Doctoral Thesis

On the asymptotic velocity of diffusions in random environment

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On the asymptotic velocity of diffusions in random environment

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presented by
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0.2 Abstract

Diffusions in random environment can be described by the solutions of a stochastic differential equation whose coefficients, i.e. the diffusion matrix and the drift, are stationary random variables on the space of environments $\Omega$. For a given realisation of the environment, the solution is a continuous Markov process whose law is called “quenched measure”. The law of the process studied in this thesis, called “annealed measure”, is then the average over $\Omega$ of the quenched measures. Our goal is to prove the existence of a limiting velocity and to explore certain conditions which ensure a deterministic non-vanishing velocity or, in other words, ballistic behaviour.

More precisely, we prove in the first part of this work that multi-dimensional diffusions in random environment have a limiting velocity which takes at most two values which are then necessarily opposite each other. In the two-dimensional case, we show that for any direction, the probability to escape to infinity in this direction equals either zero or one. Combined with our results on the limiting velocity, this implies a strong law of large numbers in two dimensions. Our arguments are partly inspired by M. Zerner’s work on the zero-one law and the non-ballistic behaviour of random walks in random environment, and we use the powerful renewal structure introduced by Shen.

In the second part, we continue the investigation of condition (T1), which was introduced by Sznitman and by Schmitz for random walks in random environment and for diffusions in random environment respectively, and which implies ballistic behaviour as well as a central limit theorem. Specifically, we prove that when $d \geq 2$, (T1) is equivalent to an effective condition that can be checked by local inspection of the environment. When $d = 1$, we prove that condition (T1) is merely equivalent to almost sure transience. As an application of the effective criterion, we show that when $d \geq 4$ a certain class of randomly perturbed Brownian motion satisfies condition (T1). This class contains examples whose ballistic nature was unknown before since they violate a drift condition previously established by Schmitz, related to Kalikow’s condition and implying (T1).
0.3 Résumé

Les diffusions en milieu aléatoire peuvent être décrites par les solutions d’une équation stochastique dont les coefficients, c.-à-d. la matrice de diffusion et la dérive, sont des variables aléatoires stationnaires sur un espace $\Omega$. Pour une réalisation $\omega \in \Omega$ donnée du milieu aléatoire, la solution est un processus de Markov continu dont la loi s’appelle mesure “quenched”. La loi du processus étudié dans cette thèse, appelée la mesure “annealed”, est alors la moyenne des mesures “quenched” sur $\Omega$. Notre but est de prouver l’existence d’une vitesse limite et d’étudier certaines conditions qui assurent une vitesse déterministe non nulle, ou en d’autres termes, un comportement balistique.

Plus précisément, nous montrons dans la première partie de ce travail que les diffusions multidimensionnelles en milieu aléatoire ont une vitesse limite qui prend tout au plus deux valeurs qui sont alors nécessairement parallèles et opposées. Dans le cas bidimensionnel, nous prouvons que pour toute direction, la probabilité d’échaper vers l’infini dans cette direction vaut zéro ou un. En liaison avec notre résultat sur la vitesse limite, ceci implique une loi des grands nombres en deux dimensions. Nos arguments s’inspirent en partie des travaux de M. Zerner sur la loi du zéro-un et sur le comportement non-balistique des marches aléatoires en milieu aléatoire, et utilisent la puissante structure de renouvellement établie par Shen.

Dans la deuxième partie, nous faisons progresser l’étude de la condition $(T')$, que Sznitman et Schmitz ont respectivement introduite dans le cas des marches aléatoires en milieu aléatoire et dans le cas des diffusions en milieu aléatoire, et qui implique un comportement balistique aussi bien qu’un théorème de limite centrale. Notamment, nous montrons que si $d \geq 2$, $(T')$ est équivalent à une condition “effective” qui peut être vérifiée par une inspection locale du milieu aléatoire. Quand $d = 1$, nous prouvons que la condition $(T')$ équivaut à la transience presque sûre. Finalement, nous utilisons le critère effectif pour montrer que quand $d \geq 4$ une certaine classe de perturbations du mouvement Brownien satisfait la condition $(T')$. Cette classe contient des exemples dont le caractère balistique était inconnu auparavant parce qu’ils violent une condition établie précédemment par Schmitz qui découle de la condition de Kalikow et qui implique $(T')$. 
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Chapter 1

Introduction

Before we introduce the model studied in this thesis, we attempt to describe its origins from physical sciences.

1.1 A probabilistic approach to diffusion phenomena in disordered media

Diffusions are stochastic processes that describe the erratic motion of a particle constantly exposed to collisions with other smaller particles and evolving in a possibly inhomogeneous medium. In physics, they are used to model on a microscopic level the processes involved in the propagation of heat, the evolution of concentrations and other transport phenomena usually described by parabolic PDEs. Possible fields of application are for instance plasma physics, oceanography and chemistry, see [4], [32], [35]. Roughly speaking, the density of infinitely many particles evolving independently according to some diffusion process corresponds to the concentration or temperature of physical interest. When the medium in which the diffusion takes place is strongly disordered, a possible approach is to describe the medium’s irregular effects on the particle by a spatial stochastic process, thus adding a
second level of randomness to the model. Despite the apparent complete randomness of the particle’s trajectory on small scales, its motion may exhibit a certain regular and deterministic behaviour on larger scales. In particular, its speed may converge to some constant and the distribution of its properly rescaled location may approach a Gaussian. In this thesis, we are mainly concerned with the convergence of the speed, also called law of large numbers.

The most fundamental example of a diffusion is Brownian motion, which describes the random motion of a particle in an homogeneous isotropic medium and is named in honor of the botanist Robert Brown who in 1827 was allegedly the first to examine the trepidation of particles like pollen suspended in water. But it was in Louis Bachelier’s thesis [2] published in 1900 where the first mathematical model for such a chaotic motion was developed, with the purpose to model stock prices. In 1905, Einstein, see [15], found the correct physical explanation for the phenomenon observed by Brown and described the essential properties of a mathematical model independently of Bachelier.

To allow for inhomogeneous materials and external forces, one introduces a diffusion matrix \( \sigma(x) \), which is a symmetric \( d \times d \) matrix, and a drift vector \( b(x) \in \mathbb{R}^d \). The diffusion \( X \) with coefficients \( \sigma \) and \( b \) capturing the inhomogeneity is then the solution of the stochastic differential equation (SDE):

\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt,
\]

where \( B_t \) is a \( d \)-dimensional Brownian motion.

1.2 Diffusions in random environment

In the case of a disordered medium where only certain statistical properties of the diffusion matrix and drift vector are known, it is natural to replace the coefficients by random variables \( \sigma(x, \omega), b(x, \omega) \), \( x \in \mathbb{R}^d \), where \( \omega \) is an element of a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) describing all possible environments. Often, the physical properties of a material are roughly the same throughout space. Therefore we assume that the distributions of \( \sigma(x, \cdot) \) and \( b(x, \cdot) \) are
invariant under translations, i.e. do not depend on $x$. Moreover the exact configurations of the medium in two remote regions should not influence each other. We therefore assume finite range dependence, that is if $\mathcal{H}_A, A \subset \mathbb{R}^d$ denotes the sigma field $\sigma(\{\sigma(x, \cdot), b(x, \cdot), x \in A\})$, we assume the existence of a constant $R$ such that $\mathcal{H}_A$ and $\mathcal{H}_B$ are independent whenever $\text{dist}(A, B) \geq R$. For technical reasons, we also make a few regularity assumptions on $\sigma$ and $b$, see (2.1.1)-(2.1.7) for a precise definition of the model under consideration. Then the law of the diffusion in a particular environment $\omega \in \Omega$ with starting point $x \in \mathbb{R}^d$ is determined by the SDE

$$dX_t = \sigma(X_t, \omega) dB_t + b(X_t, \omega) dt, \quad X_0 = x,$$

and is called quenched law $P_{x, \omega}$. Since a particular realization $\omega$ is not representative and can lead to atypical behaviour, and in order to restore some stationarity, we consider the annealed law $P_x$ which is the average of $P_{x, \omega}$ over $\omega$ with respect to the law of the environment $\mathbb{P}$. One of the mathematical difficulties of this model stems from the fact that, in general, the diffusion is no longer a Markov process under $P_x$.

To help the intuition, we briefly present the discrete analogue of our model, which is called random walk in random environment on $\mathbb{Z}^d$. Again there are two levels of randomness involved. First, one chooses a vector of transition probabilities $\omega(x, e), |e| = 1$ with $\sum_{|e|=1} \omega(x, e) = 1$, for every vertex $x \in \mathbb{Z}^d$ independently of each other according to a probability measure $\mathbb{P}$. These transition probabilities then define a Markov chain with “quenched” measure $P_{x, \omega}$, according to

$$P_{x, \omega}[X_{n+1} = y + e | X_n = y] = \omega(y, e), \text{ if } |e| = 1, \text{ and } P_{x, \omega}[X_0 = x] = 1.$$

The “annealed” measure is then again obtained by averaging over all $\omega$ under $\mathbb{P}$. In the sequel, we shall refer to this model as the “discrete setting”.

### 1.3 Short overview of past results

In the past, the method of the environment viewed from the particle was successfully used to study a certain class of examples, see for instance [27], [37], [40]. This method relies on the process with state space $\Omega$ obtained by
translating the environment in such a way that the moving particle always sits at the origin. This process is Markovian also under the annealed measure and becomes useful if it has an invariant ergodic measure $Q$ that is absolutely continuous with respect to $P$. Using ergodicity, it is then easy to deduce an explicit expression for the limiting velocity:

$$\lim_{t \to \infty} \frac{X_t}{t} = \int_{\Omega} b(0, \omega) dQ(\omega), \ P_0\text{-a.s.}$$

This approach, if successful, reveals that a naive guess for the velocity as for instance $\int_{\Omega} b(0, \omega) dP(\omega)$ is wrong in general. Typically, the particle is slowed down by random traps in the environment and in higher dimensions, its limiting velocity can even point in a direction opposite to the expected drift, see [8], Section 5 therein.

Unfortunately, the existence of an invariant measure that is absolutely continuous with respect to $P$ is only established in a few specific cases, for instance in the absence of a drift, see [40] or when $\sigma = 1$ and $b$ is divergence free, see [33] or the gradient of a stationary potential, see [37]. The environment viewed from the particle can also be used in the one-dimensional discrete setting to infer a part of Solomon’s famous result, see [50]. Indeed, this method leads to a condition for ballistic behaviour, i.e. the existence of a deterministic non-zero limiting velocity, and allows to find an explicit formula for this velocity.

In contrast to the one-dimensional setting, where a rather detailed analysis has been achieved, see for instance [7],[9],[10],[13],[20],[25],[26],[21], the general multi-dimensional setting is not well understood, since many basic questions as for instance the validity of a law of large numbers and a central limit theorem, transience-recurrence criteria, zero-one laws for directional transience remain partially unanswered despite the more recent advances especially in the discrete setting by Comets, Rassoul-Agha, Sznitman, Varadhan, Zeitouni, Zerner and others. First steps to build up on their methods and make progress also in the continuous framework were taken by Shen, see [48],[49] and Schmitz, [46],[47]. In our thesis, we carry on this endeavour.

A cornerstone of these new developments is a renewal structure based on so called regeneration times $\tau_k, \ k = 1, 2, \ldots$ introduced by Sznitman and
Zerner, [57] in the discrete setting. For a trajectory tending to infinity in direction \( \ell \), \( \tau_1 \) is defined as the first time when the projection of the walk on \( \ell \) exceeds a previous maximum and always stays above it from then on, see Figure 1. The subsequent times \( \tau_k, k \geq 2 \) are defined iteratively. Due to the i.i.d. structure of the environment, the pieces of trajectory between \( \tau_k \) and \( \tau_{k+1}, k \geq 1 \) become i.i.d. random variables under \( P_0 \).

In the continuous setting however, finite range dependence prevents two parts of a trajectory evolving on different sides of a hyper-plane to become independent. Therefore the implementation of the regeneration times for diffusions, carried out by Shen [48], is quite involved and uses a coupling construction of a similar flavour as in [11]. We present the details in Sections 2.3.1 and 2.3.2. Let us write \( \hat{P}_0 \) for the annealed “coupled” measure and \( D = \left\lfloor \inf\{t > 0 : (X_t - X_0) \cdot \ell \leq -R\} \right\rfloor \) for the rounded up value of the first time when the projection on \( \ell \) of the diffusion drops a distance \( R \) below its starting point. Then Shen proves the following renewal structure in Theorem 2.5 in [48].

**Theorem I.** Assume that \( \hat{P}_0 \)-a.s., \( \tau_1 < \infty \). Then under the measure \( \hat{P}_0 \), the random variables \( Z_k = (X_{(t_k+1)\wedge(t_{k+1}+1)} - X_{t_k}; X_{t_{k+1}} - X_{t_k}; t_{k+1} - t_k), k \geq 0 \), are independent. Furthermore, \( Z_k, k \geq 1 \), under \( \hat{P}_0 \), have the distribution of \( Z_0 = (X_{\wedge(\tau_1+1)} - X_0; X_{\tau_1} - X_0; \tau_1) \) under \( \hat{P}_0[\cdot | D = \infty] \).

Moreover he shows that \( \tau_1 \) is \( \hat{P}_0 \)-a.s. finite if and only if \( P_0 \)-a.s. the trajectories tend to infinity in direction \( \ell \), see Proposition 2.7 in [48]. In fact,
we will see that the events \( \{ \tau_1 < \infty \} \) and \( \{ X_t \cdot \ell \to \infty \} \equiv A_\ell \) are \( \hat{P}_0 \)-a.s. equal, cf. Theorem 2.3.5. With the help of the above theorem, Shen finds conditions under which a diffusion obeys a law of large numbers with non-vanishing velocity, i.e. exhibits a ballistic behaviour, and verifies a central limit theorem or invariance principle: (see Theorems 3.2 and 3.3 in [48])

**Theorem II.** Assume that \( P_0 \)-a.s., \( \lim_{t \to \infty} X_t \cdot \ell = \infty \).

1. If \( \hat{E}_0[\tau_1|D = \infty] < \infty \), then
   \[
   P_0\text{-a.s. } \lim_{t \to \infty} \frac{X_t}{t} = v = \frac{\hat{E}_0[X_{\tau_1}|D = \infty]}{\hat{E}_0[\tau_1|D = \infty]}, \quad \text{and } v \cdot \ell > 0.
   \]

2. If \( \hat{E}_0[\tau_1^2|D = \infty] < \infty \), then under \( P_0 \), the continuous processes
   \[
   \frac{X_s \cdot sv}{\sqrt{s}}, s > 0
   \]
   converge weakly as \( s \to \infty \) to \( d \)-dimensional Brownian motion with non-degenerate covariance matrix
   \[
   \frac{\hat{E}_0[(X_{\tau_1} - v\tau_1)(X_{\tau_1} - v\tau_1)^t|D = \infty]}{\hat{E}_0[\tau_1|D = \infty]}.
   \]

1.4 Results

In the first part of this thesis, we aim at proving a law of large numbers without imposing ballistic behaviour through the assumptions \( X_t \to \infty \), \( P_0 \)-a.s. and \( \hat{E}_0[\tau_1|D = \infty] < \infty \) as in Theorem II(i). When the space dimension is strictly larger than two, we reach this goal only in part, since we obtain a random velocity taking at most two values which are then necessarily opposite each other. More precisely, we find (see Theorem 2.3.8):

**Theorem III.** There exist a deterministic direction \( \ell_\ast \in S^{d-1} \) and two numbers \( v_+, v_- \geq 0 \), such that

\[
(1) \quad P_0\text{-a.s., } \lim_{t \to \infty} \frac{X_t}{t} = (v_+1_{A_{\ell_\ast}} - v_-1_{A_{-\ell_\ast}})\ell_\ast,
\]

and \( P(A_{\ell_\ast} \cup A_{-\ell_\ast}) \in \{0, 1\} \). (If this last quantity is 0, the velocity is 0 and thus the values of \( v_+, v_- \) are immaterial.)
1. Introduction

Let us mention that a similar result was obtained previously in the case of random walks in i.i.d. random environment, see [65], and mixing environment, see [42]. On the other hand, when $d = 2$, we achieve our goal and prove a law of large numbers:

**Theorem IV.** When $d = 2$, there exists a deterministic $v \in \mathbb{R}^2$ such that

$$P_0\text{-a.s., } \lim_{t \to \infty} \frac{X_t}{t} = v.$$ 

The latter result is an immediate consequence of Theorem III and the following zero-one law, see Theorem 2.4.2 and also [64] in the discrete setting, which enables us to eliminate one of the directions $\ell_*, -\ell_*$.

**Theorem V.** When $d = 2$, for any $\ell \in \mathbb{R}^2$, $P_0(A_\ell) \in \{0, 1\}$.

Note that for general dimensions, we are only able to obtain a weak zero-one law (Theorem 2.3.6), saying that $P_0(A_\ell \cup A_{-\ell}) \in \{0, 1\}$, while the strong version remains a intensively studied open question. In the discrete setting, Berger [5] recently made some progress towards a law of large numbers with deterministic speed when $d \geq 5$ by proving that in the notation of Theorem III, one of $v_+$ and $v_-$ has to vanish.

Let us now make a few comments on the proof of Theorem III. It relies in part on the renewal techniques presented above. If $P_0(A_\ell)$ is merely positive (but not necessarily 1), then a modification of the arguments leading to Theorem I shows that conditionally on $A_\ell$, the increments $Z_k$ are still i.i.d.. Under the only assumption that $P_0(A_\ell) > 0$, Theorem 2.3.7 provides the finiteness of the first moment of $X_{\tau_\ell} \cdot \ell$, which is necessary for deducing the existence of a limiting velocity from the usual law of large numbers. As a consequence, we find that on $A_\ell$, $X_t \cdot \ell/t$ converges $P_0$-a.s. to a deterministic limit (and similarly on $A_{-\ell}$, if $P_0(A_{-\ell}) > 0$). Of course, if the first moment of $\tau_\ell$ is infinite, than this limit vanishes. Due to the above mentioned weak zero-one law, we are left with the case where $P_0(A_\ell \cup A_{-\ell}) = 0$. Under this assumption, we prove, see Corollary 2.2.6, that the speed in direction $\ell$ vanishes by using a strategy similar to [65]. Assembling the results from the two cases where $P_0(A_\ell \cup A_{-\ell})$ equals 1 respectively 0, and applying them to an orthonormal basis $\ell_1, \ldots, \ell_d$, we deduce Theorem III after excluding the possibility of observing several non-parallel velocities.
In the second part of this thesis, we carry on the investigation, initiated by Sznitman [54], [55], [56] and Schmitz [46], [47] respectively in the discrete and continuous setting, of conditions that ensure the moment assumptions of Theorem II. At the center of these articles stands the so-called conditions \((T)\) and \((T')\) which seem to be among the most general assumptions that guarantee ballistic behaviour and an invariance principle as in Theorem II when \(d \geq 2\). We provide an equivalent formulation of condition \((T')\) that can be checked by examination of the law of the environment in a finite box and study its relationship with another condition resulting from what was called Kalikow’s condition in [46], Definition 5.5.

Before we explain our contribution to this topic in more details, let us define conditions \((T)\) and \((T')\) as stated in [46] and explore some of their implications. These conditions are expressed in terms of another condition \((T)_\gamma\) defined as follows. For a unit vector \(\ell\) of \(\mathbb{R}^d, d \geq 1\) and any \(u \in \mathbb{R}\), consider the stopping times

\[
T_u^\ell = \inf\{t \geq 0; X_t \cdot \ell \geq u\}, \quad \tilde{T}_u^\ell = \inf\{t \geq 0; X_t \cdot \ell \leq u\}.
\]

For \(\gamma \in (0, 1]\), we say that condition \((T)_\gamma\) holds relative to \(\ell\), in shorthand notation \((T)_\gamma | \ell\) if for all unit vectors \(\ell'\) in some neighbourhood of \(\ell\) and for

![Figure 2. A realisation of the event appearing in (3).](image-url)
all $b > 0$,

$$\limsup_{L \to \infty} L^{-\gamma} \log P_0 \left[ \tilde{T}_{-bL}^\ell < T_L^\ell \right] < 0.$$  

Condition $(T')$ relative to $\ell$ is then the requirement that

$$\text{(3) holds for all } \gamma \in (0,1),$$

and condition $(T)$ relative to $\ell$ refers to the case where

$$\text{(3) holds for } \gamma = 1.$$  

It is clear that $(T)$ implies $(T')$ and we shall see in Theorem 3.2.6 that $(T')$ is equivalent to $(T)_\gamma$ when $\gamma \in \left(\frac{1}{2},1\right)$. Moreover, it is conjectured that the conditions $(T)_\gamma$, $\gamma \in (0,1]$ are all equivalent.

