the limiting cases $n = 0$ and $n = 4$ are maximally supersymmetric.

In the present chapter we consider a different generalisation of the previously described branes by allowing for boundary magnetic fields, or analogously, for nontrivial $B-$field backgrounds. In contradistinction to the situation in flat space covered for example in [57, 39], it will be seen to be impossible to turn on boundary condensates on some supersymmetric branes without reducing the amount of conserved supersymmetries.

There are only two classes of branes in the plane wave background which remain maximally supersymmetric in the presence of boundary magnetic fields. The condensates on these branes of structure $(2, 0), (4, 2)$ give rise to new continuous D-brane families\(^1\). In addition, these families will be shown to interpolate smoothly between the mentioned class I branes and the $(0, 0), (4, 0)$ class II branes, connecting these classes therefore in a natural fashion. It is worth mentioning that our results from [90] presented in this chapter have been confirmed later on in [82] and more recently in a slightly different context in [114].

Using worldsheet methods we provide brane constructions from the open and closed string point of view. It is important to notice that the standard light-cone gauge

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\(^1\)We do not count the analogous constructions of $(0, 2), (2, 4)$ branes as separate classes as they can be deduced by an obvious relabelling / interchange of coordinates.
condition
\[ X^+ = 2\pi P^+ \tau \] (3.50)
gives rise to branes of different nature in these two sectors. In the open sector the light-cone directions with (3.50) are of Neumann type, whereas they become Dirichlet-like in the closed string case, [57]. The boundary states therefore correspond to instantonic branes instead of the more customary Dp-branes as usually constructed in the open-string setting.

To have a straightforward comparison between the two pictures and to furthermore conduct consistency tests, one has to use different light-cone gauge conditions in the two settings. An appropriate new gauge for open strings was introduced in [19]. We refer to this publication and [51] (but see also [119]) for a detailed discussion on this point.

As mentioned in section 3.1.1, the mass parameter appearing in the equations of motion (3.8) and (3.9) is gauge-dependent. With the standard gauge (3.50) it takes on the form
\[ m = 2\pi \mu P^+ \] (3.51)
in the closed string sector. The open string on the other hand acquires with the new gauge from [19, 51] the mass parameter
\[ \hat{m} = \mu X^+ \] (3.52)
to be substituted for \( m \) in the equations of motion (3.8) and (3.9). The parameter \( \mu \) is in both cases taken from the metric (3.1).

The different masses play a crucial rôle in the important consistency check known from conformal field theory as the Cardy condition. It requires the identity of certain boundary state overlaps with one-loop partition functions in the corresponding open string descriptions. In the context of plane wave physics this test was first studied in [19, 51] by defining \( m \) - dependent deformations of the well known \( f \)-functions of Polchinski and Cai from [107].

We extend and connect the results of [19, 51] by introducing gauge field dependent generalisations of the special functions \( f_i^{(m)} g_i^{(m)} \). It is worth mentioning that similar deformations of the standard \( \Gamma \)-function have appeared in the context of plane wave physics in [84, 85].

This chapter is organised as follows. After briefly summarising general aspects of boundary condensates on D-branes in flat space in section 3.3, we present a closed string boundary state description of maximally supersymmetric branes in the plane wave background in section 3.4. We deduce a general condition for the gluing matrices whose solutions are discussed in section 3.5. After constructing the corresponding boundary states in section 3.6, the open string treatment starts with section 3.7. Further technical details are to be found in the appendices C and D.
3.3 Boundary Condensates on Dp-branes

Let us begin by reviewing some well-known facts about boundary condensates on D-branes in flat space. As described for example in [1, 57, 56, 115] and references therein, the introduction of boundary condensates with (constant) gauge potentials $A_I$ on the world volume of a D-brane gives rise to the following boundary action

$$\int ds \left( A_I \partial_\sigma X^I - \frac{1}{2} F_{IJ} \gamma^{IJ} S \right)$$

(3.53)

with the (abelian) field strength $F = dA$. In case of a constant $B$-field background the bosonic bulk term proportional to

$$\epsilon^{\alpha\beta} B_{rs} \partial_\alpha X^r \partial_\beta X^s$$

(3.54)

and the corresponding fermionic couplings become total derivatives and we obtain for a constant gauge field with the appropriate potential $A_I = -\frac{1}{2} F_{IJ} X^J$ the combined surface action

$$\int ds \mathcal{F}_{IJ} \left( X^I \partial_\sigma X^J - S \gamma^{IJ} S \right).$$

(3.55)

Here we have furthermore used the standard gauge invariant quantity $\mathcal{F} = F - B$.

For the Neumann directions the boundary action (3.55) leads to the modified boundary conditions

$$\partial_\sigma X^I + \mathcal{F}^{IJ} \partial_\tau X^J = 0$$

(3.56)

at $\sigma = 0, \pi$, compare for example with [57]. The parameter $\nu = 1, 2$ distinguishes possibly different condensates on branes at $\sigma = 0$ and $\sigma = \pi$. For simplicity we will, however, concentrate on the case $\nu = 1$.

As a relation between $\partial_+ X^r$ and $\partial_- X^r$, the equation (3.56) reads

$$(\partial_+ X^I + N^{IJ} \partial_- X^J) = 0$$

(3.57)

along the Neumann directions with

$$N^{IJ} = - \left[ \frac{1 - \mathcal{F}}{1 + \mathcal{F}} \right]^{IJ}, \quad \mathcal{F}^{IJ} = \left[ \frac{1 + N}{1 - N} \right]^{IJ}.$$

(3.58)

For the Dirichlet directions we impose (3.57) with $N = 1$, that is

$$(\partial_+ X^i + \partial_- X^i) = \partial_\tau X^i = 0$$

(3.59)

on the boundary.\footnote{We use upper case letters $I, J...$ for Neumann - and lower case letters $i, j...$ for Dirichlet directions in this chapter.}

The implementation of a nonzero gauge field condensate in the light-cone gauge
boundary state description in flat space is discussed in [57]. It uses the conservation of space-time supersymmetries expressed by

\[
\left( Q_a + i \gamma^\alpha M_{ab} \tilde{Q}_b \right) \| B \rangle = 0, \quad \left( Q_a + i \eta M_{ab} \tilde{Q}_b \right) \| B \rangle = 0
\] (3.60)
as a guiding principle. With the usual fermionic gluing relations

\[
\left( S_a^\alpha + i \gamma^\beta M_{ab} \tilde{S}_b^\beta \right) \| B \rangle = 0
\] (3.61)
and the bosonic conditions (3.59) and (3.57) described beforehand, the constraints (3.60) are fulfilled iff the orthogonal matrices $N^{IJ}, M_{ab}, M_{ab}$ obey

\[
M_{ab} \gamma^I M_{cd} = \gamma^J N^{IJ}.
\] (3.62)

By this, the matrices $N^{IJ}, M_{ab}, M_{ab}$ are equivalent elements in the three different irreducible eight dimensional $SO(8)$—representations. The corresponding explicit solutions are provided in section 3.5. As there are no further conditions for maximally supersymmetric branes, arbitrary constant boundary condensates can be turned on on any even dimensional world-volume subspace of a supersymmetric brane in flat space without changing the amount of conserved supersymmetries.

### 3.4 Gluing conditions for maximally supersymmetric branes in the plane wave background

From now on we consider branes in the plane wave background with a focus on (static) maximally supersymmetric configurations. In terms of boundary states, the conservation of dynamical supersymmetries is encoded as in flat space by

\[
\left( Q_a + i \eta M_{ab} \tilde{Q}_b \right) \| B \rangle = 0,
\] (3.63)
using the supercharges presented in (3.41) and (3.42). The matrix $M$ is as before a constant $SO(8)$—spinor matrix in the appropriate representation.

Assuming the standard Dirichlet conditions (3.59) for a flat D-brane, the requirement (3.63) uniquely determines the fermionic boundary conditions, compare with [51]. With the mode expansions (3.41), (3.42) and the bosonic conditions

\[
(\alpha_n^a - \tilde{\alpha}_{-n}^a) \| B \rangle = 0
\] (3.64)
from (3.59), the fermionic gluing conditions are determined to

\[
\left( S_n^\alpha + i \eta K_{ab} \tilde{S}_{-n}^b \right) \| B \rangle = 0 \quad (n \neq 0)
\] (3.65)
with

\[
K_n = \frac{1 - \frac{\eta_m}{2\lambda_n \lambda_n} \Pi M^t}{1 + \frac{\eta_m}{2\lambda_n \lambda_n} \Pi M} = \frac{1 - \eta_n \lambda_n}{\lambda_n} \Pi M^t
\] (3.66)
3.4. GLUING CONDITIONS FOR MAXIMALLY SUPERSYMMETRIC BRANES

In deriving (3.66), we implicitly used
\[ [M, \gamma^I] = 0 \] (3.67)
along the Dirichlet directions. This condition will be further discussed in the open string setting in section 4.4.

After determining the fermionic boundary conditions from contributions in (3.63) along the Dirichlet directions, we discuss next terms along the Neumann directions. The bosonic open string boundary conditions (3.56) translate by the standard methods into
\[ \left( \partial_\tau X^I - F^{IJ} \partial_\sigma X^J \right) \vert \vert B) \rangle = 0 \quad (\tau = 0). \] (3.68)
This yields the mode-relations
\[ (\alpha^I_n - N^{IJ}_n \tilde{\alpha}^J_n) \vert \vert B) \rangle = 0 \] (3.69)
with
\[ N_n = \frac{F - \frac{w_n}{n}}{F + \frac{w_n}{n}} = -\frac{(-\omega_n - n) + (\omega_n + n)N}{(\omega_n + n) - (\omega_n - n)N} \] (3.70)
and
\[ N_{-n} = N_n; \quad N^t_n N_{-n} = N^t_n N_n = 1. \] (3.71)
The last relations make the bosonic gluing conditions self-consistent. By using (3.69) in (3.63), we obtain
\[ 0 = \sum_{n \in \mathbb{Z}} c_n \left[ \left( \gamma^I + \frac{m\eta}{2\omega_n c_n^2} M \gamma^I \Pi \right) \alpha^I_{-n} S_n + i\eta \left( M \gamma^I - \frac{m\eta}{2\omega_n c_n^2} \gamma^I \Pi \right) \tilde{\alpha}^I_n \bar{S}_{-n} \right] \vert \vert B) \rangle \] (3.72)
which simplifies to
\[ \left( \gamma^I + \frac{m\eta}{2\omega_n c_n^2} M \gamma^I \Pi \right) N^{IJ}_{-n} K_n - \left( M \gamma^I - \frac{m\eta}{2\omega_n c_n^2} \gamma^I \Pi \right) = 0. \] (3.73)
With (3.66) and
\[ \frac{m\eta}{2\omega_n c_n^2} = \frac{\omega_n - n}{m}, \] (3.74)
we finally deduce
\[ M \gamma^I M^t - \frac{\omega_n - n}{m} \left( \gamma^I \Pi M^t - M \gamma^I \Pi \right) - \frac{(\omega_n - n)^2}{m^2} \gamma^I = N^{IJ}_{-n} \left[ \gamma^I - \frac{\omega_n - n}{m} \left( \gamma^I \Pi M^t - M \gamma^I \Pi \right) - \frac{(\omega_n - n)^2}{m^2} M \gamma^I M^t \right]. \] (3.75)
Under the assumption of a \( n \)-independent gluing matrix \( N_n = N \), the condition (3.75) leads to the requirement
\[ M \gamma^I M^t = N^{IJ} \gamma^I; \quad \gamma^I = N^{IJ} M \gamma^I M^t \] (3.76)
and therefore in particular to $N^2 = 1 \rightarrow N = -1$ along the Neumann directions. The first publications on branes in the plane wave background were focussing on mode-independent gluing conditions. In this sense, branes with boundary condensates were first missed out for the same reason as the (0,0)-instanton from [51, 117] with its mode dependent fermionic gluing conditions.

Applying in our case the (generically) mode-dependent matrix (3.70) to the condition (3.75), we have

$$0 = (M \gamma^L M^t - \gamma^J N^{JL}) + \frac{m}{2n} (\gamma^J \Pi M^t - M \gamma^J \Pi) (N^{JL} - \delta^{JL}),$$

implying

$$M \gamma^J M^t = \gamma^J N^{JL}$$

(3.78)

and

$$(\delta^{KR} - N^{KR}) [\gamma^K \Pi M^t - M \gamma^K \Pi] = 0$$

(3.79)

along the Neumann directions. In the present situation the matrix $1 - N$ is invertible. Thus, we can rewrite the previous conditions as

$$M \gamma^J M^t = \gamma^J N^{JL}$$

(3.80)

and

$$\gamma^K = M \gamma^K \Pi M \Pi = \gamma^J N^{JK} M \Pi M \Pi.$$  

(3.81)

The conditions in this final form will be interpreted and solved in the next section.

So far we only considered nonzero mode contributions to (3.63). As the ‘zero-modes’ in (3.17) and (3.18), (3.19) do not contain a $\sigma$-dependency, $F$ drops out for these terms and the previous considerations for a vanishing $F$-gauge field to be found for example in [51] remain unaltered: Commuting (3.63) with $x_0^d$ one obtains

$$\left( S^a_0 + i \eta M^{ab} \tilde{S}^b_0 \right) || B \rangle = 0,$$

(3.82)

that is, the boundary state preserves eight kinematical supersymmetries. Applying this to (3.63), we are left with the condition

$$\left( -i \eta P_0^d (\gamma^J M - \gamma^J) \tilde{S}_0 - mx_0^d (\gamma^J \Pi - M \gamma^J \Pi M) \tilde{S}_0 \right) || B \rangle = 0.$$  

(3.83)

For the Neumann directions this is solved with the standard requirement

$$P_0^d || B \rangle = 0.$$  

(3.84)

For the Dirichlet directions, however, one has to have either

$$M \Pi M \Pi = 1,$$

(3.85)

corresponding to a class II brane without gauge field excitations or

$$x_0^d || B \rangle = 0,$$

(3.86)

that is, a brane placed at the origin of the transverse space.
3.5 Supersymmetric Branes with nontrivial $\mathcal{F}$ - field

The first condition (3.80) for maximally supersymmetric branes in the plane wave background is identical to (3.62) from the flat space treatment. As already mentioned beforehand, it states that the three matrices $M_{ab}, M_{ab}$ and $N_{IJ}$ are related by SO(8)-triality. It is explicitly solved by the formulas presented in [57, 56], that is,

$$N_{IJ} = e^{\frac{1}{2}\Omega_{MN}^{\gamma MN}_{IJ}}$$

and

$$M_{ab} = e^{\frac{1}{2}\Omega_{MN}^{\gamma MN}_{ab}}; \quad M_{ab} = e^{\frac{1}{2}\Omega_{MN}^{\gamma MN}_{ab}}$$

with

$$\Sigma^{MN}_{IJ} = \delta^M_I \delta^N_J - \delta^M_J \delta^N_I; \quad \gamma^{mn} = \frac{1}{2} \gamma^{[m} \gamma^{n]}.$$  
(3.87)

(3.88)

(3.89)

The second condition (3.81) has no flat space analogue. For maximally supersymmetric branes it gives rise to some qualitative differences compared to flat space, where a nonzero boundary condensate does not give rise to any new constraints.

Before discussing cases with nonzero magnetic fields in detail, it is easy to see that all the considerations so far are consistent with the results on branes in the plane wave background as summarised in section 3.2. Assuming mode independent fermionic gluing condition as for example in [23, 19], one needs $M_{IM} M_{II} = -1$ as deduced from (3.66). By (3.81), this furthermore leads to $N = -1$ and the bosonic gluing conditions finally reduce to the standard $N_a = N = -1$. This is the setup of a class I brane as first discussed in [23, 41].

The maximally supersymmetric class II branes with $M_{II} M_{II} = 1$ beyond the $(0,0)-$instanton are not contained in the previous treatment. As discussed in [116, 121], the $(4,0)$, $(0,4)$ branes couple necessarily to the longitudinal flux $\mathcal{F}_{+I}$. This possibility was not included in the previous discussion and will be covered in the context of the $(4,2)$-brane with boundary condensate later on.

3.5.1 The $(2,0)$, $(0,2)$ branes

The cases of the $(2,0)$ or $(0,2)$ branes are solved as follows. Without loss of generality we choose the first two coordinates $I = 1, 2$ as Neumann directions. We obtain

$$N = \exp \left[ \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right] = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

and

$$M_{ab} = \exp \left[ \frac{\theta}{2} \gamma_{12}^{ab} \right] = l_{ab} \cos \frac{\theta}{2} + \gamma_{12}^{ab} \sin \frac{\theta}{2}.$$  
(3.90)

(3.91)

\footnote{For branes with non-zero magnetic fields we use the same labelling as for their $\mathcal{F} \rightarrow 0$ limits from [117, 51].}
We deduce

\[ [M, \Pi] = 0, \] (3.92)

such that the second condition (3.80) reads

\[ \gamma^J = \gamma^J N^{IJ} M^2 = M \gamma^J M \leftrightarrow M' \gamma^J = \gamma^J M \quad (J = 1, 2). \] (3.93)

For the (2,0) case this is an identity and we therefore can have arbitrary constant boundary condensates on branes of this type.

The new class of branes with nonzero boundary condensate \( \mathcal{F} \) interpolates smoothly between the usual class I (2,0) brane and the (class II) (0,0) instanton. By choosing the parameter \( \theta \) in (3.90) to \( \theta = 0 \), the gluing matrices from (3.65) and (3.69) reduce to

\[ K_n = \frac{\omega_n - \eta m \Pi}{n}; \quad N_n = 1. \] (3.94)

These are the conditions for the D-instanton from [51]. For \( \theta = \pi \) we obtain on the other hand

\[ K_n = \gamma^{12}; \quad N_n = -1, \] (3.95)

that is, the conditions for the standard (2,0) brane.

The boundary state of the (2,0) brane and its consistency with the open string channel description will be discussed in section 3.8.

### 3.5.2 The (3,1), (1,3) branes

In the plane wave background it is impossible to turn on a boundary condensate on a (true) subspace of a brane worldvolume and still maintain maximal supersymmetry. This is in clear contradistinction to the situation in flat space. It comes immediately from the observation that an eigenvalue \(-1\) of \( N \) in (3.81) gives rise to the condition of class I branes. This implies in particular \( N \equiv -1 \).

But even for a non-degenerate \( \mathcal{F} \) the condition (3.81) is in general not solvable as the example of the (3,1) brane shows\(^4\). It is convenient to choose a coordinate system in which the antisymmetric \( \Omega \) in (3.88) takes on a particularly simple form. From the \( SO(4) \times SO(4) \) background symmetry there is in this case only a \( SO(3) \) symmetry on the worldvolume. It allows us to transform \( \Omega \) to\(^5\)

\[ \Omega = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & b \\ 0 & 0 & 0 & c \\ 0 & -b & -c & 0 \end{pmatrix}. \] (3.96)

By using (3.81) we can prove that a nontrivial boundary condensate on a flat (3,1)-brane is not consistent with maximal supersymmetry. Aligning the brane along the

\(^4\)The following analysis extends immediately to the case of (4,2), (2,4) branes without additional \( \mathcal{F}^{-1} \) condensates.

\(^5\)To block-diagonalise \( \Omega \), that is, to set \( b = 0 \), a full \( SO(4) \)-rotation is generically necessary.
1,2,3,5 directions and using $\Omega$ as presented in (3.96), we derive

$$M = \exp \left[ \frac{a}{2} \gamma^{12} + \frac{b}{2} \gamma^{25} + \frac{c}{2} \gamma^{35} \right].$$ \hspace{1cm} (3.97)

The straightforward evaluation of (3.81) along the $I = 5$ direction leads to $1 = \Pi M^2 \Pi = M^2$ and thus with (3.80) to $N^2 = 1$. This is the case of a trivial boundary condensate $F = 0$.

The observation that it is impossible to turn on a gauge field condensate on a (3,1) brane and still maintain maximal supersymmetry is related to the previous finding of [23, 41] that class I branes break all dynamical supersymmetries if removed from the origin of transverse space. Indeed, from an open string bosonic mode expansion analogous to (3.135) as to be derived below for the (2,0)—brane, it can be seen that the zero modes along the Neumann directions with nontrivial $F$ tend in the (well behaved) $F \rightarrow \infty \leftrightarrow \theta \rightarrow 0$ limit to the Dirichlet zero modes describing a brane removed from the origin.

From this it follows that only static (Euclidean) branes related to the instanton (as discussed beforehand) or the (4,0), (0,4) branes (to be considered in the next section) can preserve 8 dynamical supersymmetries in the presence of boundary condensates.

The probe brane approach of [116] actually suggests that the coupling of boundary condensates with the non-trivial background flux in the plane wave world volume action requires in a generic (especially non-static) case the introduction of deformed Dirichlet - conditions / deformed D-brane embeddings. An example along these lines is the possibility to shift a Lorentzian $(+,−,3,1)$ brane away from the origin if an appropriate $F_{\pm I}$-flux is switched on, [116]. It would be interesting to understand whether a comparable deformation of the Dirichlet condition (3.64) allows for supersymmetric $(3,1)$ -branes with nontrivial boundary condensates.

### 3.5.3 The $(4,2)$, $(2,4)$ branes with flux

As a second example of branes with nontrivial gauge condensates $F_{IJ} \neq 0$ in the plane wave background, we consider in this section the (4,2)-brane. In the limiting process $F \rightarrow \infty$ discussed above, it is connected to the class II $(4,0)$ brane. As first discussed in [116], this $(4,0)$-brane couples necessarily to the nontrivial $F_5$ background flux (3.2). Due to this, a nonzero boundary condensate $F_{+I} = F^{-I}$ ($F_{-I} = 0$) is required to obey the corresponding field - equations of motion, compare with [116].

The $F^{-I}$-coupling alters the bosonic gluing conditions along the $I = 1,\ldots,4$ Neumann directions, but leaves the other boundary conditions as discussed in section 3.4 above.

For a $(4,2)$ -brane with flux we switch on a boundary condensate along the $A = 5,6$ Neumann directions. Using the fermionic gluing matrix

$$\tilde{M} = \Pi \exp \left[ \frac{\theta}{2} \gamma^{56} \right]$$ \hspace{1cm} (3.98)
and employing the gluing conditions (3.64) and (3.66) as before, the condition (3.63) uniquely determines the bosonic gluing conditions along the \( I = 1, \ldots, 4 \) directions. One obtains\(^6\)

\[
\left( \partial_+ X^I + \partial_- X^I - i m \cos \frac{\theta}{2} X^I \right) \| (4, 2), \mathcal{F}^{-I}, \theta \rangle \rangle = 0 \quad (\tau = 0)
\]  
(3.99)

which in terms of modes reads

\[
\left[ \tilde{a}_0^I + \frac{1 - \cos \frac{\theta}{2}}{1 + \cos \frac{\theta}{2}} a_0^I \right] \| (4, 2), \mathcal{F}^{-I}, \theta \rangle \rangle = 0
\]  
(3.100)

and

\[
\left[ a_0^I + \left( \frac{\omega_n - m \cos \frac{\theta}{2}}{\omega_n + m \cos \frac{\theta}{2}} \right) \tilde{a}_0^I \right] \| (4, 2), \mathcal{F}^{-I}, \theta \rangle \rangle = 0.
\]  
(3.101)

The gluing conditions (3.101) are in direct analogy to the (4,0) case discussed in [117, 51].

The boundary condensate \( \mathcal{F} \) interpolates smoothly between the (4,2) and the (4,0) brane. It is worth noting that under the \( \theta \to \pi \) limit not only \( \mathcal{F}^{AB} \), but also \( \mathcal{F}^{-I} \) tends to zero to exactly reproduce the class I setting of [23, 117].

3.6 Boundary states and cylinder diagrams

In this section we formulate the (2,0)-brane boundary state in the presence of transverse fluxes. As an application, cylinder diagrams are determined as appropriate boundary state overlaps. This prepares the ground for the comparison with open string results from section 3.8 and generalises the findings of [19, 51].