This rather geometrical definition has proven useful for the construction of certain examples, see for instance [46] Proposition 5.1. However, it is important to relate these conditions to the crucial renewal structure in view of an application of Theorem II. Following [55], Schmitz showed that $(T)_\gamma | \ell$ is equivalent to

$$\begin{cases} P_0\text{-a.s., } \lim_{t \to \infty} X_t \cdot \ell = \infty, \\ \text{and for some } \mu > 0, \quad \hat{E}_0 \left[ \exp \left\{ \mu \sup_{0 \leq t \leq \tau_1} |X_t|^\gamma \right\} \right] < \infty. \end{cases}$$

With the help of a renormalisation procedure, he then obtained the following bound on the tail of $\tau_1$ under the assumption of $(T')|\ell$ when $d \geq 2$:

$$\limsup_{u \to \infty} (\log u)^{-\alpha} \log \hat{P}_0[\tau_1 > u] < 0, \quad \text{for } \alpha < 1 + \frac{d-1}{d+1}.$$  

Since $\alpha$ can be chosen larger than 1, the assumptions of Theorem II are obviously satisfied, and hence condition $(T')$ implies ballistic behaviour.

As announced, we derive in Chapter 3 another equivalent formulation of condition $(T')$ called effective criterion in the spirit of Sznitman [55]. In contrast to the geometrical definition, this formulation has the merit not to rely on an asymptotic condition on exit probabilities out of growing infinite
slabs. Instead, it can be checked by inspection of the environment in some finite box.

We now briefly sketch the effective criterion. For a detailed description, we refer the reader to the introduction of Chapter 3, see (3.1.14) and to Theorem 3.2.6. If $B$ is a box almost centered around 0 with two faces perpendicular to a direction $\ell \in \mathbb{R}$, we write $\rho_B(\omega)$ for the inverse of the ratio of the probability under $P_{0,\omega}$ of exiting $B$ through the face perpendicular to $\ell$ on the “positive side of $\ell$” divided by the probability of exiting through any other face, see also (3.1.15). Then we prove that $(T')|\ell$ is in essence equivalent to the existence of a box $B$ of this type such that some moment of $\rho_B$ is smaller than a certain deterministic polynomial in the size of the box $B$.

When $d = 1$, the box is replace by an interval $[-L, L]$ and $\rho_B$ becomes the quotient of the probability under $P_{0,\omega}$ of exiting the interval through $-L$ divided by the probability of exiting through $L$. The moment condition on $\rho$ then reduces to the assumption that for some $L > 0$, $E[\log \rho_B] < 0$. The latter expression reminds us very much of the condition in Solomon’s famous theorem from 1974, see [50], which characterizes recurrence and transience in the one-dimensional discrete setting. In fact, we shall prove, see Section 3.2.1, that in contrast to the multi-dimensional case, condition $(T')$ is equivalent to transience and does not necessarily imply ballistic behaviour when $d = 1$, since one can construct examples of one-dimensional diffusions tending to infinity with vanishing velocity.

Let us come back to the multi-dimensional setting. Another merit of the effective criterion is that it sheds some light on the relationship between the conditions $(T)_\gamma$ for different $\gamma$. As noted before, the strength of $(T)_\gamma$ obviously increases with $\gamma \in (0, 1]$. With the help of the effective criterion, we can prove that these conditions are equivalent for all $\gamma \in (\frac{1}{2}, 1)$, and it is conjectured that this equivalence extends to $\gamma \in (0, 1)$. Moreover an intermediate result, see Remark 3.2.1, strengthens the belief that $(T')$ and $(T)$ are equivalent too.

Despite the different equivalent formulations, condition $(T')$ remains difficult to check, since it involves an exit problem under a non Markovian measure, respectively the intricate regeneration time $\tau_1$. For the construction of examples, one is therefore interested in other (stronger) conditions, implying
1. Introduction

(T'), that can be verified more easily. In this sense, Schmitz [46] introduces the continuous analogue of what was called Kalikow’s condition in Szmitman-Zerner [57] and whose origin goes back to [23]. It involves the construction of an auxiliary Markov process. Indeed, one can show, see Proposition 3 in [47], that for every bounded domain $U$ containing 0, there exists a diffusion with coefficients independent of the environment, that has the same exit distribution from $U$ as the annealed diffusion in random environment. Then Kalikow’s condition is the requirement that for some $\ell \in \mathbb{R}^d$, $\inf \hat{b}_U(x) \cdot \ell > \epsilon$, where $\hat{b}_U$ is the drift of the auxiliary diffusion in $U$ and where the infimum runs over all bounded domains $U \ni 0$ and all $x \neq 0$ in $U$ not too close to $\partial U$. In addition, Schmitz provides a convenient criterion to check Kalikow’s condition, cf. Theorem 1 in [47]:

**Theorem VI.** There exists a constant $c_\epsilon > 1$, such that for $\ell \in \mathbb{R}^d$, the inequality

$$(7) \quad \mathbb{E}[\langle b(0, \omega) \cdot \ell \rangle_+] > c_\epsilon \mathbb{E}[\langle b(0, \omega) \cdot \ell \rangle_-]$$

implies Kalikow’s condition and hence (T)$|\ell$.

One should however not expect that (7) characterizes condition (T') or ballistic behaviour, as we shall see in the last part of this thesis. Using the effective criterion presented above, we are able to show that when $d \geq 4$ certain perturbations of Brownian motion violating (7) satisfy (T'), see Theorem 3.3.1 and Remark 3.3.1. We hence provide examples of diffusions whose ballistic nature had not been established before. In the discrete setting, Szmitman ([56], Theorem 5.1) proves a stronger version of this result. He obtains examples for condition (T') that fail to satisfy even Kalikow’s condition. We believe, that with an improved effective criterion, one should be able to produce such examples also in the continuous framework.

**Organization of the thesis** The second chapter corresponds to our article [18], that appeared in the Annals of Applied Probability. We establish the existence of a limiting velocity, see Theorems III, IV, and prove the strong zero-one law for directional transience when the dimension is two, see Theorem V. The third chapter consists of our article [19] accepted for publication in the Annals of Probability. We prove an effective criterion for condition (T') and use it to construct new examples of ballistic diffusions which fail to satisfy (7).
Chapter 2

Limit velocity and zero-one laws for diffusions in random environment

Abstract: We prove that multi-dimensional diffusions in random environment have a limiting velocity which takes at most two different values. Further, in the two-dimensional case we show that for any direction, the probability to escape to infinity in this direction equals either zero or one. Combined with our results on the limiting velocity, this implies a strong law of large numbers in two dimensions.

2.1 Introduction

Over the last twenty five years, diffusions in a random medium have been the object of many studies. They came as a natural way to generalize homogenization in a periodic medium and model disorder at a microscopic scale,
2. Limit velocity and zero-one laws

cf. [4], [35]. In spite of a large literature, see for instance [27], [29], [32], [33], [36], [37], [39], [40], [46], [48], [58], [61] only partial results are known on such basic questions as zero-one laws, recurrence-transience, the law of large numbers and central limit theorems.

The method of the environment viewed from the particle has been a powerful tool in the study of diffusions in a random medium, but many examples fall outside its scope. Recently in the discrete setting, other methods, for instance exploring renewal-type arguments, have contributed to a revival of the subject, cf. [11], [12], [41], [57], [53], [54], [55], [60], [62], [64], [65]. It is natural, but not straightforward to try to build up on these ideas and make progress in the continuous framework. This approach has proved successful notably in the ballistic case, i.e. when the diffusion has a non-vanishing limiting velocity, cf. for instance [31], [46], [48]. The present article follows a similar endeavour. We prove in the general framework of diffusions in a random environment, see below, the existence of a limiting velocity as well as certain zero-one laws. Corresponding results are known in the discrete framework, cf. [42], [60], [64], [65]. Our work is closer in spirit to the last two references. It also draws on the renewal structure constructed by Shen [48] which is more intricate than its discrete counterpart in [57].

Before we discuss our results any further, we first describe the model. The random environment is specified by a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) on which acts a jointly measurable group \(\{t_x; x \in \mathbb{R}^d\}\) of \(\mathbb{P}\)-preserving transformations, with \(d \geq 1\). The diffusion matrix and the drift of the diffusion in random environment are stationary functions \(a(x, \omega), b(x, \omega), x \in \mathbb{R}^d, \omega \in \Omega\), with respective values in the space of non-negative \(d \times d\) matrices and in \(\mathbb{R}^d\), i.e.,

\[
\begin{align*}
(2.1.1) \quad a(x+y, \omega) &= a(x, t_y \omega), \quad b(x+y, \omega) = b(x, t_y \omega), \quad \text{for } x, y \in \mathbb{R}^d, \omega \in \Omega.
\end{align*}
\]

We assume that these functions are bounded and uniformly Lipschitz i.e. there is a \(\bar{K} > 1\), such that for \(x, y \in \mathbb{R}^d, \omega \in \Omega\),

\[
\begin{align*}
(2.1.2) \quad |b(x, \omega)| + |a(x, \omega)| &\leq \bar{K}, \quad |b(x, \omega) - b(y, \omega)| + |a(x, \omega) - a(y, \omega)| \leq \bar{K}|x-y|,
\end{align*}
\]

where \(|\cdot|\) denotes the Euclidean norm for vectors and matrices. Further we assume that the diffusion matrix is uniformly elliptic, i.e. there is a \(\nu > 1\) such that for all \(x, y \in \mathbb{R}^d, \omega \in \Omega\):

\[
\begin{align*}
(2.1.3) \quad \frac{1}{\nu} |y|^2 &\leq y \cdot a(x, \omega)y \leq \nu |y|^2.
\end{align*}
\]
The coefficients $a, b$ satisfy a condition of finite range dependence: for $A \subset \mathbb{R}^d$, we define

\begin{equation}
\mathcal{H}_A = \sigma(a(x, \cdot), b(x, \cdot); x \in A),
\end{equation}

and assume that for some $R > 0$,

\begin{equation}
\mathcal{H}_A \text{ and } \mathcal{H}_B \text{ are independent under } \mathbb{P} \text{ whenever } d(A, B) \geq R,
\end{equation}

where $d(A, B)$ is the mutual Euclidean distance between $A$ and $B$. With the above regularity assumptions on $a$ and $b$, for any $\omega \in \Omega, x \in \mathbb{R}^d$, the martingale problem attached to $x$ and the operator

\begin{equation}
\mathcal{L}_\omega = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\cdot, \omega) \partial_{ij}^2 + \sum_{i=1}^{d} b_i(\cdot, \omega) \partial_i
\end{equation}

is well posed, see [52] or [3], page 130. The corresponding law $P_{x,\omega}$ on $C(\mathbb{R}_+, \mathbb{R}^d)$, unique solution of the above martingale problem, describes the diffusion in the environment $\omega$ and starting from $x$. We write $E_{x,\omega}$ for the expectation under $P_{x,\omega}$ and we denote the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$ with $(X_t)_{t \geq 0}$. Observe that $P_{x,\omega}$ is the law of the solution of the stochastic differential equation

\begin{equation}
\begin{cases}
\frac{dX_t}{dt} = \sigma(X_t, \omega)d\beta_t + b(X_t, \omega)dt, \\
X_0 = x, \quad P_{x,\omega}-\text{a.s.},
\end{cases}
\end{equation}

where for instance $\sigma(\cdot, \omega)$ is the square root of $a(\cdot, \omega)$ and $\beta$ is some $d$-dimensional Brownian motion under $P_{x,\omega}$. The laws $P_{x,\omega}$ are usually called “quenched laws” of the diffusion in random environment. To restore translation invariance, we consider the so-called “annealed laws” $P_x, x \in \mathbb{R}^d$, which are defined as semi-direct products:

\begin{equation}
P_x \overset{\text{def}}{=} \mathbb{P} \times P_{x,\omega}.
\end{equation}

Of course the Markov property is typically lost under the annealed laws.

The goal of this article is to show the existence of a limiting velocity as well as certain zero-one laws for this process. For any unit vector $l \in \mathbb{R}^d$, denote with

\begin{equation}
A_l = \left\{ \lim_{t \to \infty} l \cdot X_t = +\infty \right\},
\end{equation}

2.1. Introduction
the event that the diffusion escapes to infinity in direction $l$. We prove a weak zero-one law saying that for any direction $l$, $P_0(A_l \cup A_{-l})$ equals either zero or one, see Proposition 2.3.6. Then our main result for general dimension $d \geq 1$ (cf. Theorem 2.3.8) shows the existence of a deterministic unit vector $l_*$ and two deterministic numbers $v_+, v_- \geq 0$, such that

\[
\lim_{t \to \infty} \frac{X_t}{t} = (v_1^{1_{A_{l_*}}} - v_-^{1_{A_{-l_*}}})l_*, \quad P_0\text{-almost surely.}
\]

When $d = 2$, we also prove the following stronger zero-one law, cf. Theorem 2.4.2:

\[
\text{for any } l \in S^1, \quad P_0(A_l) \in \{0, 1\},
\]

which together with (2.1.10) implies the following strong law of large numbers:

\[
\text{When } d = 2, \text{ there is a } v \in \mathbb{R}^2 \text{ such that } P_0\text{-a.s., } \lim_{t \to \infty} \frac{X_t}{t} = v.
\]

In the context of random walks in ergodic environments, Zerner and Merkl give in [64] an example, where in the statement corresponding to (2.1.10) two opposite velocities occur with probability $\frac{1}{2}$ each. This signals that an independence assumption on the environment is of importance for the validity of the zero-one law (2.1.11) or the law of large numbers (2.1.12). These questions remain open problems when $d \geq 3$.

To prove (2.1.10), we consider an arbitrary direction $l$ and proceed differently depending on the value of $P_0(A_l \cup A_{-l})$. In oscillating case where $P_0(A_l \cup A_{-l}) = 0$, we show in Section 2.2, that $\lim_{t \to \infty} \frac{t}{t} X_t = 0$, $P_0$-a.s., cf. Corollary 2.2.6. The argument relies on the fact that for any direction $l \in S^{d-1}$,

\[
P_0\left(\limsup_{t \to \infty} \frac{t}{t} X_t > 0\right) > 0 \quad \text{implies} \quad P_0(A_l) > 0,
\]

see Theorem 2.2.4. The strategy used to derive (2.1.13) is similar to the article [65] by Zerner. However, because of finite range dependence and space-time continuity, the arguments are more involved. Nevertheless, we believe that we achieved some simplifications, as our proof avoids infinite products of independent processes (cf. [65] and equation (13) therein). In
the context of random walks in a discrete mixing environment, an alternative way to handle the oscillating case can be found in [42].

In order to analyze the case $P_0(A_t \cup A_{-t}) = 1$, we use a renewal structure in the spirit of Shen [48], see Section 2.3, and prove that $P_0(A_t) > 0$ implies that on $A_t$, $P_0$-a.s., $\lim_{t \to \infty} \frac{t\cdot X_t}{t} = v_t$. The number $v_t$ is either 0 or expressed in terms of a certain regeneration time $\tau_1$, cf. (2.3.41). As in [48], we construct the successive regeneration times $\tau_k, k \geq 1$, on an enlarged probability space which is obtained by coupling the diffusion with a suitable sequence of auxiliary i.i.d. Bernoulli variables, cf. Subsection 2.3.1. The quenched measure on the enlarged space, which couples the diffusion to the Bernoulli variables, is denoted with $\hat{P}_{x,\omega}$. In essence, $\tau_1$ is the first time when the trajectory reaches a local maximum in direction $l$, some auxiliary Bernoulli variable takes value one and from then on the diffusion never backtracks, cf. Subsection 2.3.2. We generalize the results of Shen to the case where $0 < P_0(A_t) \leq 1$, (instead of assuming $P_0(A_t) = 1$), cf. Proposition 2.3.4 and Theorem 2.3.5. In the discrete setting, couplings were first used by Zeitouni, cf. for instance [62], Section 3, with the purpose to overcome the dependence structure of a mixing environment. Another important ingredient for an effective application of the renewal structure is a control on the first moment of $l \cdot X_{\tau_1}$:

\begin{equation}
(2.1.14) \quad \text{If } P_0(A_t) > 0, \text{ then } \hat{E}_0[l \cdot X_{\tau_1} | D = \infty] < \infty,
\end{equation}

where $\hat{E}_0$ is the expectation under $\mathbb{P} \times \hat{P}_{0,\omega}$ and $\{D = \infty\}$ is the event that the diffusion never backtracks a distance $R$ below its starting-point. In the discrete setting, a related result due to Zerner can be found in [62], Lemma 3.2.5. The argument we provide here however does not require Blackwell’s renewal theorem, see also the comments preceding Proposition 2.3.7.

In the last section, we prove the zero-one law (2.1.11) in two dimensions. Our strategy is similar to [64] in the discrete case. We consider two diffusion processes under the law $\mathbb{E}(P_{0,\omega} \times P_{y_L,\omega})$, where $l \cdot y_L \geq 3L$ and $L$ is large. We assume that $P_0(|l \cdot X_t| \to \infty) = 1$ and deduce that the probability of a close encounter of the two diffusions between 0 and $y_L$ vanishes as $L \to \infty$, see Lemma 2.4.1. This result holds in all dimensions. On the other hand, when $d = 2$, if we assume by contradiction that $P_0(A_t)P_0(A_{-t}) > 0$, we can choose $y_L$ such that for large $L$, the two diffusions intersect “between 0 and $y_L$” with non vanishing probability, see Theorem 2.4.2. Then the zero-one
The article is organised as follows:
In Section 2, we prove (2.1.13), cf. Theorem 2.2.4. This yields with Corollary 2.2.6 the main ingredient to prove (2.1.10) when $P_0(A_t \cup A_{-t}) = 0$.
In Section 3, we recall the coupling construction leading to the measures $\bar{P}_x,\omega$, define the regeneration times $\tau_k, k \geq 1$, cf. Subsection 2.3.2 and develop the theorems describing the renewal structure, cf. Subsection 2.3.3. We also prove a weak zero-one law, cf. Proposition 2.3.6, as well as (2.1.14), cf. Proposition 2.3.7. Our main result shows for all $d \geq 1$ the existence of a limiting velocity, cf. (2.1.10) or Theorem 2.3.8.
In Section 4, we prove the two-dimensional zero-one law (2.1.11), cf. Theorem 2.4.2.
In the Appendix, we provide for the reader’s convenience the proof of a variation of Theorem 2.7 of [48] stated in Lemma 2.3.3.

Convention on constants Unless otherwise stated, constants only depend on the quantities $d, K, \nu, R$. We denote with $c$ positive constants with values changing from place to place and with $c_0, c_1, \ldots$ positive constants with values fixed at their first appearance. Dependence on additional parameters appears in the notation.

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2.2 Oscillations and null directional speed

In this section we first introduce some additional notations and then we start with the study of the case, where the trajectory oscillates in some direction $l \in S^{d-1}$. This case corresponds to $P_0[A_t \cup A_{-t}] = 0$ and we will see later, that $P_0[A_t \cup A_{-t}]$ equals either zero or one, cf. Proposition 2.3.6. The main result is Theorem 2.2.4: Under the assumption $P_0\left[ \limsup_{t \to \infty} \frac{L_{X_t}}{t} > 0 \right] > 0$, the trajectories will not backtrack below a certain level with positive probability and with Lemma 2.2.5, we deduce that $P[A_t] > 0$. It follows then easily that $P_0[A_t \cup A_{-t}] = 0$ implies zero asymptotic speed in the direction $l$ (see
Corollary 2.2.6).

We now introduce some notations used throughout the article. We denote with \( \mathbb{N} \) the set of non-negative integers. The integer part of a real \( t \geq 0 \) and the smallest integer larger than \( t \) are respectively denoted with \([t]\) and \([t]\).

Let \( S^{d-1} \) stand for the Euclidean unit sphere of \( \mathbb{R}^d \) and \( B(x, r) \) for the open Euclidean ball with radius \( r \) centered at \( x \). For \( a < b \) two reals and \( l \in S^{d-1} \), we define

\[
S(a, b) = \{ x \in \mathbb{R}^d ; a < x \cdot l < b \}, \quad \bar{S}(a, b) = \{ x \in \mathbb{R}^d ; a \leq x \cdot l \leq b \},
\]

the open and closed slabs between \( a \) and \( b \) in the direction \( l \). If \( A \) is a Borel set of \( \mathbb{R}^d \), \(|A|\) stands for its Lebesgue-measure.

For an open or closed set \( A \subset \mathbb{R}^d \), we denote with \( H_A = \inf \{ t \geq 0 ; X_t \in A \} \) the entrance time into \( A \) and with \( T_A = \inf \{ t \geq 0 ; X_t \notin A \} \) the exit time from \( A \). We will also use the following stopping times measuring absolute and relative displacements of the trajectory. For \( u \in \mathbb{R} \),

\[
T_u = H_{\{ z \in \mathbb{R}^d ; z \cdot l < u \}}, \quad \tilde{T}_u = H_{\{ z \in \mathbb{R}^d ; z \cdot l \leq u \}},
\]

\[
T_u^{rel} = \inf \{ t \geq 0 ; l \cdot (X_t - X_0) \geq u \}, \quad \tilde{T}_u^{rel} = \inf \{ t \geq 0 ; l \cdot (X_t - X_0) \leq u \}.
\]

We write \((\mathcal{F}_t)_{t \geq 0}\) and \((\theta_t)_{t \geq 0}\) for the canonical right continuous filtration and for the canonical time-shift on \( C(\mathbb{R}_+, \mathbb{R}^d) \) respectively.

We turn now to the construction of the objects appearing in Proposition 2.2.1. We consider some number \( L = 3L' > 3L \) and define the successive times of entrance in \( S(mL+L', mL+2L') \) and departure from \( S(mL, (m+1)L) \), cf. Figure 1: for \( m \in \mathbb{N} \),

\[
R_1^{(m)} = H_{S(mL+L', mL+2L')}, \quad S_1^{(m)} = T_{S(mL, (m+1)L)} \circ \theta_{R_1^{(m)}} + R_1^{(m)},
\]

and by induction for \( k \geq 2 \),

\[
R_k^{(m)} = R_1^{(m)} \circ \theta_{S_1^{(m)}} + S_k^{(m)}, \quad S_k^{(m)} = S_1^{(m)} \circ \theta_{S_k^{(m)}} + S_k^{(m)}.
\]

We define, for integer \( \alpha \geq 2 \), (this integer will typically be large in the sequel)

\[
N_{\alpha}^{(m)} = \sum_{k \geq 1} \mathbf{1}_{\{ R_k^{(m)} + 1 \leq S_k^{(m)} < T_{(m+\alpha)L} < \infty \}}.
\]
the number of entrances in $\bar{S}(mL+L', mL+2L')$ after which the trajectory stays at least one time unit in $S(mL, (m+1)L)$. Moreover, we consider:

$$(2.2.5) \quad k^{(m)}_{\alpha} = \max\{k \geq 1 : R_k^{(m)} + 1 \leq S_k^{(m)} < T_{(m+\alpha)L} \} \mathbb{1}_{\{T_{(m+\alpha)L} < \infty\}},$$

where the maximum over the empty set is defined to be 0. With the convention $S_0^{(m)} = T_{mL}$, we then set:

$$(2.2.6) \quad h_{\alpha}^{(m)} = \begin{cases} S_k^{(m)} - T_{mL}, & \text{if } T_{(m+\alpha)L} < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

The quantity $h_{\alpha}^{(m)}$ is the time duration, beginning at $T_{mL}$, after which the trajectory does not make “long visits” to the slab $S(mL+L', mL+2L')$ anymore. Note that $h_{\alpha}^{(m)}$ is non-decreasing in $\alpha$.

Let us give an outline of the steps leading to the main result of this section, i.e. Theorem 2.2.4. In Proposition 2.2.1 we show that a continuous path $w$ satisfying $\limsup_{t \to \infty} \frac{l \cdot w(t)}{t} > 0$ has the property that there is a large asymptotic fraction of slabs among the $S(mL, (m+1)L)$, $m \geq 1$, around which the oscillations of $w$ that occur before reaching a level at a distance $\alpha L$ in direction $l$, last only some finite time $h$ independent of $\alpha$. An analogous result for discrete path is stated in [65], Lemma 3. In the next step, we deduce the existence of an $h > 0$ such that with positive probability the following events, later called $C_m$, cf. (2.2.20), happen with a large asymptotic frequency: on $C_m$, the particle at time $H_{S(mL, (m+1)L)} + h$ is located in a narrow slab “to the
right of $S(mL,(m+1)L)$ and then moves to a level at a distance $\alpha L$ without backtracking, cf. Lemma 2.2.3. Then we extract the crucial information about the absence of backtracking. In essence for this purpose, we condition each event $C_m$ on the information prior to $H_S(mL,(m+1)L) + h$, and transfer our control on the asymptotic frequency of the $C_m$’s, to a control on the asymptotic mean of the conditional probabilities. This is done with the help of certain martingales and Azuma’s inequality, see (2.2.33). Finally we dominate these conditional probabilities by a sequence of i.i.d. variables under $\mathbb{P}$, apply the law of large numbers and conclude that the probability to never backtrack is positive by letting $\alpha$ tend to infinity. This method bypasses the technique of infinite products of probability spaces in [65], cf. (13) therein, which is hard to implement in the continuous setting.

**Proposition 2.2.1.** Let $w(\cdot)$ be a continuous path in $\mathbb{R}^d$ starting at 0 and satisfying $\limsup_{t \to \infty} \frac{l \cdot w(t)}{t} > 0$, then there exists an integer $h \geq 1$ such that for all integers $\alpha \geq 2$,

\begin{equation}
(2.2.7) \quad \limsup_{M \to \infty} \frac{1}{M + 1} \sum_{m=0}^{M} 1_{\{h^{(m)}(w) \leq h\}} \geq \frac{1}{3}.
\end{equation}

Proof. We choose $\delta > 0$ such that $\limsup_{t \to \infty} \frac{l \cdot w(t)}{t} \geq \delta$. There is a sequence $(t_k)_{k \geq 1}$ in $\mathbb{R}_+$ tending to infinity such that $l \cdot w(t_k) > \delta t_k$. Thus, for all $\alpha \geq 2$:

\begin{equation}
(2.2.8) \quad T_{\frac{1}{2}t_k + \alpha L}(w) \leq t_k, \quad \text{for all large enough } k \text{ (depending on } \alpha). \tag{2.2.8}
\end{equation}

(For the sake of simplicity, we will drop $w$ from the notation.) If we choose $M_k$ integer such that $M_k L \leq \frac{\delta^2}{2} t_k \leq (M_k + 1)L$, $k \geq 1$, (2.2.8) implies that for all integers $\alpha \geq 2$:

\begin{equation}
(2.2.9) \quad T_{(M_k+\alpha)L} \leq \frac{2(M_k + 1)L}{\delta}, \quad \text{for all large enough } k \text{ (depending on } \alpha). \tag{2.2.9}
\end{equation}

If $R_k^{(m)} + 1 \leq S_k^{(m)} < T_{(m+\alpha)L}$ and since $T_{(m+\alpha)L}$ is finite for all $m$, the path $w$ spends at least one unit of time entirely in the slab $S(mL,(m+1)L)$ before reaching level $(m+\alpha)L$. Hence, for all $k$ large enough, we deduce from (2.2.9) that, (cf. (2.2.4) for the notation)

\begin{equation}
(2.2.10) \quad \sum_{m=0}^{M_k} N^{(m)} \leq T_{(M_k+\alpha)L} \leq \frac{2(M_k + 1)L}{\delta}, \quad \text{and}
\end{equation}
2. Limit velocity and zero-one laws

(2.2.11)

\[
\sum_{m=0}^{M_k} h^{(m)}_{\alpha} \leq \sum_{j=0}^{\alpha-1} \sum_{m \mod \alpha = j}^{m \leq M_k} (T_{(m+\alpha)L} - T_{mL}) \leq \alpha T_{(M_k+\alpha)L} \leq \frac{2\alpha(M_k+1)L}{\delta},
\]

for all large enough \( k \). Assume now that (2.2.7) with real \( h \) does not hold, i.e.:

for all \( h \geq 1 \), there is an integer \( \alpha \geq 2 \) such that

\[
\limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1\{h^{(m)}_{\alpha} \leq h\} < \frac{1}{3}.
\]

We can construct inductively \( h_0 = 1, \alpha_1 \geq 2, h_i = \frac{6\alpha_i L}{\delta}, \alpha_{i+1} > \alpha_i \) using that \( h^{(m)}_{\alpha} \) is non-decreasing in \( \alpha \), such that

(2.2.12)

for all \( i \geq 1 \), \( \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1\{h^{(m)}_{\alpha_{i+1}} \leq h_i\} < \frac{1}{3} \).

On the other hand, (2.2.11) and the choice \( h_i = \frac{6\alpha_i L}{\delta} \) imply that

(2.2.13)

for all \( i \geq 1 \), \( \limsup_{k \to \infty} \frac{1}{M_k+1} \sum_{m=0}^{M_k} 1\{h^{(m)}_{\alpha_{i}} > h_i\} \leq \frac{1}{3} \).

Observe that for all \( i, k \geq 1 \),

\[
1 \leq \frac{1}{M_k+1} \sum_{m=0}^{M_k} 1\{h^{(m)}_{\alpha_{i+1}} \leq h_i\} + 1\{h^{(m)}_{\alpha_{i}} > h_i\} + 1\{h^{(m)}_{\alpha_{i}} < h^{(m)}_{\alpha_{i+1}}\}.
\]

This inequality together with (2.2.12) and (2.2.13) yields:

(2.2.14)

for all \( i \geq 1 \), \( \frac{1}{3} \leq \frac{1}{M_k+1} \sum_{m=0}^{M_k} 1\{h^{(m)}_{\alpha_{i}} < h^{(m)}_{\alpha_{i+1}}\} \), for all large enough \( k \).

If \( h^{(m)}_{\alpha_{i}} < h^{(m)}_{\alpha_{i+1}} \), the trajectory, after reaching level \( (m+\alpha_i)L \) has to return to the slab \( S((mL+L'), (mL+2L')) \) and stay in the slab \( S((mL, (m+1)L)) \) for at least one unit of time, all this before reaching level \( (m+\alpha_{i+1})L \). Therefore we see that

\[
1\{h^{(m)}_{\alpha_{i}} < h^{(m)}_{\alpha_{i+1}}\} \leq N^{(m)}_{\alpha_{i+1}} - N^{(m)}_{\alpha_i}.
\]
and hence for arbitrary \( i_0 \geq 1 \) and large \( k \), we obtain:

\[
\frac{i_0}{3} \leq \frac{1}{M_k + 1} \sum_{m=0}^{M_k} \sum_{i=1}^{i_0} 1\{k^{(m)}_{\alpha_i} < k^{(m)}_{\alpha_{i+1}}\} \leq \frac{1}{M_k + 1} \sum_{m=0}^{M_k} N^{(m)}_{\alpha_{i_0}+1} \leq \frac{2L}{\delta},
\]

a contradiction. We have thus proved the existence of a real \( h \geq 1 \) such that (2.2.7) holds. By monotonicity, we can increase \( h \) to be an integer.

The next Lemma comes as a preparation for the main result of this section, namely Theorem 2.2.4. If \( S \) is any stopping time, we write \( S_k; k \geq 0 \), for the iterates of \( S \) namely,

\[
(2.2.15) \quad S_0 = 0, S_1 = S, \text{ and } S_{k+1} = S \circ \theta_{S_k} + S_k \leq \infty.
\]

**Lemma 2.2.2.** For every \( k \geq 1 \), let \( U^k \) be a \((\mathcal{F}_t)_{t \geq 0}\)-stopping time and \( \Delta_k \in \mathcal{F}_{U^k} \). Denote with \( U^k_m, m \geq 0 \), the iterates of \( U^k \). If there exists numbers \( \gamma_1, \gamma_2 > 0 \), such that

\[
(2.2.16) \quad \text{for all } k \geq 1, x \in \mathbb{R}^d, \omega \in \Omega : \quad P_{x,\omega}(\Delta_k) \leq \gamma_1 e^{-\gamma_2 k},
\]

then for each \( \epsilon > 0 \), there is a \( k_0(\epsilon, \gamma_1, \gamma_2) \geq 1 \), such that

\[
(2.2.17) \quad \text{for } k \geq k_0, \quad P_0\text{-a.s., } \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\Delta_k \circ \theta_{U^k_m}} \leq \epsilon,
\]

with the convention that \( 1_{\Delta_k \circ \theta_{U^k_m}} = 0 \), if \( U^k_m = \infty \).

**Proof.** Note that \( 1_{\Delta_k \circ \theta_{U^k_m}} \) is \( \mathcal{F}_{U^k_{m+1}} \)-measurable. The strong Markov-property yields for \( M \geq 1, k \geq 1 \):

\[
(2.2.18) \quad E_{x,\omega}[\exp\left(\sum_{m=0}^{M} 1_{\Delta_k \circ \theta_{U^k_m}}, U^k_M < \infty\right)]
\]

\[
= E_{x,\omega}\left[\exp\left(\sum_{m=0}^{M-1} 1_{\Delta_k \circ \theta_{U^k_m}}\right) E_{X_{U^k_M}}[\exp(1_{\Delta_k}), U^k_M < \infty]\right]
\]

\[
= E_{x,\omega}\left[\exp\left(\sum_{m=0}^{M-1} 1_{\Delta_k \circ \theta_{U^k_m}}\right)((e - 1) P_{X_{U^k_M}}[\Delta_k] + 1), U^k_M < \infty\right].
\]
Using (2.2.16) and iteration, we obtain for $M \geq 1, k \geq 1$:

$$E_{x,\omega}[\exp\left(\sum_{m=0}^{M} 1_{\Delta_k \circ \theta_{U_m^k}}, U_M^k < \infty\right)] \leq (\gamma_1 e^{-\gamma_2 k} (e - 1) + 1)^{M+1}.$$ 

Therefore, using Chebychev's inequality, we find:

$$P_{x,\omega}\left[\frac{1}{M} \sum_{m=0}^{M-1} 1_{\Delta_k \circ \theta_{U_m^k} > \epsilon}\right] \leq e^{-\epsilon M} (\gamma_1 e^{-\gamma_2 k} (e - 1) + 1)^{M} \leq e^{M\left(-\epsilon + (e-1)\gamma_1 e^{-\gamma_2 k}\right)}.$$ 

If $k$ is large enough, the argument of the exponential becomes negative and our claim follows from Borel-Cantelli’s lemma.

In the next lemma with two successive reduction steps, we replace $\{h^{(m)}_\alpha \leq h\}$ appearing in (2.2.7) by an event $C_m$ that has the following meaning: At the stopping-time $T_{mL} + h_0$, for some $h_0 \geq 1$, the position of the diffusion is located in the slab $S(mL+2L', (m+K)L)$ and after this stopping-time, the trajectory reaches level $(m + \alpha)L$ without going below level $mL + 2L'$, cf. (2.2.20).

**Lemma 2.2.3.** Assume that $P_0(\limsup_{t \to \infty} \frac{1}{t} X_t > 0) > 0$, then there exists an integer $h_0 \geq 1$ and constants $L = 3L' > 3R, K = K(h_0) \geq 1$, such that

$$P_0\left[\inf_{\alpha \geq K} \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{C_m} \geq \frac{1}{12}\right] > 0,$$

where

$$C_m = \{X_{T_{mL}+h_0} \in S(ml+2L', (m+K)L) \cap \{T_{mL} + h_0 < T_{(m+\alpha)L}\} \cap \theta_{T_{mL}+h_0}^{-1}\{T_{mL+2L'} > T_{(m+\alpha)L}\}, \text{ for } m \geq 0.\right.$$ 

**Proof.** With our assumption, Proposition 2.2.1 yields that for some integer $h_0$:

$$P_0\left[\inf_{\alpha \geq 2} \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{h^{(m)}_\alpha \leq h_0} \geq \frac{1}{3}\right] > 0.$$
In a first reduction step, we want to keep only those slabs \( S(\mu L, (\mu+1)L) \), where after time \( T_{\mu L} + \rho_0 \) and before reaching level \( (\mu + \alpha)L \), the paths do not return to the inner part \( S(\mu L + L', mL + 2L') \) of the slab. More precisely we claim that if \( L' > R \) is large enough, then we obtain from (2.2.21):

\[
P_0 \left[ \inf_{\mu \geq 2} \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbb{1}_{\{(h^{(m)}_{\alpha} \leq \rho_0, R' \circ \theta_{T_{\mu L} + T_{\mu L} > T_{(\mu+\alpha)L}} \geq \frac{1}{6}\}} > 0, \right]
\]

where we used the notation:

\[
R' = \inf \{ t \geq \rho_0 : l \cdot (X_t - X_0) \in \{L', 2L\} \}.
\]

Indeed, consider for fixed \( \alpha \geq 2, m \geq 0 \), a trajectory \( w \) starting in 0 and satisfying \( h^{(m)}_{\alpha} \leq \rho_0 \). If \( w \) visits the inner slab \( S(\mu L + L', mL + 2L') \) between time \( T_{\mu L} + \rho_0 \) and \( T_{(\mu+\alpha)L} \), then it must exist from the outer slab \( S(\mu L, (\mu+1)L) \) within time 1 as otherwise \( h^{(m)}_{\alpha} \) becomes larger than \( \rho_0 \). Note also that by definition, \( h^{(m)}_{\alpha} \leq \rho_0 \) implies \( T_{(\mu+\alpha)L} < \infty \), \( P_0 \)-a.s. Hence \( P_0 \)-a.s. we have:

\[
\{h^{(m)}_{\alpha} \leq \rho_0, R' \circ \theta_{T_{\mu L} + T_{\mu L} \leq T_{(\mu+\alpha)L}} \} \subset \theta_{T_{\mu L}}^{-1}(\Delta_{L', \alpha}) \cap \{T_{\mu L} < \infty\},
\]

where we defined \( \Delta_{L', \alpha} = \{\sup_{s \leq 1} |X_s - X_0| \geq R', R' \leq T_{\alpha L} < \infty\} \). By the Markov property and Bernstein’s inequality (see [3], Proposition 8.1 p.23), we have for all \( x \in \mathbb{R}^d, \omega \in \Omega \):

\[
P_{x,\omega}[\Delta_{L', \alpha}] \leq E_{x,\omega}[R' < \infty, P_{X_{R'}, \omega}[^{\sup_{s \leq 1} |X_s - X_0| \geq L' \}] \leq c_1 e^{-c_2 L'^2}.
\]

We decompose the indicator-function \( \mathbb{1}_{\{h^{(m)}_{\alpha} \leq \rho_0\}} \) appearing in (2.2.21) as follows:

\[
\limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbb{1}_{\{h^{(m)}_{\alpha} \leq \rho_0\}} \leq \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\Delta_{L', \alpha} \circ \theta_{T_{\mu L}}}(R' < \infty, P_{X_{R'}, \omega}[^{\sup_{s \leq 1} |X_s - X_0| \geq L' \}]
\]

\[
+ \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbb{1}_{\{h^{(m)}_{\alpha} \leq \rho_0\}} 1_{\{R' \circ \theta_{T_{\mu L} + T_{\mu L} > T_{(\mu+\alpha)L}} \}}.
\]

In order to apply Lemma 2.2.2 to the first term of (2.2.25), since \( \theta_{T_{\mu L}}^{-1}\Delta_{L', \alpha} \in \mathcal{F}_{T_{(\mu+\alpha)L} \}, \) for \( m \geq 0 \), we rewrite the sum in the first term as a double sum.
running over all residue classes modulo $\alpha$ and obtain as an upper bound

\[
(2.2.26) \quad \frac{1}{\alpha} \sum_{j=0}^{\alpha-1} \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\Delta_{L',\alpha} \cap \theta_{T_{(m\alpha+j)3L'}}}.
\]

With (2.2.24), we can apply Lemma 2.2.2 for every $j = 0, \ldots, \alpha - 1$. The parameter $L'$ is chosen integer and plays the role of $k$ in the lemma. Moreover we respectively substitute $T_{\alpha3L'}^r \circ \theta_{T_{jL}} + T_{jL}$ and $\theta_{T_{j3L'}}^{-1}(\Delta_{L',\alpha}) \cap \{T_{j3L'} < \infty\}$ for $U^k$ and $\Delta_k$, and use $\epsilon = \frac{1}{6}$. Note that $P_0$-a.s., for $m \geq 1$, the $m$-th iterate of $T_{\alpha3L'}^r \circ \theta_{T_{jL}} + T_{jL}$ is $T_{(m\alpha+j)3L'}$. As the lower bound for $L'$ provided by Lemma 2.2.2 only depends on the constants $c_1, c_2$ in (2.2.24), there exists a constant $L' > R$, such that for all $\alpha \geq 2$ and all $j = 0, \ldots, \alpha - 1$, we obtain:

\[
\limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\Delta_{L',\alpha} \cap \theta_{T_{(m\alpha+j)3L'}}} < \frac{1}{6}, \quad P_0\text{-a.s.}
\]

Hence the first term of (2.2.25) is $P_0$-a.s. for all integers $\alpha \geq 2$ smaller than $\frac{1}{6}$. In view of (2.2.21), this estimate proves the claim (2.2.22) of the first reduction step.