3.6.1 The (2,0)-brane boundary state

From the gluing conditions presented in section 3.4 and the (anti-) commutation relations as summarised in section 3.1.1, one can immediately write down the boundary state of the (2,0) - brane in the presence of a nontrivial boundary condensate. For a brane placed at \( y = 0 \) in transverse space we have (compare for example with [23, 51, 119])

\[
\| (2, 0), 0, \eta, P^+ \rangle \rangle = \mathcal{N}_\theta^{(2,0)} \exp \left[ \sum_{k=1}^{\infty} \left( \frac{1}{\omega_k} \alpha^i_{-k} \tilde{\alpha}^i_{-k} + \frac{N_k^{ij}}{\omega_k} \alpha^{I}_{-k} \tilde{\alpha}^{J}_{-k} - i \eta R_k^{ab} S_h^a \tilde{S}_h^b \right) \right]
\exp \left[ -\frac{1}{2} \left[ \frac{1 + \eta M}{1 - \eta M} \right]_{ab} \theta^a \theta^b - \frac{1}{2} \left[ \frac{1 - \eta M}{1 + \eta M} \right]_{ab} \tilde{\theta}^a \tilde{\theta}^b \right] e^{\frac{1}{2} a^a_0 a^a_0 - \frac{1}{2} a^a_0 \theta^a_0} |0\rangle.
\]  
(3.102)

\(^6\)We present a more detailed derivation of this result in the context of open strings in section 3.7.
The ground state \( |0\rangle \) is given by the usual Fock space vacuum corresponding to the fixed lightcone momentum \( P^+ \). It is in particular annihilated by the fermionic zero modes \( \Theta_L^\perp \) and \( \Theta_R \) as defined in section 3.1.1. The normalisation factor \( \mathcal{N}_\theta^{(2,0)} \) is still to be identified by a comparison with the open string one-loop calculation to be carried out in section 3.8.

**Cylinder Diagrams**

As described in the context of plane wave physics in [19, 51], cylinder diagrams are given in terms of boundary states by overlaps of the following type

\[
\mathcal{A}_{\eta,\eta,\theta} = \langle \langle (2,0), 0, \eta, -P^+, \theta | e^{-2\pi t H_{P^+}} | (2,0), 0, \eta, P^+, \theta \rangle \rangle. \tag{3.103}
\]

Keeping in mind the different momenta \( P^+ \) for the in- and out-going boundary states, the overlap (3.103) can be evaluated by standard algebraic methods. One obtains for the brane / brane case \( \eta = \bar{\eta} \)

\[
\mathcal{A}_{\eta,\eta,\theta} = \left( \mathcal{N}_\theta^{(2,0)} \right)^* \mathcal{N}_\theta^{(2,0)} \tag{3.104}
\]

and for the brane / anti-brane combination with \( \eta = -\bar{\eta} \)

\[
\mathcal{A}_{\eta,-\eta,\theta} = \frac{\left( \mathcal{N}_\theta^{(2,0)} \right)^* \mathcal{N}_\theta^{(2,0)} (g_2^{(m)}(q, \theta))^4}{(2 \sinh \left[ m\pi \sin \frac{\theta}{2} \right])^4 \left( f_1^{(m)}(q) \right)^8}. \tag{3.105}
\]

The function \( f_1^{(m)}(q) \) from [19] is provided in the appendix C in equation (C.1). Furthermore, \( g_2^{(m)}(q, \theta) \) is the following deformation of the function \( g_2^{(m)}(q) \) from [51] (compare with (C.10)):

\[
g_2^{(m)}(q, \theta) = 2 \sinh \left[ m\pi \sin \frac{\theta}{2} \right] q^{-2\Delta_m} \left( 1 + \frac{\sin^2 \frac{\theta}{2}}{(1 - \cos \frac{\theta}{2})^2} q^m \right) \left( 1 + \frac{\sin \frac{\theta}{2}}{(1 + \cos \frac{\theta}{2})^2} q^m \right) \prod_{n=1}^{\infty} \frac{1 + q^{\frac{\omega_n - m \cos \frac{\theta}{2}}{\omega_n + m \cos \frac{\theta}{2}}} \left( 1 + q^{\frac{\omega_n + m \cos \frac{\theta}{2}}{\omega_n - m \cos \frac{\theta}{2}}} \right)}{\left( 1 + q^{\frac{\omega_n - m \cos \frac{\theta}{2}}{\omega_n + m \cos \frac{\theta}{2}}} \right) \left( 1 + q^{\frac{\omega_n + m \cos \frac{\theta}{2}}{\omega_n - m \cos \frac{\theta}{2}}} \right)}. \tag{3.106}
\]

The zero mode contributions in (3.106) might alternatively be written as

\[
q^{-2\Delta_m} \frac{4 \sinh \left[ m\pi \sin \frac{\theta}{2} \right]}{\sin \frac{\theta}{2}} \sqrt{\left( \sin^2 \frac{\theta}{4} + \cos^2 \frac{\theta}{4} q^m \right) \left( \cos^2 \frac{\theta}{4} + \sin^2 \frac{\theta}{4} q^m \right)}. \tag{3.107}
\]
which will be used below to determine the $\theta \to 0$ limit.

Using the open-string result (3.161) to be derived later on, the boundary state
normalisation factor $\mathcal{N}_\theta^{(2,0)}$ is given up to a phase by

$$\mathcal{N}_\theta^{(2,0)} = 2 \sinh \left[ m\pi \sin \frac{\theta}{2} \right]. \quad (3.109)$$

This again reproduces the $(2,0)$ result of [19], but vanishes in the instanton limit.
As the fermionic part of the boundary state (3.102) diverges in this limit, this is,
however, not surprising. Altogether it yields a smooth behaviour of the different
overlaps in both limiting cases.

The behaviour under modular transformations of the $\theta$-dependent function (3.107)
will be discussed in the appendix C. It is straightforward to see that (3.107) connects
the functions $f_2^{(m)}(q)$ and $g_2^{(m)}(q)$ from [19, 51] via

$$\lim_{\theta \to \pi} g_2^{(m)}(q, \theta) = 2 \sinh \left[ m\pi \right] \left( f_2^{(m)}(q) \right)^2. \quad (3.110)$$

With this, (3.104) and (3.105) reproduce the (closed string) results of [19, 51]. Further
details are provided in the appendix C.

### 3.6.2 The $(4,2) - (0,2)$ - overlap

As an example of an overlap containing the $(4,2)$ boundary state with nonzero fluxes
$\mathcal{F}^{AB}$ ($A, B = 5, 6$) and $\mathcal{F}^{-I}$ ($I = 1 \ldots 4$) we consider here the cylinder diagram with
a $(0,2)$ - (anti-) brane. Both branes are assumed to have the same transverse gauge
field strength $\mathcal{F}^{AB}$ on their worldvolumes:

$$B_{\eta,0} = \langle \langle (0,2), 0, \eta, -P^+, \theta | e^{-2\pi i H P^+} | (4,2), 0, P^+, \theta, \mathcal{F}^{-1} \rangle \rangle. \quad (3.112)$$

For the overlap with the $(0,2)$ - brane ($\eta = 1$) we deduce

$$B_{(\eta=1),\theta} = \mathcal{N}_\theta^{(0,2)} \mathcal{N}_\theta^{(4,2)} \left( \frac{\prod_{n=1}^{\infty} \left( 1 - q^{\omega_n} \right)}{\prod_{n=1}^{\infty} \left( 1 + \frac{\omega_n - m \cos \frac{\theta}{2}}{\omega_n + m \cos \frac{\theta}{2}} q^{\omega_n} \right)} \right)^4. \quad (3.113)$$

and for the antibrane ($\eta = -1$)

$$B_{(\eta=-1),\theta} = \mathcal{N}_\theta^{(0,2)} \mathcal{N}_\theta^{(4,2)} \left( \frac{\prod_{n=1}^{\infty} \left( 1 + \frac{\omega_n + m \cos \frac{\theta}{2}}{\omega_n - m \cos \frac{\theta}{2}} q^{\omega_n} \right)}{\prod_{n=1}^{\infty} \left( 1 - q^{\omega_n} \right)} \right)^4. \quad (3.114)$$

In both cases, the zero-mode contributions as for example from the bosons ($I = 1, \ldots, 4$)

$$\left( 1 + \frac{1 - \cos \frac{\theta}{2}}{1 + \cos \frac{\theta}{2}} q^m \right)^{-2} \quad (3.115)$$
cancel out with the corresponding fermionic terms.

Besides the product representation of the $f_1^{(m)}$ function from [19], the special functions in (3.113) and (3.114) are essentially given by the different halves of $g_2^{(m)}$ from (3.107) which themselves have a good behaviour under modular transformations. A comparable result was found in [51] for the $(4,0) - (2,0)$ overlap without gauge field excitations.

### 3.7 The open string description

In this section we consider the previously introduced branes from an open string point of view. First, we study the open string (dynamical) supersymmetries in general and reproduce the conditions (3.80) and (3.81) from the boundary state approach. Thereafter, we present a detailed treatment of open strings ending on $(2,0)$ branes with $\mathcal{F} \neq 0$ in section 3.8. Finally, we supply strong evidence for the consistency of the open and closed string results by establishing the ‘Cardy condition’ as explained in the last section.

Using the results reviewed in section 3.1.1, the open string theory in the plane wave background is still described by the equations of motion (3.8) and (3.9) with the new mass parameter $\tilde{m}$ as introduced in section 3.2.

As described in section 3.3, the bosonic boundary conditions for the case of a non-vanishing boundary condensate $\mathcal{F}^{IJ}$ are given by

$$\partial_\sigma X^I + \mathcal{F}^{IJ} \partial_\tau X^J = 0; \quad \sigma = 0, \pi \quad (3.116)$$

for the Neumann and

$$X^i = g_i^\sigma \quad \sigma = 0, \pi \quad (3.117)$$

for the Dirichlet directions. For the fermions we furthermore have

$$S(\tau, \sigma = 0) = \bar{M}\bar{S}(\tau, \sigma = 0), \quad S(\tau, \sigma = \pi) = \eta \bar{M}\bar{S}(\tau, \sigma = \pi). \quad (3.118)$$

As before, the parameter $\eta = \pm 1$ distinguishes between the case of a brane / brane or a brane / antibrane pair.

#### 3.7.1 Open string supersymmetries

As explained in section 3.1.1, the dynamical supersymmetries in the closed string sector can be derived from the conserved currents (3.36) and (3.37) and are calculated by expressions as

$$\sqrt{2P^+}Q_\alpha = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left( \partial_- \gamma^s \Sigma - \gamma^s \gamma^s \Pi \bar{S} \right)_\alpha. \quad (3.119)$$

Using the same formulae in the open sector leads to time-dependent expressions. Possibly conserved supersymmetries with cancelling time dependencies are given by suitable linear combinations

$$Q_{\text{open}} = Q - K\bar{Q} \quad (3.120)$$
as already explained in chapter 2.

From the definition of (3.120) in terms of (3.36) and (3.37) it follows that the open string supercharges (3.120) are time independent in case of a vanishing boundary term

\[
\left( \partial_+ X^a \gamma^a \bar{S} + \tilde{m} X^a \gamma^a \Pi \bar{S} \right) - K \left[ -\partial_+ X^a \gamma^a \bar{S} + \tilde{m} X^a \gamma^a \Pi \bar{S} \right] \right|_{\sigma=0} = 0. \tag{3.121}
\]

Separating (3.121) into contributions from Neumann and Dirichlet directions, we obtain with (3.57) and (3.118)

\[
0 = \partial_+ X^I \left( \gamma^I \tilde{M} - N^IJ K \gamma^J \right) \bar{S} + \tilde{m} X^I \left( \gamma^I \Pi - K \gamma^I \Pi \tilde{M} \right) \bar{S} \right|_{\sigma=0} \tag{3.122}
\]

from the Neumann and

\[
0 = \partial_+ X^I \left( \gamma^I \tilde{M} - K \gamma^I \tilde{M} \right) \bar{S} + \tilde{m} X^I \left( \gamma^I \Pi - K \gamma^I \Pi \tilde{M} \right) \bar{S} \right|_{\sigma=0} \tag{3.123}
\]

from the Dirichlet directions. These conditions require in particular

\[
\gamma^I N^IJ = K^I \gamma^J \tilde{M}; \quad \gamma^I = K^I \gamma^I \Pi \tilde{M} \Pi; \quad \gamma^I \tilde{M} - K \gamma^I = 0 \tag{3.124}
\]

and a comparison with (3.80) and (3.81) reveals

\[
K = \tilde{M} = M^t. \tag{3.125}
\]

As long as $\Pi \tilde{M} \Pi \tilde{M} \neq 1$ (that is, for non-class II branes), we furthermore have to set $y_\sigma = 0$, that is, to place the branes to the origin of transverse space.

For a $(2,0)$ brane with $F \neq 0$ the previous considerations lead to the gluing matrix

\[
\tilde{M} = M^t = \exp \left[ -\frac{\theta}{2} \gamma^{12} \right]. \tag{3.126}
\]

In the conventions of [41, 116] this actually corresponds in the limit of a vanishing boundary condensate to the anti-$(2,0)$ brane, as

\[
\tilde{M} \rightarrow -\gamma^{12} \tag{3.127}
\]

for $\theta \rightarrow \pi$. But this is simply the usual sign ambiguity in between the open- and closed string picture quantities and we will still refer to (3.126) as the $(2,0)$ brane’s gluing matrix.

The conserved supercharges corresponding to the choice (3.126) again interpolate between the instanton and the $(2,0)$ supercharges appearing respectively in [51] and [41, 117]. The limits are the combinations $Q_{(0,0)} = Q - \bar{Q}$ for the instanton and $Q_{(2,0)} = Q + \gamma^{12} \bar{Q}$ for the $(2,0)$-brane.

\footnote{As explained in chapter 2, this is a sufficient, but not necessary condition. See also the next chapter 4 for a further discussion.}
3.8. THE OPEN STRING DESCRIPTION OF THE (2, 0)-BRANE

The (4, 2) brane with $F^{-I} \neq 0$

For the open string description of the (4, 2) brane with boundary condensates we have the following boundary conditions at $\sigma = 0, \pi$:

\[
\begin{align*}
\partial_+ X^i + \partial_- X^i &= 0; \quad X^i = 0; \quad i = 7, 8 \\
\partial_+ X^A + N^{AB} \partial_- X^B &= 0; \quad A, B = 5, 6 \\
\partial_+ X^I - \partial_- X^I + \alpha \tilde{m} X^I &= 0; \quad I = 1, \ldots, 4 \\
S &= \tilde{M} \tilde{S}.
\end{align*}
\]

From (3.121) we obtain with $K = \tilde{M}$ the same conditions as previously derived in (3.80) and (3.81) for the Dirichlet ($i = 7, 8$) and the Neumann ($A = 5, 6$) directions. These conditions are solved by the matrix

\[
\tilde{M} = \Pi \exp \left[-\frac{\theta}{2} \gamma^{56}\right]
\]

which also reproduces the correct gluing conditions for the (4, 2) brane in the $\theta \to \pi$ limit.

For the remaining $I = 1, \ldots, 4$ directions the requirement for eight conserved dynamical supersymmetries becomes

\[
0 = \partial_- X^I \left( \gamma^I M + M \gamma^I \right) \tilde{S} + \tilde{m} X^I M \left( M \gamma^I \Pi - \gamma^I \Pi M - \alpha \gamma^I \right)_{\sigma = 0}.
\]

With (3.132) the last condition is obeyed in case of $\alpha = 2m \cos \frac{\theta}{2}$. This determines the strength of the $F^{-I}$ components as functions of the ‘transverse’ field-strength $F^{AB}$.

3.8 The open string description of the (2, 0)-brane

In this section we present a detailed treatment of the open string theory for the (2, 0) brane with nontrivial boundary condensate. In a first step, we derive the relevant bosonic and fermionic mode expansions and determine the corresponding lightcone gauge Hamiltonian. After a brief summary of the results from the canonical quantisation carried out in the appendix D, we finally calculate some open string partition functions and relate them to closed string boundary state overlaps from section 3.6. It is worth mentioning that the analogous case of a (4,2) brane with flux discussed already in the closed string sector could be dealt with by essentially the same methods.

3.8.1 Bosons

Using the most general solution to the bosonic equations of motion (3.11), the mode expansion along the Dirichlet directions for strings ending on branes at the origin
of transverse space becomes
\[ X^i(\tau, \sigma) = -2 \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{-i\omega_n \tau}}{\omega_n} \alpha^i_n \sin(n \sigma); \quad \omega_n = \text{sgn}(n) \sqrt{n^2 + \tilde{m}^2}, \] (3.134)

compare for example with \[41, 19\]. For the more interesting case of Neumann directions we obtain
\[ X^I(\sigma) = e^{-i\tilde{m} \sin \frac{\theta}{2} \tau} \exp \left[ i J \tilde{m} \cos \frac{\theta}{2} \right]^{I^J} a^I + e^{i\tilde{m} \sin \frac{\theta}{2} \tau} \exp \left[ -i J \tilde{m} \cos \frac{\theta}{2} \right]^{I^J} a^I \]
\[ + i \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{-i\omega_n \tau}}{\omega_n} \left[ \alpha^I_n e^{in\sigma} + \tilde{\alpha}^I_n e^{-in\sigma} \right] \] (3.135)

with
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \tilde{\alpha}^I_n = -\frac{F - \frac{n}{\omega_n} \alpha_n}{F + \frac{n}{\omega_n}}; \quad F = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}; \quad f = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}. \] (3.136)

The last equation follows by comparing (3.58) with (3.90), yielding in particular
\[ N = \frac{1}{1 + f^2} \begin{pmatrix} (1 - f^2) & 2f \\ -2f & (1 - f^2) \end{pmatrix}. \] (3.137)

The first two terms in (3.135) are the substitutes for the otherwise absent zero modes in case of nontrivial boundary condensates. They correspond to modes \( n \) for which the matrix \( F \pm \frac{n}{\omega_n} \) appearing in (3.136) is degenerate, that is, for modes with
\[ 0 = \det \left( F \pm \frac{n}{\omega_n} \right) = f^2 + \left( \frac{n}{\omega_n} \right)^2 \iff n = \pm i\tilde{m} \cos \frac{\theta}{2}. \] (3.138)

The bosonic mode expansion (3.135) was first written down in \[38\] in conventions more closely resembling the usual flat space description, compare for example with \[37\]. Their modings, however, display a singular behaviour in the \( \theta \to 0 \) limit.

The two extra ‘special’ terms in (3.135) fulfill the boundary condition (3.56) for all \( \sigma \) and not only on the boundary. In the limits \( \theta \to \pi, 0 \) these terms tend to (redefinitions of) the usual ‘zero’-modes of the \((2,0)\)-brane or the instanton.

### 3.8.2 Fermions

In this section we determine the fermionic mode expansions using the general solutions (3.12), (3.13) for the equations of motion (3.9). We enforce the boundary conditions (3.118) with the gluing matrix \( \tilde{M} = \exp \left[ -\frac{\theta}{2} \gamma^{12} \right] \).

For a nontrivial \( \tilde{M} \) with \( \theta \in (0, \pi) \) the boundary conditions (3.118) lead for both choices \( \eta = \pm 1 \) to vanishing ‘zero’-modes \( S = T = \tilde{S} = \tilde{T} = 0 \). The conditions for the nonzero modes read
\[ \left( \mathbb{1} + \frac{i}{\tilde{m}} (\omega_n - n) \tilde{M} \Pi \right) S_n = \left( \tilde{M} - \frac{i}{\tilde{m}} (\omega_n - n) \Pi \right) \tilde{S}_n \] (3.139)
\[ \left( \mathbb{1} + \frac{i}{\tilde{m}} (\omega_n - n) \tilde{M} \Pi \right) S_n = \left( \eta \tilde{M} - \frac{i}{\tilde{m}} (\omega_n - n) \Pi \right) e^{-2\pi in} \tilde{S}_n. \] (3.140)
The brane - brane configuration: $\eta = 1$

Using (3.139) and (3.140), the mode expansions for strings stretching between a brane - brane pair are determined to be

$$S(\tau, \sigma) = \frac{1 + \Pi}{2} \exp \left[ + \tilde{m} \sin \frac{\theta}{2} \tau \gamma^{12} \right] S_0 e^{\tilde{m} \cos \frac{\theta}{2} \sigma}$$

$$+ \frac{1 - \Pi}{2} \exp \left[ - \tilde{m} \sin \frac{\theta}{2} \tau \gamma^{12} \right] \tilde{M} S_0 e^{-\tilde{m} \cos \frac{\theta}{2} \sigma}$$

$$+ \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left[ S_n e^{-i(\omega_n \tau - n \sigma)} + \frac{i}{\tilde{m}} (\omega_n - n) \Pi S_n e^{-i(\omega_n \tau + n \sigma)} \right] \tag{3.141}$$

and

$$\tilde{S}(\tau, \sigma) = \frac{1 + \Pi}{2} \exp \left[ + \tilde{m} \sin \frac{\theta}{2} \tau \gamma^{12} \right] \tilde{M} S_0 e^{\tilde{m} \cos \frac{\theta}{2} \sigma}$$

$$+ \frac{1 - \Pi}{2} \exp \left[ - \tilde{m} \sin \frac{\theta}{2} \tau \gamma^{12} \right] S_0 e^{-\tilde{m} \cos \frac{\theta}{2} \sigma}$$

$$+ \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left[ \tilde{S}_n e^{-i(\omega_n \tau + n \sigma)} - \frac{i}{\tilde{m}} (\omega_n - n) \Pi \tilde{S}_n e^{-i(\omega_n \tau - n \sigma)} \right]. \tag{3.142}$$

We furthermore have

$$\omega_n = \text{sgn}(n) \sqrt{n^2 + \tilde{m}^2}, c_n = \frac{\tilde{m}}{\sqrt{2 \omega_n (\omega_n - n)}}, -2 \sin \frac{\theta}{2} \gamma^{12} = \tilde{M} - \tilde{M}^t \tag{3.143}$$

and the operator identifications

$$\tilde{S}_n = \left( \frac{1 + \frac{i}{\tilde{m}} (\omega_n - n) \Pi \Pi}{\tilde{M} - \frac{i}{\tilde{m}} (\omega_n - n) \Pi} \right) S_n. \tag{3.144}$$

As in the bosonic case, the first two terms in each expansion correspond to modes with $n = \pm i \tilde{m} \cos \frac{\theta}{2}$ for which some matrices in (3.139) and (3.140) are degenerate. These special ‘zero’-modes again fulfils the fermionic boundary conditions for all $\sigma \in [0, \pi]$ and not only on the boundaries.

The brane - antibrane configuration: $\eta = -1$

In distinction to the situation described beforehand, there are no extra ‘special’-zero mode contributions in the case of an open string joining a brane - antibrane -pair with the same gauge condensates $\mathcal{F}$. This follows immediately by combining (3.139) and (3.140) to

$$\left( n - i \tilde{m} \cos \frac{\theta}{2} \Pi \right) \tilde{S}_n = - \left( n + i \tilde{m} \cos \frac{\theta}{2} \Pi \right) e^{-2i \pi n} \tilde{S}_n, \tag{3.145}$$
yielding \( \tilde{S}_n = \pm \tilde{m} \cos \frac{\theta}{2} = 0 \).

From (3.139) the identification between the nonzero modes \( \tilde{S}_n \) and \( S_n \) is still given by (3.144). To simultaneously fulfill the second condition (3.140), the modings \( n \) have to obey

\[
\begin{align*}
n &\in P_\theta^+ : \quad \frac{n + i \tilde{m} \cos \frac{\theta}{2}}{n - i \tilde{m} \cos \frac{\theta}{2}} = e^{2\pi i n^a}; \quad n \neq 0 \\
n &\in P_\theta^- : \quad \frac{n - i \tilde{m} \cos \frac{\theta}{2}}{n + i \tilde{m} \cos \frac{\theta}{2}} = e^{2\pi i n^a}; \quad n \neq 0
\end{align*}
\]

for the \( S^+ = \frac{1+\Pi}{2} S_n \) modes and

\[\begin{align*}
n &\in P_\theta^+ : \quad \frac{n + i \tilde{m} \cos \frac{\theta}{2}}{n - i \tilde{m} \cos \frac{\theta}{2}} = e^{2\pi i n^a}; \quad n \neq 0 \\
n &\in P_\theta^- : \quad \frac{n - i \tilde{m} \cos \frac{\theta}{2}}{n + i \tilde{m} \cos \frac{\theta}{2}} = e^{2\pi i n^a}; \quad n \neq 0
\end{align*}\]

for the \( S^- = \frac{1-\Pi}{2} S_n \) modes. This is in direct analogy to the \((0,0) - \overline{(0,0)}\) configuration described in [51]. Both equations have infinitely many solutions on the real axis and for small \( \tilde{m} \) all of them are close to the flat space case of half integer numbers. From \( \tilde{m} \cos \frac{\theta}{2} > \frac{1}{2} \pi \) on, however, two solutions of \( P_\theta^- \) become imaginary. This might be interpreted in analogy to the additional 'zero-modes' for a string stretching between two branes of the same kind with nonzero flux.