In the second reduction step, we would like to keep only those slabs where the trajectory stays in a big ball during time $h_0$ after $T_{mL}$. We claim that there exists a constant $K = K(h_0, L) \geq 1$, such that

\[
(2.2.27) \quad 0 < P_0 \left[ \inf_{\alpha > K} \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\{h_0^{(m)} \leq h_0, R' \circ \theta_{T_{mL}} + T_{mL} > T_{(m\alpha+j)L}\}} \times 1_{\{\sup_{s \leq h_0} |X_s - X_0| \circ \theta_{T_{mL}} < KL\} \geq \frac{1}{12}\} \right].
\]

Indeed, define $\Delta_k = \{\sup_{s \leq h_0} |X_s - X_0| \geq kL\}$, for $k \geq 1$. From Bernstein's inequality (see [3], Proposition 8.1 p.23), there exist positive constants $c_3(h_0)$, $c_4(h_0)$, such that

\[
(2.2.28) \quad P_{x,\omega}[\Delta_k'] \leq c_3e^{-c_4(kL)^2}, \quad \text{for all } x \in \mathbb{R}^d, \omega \in \Omega.
\]

As before we decompose the indicator-function appearing in (2.2.22) according to $\theta_{T_{mL}}^{-1}(\Delta_k')$ and $\theta_{T_{mL}}^{-1}(\Delta_k')$. In view of an application of Lemma 2.2.2,
since $\theta_{T_{mL}}^{-1} \Delta'_k \in \mathcal{F}_{T_{(k+m)L}}$, $m \geq 0$, we rewrite the sum $\sum_{m=0}^{M} 1_{\Delta'_k} \circ \theta_{T_{mL}}$ as a double sum running over all the residue classes modulo $k$. As a result:

\[(2.2.29) \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\Delta'_k} \circ \theta_{T_{mL}} \leq \frac{1}{k} \limsup_{M \to \infty} \frac{1}{M+1} \sum_{m=0}^{M} 1_{\Delta'_k} \circ \theta_{T_{(mk+j)L}}.\]

With (2.2.28), we apply Lemma 2.2.2 for every $j = 0, \ldots, k-1$: $\theta_{T_{jL}}^{-1}(\Delta'_k) \cap \{T_{jL} < \infty\}$ and $T_{kL}^\text{rel} \circ \theta_{T_{jL}} + T_{jL}$ play the role of $\Delta_k$ and $U^k$ and we choose $\epsilon = \frac{1}{12}$ in Lemma 2.2.2. $P_0$-a.s., for $m \geq 1$, the $m$-th iterate of $T_{kL}^\text{rel} \circ \theta_{T_{jL}} + T_{jL}$ is $T_{(mk+j)L}$. Hence there is a constant $K = K(h_0, L) \geq 1$, such that for $k = K$ the left-hand side of (2.2.29) is $P_0$-a.s. smaller than $\frac{1}{12}$. This proves (2.2.27). We conclude the proof by noting that $\{h_\alpha^{(m)} \leq h_0, \sup_{s \leq h_0} |X_s - X_0| \circ \theta_{T_{mL}} < KL, R' \circ \theta_{T_{mL}} \geq T_{(m+\alpha)L} - T_{mL}\}$ is $P_0$-a.s. included in $C_m$. $\square$

**Theorem 2.2.4.** ($d \geq 1$). For any $l \in S^{d-1}$, $P_0(\limsup_{t \to \infty} \frac{tX_t}{t} > 0) > 0$ implies that there exists a number $r_0 > 0$ such that $P_0\left(\hat{T}_r^\text{rel} = \infty\right) > 0$ and as a consequence $P_0(A_l) > 0$.

(Note that $r_0$ only depends on the quantities $h_0, L, K$ from Lemma 2.2.3.)

**Proof.** Let us briefly outline the argument: We would like to apply the law of large numbers to the sum $\sum_{m=0}^{M} 1_{G_m}$ appearing in (2.2.19), but the dependence structure of the sequence $(G_m)_{m \geq 0}$ seems to be complicated. Therefore we will replace this sequence by one that is iid with respect to $\mathbb{P}$. This will be achieved by constructing an appropriate martingale and using Azuma’s inequality.

We pick $L = 3L', K$ and $h_0$ as in Lemma 2.2.3. We introduce the following filtrations: for integer $\alpha > K$ and $j = 0, \ldots, \alpha - 1$,

\[(2.2.30) \begin{align*}
G_m^j &= \mathcal{F}_{T_{(m\alpha+j)L}}, \quad m \geq 1, \quad G_0^j = \mathcal{F}_0; \\
\tilde{G}_m^j &= \mathcal{F}_{T_{(m+1)\alpha+jL} \wedge (T_{(m\alpha+j)L} + h_0)}; \quad m \geq 0.
\end{align*}\]

Recall the definition of $C_m$ (2.2.20) and observe that for $\alpha > K$, $j = 0, \ldots, \alpha - 1$, $m \geq 0$:

\[(2.2.31) C_{m\alpha+j} \in G_{m+1}^j \quad \text{and} \quad G_m^j \subset \tilde{G}_m^j \subset G_{m+1}^j,\]
because: $T_1^{m\alpha+j} \leq T_1^{(m+1)\alpha+j} \land (T_1^{m\alpha+j} + h_0) \leq T_1^{((m+1)\alpha+j)}$.

We define for $j = 0, \ldots, \alpha - 1$ and $n \geq 1$:

$$M_n^j = \sum_{m=0}^{n-1} 1_{C_{m\alpha+j}} - E_0,\omega[1_{C_{m\alpha+j}} | \bar{G}_m^j], \quad M_0^j = 0.$$  

By (2.2.31), $M_n^j$ is $G_n^j$ measurable, for $n \geq 0$, and integrable. It is a $G_n^j$ martingale under $P_0,\omega$, for any $\omega \in \Omega$, because $P_0,\omega$-a.s., for $n \geq 1$ we have

$$E_0,\omega[1_{C_{(n-1)\alpha+j}} | G_n^{j-1}] = E_0,\omega[E_0,\omega[1_{C_{(n-1)\alpha+j}} | G_{n-1}^j] | G_{n-1}^j].$$

Since $M_n^j$ has bounded increments Azuma’s inequality, see [1], Theorem 2.1, applies and we find:

$$P_0,\omega[\lim_{n \to \infty} M_n^j/n = 0] = 1, \quad \text{for all } j = 0, \ldots, \alpha - 1.$$

Hence for any $\omega \in \Omega$, $P_0,\omega$-a.s., for all $\alpha > K$ and $j = 0, \ldots, \alpha - 1$,

$$\limsup_{M \to \infty} \frac{1}{M + 1} \sum_{m=0}^{M} 1_{C_{m\alpha+j}} = \limsup_{M \to \infty} \frac{1}{M + 1} \sum_{m=0}^{M} E_0,\omega[1_{C_{m\alpha+j}} | \bar{G}_m^j].$$

The strong Markov-property yields that $P_0,\omega$-a.s., for $\alpha > K$, $j = 0, \ldots, \alpha - 1$ and $m \geq 0$:

$$E_0,\omega[1_{C_{m\alpha+j}} | \bar{G}_m^j] = \{X_{T_1^{(m\alpha+j)} + h_0} \in I_{m,j}, T_1^{(m\alpha+j)} + h_0 < T_1^{((m+1)\alpha+j)} \}$$

$$\times P_{X_{T_1^{(m\alpha+j)} + h_0},\omega}[\bar{T}_1^{(m\alpha+j) + 2L'} > T_1^{((m+1)\alpha+j)}]$$

$$\leq \sup_{y \in I_{m,j}} P_{y,\omega}[\bar{T}_1^{(m\alpha+j) + 2L'} > T_1^{((m+1)\alpha+j)}],$$

where $I_{m,j} \overset{\text{def}}{=} S((m\alpha+j) + 2L', (m\alpha+j + K)L)$.

Therefore, (2.2.19) together with (2.2.33) and (2.2.34) imply that

$$\mathbb{P}\left[\inf_{\alpha > K} \alpha \sum_{j=0}^{\alpha-1} \limsup_{M \to \infty} \frac{1}{M + 1} \sum_{m=0}^{M} \sup_{y \in I_{m,j}} P_{y,\omega}[\bar{T}_1^{(m\alpha+j) + 2L'} > T_1^{((m+1)\alpha+j)}] > \frac{1}{12}\right] > 0.$$
2.2. Oscillations and null directional speed

With respect to $\mathbb{P}$, the variables $f_{m,j} \overset{\text{def}}{=} \sup_{y \in I_{m,j}} P_{y,\omega} [\bar{T}_{(m+1)\alpha+j} > T_{((m+1)\alpha+j)L}, m \geq 0]$ are i.i.d. for every $j = 0, \ldots, \alpha - 1$. Indeed the respective slabs $S((m+1)\alpha+j)L+2L',((m+1)\alpha+j)L)$ as $m$ varies are separated by at least $2L' > R$, and one applies (2.1.5), as well as translation invariance. Hence, from the law of large numbers and from (2.2.35) we deduce that

$$\inf_{\alpha > K} \mathbb{E} [\sup_{y \in I_{0,0}} P_{y,\omega} [\bar{T}_{2L'} > T_{\alpha L}]] > \frac{1}{12},$$

and by dominated convergence for $\alpha \to \infty$:

$$\frac{1}{12} < \mathbb{E} [\sup_{y \in I_{0,0}} P_{y,\omega} [\bar{T}_{2L'} = \infty]] \leq \mathbb{E} [\sup_{y \in I_{0,0}} P_{y,\omega} [\bar{T}^{\text{rel}}_{(KL-2L')} = \infty]].$$

We define $\frac{r_0}{2} = KL - 2L'$. Assume now that $P_0[\bar{T}^{\text{rel}}_{-r_0} = \infty] = 0$. It follows from Fubini’s Theorem, that there is a $\mathbb{P}$-null set $\Gamma \subset \Omega$, such that

$$\text{for } \omega \in \Gamma^c, P_{x,\omega} [\bar{T}^{\text{rel}}_{-r_0} = \infty] = 0,$$

except on a Lebesgue-negligible subset of $\mathbb{R}^d$.

But then, for any $y \in \mathbb{R}^d, \omega \in \Omega$:

$$P_{y,\omega} [\bar{T}^{\text{rel}}_{-\frac{r_0}{2}} = \infty] = P_{y,\omega} [\bar{T}^{\text{rel}}_{-\frac{r_0}{2}} = \infty, \sup_{s \leq \frac{1}{n}} |X_s - X_0| \leq \frac{r_0}{4}]$$

$$+ P_{y,\omega} [\bar{T}^{\text{rel}}_{-\frac{r_0}{2}} = \infty, \sup_{s \leq \frac{1}{n}} |X_s - X_0| > \frac{r_0}{4}]$$

$$\leq P_{y,\omega} [\bar{T}^{\text{rel}}_{-r_0} \circ \theta_{\frac{1}{n}} = \infty] + P_{y,\omega} [\sup_{s \leq \frac{1}{n}} |X_s - X_0| > \frac{r_0}{4}].$$

By the Markov property, the first term on the right-hand side equals

$$\int_{\mathbb{R}^d} p^{(\omega)}(y, x, \frac{1}{n}) P_{x,\omega} [\bar{T}^{\text{rel}}_{-r_0} = \infty] \, dx,$$

where $p^{(\omega)}(y, \cdot, \frac{1}{n})$ denotes the transition density of the diffusion starting in $y$ in the environment $\omega$ at time $1/n$ with respect to the Lebesgue measure. This density exists under the assumptions (2.1.2), (2.1.3), see [16], Theorem 4.5. Hence using (2.2.37), this term equals 0 for all $\omega \in \Gamma^c, y \in \mathbb{R}^d$. The second term on the right hand side of (2.2.38) converges to 0 as $n \to \infty$ by continuity of the trajectories. And so it would follow that for all $\omega \in \Gamma^c, y \in \mathbb{R}^d, P_{y,\omega} [\bar{T}^{\text{rel}}_{\frac{r_0}{2}} = \infty] = 0$. But this contradicts (2.2.36) and hence:

$$P_0 [\bar{T}^{\text{rel}}_{-r_0} = \infty] > 0.$$
To show that \( P_0(A_l) > 0 \), we need the following useful lemma:

**Lemma 2.2.5.** Consider \( l \) in \( S^{d-1} \). For \( u, v \in \mathbb{R} \), \( u < v \), define the stopping times \( \beta^{l,u}_t = \inf\{ t \geq 1; l \cdot X_t \geq u \} \) and denote its iterates with \( \beta^{l,u}_k, k \geq 0 \). Then one has for all \( x \in \mathbb{R}^d, \omega \in \Omega \):

\[
(2.2.40) \quad P_{x,\omega}[\beta^{l,u}_k < \infty, \text{ for all } k \geq 0 \text{ and } T_v = \infty] = 0.
\]

**Proof.** By the support theorem (see [3] p.25), there is a constant \( c = c(v - u) > 0 \) such that for all \( x \in \mathcal{S}(u,v), \omega \in \Omega : P_{x,\omega}[T_v \leq 1] > c \).

Then the Markov-property shows that for all \( x \in \mathbb{R}^d, \omega \in \Omega, k \geq 1, \)

\[
P_{x,\omega}[\beta^{l,u}_k < \infty, T_v = \infty] \leq P_{x,\omega}[0 \leq \beta^{l,u}_k < T_v] \\
\leq P_{x,\omega}[0 \leq \beta^{l,u}_{k-1} < T_v, T_v \circ \theta_{\beta^{l,u}_{k-1}} > 1] \leq (1 - c) P_{x,\omega}[0 \leq \beta^{l,u}_{k-1} < T_v].
\]

After iteration and letting \( k \) tend to infinity, we obtain the claim. \( \Box \)

We are now ready to finish the proof of Theorem 2.2.4. We observe that for any \( v > 0 \):

\[
(2.2.41) \quad \{ \tilde{T}_{-v} = \infty \} \subset A_l, \ P_0\text{-a.s.}\.
\]

Indeed, we have in view of Lemma 2.2.5 with \(-l\) in the role of \( l\):

\[
P_0[A^c_l, \tilde{T}_{-v} = \infty] = P_0[\exists u \in \mathbb{Z}, u < v : \beta^{-l,u}_k < \infty \text{ and } \tilde{T}_{-v} = \infty] = 0.
\]

It thus follows from (2.2.39) and (2.2.41) that \( P_0(A_l) > 0 \). \( \Box \)

**Corollary 2.2.6.** \((d \geq 1)\).

Let \( l \in S^{d-1} \). If \( P_0[A_l \cup A_{-l}] = 0 \) then \( P_0\text{-a.s.}, \lim_{t \to \infty} \frac{L_X_t}{t} = 0 \).

**Proof.** If \( P_0[A_l] = 0 \), Theorem 2.2.4 implies that \( \limsup_t \frac{L_X_t}{t} \leq 0 \), \( P_0\text{-a.s.} \).

The same argument for \(-l\) implies that \( \liminf_t \frac{L_X_t}{t} \geq 0 \), \( P_0\text{-a.s.} \), and the claim follows. \( \Box \)
2.3 Limit velocity

The aim of this section is to prove the existence of a possibly non deterministic asymptotic velocity, cf. Theorem 2.3.8. As a preparation we need to revisit some of the theorems proven in Shen [48], now in the absence of the assumption $P_0(A_l) = 1$ made in [48]. For this reason we will consider in the following probabilities conditioned on the event that the diffusion is unbounded in a direction $l$ or that it escapes to infinity in a direction $l$. But first we recall the definitions of the regeneration times $\tau_k$, $k \geq 1$ and the coupling measure $\tilde{P}_{x,\omega}$ introduced in [48].

2.3.1 The coupling measure

For $x \in \mathbb{R}^d$ and $l \in S^{d-1}$, we consider

\[(2.3.1) \quad B^x = B(x+9Rl, R), \quad U^x = B(x+5Rl, 6R),\]

where $R$ is the range of dependence of the environment.

We denote by $\lambda_j$ the canonical coordinates on $\{0,1\}^\mathbb{N}$. Further, we let $(S_m)_{m \geq 0}$, denote the canonical filtration on $\{0,1\}^\mathbb{N}$ and $S$ the canonical $\sigma$-algebra. On the enlarged space $C(\mathbb{R}_+, \mathbb{R}^d) \times \{0,1\}^\mathbb{N}$, we consider the fol-
lowing $\sigma$-fields:

\[(2.3.2) \quad Z_t \overset{\text{def}}{=} \mathcal{F}_t \otimes \mathcal{S}_{[t]}, \text{with } t \geq 0, \quad \text{and } Z \overset{\text{def}}{=} \mathcal{F} \otimes \mathcal{S} = \sigma \left( \bigcup_{m \in \mathbb{N}} Z_m \right).\]

On the enlarged space, the shift operators $\hat{\theta}_m$, $m \geq 0$ are defined so that $\hat{\theta}_m (X, \lambda) = (X_{m+}, \lambda_{m+})$. Then from Theorem 2.1 in Shen [48], one has the following measures, coupling the diffusion in random environment with a sequence of Bernoulli variables:

**Proposition 2.3.1.** There exists $\epsilon > 0$, such that for every $l \in S^{d-1}$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$, there exists a probability measure $\hat{P}_{x,\omega}$ on $(C(\mathbb{R}_+, \mathbb{R}^d) \times \{0, 1\}^N, \mathcal{Z})$ depending measurably on $\omega$ and $x$, such that:

\[(2.3.3) \quad \text{Under } \hat{P}_{x,\omega}, (X_t)_{t \geq 0} \text{ is } P_{x,\omega}\text{-distributed, and the } \lambda_m, m \geq 0, \text{ are i.i.d. Bernoulli variables with success probability } \epsilon.\]

\[(2.3.4) \quad \text{For } m \geq 1, \lambda_m \text{ is independent of } \mathcal{F}_m \otimes \mathcal{S}_{m-1} \text{ under } \hat{P}_{x,\omega}.\]

Conditioned on $Z_m$, $X \circ \hat{\theta}_m$ has the same law as $X$ under $\hat{P}_{x,\omega}$, where for $y \in \mathbb{R}^d$, $\lambda \in \{0, 1\}$, $\hat{P}_{y,\omega}^\lambda$ denotes the law $P_{y,\omega}[\cdot | \lambda_0 = \lambda]$.

\[(2.3.5) \quad \hat{P}_{x,\omega}^1 \text{ almost surely, } X_s \in U_x \text{ for } s \in [0, 1] \text{ (recall (2.3.1)).}\]

\[(2.3.6) \quad \text{Under } \hat{P}_{x,\omega}, X_1 \text{ is uniformly distributed on } B_x \text{ (recall (2.3.1)).}\]

The new annealed measures on $(\Omega \times C(\mathbb{R}_+, \mathbb{R}^d) \times \{0, 1\}^\mathbb{N}, \mathcal{A} \otimes \mathcal{Z})$ are:

\[(2.3.7) \quad \hat{P}_x \overset{\text{def}}{=} \mathbb{P} \times \hat{P}_{x,\omega} \quad \text{and} \quad \hat{E}_x \overset{\text{def}}{=} \mathbb{E} \times \hat{E}_{x,\omega}.\]

### 2.3.2 The regeneration times $\tau_k$

We follow [48] and [46] to define the first regeneration time $\tau_1$. To this end, we introduce a sequence of integer-valued $(Z_t)_{t \geq 0}$-stopping times $N_k$, for which
the condition \( \lambda_{N_k} = 1 \) holds, and at these times the process \((l \cdot X_s)_{s \geq 0}\) reaches essentially a local maximum (within a small variation). Then \( \tau_1 \), when finite, is the first \( N_k + 1, k \geq 1 \), such that the process \((l \cdot X_t)_{t \geq 0}\) never goes below \( l \cdot X_{N_k + 1} - R \) after time \( N_k + 1 \). In fact, the precise definition of \( \tau_1 \) relies on several sequences of stopping times. First, for \( a > 0 \), introduce the \( (\mathcal{F}_t)_{t \geq 0}\)-stopping times \( V_k(a), k \geq 0 \), (recall \( T_u \) in (2.2.2)):

\[
V_0(a) \overset{\text{def}}{=} T_{M(0) + a} \leq \infty, \quad V_{k+1}(a) \overset{\text{def}}{=} T_{M\left(\lfloor V_k(a) \rfloor\right) + R} \leq \infty,
\]

where \( M(t) \overset{\text{def}}{=} \sup\{l \cdot X_s : 0 \leq s \leq t\} \).

In view of the Markov property, see (2.3.4), we require the stopping times \( N_k(a), k \geq 1 \), to be integer-valued and with this in mind, introduce as an intermediate step the (integer-valued) stopping times \( \tilde{N}_k(a) \) where the process \( X_t \cdot l \) essentially reaches a maximum:

\[
\tilde{N}_1(a) \overset{\text{def}}{=} \inf \left\{ \left[ V_k(a) \right] : k \geq 0, \sup_{s \in [V_k, \lfloor V_k \rfloor]} |l \cdot (X_s - X_{V_k})| < \frac{R}{2} \right\},
\]

\[
\tilde{N}_{k+1}(a) \overset{\text{def}}{=} \tilde{N}_1(3R) \circ \hat{\theta}_{\tilde{N}_k(a)} + \tilde{N}_k(a), \quad k \geq 1,
\]

By convention we set \( \tilde{N}_0 = 0 \) and \( \tilde{N}_{k+1} = \infty \) if \( \tilde{N}_k = \infty \) and then define \( N_1(a) \) as

\[
N_1(a) \overset{\text{def}}{=} \inf \left\{ \tilde{N}_k(a) : k \geq 1, \lambda_{\tilde{N}_k(a)} = 1 \right\}.
\]

Now we can define the \( (\mathcal{Z}_t)_{t \geq 0}\)-stopping times:

\[
S_1 \overset{\text{def}}{=} N_1(3R) + 1, \quad R_1 \overset{\text{def}}{=} S_1 + D \circ \hat{\theta}_{S_1},
\]

\[
S_{k+1} \overset{\text{def}}{=} N_{k+1} + 1, \quad R_{k+1} \overset{\text{def}}{=} S_{k+1} + D \circ \hat{\theta}_{S_{k+1}}.
\]

(By convention we set \( R_0 = 0 \).) The \( (\mathcal{Z}_t)_{t \geq 0}\)-stopping times \( N_{k+1}, S_{k+1} \) and \( R_{k+1} \) are defined in an iterative way for \( k \geq 1 \):

\[
N_{k+1} \overset{\text{def}}{=} R_k + N_1(a_k) \circ \hat{\theta}_{R_k} \text{ with } a_k \overset{\text{def}}{=} M(R_k) - X_{R_k} \cdot l + R,
\]

\[
S_{k+1} \overset{\text{def}}{=} N_{k+1} + 1, \quad R_{k+1} \overset{\text{def}}{=} S_{k+1} + D \circ \hat{\theta}_{S_{k+1}}.
\]

(the shift \( \hat{\theta}_{R_k} \) is not applied to \( a_k \) in the above definition).

For \( k \geq 1 \), observe that on the event \( \{N_k < \infty\} \), \( \lambda_{N_k} = 1 \) and \( \sup_{s \leq N_k} X_s \cdot l \).
l \leq X_{N_k} \cdot l + R; Notice that for all k \geq 1, the \((Z_t)_{t \geq 0}\)-stopping times \(N_k, S_k\) and \(R_k\) are integer-valued, possibly equal to infinity, and we have

\[ 1 \leq N_1 \leq S_1 \leq R_1 \leq N_2 \leq S_2 \leq R_2 \cdots \leq \infty. \]

The first regeneration time \(\tau_1\) is defined, as in [48] and [46], (see also [57]) by

\[ \tau_1 \overset{\text{def}}{=} \inf\{S_k : S_k < \infty, R_k = \infty\} \leq \infty. \]

### 2.3.3 Renewal structure and limit velocity

We first develop the main theorems describing the renewal structure and then present a weak zero-one law, which says that for any unit vector \(l\), \(P_0(A_l \cup A_{-l})\) is either 0 or 1, cf. Proposition 2.3.6. We then prove finiteness of \(\hat{E}_0[l \cdot X_{\tau_1} | D = \infty]\) under the condition \(P_0(A_l) > 0\), cf. Proposition 2.3.7 and derive the existence of a possibly random asymptotic velocity in Theorem 2.3.8. We begin with an easy lemma which refines (2.2.41).

**Lemma 2.3.2.** For any \(l \in S^{d-1}\), \(P_0(A_l) > 0 \iff P_0(\hat{T}_{-R} = \infty) > 0\).

**Proof.** In view of (2.2.41), we only need to prove that \(P_0(A_l) > 0\) implies \(P_0(\hat{T}_{-R} = \infty) > 0\). Assume by contradiction that \(P_0(\hat{T}_{-R} = \infty) = 0\). Using translation invariance of \(\mathbb{P}\), and Fubini’s theorem, we see that for almost all \(\omega \in \Omega:\)

\[ P_{x,\omega}(\hat{T}_{-R} = \infty) = 0, \quad \text{except on a Lebesgue-negligible subset of} \ \mathbb{R}^d. \]

A calculation similar to (2.2.38) shows that for almost all \(\omega\) and every \(x \in \mathbb{R}^d\), we have that \(P_{x,\omega}(\hat{T}_{-R} < \infty) = 1\). The strong Markov property implies at once that \(P_{0,\omega}(\hat{T}_{-R} < \infty) = 1\), \(\mathbb{P}\)-a.s., for all \(k \geq 1\). This contradicts \(P_0(A_l) > 0\). \(\square\)

In the sequel, we will use the following additional notation. For any \(l \in S^{d-1},\)

\[ B_l \overset{\text{def}}{=} \{\sup_{s \geq 0} l \cdot X_s = \infty\}. \]
From (2.2.41) and the definition of $D$ (see (2.3.12)), we have of course for any $l \in S^{d-1}$:

$\{D = \infty\} \subset A_l \subset B_l$, $P_0$-a.s.

We will see later that if $P_0(A_l) > 0$, then $A_l = B_l$, $P_0$-a.s., cf. Theorem 2.3.5. The next lemma shows that the first renewal time $\tau_1$ is finite on the event $B_l$, if $P_0(A_l) > 0$.

**Lemma 2.3.3.** Consider $l \in S^{d-1}$ and assume $P_0(A_l) > 0$, then $B_l \subset \{\tau_1 < \infty\}$, $P_0$-a.s., with the notation (2.3.15).

The proof is similar to the proof of Proposition 2.7 in [48] and is included in the Appendix for the convenience of the reader.

On the space $\Omega \times C(\mathbb{R}_+, \mathbb{R}^d) \times \{0, 1\}^\mathbb{N}$, we introduce the sub-$\sigma$-algebra $\mathcal{G}$ of $A \otimes \mathbb{Z}_\infty$ that is generated by sets of the form:

$\{\tau_1 = m\} \cap O_{m-1} \cap \{l \cdot X_{m-1} > a\} \cap \{X_m \in G\} \cap F_a$,

where $m \geq 2$, $a \in \mathbb{R}$, $O_{m-1} \in \mathbb{Z}_{m-1}$, $G \subset \mathbb{R}^d$ open, $F_a \in \mathcal{H}\{z \in \mathbb{R}^d : l \cdot z \leq a + R\}$.

Loosely speaking, $\mathcal{G}$ contains information on the trajectories up to time $\tau_1 - 1$ and at time $\tau_1$ as well as information on the environment that has possibly been visited by the diffusion up to time $\tau_1 - 1$. Note that no information between time $\tau_1 - 1$ and $\tau_1$ is included. This is crucial when one exploits the finite range dependence property of the environment with the help of the coupling measure $\hat{P}_0$, as we saw already in the proof of Lemma 2.3.3, cf. (2.5.2).

The next proposition is a variation on Theorem 2.4 in [48], and provides the base for the renewal structure presented in Theorem 2.3.5.

**Proposition 2.3.4.** Consider $l$ in $S^{d-1}$ and assume $P_0(A_l) > 0$. Then for any $x \in \mathbb{R}^d$, any bounded functions $f, g, h$ respectively $Z, \mathcal{H}\{z \in \mathbb{R}^d : l \cdot z \leq -R\}$, $\mathcal{G}$-measurable, one has:

$\mathbb{E}_x[f(X_{\tau_1} - X_{\tau_1}, \lambda_{\tau_1})g \circ t_{X_{\tau_1}} h \mid B_l] = \mathbb{E}_x[h \mid B_l] \mathbb{E}_0[f(X, \lambda)g \mid D = \infty]$.

with $B_l$ as in (2.3.15) and $t_y$ the spatial shift, cf. the beginning of the introduction.
Proof. We only discuss the salient features of the proof which is a variation on that of Theorem 2.4 in [48]. As in the proof of this theorem, it suffices to prove (2.3.18) for \( h = 1_{\{\tau_1 = m\}} 1_{F_a} 1_{O_{m-1}} 1_{\{X_m \in G\}} 1_{\{l \cdot X_{m-1} > a\}} \), with \( m \geq 2, a \in \mathbb{R}, O_{m-1} \in \mathcal{Z}_{m-1}, G \subset \mathbb{R}^d \) open, \( F_a \in \mathcal{H}_{\{z \in \mathbb{R}^d : l \cdot z \leq a + R\}} \), since (2.3.17) constitutes an \( \pi \)-system. Note that there is an \( O_{m-1} \in \mathcal{Z}_{m-1} \), such that

\[
\{\tau_1 = m\} \cap O_{m-1} \cap B_l = \hat{O}_{m-1} \cap \{D \circ \hat{\theta}_m = \infty\} \cap \hat{\theta}_m^{-1}(B_l)
\]

where the last step follows from (2.3.16). As \( B_l \) disappears from the calculations, the rest of the argument is identical to the proof of Theorem 2.4 in [48], see also [49].

On the event \( \{\tau_1 < \infty\} \), we define inductively a non-decreasing sequence of random variables \( \tau_k \leq \infty \), via:

\[
(2.3.19) \quad \tau_{k+1}((X, \lambda_\cdot)) \overset{\text{def}}{=} \tau_1 ((X, \lambda_\cdot)) + \tau_k ((X_{\tau_1 + \cdot}, \lambda_{\tau_1 + \cdot})), \quad k \geq 1.
\]

We are able to reconstruct in our context an analogue of the renewal structure of Theorem 2.5 in [48].

**Theorem 2.3.5.** Consider \( l \in S^{d-1} \) and assume \( P_0(A_l) > 0 \). Then \( \hat{P}_0 \)-a.s., \( \{D = \infty\} \subset A_l = B_l = \{\tau_k < \infty, \text{ for all } k \geq 1\} \) (recall (2.3.12), (2.1.9), (2.3.16)) and under \( \hat{P}_0[\cdot | A_l] \), the random variables

\[
(2.3.20) \quad Z_k \overset{\text{def}}{=} (X_{(\tau_{k+\cdot}) \land (\tau_{k+1} - 1)} - X_{\tau_k}, X_{\tau_{k+1}} - X_{\tau_k}, \tau_{k+1} - \tau_k), \quad k \geq 0,
\]

are independent. Moreover under \( \hat{P}_0[\cdot | A_l] \), the random variables \( Z_k, k \geq 1 \) have the same distribution as \( Z_0 \) under \( \hat{P}_0[\cdot | D = \infty] \).

**Proof.** We use induction over the index \( n \geq 0 \), of the filtration \( G_n \overset{\text{def}}{=} \sigma(Z_0, \ldots, Z_n) \). From Proposition 2.3.4 and the fact that \( G_0 \subset G \), cf. (2.3.17), we know that for any \( C \) in the product \( \sigma \)-algebra on \( C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}_+ \) and any bounded \( G_0 \)-measurable \( h_0 \):

\[
(2.3.21) \quad \hat{E}_0[1_{\{Z_1 \in C\}} h_0 | B_l] = \hat{E}_0[h_0 | B_l] \hat{P}_0[Z_0 \in C | D = \infty].
\]
2.3.3. Renewal structure and limit velocity

It follows that on $B_t$, $\tau_2$ is $\hat{P}_0$-a.s. finite, because $\hat{P}_0[\tau_2 < \infty \mid B_t] = \hat{P}_0[\tau_1 < \infty \mid D = \infty] = 1$ by Lemma 2.3.3 and (2.3.16). Assume now that for some $n \geq 1$, $\tau_n < \infty$, on $B_t$ and that for any $C$ as above and any bounded $\mathcal{G}_{n-1}$-measurable $h_{n-1}$:

$$(2.3.22) \quad \hat{E}_0[1_{\{Z_n \in C\}} \ h_{n-1} \mid B_t] = \hat{E}_0[h_{n-1} \mid B_t] \hat{P}_0[Z_0 \in C \mid D = \infty].$$

As above we see that $\hat{P}_0[\tau_{n+1} < \infty \mid B_t] = 1$. We will prove a similar identity as (2.3.22) with $(n + 1)$ in place of $n$. By the definition of $\tau_{n+1}$, $\mathcal{G}_n \cap \{\tau_1 < \infty\}$ is generated by a $\pi$-system consisting of intersections between events in $\mathcal{G}_0 \cap \{\tau_1 < \infty\}$ and $\theta_{\tau_1}^{-1} \mathcal{G}_{n-1}$. With Dynkin’s Lemma, see [14], p.447, it suffices therefore to consider bounded, $\mathcal{G}_n$-measurable functions $h_n$ satisfying

$$(2.3.23) \quad h_n = h_0 \cdot h_{n-1} \circ \theta_{\tau_1}, \quad \hat{P}_0$$

for some bounded, $\mathcal{G}_0$- respectively $\mathcal{G}_{n-1}$-measurable functions $h_0$ and $h_{n-1}$. Let us now prove the induction step with $h_n$ as in (2.3.23). By Proposition 2.3.4, we have for any $C$ as above:

$$
\hat{E}_0[1_{\{Z_{n+1} \in C\}} \ h_n \mid B_t] = \hat{E}_0[(h_{n-1}1_{\{Z_n \in C\}}) \circ \theta_{\tau_1} h_0 \mid B_t]
\quad = \frac{\hat{E}_0[h_0 \mid B_t]}{\hat{P}_0[D = \infty]} \hat{E}_0[1_{\{Z_n \in C\}} h_{n-1}1_{\{D = \infty\}}].

(2.3.24)
$$

Let us admit for the time being that

$$(2.3.25) \quad h_{n-1}1_{\{D = \infty\}} \text{ is indistinguishable from a } \mathcal{G}_{n-1}\text{-measurable variable}$$

and conclude the induction step. It follows from (2.3.22), (2.3.25) and the fact $\hat{P}_0$-a.s., $\{D = \infty\} \subset B_t$, cf. (2.3.16), that the left-hand side of (2.3.24) equals

$$
(2.3.26) \quad \hat{E}_0[h_0 \mid B_t] \hat{E}_0[h_{n-1} \mid D = \infty] \hat{P}_0[Z_0 \in C \mid D = \infty].
$$

Replacing $C$ with $C(\mathbb{R}_+, \mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}$, we obtain:

$$
\hat{E}_0[h_n \mid B_t] = \hat{E}_0[h_0 \mid B_t] \hat{E}_0[h_{n-1} \mid D = \infty].
$$

Inserting this into (2.3.26) yields:

$$
\hat{E}_0[1_{\{Z_{n+1} \in C\}} h_n \mid B_t] = \hat{E}_0[h_n \mid B_t] \hat{P}_0[Z_0 \in C \mid D = \infty].
$$
In other words, (2.3.22) holds with \((n + 1)\) in place of \(n\). Note that the induction argument shows that if \(P_0(A_l) > 0\), then \(P_0\)-a.s., \(B_l \subset \{\tau_k < \infty, \text{for all } k \geq 0\}\) and thus \(\hat{P}_0\)-a.s., \(B_l = A_l = \{\tau_k < \infty, \text{for all } k \geq 0\}\). (We will see later that in fact \(\hat{P}_0\)-a.s., \(A_l = \{\tau_1 < \infty\}\), if \(P_0(A_l) > 0\), cf. Proposition 2.3.6)

There remains to prove (2.3.25):

Observe that \(\hat{P}_0\)-a.s., \(\{D = \infty\} = \{\hat{T}_R = \infty\} = \{\hat{T}_R \geq \tau_1\}\). Further it is clear that the last event is included in \(\{D \geq \tau_1\}\), cf. (2.3.12). They are in fact equal \(\hat{P}_0\)-a.s., because the converse inclusions stems from the following facts:

\[\{D \geq \tau_1\} \cap \{\tau_1 = \infty\}\]

is a \(\hat{P}_0\) null-set by (2.3.16) and Lemma 2.3.3, and \(\{D \geq \tau_1\} \cap \{\tau_1 < \infty\} \subset \{\hat{T}_R > \tau_1 - 1\}\), \(\hat{P}_0\)-a.s.. But on \(\{\tau_1 < \infty\}\), \(\hat{P}_0\)-a.s., \(l \cdot X_{\tau_1 - 1 + s} \geq 2R\), for all \(s \geq 0\), by construction of \(\tau_1\). Therefore \(\{\hat{T}_R > \tau_1 - 1\} \subset \{\hat{T}_R = \infty\} \subset \{\hat{T}_R \geq \tau_1\}\), \(\hat{P}_0\)-a.s..

We thus see that \(\hat{P}_0\)-a.s., \(\{D = \infty\} = \{D \geq \tau_1\} = \{D \leq \tau_1 - 1\}\) which is \(\mathcal{G}_0\)-measurable and thus \(h_{n-1|D=\infty}\) is indistinguishable from a \(\mathcal{G}_{n-1}\)-measurable variable.

**Proposition 2.3.6. (weak zero-one law, \(d \geq 1\))**

For any \(l \in S^{d-1}\), \(P_0(A_l \cup A_{-l}) \in \{0, 1\}\). Moreover if \(P_0(A_l) > 0\), then \(\hat{P}_0\)-a.s., \(B_l = A_l = \{\tau_1 < \infty\}\), where \(B_l\) is defined in (2.3.15).

**Proof.** Assume that \(P_0(A_l) > 0\), and consider any \(L > 0\). Let \(H_k, k \geq 0\), be the iterates of \(H_{\mu(L)} \circ \theta_1 + 1\). We claim that \(P_0[H_k < \infty, \text{for all } k \geq 0] = 0\). Indeed, using the notations from Lemma 2.2.5, we see that

\[P_0[\{H_k < \infty, \text{for all } k \geq 0\} \cap B^c_l] \leq P_0 \left[ \bigcup_{v \in \mathbb{N}} \{\beta_k^{L-v} < \infty, \text{for all } k \geq 0 \text{ and } T_v = \infty\} \right] = 0.

From Theorem 2.3.5, we know that \(B_l = A_l, P_0\)-a.s. and therefore we find:

\[P_0[\{H_k < \infty, \text{for all } k \geq 0\} \cap B_l] = 0.