3.8.3 The lightcone gauge Hamiltonian

The light-cone gauge Hamiltonian is obtained by the integral

\[
\frac{X^+}{2\pi} H^{\text{open}} = \frac{1}{4} \int_0^\pi d\sigma \left( \dot{X}^2 + X'^2 + \tilde{m}^2 X^2 \right) + \frac{i}{2} \int_0^\pi d\sigma \left( S\dot{S} + \tilde{S}\dot{\tilde{S}} \right), \tag{3.148}
\]

compare with [94, 95]. By using the conventions of Landau-Ginzburg models a derivation of the corresponding currents is presented in the appendix A.

With the mode expansions (3.134), (3.135) and (3.141), (3.142) we obtain from (3.148)

\[
\frac{X^+}{2\pi} H^{\text{open}} = \frac{\tilde{m}}{2 \cos \frac{\theta}{2}} \sinh \left[ \tilde{m} \pi \cos \frac{\theta}{2} \right] \left( a e^{-i \pi \tilde{m} \cos \frac{\theta}{2}} a^+ + a^+ e^{i \pi \tilde{m} \cos \frac{\theta}{2}} a \right)
\]

\[
+ i \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \sinh \left[ \tilde{m} \pi \cos \frac{\theta}{2} \right] \left( S_0 \frac{1 - \Pi}{2} \gamma^2 S_0 e^{-\tilde{m} \cos \frac{\theta}{2} \pi} + \pi \sum_{n \neq 0} \left( \alpha^I_n \alpha_n^+ + \alpha_n^+ \alpha_n^I + \omega_n S_n S_n^\dagger \right) \right)
\]

for open strings stretching between a brane - brane configuration at transverse position \( y = 0 \).

For the Hamiltonian in the brane - antibrane case the fermionic zero modes \( S_0 \) are absent and the fermionic nonzero modes have to fulfil either (3.146) or (3.147). In this case there is furthermore a nontrivial normal ordering constant to be discussed briefly below. Apart from possible zero mode contributions, normal ordering terms are absent in the first case due to supersymmetry.
3.8.4 Quantisation

The canonical quantisation proceeds in the usual way. By stressing that we have \( F = f, B = 0 \), it is clear that the fermionic canonical conjugated momenta are unaffected by the boundary condensates. Requiring therefore the equal time (anti-) commutation relations (3.26) - (3.31), each to be evaluated for values \( 0 < \sigma, \bar{\sigma} < \pi \), one can derive the corresponding mode relations as explained in detail in the appendix D.

To summarise the results, one obtains for the bosons

\[
[\alpha^i_n, \alpha^j_m] = \omega_n \delta_{m+n} \delta^{ij} \tag{3.150}
\]

\[
[\alpha^I_n, \alpha^J_m] = \omega_n \delta_{m+n} \delta^{IJ} \tag{3.151}
\]

\[
[a^I, a^{IJ}] = \pi \sin \theta \left[ \frac{\cosh(\bar{m}_n \cos \frac{\theta}{2})}{\sinh(\bar{m}_n \cos \frac{\theta}{2})} - iJ \right] \tag{3.152}
\]

\[
= \frac{\pi \sin \theta}{\sinh(\bar{m}_n \cos \frac{\theta}{2})} \exp \left[ -iJ \bar{m}_n \cos \frac{\theta}{2} \right] \tag{3.153}
\]

and for the fermions

\[
\{s^a_0, s^b_0\} = \frac{\bar{m}_n \cos \frac{\theta}{2}}{\sinh(\bar{m}_n \cos \frac{\theta}{2})} \left( \frac{1 + \Pi}{2} e^{\pi \bar{m}_n \cos \frac{\theta}{2}} + \frac{1 - \Pi}{2} e^{-\pi \bar{m}_n \cos \frac{\theta}{2}} \right)^{ab} \tag{3.154}
\]

\[
\{s^a_n, s^b_m\} = \delta_{n+m} \delta^{ab}. \tag{3.155}
\]

For further details on the quantisation in the bosonic case and for a discussion of the relation with noncommutative field theories, we refer to [38, 37].

3.8.5 Partition Functions

In this final part of our open string treatment we calculate partition functions for strings stretching between \((2,0)\) branes with flux for the cases of brane - brane and brane - antibrane pairs.

As discussed in [107, 19], these partition functions are given by

\[
Z(\tilde{t}) = \text{Tr} \exp \left[ -\frac{X^+}{2\pi} H^{\text{open}}_t \right], \tag{3.156}
\]

with a trace running over the open string Hilbert spaces as (implicitly) determined in the last section.

For the brane - brane pair, the normal ordered contributions of the bosonic zero modes in (3.149) are given by

\[
\bar{m}_n \sinh(\bar{m}_n \cos \frac{\theta}{2}) a^+ \exp \left[ i\pi \bar{m}_n \cos \frac{\theta}{2} \right] a + 2\pi \bar{m}_n \sin \frac{\theta}{2} \tag{3.157}
\]
and are therefore leading to a factor
\[
\frac{1}{(2 \sinh \left[ \frac{\bar{m} \pi \bar{t} \sin \frac{\theta}{2}}{2} \right])^2}
\]
as contribution to (3.156).

Requiring the fermionic vacuum to be annihilated by the combinations
\[
\lambda^l = S_0^{l+2} + i S_0^{2+l}, \quad l = 0, \ldots, 3
\]
the fermionic zero modes result in the factor
\[
\left( 2 \sinh \left[ \frac{\bar{m} \pi \bar{t} \sin \frac{\theta}{2}}{2} \right] \right)^4.
\]
Together with the nonzero-mode contributions, calculated as for example discussed in [51], one obtains the open string partition functions
\[
Z_{\eta, \lambda, \theta}(\bar{t}) = \left( 2 \sinh \left[ \frac{\bar{m} \pi \bar{t} \sin \frac{\theta}{2}}{2} \right] \right)^2
\]
and
\[
Z_{\eta, -\lambda, \theta}(\bar{t}) = \frac{1}{(2 \sinh \left[ \frac{\bar{m} \pi \bar{t} \sin \frac{\theta}{2}}{2} \right])^2} \left( \frac{g_4^{(\bar{m})}(\bar{t}, \theta)}{f_4^{(\bar{m})}(\bar{t})} \right)^4.
\]
Here, the first equation (3.161) describes the situation of a brane-brane pair and the second equation (3.162) is obtained for a brane-antibrane configuration.

The function \( f_4^{(m)}(\bar{t}) \) is taken from [19, 51] and can be found in equation (C.1). Furthermore, we have set
\[
\tilde{g}_2^{(\bar{m})}(\bar{t}, \theta) = 2 \sinh \left[ \frac{\bar{m} \pi \bar{t} \sin \frac{\theta}{2}}{2} \right] \tilde{g}^{\Delta_{\eta, \lambda, \theta} \frac{\theta}{2}(1-\sin \frac{\theta}{2})} \times \prod_{\lambda \in \mathcal{P}^+} \sqrt{(1 - \tilde{q}^{(x^2+\bar{m}^2)})} \prod_{\lambda \in \mathcal{P}^-} \sqrt{(1 - \tilde{q}^{(x^2+\bar{m}^2)})}
\]
as \( \theta \)-dependent generalisation of the function \( \tilde{g}_2^{(\bar{m})} \) from (C.14).

The offset \( \Delta_{\eta, \lambda, \theta} \) is essentially determined by the normal ordering constant in the light-cone gauge Hamiltonian (3.149). Its explicit form will be presented in the appendix C where also the identity of (3.161) and (3.162) with the corresponding closed string results will be established. By this, the branes with boundary condensates pass this important consistency check.

As in the closed string picture, the family of functions (3.163) reproduces the results of [51] in the limits \( \theta \to 0, \pi \):
\[
\lim_{\theta \to 0} \tilde{g}_4^{(\bar{m})}(\bar{t}, \theta) = \tilde{g}_4^{(\bar{m})}(\bar{t})
\]
\[
\lim_{\theta \to \pi} \tilde{g}_4^{(\bar{m})}(\bar{t}, \theta) = 2 \sinh |m\pi| \left( f_4^{(\bar{m})}(\bar{q}) \right)^2.
\]
This is in particular also consistent with the modular transformation properties discussed in [19, 51].
Chapter 4

Boundary fermions and the plane wave

4.1 Introduction

In this chapter we consider branes in the plane wave background (3.1) in the context of boundary fermions as introduced in chapter 2. The boundary excitation give rise to deformed boundary conditions in the bosonic and fermionic sectors, modifying the previously used conditions (3.64), (3.65) and (3.69) from chapter 3.

In the classification reviewed in section 3.2, the resulting new supersymmetric boundary configurations are of type $(n,n)$. As a main result, the limiting case of the spacetime filling $(4, 4)$-brane with only Neumann directions in the transverse space is shown to be maximally spacetime supersymmetric.

The relation of plane wave physics to boundary fermionic fields is most easily derived in the context of Maldacena - Maoz backgrounds. As explained in section 1.1, the plane wave background is obtained in this setting as a Landau - Ginzburg model with superpotential

\[ W(z) = -im \sum_{j=1}^{4} (z^j)^2; \quad \overline{W}({\bar{z}}) = im \sum_{j=1}^{4} (\bar{z}^j)^2. \]  

(4.1)

The relation to the standard formulation using Green - Schwarz spinors will be discussed in detail in the next section.

In addition to the information presented in the review section 3.2, we want to stress that all the maximally supersymmetric branes in the plane wave background are also integrable, that is, they preserve the integrable structure of the closed string theory in the sense of [54]. As this spacetime gives rise to a free worldsheet theory, relatively little attention is usually payed to this point. However, the inclusion of boundary fermions modifies this situation, as they generically give rise to an interacting boundary field theory, which in most cases is also incompatible with integrability. The requirement of conserved higher spin currents in the boundary theory will lead to strong constraints on admissible boundary couplings. It is worth
mentioning that the massive Ising model, appearing as the fermionic part of the plane wave worldsheet theory on \((3.1)\), has been intensively discussed in the literature on integrable (boundary) models, see for example \([54, 36]\) and references therein.

Using the methods explained in 2, we are first of all aiming at integrable branes with a preserved \(\mathcal{N} = 2\) (worldsheet) supersymmetry structure. The bosonic boundary conditions of these new branes are expressible as a standard coupling to a nonzero longitudinal flux \(\mathcal{F}_{+f}\). The fermionic bulk and boundary fields, on the other hand, are initially determined by a coupled system of differential equations on the boundary. The on-shell elimination of the boundary fermions from this system leads to an expression for the bulk field boundary conditions in terms of a linear differential equation in the boundary parametrising coordinate \(\tau\). As an interesting result, the boundary fermions can finally be expressed as a function of the bulk fermionic fields without including additional degrees of freedom. In the quantum theory the corresponding expressions also correctly reproduce the required quantum mechanical anticommutation relations for the boundary fields.

This chapter is organised as follows. In the starting section 4.2 we collect background information on the plane wave theory formulated as a Landau-Ginzburg model and derive the relation between the \(\mathcal{N} = (2, 2)\) worldsheet supercharges and the maximal spacetime supersymmetry from \([25, 94, 95]\). In section 4.3, we initiate our study of boundary fermions in the context of plane wave physics and derive the conditions for integrable and \(\mathcal{N} = 2\) supersymmetric branes. The branes solving these conditions are then studied in detail in section 4.4 by constructing and quantising the corresponding open string theory along the lines of chapter 3. In the subsequent section 4.5, we construct the corresponding boundary states and consider the conservation of spacetime supersymmetries beyond the \(\mathcal{N} = 2\) subalgebra. In the final section 4.6, the equivalence of the open and closed string constructions is discussed by establishing the equality of certain open string partition functions with corresponding closed string boundary state overlaps.

### 4.2 The plane wave as a Landau-Ginzburg model

In this section we collect some information about the worldsheet theory for strings in the maximally supersymmetric plane wave background of \([25]\) formulated as a \(\mathcal{N} = (2, 2)\) supersymmetric Landau-Ginzburg model. In particular, we explain the relation between the Landau-Ginzburg and Green-Schwarz fermions following the layout of \([88]\). This especially leads to expressions for the \(\mathcal{N} = (2, 2)\) supercharges as linear combinations of spacetime supersymmetries from \([25, 94, 95]\). Our conventions regarding Landau-Ginzburg models are those summarised for example in \([62]\).

As already mentioned above, the plane wave theory from \([25, 94, 95]\) is obtained as a Landau-Ginzburg model with component Lagrangian \((2.11)\) by choosing the
4.2. **THE PLANE WAVE AS A LANDAU-GINZBURG MODEL**

superpotential to

\[ W(z) = -im \sum_{i=1}^{4} (z^i)^2; \quad \bar{W}(\bar{z}) = im \sum_{j=1}^{4} (\bar{z}^j)^2. \] (4.2)

This choice gives rise to the equations of motion

\[ (\partial_+ \partial_- + m^2) z^i = 0 = (\partial_+ \partial_- + m^2) \overline{z}^i \] (4.3)

for the bosons and

\[ 0 = \partial_- \psi^i_- + m \overline{\psi}^i_- \quad \quad 0 = \partial_- \overline{\psi}^i_+ + m \psi^i_+ \]
\[ 0 = \partial_+ \psi^i_- - m \overline{\psi}^i_+ \quad \quad 0 = \partial_+ \overline{\psi}^i_+ - m \psi^i_- \] (4.4)

for the fermions.

The relation between the fermions in (2.11) and the standard Green-Schwarz fields \( S, \tilde{S} \) was pointed out in [88] and is given by

\[ S^a = \psi^a \Gamma^a \eta^b + \overline{\psi^a} \Gamma^a \eta^b \] (4.5)
\[ \tilde{S}^a = \psi^a \Gamma^a \eta^b + \overline{\psi^a} \Gamma^a \eta^b \] (4.6)

with a constant spinor \( \eta \) fulfilling

\[ 0 = \Gamma \eta; \quad \eta \eta^* = 1; \quad \Pi \eta = -\eta^*. \] (4.7)

The (new) Majorana type requirement \( \Pi \eta = -\eta^* \) contains the real matrix

\[ \Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \] (4.8)

from [94, 95] and is consistent due to \( \Pi^2 = 1 \). It is chosen to determine an up to a sign unique spinor \( \eta \) and it correctly reproduces the equations of motion

\[ \partial_+ S = m \Pi \tilde{S}; \quad \partial_- \tilde{S} = -m \Pi S \] (4.9)

for the GS fields by starting from (4.4).

The identifications (4.5), (4.6) or the inverted expressions

\[ \psi^i_+ = \frac{1}{2} \eta^{*a} \Gamma^a_{ob} S^b \quad \quad \overline{\psi}^i_+ = \frac{1}{2} \eta^{*a} \Gamma^a_{ob} \overline{S}^b \] (4.10)
\[ \psi^i_- = \frac{1}{2} \eta^{*a} \Gamma^a_{ob} \tilde{S}^b \quad \quad \overline{\psi}^i_- = \frac{1}{2} \eta^{*a} \Gamma^a_{ob} \overline{\tilde{S}}^b \] (4.11)

can be geometrically interpreted as follows [88]. The choice of a complex structure in the definition of the Landau-Ginzburg Lagrangian (2.11) singles out a \( SU(4) \) subgroup of the \( SO(8) \) in whose spinor representations the standard Green-Schwarz spinors reside. Under this subgroup these representations decompose into

\[ 8_- \rightarrow 4 + \overline{4} \] (4.12)
and the summands correspond to the spinor fields in (2.11) carrying a vector index. As the superpotential (4.2) already breaks the $SO(8)$-background symmetry present in flat space down to $SO(4) \times SO(4) \times \mathbb{Z}_2$, the complex structure used in the previous argument actually picks out the diagonal $SO(4)$ subgroup of this product group. For this reduced symmetry group the fields $\psi^i_{\pm}, \overline{\psi}^j_{\pm}$ transform in the same representation, explaining the seemingly strange index structure of the equations of motion (4.4).

Before discussing the $\mathcal{N} = (2,2)$ worldsheet supersymmetry, we briefly establish the existence of the spinor $\eta$ with the requirements (4.7). Using the standard properties of the complex Dirac matrices $\Gamma^i, \Gamma^\nu$, the spinor $\eta$ is determined to be

$$\eta = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 (\mathbb{I} - \Pi) \zeta$$

with a constant real spinor $\zeta = \zeta^*$ of appropriate norm. For example by employing the explicit spinor representation presented in chapter 5 of [58], one can show that the matrix $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 (\mathbb{I} - \Pi)$ is of real rank one, that is, $\eta$ is actually unique up to a sign. Finally, by using

$$\Pi = \gamma^1 \gamma^2 \gamma^2 \gamma^4 = \prod_{i=1}^{4} \frac{\Gamma^i + \Gamma^\nu}{\sqrt{2}}$$

the condition $\eta^* = -\Pi \eta$ becomes

$$\eta^* = -\frac{1}{4} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \zeta = -\frac{1}{4} \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \eta.$$  

The lowest weight $su(4)$ state $\eta$ is therefore essentially related to the corresponding highest weight state by complex conjugation.

**4.2.1 $\mathcal{N} = (2,2)$ supersymmetry**

In the following we derive relations between the $\mathcal{N} = (2,2)$ worldsheet supersymmetry and the spacetime supercharges from the Green-Schwarz formulation. This will in particular also lead to an explicit confirmation of the related group theoretical discussion in [51].

The supercurrents for the plane wave Landau-Ginzburg model with superpotential (4.2) are obtained from (2.31) and (2.32) to

$$G^1_\pm = g_{ij} \partial_\pm \bar{z}^i \psi^j_\pm \pm m \bar{\psi}^j_\pm \bar{z}^i$$

$$\overline{G}^1_\pm = g_{ij} \partial_\pm \bar{z}^i \psi^j_\pm \mp m \bar{\psi}^j_\pm \bar{z}^i.$$  

They lead to the conserved charges

$$Q_\pm = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left( g_{ij} \partial_\pm \bar{z}^i \psi^j_\pm \pm m \bar{\psi}^j_\pm \bar{z}^i \right)$$

$$\overline{Q}_\pm = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \left( g_{ij} \psi^j_\pm \partial_\pm \bar{z}^i \pm m \psi^j_\pm \bar{z}^i \right).$$
representing the $\mathcal{N} = (2, 2)$ worldsheet supersymmetry. By using the identifications (4.5), (4.6) and
\[ \gamma^\tau \eta = -i \gamma^{\tau+4} \eta, \quad \gamma^\tau \eta^* = i \gamma^{\tau+4} \eta^* \] (4.20)
from (4.7), we deduce
\[ Q_+ = \frac{1}{4\pi} \int_0^{2\pi} d\sigma \left( g_{\sigma \sigma} \partial_+ \bar{z}^\dagger \left( \eta^* \gamma^4 \bar{S} \right) + m \bar{z}^\dagger \left( \eta \gamma^4 S \right) \right) \] (4.21)
\[ = \frac{\eta^*}{2\pi} \int_0^{2\pi} d\sigma \left( \partial_+ x^I \gamma^I \bar{S} + mx^I \gamma^I \Pi S \right). \] (4.22)
Comparing this with the expressions for the dynamical spacetime supercharges (3.40), we obtain
\[ \frac{Q_+}{\sqrt{2p^+}} = \eta^* \tilde{Q} = (\eta^*)_\alpha \tilde{Q}_\alpha, \] (4.23)
by implicitly using the negative $SO(8)$ chiralities of the spinors $S, \bar{S}$. In a similar way one expresses the remaining supersymmetries as linear combinations of the spacetime charges to
\[ \frac{Q_+}{\sqrt{2p^+}} = \eta^* \tilde{Q} = -\eta \Pi \tilde{Q}, \quad \frac{Q_-}{\sqrt{2p^+}} = \eta^* Q = -\eta \Pi Q \] (4.24)
\[ \frac{Q_+}{\sqrt{2p^+}} = \eta \tilde{Q} = -\eta^* \Pi \tilde{Q}, \quad \frac{Q_-}{\sqrt{2p^+}} = \eta Q = -\eta^* \Pi Q. \] (4.25)

To relate this result to the discussion presented in section 5 in [51], we only have to note that the condition (4.7) requires $\eta$ to be the bottom state discussed in [51] whereas $\eta^*$ is the corresponding top-state as previously established at the end of the last section.

### 4.3 Boundary fermions: Supersymmetry and integrability

With this section we start to construct branes in the plane wave background under the inclusion of boundary fermionic fields. In a first step, we define a suitable boundary Lagrangian and derive the corresponding boundary conditions for the bulk fields and the equations of motion for the boundary fermions.

Using these data, we can thereafter calculate the determining equations for the boundary fields under the requirement of conserved $\mathcal{N} = 2$ supersymmetry and integrability in the boundary theory. Further information about the integrable structure and calculational details omitted in this section can be found in the appendix E.

For all our branes, two directions combined to a complex variable
\[ z^j = x^j + ix^{j+4}, \quad j = 1, \ldots, 4 \] (4.26)
are chosen to have the same type of boundary conditions. For later convenience we furthermore define sets $\mathcal{D}_-, \mathcal{N}_-$ containing the Dirichlet and Neumann directions ranging in $r = 1, \ldots, 4$ and correspondingly $\mathcal{D}_+, \mathcal{N}_+$ with elements in $r = 5, \ldots, 8$.

### 4.3.1 Boundary conditions

By mildly extending the boundary Lagrangians defined in [54, 125, 99], see also [33, 73], to include matrix valued boundary fields, we work subsequently with

$$L_{\text{boundary}}^{\sigma=\pi} = \frac{i}{2} g_{ij} \left( e^{-i\beta} \bar{\psi}_+^j \psi_+^i - e^{i\beta} \bar{\psi}_-^j \psi_-^i \right) - \frac{i}{2} \text{tr} \left[ A \overline{\partial}_{\pi} A^\dagger \right] + B(z, \bar{z})$$

$$+ \frac{i}{2} \text{tr} \left[ \partial_j F^\dagger(z) A^\dagger + \partial_j G(z) A \right] \left( \bar{\psi}_+^j + e^{i\beta} \bar{\psi}_-^j \right)$$

$$+ \frac{i}{2} \text{tr} \left[ \partial_j G(z) A^\dagger + \partial_j F(z) A \right] \left( \psi_+^j + e^{-i\beta} \psi_-^j \right),$$

(4.27)

defined along the Neumann directions at the boundary $\sigma = \pi$. The square matrix $A = (a_{rs})$ contains the boundary fermions and $F, G$ are matrix valued functions of the bosonic bulk fields evaluated on the boundary.

The boundary conditions along the Neumann directions following from the variations of (2.11) and (4.27) are derived along the lines presented in chapter 2 to

$$\partial_{\sigma} z^j = g_{ij} \left( \partial_j B + i \text{tr} \left[ \partial_i \partial_j F^\dagger A^\dagger + \partial_i \partial_j G A^\dagger \right] \bar{\theta}_+^j \right)$$

(4.28)

$$\partial_{\sigma} \bar{z}^i = g_{ij} \left( \partial_j B + i \text{tr} \left[ \partial_i \partial_j G A^\dagger + \partial_i \partial_j F A \right] \theta_+^i \right)$$

(4.29)

$$\theta_-^i = \frac{1}{2} g_{ij} \text{tr} \left[ \partial_j F^\dagger A^\dagger + \partial_j G A \right]$$

(4.30)

$$\bar{\theta}_-^i = \frac{1}{2} g_{ij} \text{tr} \left[ \partial_j G A^\dagger + \partial_j F A \right]$$

(4.31)

$$\partial_{\tau} A = \partial_j F^\dagger \bar{\theta}_+^j + \partial_j G \theta_+^j$$

(4.32)

$$\partial_{\tau} A^\dagger = \partial_j G^\dagger \bar{\theta}_+^j + \partial_j F \theta_+^j.$$  

(4.33)

All equations are understood to be evaluated at $\sigma = \pi$. Furthermore, we have used the convenient combinations

$$\theta_+^i = \frac{1}{2} \left( \psi_+^i + e^{-i\beta} \psi_-^i \right)$$

$$\bar{\theta}_+^i = \frac{1}{2} \left( \bar{\psi}_+^i + e^{i\beta} \bar{\psi}_-^i \right)$$

$$\theta_-^i = \frac{1}{2} \left( \psi_+^i - e^{-i\beta} \psi_-^i \right)$$

$$\bar{\theta}_-^i = \frac{1}{2} \left( \bar{\psi}_+^i - e^{i\beta} \bar{\psi}_-^i \right),$$

(4.34)

for the bulk fermions. By setting

$$L_{\text{boundary}}^{\sigma=0} = - L_{\text{boundary}}^{\sigma=\pi}$$

(4.35)

one obtains functionally the same boundary conditions at $\sigma = 0$ as derived beforehand for $\sigma = \pi$ with, however, possibly different matrices $F, G$ at the two boundaries. Although the constraints on $F$ and $G$ to be derived below are also
valid in the case of different boundary fields, we focus for simplicity on the case of equal boundary conditions up to different choices for \( \beta \), corresponding to brane / antibrane configurations.