This proves the claim and as \(L\) is arbitrary, we see that \(P_0[\lim_{k \to \infty} |l \cdot X_l| = \infty] = 1\), and hence \(P_0(A_l \cup A_{-l}) = 1\), under the assumption \(P_0(A_l) > 0\).
The case where \( P_0(A_l) > 0 \) is treated analogously and the 0-1 law follows. Finally observe that under the assumption \( P_0(A_l) > 0 \), we have that \( \hat{P}_0 \)-a.s., \( \{\tau_1 < \infty\} = (\{\tau_1 < \infty\} \cap A_l) \cup (\{\tau_1 < \infty\} \cap A_{-l}) \), where the second set in the union is empty. Hence \( \{\tau_1 < \infty\} \subset A_l \), \( \hat{P}_0 \)-a.s.. The converse inclusion follows from Lemma 2.3.3.

The next proposition proves that \( P_0(A_l) > 0 \) implies that \( l \cdot X_{\tau_1} \) has a finite first moment under \( \hat{P}_0[\cdot \mid D = \infty] \). In the discrete i.i.d. setting where the renewal structure is technically less intricate, cf. for instance \([57]\), and under the assumption that \( l \) is a coordinate direction, one can show a stronger result, namely the equality \( E_0[l \cdot X_{\tau_1} \mid D = \infty] = P_0[D = \infty]^{-1} \), see for instance \([62]\), Lemma 3.2.5. Let us now give an outline of the argument we use. We find an \( L > 0 \) for which there is a positive lower bound, uniform in \( r > 0 \), for the annealed probability that the interval \([r, r + L]\) contains one of the \( l \cdot X_{\tau_m}, m \geq 1 \). This yields a positive lower bound on the linear growth in \( M \) of the expected number of renewal points \( l \cdot X_{\tau_m} \) smaller than \( M \). But by the elementary renewal theorem this linear growth coincides with \( \hat{E}_0[l \cdot X_{\tau_1} \mid D = \infty]^{-1} \). We thus obtain the desired upper bound on \( \hat{E}_0[l \cdot X_{\tau_1} \mid D = \infty] \). Whereas the construction of an \( L \) as above is relatively straightforward in the discrete setup, it is somewhat involved in the continuous setting because of the more delicate nature of the regeneration times. Let us incidentally point out that the use of the elementary renewal theorem bypasses the arithmeticity conditions of Blackwell’s renewal theorem used in \([62]\). This is an advantage when working with a general direction \( l \), (both in the discrete and continuous setups).

**Proposition 2.3.7.** Consider \( l \) in \( S^{d-1} \) and assume that \( P_0(A_l) > 0 \). Then there is a constant \( c_0 > 0 \) such that if \( L \) is large enough, for any \( r \geq 0 \) one has:

\[
\hat{P}_0[\text{for some } m \geq 1, \ l \cdot X_{\tau_m} \in [r, r + L] \mid A_l] > c_0 \quad \text{and} \\
\hat{E}_0[l \cdot X_{\tau_1} \mid D = \infty] \leq \frac{L}{c_0}.
\]

**Proof.** We first prove (2.3.27). Consider any \( r \geq 0, 0 < \delta < \frac{R}{m} \) and define \( T = T_{r+\delta} \), cf. (2.2.2). The heart of the matter is to construct an event \( E \), cf. (2.3.30) below, forcing the occurrence of some \( l \cdot X_{\tau_m}, m \geq 1 \), in an
interval. More precisely we will show that

\[ P_0\text{-a.s., on the event } E, \text{ some } l \cdot X_{\tau_m}, m \geq 1, \text{ belongs to } [r, r + 18R], \]

where \( E \) is defined as:

\[ E = \{ T < \infty, \sup_{s \in [T, [T] + 1]} |X_s - X_T| \leq \frac{R}{4}, \sup_{0 \leq s \leq 2} |X_s - X_0 - \psi(s) \circ \hat{\theta}_{[T]}| < \delta, \lambda_{[T] + 2} = 1, D \circ \hat{\theta}_{[T] + 3} = \infty \}, \]

and \( \psi : \mathbb{R}_+ \to \mathbb{R}^d \) is the function \( \psi(s) = \begin{cases} 5Rls, & s \leq 1, \\ 5Rl + (s - 1)\frac{R}{4}Rl, & 1 < s \leq 2. \end{cases} \)

Figure 3. A realisation of the event \( E \), cf. (2.3.30) and the corresponding speed of the trajectory.

The intuitive idea behind the construction of \( E \) is the following: in essence after first reaching level \( r + R/4 \) at time \( T \), the trajectory is forced -in the next unit of time after \( [T] \)- to move \( 5R \) “to the right” and -in the subsequent unit of time- to move an additional distance \( R \) “to the right”. Then either \( [T] \) coincides with a regeneration time, or as we will see, some time “of type V” (after suitable time shift, see (2.3.8)) occurs during the first interval \( [[T], [T] + 1] \). Rounding up this time to the next integer yields \( [T] + 1 \) and the constraints imposed on the trajectory during the second unit of time \( [[T] + 1, [T] + 2] \) as well as on the Bernoulli variables, ensure that \( [T] + 2 \) is “of type N”, cf. (2.3.10). Because of the no-backtracking condition in \( E \), \( [T] + 3 \) is then a regeneration time and we have a good control on how far “to the right” the trajectory has moved at that time.
We now proceed with the proof of (2.3.29). Let $\tau_m < [T], m \geq 0$, be the last regeneration time strictly before $[T]$, with $m = 0$ by convention when $[T] = 0$, which is a $\hat{P}_0$-negligible event. We define

$$k = \sup\{n \geq 0 : R_n \circ \hat{\theta}_{\tau_m} + \tau_m \leq [T]\},$$

see (2.3.11), (2.3.13) for the notations. On the event $E$, the following two cases can occur:

- Either $N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m < [T]$, then we claim that

$$N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m + 1 = \tau_{m+1} = [T].$$

Indeed, according to the definition (2.3.14), (2.19) of $\tau_{m+1}$, the first equality in (2.3.32) automatically holds if $R_{k+1} \circ \hat{\theta}_{\tau_m}$ is infinite. Assume by contradiction, that $R_{k+1} \circ \hat{\theta}_{\tau_m} < \infty$. By the definition of $k$, $R_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m > [T]$ and by the definition of $R_{k+1}$ (cf. (2.3.11)), the trajectory would have to return to level $u^* = I_{X^*} x$, $R$ strictly after time $[T]$. But under our assumption, $N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m + 1 < [T]$ and hence with the second condition in the definition of $E$, $u^* \leq l \cdot X[T] - \frac{3R}{4}$. On $E$ however, after time $[T]$, the trajectory always stays strictly above level $l \cdot X[T] - \frac{R}{2}$. This contradiction proves that $R_{k+1} \circ \hat{\theta}_{\tau_m}$ is infinite and hence the first equality of (2.3.32) follows. The second equality simply stems from the fact that $N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m + 1 < [T] \leq \tau_{m+1}$ in the considered case.

- Or $[T] \leq N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m$, then we first note that $[T] = N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m$ is $\hat{P}_0$-negligible as $\{\lambda[T] = 1\} \cap E$ is a $\hat{P}_0$ null-set by (2.3.6). We claim that

$$[T] + 3 = \tau_{m+1}.$$  

To see this, we first determine below a random time $\tilde{N} \leq [T]$ “of type $\tau, R$ or $\tilde{N}$”, serving as starting point for the construction of a new generation of stopping-times “of type $V^*$, cf. section 2.3.2. With $k$ as in (2.3.31), we define

$$\rho = R_k \circ \hat{\theta}_{\tau_m} + \tau_m, \quad j_\rho = \sup\{j \geq 0 : \tilde{N}_j \leq [T]\}, \quad \tilde{N}_j = \tilde{N}_j \circ \hat{\theta}_\rho + \rho, \quad j \geq 0,$$

$$\tilde{N}_j = \tilde{N}_j \circ \hat{\theta}_{\tilde{N}} + \tilde{N}$$

Since on $E$ the trajectory visits a new half plane once it reaches level $r + \frac{R}{2}$, there exists a smallest $i \geq 0$, such that $V \overset{\text{def}}{=} V_i(a) \circ \hat{\theta}_N + \tilde{N}$ (where $a$ equals
either $3R$ or $M(\rho) - X_\rho \cdot l + R$ according to the type of $N$, cf. (2.3.8), (2.3.9), (2.3.11), (2.3.13), satisfies:

$$[T] < V \leq (N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m) \wedge ([T] + 1) \text{ and } l \cdot X_{[T]} < l \cdot X_{V} < l \cdot X_{[T]} + 4R.$$ 

Note that $[V] = [T] + 1 \neq N_{j_0+1}$ because $l \cdot (X_{[V]} - X_V) > \frac{R}{2}$, see also (2.3.9). But the “next” $V$, namely $V' \overset{\text{def}}{=} V_{t+1}(a) \circ \hat{\theta}_N + N > [T] + 1$, is reached by definition of $E$ near level $l \cdot X_{[T]} + 6R$, and $[V']$ coincides with

$$N_{k+1} \circ \hat{\theta}_{\tau_m} + \tau_m = [T] + 2,$$

since $l \cdot (X_{[V']} - X_{V'}) \leq \frac{R}{2}$ and $\lambda_{[T]+2} = 1$. We obtain that $[T] + 3$ is the next regeneration time $\tau_{m+1}$, since on $E$ the trajectory never backtracks after $[T] + 3$. This proves (2.3.33).

So far we have shown (2.3.29) and there remains to prove that the probability $\hat{P}_0|E \mid A_l$ is bounded away from 0, independently of $r$. The claim (2.3.27) will then follow. To this end, we observe that:

$$\hat{P}_0[E \in A_l] = \sum_{n=0}^{\infty} E \hat{E}_0, [W] = n, \sup_{T \leq s \leq [W]} |l \cdot (X_s - X_T)| \leq \frac{R}{4},$$ 

$$\sup_{0 \leq s \leq 2} |X_s - X_0 - \psi(s)| \circ \hat{\theta}_n < \delta, \lambda_{n+2} = 1,$$

$$\hat{P}_{0, \omega}[D \circ \hat{\theta}_{n+3} = \infty, \hat{\theta}_{n+3}^{-1}(A_l) \mid Z_{n+2}].$$

With the Markov-property (2.3.4) as well as (2.3.6) and the first inclusion in (2.3.16), we find that for $\mathbb{P}$-a.e. $\omega$:

$$\hat{P}_{0, \omega}[D \circ \hat{\theta}_{n+3} = \infty, \hat{\theta}_{n+3}^{-1}(A_l) \mid Z_{n+2}] = \hat{P}_{X_{n+2}, \omega}[D \circ \hat{\theta}_1 = \infty]$$

$$= \frac{1}{|B(0, R)|} \int P_{y, \omega}[D = \infty] 1_{B_{x_{n+2}}}(y) dy.$$ 

We insert (2.3.35) into (2.3.34) and use the following facts:

- $\{\lambda_{n+2} = 1\}$ has probability $\epsilon$ and is independent of $\mathcal{F}_{n+2} \otimes S_{n+1}$, see (2.3.3) and (2.3.4).
2.3.3. Renewal structure and limit velocity

- \( \omega \rightarrow P_{0,\omega} [T] = n, \sup_{T \leq s \leq T} |l \cdot (X_s - X_T)| \leq \frac{R}{4}, \sup_{0 \leq s \leq 2} |X_s - X_0 - \psi(s)| \circ \theta_n < \delta, y \in B^{X_{n+2}} \) is \( \mathcal{H}_{\{z \in \mathbb{R}^d: z \leq y, l - 8R + \delta\}} \)-measurable, (recall the definition of \( B^x \), (2.3.1).)

- \( \omega \rightarrow P_{y,\omega} [D = \infty] \) is \( \mathcal{H}_{\{z \in \mathbb{R}^d: z \geq y, l - R\}} \)-measurable. Therefore by the finite range dependence property (2.1.5) both maps are \( \mathbb{P} \)-independent.

Moreover from Lemma 2.3.2, we have that \( P_0 [D = \infty] > c > 0 \) and hence we obtain that \( \hat{P}_0 [E \cap A_l] \) equals

\[
\frac{c}{|B(0, R)|} \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} dy \mathbb{E} P_{0,\omega} [T = n, \sup_{T \leq s \leq T} |l \cdot (X_s - X_T)| \leq \frac{R}{4}, \sup_{0 \leq s \leq 2} |X_s - X_0 - \psi(s)| \circ \theta_n < \delta, y \in B^{X_{n+2}} ] \cdot \mathbb{E} P_{y,\omega} [D = \infty] \\
\geq c c' \sum_{n=0}^{\infty} \mathbb{E} P_{0,\omega} [T = n, \sup_{T \leq s \leq T} |l \cdot (X_s - X_T)| \leq \frac{R}{4}, P_{X_n,\omega} [\sup_{0 \leq s \leq 2} |X_s - X_0 - \psi(s)| \leq \delta] ] \\
\geq c c' (\delta, \psi) P_0 [T < \infty] \geq c c' (\delta, \psi) P_0 (A_l),
\]

where the constant \( c' (\delta, \psi) \) stems from the support theorem (see [3], p.25). This proves (2.3.27).

We now prove (2.3.28). From Theorem 2.3.5, we know that \( l \cdot (X_{\tau_{m+1}} - X_{\tau_k}), m \geq 0, \) are independent under \( \hat{P}_0 [\cdot \mid A_l] \) and have for \( m \geq 1 \) the same distribution as \( l \cdot X_{\tau_1} \) under \( \hat{P}_0 [\cdot \mid D = \infty] \). Moreover \( \hat{P}_0 [\tau_1 < \infty \mid A_l] = 1 \). Thus the elementary renewal theorem in the delayed case (see [44] Theorem 3.3.3) can be applied, and yields:

(2.3.36)

\[
\hat{E}_0 [l \cdot X_{\tau_1} \mid D = \infty]^{-1} = \lim_{k \rightarrow \infty} \hat{E}_0 [\max\{m \geq 1 : l \cdot X_{\tau_m} \leq kL\} \mid A_l] \geq \frac{c_0}{L}.
\]

This proves (2.3.28). \( \square \)

We now turn to the main result in this section, that describes the limiting velocity of the diffusion process.
Theorem 2.3.8. (limit velocity, $d \geq 1$)
There exist a deterministic direction $l_* \in S^{d-1}$ and two numbers $v_+, v_- \geq 0$, such that

$$\lim_{t \to \infty} \frac{X_t}{t} = (v_+1_{A_{l_*}} - v_-1_{A_{-l_*}})l_*,$$

and $P(A_{l_*} \cup A_{-l_*}) \in \{0, 1\}$. (If this last quantity is 0, the velocity is 0 and thus the values of $v_+, v_-$ are immaterial.)

Proof. We first prove that for any fixed direction $l \in S^{d-1}$, there are non-negative numbers $v_l, v_{-l}$, such that

$$\lim_{t \to \infty} \frac{l \cdot X_t}{t} = v_l 1_{A_l} - v_{-l} 1_{A_{-l}}.$$

If $P_0(A_l \cup A_{-l}) = 0$, it follows from Corollary 2.2.6 that (2.3.38) holds with $v_l = v_{-l} = 0$. In view of the weak zero-one law, Proposition 2.3.6, we only have to consider the case $P_0(A_l \cup A_{-l}) = 1$. We assume without loss of generality that $P_0(A_l) > 0$. On $A_l$, $\hat{P}_0$-a.s., $\tau_k < \infty, k \geq 1$, cf. Theorem 2.3.5 and we define for $t > 0$, a non-decreasing, integer-valued function $k(t)$ tending to infinity $\hat{P}_0$-a.s., such that

$$\tau_k(t) \leq t < \tau_{k(t)+1},$$

with the convention $\tau_0 = 0$. Observe that on $A_l$, we have $\hat{P}_0$-a.s.,

$$\frac{l \cdot X_{\tau_k(t)}}{k(t)} - \frac{R}{\tau_{k(t)+1}} \leq \frac{l \cdot X_t}{t} \leq \frac{l \cdot X_{\tau_k(t)+1}}{k(t)+1} + \frac{3R}{\tau_k(t)}.$$

By (2.3.28), the iid structure of the increments $l \cdot (X_{\tau_k} - X_{\tau_k-1}), k \geq 2$ under $\hat{P}_0[\cdot | A_l]$ (see Theorem 2.3.5) and the usual law of large numbers, we find:

$$\lim_{k \to \infty} \frac{l \cdot X_{\tau_k}}{k} = \hat{E}_0[l \cdot X_{\tau_1} | D = \infty] < \infty, \quad \hat{P}_0[\cdot | A_l]-a.s..$$

Either $\hat{E}_0[\tau_1 | D = \infty] = \infty$, and then the positivity and the iid structure of the increments $\tau_k - \tau_{k-1}, k \geq 2$, see Theorem 2.3.5, imply that $\frac{1}{n} \sum_{k=1}^{n} \tau_k -$
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\( \tau_{k-1} \to \infty, \hat{P}_0[\cdot \mid A_i] \)-a.s. Passing to the limit in (2.3.39), we obtain in this case \( v_i = 0 \) in (2.3.38).

(2.3.41)

- Or \( \hat{E}_0[\tau_1 \mid D = \infty] < \infty \), then we obtain \( v_i = \frac{\hat{E}_0[\cdot \cdot \cdot \mid D = \infty]}{\hat{E}_0[\tau_1 \mid D = \infty]} > 0. \)

If \( P(A_{-i}) \) is also positive, then the same argument determines \( v_{-i} \), otherwise we set \( v_{-i} = 0 \). This proves (2.3.38).

Applying (2.3.38) to a basis of \( \mathbb{R}^d \), we obtain:

(2.3.42)

\[ X_t / t \to v, \quad P_0\text{-a.s.,} \]

where \( v \) is a random vector taking at most \( 2^d \) values.

In the next step we show that in fact \( v \) takes at most two parallel and opposite values. Indeed, assume that there are \( v_1, v_2 \) non-colinear, non-zero vectors with \( \hat{P}_0[v = v_i] > 0, i = 1, 2 \). Define \( e_i = \frac{v_i}{|v_i|}, i = 1, 2 \) and

\[ l_\alpha \overset{\text{def}}{=} \alpha e_1 + (1 - \alpha)e_2, \]

for \( \alpha \in (0, 1) \). From (2.3.42) and (2.3.38) we see that \( P_0\text{-a.s.,} \)

\[ v \cdot l_\alpha = v_{1\alpha} 1_{A_{l\alpha}} - v_{-1\alpha} 1_{A_{-l\alpha}}, \quad \text{for } \alpha \in (0, 1). \]

Therefore if for some \( \alpha \in (0, 1) \), \( l_\alpha \cdot v_i > 0 \), for \( i = 1, 2 \), then since \( \hat{P}_0[v = v_i] > 0 \), we find:

(2.3.43)

\[ l_\alpha \cdot v_1 = l_\alpha \cdot v_2. \]

If we can choose \( \alpha \) in a non-empty open interval such that \( l_\alpha \cdot v_i > 0, i = 1, 2 \) holds, we may take derivatives with respect to \( \alpha \) in (2.3.43) and deduce:

\[ 0 = (e_1 - e_2)(v_1 - v_2) = (e_1 - e_2)(|v_1|e_1 - |v_2|e_2) = (1 - e_1 \cdot e_2)(|v_1| + |v_2|). \]

By assumption, \( |e_1 \cdot e_2| < 1 \), which produces a contradiction. Let us check that indeed \( l_\alpha \cdot v_i > 0, i = 1, 2 \), is true for \( \alpha \) in a non-empty open interval.

\[ l_\alpha \cdot v_1 > 0 \iff l_\alpha \cdot e_1 > 0 \iff \alpha > \frac{-e_1 \cdot e_2}{1 - e_1 \cdot e_2}, \]

\[ l_\alpha \cdot v_2 > 0 \iff l_\alpha \cdot e_2 > 0 \iff \alpha < \frac{1}{1 - e_1 \cdot e_2}. \]
Both bounds define a non-empty open interval as $|c_1 \cdot c_2| < 1$. As a result, there is an $l_* \in S^{d-1}$ such that $P_0[v \in \mathbb{R}l_*] = 1$. The application of (2.3.38) with $l_*$ together with (2.3.42) finishes the proof.

\[\square\]

2.4 Zero-one Law when $d = 2$

In this section, we prove that in the two-dimensional case, for any direction $l$, $P_0(A_l)$ is either 0 or 1. Note that this result combined with Theorem 2.3.8 implies at once a law of large numbers, i.e. $\frac{X_t}{t}$ converges $P_0$-a.s. to a deterministic velocity, which is possibly 0. Our strategy is inspired by that of M.Zerner and F.Merkl in [64], where they proved an analogous zero-one law for random walks in two-dimensional iid environments. Note that Lemma 2.4.1 and the beginning of the proof of Theorem 2.4.2 are valid for all dimensions.

We will use the following notations: for every environment $\omega$, we consider two independent diffusions, called $X$ and $Y$. Stopping times with superscript 1 respectively 2 refer to $X$ resp. $Y$. We define for $\omega \in \Omega, x, y \in \mathbb{R}^d$ the product measure $P^\omega_{x,y} = P^\omega_x \times P^\omega_y$ as well as $P_{x,y} = \mathbb{E}P^\omega_{x,y}$. We recall that the first entrance time in a set $B$ is called $H_B$, cf. above (2.2.2). For every $\omega \in \Omega, x \in \mathbb{R}^d$ we write:

\[r(x,\omega) = P_{x,\omega}(l \cdot X_t \to \infty).\]

The basic idea is to first show that under the assumption $P_0(A_l \cup A_{-l}) = 1$, the two diffusions starting respectively in 0 and $y_L$, with $l \cdot y_L$ large, are unlikely to visit a same small ball located between their starting points, see Lemma 2.4.1. On the other hand, when $d = 2$, if we assume that $P_0(A_l)P_0(A_{-l}) > 0$, we can choose $y_L$ such that for large $L$, the two diffusions intersect "between 0 and $y_L$" with non vanishing probability, thus leading to a contradiction, see Theorem 2.4.2.

Lemma 2.4.1 relies on the fact that for every $\omega$, $r(X_t,\omega)$ and $r(Y_t,\omega)$ are $P_{x,\omega}$-martingales by the Markov-property, and they converge to $1_{A_l}$, $P_{x,\omega}$-a.s.. Loosely speaking, $X$ and $Y$ cannot meet in a region between their respective starting points if they are far apart, because $r(X_t,\omega)$ and $r(Y_t,\omega)$ would have to approach 1 respectively 0 in the same region.
Lemma 2.4.1. \((d \geq 1)\)

Consider \(l \in S^{d-1}\) and assume \(P_0(A_l \cup A_{-l}) = 1\). Then for any sequence \(y_L, L \geq 4R, \) satisfying \(l \cdot y_L \geq 3L\), we have

\[
\lim_{L \to \infty} P_{0, y_L}[\text{there exists } z \in S(L, l \cdot y_L - L) : H^1_B(z, R) < \infty, H^2_B(z, R) < \infty] = 0.
\]

**Proof.** The considered set in (2.4.2) is measurable because it suffices to consider a countable, dense subset of \(S(L, l \cdot y_L - L)\) in the union, since we use entrance times into open balls. For any integer \(L \geq 4R\), its probability is bounded from above by:

\[
P_0[\exists z \in S(L, l \cdot y_L - L) : H^1_B(z, R) < \infty, \sup_{y \in B(z, R)} r(y, \omega) < \frac{1}{2}]
\]

\[+ P_{y_L}[\exists z \in S(L, l \cdot y_L - L) : H^2_B(z, R) < \infty, \sup_{y \in B(z, R)} r(y, \omega) \geq \frac{1}{2}]. \tag{2.4.3}
\]

By Harnack’s inequality (see [17], p.250), we have \(\inf_{y \in B(z, R)} r(y, \omega) \geq c \sup_{y \in B(z, R)} r(y, \omega)\) and since \(P_0(A_l \cup A_{-l}) = 1\), the expression in (2.4.3) is smaller than

\[
P_0[\exists z \in S(L, l \cdot y_L - L) : H^1_B(z, R) < \infty, \sup_{y \in B(z, R)} r(y, \omega) < \frac{1}{2}, A_l]
\]

\[+ P_0[T_{L-R} < \infty, A_{-l}]
\]

\[+ P_{y_L}[\exists z \in S(L, l \cdot y_L - L) : H^2_B(z, R) < \infty, \inf_{y \in B(z, R)} r(y, \omega) \geq \frac{c}{2}, A_{-l}]
\]

\[+ P_{y_L}[\hat{T}_{L+R}^{rel} < \infty, A_l]. \tag{2.4.4}
\]

The second and last terms converge to 0 as \(L \to \infty\). The first term is smaller than

\[
P_0[H^1_{\{z \in S(L-R, l \cdot y_L - L + R) : r(z, \omega) \leq \frac{1}{2}\}} < \infty, A_l, T_{L-R} > \frac{L-R}{2K}]
\]

\[+ P_0[T_{L-R} \leq \frac{L-R}{2K}], \tag{2.4.5}
\]

where \(\hat{K}\), defined in (2.1.2), denotes a uniform bound on the drift. From the martingale convergence theorem we know that \(\lim_{t \to \infty} r(X_t, \omega) = 1_{A_l}, P_{0, \omega}-a.s..\)
This implies that the first term of (2.4.5) tends to 0 as \( L \to \infty \) since \( P_{0,\omega} \)-a.s.,

\[
\left\{ \frac{L-R}{2K} < H^1_{\{ z \in S(L-R, l \cdot y_L - L + R) : r\left( z, \omega \right) < \frac{1}{2} \}} \right\} < \infty, A_i \}
\]
\[\subset \{ \inf_{s \geq \frac{L-R}{2K}} r(X_s, \omega) < \frac{1}{2}, \lim_{s \to \infty} r(X_s, \omega) = 1 \} .
\]

The second term of (2.4.5) tends to 0 by Bernstein's inequality (see [3], Proposition 8.1, p.23). Using translation invariance, the third term in (2.4.4) is treated similarly.

\[\square\]

**Theorem 2.4.2.** \((d = 2)\). For any direction \( l \in S^1 \),

\[
(2.4.6) \quad P_0[A_i] \in \{0, 1\} .
\]

**Proof.** Assume by contradiction that \( P_0(A_i)P_0(A_{-i}) > 0 \). For any integer \( L \geq 4R \), we denote with \( \Gamma_L \) the probability in (2.4.2) and recall that \( R \) is defined in (2.1.5). We claim that there exists a sequence \( y_L \geq 3L \), with \( L \geq 4R \) such that

\[
(2.4.7) \quad \liminf_{L \to \infty} \Gamma_L > 0 .
\]

This with (2.4.2) yields a contradiction and Theorem 2.4.2 will follow. We already specify that \( y_L \cdot l = 3L + 22R \). The component orthogonal to \( l \) will be chosen in Lemma 2.4.3 below, see (2.4.29) and (2.4.32). In the first step we will use independence to separate the inner slab \( TS_L \overset{\text{def}}{=} S(L + \frac{R}{2}, l \cdot y_L - L - \frac{R}{2}) \) from the half-spaces \( \{ x \in \mathbb{R}^d : x \cdot l \leq L \} \) and \( \{ x \in \mathbb{R}^d : x \cdot l \geq L \cdot y_L - L \} \).

To achieve this, we use the coupling measure \( \hat{P}_x^\omega \) starting at \( x \) for the direction \( l \) on the enlarged path-space \( C(\mathbb{R}_+, \mathbb{R}^d) \times \{0, 1\}^N \), cf. section 2.3.1. For the direction \(-l\), we denote the coupling measure starting at \( y \) with \( \hat{P}_{y,\omega} \). We introduce the product measure \( \hat{P}_x^\omega = \hat{P}_x^\omega \times \hat{P}_{y,\omega} \). The Bernoulli-variables respectively associated with \( X \) and \( Y \) are called \( \lambda^1 \) and \( \lambda^2 \), and \( \hat{P}_x^\omega(\lambda^1 = 1) = \hat{P}_{y,\omega}(\lambda^2 = 1) = \epsilon \). For any \( L \geq 4R \), we define the events

\[
(2.4.8) \quad D^1 = \{ T^{rel,1}_L < \hat{T}^{rel,1}_R, \sup_{T^{rel,1}_L \leq s \leq [T^{rel,1}_L]} \left| X_s - X_{T^{rel,1}_L} \right| \leq \frac{R}{2} \} , \quad \text{and}
\]
\[
D^2 = \{ \hat{T}^{rel,2}_L < T^{rel,2}_R, \sup_{\hat{T}^{rel,2}_L \leq s \leq [\hat{T}^{rel,2}_L]} \left| Y_s - Y_{\hat{T}^{rel,1}_L} \right| \leq \frac{R}{2} \} ,
\]
and recall that \( T_{IS_L} \) denotes the exit time from \( IS_L \). For any \( L \geq 4R \), we have the following lower bound for \( \Gamma_L \) obtained by controlling the trajectories of \( X \) and \( Y \) in a symmetric way before we (almost surely) send them into the inner slab \( IS_L \) by requiring \( \lambda^1_{[T^1_L]}, \lambda^2_{[T^2_L]} \) to equal one, cf. (2.3.6):

\[
(2.4.9) \quad \Gamma_L \geq \mathbb{E}^{\hat{P}_{0,y_L}} \left[ D^1, \lambda^1_{[T^1_L]}, D^2, \lambda^2_{[T^2_L]} \right] = 1, \quad \exists z \in IS_L,
\]

\[
H^1_{B(z,R)} \circ \hat{\theta}_{[T^1_L]} + 1 < T^1_{IS_L} \circ \hat{\theta}_{[T^1_L]} + 1, \quad H^2_{B(z,R)} \circ \hat{\theta}_{[T^2_L]} + 1 < T^2_{IS_L} \circ \hat{\theta}_{[T^2_L]} + 1.
\]

With property (2.3.4), the latter expression equals

\[
(2.4.10) \quad \mathbb{E}^{\hat{P}_{0,y_L}} \left[ D^1, D^2, g(\omega, X_{[T^1_L]}, Y_{[T^2_L]}) \right],
\]

where for \( \omega \in \Omega, \ u, v \in \mathbb{R}^d \), we have defined

\[
(2.4.11) \quad g(\omega, u, v) = \mathbb{P}^L_{u,v} \left[ \exists z \in IS_L, H^1_{B(z,R)} \circ \hat{\theta}_{[T^1_L]} < T^1_{IS_L} \circ \hat{\theta}_{[T^1_L]} \right].
\]

Using the fact that under \( \hat{P}^1_{u,\omega} \), \( X_1 \) is uniformly distributed on the ball \( B^u = B(u + 9Rl, R) \), and accordingly under \( \hat{P}^1_{v,\omega} \), \( Y_1 \) is uniformly distributed on the ball \( B^v = B(v - 9Rl, R) \), cf. (2.3.6), we obtain from (2.4.11), for any \( \omega \in \Omega, \ u, v \in \mathbb{R}^d \),

\[
(2.4.12) \quad g(\omega, u, v) = \frac{1}{|B((0,0),R)^2|} \int \int h(\omega, x, y) 1_{x \in B^u} 1_{y \in B^v} \, dx \, dy,
\]

where for \( \omega \in \Omega, \ x, y \in \mathbb{R}^d \), we have defined

\[
(2.4.13) \quad h(\omega, x, y) = P^\omega_{x,y} \left[ \exists z \in IS_L, H^1_{B(z,R)} < T^1_{IS_L}, \ H^2_{B(z,R)} < T^2_{IS_L} \right].
\]

For \( x \in S(L + \frac{15}{2}R, L + \frac{21}{2}R) \) and \( y \in S(t_2, y - \frac{21}{2}R, t_1 y - L, y - \frac{15}{2}R) \), \( \omega \mapsto h(\omega, x, y) \) is \( H_{IS_L} \subset H_{S(t_1 x - R, t_1 y + 4R)} \) measurable. On the other hand, the map

\[
\omega \mapsto P^\omega_{0,y_L} \left[ D^1, D^2, x \in B^X_{[T^1_L]}, y \in B^Y_{[T^2_L]} \right]
\]

is \( H_{\{z \in \mathbb{R}^d : l \leq l \cdot x - 7R\} \cup \{z \in \mathbb{R}^d : l \cdot z \geq l \cdot y + 7R\}} \) measurable. Hence, when we insert
(2.4.12) into (2.4.10), finite range dependence (see (2.1.5)) yields:

\[ \Gamma_L \geq \frac{e^2}{|B(0, R)|^2} \times \int \int P_0[D^1, x \in B \chi_{[T^1_L]}] P_{y_L}[D^2, y \in B \chi_{[\tilde{T}^{rel,2}_L]}] \mathbb{E}h(\omega, x, y) \, dx \, dy, \]

where the double integral in fact is only over \( \tilde{S}(L+\frac{15}{2}R, L+\frac{21}{2}R) \times \tilde{S}(l, y_L-L-\frac{21}{2}R, l, y_L-L-\frac{15}{2}R) \). This stems from the definition of \( B^u \) and \( \tilde{B}^v \), and the fact that on the event \( D^1, L - \frac{R}{2} \leq l \cdot X_{[T^1_L]} \leq L + \frac{R}{2} \) and similarly on \( D^2 \) for \( Y_{[\tilde{T}^{rel,2}_L]} \). Observe that for any \( x \in \tilde{S}(L+\frac{15}{2}R, L+\frac{21}{2}R), y \in \tilde{S}(l, y_L-L-\frac{21}{2}R, l, y_L-L-\frac{15}{2}R) \), we have:

\[ \mathbb{E}h(\omega, x, y) = 1 - \mathbb{E}P^\omega_{x,y} \left[ \inf_{0 \leq s \leq T^1_{L,S_L}, 0 \leq t \leq T^2_{L,S_L}} |X_s - Y_t| \geq 2R \right]. \]

Using a discretisation with cubes of side-length \( \frac{R}{2\sqrt{d}} \) of the sets \( X_{[0, T^1_{L,S_L}]} \) and \( Y_{[0, T^2_{L,S_L}]} \) and with the help of finite range dependence, we see that (2.4.15) is larger than

\[ \tilde{h}(x, y) \overset{\text{def}}{=} P_x \times P_y \left[ \inf_{0 \leq s \leq T^1_{L,S_L}, 0 \leq t \leq T^2_{L,S_L}} |X_s - Y_t| < R \right]. \]

In view of (2.4.14) and (2.4.16), we have thus obtained the following lower bound for the initial probability: for any \( L \geq 4R \),

\[ \Gamma_L \geq \frac{e^2 c P_0[T_R = \infty] P_0[T^1_L = \infty]}{|B(0, R)|^2} \times \int \int \mu^+_L(B(x - 9lR, R)) \mu^-_L(B(y - y_L + 9lR, R)) \tilde{h}(x, y) \, dx \, dy, \]

(2.4.18)

where \( \mu^+_L(\cdot) = P_0[X_{[T^1_L]} \in \cdot | D^1] \) and \( \mu^-_L(\cdot) = P_0[Y_{[T^2_{L,S_L}]} \in \cdot | D^2] \),

with \( D^1, D^2 \) defined in (2.4.8) and where the positive constant \( c \) is a lower bound for \( P_{z,\omega} \left[ \sup_{T_L \leq s \leq [T_L]} |X_s - X_{T_L}| \leq \frac{R}{2} \right] P_{z,\omega} \left[ \sup_{\tilde{T}_L \leq s \leq [\tilde{T}_L]} |X_s - X_{\tilde{T}_L}| \leq \frac{R}{2} \right], \) stemming from the support theorem (see [3], p.25).

The conclusion of the proof relies on the following lemma.
Lemma 2.4.3. \((d = 2)\). If \(P_0(A_i)P_0(A_{-i}) > 0\), then there exists \(p \in (0, 1)\), such that for any integer \(L \geq 4R\), there are two measurable sets \(A^+, A^- \subset \mathbb{R}^2\) and a point \(y_L \in \mathbb{R}^2\), with \(l \cdot y_L \geq 3L\), for which:

\[
(2.4.19) \quad h(x, y) \geq p \quad \text{whenever } x \in A^+ \text{ and } y \in A^-; \quad \text{and}
\]

\[
(2.4.20) \quad \int_{A^+} \mu_L^+(B(x-9lR, R)) \, dx > p, \quad \int_{A^-} \mu_L^-(B(y-y_L+9lR, R)) \, dy > p.
\]

\(\mu_L^+, \mu_L^-\) are defined in (2.4.18).

Proof. Choose \(e_2 \in S^1\) with \(e_2 \cdot l = 0\). Let \(a_k \in \mathbb{R}, k = 1, 2, 3\), be respective \(k^4\) quantiles of the “second marginal” of \(\mu_L^+\), chosen to be the smallest number such that \(\mu_L^+(l + (-\infty, a_k) e_2) \geq \frac{k}{4}\). Let \(b_k \in \mathbb{R}, k = 1, 2, 3\), be the corresponding quantiles for \(\mu_L^-\). Define \(A_k = [a_{k-1}, a_k], B_k = [b_{k-1}, b_k], k = 2, 3\). Choose \(i, j \in \{2, 3\}\) such that \(|A_i| = \min(|A_2|, |A_3|), |B_j| = \min(|B_2|, |B_3|)\). We define for integer \(L \geq 4R\),

\[
(2.4.21) \quad A^+ = [L + \frac{15}{2}R, L + \frac{21}{2}R]l + [a_{i-1} - U, a_i + U] e_2,
\]

\[
(2.4.22) \quad A^- = [-L - \frac{21}{2}R, -L - \frac{15}{2}R]l + [b_{j-1} - U, b_j + U] e_2 + y_L,
\]

where \(U = \frac{\sqrt{9}}{2}R\) (half the side-length of a square fitting into a ball of radius \(R\)) and where we recall that \(y_L \cdot e_1 = 3L + 22R\). The component \(y_L \cdot e_2\) will be chosen below (2.4.29). It is easy to check that:

\[
(2.4.23) \quad \int_{A^+} \mu_L^+(B(x-9lR, R)) \, dx 
\geq 4U^2 \mu_L^+(|L - \frac{9}{2}R + U, L + \frac{9}{2}R - U|l + [a_{i-1}, a_i] e_2) \geq U^2,
\]

where we recall that the first marginal of \(\mu_L^+\) is supported by \(\mathcal{S}(L - \frac{R}{2}, L + \frac{R}{2})\). The same lower bound holds for \(\int_{A^-} \mu_L^-(B(y-y_L+9lR, R)) \, dy\). This proves (2.4.20).

We next show (2.4.19). Adding the following two inequalities \(a_3 - a_1 \geq 2|A_i|, b_3 - b_1 \geq 2|B_j|\) yields \((a_3 + b_3) + (-a_1 - b_1) \geq 2|A_i| + 2|B_j|\). Therefore at least one of the two following inequalities must hold:

\[
(2.4.24) \quad a_3 + b_3 \geq |A_i| + |B_j|, \quad \text{case I},
\]

\[
(2.4.25) \quad a_1 + b_1 \leq -(|A_i| + |B_j|), \quad \text{case II}.
\]
Let us now examine case I. We derive a lower bound for $\tilde{h}(x, y)$ defined in (2.4.16) by producing a crossing of the trajectories of $X$ and $Y$ in a way that brings into play $D^1$ and $D^2$ (defined in (2.4.8)). This allows us to use the measures $u^+_L, u^-_L$ and their quantiles to estimate the crossing probability. For $L \geq 4R$, $x \in \mathcal{S}(L + \frac{15}{2}R, l_y)$ and $y \in \mathcal{S}(l_y - \frac{21}{2}R, l_y - L - \frac{15}{2}R)$:

\begin{equation}
\tilde{h}(x, y) \geq \mathbb{P}_x \times \mathbb{P}_y \left[ \begin{array}{ll}
D^1, X_{[T^\cdot \text{rel.1}, \cdot]} \cdot e_2 > y \cdot e_2 + R, \\
\sup_{s \leq 1} |X_s - X_0 - 9lRs| \circ \theta_{[T^\cdot \text{rel.1}]} \leq \frac{R}{2}, \\
D^2, Y_{[T^\cdot \text{rel.2}, \cdot]} \cdot e_2 > x \cdot e_2 + R, \\
\sup_{s \leq 1} |Y_s - Y_0 - (-9lRs)| \circ \theta_{[T^\cdot \text{rel.2}]} \leq \frac{R}{2} \end{array} \right].
\end{equation}

Indeed, on the above event (see Figure 4), the set $\mathcal{H}S_x \overset{\text{def}}{=} [L + \frac{13}{2}R, L + \frac{21}{2}R]l + (-\infty, e_2 \cdot x + \frac{R}{2}]e_2$ is connected to the line $\{z \in \mathbb{R}^2 : l \cdot z = l \cdot y_y - l - \frac{13}{2}R\}$ by a part of the trajectory of $X$, that leaves the slab $\mathcal{I}S_L = \mathcal{S}(L + \frac{13}{2}R, y_y - l - \frac{13}{2}R)$ through the “right” boundary without entering the set $\mathcal{H}S_y \overset{\text{def}}{=} [y_y - l - \frac{21}{2}R, y_y - l - \frac{13}{2}R]l + (-\infty, e_2 \cdot y + \frac{R}{2}]e_2$ containing $y$. This part of the trajectory divides the set $\mathcal{I}S_L \setminus \mathcal{H}S_x$ and gives rise to two connected, unbounded components, the lower one containing $y$. As the trajectory of $Y$ leaves the slab $\mathcal{I}S_L$ through the “left” boundary without
entering \( \mathcal{HS}_x \), it has to intersect the part of the \( X \)-trajectory separating the two connected components.