Along the Dirichlet directions we use the standard boundary conditions as for example discussed in \[63\]. These are in particular independent of the previously introduced boundary fermions and read explicitly

\[
z^i = y^i_{0,\sigma}, \quad \bar{z}^i = \bar{y}^i_{0,\sigma} \tag{4.36}
\]

\[
0 = \theta^i_+, \quad 0 = \bar{\theta}^i_+. \tag{4.37}
\]

All fields are again understood to be evaluated at \( \sigma = 0, \pi \).

### 4.3.2 B-Type Supersymmetry

First of all, we are aiming at boundary configurations with two conserved B-type supersymmetries. As pointed out in \[54\] in a different context, the open string conservation of quantities deducing from local conserved currents amounts to the time independence of (in our case) the following combinations

\[
Q = \bar{Q}_+ + e^{i\beta} \bar{Q}_- + \Sigma_\pi(\tau) - \Sigma_0(\tau) \tag{4.38}
\]

\[
Q^\dagger = Q_+ + e^{-i\beta} Q_- + \bar{\Sigma}_\pi(\tau) - \bar{\Sigma}_0(\tau) \tag{4.39}
\]

with generically nonzero \((local)\) contributions of boundary fields \( \Sigma_\sigma(t) \) at \( \sigma = \pi \) and \( \sigma = 0 \). This might also be compared with the slightly extended discussion in chapter 2.

By using the supercurrents \( (4.16) \) and \( (4.17) \) presented in section 4.2, the quantities \( (4.38) \) and \( (4.39) \) are time independent in case of

\[
0 = \bar{G}_+ + e^{i\beta} \bar{G}_- \bigg|_{\sigma=\pi} - \bar{\Sigma}_\pi(\tau) \tag{4.40}
\]

\[
0 = \bar{G}_+ + e^{i\beta} \bar{G}_- \bigg|_{\sigma=0} - \bar{\Sigma}_0(\tau). \tag{4.41}
\]

Along the Dirichlet directions these conditions are trivially fulfilled with the boundary conditions \( (4.36) \) and \( (4.37) \) together with a vanishing field \( \Sigma_\sigma \) in these directions. In the case of Neumann directions with boundary conditions \( (4.28)-(4.33) \) the situation is more interesting. For a single Neumann direction the solution to \( (4.40) \) and \( (4.41) \) is discussed in detail in chapter 2 and that treatment extends immediately to the present case including matrix valued boundary fields. Suppressing the calculational details, we obtain the conditions

\[
B = \frac{1}{2} \mathrm{tr} \left[ G G^\dagger + F F^\dagger \right] + \text{const} \tag{4.42}
\]

\[
W = i e^{-i\beta} \mathrm{tr} \left[ FG \right] + \text{const}. \tag{4.43}
\]

The second equation \( (4.43) \) is understood to be valid along the Neumann directions, only. For the local boundary field \( \Sigma_\pi \) we furthermore have

\[
\Sigma_\pi(\tau) = -2 g_{ij} \partial_\tau z^i + \mathrm{tr} \left[ (z^j \partial_j F - F) A + (z^j \partial_j G - G) A^\dagger \right], \tag{4.44}
\]

compare for example with \((2.46)\).
4.3.3 Integrability

Although arbitrary boundary fields obeying (4.42) and (4.43) already give rise to $\mathcal{N} = 2$ supersymmetrical settings, we are here interested in the more restricted case of integrable boundary conditions, that is, branes which also respect the integrable structure present in the bulk theory. As mentioned in the introduction, all known maximally supersymmetric branes in the plane wave theory are actually also integrable\(^1\). By the inclusion of boundary fields as in (4.27), this integrability conservation is a priori no longer guaranteed and leads, if enforced, to further constraints on admissible boundary conditions.

In this section we present the explicit expression of two higher spin bulk currents and state the conditions for their conservation in the presence of boundaries. This conservation gives strong evidence for the integrability of the boundary theory. To further underpin the actual presence of such a structure one might use the explicit mode expansions to be derived in the next section and compare them with the requirements derived in [54] for integrable boundary field theories. We will briefly comment on this in the appendix E.

Local conserved higher spin currents for the massive Ising model were written down in [127]. Here we will focus on combinations which, for a single Neumann direction, appear as limiting cases of the first nontrivial higher spin currents in the $\mathcal{N} = 2$ sine-Gordon model as discussed in the appendix A. We defer a more detailed discussion of this point to the appendix E where we also supply the infinite series of conserved fluxes from [127].

In manifestly real form the currents of present interest are given by

\[
T_4 = g_{\text{in}} \left( \partial_+^2 \bar{z}^i \partial_+ \bar{z}^i + \frac{i}{2} \partial_+ \bar{\psi}^i_+ \partial_+^2 \psi^i_+ - \frac{i}{2} \partial_+^2 \bar{\psi}^i_+ \partial_+ \psi^i_+ \right) \tag{4.45}
\]

and

\[
\bar{\theta}_2 = g_{\text{in}} \left( -m^2 \partial_+ \bar{z}^i \partial_+ z^i - \frac{im^2}{2} \partial_+^2 \bar{\psi}^i_+ \partial_+ \psi^i_+ + \frac{im^2}{2} \partial_+ \bar{\psi}^i_+ \partial_+ \psi^i_+ \right) \tag{4.46}
\]

On-shell, they fulfil the conservation equation

\[
\partial_- T_4 = \partial_+ \theta_2; \quad \partial_+ \bar{T}_4 = \partial_- \bar{\theta}_2. \tag{4.49}
\]

In the bulk theory both fluxes give rise to conserved spin 3 charges. The conservation of a suitable combination of the previous operators in the presence of boundaries is

\(^1\)This claim can be proven by employing the methods of [54] to be briefly mentioned in the appendix E. For the $(0,0)$-instanton as a particular $(n,n)$-brane this result will be established in due course.
discussed in the appendix. There the conditions for integrability are found to be

\[
\begin{align*}
\partial_i \partial_j \partial_k B &= 0 & \partial_i \partial_j \partial_k B &= 0 \\
\partial_i \partial_j \partial_k B &= 0 & \partial_i \partial_j \partial_k B &= 0
\end{align*}
\] (4.50)

for the boundary potential and

\[
\begin{align*}
0 &= \text{tr} \left( \partial_i \partial_j G A_i^\dagger + \partial_i \partial_j F A \right) \\
0 &= \text{tr} \left( \partial_i \partial_j G^i A + \partial_i \partial_j F^i A^\dagger \right)
\end{align*}
\] (4.51)

for the matrices \( F \) and \( G \) of the boundary Lagrangian (4.27).

Having presented the conditions for \( \mathcal{N} = 2 \) supersymmetry (4.42), (4.43) in the last section and for integrability in (4.50) and (4.51), it is now straightforward to write down the corresponding solutions. They are given by the linear functions

\[
F = A_i z^i + C \quad G = B_i z^i + D
\] (4.52)

along the Neumann directions with

\[
\text{tr} \left( A_i B_j \right) = -e^{ij} \tilde{m} \delta_{ij}; \quad \text{tr} \left( A_i D + B_i C \right) = 0.
\] (4.53)

The resulting boundary potential becomes up to an irrelevant constant

\[
B(z, \bar{z}) = \frac{1}{2} \text{tr} \left( A_i A_i^\dagger + B_i B_i^\dagger \right) z^i \bar{z}^i + \text{tr} \left( A_i C_i^\dagger + B_i D_i^\dagger \right) z^i + \text{tr} \left( C A_i^\dagger + D B_i^\dagger \right) \bar{z}^i,
\] (4.54)

again extending only along the Neumann directions.

### 4.4 The open string with boundary fermions

In this section we present a detailed discussion of \((n,n)\)-branes with \( n = 0, \ldots, 4 \) from an open string point of view by enforcing Neumann boundary conditions as introduced in the last section. With the equations of motion for the boundary fermions we can eliminate these extra fields from the remaining boundary conditions. Although the resulting boundary conditions on the fermionic bulk fields differ clearly from the standard settings, the corresponding solutions can be found and quantised by standard methods as already used in chapter 3.

As stated in the introduction, the boundary fermions can be expressed in terms of the bulk fields restricted to the boundary without including additional degrees of freedom. We explain in detail how this solution reproduces the expected anticommutators of the boundary fermions in the quantum theory. The section closes with a derivation of the \( \mathcal{N} = 2 \) superalgebra of the boundary theory. These results will be needed in the discussion of the open-closed duality in section 4.6.
For the boundary fields appearing in the Neumann directions we work with a particular solution of type (4.52) given by

\[ F = \text{diag}(A\tilde{z}^3 + C^3); \quad G = \text{diag}(B\tilde{z}^3 + D^3) \]  

with no sum over hatted indices. The solution (4.55) allows us to treat the fields along any complex direction \( z^i \) separately and construct \((n, n)\)-type branes for all \( n \) in a single approach.

We consider only strings spanning between branes with the same type of boundary fields and restrict the parameter \( \beta \) appearing in (4.27) to the values 0 and \( \pi \). This choice corresponds to (‘pure’) brane or antibrane settings. The more general situation of \( \beta \in (0, \pi) \) can be dealt with with the methods explained in chapter 3 in the context of boundary magnetic fields in the plane wave background.

For future reference we present the equations of motion (4.3) and (4.4) by using real coordinates. For the bosons we have\(^2\)

\[ 0 = (\partial_+ \partial_- + \tilde{m}^2) X^s \]  

with \( s = 1, \ldots, 8 \). For the fermions

\[ \begin{align*}
\partial_- \psi_+^t &= -\tilde{m}\psi_-^t \quad &\partial_+ \psi_-^t &= \tilde{m}\psi_+^t \\
\partial_- \psi_+^{t+4} &= +\tilde{m}\psi_-^{t+4} \quad &\partial_+ \psi_-^{t+4} &= -\tilde{m}\psi_+^{t+4}
\end{align*} \]  

with \( t = 1, \ldots, 4 \). The fermionic fields along the \( s = 5, \ldots, 8 \) directions are obtained from those along the \( s = 1, \ldots, 4 \) directions by interchanging \( \tilde{m} \leftarrow -\tilde{m} \), reflecting the different eigenvalues of the matrix \( \Pi \) introduced in chapter 3.

The most general solutions to these equations of motion can be found in (3.11) and (3.12), (3.13) by setting \( \Pi \rightarrow -1 \) in the fermionic expansions.

### 4.4.1 Dirichlet directions

In this section we consider the bulk fields spanning along a Dirichlet direction with boundary conditions

\[ X^s(\tau, \sigma = 0) = y_0^s; \quad X^s(\tau, \sigma = \pi) = y_\pi^s \]  

and

\[ 0 = (\psi_+^s + \rho\psi_-^s)(\tau, \sigma = 0, \pi). \]  

As before, \( \rho = \pm 1 \) distinguishes between the brane / antibrane configurations. Our discussion proceeds in this part along the lines of the \((0, 0)\)–instanton construction

\(^2\)As already explained in chapter 3, the mass parameter \( m \) takes on different values in the open and closed string channels. The same effect is observed for the parameters \( b, k \) of the boundary potential to be introduced in the next section. A more detailed discussion on this point is presented in section 4.5.3.
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from [51], but differs mildly in the fermionic sector due to our choice of LG-fermions, compare with section 4.2.

From (3.11) the boundary conditions (4.59) and (4.60) lead to the bosonic mode expansion

\[
X^s(\tau, \sigma) = x_0^s \cosh(\tilde{m}\sigma) + \frac{x_0^s - \bar{x}_0^s \cosh(\tilde{m}\pi)}{\sinh(\tilde{m}\pi)} \sinh(\tilde{m}\sigma) \nonumber
\]

\[
-\sqrt{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{\omega_n} e^{-i\omega_n\tau} a_n^s \sin(n\sigma) \tag{4.61}
\]

with \(\omega_n = \text{sgn}(n)\sqrt{n^2 + \tilde{m}^2}\), compare for example with [51].

For the fermions spanning between a brane-brane configuration we deduce for \(t \in \mathcal{D}_-\)

\[
\psi^t_A(\tau, \sigma) = -\psi^t e^{-\tilde{m}\sigma} + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left( \tilde{\psi}_n^t e^{-i(\omega_n\tau + n\sigma)} - i\frac{\omega_n - n}{\tilde{m}} \psi_n^t e^{-i(\omega_n\tau - n\sigma)} \right) \tag{4.62}
\]

\[
\psi^t_B(\tau, \sigma) = \psi^t e^{-\tilde{m}\sigma} + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left( \psi_n^t e^{-i(\omega_n\tau - n\sigma)} + i\frac{\omega_n - n}{\tilde{m}} \tilde{\psi}_n^t e^{-i(\omega_n\tau + n\sigma)} \right) \tag{4.63}
\]

with the identifications

\[
\tilde{\psi}_n^t = -\frac{n - i\tilde{m}}{\omega_n} \psi_n^t. \tag{4.64}
\]

As explained before, the solutions along the directions \(t \in \mathcal{D}_+\) are obtained from (4.62) and (4.63) by using \(\tilde{m} \rightarrow -\tilde{m}\).

The fermionic fields spanning between a brane / antibrane combination have the same structure as presented in (4.62) and (4.63). In this case, however, the zero modes \(\psi^t\) are absent and the nonzero modes have to fulfll either

\[
e^{2\pi i n} = \frac{n - i\tilde{m}}{n + i\tilde{m}} \text{ or } e^{2\pi i n} = \frac{n + i\tilde{m}}{n - i\tilde{m}}, \quad n \neq 0, \tag{4.65}
\]

depending on whether \(t \in \mathcal{D}_-\) or \(t \in \mathcal{D}_+\). This is analogous to the findings for the (0,0) instanton in [51] and the related results in chapter 3.

Quantisation

By requiring the standard canonical commutators as summarised in the appendix D, we obtain the commutation relations for the modes introduced in the last section to

\[
[a^t_m, a^j_n] = \omega_m \delta^{ij} \delta_{m+n} \tag{4.66}
\]

\[
\{\psi^t_m, \psi^s_n\} = \delta^{rs} \delta_{m+n} \tag{4.67}
\]

\[
\{\psi^t, \psi^s\} = \frac{2\pi\tilde{m}}{1 - e^{-2\pi\tilde{m}}} \delta^{rs} = -\frac{\pi\tilde{m} e^{\pi\tilde{m}}}{\sinh(\pi\tilde{m})} \delta^{rs}. \tag{4.68}
\]

The anticommutators are written down for parameters \(r, s\) ranging in \(\mathcal{D}_-\). Some details of the derivations can be found in the appendix D.
4.4.2 Neumann directions

In this part we consider the mode expansions for the new Neumann type boundary conditions as introduced above. We work with the boundary fields presented in equation (4.55), whose parameters fulfil

\[ A_i^j B_i^j = -\epsilon_i^j \tilde{m} \; \; \; A_i^j \tilde{D}_i^j + C_i^j \tilde{B}_i^j = 0 \]  

(4.69)

to obey (4.43). This ensures in particular the conservation of a \( \mathcal{N} = 2 \) supersymmetry structure. As before, there is no sum over hatted indices. With (4.55), the boundary potential \( B \) from (4.42) takes on the structure

\[ B(z, \bar{z}) = \sum_{i \in \mathcal{N}} \left( \tilde{b}_i^j z_i^j \bar{z}_i^j + \tilde{k}_i^j z_i^j \bar{z}_i^j \right) + \text{const} \]  

(4.70)

by using the convenient combinations

\[ \tilde{b}_i^j = \tilde{b}_i^j = \tilde{b}_i^{j+4} = \frac{A_i^j \tilde{A}_i^j + B_i^j \tilde{B}_i^j}{2} \]  

(4.71)

\[ \tilde{k}_i^j = \frac{C_i^j \tilde{A}_i^j + D_i^j \tilde{B}_i^j}{2} \]  

(4.72)

With (4.69) we furthermore have

\[ A_i^j \tilde{A}_i^j = \tilde{b}_i^j \pm \sqrt{(\tilde{b}_i^j)^2 - \tilde{m}_i^2}; \; \; \; k_i^j = \pm \frac{C_i^j}{A_i^j} \sqrt{(\tilde{b}_i^j)^2 - \tilde{m}_i^2} \]  

(4.73)

for

\[ 0 < \tilde{m}_i \leq \tilde{b}_i \; \; \text{and} \; \; i \in \mathcal{N}_- \]  

(4.74)

In this section we assume throughout \( m < b_i \) and comment on the limiting cases \( b_i = m \) and their relation in the bosonic sector to previously known branes later on in section 4.5.4.

From (4.28)-(4.31) the boundary conditions for the bosons at \( \sigma = 0, \pi \) become

\[ \partial_\sigma X_I^j = \tilde{b}_i^j X_I^I + \tilde{k}_i^j \]  

(4.75)

with \( I \in \mathcal{N} \). For the fermionic boundary conditions we use the boundary equations of motion (4.32) and (4.33) to eliminate the boundary fermions from (4.30) and (4.31). We derive

\[ \partial_\tau \left( \psi_+^I - \rho \psi_-^I \right) = \left( \tilde{b}_I^j - \rho \tilde{m} \right) \left( \psi_+^I + \rho \psi_-^I \right) \]  

(4.76)

\[ \partial_\tau \left( \psi_{+I}^{I+4} - \rho \psi_{-I}^{I+4} \right) = \left( \tilde{b}_I^j + \rho \tilde{m} \right) \left( \psi_{+I}^{I+4} + \rho \psi_{-I}^{I+4} \right) \]  

(4.77)

for the fermionic bulk fields with \( \sigma = 0, \pi \) and \( I \in \mathcal{N}_- \). Both cases are formulated in a real basis and the parameter \( \rho \) distinguishes as before between the brane /
antibrane boundary conditions.

Using the general solution (3.11) together with the boundary conditions (4.75), the bosonic mode expansions along the Neumann directions are found to be

\[ X^f(\tau, \sigma) = N^f \cosh(\tilde{m}\sigma) + \tilde{N}^f \sinh(\tilde{m}\sigma) + P^f e^{\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} e^{i\tilde{b}^f \sigma} + Q^f e^{-\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} e^{-i\tilde{b}^f \sigma} \]

\[ + \frac{i}{\sqrt{2}} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{\omega_n} \left( a^+_n e^{-i(\omega_n \tau - n\sigma)} + a^-_n e^{-i(\omega_n \tau + n\sigma)} \right) \]

with

\[ a^+_n = n + i \tilde{b}^f \alpha_n \]

and

\[ N^f = \frac{\tilde{b}^f c \cosh \frac{\tilde{m}}{2} - \tilde{m} \sinh \frac{\tilde{m}}{2} \tilde{k}^f}{(\tilde{m}^2 - (\tilde{b}^f)^2) \cosh \frac{\tilde{m}}{2}} \]

\[ \tilde{N}^f = \frac{\tilde{m} \cosh \frac{\tilde{m}}{2} - \tilde{b}^f \sinh \frac{\tilde{m}}{2} \tilde{n}^f}{(\tilde{m}^2 - (\tilde{b}^f)^2) \cosh \frac{\tilde{m}}{2}} \]

The special modes \( P^f, Q^f \) with a time dependence proportional to \( e^{\pm \sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} \) are of the same type as those appearing in chapter 3 in the treatment of open strings in the plane wave background under the inclusion of a nontrivial \( \mathcal{F}^{1J} \)-field. They play a crucial rôle in the quantisation to be discussed in the next section.

For the fermions spanning between a brane / brane pair with \( \rho = 1 \) we obtain from (3.12), (3.13) and the boundary conditions (4.76) the solutions

\[ \psi^{I\dagger}_+(\tau, \sigma) = - \psi^{I\dagger} e^{-\tilde{m}\sigma} + e^{-\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} e^{i\tilde{b}^f \sigma} \chi^f + e^{-\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} e^{i\tilde{b}^f \sigma} \frac{\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2 + \tilde{b}^f}}{\tilde{m}} \chi^f \]

\[ + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left( \tilde{\psi}^I_n e^{-i(\omega_n \tau + n\sigma)} - \frac{i}{m} \omega_n - n \psi^{I\dagger} e^{-i(\omega_n \tau - n\sigma)} \right) \]

\[ \psi^I_-(\tau, \sigma) = \psi^I e^{-\tilde{m}\sigma} + e^{-\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} e^{i\tilde{b}^f \sigma} \chi^f + e^{-\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2}\tau} e^{i\tilde{b}^f \sigma} \frac{\sqrt{(\tilde{b}^f)^2 - \tilde{m}^2 + \tilde{b}^f}}{\tilde{m}} \chi^f \]

\[ + \sum_{n \in \mathbb{Z} \setminus \{0\}} c_n \left( \tilde{\psi}_n e^{-i(\omega_n \tau - n\sigma)} + \frac{i}{m} \omega_n - n \psi^I e^{-i(\omega_n \tau + n\sigma)} \right) \]

with

\[ \tilde{\psi}_n = \frac{\omega_n}{n + i \tilde{b}^f} \psi^I_n \]

and \( I \in \mathcal{N}_- \). The modes \( \chi^f \) and \( \tilde{\chi}^f \) correspond to the bosonic operators \( P^f, Q^f \), compare for example with [90]. As described there, the terms in (4.82) and (4.83)
containing these special modes fulfil the conditions (4.76), (4.77) for all \( \sigma \) and not only on the boundary.

The remaining fermionic solutions along the \( I \in \mathcal{N}_+ \) directions are again deduced by sending \( \tilde{m} \to -\tilde{m} \) in (4.82) and (4.83). In particular, one obtains the mode identifications for the nonzero modes in this case to

\[
\tilde{\psi}_n^{\bar{I}+4} = \frac{\omega_n}{n-i\tilde{m}} \frac{n+ib^I}{n-i\tilde{b}^I} \tilde{\psi}_n^{\bar{I}+4}.
\] (4.85)

As for the Dirichlet directions, the mode expansion for strings stretching between a brane / antibrane pair follows from (4.82) and (4.83) by dropping the zero modes \( \psi^I \), but retaining the special modes \( \chi^I \) and \( \tilde{\chi}^I \). Furthermore, the moding for the nonzero modes again has to fulfil either

\[
e^{2\pi in} = -\frac{n-im}{n+im} \quad \text{or} \quad e^{2\pi in} = -\frac{n+im}{n-im},
\] (4.86)

depending on whether \( I \in \mathcal{N}_- \) or \( I \in \mathcal{N}_+ \).

Quantisation

The standard canonical conditions (D.1)-(D.6) lead in the Neumann case to the following commutators. For the bosons we obtain

\[
[P^I, Q^J] = \delta^{IJ} \frac{2\pi \tilde{b}^I}{\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2}} \frac{1}{1 - e^{2\pi b^I}}
\] (4.87)

\[
[a^I_m, a^J_n] = 2\pi \delta^{IJ} \delta_{m+n}
\] (4.88)

and for the fermions

\[
\{\psi_m^I, \psi_n^J\} = \delta^{IJ} \delta_{m+n}
\] (4.89)

\[
\{\psi^I, \psi^J\} = 2\pi \tilde{m} \delta^{IJ} \frac{\tilde{m} - \tilde{b}^I}{1 - e^{-2\pi \tilde{m}}} 
\] (4.90)

\[
\{\chi^I, \tilde{\chi}^J\} = 2\pi \tilde{b}^I \delta^{IJ} \frac{\sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} - \tilde{b}^I}{1 - e^{2\pi b^I}} .
\] (4.91)

The fermionic relations are formulated for \( I, J \in \mathcal{N}_- \), only. Some details of the derivations are presented in the appendix D.

4.4.3 Boundary fermions

In the last section the boundary fermionic fields were eliminated from the remaining boundary conditions by using their equations of motion. In this section we reconsider this situation and present the explicit solution for the boundary fermions as a suitable combinations of fermionic bulk fields evaluated on the boundary.
4.4. THE OPEN STRING WITH BOUNDARY FERMIONS

For our choice of diagonal matrices $F, Q$ all non-diagonal elements of $A, A^\dagger$ in (4.27) decouple from the remaining fields and we can therefore concentrate on the diagonal components. For these elements we have to solve the equations of motion (4.32) and (4.33) by using (4.55). For notational simplicity we write down only expressions for fermions corresponding to the $z_1$ direction and suppress further, for this case irrelevant, indices.

Using (4.76) and (4.77) in the equations of motion (4.32) and (4.33), we obtain the boundary fermions to

$$a(t) = a_0 + \frac{A + B}{2(b - \tilde{m})} (\psi_1^+ - \psi_1^-) - i \frac{A - B}{2(b + \tilde{m})} (\psi_2^+ - \psi_2^-) \tag{4.92}$$

$$\bar{a}(t) = \bar{a}_0 + \frac{A + B}{2(b - \tilde{m})} (\psi_1^+ - \psi_1^-) + i \frac{A - B}{2(b + \tilde{m})} (\psi_2^+ - \psi_2^-) \tag{4.93}$$

with constant fermions $a_0, \bar{a}_0$. Using so far only the differentiated boundary conditions (4.76) and (4.77), we have to test whether there are additional constraints on these extra fermions. From the (undifferentiated) conditions (4.30) and (4.31) we deduce

$$0 = B\bar{a}_0 + Aa_0. \tag{4.94}$$

This amounts to $a_0 = \bar{a}_0 = 0$ by using the explicit expressions for $A$ and $B$ from (4.69) and (4.73) with $b > \tilde{m}$. For our solution (4.55) all boundary fermions in (4.27) therefore either decouple from the remaining fields or are expressible in terms of bulk functions restricted to the boundary.

For consistency of the last result, the fermionic anticommutation relations for the bulk fields derived in section 4.4.2 should reproduce the expected anticommutators for the boundary fermionic fields $a(t)$ and $\bar{a}(t)$. To determine these relations we have to evaluate expressions like

$$(*) = \{ \psi_1^+(\tau, \sigma) - \psi_1^+(\tau, \bar{\sigma}), \psi_1^+(\tau, \bar{\sigma}) - \psi_1^-(\tau, \bar{\sigma}) \} \tag{4.95}$$

on the boundaries. This is, different to the bulk, relatively subtle due to potential divergences. Using the anticommutators (4.90) and (4.91), we obtain

$$(*) = -\frac{8\pi\tilde{m}}{1 - e^{-2\pi\tilde{m}}} \left( \tilde{m} - \tilde{b} e^{-\tilde{m}(\sigma + \bar{\sigma})} - \frac{8\pi\tilde{b}}{1 - e^{2\pi\tilde{m}}} e^{-\tilde{m}(\sigma + \bar{\sigma})} \right) + 2 \sum_{n \neq 0} \left( e^{i\sigma(n + i\tilde{m})} - e^{i\sigma(n - i\tilde{m})} e^{i\sigma(n + i\tilde{m})} \right). \tag{4.96}$$

After setting one of the arguments $\sigma, \bar{\sigma}$ equal to the boundary values 0 or $\pi$, we obtain for the infinite sum

$$4i(\tilde{b} - \tilde{m}) \sum_{n \neq 0} \frac{ne^{in(\sigma + \bar{\sigma})}}{(n - i\tilde{m})(n + i\tilde{m})} = -4i(\tilde{b} - \tilde{m}) \int_c d\bar{z} \frac{e^{iz(\sigma + \bar{\sigma})}}{1 - e^{2\pi iz}} (z - i\tilde{m})(z + i\tilde{b}). \tag{4.97}$$
Here, $C$ is a contour running infinitesimally above and below the real axis, compare with [51]. By closing the contours the residues cancel out with the first terms in (*) and we finally obtain

$$\{\psi^1_+ (\tau, \sigma) - \psi^1_+ (\tau, \bar{\sigma}), \psi^1_+ (\tau, \bar{\sigma}) - \psi^1_+ (\tau, \sigma)\} = \pm 4\pi (\bar{b} - \bar{m}) \quad (4.98)$$

$$\{\psi^2_+ (\tau, \sigma) - \psi^2_+ (\tau, \bar{\sigma}), \psi^2_+ (\tau, \bar{\sigma}) - \psi^2_+ (\tau, \sigma)\} = \pm 4\pi (\bar{b} + \bar{m}) \quad (4.99)$$

at $\sigma = \bar{\sigma} = 0$ and $\sigma = \bar{\sigma} = \pi$, respectively. With (4.92) and (4.93) this leads to the anticommutators

$$\{a(t), a(t)\} = 0 \quad (4.100)$$

$$\{\bar{a}(t), \bar{a}(t)\} = 0 \quad (4.101)$$

$$\{a(t), \bar{a}(t)\} = \pm \pi, \quad (4.102)$$

which are the required results. The signs originate in different overall signs appearing in the boundary Lagrangian (4.27) at the two boundaries.

### 4.4.4 The $\mathcal{N} = 2$ superalgebra

In this final section of our open string treatment we determine the Hamiltonians of the previously discussed configurations and for the brane / brane situation also the resulting $\mathcal{N} = 2$ supercharges. The expressions for the Hamiltonians are particularly needed in section 4.6.

The conserved supercharges are calculated from the equations (4.38) and (4.39) established in section 4.3.2. The Hamiltonians follow from the (closed string) conserved currents

$$T_2 = g_{jj} \left( \partial_+ \bar{z}^j \partial_+ z^j + \frac{i}{2} \bar{\psi}_j^{+*} \partial_+ \psi^j_+ \right) \quad \theta_0 = g_{jj} \left( -m^2 \bar{z}^j \bar{z}_j - \frac{i}{2} \bar{\psi}_j^{+*} \partial_+ \psi^j_+ \right) \quad (4.103)$$

$$\bar{T}_2 = g_{jj} \left( \partial_- \bar{z}^j \partial_- z^j + \frac{i}{2} \bar{\psi}_j^{+*} \partial_- \psi^j_- \right) \quad \bar{\theta}_0 = g_{jj} \left( -m^2 \bar{z}^j \bar{z}_j - \frac{i}{2} \bar{\psi}_j^{+*} \partial_- \psi^j_- \right) \quad (4.104)$$

which fulfil on-shell

$$\partial_- T_2 = \partial_+ \theta_0; \quad \partial_+ \bar{T}_2 = \partial_- \bar{\theta}_0. \quad (4.105)$$

### Dirichlet directions

Using the currents (4.103) and (4.104), the open string Hamiltonian along the Dirichlet directions for a brane / brane configuration becomes with the mode expansions (4.61)-(4.63) in the overall normalisation explained in detail in [51]

$$\frac{X}{2\pi} H^{\text{open}} = \frac{\bar{m}}{2 \sinh(\bar{m} \pi)} \sum_{a \in \mathcal{D}} (\cosh(\bar{m} \pi)(x_0^a x_0^a + x_2^a x_2^a) - 2 x_0^a x_1^a)$$

$$+ 2\pi \sum_{n > 0} (a_n^a \bar{a}_{-n}^a + \omega_n \psi_0^a \bar{\psi}_n^a) \quad (4.106)$$
4.4. THE OPEN STRING WITH BOUNDARY FERMIONS

The summation index $a$ is understood to range over all Dirichlet directions. The Hamiltonian for the brane / antibrane configuration has the same structure with a fermionic nonzero moding as given in (4.65). In this case there is also an overall normal ordering constant which will be implicitly determined in section 4.6.

The contribution to the overall $\mathcal{N} = 2$ supercharges for a brane / brane configuration with $\rho = 1$ becomes

$$Q = 2 \sum_{a \in D_-} \left( \left( \psi^a - i\psi^{a+4} \right) \left( x_0^a + ix_0^{a+4} \right) - \left( e^{-m_\pi} \psi^a - i e^{-m_\pi} \psi^{a+4} \right) \left( x_0^a + ix_0^{a+4} \right) \right)$$

$$+ 2\pi \sqrt{2} \sum_{n \neq 0} c_n \left[ \left( 1 - i \frac{\omega_n - n}{m} \right) \psi^a_n - i \left( 1 + i \frac{\omega_n - n}{m} \right) \psi^{a+4}_n \right] \left( e^{-m_\pi} \psi^a - i e^{-m_\pi} \psi^{a+4} \right)$$

(4.107)

with the corresponding complex conjugated expression for $Q^\dagger$.

**Neumann directions**

Along the Neumann directions the currents (4.103) and (4.104) require the inclusion of boundary currents in the open string sector as discussed in section 4.3.2 for the supercharges and in the appendix E for the higher spin currents of the integrable structure. In the present case, the local boundary field is of the form

$$\Sigma^{(l)}(z) = 2 \left( B(z, \bar{z}) + ig_{\bar{j}l} \left( \bar{\theta}_-^l \theta_+^l - \bar{\theta}_+^l \theta_-^l \right) \right)$$

(4.108)

and the suitable normalised Hamiltonian becomes for open strings stretching between a brane / brane pair

$$\frac{X^+}{2\pi} H^{\text{open}} = H_0 + \sum_{l \in \mathcal{N}} \left[ \tilde{m}^2 - \tilde{b}^l \right] \left( e^{2\pi \tilde{b}^l} - 1 \right) Q^l P^l + 2\pi \sum_{n \geq 0} \left( a_n^l \alpha_n^l + \omega_n \psi_n^l \psi_n^l \right)$$

$$+ i \sum_{l \in \mathcal{N}} \left[ \left( e^{2\pi \tilde{b}^l} - 1 \right) \left( \tilde{b}^l \right)^2 \tilde{m}^2 \tilde{\bar{d}}^l \tilde{\bar{m}}^2 \left( \tilde{b}^l \tilde{\bar{m}} - \tilde{b}^l \tilde{m} \right) + \tilde{b}^l \right] \left( \frac{\chi_l \overline{\chi}_l}{\tilde{b}^l - \tilde{m}} + \frac{\chi_{l+1} \overline{\chi}_{l+1}}{\tilde{b}^l + \tilde{m}} \right)$$

(4.109)

with

$$2H_0 = \sum_{l \in \mathcal{N}} \left[ \left( \tilde{m} \cosh(\tilde{m} \pi) - \tilde{b}^l \sinh(\tilde{m} \pi) \right) \sinh(\tilde{m} \pi) \left( N^l \tilde{N}^l + \tilde{N}^l N^l \right) \right]$$

$$+ 2 \left( \tilde{m} \sinh(\tilde{m} \pi) - \tilde{b}^l \cosh(\tilde{m} \pi) \right) \sinh(\tilde{m} \pi) \left( N^l \tilde{N}^l ight)$$

$$- 2\tilde{b}^l \left( N^l (\cosh(\tilde{m} \pi) - 1) + \tilde{N}^l \sinh(\tilde{m} \pi) \right)$$

(4.110)
as contribution from the bosonic zero modes. By using (4.80) and (4.81), this simplifies to

$$H_0 = \tilde{m} \sum_{l \in \mathbb{N}} \frac{\tanh \frac{\tilde{m} \pi}{2}}{(\tilde{b}^l)^2 - \tilde{m}^2} k^l \tilde{k}^l. \quad (4.111)$$

The Hamiltonian (4.109) is already presented in its normal ordered form by implicitly defining $P^l$ and $\tilde{\chi}^l$ as annihilation operators for the special zero-modes. With these choices the corresponding normal ordering constants cancel.

The Hamiltonian for open strings between a brane / antibrane pair also has the form (4.109). In that case, however, the fermionic moding has to fulfil (4.65) and there also appears a nonzero normal ordering constant. It solely originates from the nonzero modes and takes on the same value as in the previously discussed Dirichlet case.

The contribution to the supercharge in the case of strings between two branes with $\rho = 0$ is finally obtained to be

$$Q = \sum_{l \in \mathbb{N}} \left[ 2 \left( k^l + i \tilde{k}^{l+4} \right) \left( \frac{e^{-\tilde{m} \pi} - 1}{\tilde{b}^l - \tilde{m}} - i \frac{\tilde{m} \pi}{\tilde{b}^l + \tilde{m}} \right) \Psi^l - i \frac{\tilde{m} \pi}{\tilde{b}^l + \tilde{m}} \Psi^{l+4} \right]$$

$$+ \frac{e^{2\pi \tilde{b}^l} - 1}{\tilde{m} \tilde{b}^l} \sqrt{(\tilde{b}^l)^2 - \tilde{m}^2} \left( \sqrt{\tilde{b}^l + \tilde{m}} + \sqrt{\tilde{b}^l - \tilde{m}} \right)$$

$$\times \left( \sqrt{\tilde{b}^l + \tilde{m}} \chi^l + i \sqrt{\tilde{b}^l - \tilde{m}} \chi^{l+4} \right) \left( P^l + i P^{l+4} \right)$$

$$- \frac{e^{2\pi \tilde{b}^l} - 1}{\tilde{m} \tilde{b}^l} \sqrt{(\tilde{b}^l)^2 - \tilde{m}^2} \left( \sqrt{\tilde{b}^l + \tilde{m}} + \sqrt{\tilde{b}^l - \tilde{m}} \right)$$

$$\times \left( \sqrt{\tilde{b}^l + \tilde{m}} \tilde{\chi}^l + i \sqrt{\tilde{b}^l - \tilde{m}} \tilde{\chi}^{l+4} \right) \left( Q^l + i Q^{l+4} \right) \right]$$

$$+ 2\pi \sqrt{2} \sum_{n, \rho, \nu, \lambda \in \mathbb{N}_-} c_n \left[ \left( 1 - i \frac{\omega_n - \nu}{\tilde{m}} \right) \psi_n - i \left( 1 + i \frac{\omega_n - \nu}{\tilde{m}} \right) \psi^{l+4}_n \right] \left( a^l_n + i a^{l+4}_n \right)$$

with the corresponding complex conjugated expression for $Q^l$.

**The superalgebra**

Adding up the appropriate contributions from (4.107) and (4.112) corresponding to the particular $(n, n)$-brane under consideration, one obtains the supercharges representing the conserved $\mathcal{N} = 2$ supersymmetry.

The relevant anticommutators are found to be

$$\{Q, Q^l\} = (8X^+) \quad (4.113)$$

$$\{Q, Q\} = 8\pi \tilde{m} \sum_{i \in \mathcal{D}_-} \left( (z_0^i)^2 - (z^i_\rho)^2 \right) \quad (4.114)$$
which completes our discussion of the open string superalgebra.

4.5 Boundary states and spacetime supersymmetry

In this section we study the branes introduced in sections 4.3 and 4.4 from a closed string perspective by formulating them in terms of boundary states. This will on the one hand confirm our previous results, but is on the other hand also particularly suitable for a discussion of preserved spacetime supersymmetries. As a main result, the spacetime filling (4, 4)-brane is seen to be maximally spacetime supersymmetric. This can be understood in direct analogy to the other limiting case of the (0, 0)-instanton. To have a more straightforward comparison with the constructions known for example from [51], we use a formulation based on Green-Schwarz spinors in the closed string channel.

4.5.1 Gluing conditions

By using the standard procedure as for example explained in [54] or in the context of branes in the plane wave background in [19, 51], one translates the open string boundary to the corresponding closed string gluing conditions. For the bosonic fields we obtain from (4.59) and (4.75)

\[ 0 = (s'(r, \sigma) - y_0)|_{r=0} || \mathcal{B} \rangle \]

\[ 0 = \left( \partial_\tau x^I(\tau, \sigma) + i \left( b_I x^I(\tau, \sigma) + k^I \right) \right)|_{\tau=0} || \mathcal{B} \rangle \]

with \( r \in \mathcal{D} \) and \( I \in \mathcal{N} \). For the fermions, on the other hand, we have along the Dirichlet directions with \( r \in \mathcal{D} \)

\[ 0 = (\psi^t_+(\tau, \sigma) - i \rho \psi_-^t(\tau, \sigma))|_{\tau=0} || \mathcal{B} \rangle \]

and for the Neumann directions

\[ 0 = \partial_\sigma \left( \psi^f_+(\tau, \sigma) + i \rho \psi^-_f(\tau, \sigma) \right) + i(b^I - \rho m) \left( \psi^f_+(\tau, \sigma) - i \rho \psi^-_f(\tau, \sigma) \right)|_{\tau=0} || \mathcal{B} \rangle \]

with \( I \in \mathcal{N}_- \). For \( I \in \mathcal{N}_+ \) one has to interchange \( m \leftrightarrow -m \) and the parameter \( \rho = \pm 1 \) distinguishes as before between the brane / antibrane cases.

Translating these conditions to relations between Green-Schwarz fermionic fields by applying the results mentioned in section 4.2, one deduces the gluing conditions

\[ 0 = \eta^t \Gamma^j \left( \tilde{S}(\tau, \sigma) - i \rho S(\tau, \sigma) \right)|_{\tau=0} || \mathcal{B} \rangle; \quad 0 = \eta \Gamma^j \left( \tilde{S}(\tau, \sigma) - i \rho S(\tau, \sigma) \right)|_{\tau=0} || \mathcal{B} \rangle \]

(4.119)
along the Dirichlet directions with \( j, \bar{j} \in \mathcal{D}_- \) and
\[
0 = \eta^* \Gamma^{\bar{j}} \left( \partial_\tau \left( \tilde{S} + \mathbf{i} \rho S \right) (\tau, \sigma) + \mathbf{i} \left( b^{\bar{j}} - m \rho \Pi \right) \left( \tilde{S} - \mathbf{i} \rho S \right) (\tau, \sigma) \right) \bigg|_{\tau = 0} ||\mathcal{B}|| \quad (4.120)
\]
\[
0 = \eta^* \Gamma^{\bar{j}} \left( \partial_\tau \left( \tilde{S} + \mathbf{i} \rho S \right) (\tau, \sigma) + \mathbf{i} \left( b^{\bar{j}} - m \rho \Pi \right) \left( \tilde{S} - \mathbf{i} \rho S \right) (\tau, \sigma) \right) \bigg|_{\tau = 0} ||\mathcal{B}|| \quad (4.121)
\]
along the Neumann directions with \( j, \bar{j} \in \mathcal{N}_- \) and \( b^{\bar{j}} = b^{\bar{j}} \) as before.

To combine the fermionic gluing conditions to a single formula we define matrices \( \mathcal{R}, \mathcal{T} \) by the following requirements
\[
\gamma^* \gamma = \gamma^* \gamma; \quad \gamma \gamma^* \mathcal{R} = \gamma \gamma^* \quad (4.122)
\]
\[
\gamma^* \gamma \mathcal{T} = b^\gamma \gamma^* \gamma; \quad \gamma \gamma^* \mathcal{T} = b^\gamma \gamma^* \gamma \quad (4.123)
\]
along the Neumann directions with \( i, \bar{i} \in \mathcal{N}_- \) and
\[
\eta^* \Gamma^{\bar{i}} \mathcal{R} = \eta^* \Gamma^{\bar{i}} \mathcal{T} = 0; \quad \eta \Gamma^{\bar{i}} \mathcal{R} = \eta \Gamma^{\bar{i}} \mathcal{T} = 0 
\quad (4.124)
\]
for the Dirichlet directions with \( r, \bar{r} \in \mathcal{D}_- \). These matrices fulfil in particular
\[
\mathcal{R}^2 = \mathcal{R}; \quad [\mathcal{R}, \mathcal{T}] = [\mathcal{R}, \Pi] = [\mathcal{T}, \Pi] = 0. \quad (4.125)
\]
Using \( \mathcal{R} \) and \( \mathcal{T} \), the fermionic gluing conditions simplify to the single expression
\[
0 = \left( \partial_\tau \left( \tilde{S} + \mathbf{i} \rho S \right) (\tau, \sigma) + \mathbf{i} \left( \mathcal{T} - m \rho \Pi \right) \left( \tilde{S} - \mathbf{i} \rho S \right) (\tau, \sigma) \right) \bigg|_{\tau = 0} ||\mathcal{B}||. \quad (4.126)
\]

**The boundary state of the \((n,n)\)-brane**

By using the closed string mode expansions as derived in section 3.1.1, the previously established field-gluing conditions translate into relations between closed string modes acting on the boundary states.

The bosonic conditions become
\[
0 = \left( x_0^i - y_0^i \right) ||\mathcal{B}||; \quad 0 = \left( \alpha_n^i - \bar{\alpha}_{-n}^i \right) ||\mathcal{B}|| 
\quad (4.127)
\]
for \( i \in \mathcal{D} \) and
\[
0 = \left( p_0^I + \mathbf{i} \left( b_I x_0^I + b_I k^I \right) \right) ||\mathcal{B}||; \quad 0 = \left( \alpha_n^I + \frac{\omega_n + \mathbf{i} b_I \omega_n}{\omega_n - \mathbf{i} b_I} \bar{\alpha}_{-n}^I \right) ||\mathcal{B}|| 
\quad (4.128)
\]
with \( I \in \mathcal{N} \). The fermionic gluing conditions translate into
\[
0 = \left( \tilde{S}_0 - \mathbf{i} \rho S_0 \right) ||\mathcal{B}|| 
\quad (4.129)
\]
\[
0 = \left( \tilde{S}_n - \mathbf{i} \rho \frac{\omega_n - m \rho \Pi}{n} \left( 1 - \frac{2 \omega_n}{\omega_n - \mathcal{T}} \mathcal{R} \right) S_{-n} \right) ||\mathcal{B}||. \quad (4.130)
\]
Finally, by using the zero mode combinations (3.22), the bosonic zero mode gluing conditions furthermore take on the structure

\[ 0 = \left( \alpha_0^i - \alpha_0^i + i\sqrt{2m}y^i \right) ||B|| \]  
(4.131)

\[ 0 = \left( \alpha_0^i + \frac{m + b^j}{m - b^j} \alpha_0^i + i\sqrt{2mk^j} \right) ||B||. \]  
(4.132)

With the closed string gluing conditions (4.127)-(4.132) it is now straightforward to write down the corresponding boundary state up to an overall normalisation. This normalisation \( \mathcal{N}_{(n,n)} \) is obtained from the results presented in section 4.6 in the standard procedure by comparing a suitable closed string boundary state overlap with the corresponding open string one loop partition function. As in the instanton case from [51], the normalisation \( \mathcal{N}_{(n,n)} \) turns out to be

\[ \mathcal{N}_{(n,n)} = (4\pi m)^2 \]  
(4.133)

up to an irrelevant overall constant phase. Using the gluing conditions (4.127)-(4.132) the boundary state takes on the form

\[ ||B|| = \mathcal{N}_{(n,n)} \exp \left[ \sum_{r=1}^{\infty} \sum_{i \in D} \frac{1}{\omega_r} \alpha_{-r}^i \tilde{\alpha}_r^i + \sum_{r=1}^{\infty} \sum_{i \in N} \frac{1}{\omega_r} \frac{b^j}{\omega_r - b^j} \alpha_{-r}^i \tilde{\alpha}_r^i \right] \right] ||B_0|| \]  
(4.134)

with

\[ ||B_0|| = \prod_{l \in N} \prod_{i \in D} (B_0^l B_0^i) |0, \rho \rangle_f \]  
(4.135)

and

\[ B_0^i = \exp \left[ \left( \frac{1}{2} a_0^i a_0^i - i\sqrt{2m}y^i a_0^i \right) \right] e^{-\frac{m}{2} y^i y^i} \]  
(4.136)

\[ B_0^j = \exp \left[ - \left( \frac{1}{2} \frac{m + b^j}{m - b^j} a_0^i a_0^j + i\sqrt{2mk^j} \right) \right] e^{-\frac{m}{2} \frac{k^j b^j}{(\alpha^2 - m^2)}}. \]  
(4.137)

The fermionic vacuum state \( |0, \rho \rangle_f \) is finally determined by the condition (4.129), compare for example with [51].

### 4.5.2 Spacetime supersymmetry

In this section we determine the preserved (spacetime) supersymmetries of the boundary state (4.134). Our discussion from the open string point of view in section 4.4 together with the considerations from section 4.2 ensure at least two preserved supersymmetries on (4.134). Under certain conditions, however, some \((n, n)\)-branes
preserve additional supercharges. A certain class of (4, 4)-branes will be seen to be even maximally spacetime supersymmetric.

With the supercharges (3.41) and (3.42), the conservation of supersymmetries by the boundary state (4.134) is expressed by

$$0 = P \left( Q + i \rho M \tilde{Q} \right) \langle |\mathcal{B}\rangle \rangle$$

(4.138)

with a constant $SO(8)$–spinor matrix $M$ and a suitable projector $P$. The (maximal) rank of $P$ equals the number of preserved (dynamical) supersymmetries.

By using (4.127) - (4.130), we derive conditions for the matrices $M$ and $P$ as follows. From the zero modes along either the Dirichlet or Neumann directions from (4.127) and (4.128) we obtain

$$0 = P (1 - M) \iff PM = P. \quad (4.139)$$

From the nonzero modes along the Dirichlet directions from (4.127) we have

$$0 = P \gamma^i \left( \left( 1 + \rho \frac{\omega_n - n}{m} \right) S_n + i \rho \left( 1 - \rho \frac{\omega_n - n}{m} \right) \tilde{S}_n \right) \langle |\mathcal{B}\rangle \rangle \quad (4.140)$$

and with (4.130)

$$0 = P \gamma^i \mathcal{R}. \quad (4.141)$$

The Neumann directions with gluing conditions (4.128) furthermore give rise to

$$0 = P \gamma^i \left( \frac{\omega_n + \rho b^j}{\omega_n - b^j} \left( 1 - \rho \frac{\omega_n - n}{m} \right) S_n - i \rho \left( 1 + \rho \frac{\omega_n - n}{m} \right) \tilde{S}_n \right) \langle |\mathcal{B}\rangle \rangle \quad (4.142)$$

from which

$$0 = P \gamma^i \left[ \omega_n (1 - \mathcal{R}) + \mathcal{R} \left( b^j - \mathcal{T} \right) \right] \quad (4.143)$$

results by using (4.130). As (4.143) is required to hold for all $n$, the conditions for preserved supersymmetries finally turn out to be

$$0 = P \gamma^i (1 - \mathcal{R}); \quad 0 = P \gamma^i \left( b^j - \mathcal{T} \right)$$

$$0 = P \gamma^i \mathcal{R} \quad (4.144)$$

with $I \in \mathcal{N}$ and $i \in \mathcal{D}$. From (4.144) we can read off the number of conserved supersymmetries for $(n,n)$-branes with the present boundary conditions. To start with, for $n = 0$ one obtains the (0,0)-instanton from [117, 51]. It has only Dirichlet directions and from (4.124) we furthermore have $\mathcal{R} = 0$. That is, $P$ is of maximal rank, implying a maximally supersymmetric brane. This is of course exactly the result of [117, 51].
4.5. BOUNDARY STATES AND SPACETIME SUPERSYMMETRY

The remaining branes preserve at least the $\mathcal{N} = 2$ supersymmetry structure discussed in section 4.4 from an open string point of view. Here this subalgebra is obtained by the projector

$$ P = |\eta\rangle \langle \eta^* | + |\eta^*\rangle \langle \eta | $$

(4.145)

using the constant spinor $\eta$ defined in section 4.2. For the $(1, 1)$ and $(3, 3)$-branes and in case of pairwise different $b^i$ for the $(2, 2)$ and $(4, 4)$-branes these are the complete number of conserved supersymmetries.

For homogeneous boundary conditions along the Neumann directions, that is, by using the same parameter $b$ for all Neumann blocks, there, however, appear additional supersymmetries for the $(2, 2)$ and the $(4, 4)$ branes beyond the $\mathcal{N} = 2$ subalgebra. Employing the matrices $\mathcal{R}, \mathcal{T}$, the situation of homogenous boundary conditions translates into

$$ \mathcal{T} = b \mathcal{R}, $$

(4.146)

simplifying (4.144) accordingly. Evaluating these conditions with (4.146), the $(2, 2)$ brane is found to be quarter supersymmetric, that is, it preserves 4 supersymmetries and the $(4, 4)$ brane with $\mathcal{R} = 1$ and no Dirichlet directions along the transverse coordinates becomes finally even maximally supersymmetric.

In the classification of [51, 117] presented in section 3.2, the $(n, n)$-branes all belong to the class II branes. Our $(4, 4)$-brane therefore adds a maximally supersymmetric brane to this family, containing so far only the other extremal case of the $(0, 0)$ instanton and the $(4, 0), (0, 4)$ branes as half supersymmetric branes.

4.5.3 Boundary conditions with longitudinal flux $\mathcal{F}_{I+}$

In this section we briefly discuss how to realise the deformed Neumann boundary conditions (4.75) by switching on a nonzero flux $\mathcal{F}_{I+}$. In the context of plane wave physics this has been first discussed in [116] and later on applied in particular in [117, 51, 35].

In the presence of a boundary condensate general Neumann conditions read

$$ \partial_\sigma X^r = \mathcal{F}_i^r \partial_\tau X^s $$

(4.147)

at $\sigma = 0, \pi$, compare with (3.56). By switching on only particular longitudinal components of $\mathcal{F}$, one obtains

$$ \partial_\sigma x^r = \mathcal{F}_+^r \partial_\tau X^+ \sim \mathcal{F}_+^r P^+ $$

(4.148)

by using the standard lightcone gauge condition on $X^+$. Choosing the flux $\mathcal{F}_{I+}$ as a general affine function in $X^I$ with appropriate constant factors one obtains from (4.148) the boundary conditions (4.75), compare again with [116].

For boundary fields $F, G$ fulfilling the requirements for $\mathcal{N} = 2$ supersymmetry and
integrability, the boundary conditions (4.28) and (4.29) were seen in section 4.3 to be independent of the fermionic fields. As shown above, they take on the standard form for Neumann boundary conditions in the presence of a particular boundary condensate. Nevertheless, the fermionic boundary conditions (4.30), (4.31) respectively (4.76) and (4.77) differ clearly from the conditions usually employed for the fermionic fields in the presence of boundaries. It would be very interesting to obtain a deeper understanding of these conditions and their relation to the flux $\mathcal{F}_{t+}$ from (4.148), for example by considerations along the lines of [68].

Before discussing the open/closed duality in section 4.6, we use (4.148) to explain the relation between the open string quantities $\tilde{b}, \tilde{m}$ and $\tilde{k}$ and their closed string relatives $b, m, k$. As discussed section 3.2, one needs to apply different lightcone gauge conditions in the open respectively closed string sectors to deal with branes of the same structure in both cases. In (4.148) this effectively amounts to interchange the roles of $P^+$ and the lightcone separation $X^+$ of the branes under consideration. As discussed in [19], it follows immediately that $m, b, k$ are related to the corresponding open string quantities by

$$\tilde{m} = mt; \quad \tilde{b} = bt; \quad \tilde{k} = kt$$

with

$$t = \frac{X^+}{2\pi P^+}.$$  

The number $t$ is the modular parameter to appear in section 4.6, where also the relations (4.149) will be of further use.

### 4.5.4 The $b \to m$ limit.

To discuss the limiting situation of $b = m$ excluded in the previous discussion we briefly reconsider the local boundary field $\Sigma_\sigma(\tau)$ introduced in section 4.3. This will especially also establish the maximal supersymmetry of the $(4,4)$ brane in the open string sector. So far this was only secured for the particular $\mathcal{N} = 2$ subalgebra studied in section 4.2.

From the supercurrents (3.36) and (3.37), the condition for conserved spacetime supercharges in the open sector is given by

$$\partial_\tau \Sigma_\sigma = P \left[ \left( \partial_\tau x^\sigma \gamma^\alpha S + \tilde{m} x^\sigma \gamma^\alpha \bar{S} \right) + M \left( -\partial_\tau x^\sigma \gamma^\alpha \bar{S} + \tilde{m} x^\sigma \gamma^\alpha S \right) - \tilde{b} x^\sigma \gamma^\alpha \bar{S} - \tilde{m} x^\sigma \gamma^\alpha S \right].$$

It is again understood to be evaluated at the boundaries $\sigma = 0, \pi$ and for the case of the $(4,4)$ brane, to which we restrict attention here, one furthermore has $P = M = 1$.

By using the bosonic boundary conditions (4.75) and

$$0 = \left( \partial_\tau \left( \tilde{S} - S \right) - \left( \tilde{b} - \tilde{m} \Pi \right) \left( \tilde{S} + S \right) \right) \bigg|_{\sigma = 0, \pi}$$

(4.149)
corresponding to (4.126), we derive the following local boundary field

$$\Sigma_\pi(\tau) = \sum I \left( \left( X^I + \frac{\tilde{b}}{b - \tilde{m}} \right) \gamma^I \left( S - \tilde{S} \right) \right) \bigg|_{\sigma = \pi}.$$  \hspace{1cm} (4.153)

As it fulfills (4.151), the open string theory for the (4, 4) brane preserves the maximal supersymmetry as expected from the boundary state treatment. From (4.153) it is furthermore apparent that the (4, 4) remains maximally supersymmetric in the $\tilde{b} \to \tilde{m}$ limit in case of $\tilde{k} = 0$. The last condition corresponds to the choice $C^a = 0$ in (4.73).

It is worth pointing out that the bosonic boundary conditions (4.75) take on in this limit the structure used in [35] in an alternative construction of $(n, n)$-branes. There the authors show from an open string point of view that the common fermionic boundary conditions

$$0 = \left( \tilde{S} - MS \right) \bigg|_{\sigma = 0, \pi}$$  \hspace{1cm} (4.154)

with a matrix $M$ as defined in section 3.2 together with the bosonic boundary conditions

$$\partial_{\nu} X^I = \pm m X^I; \quad \partial_{\nu} X^{I+4} = \mp m X^{I+4}$$  \hspace{1cm} (4.155)

with $I \in \mathcal{N}_-$ lead to $(n, n)$-branes ($n = 1, \ldots 4$) preserving four spacetime supersymmetries. This is expressed by the projectors

$$P = \frac{1 \pm M \Pi}{2}$$  \hspace{1cm} (4.156)

in the conditions (4.151) and (4.138).

### 4.6 Open-Closed duality

In this section we consider an important consistency check for the $(n, n)$-boundary states constructed in section 4.5 by testing the equality of the closed string boundary state overlap

$$\mathcal{A}(t) = \langle (b, k, y_2) | e^{-2\pi i H_{\text{closed}}^+ P^+} | (b, k, y_1) \rangle$$  \hspace{1cm} (4.157)

and the one loop open string partition function

$$Z(\tilde{t}) = \text{Tr} \left[ e^{-\frac{X^{+}}{2\Sigma} H_{\text{open}}^2} \right],$$  \hspace{1cm} (4.158)

as already discussed in chapter 3. The trace in (4.158) runs over the states of an open string spanning between branes with boundary conditions corresponding to
the boundary states in (4.157). Furthermore, we note that the modular parameters are related by

$$\tilde{t} = \frac{1}{t}$$

(4.159)

and the field parameters $b^i, k^i, m$ translate as discussed in section 4.5.3.

We express (4.157) and (4.158) in terms of special functions from [19, 51] as defined in the appendix C.

For open strings spanning between two $(n, n)$-branes of the same type there are fermionic zero modes commuting with the corresponding open string Hamiltonian. As explained for example in [19], these modes lead to vanishing open string partition functions. In the closed string sector this result is confirmed by considering the zero mode part overlap which is also found to vanish, see again [59, 19, 51].

To obtain a nontrivial behaviour we consider the situation of a brane-antibrane configuration. From (4.158) we have for the open string partition function along each complex pair of Dirichlet directions

$$Z_{x^i, x^{i+4}}(\tilde{t}) = e^{-\frac{\tilde{m}}{2} \tanh(m\pi)} \sum_{j=1,4} \left( \cosh(m\pi)(y_1 y_2 + y_2 y_1) - 2y_1 y_2 \right) \frac{g_4^{(\tilde{m})}(\tilde{q})}{\left(f_1^{(\tilde{m})}(\tilde{q})\right)^2}$$

(4.160)

with $\tilde{q} = e^{-2\pi \tilde{t}}$. For a pair of Neumann directions we deduce analogously

$$Z_{x^i, x^{i+4}}(\tilde{t}) = e^{-\frac{\tilde{m}}{2} \tanh(m\pi)} \sum_{j=1,4} \left( \frac{\tanh(m\pi)}{(b^j)^2 - m^2} k^j k^j \right) \frac{g_4^{(\tilde{m})}(\tilde{q})}{\left(f_1^{(\tilde{m})}(\tilde{q})\right)^2}.$$

(4.161)

For the boundary state overlap (4.157) one derives

$$\mathcal{A}_{x^i, x^{i+4}}(t) = \exp \left[ - \sum_{j=1,4} \left( m (1 + q^m) \left( y_1 y_2 + y_2 y_1 \right) + \frac{2mq^m y_1 y_2}{1 - q^m} \right) \right] \frac{g_2^{(m)}(q)}{\left(f_1^{(m)}(q)\right)^2}$$

(4.162)

along each pair of Dirichlet directions and

$$\mathcal{A}_{x^i, x^{i+4}}(t) = \exp \left[ - \sum_{j=1,4} \frac{mk^j k^j}{(b^j)^2 - m^2} \frac{1 - q^m}{1 + q^m} \right] \frac{g_2^{(m)}(q)}{\left(f_1^{(m)}(q)\right)^2}$$

(4.163)

along a pair of Neumann directions by using in both cases the normalisation (4.133).

The zero mode prefactors in (4.162), (4.163) are for example calculated by inserting a complete set of coherent states as explained in [58, 59].

From the modular transformations properties

$$f_1^{(m)}(q) = f_1^{(\tilde{m})}(\tilde{q}); \quad g_2^{(m)}(q) = g_4^{(\tilde{m})}(\tilde{q})$$

(4.164)

the open string partition functions (4.160) and (4.161) are seen to be equal to the corresponding closed string boundary state overlaps (4.162) and (4.163). By this, the $(n, n)$-branes pass this important consistency check.
Chapter 5

Summary

In this thesis we presented a detailed study of branes with boundary structure in specific Maldacena-Maoz backgrounds with nontrivial Ramond-Ramond fluxes. In particular, we constructed integrable and supersymmetric boundary configurations for the $\mathcal{N} = 2$ supersymmetric sine-Gordon and the maximally supersymmetric plane wave worldsheet theory. We conclude this dissertation by mentioning some open questions. They might give rise to some interesting future research.

First of all, a more detailed understanding of the geometric significance of boundary fermions is desirable. This applies to their appearance in the context of Maldacena-Maoz backgrounds in chapters 2 and 4, but also to the study of B-type branes by using matrix factorisation techniques as initiated in [33, 73].

For the case of integrable branes in the plane wave background, we have shown that the fermionic boundary excitations can be expressed as combinations of bulk fields restricted to the boundaries. In this context, the search for the geometric meaning of the boundary fermions is therefore superseded by the question of how to interpret the resulting deformed boundary conditions. For the bosonic degrees of freedom we have proved that these conditions originate from a particular coupling to a longitudinal boundary flux. An analogous spacetime interpretation of the fermionic boundary conditions nevertheless remains elusive.

A different possibility for future research is to further utilise methods from integrable field theories to derive additional information for the (interacting) worldsheet theories and compare them to results from supergravity based calculations. In particular the thermodynamic Bethe-Ansatz is to be mentioned in this context. Bethe Ansatz results for supersymmetric integrable field theories can be found in [81, 97, 17, 45, 44] and a first comparison with particular supergravity results from a specific pp-wave background solution is presented in [122].
Appendix A

Conserved currents in Landau-Ginzburg models

In this appendix we explain in some detail how to deduce (higher spin) conserved currents in Landau-Ginzburg models as used in chapters 2 and 4. Following [99], we rely on a superfield formulation by using the conventions of [63] and [62]. For comprehensive reviews of this formalism in the presence of $\mathcal{N} = (2, 2)$ supersymmetry we refer to [126, 123].

As explained in [63, 123], the supertranslation generators acting on the superfields $\Phi^i, \bar{\Phi}^\dagger$ are of structure

\[ Q_\pm = \frac{\partial}{\partial \theta^\pm} + i \bar{\theta}^\pm \partial_\pm; \quad \bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - i \theta^\pm \bar{\partial}_\pm, \]

while the supercovariant derivatives are given by

\[ D_\pm = \frac{\partial}{\partial \theta^\pm} - i \bar{\theta}^\pm \partial_\pm; \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i \theta^\pm \bar{\partial}_\pm. \]

The only nontrivial anticommutators of these differential operators are

\[ \{Q_\pm, Q_\pm\} = -2i \partial_\pm; \quad \{D_\pm, \bar{D}_\pm\} = 2i \partial_\pm \]

and the conditions for chiral superfields, with which we will work throughout, read

\[ \bar{D}_\pm \Phi^i = 0; \quad D_\pm \bar{\Phi}^\dagger = 0. \]

Using (A.4) and (A.2), the component expansions of chiral superfields can be written as

\[ \Phi^i = z^i + \sqrt{2} \theta^+ \psi_+^i + \sqrt{2} \theta^- \psi_-^i + 2 \theta^+ \theta^- F^i - \theta^+ \theta^- \bar{\bar{\partial}}_\pm z^i - i \theta^+ \bar{\theta}^- \bar{\partial}_- z^i \]

\[ -i \sqrt{2} \theta^+ \theta^- \bar{\bar{\partial}}_\pm \psi_+^i - i \sqrt{2} \theta^+ \bar{\theta}^- \bar{\bar{\partial}}_+ \psi_-^i - \theta^+ \bar{\theta}^- \bar{\bar{\partial}}_+ \partial_+ z^i, \]

\[ \bar{\Phi}^\dagger = \bar{z}^\dagger - \sqrt{2} \theta^+ \bar{\psi}_+^\dagger - \sqrt{2} \theta^- \bar{\psi}_-^\dagger + 2 \theta^+ \bar{\theta}^- \bar{F}^\dagger + \theta^+ \bar{\theta}^- \bar{\bar{\partial}}_+ \bar{z}^\dagger + i \theta^+ \bar{\theta}^- \bar{\partial}_- \bar{z}^\dagger \]

\[ -i \sqrt{2} \theta^+ \bar{\theta}^- \bar{\bar{\partial}}_+ \bar{\psi}_+^\dagger - i \sqrt{2} \theta^+ \theta^- \bar{\bar{\partial}}_\pm \bar{\psi}_-^\dagger - \theta^+ \theta^- \bar{\bar{\partial}}_+ \partial_+ \bar{z}^\dagger. \]
From this, the component fields are determined to

\[
\begin{align*}
z^i &= \Phi^i \\
\psi^i_+ &= \frac{1}{\sqrt{2}} D_+ \Phi^i \\
\psi^i_- &= \frac{1}{\sqrt{2}} D_- \Phi^i
\end{align*}
\] (A.7) (A.8) (A.9)

The notation \( .. \) is understood as abbreviation for the projection \( ..|_{\theta_+ = 0, \bar{\theta}_+ = 0} \).

In a background with flat transverse metric and a zero Killing vector term, the \( \mathcal{N} = (2, 2) \) supersymmetric Landau-Ginzburg bulk Lagrangian has in superfield notation the form [63, 123]

\[
\mathcal{L} = \int d^4\theta g_{\alpha\beta} \Phi^\dagger \Phi + \frac{1}{2} \left( \int d^2\theta W(\Phi) + (c.c) \right).
\] (A.10)

It results in the equations of motion

\[
\begin{align*}
D_+ D_- \Phi^i &= g^{ij} \partial_\theta W(\Phi) \\
\overline{D}_+ \overline{D}_- \Phi^i &= -g^{ij} \partial_{\bar{\theta}} W(\Phi).
\end{align*}
\] (A.11) (A.12)

Following [78, 99], we discuss next how to apply the superfield formulation in the search for conserved (component) fluxes for (bulk) Landau-Ginzburg theories. Explicit examples will be spelled out in the subsequent sections.

Any triple \( \{X, Y, Z\} \) of superfields fulfilling

\[
\begin{align*}
D_- X &= D_+ Y; \\
D_- X &= D_+ Z
\end{align*}
\] (A.13)

gives rise to the equation

\[
\partial_- (D_+ D_- X) = \partial_+ (D_- D_+ Y)
\] (A.14)

by acting on both sides of the first equation in (A.13) with the differential operator \( D_+ \overline{D}_+ D_- \) and then using the anticommutation relations (A.3). After projecting (A.14) to the component fields, it describes a conserved flux in the bulk theory.

Analogously, one finds for a triple \( \tilde{X}, \tilde{Y}, \tilde{Z} \) fulfilling

\[
\begin{align*}
\overline{D}_+ \tilde{X} &= D_- \tilde{Y}; \\
D_+ \tilde{X} &= \overline{D}_- \tilde{Z}
\end{align*}
\] (A.15)

the conserved current

\[
\partial_+ \left( D_- \overline{D}_- \tilde{X} \right) = \partial_- \left( D_+ D_- \tilde{Y} \right)
\] (A.16)

by acting this time with \( D_- \overline{D}_- D_+ \) on the first expression in (A.15).

It is obvious that the fluxes (A.14) and (A.16) depend only on the corresponding \( D_\pm \)-cohomology classes of \( Y \) and \( \tilde{Y} \) and not on the particular representatives. We will see an example of this below.
A.1 The energy-momentum tensor

Every Landau-Ginzburg model possesses a conserved spin two flux, whose charge corresponds to the standard energy-momentum tensor. To commence with a simple example, we explain how to derive this current by the superfield formalism sketched beforehand. For notational simplicity we concentrate on a Landau-Ginzburg model with a single complex coordinate.

By using the equations of motion (A.11) and (A.12), one can show that the following triples

\[ X = \overline{D}_+ \Phi D_+ \Phi; \quad Y = W(\Phi); \quad Z = \overline{W}(\Phi) \tag{A.17} \]

and

\[ \tilde{X} = \overline{D}_- \Phi D_- \Phi; \quad \tilde{Y} = -W(\Phi); \quad \tilde{Z} = -\overline{W}(\Phi) \tag{A.18} \]

fulfil (A.13) and (A.15), respectively. By using the superpotential of the plane wave background as presented in equation (4.1), one obtains from (A.17) the component current

\[ T_2 = 4 \left( \partial_+ \overline{z} \partial_+ z + \overline{\psi}_+ \partial_+ \psi_+ \right) \tag{A.19} \]

\[ \theta_0 = 4 \left( -m^2 z \overline{z} + i m \psi_+ \overline{\psi}_- \right). \tag{A.20} \]

On-shell, this conserved current agrees with the manifestly real expression used in chapter 4 modulo a shift

\[ T_2 \rightarrow T_2 + \partial_+ H \]

\[ \theta_0 \rightarrow \theta_0 + \partial_- H. \tag{A.21} \]

In the closed string sector the shifted equations give rise to the same charges. In the open string sector the resulting charges differ only by boundary contributions. In particular, both equivalent fluxes in (A.21) are conserved under the same conditions in the presence of boundaries, compare for example with section 2.

A.2 Conserved higher spin currents of the $N = 2$ sine-Gordon model

In this section we derive the higher spin conserved currents of the $N = 2$ sine-Gordon model on which our integrability treatment in chapter 2 is based on. As described there, the Landau-Ginzburg model of present interest is described by the superpotential

\[ W(z) = -2g i \cos(z). \tag{A.22} \]

It leads to the superfield equations of motion

\[ D_+ D_- \Phi = -2g i \sin(\overline{\Phi}) \tag{A.23} \]

\[ \overline{D}_+ D_- \Phi = 2g i \sin \Phi \tag{A.24} \]
from (A.11) and (A.12). The first nontrivial higher spin currents of this model were described in [78] and our present discussion follows [99]. A suitable superfield triple fulfilling (A.13) is given by

\[ X = \overline{D}_+ \Phi D_+ \Phi \left( (\partial_+ \Phi)^2 + (\partial_+ \Phi)^2 \right) - 2\partial_+ \overline{D}_+ \Phi \partial_+ D_+ \Phi \]
\[ Y = 2g_i \cos \Phi \left( (\partial_+ \Phi)^2 + (\partial_+ \Phi)^2 \right) \]
\[ Z = 2g_i \cos \Phi \left( (\partial_+ \Phi)^2 + (\partial_+ \Phi)^2 \right) \]  
\text{(A.25)}

and correspondingly for (A.15)

\[ \tilde{X} = \overline{D}_- \Phi D_- \Phi \left( (\partial_- \Phi)^2 + (\partial_- \Phi)^2 \right) - 2\partial_- \overline{D}_- \Phi \partial_- D_- \Phi \]
\[ \tilde{Y} = -2g_i \cos \Phi \left( (\partial_- \Phi)^2 + (\partial_- \Phi)^2 \right) \]
\[ \tilde{Z} = -2g_i \cos \Phi \left( (\partial_- \Phi)^2 + (\partial_- \Phi)^2 \right) . \]  
\text{(A.26)}

Our choices of \( Y, Z \) and \( \tilde{Y}, \tilde{Z} \) have a somewhat simpler structure than those used in [99]. They are, however, only different representatives of the same \( D_\pm, D_\pm \) cohomology classes and therefore lead to identical conserved currents. Using (A.25) and (A.26), the corresponding component versions of (A.14) and (A.16) were derived in [99] and are given (in our conventions) by

\[ T_4 = - (\partial_+ z)^3 \partial_+ z - (\partial_+ z)^3 \partial_+ \bar{z} + 2\partial_+^2 \bar{z} \partial_+^2 z \\
- i\bar{\psi}_+ \partial_+ \psi_+ \left( (\partial_+ \bar{z})^2 + 3(\partial_+ z)^2 \right) - 2i \bar{\psi}_+ \partial_+ \psi_+ (\partial_+ \bar{z})^2 \\
- 2i \bar{\psi}_+ \partial_+ \bar{\psi}_+ \partial_+ \bar{z} \bar{\partial}_+^2 \bar{z} + 2i \bar{\partial}_+ \bar{\psi}_+ \partial_+ \bar{\psi}_+ \]  
\text{(A.27)}

\[ \theta_2 = g^2 \sin z \sin \bar{z} \left[ (\partial_+ \bar{z})^2 + (\partial_+ z)^2 \right] - 2g^2 \cos z \cos \bar{z} \partial_+ z \partial_+ \bar{z} \\
- ig \cos z \psi_+ \bar{\psi}_+ \left[ (\partial_+ \bar{z})^2 + (\partial_+ z)^2 \right] \\
- 2i g \sin z \partial_+ z \left[ -\psi_+ \partial_+ \psi_- + \psi_- \partial_+ \psi_+ \right] + 2ig \cos z \partial_+ \psi_- \partial_+ \psi_+ \\
= g^2 \sin z \sin \bar{z} \left[ (\partial_+ \bar{z})^2 + (\partial_+ z)^2 \right] - 2g^2 \cos z \cos \bar{z} \partial_+ z \partial_+ \bar{z} \\
- 2i g^2 \sin z \cos \bar{z} \psi_+ \bar{\psi}_+ + 2i g^2 \cos z \cos \bar{z} \bar{\psi}_+ \partial_+ \psi_+ \\
- ig \cos z \psi_+ \bar{\psi}_+ \left[ (\partial_+ \bar{z})^2 + (\partial_+ z)^2 \right] - 2ig \sin z \partial_+ z \bar{\psi}_+ \partial_+ \psi_+ \]  
\text{(A.28)}

and

\[ T_4 = - (\partial_- z)^3 \partial_- z - (\partial_- z)^3 \partial_- \bar{z} + 2\partial_-^2 \bar{z} \partial_-^2 z \\
- i\bar{\psi}_- \partial_- \psi_- \left( (\partial_- \bar{z})^2 + 3(\partial_- z)^2 \right) - 2i \bar{\psi}_- \partial_- \psi_- (\partial_- \bar{z})^2 \\
- 2i \bar{\psi}_- \partial_- \bar{\psi}_- \partial_- \bar{z} \bar{\partial}_-^2 \bar{z} + 2i \bar{\partial}_- \bar{\psi}_- \partial_- \bar{\psi}_- \]  
\text{(A.29)}

\[ \overline{\theta}_2 = g^2 \sin z \sin \bar{z} \left[ (\partial_- \bar{z})^2 + (\partial_- z)^2 \right] - 2g^2 \cos z \cos \bar{z} \partial_- z \partial_- \bar{z} \\
- ig \cos z \psi_- \bar{\psi}_- \left[ (\partial_- \bar{z})^2 + (\partial_- z)^2 \right] \\
- 2i g \sin z \partial_- z \left[ -\psi_- \partial_- \psi_+ + \psi_+ \partial_- \psi_- \right] + 2ig \cos z \partial_- \psi_+ \partial_- \psi_- \\
= g^2 \sin z \sin \bar{z} \left[ (\partial_- \bar{z})^2 + (\partial_- z)^2 \right] - 2g^2 \cos z \cos \bar{z} \partial_- z \partial_- \bar{z} \\
- 2i g^2 \sin z \cos \bar{z} \psi_- \bar{\psi}_- + 2i g^2 \cos z \cos \bar{z} \bar{\psi}_- \partial_- \psi_- \\
- ig \cos z \psi_- \bar{\psi}_- \left[ (\partial_- \bar{z})^2 + (\partial_- z)^2 \right] + 2ig \sin z \partial_- z \psi_+ \partial_- \psi_- . \]  
\text{(A.30)}
From (A.14) and (A.16) they fulfil on-shell
\[
\partial_- T_4 = \partial_+ \theta_2 \quad \text{(A.31)}
\]
\[
\partial_+ T_4 = \partial_- \theta_2. \quad \text{(A.32)}
\]

A slightly disturbing feature appearing in (A.27)-(A.30) is the manifestly real bosonic structure with no obvious analog in the fermionic part. By using a redefinition with a suitable shift as in (A.21) we can, however, remedy this and establish (manifest) reality throughout. The equivalent currents are

\[
T'_4 = -(\partial_+ \bar{\psi})^3 \partial_+ \psi - (\partial_+ \bar{\psi})^3 \partial_+ \bar{\psi} + 2i \bar{\psi} \bar{\psi} \partial_+ \bar{\psi}
+ 3i \partial_+ \bar{\psi} \psi_+ (\partial_+ \bar{\psi})^2 - 3i \bar{\psi}_+ \partial_+ \bar{\psi} (\partial_+ \bar{\psi})^2
+ i \partial_+ \bar{\psi} (\partial_+ \bar{\psi}^2 \partial_+ \psi_+ - i \partial_+ \bar{\psi} \partial_+ \psi_+) \quad \text{(A.33)}
\]

\[
\theta'_2 = g^2 \sin \bar{z} \sin \bar{z} \left[ (\partial_+ \bar{\psi})^2 + (\partial_+ \bar{\psi})^2 \right] - 2g^2 \cos \bar{z} \cos \bar{z} \partial_+ \bar{\psi} \partial_+ \bar{\psi}
+ ig^2 \cos \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- (\partial_+ \bar{\psi})^2}{\partial_+ \bar{\psi}} - ig^2 \cos \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- (\partial_+ \bar{\psi})^2}{\partial_+ \bar{\psi}}
+ ig \sin \bar{z} \partial_+ \bar{\psi} \bar{\psi}_- \partial_+ \bar{\psi} - ig \sin \bar{z} \bar{\psi}_- \partial_+ \bar{\psi} \partial_+ \bar{\psi}
+ 2ig^2 \sin \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- \partial_+ \bar{\psi}}{\partial_+ \bar{\psi}} - 2ig^2 \sin \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- \partial_+ \bar{\psi}}{\partial_+ \bar{\psi}} \quad \text{(A.34)}
\]

and analogously

\[
\bar{T}'_4 = - (\partial_- \bar{\psi})^3 \partial_- \psi - (\partial_- \bar{\psi})^3 \partial_- \bar{\psi} + 2i \bar{\psi} \bar{\psi} \partial_- \bar{\psi}
+ 3i \partial_- \bar{\psi} \psi_- (\partial_- \bar{\psi})^2 - 3i \bar{\psi}_- \partial_- \bar{\psi} (\partial_- \bar{\psi})^2
+ i \partial_- \bar{\psi} (\partial_- \bar{\psi}^2 \partial_- \psi_- - i \partial_- \bar{\psi} \partial_- \psi_-) \quad \text{(A.35)}
\]

\[
\bar{\theta}'_2 = g^2 \sin \bar{z} \sin \bar{z} \left[ (\partial_- \bar{\psi})^2 + (\partial_- \bar{\psi})^2 \right] - 2g^2 \cos \bar{z} \cos \bar{z} \partial_- \bar{\psi} \partial_- \bar{\psi}
+ ig^2 \cos \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- (\partial_- \bar{\psi})^2}{\partial_- \bar{\psi}} - ig^2 \cos \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- (\partial_- \bar{\psi})^2}{\partial_- \bar{\psi}}
+ ig \sin \bar{z} \partial_- \bar{\psi} \bar{\psi}_- \partial_- \bar{\psi} - ig \sin \bar{z} \bar{\psi}_- \partial_- \bar{\psi} \partial_- \bar{\psi}
+ 2ig^2 \sin \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- \partial_- \bar{\psi}}{\partial_- \bar{\psi}} - 2ig^2 \sin \bar{z} \cos \bar{z} \frac{\bar{\psi}_+ \bar{\psi}_- \partial_- \bar{\psi}}{\partial_- \bar{\psi}} \quad \text{(A.36)}
\]

The existence of these equivalent real forms of (A.27) - (A.30) explains why all conditions derived in chapter 2 came in complex conjugated pairs.
Appendix B

The boundary current $\Sigma^{(3)}_\pi(t)$ of the sine-Gordon model

In this appendix we collect some technical details left out in chapter 2. In particular, we present the explicit boundary current $\Sigma^{(3)}_\pi(t)$ that arises from the conserved currents (A.27)-(A.30) discussed in the previous appendix A.

Our integrability treatment of the $\mathcal{N} = 2$ boundary sine-Gordon model and the determination of $\Sigma^{(3)}_\pi(t)$ rely on the following identities at $\sigma = \pi$:

\begin{align}
\partial_\tau \bar{\theta}_- &= \frac{1}{2} \partial_\tau \bar{A}(z) = \frac{1}{2} \bar{A}'(z) \partial_\tau \bar{z} + \bar{F}'G' \bar{\theta} + \partial_z \partial_z B \bar{\theta} + \partial_z \partial_z B \bar{\theta} + \partial_z \partial_z B \bar{\theta} \tag{B.1} \\
\partial_\tau \theta_- &= \frac{1}{2} \partial_\tau A(z) = \frac{1}{2} A'(z) \partial_\tau z + \partial_z \partial_z B \bar{\theta} + \partial_z \partial_z B \bar{\theta} + \partial_z \partial_z B \bar{\theta} \tag{B.2} \\
\partial_\tau A'(z) &= A''(z) \partial_\tau z + 2 \partial_\tau \partial_\tau B \bar{\theta} + \partial_z (G'F') \theta \tag{B.3} \\
\partial_\tau \bar{A}'(z) &= \bar{A}''(z) \partial_\tau \bar{z} + \partial_\bar{z} \left(\bar{F}'\bar{G}'\right) \bar{\theta} + 2 \partial_\bar{z} \partial_\bar{z} \bar{\theta} \theta \tag{B.4}
\end{align}

and

\begin{align}
\partial_\tau A''(z) &= A'''(z) \partial_\tau z + \left(G'''G' + F'''F'\right) \bar{\theta} + \left(G'''F' + F'''G'\right) \theta \tag{B.5} \\
\partial_\tau \bar{A}''(z) &= \bar{A}'''(z) \partial_\tau \bar{z} + \left(F'''G' + \bar{G}'''\bar{F}'\right) \bar{\theta} + \left(\bar{F}'''F' + \bar{G}'''G'\right) \theta. \tag{B.6}
\end{align}

All previous identities follow from the boundary conditions (2.25) - (2.30) upon (partial) differentiation.

Quadratic fermionic terms as $A(z)\bar{A}(\bar{z})$ furthermore lead to relations as

\begin{align}
A(z)\bar{A}(\bar{z}) &= \left(G'(z)\bar{G}' - F'(z)\bar{F}'(\bar{z})\right) \bar{a}a \tag{B.7}
\end{align}

and

\begin{align}
A(z)A'(z) &= \left(G'(z)F'''(z) - F'(z)G'''(z)\right) \bar{a}a. \tag{B.8}
\end{align}
Using the prior identities in the calculational steps explained in section 2.4, the local boundary term $\Sigma_{\tau}^{(3)}(\tau)$ appearing in the conserved quantity (2.54) is determined to be

$$
\Sigma_{\tau}^{(3)}(\tau) = 16i \partial_{\tau}^2 z A'(z) + 16i \partial_{\tau}^2 \bar{z} \bar{A}'(\bar{z}) \bar{\theta}_+ \\
+8 \partial_{\tau} \partial_{\bar{z}} B(\partial_{\tau} z)^2 + 8 \partial_{\bar{z}} \partial_{\bar{z}} B(\partial_{\bar{z}} \bar{z})^2 + 16 \partial_{\tau} \partial_{\bar{z}} B \partial_{\tau} z \partial_{\tau} \bar{z} \\
+8ig^2 \sin z \cos \bar{z} A'(z) \theta_+ + 8ig^2 \sin \bar{z} \cos z \bar{A}'(\bar{z}) \bar{\theta}_+ \\
-4i \bar{\theta}_- \theta_+ ((\partial_{\tau} z)^2 + 3(\partial_{\tau} \bar{z})^2) - 4i \bar{\theta}_- \theta_+ ((\partial_{\tau} \bar{z})^2 + 3(\partial_{\tau} z)^2) \\
-8i \theta_{-\bar{\theta}_+} ((\partial_{\tau} z)^2 + (\partial_{\tau} \bar{z})^2) \\
-16ig \cos z e^{ig} (\theta_+ \partial_{\tau} \theta_- - \theta_- \partial_{\tau} \theta_+) + 32i \partial_{\tau} \bar{\theta}_- \partial_{\tau} \theta_+ \\
+16ig^2 \cos z \cos \bar{z} \partial_{\tau} \theta_- - 8ig \sin \bar{z} e^{-ig} \partial_{\tau} \bar{B} \bar{\theta}_+ \bar{\theta}_- \\
+16i \bar{\theta}_+ \bar{A}'(\bar{z}) \partial_{\bar{z}}^2 \bar{\theta}_+ - 16i A'(z) \partial_{\tau}^2 z \theta_+ + 32i \partial_{\tau} \partial_{\bar{z}} \partial_{\bar{z}} B \partial_{\tau} \bar{z} \bar{\theta}_- \theta_+ \\
+16i (\partial_{\tau} z)^2 \bar{\theta}_+ \bar{A}'(\bar{z}) - 16i (\partial_{\tau} \bar{z})^2 A'(z) \theta_+ \\
-8ig \cos z \cos \bar{z} \bar{\theta}_+ \theta_- + 8ig \sin z e^{ig} \partial_{\tau} \bar{B} \theta_- \theta_+ \\
-8i \partial_{\tau} \bar{B} \partial_{\tau} \bar{z} \bar{\theta}_- \theta_- + 16 \partial_{\tau} \bar{B} A'(z) \theta_- \bar{\theta}_+ \theta_+ \\
+H_1(z, \bar{z}) + H_2(z, \bar{z}) \quad \text{(B.9)}
$$

with

$$
\partial_{\tau} H_1(z, \bar{z}) = (-2(\partial_{\tau} z)^3 - 6\partial_{\tau} \bar{B} \partial_{\tau} (\partial_{\bar{z}} \bar{B})) \quad \text{(B.10)} \\
\partial_{\bar{z}} H_1(z, \bar{z}) = (-2(\partial_{\bar{z}} \bar{B})^3 - 6\partial_{\bar{z}} \bar{B} \partial_{\bar{z}} (\partial_{\tau} z)) \quad \text{(B.11)}
$$

and

$$
\partial_{\tau} H_2(z, \bar{z}) = +4g^2 [\sin z \sin \bar{z} \partial_{\tau} B - \cos z \cos \bar{z} \partial_{\tau} B] + 2 \sin z \cos \bar{z} \partial_{\tau} \partial_{\bar{z}} B + 2 \cos z \sin \bar{z} \partial_{\bar{z}} \partial_{\tau} B] \quad \text{(B.12)} \\
\partial_{\bar{z}} H_2(z, \bar{z}) = +4g^2 [\sin z \sin \bar{z} \partial_{\bar{z}} B - \cos z \cos \bar{z} \partial_{\bar{z}} B] + 2 \sin z \cos \bar{z} \partial_{\bar{z}} \partial_{\tau} B + 2 \cos z \sin \bar{z} \partial_{\tau} \partial_{\bar{z}} B] \quad \text{(B.13)}
$$

For the choice $B(z, \bar{z}) = \alpha \sin \frac{z}{2} \sin \frac{\bar{z}}{2}$ these functions read for example

$$
H_1(z, \bar{z}) = \frac{1}{12} \alpha^2 (-4 + \cos z + \cos \bar{z} + 2 \cos z \cos \bar{z}) B(z, \bar{z}) \quad \text{(B.14)} \\
H_2(z, \bar{z}) = 4g^2 B(z, \bar{z}) \quad \text{(B.15)}
$$
Appendix C

Modular functions

In this appendix we present the various modular functions appearing in chapters 3 and 4. The first family of modular functions $f_{i}^{(m)}(q)$ from [19] is given as a mass-dependent generalisation of the $f$-functions of Polchinski and Cai from [107]. The functions $g_{i}^{(m)}(q)$ from [51] involve the transcendent modings as introduced in chapter 3 and have a considerably more complex structure. In particular, they are without a straightforward analogue in flat space-physics. We close this appendix by recapitulating our generalisation of $g_{2}^{(m)}(q)$ from chapter 3 and also discuss the interpolation between functions from [19] and [51].

The elements from the first class of modular functions from [19] are defined by

\[
\begin{align*}
\tilde{A}_{m}(q) & = q^{-\Delta_{m}} (1 - q^{m})^{\frac{1}{2}} \prod_{n=1}^{\infty} \left( 1 - q^{\sqrt{m^2 + n^2}} \right), \\
A_{m}(q) & = q^{-\Delta_{m}} (1 + q^{m})^{\frac{1}{2}} \prod_{n=1}^{\infty} \left( 1 + q^{\sqrt{m^2 + n^2}} \right), \\
\tilde{A}_{m}'(q) & = q^{-\Delta_{m}'} \prod_{n=1}^{\infty} \left( 1 + q^{\sqrt{m^2 + (n-\frac{1}{2})^2}} \right), \\
A_{m}'(q) & = q^{-\Delta_{m}'} \prod_{n=1}^{\infty} \left( 1 - q^{\sqrt{m^2 + (n-\frac{1}{2})^2}} \right)
\end{align*}
\]

with

\[
\Delta_{m} = -\frac{1}{(2\pi)^{2}} \sum_{p=1}^{\infty} \int_{0}^{\infty} ds \ e^{-p^{2}s} e^{-\frac{s^{2}m^{2}}{s}}
\]

and

\[
\Delta_{m}' = -\frac{1}{(2\pi)^{2}} \sum_{p=1}^{\infty} (-1)^{p} \int_{0}^{\infty} ds \ e^{-p^{2}s} e^{-\frac{s^{2}m^{2}}{s}}.
\]

By using

\[
q = e^{-2\pi i}; \quad q = -2\pi i; \quad \tilde{t} = \frac{1}{t}; \quad \tilde{m} = mt
\]
one obtains the modular properties
\[ f_1^{(m)}(q) = f_1^{(\bar{m})}(\bar{q}); \quad f_2^{(m)}(q) = f_1^{(\bar{m})}(\bar{q}); \quad f_3^{(m)}(q) = f_3^{(\bar{m})}(\bar{q}), \] (C.8)
as shown in detail in [19]. The second class of mass dependent modular functions from [51] is given by
\[
g_1^{(m)}(q) = 4\pi m q^{-2\Delta_m} q^n \prod_{n=1}^{\infty} \left(1 - \frac{(\omega_n + m)}{\omega_n - m} q^{\omega_n}\right) \left(1 - \frac{(\omega_n - m)}{\omega_n + m} q^{\omega_n}\right) \] (C.9)
\[
g_2^{(m)}(q) = 4\pi m q^{-2\Delta_m} q^n \prod_{n=1}^{\infty} \left(1 + \frac{(\omega_n + m)}{\omega_n - m} q^{\omega_n}\right) \left(1 + \frac{(\omega_n - m)}{\omega_n + m} q^{\omega_n}\right) \] (C.10)
\[
g_3^{(m)}(q) = 2q^{-2\Delta_m} \prod_{n=1}^{\infty} \left(1 + \frac{(\omega_n - \frac{1}{2} + m)}{\omega_n - \frac{1}{2} - m} q^{\omega_n - \frac{1}{2}}\right) \left(1 + \frac{(\omega_n - \frac{1}{2} - m)}{\omega_n - \frac{1}{2} + m} q^{\omega_n - \frac{1}{2}}\right) \] (C.11)
\[
g_4^{(m)}(q) = 2q^{-2\Delta_m} \prod_{n=1}^{\infty} \left(1 - \frac{(\omega_n - \frac{1}{2} + m)}{\omega_n - \frac{1}{2} - m} q^{\omega_n - \frac{1}{2}}\right) \left(1 - \frac{(\omega_n - \frac{1}{2} - m)}{\omega_n - \frac{1}{2} + m} q^{\omega_n - \frac{1}{2}}\right) \] (C.12)
and
\[
g_1^{(\bar{m})}(\bar{q}) = \bar{q}^{-\bar{\Delta}_{\bar{m}}} \prod_{l \in \mathcal{M}^+} \left(1 - \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \prod_{l \in \mathcal{M}^{-}} \left(1 - \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \] (C.13)
\[
g_2^{(\bar{m})}(\bar{q}) = \bar{q}^{-\bar{\Delta}_{\bar{m}}} \prod_{l \in \mathcal{M}^+} \left(1 + \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \prod_{l \in \mathcal{M}^{-}} \left(1 + \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \] (C.14)
\[
g_3^{(\bar{m})}(\bar{q}) = \bar{q}^{-\bar{\Delta}_{\bar{m}}} \prod_{l \in \mathcal{P}^+} \left(1 + \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \prod_{l \in \mathcal{P}^{-}} \left(1 + \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \] (C.15)
\[
g_4^{(\bar{m})}(\bar{q}) = \bar{q}^{-\bar{\Delta}_{\bar{m}}} \prod_{l \in \mathcal{P}^+} \left(1 - \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \prod_{l \in \mathcal{P}^{-}} \left(1 - \bar{q}^{\bar{\omega}_n}\right)^{\frac{1}{2}} \] (C.16)

with \(\bar{\omega}_n = \sqrt{n^2 + m^2}\) and
\[
\bar{\Delta}_{\bar{m}} = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \sum_{r=0}^{\infty} c_p^r \bar{m} \frac{\partial^r}{(\partial \bar{m})^2} \frac{1}{\bar{m}} \int_0^\infty ds \left(\frac{-s}{\pi^2}\right)^r e^{-p^2 s - \pi^2 \bar{m}^2 s}, \] (C.17)
\[
\bar{\Delta}_{\bar{m}}' = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} (-1)^p \sum_{r=0}^{\infty} c_p^r \bar{m} \frac{\partial^r}{(\partial \bar{m})^2} \frac{1}{\bar{m}} \int_0^\infty ds \left(\frac{-s}{\pi^2}\right)^r e^{-p^2 s - \pi^2 \bar{m}^2 s}. \] (C.18)

The Taylor coefficients \(c_p^r\) are taken from the expansion
\[
\left(\frac{x+1}{x-1}\right)^{p} + \left(\frac{x-1}{x+1}\right)^{p} = \sum_{r=0}^{\infty} c_p^r x^{2r}, \] (C.19)
and we furthermore defined sets $\mathcal{P}_\pm$ and $\mathcal{M}_\pm$ by

$$
l \in \mathcal{P}_+ \iff 0 = \frac{l + i\tilde{m}}{l - i\tilde{m}} + e^{2\pi \bar{u}}
$$

$$
l \in \mathcal{P}_- \iff 0 = \frac{l - i\tilde{m}}{l + i\tilde{m}} + e^{2\pi \bar{u}}
$$

$$
l \in \mathcal{M}_+ \iff 0 = \frac{l + i\tilde{m}}{l - i\tilde{m}} - e^{2\pi \bar{u}}
$$

$$
l \in \mathcal{M}_- \iff 0 = \frac{l - i\tilde{m}}{l + i\tilde{m}} - e^{2\pi \bar{u}}.
$$

The behaviour under modular transformations is in this case given by

$$
g_1^{(m)}(q) = \tilde{g}_1^{(\tilde{m})}(\tilde{q}); \quad g_2^{(m)}(q) = \tilde{g}_2^{(\tilde{m})}(\tilde{q}); \quad g_4^{(m)}(q) = \tilde{g}_4^{(\tilde{m})}(\tilde{q}); \quad g_3^{(m)}(q) = \tilde{g}_3^{(\tilde{m})}(\tilde{q})
$$

(C.24)

as established in [51].

In chapter 3 we have introduced a generalisation of the previous special functions appearing in cylinder diagrams of branes with gauge field condensates. We fill in some previously left out details here and concentrate as before on the generalisations of $g_2^{(m)}$ and $\tilde{g}_4^{(\tilde{m})}$. It is worth pointing out that a similar treatment is also possible for the remaining functions.

From section 3.6 we have for the first function

$$
g_2^{(m)}(q, \theta) = 2\sinh \left[ m\pi \sin \left( \frac{\theta}{2} \right) \right] q^{-2\Delta_m} \sqrt{1 + \sin^2 \left( \frac{\theta}{2} \right) \left( 1 - \cos \frac{\theta}{2} \right)^2 q^m} \prod_{n=1}^{\infty} \left( 1 + q^{\omega_n - m \cos \frac{\theta}{2}} \right) \left( 1 + q^{\omega_n + m \cos \frac{\theta}{2}} \right).
$$

(C.25)

By using (C.10) and (C.2) we obtain the limits

$$
\lim_{\theta \to 0} g_2^{(m)}(q, \theta) = g_2^{(m)}(q)
$$

(C.26)

$$
\lim_{\theta \to \pi} g_2^{(m)}(q, \theta) = 2 \sinh \left[ m\pi \right] \left( f_2^{(m)}(q) \right)^2.
$$

(C.27)

The generalisation of $\tilde{g}_4^{(\tilde{m})}$ from section 3.8 reads

$$
\tilde{g}_4^{(\tilde{m})}(\tilde{t}, \theta) = 2\sinh \left[ \tilde{m}\pi \tilde{t} \right] \tilde{q}^{-\Delta_{\tilde{m}, \theta} + \frac{\tilde{m}}{2} (1 - \sin \frac{\theta}{2})} \prod_{\lambda \in P_+} \sqrt{1 - \tilde{q}^{\lambda^2 + \tilde{m}^2}} \prod_{\lambda \in P_-} \sqrt{1 - \tilde{q}^{\lambda^2 + \tilde{m}^2}},
$$

(C.28)

with the offset

$$
\Delta_{\tilde{m}, \theta} = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} (-1)^p \sum_{r=0}^{\infty} \tilde{c}_r \tilde{m} \left( \frac{\partial}{\partial \tilde{m}^2} \right)^r \frac{1}{\tilde{m}} \int_0^{\infty} ds \left( \frac{-s}{\pi^2 \cos^2 \frac{\theta}{2}} \right)^r e^{-s \theta^2 - \frac{\tilde{m}^2 s^2}{s}}.
$$

(C.29)
and
\[
\begin{align*}
&n \in P_\theta^+ : \quad \frac{n + i \tilde{m} \cos \frac{\vartheta}{2}}{n - i \tilde{m} \cos \frac{\vartheta}{2}} = -e^{2\pi in}, \quad n \neq 0 \tag{C.30} \\
n \in P_\theta^- : \quad \frac{n - i \tilde{m} \cos \frac{\vartheta}{2}}{n + i \tilde{m} \cos \frac{\vartheta}{2}} = -e^{2\pi in}, \quad n \neq 0. \tag{C.31}
\end{align*}
\]

The coefficients \(c_r^\theta\) in (C.29) are again taken from the power series expansion (C.19) and the relation under modular transformations between these \(\theta\)-deformed functions is the expected
\[
\tilde{g}_2^{(m)}(t, \vartheta) = \tilde{g}_1^{(\tilde{m})}(-t, \vartheta). \tag{C.32}
\]

By setting \(m_1 = m \cos \frac{\vartheta}{2}\) the proof of (C.32) carries over from the discussion presented in appendix D of [51], such that we will not reproduce it here in detail.

To discuss the limits \(\vartheta \to 0, \pi\) one needs to treat carefully the power series expansion (C.19) or analogously
\[
\left(\frac{\omega_n + m_1}{\omega_n - m_1}\right)^p + \left(\frac{\omega_n - m_1}{\omega_n + m_1}\right)^p = \sum_{r=0}^{\infty} c_r^\theta \left(\frac{\omega_n}{m_1}\right)^r. \tag{C.33}
\]

As it should, the limit \(\vartheta \to 0\) in (C.28) and (C.29) reproduces the instanton result. The limit \(\vartheta \to \pi\), however, is singular, as the expansion (C.33) as used in the derivation of (C.29) strictly makes sense only when understood as an analytic continuation for the meromorphic function on the left hand side of (C.33) to values
\[
m_1 = m \cos \frac{\vartheta}{2} < \omega_n = \sqrt{n^2 + m^2}. \tag{C.34}
\]

Doing this, only the \(r = 0\) term in (C.29) contributes with a factor \(c_0^\theta \to 2\), leading to
\[
\overline{\Delta_{m, \vartheta}} \to 2\Delta'_m \tag{C.35}
\]
with \(\Delta'_m\) from (C.6). Altogether, this results in the limits
\[
\lim_{\vartheta \to 0} \tilde{g}_1^{(\tilde{m})}(\tilde{t}, \vartheta) = \tilde{g}_1^{(\tilde{m})}(\tilde{t}) \tag{C.36}
\]
\[
\lim_{\vartheta \to \pi} \tilde{g}_1^{(\tilde{m})}(\tilde{t}, \vartheta) = 2 \sinh [m \pi] \left(\int_1^{(\tilde{m})}(\tilde{q})\right)^2. \tag{C.37}
\]

Together with (C.26) and (C.27) they are of course in accordance with the results from [19] and [51].

As in the previous steps, in parts of the proof of (C.32) use is made of an analytic continuation in \(m_1\). It would be interesting to see whether there is a more direct approach as for example found by Gannon in [53] for the \(f_i^{(m)}\) functions by applying certain \(\theta\)-function identities.
Appendix D

Quantisation

In this appendix we provide some details for the canonical quantisation of open strings in the settings covered in chapters 3 and 4. Using the conventions introduced in chapter 4, the required (anti-) commutation relations in the quantum theory take on the standard form

\[
[x^r(r,\sigma), p^a(r,\bar{\sigma})] = 4\pi i \delta^{ra} \delta(\sigma - \bar{\sigma}) \tag{D.1}
\]
\[
[x^r(r,\sigma), x^a(r,\bar{\sigma})] = 0 \tag{D.2}
\]
\[
[p^r(r,\sigma), p^a(r,\bar{\sigma})] = 0 \tag{D.3}
\]

for the bosonic fields and

\[
\{\psi^+_a(r,\sigma), \psi^+_b(r,\bar{\sigma})\} = 2\pi \delta^{ab} \delta(\sigma - \bar{\sigma}) \tag{D.4}
\]
\[
\{\psi^-_a(r,\sigma), \psi^-_b(r,\bar{\sigma})\} = 2\pi \delta^{ab} \delta(\sigma - \bar{\sigma}) \tag{D.5}
\]
\[
\{\psi^+_a(r,\sigma), \psi^-_b(r,\bar{\sigma})\} = 0 \tag{D.6}
\]

for the fermions. All relations are understood to be evaluated for \(0 < \sigma, \bar{\sigma} < \pi\). For fermionic fields in the Green-Schwarz formulation the analogous relations can be found in the equations (3.27)-(3.31) from chapter 3\(^1\).

Choosing appropriate normalisations for the nonzero modes in the field expansions, the corresponding commutation relations take on the canonical form presented in sections 3.8, 4.4.1 and 4.4.2. The zero mode relations follow from them by using the contour integral method introduced in [51] and briefly sketched in section 4.4.3 beforehand. We start by discussing open strings as appearing in chapter 4 and consider the situation of boundary magnetic fields from chapter 3 thereafter in section D.2.

\(^1\)The overall factor between the bosonic conditions (D.1) and (3.27) comes from different normalisation conventions for the canonical conjugated momenta \(p^a(\tau,\sigma)\) in the chapters 3 and 4. They are of no further significance.
D.1 Open strings with boundary fermions

For the fermions along Dirichlet directions with \( a, b \in \mathbb{D} \), we have from (4.62)

\[
\{\psi^a_+, \psi^b_+ (\tau, \sigma)\} = \{\psi^a, \psi^b\} e^{-\tilde{m}(\sigma+\bar{\sigma})} + \sum_{r \neq 0} e^{ir(\sigma-\bar{\sigma})} \\
+ 2i \sum_{r \neq 0} c_r^2 r + i \tilde{m} \omega_r - r \tilde{m} e^{ir(\sigma+\bar{\sigma})} \\
= \{\psi^a, \psi^b\} e^{-\tilde{m}(\sigma+\bar{\sigma})} + \sum_{r \in \mathbb{Z}} e^{ir(\sigma-\bar{\sigma})} + \sum_{r \neq 0} \frac{i \tilde{m}}{r - i \tilde{m}} e^{ir(\sigma+\bar{\sigma})}
\]

with

\[
\sum_{r \in \mathbb{Z}} \frac{i \tilde{m}}{r - i \tilde{m}} e^{ir(\sigma+\bar{\sigma})} = - \int_\mathbb{C} dz \frac{e^{iz(\sigma+\bar{\sigma})} i \tilde{m}}{1 - e^{2\pi iz} z - i \tilde{m}} = \frac{-2 \pi \delta e^{-i \tilde{m}(\sigma+\bar{\sigma})}}{1 - e^{-2\pi \delta}}
\]

and

\[
\sum_{r \in \mathbb{Z}} e^{ir(\sigma-\bar{\sigma})} = 2\pi \delta(\sigma - \bar{\sigma}), \quad 0 < \sigma, \bar{\sigma} < \pi.
\]

By using the zero mode anticommutators (4.67) one obtains from (D.7) the required result (D.4).

For the bosons along a Neumann direction we deduce analogously from (4.78)

\[
[x^I(\tau, \sigma), p^J(\tau, \bar{\sigma})] = 2 \left( - \left[ P^J, Q^I \right] + \left[ Q^J, P^I \right] \right) \sqrt{(\tilde{b}^I)^2 - \tilde{m}^2} e^{\tilde{b}^I(\sigma + \bar{\sigma})} \\
+ 2i \delta \sum_{r \in \mathbb{Z} \setminus \{0\}} e^{ir(\sigma-\bar{\sigma})} + 2i \delta \sum_{r \in \mathbb{Z} \setminus \{0\}} \frac{r - i \tilde{b}^I}{r + i b^I}
\]

with

\[
\sum_{r \in \mathbb{Z}} \frac{r - i b^I}{r + i b^I} = - \int_\mathbb{C} dz \frac{e^{iz(\sigma+\bar{\sigma})} z - i b^I}{1 - e^{2\pi iz} z + i b^I} = 4\pi b^I \frac{e^{b^I(\sigma+\bar{\sigma})}}{1 - e^{2\pi b^I}}.
\]

From (D.1) we derive (4.87).

Finally, for fermionic fields spanning along Neumann directions with \( I, J \in \mathcal{N}_- \) we obtain from (4.82) the equation

\[
\{\psi^I_+, \psi^J_+ (\tau, \sigma)\} = \{\psi^I, \psi^J\} e^{-\tilde{m}(\sigma+\bar{\sigma})} + 2 \left\{ \chi^I, \tilde{\chi}^J \right\} \sqrt{(\tilde{b}^I)^2 - \tilde{m}^2 + \tilde{b}^I} e^{\tilde{b}^I(\sigma + \bar{\sigma})} \\
+ \delta \sum_{r \neq 0} e^{ir(\sigma-\bar{\sigma})} - i \tilde{m} \sum_{r \neq 0} \frac{1}{r - i \tilde{m}} e^{ir(\sigma+\bar{\sigma})}
\]

(D.12)
D.2. OPEN STRINGS WITH BOUNDARY MAGNETIC FIELDS

with in this case

\[-im \sum_{r \in \mathbb{Z}} \frac{1}{r - im r + ib} e^{i r (\sigma + \sigma')} = \frac{2\pi m}{1 - e^{-2\pi m}} \frac{m - b}{m + b} e^{-m(\sigma + \sigma')} + \frac{4\pi m}{1 - e^{2\pi b}} \frac{b}{b + m} e^{b(\sigma + \sigma')},\]

confirming (4.89) and (4.90).

All other relations are either implied by the presented results or established analogously.

D.2 Open strings with boundary magnetic fields

In this section we fill in some details of the canonical quantisation for open strings in the presence of boundary magnetic fields from chapter 3. Using the standard nonzero anticommutators (3.155) as explained in the last section, we obtain with (3.141)

\[
\{S(\tau, \sigma), S(\tau, \sigma')\} = e^{\tilde{\omega} \cos \frac{\theta}{2} (\sigma + \sigma')} \frac{1 + \Pi}{2} \{S_0, S_0\} + e^{-\tilde{\omega} \cos \frac{\theta}{2} (\sigma + \sigma')} \frac{1 - \Pi}{2} \{S_0, S_0\} + \sum_{n \neq 0} \left( e^{i n (\sigma - \sigma')} - \frac{\tilde{m} i}{2\omega_n} \Pi(K_{-n} + K_{-n}) e^{i n (\sigma + \sigma')} \right),
\]

(D.13)

using the matrix \(K_n\) as defined in (3.144). By applying

\[
K_{-n} + K_{-n} = \frac{2\omega_n \cos \frac{\theta}{2}}{n + i\tilde{m} \Pi \cos \frac{\theta}{2}} = 2\omega_n \cos \frac{\theta}{2} n - i\tilde{m} \Pi \cos \frac{\theta}{2} \frac{1}{n^2 + \tilde{m}^2 \cos^2 \frac{\theta}{2}},
\]

(D.14)

the infinite sum in (D.13) becomes

\[
\sum_{n \in \mathbb{Z}} \left( e^{i n (\sigma - \sigma')} - \frac{\tilde{m} i}{2\omega_n} \Pi \frac{n e^{i n (\sigma + \sigma')}}{n^2 + \tilde{m}^2 \cos^2 \frac{\theta}{2}} - \tilde{m}^2 \cos^2 \frac{\theta}{2} \frac{1}{n^2 + \tilde{m}^2 \cos^2 \frac{\theta}{2}} e^{i n (\sigma + \sigma')} \right).
\]

(D.15)

Evaluating (D.15) with contour integrals, we deduce

\[
\sum_{n \in \mathbb{Z}} \frac{n e^{i n (\sigma + \sigma')}}{n^2 + \tilde{m}^2 \cos^2 \frac{\theta}{2}} = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{z}{1 - e^{2\pi i z}} e^{i z (\sigma + \sigma')} dh \quad (D.16)
\]

and therefore

\[
\sum_{n \in \mathbb{Z}} \frac{n e^{i n (\sigma + \sigma')}}{n^2 + \tilde{m}^2 \cos^2 \frac{\theta}{2}} = \pi i \left( e^{-\tilde{m} \cos \frac{\theta}{2} (\sigma + \sigma')} + e^{-\tilde{m} \cos \frac{\theta}{2} (\sigma + \sigma')} \right) \frac{1 - e^{-2\pi i \tilde{m} \cos \frac{\theta}{2}}}{1 - e^{-2\pi i \tilde{m} \cos \frac{\theta}{2}}} e^{i \pi \tilde{m} \cos \frac{\theta}{2}}
\]

\[
= \pi i \frac{\sinh \left[ \tilde{m} \cos \frac{\theta}{2} (\pi - \sigma - \sigma') \right]}{\sinh \left[ \tilde{m} \pi \cos \frac{\theta}{2} \right]}.
\]

(D.17)
For the second sum in (D.15) we have analogously
\[
\sum_{n \in \mathbb{Z}} e^{in(\sigma + \sigma')/2} = \frac{\pi}{\tilde{m} \cos \frac{\theta}{2}} \left( \frac{e^{-\tilde{m} \cos \frac{\theta}{2}(\sigma + \sigma')/2}}{1 - e^{-2\pi \tilde{m} \cos \frac{\theta}{2}}/2} - \frac{e^{\tilde{m} \cos \frac{\theta}{2}(\sigma + \sigma')/2}}{1 - e^{2\pi \tilde{m} \cos \frac{\theta}{2}}/2} \right).
\]
By (D.13), these results determine the zero-mode relations as presented in equation (3.154).

We close this appendix by discussing the bosonic fields \( X^I \) stretching along Neumann directions in the presence of boundary magnetic fields. The canonical conjugated momenta are in this case given by
\[
\pi^I = \frac{1}{2} \left( \partial_\tau X^I + F^{IJ} \partial_{a^J} X^J \right) - i \tilde{m} \frac{\sin \frac{\theta}{2}}{2} \exp\left[i J \tilde{m} \cos \frac{\theta}{2} a^J \right] e^{i m \sin \frac{\theta}{2} a^J} - e^{i m \sin \frac{\theta}{2} a^J} \exp[-i J \tilde{m} \cos \frac{\theta}{2} a^J] \right) + \frac{1}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-i \omega_n \tau} \left( \left[ 1 - \frac{n}{\omega_n} F \right]^{1J} \alpha_n^{1J} e^{i n \sigma} + \left[ 1 + \frac{n}{\omega_n} F \right]^{1J} \alpha_n^{1J} e^{-i n \sigma} \right). \tag{D.20}
\]
Together with (3.151) and the identities (3.136), the nonzero mode contributions to the commutator (3.27) are calculated to
\[
i \sum_{n \in \mathbb{Z}} e^{in(\sigma - \sigma')} - i \sum_{n \in \mathbb{Z}} \frac{\tilde{m}^2 \cos^2 \frac{\theta}{2} - n^2}{n^2 + \tilde{m}^2 \cos^2 \frac{\theta}{2}} e^{in(\sigma + \sigma')}, \tag{D.21}
\]
which can be further evaluated to
\[
2\pi i \delta(\sigma - \sigma') - 2\pi i \tilde{m} \cos \frac{\theta}{2} \left[ \frac{e^{-\tilde{m} \cos \frac{\theta}{2}(\sigma + \sigma')/2}}{1 - e^{-2\pi \tilde{m} \cos \frac{\theta}{2}}/2} - \frac{e^{\tilde{m} \cos \frac{\theta}{2}(\sigma + \sigma')/2}}{1 - e^{2\pi \tilde{m} \cos \frac{\theta}{2}}/2} \right]. \tag{D.22}
\]
Setting \([a^I, a^{J*}] = L^{IJ}\), the zero mode contribution to (D.1) are found to be
\[
\frac{i \tilde{m}}{4 \sin \frac{\theta}{2}} \left( e^{\tilde{m} \cos \frac{\theta}{2}(\sigma + \sigma')} ((1 + iJ)L + (1 - iJ)L^T) + e^{-\tilde{m} \cos \frac{\theta}{2}(\sigma + \sigma')} ((1 - iJ)L + (1 + iJ)L^T) \right). \tag{D.23}
\]
A comparison of (D.23) with (D.22) uniquely determines \( L \) and confirms the relations presented in (3.152).
Appendix E

Integrability in the plane wave background

In this appendix we present some additional information about the integrable structure underlying the plane wave worldsheet theory. The higher spin currents responsible for the integrability of the massive Ising model were written down in [127] and are given by

\begin{align}
T_{n+1}^f &= g_{\tilde{\tau}} \overline{\psi}_i \partial_+ \psi_i^j; & \theta_{n-1}^b &= -g_{\tilde{\tau}} \psi_i^j \partial_+ \overline{\psi}_i^j
\end{align}

We focus on the relevant cases for a theory defined on $S^1 \times \mathbb{R}$. The analogous bosonic currents for the situation described in chapter 4 are found to be

\begin{align}
T_{2n}^b &= g_{\tilde{\tau}} \partial_+ z^i \partial_- z_i^j; & \theta_{2n-2}^b &= -m^2 g_{\tilde{\tau}} \partial_+ z^i \partial_- z_i^j
\end{align}

The integrable currents for the plane wave theory finally follow from these by suitably combining (E.1) and (E.2). Appearing relative prefactors can be determined by requiring the cancellation of separate normal ordering constants of (E.1) and (E.2) in the quantum theory.

In a free theory, there are obviously many different currents like (E.2). They can for instance be obtained by taking the different parts along single real directions in (E.1), (E.2) and recombining them in various ways. This leads to additional conserved higher spin bulk currents, but most choices are incompatible with the complex structure chosen in the Lagrangian (2.11).

Our decision to consider the particular combinations (4.45)-(4.48) is in particular based on the observation that these currents appear as limits of the highly nontrivial higher spin currents of the $\mathcal{N} = 2$ supersymmetric sine-Gordon model. These currents were discussed in detail in the appendix A by using a Landau-Ginzburg formulation with superpotential

\begin{align}
W = -2ig \cos z + \text{const.}
\end{align}
Reintroducing the standard parameter $\omega$ and rescaling the coupling constant to $g \to -\frac{m}{\omega^2}$, a plane wave theory with a superpotential analogous to (4.1) is obtained from (E.3) in the $\omega \to 0$ limit.

For the first higher spin currents (A.27) and (A.28) we obtain furthermore

$$\frac{T_4}{\omega^2} = 2 \left( \partial_+^2 \bar{z} \partial_+^2 z + i \partial_+ \bar{\psi}_+ \partial_+^2 \psi_+ \right) + o(\omega^2) \quad (E.4)$$

$$\frac{\theta_2}{\omega^2} = 2 \left( -m^2 \partial_+ \bar{z} \partial_+ z - im^2 \bar{\psi}_+ \partial_+ \psi_+ \right) + o(\omega^2). \quad (E.5)$$

The formulas presented in (4.45) and (4.46) and correspondingly in (4.47) and (4.48) differ from (E.4), (E.5) only in total derivative terms, included to obtain manifestly real expressions.

In the boundary theory the currents (4.45)-(4.48) give rise to the conserved charge

$$I_3 = \int_0^\pi d\sigma \left( T_4 + \bar{T}_4 - \theta_2 - \bar{\theta}_2 \right) - \Sigma_{0,\sigma}^{(3)}(t) + \Sigma_{0}^{(3)}(t) \quad (E.6)$$

with local boundary fields $\Sigma_{0,\sigma}^{(3)}(t)$. The calculational strategy to determine these fields and the corresponding differential equations for $F, G$ and the boundary potential $B$ is explained in detail in chapter 2. Here we present only the explicit form of the boundary current $\Sigma_{\sigma}(t)$ along the Neumann directions. It is given by

$$\Sigma_{\sigma}^{(3)}(t) = 4m^2 \partial_i \partial_j B z^i \bar{z}^j + 2m^2 \partial_i \partial_j B \bar{z}^i \bar{z}^j + 2m^2 \partial_i \partial_j B z^i \bar{z}^j + 4\partial_i \partial_j \partial_+ \bar{z}^i \partial_+ \bar{z}^j + 4\partial_i \partial_j B \partial_+ \bar{z}^i \partial_+ \bar{z}^j$$

$$+ 8m^2 \partial_+ \bar{\theta}_+ \partial_+ \bar{\theta}_- + 8m^2 \bar{\theta}_- \partial_+ \bar{\theta}_+$$

$$+ 8m^2 \partial_+ \bar{\theta}_+ \partial_+ \bar{\theta}_- + 8m^2 \bar{\theta}_- \partial_+ \bar{\theta}_+$$

$$+ 8m^2 \partial_+ \bar{\theta}_+ \partial_+ \bar{\theta}_- + 8m^2 \bar{\theta}_- \partial_+ \bar{\theta}_+ + 8m^2 \bar{\theta}_- \partial_+ \bar{\theta}_+ + 8m^2 \bar{\theta}_- \partial_+ \bar{\theta}_+$$

$$4m i e^{i\beta} \left( \partial_+ \bar{\theta}_+ - \partial_+ \bar{\theta}_- \right) - 4m i e^{-i\beta} \partial_+ \bar{\theta}_+ \bar{\theta}_-.$$

The conservation of a higher spin current like (E.6) leads to strong evidence for the integrability of the underlying field theory. Nevertheless, it does clearly not constitute a proof. As mentioned in section 4.3.3, one might furthermore test the mode expansions and commutation relations of section 4.4 against the requirements derived in [54] for an integrable boundary theory. These are in particular the boundary Yang-Baxter equation, the unitarity requirement and the crossing symmetry. In particular the last requirement relates the open string mode identifications with the corresponding closed string gluing conditions by a suitable analytic continuation.

We do not spell out the details here, but mention that the modings derived in section 4.4 fulfil all the requirements presented in [54]. One might also compare this with the treatment of the massive Ising model in [54] and [36].
Bibliography


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