So we can bound \( \bar{h}(x, y) \) using the conditional measures \( \mu_L^+, \mu_L^- \). Indeed with the support theorem (see [3], p.25) and translation invariance, it follows from (2.4.26) that

\[
\bar{h}(x, y) \geq c P_0[T_{\tilde{R}}^{rel} = \infty] P_0[T_{\tilde{R}}^{rel} = \infty] \\
\times \mu_L^+\left( \mathbb{R} l + ((y - x)e_2 + R, \infty) e_2 \right) \\
\times \mu_L^-\left( \mathbb{R} l + ((x - y)e_2 + R, \infty) e_2 \right) \\
\times \mu_L^+\left( \mathbb{R} l + ((y - x)e_2 + R, \infty) e_2 \right) \\
\times \mu_L^-\left( \mathbb{R} l + ((x - y)e_2 + R, \infty) e_2 \right)
\]

If we choose \( y_L \cdot e_2 \) such that for all \( x \in A^+, y \in A^- \),

(2.4.27) \( (y - x) \cdot e_2 \leq a_3 + 2U \overset{\text{def}}{=} \tilde{a}_3 \), and \( (x - y) \cdot e_2 \leq b_3 + 2U \overset{\text{def}}{=} \tilde{b}_3 \),

then we obtain for all \( x \in A^+, y \in A^- \):

\[
\bar{h}(x, y) \geq \rho \mu_L^+\left( \mathbb{R} l + (\tilde{a}_3 + R, \infty) e_2 \right) \\
\times \mu_L^-\left( \mathbb{R} l + (\tilde{b}_3 + R, \infty) e_2 \right),
\]

with \( \rho = c P_0[T_{\tilde{R}}^{rel} = \infty] P_0[T_{\tilde{R}}^{rel} = \infty] > 0 \), by Lemma 2.3.2.

It remains to be checked that (2.4.27) is possible for suitable \( y_L \cdot e_2 \) and that

\[
(2.4.28) \quad \begin{align*}
\text{i) } & \liminf_{L \to \infty} \mu_L^+\left( \mathbb{R} l + (\tilde{a}_3 + R, \infty) e_2 \right) > 0 \\
\text{ii) } & \liminf_{L \to \infty} \mu_L^-\left( \mathbb{R} l + (\tilde{b}_3 + R, \infty) e_2 \right) > 0.
\end{align*}
\]

We first see from (2.4.21) and (2.4.22) that (2.4.27) is satisfied for all \( x \in A^+, y \in A^- \) if

(2.4.29) \( y_L \cdot e_2 + b_j + U - a_{i-1} + U \leq \tilde{a}_3 \), and \( a_i + U - y_L \cdot e_2 - b_{j-1} + U \leq \tilde{b}_3 \).

Hence we have to choose \( y_L \cdot e_2 \) in \([a_i - b_{j-1} - b_3, a_{i-1} - b_j + a_3]\), which is possible since \( (a_i - a_{i-1}) + (b_j - b_{j-1}) \leq a_3 + b_3 \) in case I, cf. (2.4.24).

Finally let us check (2.4.28). For any \( L \geq 4R \), we have, cf. (2.4.18):

(2.4.30) \( \mu_L^+\left( \mathbb{R} l + (\tilde{a}_3 + R, \infty) e_2 \right) \geq c P_0[X_{T_L} \cdot e_2 \geq \tilde{a}_3 + \frac{3R}{2}, T_L < \tilde{T}_{-R}],
\]

using the support theorem and the strong Markov-property. The function \( x \mapsto P_x, \omega[X_{T_L} \cdot e_2 \geq \tilde{a}_3 + \frac{3R}{2}, T_L < \tilde{T}_{-R}] \) is \( \mathcal{L}_\omega \)-harmonic in the box
2. Limit velocity and zero-one laws

\((-\frac{3R}{4}, \frac{3R}{4})l + (-R, 2U + 3R)e_2\). Thus, Harnack’s inequality (see [17], p.250) implies, that for some constant \(c > 0\), the left-hand side of (2.4.30) is bigger than

\[
(2.4.31) \quad c\mathbb{E}P_{(2U+2R)e_2, \omega} [X_{T_L} \cdot e_2 \geq \tilde{a}_3 + \frac{3R}{2}, T_L < \tilde{T}_R] \quad \text{transl. inv.}
\]

\[
= cP_0 [X_{T_L} \cdot e_2 \geq a_3 - \frac{R}{2}, T_L < \tilde{T}_R] \geq cP_0 [X_{[T_L]} \cdot e_2 \geq a_3 | D^1] P_0 [D^1],
\]

and finally the support theorem and the definition of \(a_3\) yield:

\[
\mu_L^+ \left( \mathbb{R}l + (\tilde{a}_3 + R, \infty) e_2 \right) \geq cP_0 [\tilde{T}_R = \infty]. \quad \text{Lemma 2.3.2} \quad 0.
\]

This proves (2.4.28) i). We show (2.4.28) ii) in the same way. In case II (cf. (2.4.25)), crossings are produced by requiring instead \(X_{T^{rel,1}_L} e_2 < y \cdot e_2 - R\) and \(Y_{T^{rel,2}_L} \cdot e_2 < x \cdot e_2 - R\) in (2.4.26). Moreover \(y_L \cdot e_2\) has to be chosen in such a way that for all \(x \in A^+, y \in A^-:\)

\[
(2.4.32) \quad (y - x) \cdot e_2 \geq a_1 - 2U, \quad \text{and} \quad (x - y) \cdot e_2 \geq b_1 - 2U.
\]

These conditions are satisfied when \(y_L \cdot e_2 \in [a_1 + a_i - b_j, a_i - b_j - b_1]\), which is non-empty under (2.4.25). The rest of the argument has to be adjusted accordingly. This finishes the proof of (2.4.19). \(\square\)

We have now proved (2.4.7) and as noted before Theorem 2.4.2 follows. \(\square\)

2.5 Appendix A

We now give the proof of Lemma 2.3.3.

\textbf{Proof.} Define the event \(\Delta_0 = \{\sup_{0 \leq s \leq 1} |l \cdot (X_s - X_0)| > \frac{R}{2}\}\). The support theorem ([3], p.25) shows that there is a constant \(c > 0\), such that \(P_{x, \omega}(\Delta_0) < 1 - c\), for all \(x \in \mathbb{R}^d, \omega \in \Omega\).

On the event \(B_l\), cf. (2.3.15), for any \(a > 0\), all the stopping times \(V_k(a), k \geq 1\), are finite (recall (2.3.8)). For simplicity, we drop \(a\) from the notation. Define \(\Delta_k = \{V_k < \infty, \sup_{V_k \leq s \leq V_k} |l \cdot (X_s - X_{V_k})| > \frac{R}{2}\}\). On the event \(B_l\),
\( \tilde{N}_1 \) is finite \( P_0 \)-a.s., because for \( n \) tending to infinity,

\[
P_0 \left[ \bigcap_{k=1}^{n} \Delta_k \right] \leq \mathbb{E} E_{0, \omega} \left[ \prod_{k=1}^{n-1} 1_{\Delta_k} P_{X_{V_n}, \omega} \left[ \Delta_0 \right] \right] \leq (1 - c)^n n^{-\infty} \to 0.
\]

With the help of the strong Markov property, we obtain iteratively:

\[ P_0 [\tilde{N}_k < \infty, \text{ for all } k \geq 1 \mid B_I] = 1. \]

The next step is to observe that on the event \( B_I \), \( N_1 \) is finite \( \hat{P}_0 \)-a.s. Indeed, for any \( n \geq 1 \) using independence of \( \lambda_j \) and \( \mathcal{F}_j \otimes \mathcal{S}_j \) with respect to \( \hat{P}_{x, \omega} \), cf. (2.3.4), we obtain:

\[
P_0 \left[ B_I \cap \{ N_1 = \infty \} \right] \leq \hat{P}_0 [\tilde{N}_m < \infty, \lambda_{\tilde{N}_m} = 0, \text{ for all } m \leq n]
\]

\[
= \sum_{j \in \mathbb{N}} \hat{P}_0 \left[ \tilde{N}_m < \infty, \lambda_{\tilde{N}_m} = 0, \text{ for all } m \leq n - 1, \tilde{N}_n = j, \lambda_j = 0 \right] \in \mathcal{F}_j \otimes \mathcal{S}_{j-1}
\]

\[
\leq (1 - \epsilon)^n \to 0, \text{ as } n \to \infty.
\]

Again, by the strong Markov property, we see that on the event \( B_I \), if \( R_k < \infty \) then \( N_{k+1} = N_1(a_k) \circ \theta_{R_k} + R_k \) is finite. (\( a_k \) is not time-shifted in the formula for \( N_{k+1} \) (recall (2.3.13)). The assumption \( P_0(A_I) > 0 \) and Lemma 2.3.2 ensure that \( P_0(D = \infty) > 0 \). In the next step we show that since \( P_0(D = \infty) > 0 \), the path cannot backtrack a distance \( R \) after time \( N_k + 1 \) for every \( k \geq 1 \):

\[
\hat{P}_0 \left[ \{ R_k < \infty \} \cap B_I \right] \leq \hat{P}_0 \left[ N_k < \infty, D \circ \theta_{N_k+1} < \infty \right]
\]

(2.5.1)

\[
= \mathbb{E} \hat{P}_{0, \omega} \left[ N_k < \infty, \hat{P}_{X_{N_k}, \omega}^{1} \hat{P}_{X_{1}, \omega}^{2} [D < \infty] \right].
\]

The last equality follows from (2.3.4). From (2.3.6), we see that for any \( x \in \mathbb{R}^d, \omega \in \Omega \):

\[
\hat{P}_{x, \omega}^{1} [\hat{P}_{X_{1}, \omega}^{2} [D < \infty]] = \frac{1}{|B(0, R)|} \int_{B^{\times}} P_{y, \omega} [D < \infty] dy.
\]

Inserting this expression into (2.5.1), we find that for \( k \geq 1 \):

(2.5.2) \[
\hat{P}_0 \left[ \{ R_k < \infty \} \cap B_I \right] \leq \frac{1}{|B(0, R)|} \int \mathbb{E} \left[ \hat{P}_{0, \omega} \left[ N_k < \infty, y \in B^{X_{N_k}} \right] P_{y, \omega} [D < \infty] \right] dy.
\]
The random variable \( \omega \mapsto \hat{P}_0[\tau_1 < \infty \mid B_t] \) is measurable with respect to \( \mathcal{H}_{\{z : z \leq y \cdot t - 4R\}} \), because of [49], equation (3) therein, and the fact that for any \( m \geq 1 \), there is a \( U_m \in \mathcal{F}_m \otimes \mathcal{S}_{m-1} \), with \( U_m \subset \{ \sup_{t \leq m} l \cdot X_t \leq l \cdot y - 7R \} \), such that \( \{ N_k < \infty, y \in B^{X^k} \} = \bigcup_{m \geq 1} U_m \cap \{ \lambda_m = 1 \} \).

The random variable \( \omega \mapsto P_{y,\omega}[D < \infty] = 1 - P_0[ \omega[D = \infty] \) is measurable w.r.t. \( \mathcal{H}_{\{z : z \geq y \cdot t - R\}} \).

Thus we can use the finite range dependence property (2.1.5) and obtain:

\[
\hat{P}_0[\{R_k < \infty\} \cap B_t] \leq \hat{P}_0[N_k < \infty | P_0[D < \infty] \\
\leq \hat{P}_0[R_{k-1} < \infty | P_0[D < \infty] \overset{\text{induction}}{\leq} P_0[D < \infty]^k \to 0, \text{ as } k \to \infty.
\]

We conclude that \( \hat{P}_0[\text{for some } j \geq 1 : N_j < R_j = \infty \mid B_t] = 1 \), or in other words: \( \hat{P}_0[\tau_1 < \infty \mid B_t] = 1 \). \(\square\)
Chapter 3

An effective criterion and a new example for ballistic diffusions in random environment

Abstract: In the setting of multi-dimensional diffusions in random environment, we carry on the investigation of condition $(T')$, introduced by Sznitman in [54] and by Schmitz in [46] respectively in the discrete and continuous setting, and which implies a law of large numbers with non vanishing limiting velocity (ballistic behaviour) as well as a central limit theorem. Specifically, we show that when $d > 2$, $(T')$ is equivalent to an effective condition that can be checked by local inspection of the environment. When $d = 1$, we prove that condition $(T')$ is merely equivalent to almost sure transience. As an application of the effective criterion, we show that when $d \geq 4$ a perturbation of Brownian motion by a random drift of size at most $\epsilon > 0$ whose projection on some direction has expectation bigger than $\epsilon^{2-\eta}, \eta > 0$, satisfies condition $(T')$ when $\epsilon$ is small and hence exhibits ballistic behaviour. This class of diffusions contains new examples of ballistic behaviour which in particular do not fulfill the condition in [46], (5.4) therein, related to Kalikow’s condition, see [57].
3. Effective criterion and new example

3.1 Introduction

Diffusions in random environment have emerged about twenty five years ago from homogenization theory in the study of disordered media, see for instance [4]. Within the rich field of "random motions in random media", they are closely related to the discrete model of "random walks in random environment", see [35], [62].

In the one-dimensional discrete setting a complete characterization of ballistic behaviour which refers to the situation where the motion tends to infinity in some direction with non vanishing velocity was established already in 1975 by Solomon, [50], see also [22], [45]. In the multidimensional setting however, such a characterization has not been found yet, but a lot of progress has been made over the last seven years: the so called conditions (T') and (T'') introduced by Sznitman, see [54], [55], for random walks in random environment seem to be promising candidates for an equivalent description of ballistic behaviour when the space dimension \( d \geq 2 \). In essence, one possible formulation of condition (T'), see (3.1.12) requires exponential decay of the probability that the trajectory exits a slab of growing width through one side rather than the other. These conditions have interesting consequences such as a ballistic law of large numbers and a central limit theorem. Their analogues in the setting of diffusions have been developed by Schmitz, see [46], [47], and he used a previous result of Shen, [48], to show that they imply the same asymptotic behaviour as mentioned before in the discrete setup. The drawback of the definitions of conditions (T) or (T') as they were stated in [46], see (3.1.10), is their asymptotic nature which makes them difficult to check by local considerations. To remedy this problem, we provide in the first part of this article an effective criterion, in the spirit of [55], which is equivalent to (T'), see Theorem 3.2.6, and which can be checked by inspection of the environment in a finite box.

In the second part of this work, which is related to [56] in the discrete setting, we use the effective criterion to show that when \( d \geq 4 \), Brownian motion perturbed with a small random drift satisfying the assumption (3.1.16), fulfills condition (T'), see Theorem 3.3.1. As we will see below, this class of diffusions contains new examples for ballistic behaviour beyond prior knowledge.
Before we discuss our results any further, we first describe the model. The random environment is specified by a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) on which acts a jointly measurable group \(\{t_x; x \in \mathbb{R}^d\}\) of \(\mathbb{P}\)-preserving transformations, with \(d \geq 1\). The diffusion matrix and the drift of the diffusion in random environment are stationary functions \(a(x, \omega), b(x, \omega), x \in \mathbb{R}^d, \omega \in \Omega\), with respective values in the space of non-negative \(d \times d\) matrices and in \(\mathbb{R}^d\), i.e.,

\[
(a(x+y, \omega) = a(x, t_y \omega), \quad b(x+y, \omega) = b(x, t_y \omega), \quad \text{for } x, y \in \mathbb{R}^d, \omega \in \Omega.
\]

We assume that these functions are bounded and uniformly Lipschitz i.e. there is a \(\bar{K} > 1\), such that for \(x, y \in \mathbb{R}^d, \omega \in \Omega\),

\[
|b(x, \omega)| + |a(x, \omega)| \leq \bar{K}, \quad |b(x, \omega) - b(y, \omega)| + |a(x, \omega) - a(y, \omega)| \leq \bar{K}|x-y|,
\]

where \(|\cdot|\) denotes the Euclidean norm for vectors and matrices. Further, we assume that the diffusion matrix is uniformly elliptic, i.e. there is a \(\nu > 1\) such that for all \(x, y \in \mathbb{R}^d, \omega \in \Omega\):

\[
\frac{1}{\nu}|y|^2 \leq y \cdot a(x, \omega)y \leq \nu|y|^2.
\]

The coefficients \(a, b\) satisfy a condition of finite range dependence: for \(A \subset \mathbb{R}^d\), we define

\[
\mathcal{H}_A = \sigma(a(x, \cdot), b(x, \cdot); x \in A),
\]

and assume that for some \(R > 0\),

\[
\mathcal{H}_A \text{ and } \mathcal{H}_B \text{ are independent under } \mathbb{P} \text{ whenever } d(A, B) \geq R,
\]

where \(d(A, B)\) is the mutual Euclidean distance between \(A\) and \(B\). With the above regularity assumptions on \(a\) and \(b\), for any \(\omega \in \Omega, x \in \mathbb{R}^d\), the martingale problem attached to \(x\) and the operator

\[
\mathcal{L}_\omega = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\cdot, \omega) \partial^2_{ij} + \sum_{i=1}^{d} b_i(\cdot, \omega) \partial_i
\]

is well posed, see [52] or [3], page 130. The corresponding law \(P_{x, \omega}\) on \(C(\mathbb{R}_+; \mathbb{R}^d)\), unique solution of the above martingale problem, describes the diffusion in the environment \(\omega\) and starting from \(x\). We write \(E_{x, \omega}\) for the
expectation under $P_{x,\omega}$ and we denote the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$ with $(X_t)_{t \geq 0}$. Observe that $P_{x,\omega}$ is the law of the solution of the stochastic differential equation

\begin{equation}
\begin{cases}
  dX_t = \sigma(X_t, \omega)d\beta_t + b(X_t, \omega)dt, \\
  X_0 = x, P_{x,\omega}\text{-a.s.},
\end{cases}
\end{equation}

where for instance $\sigma(\cdot, \omega)$ is the square root of $a(\cdot, \omega)$ and $\beta$ is some $d$-dimensional Brownian motion under $P_{x,\omega}$. The laws $P_{x,\omega}$ are usually called “quenched laws” of the diffusion in random environment. To restore translation invariance, we consider the so-called “annealed laws” $P_x$, $x \in \mathbb{R}^d$, which are defined as semi-direct products:

\begin{equation}
P_x \overset{\text{def}}{=} P \times P_{x,\omega}.
\end{equation}

Of course the Markov property is typically lost under the annealed laws.

We now come back to the object of this work. We start by recalling the definition of conditions $(T)$ and $(T')$ as stated in [46]. These conditions are expressed in terms of another condition $(T)_\gamma$ defined as follows. For a unit vector $\ell$ of $\mathbb{R}^d$, $d \geq 1$ and any $u \in \mathbb{R}$, consider the stopping times

\begin{equation}
T_u^\ell = \inf\{t \geq 0; X_t \cdot \ell \geq u\}, \quad \hat{T}_u^\ell = \inf\{t \geq 0; X_t \cdot \ell \leq u\}.
\end{equation}

For $\gamma \in (0, 1]$, we say that condition $(T)_\gamma \ell$ holds relative to $\ell$, in shorthand notation $(T)_\gamma | \ell$ if for all unit vectors $\ell'$ in some neighbourhood of $\ell$ and for all $b > 0$,

\begin{equation}
\limsup_{L \to \infty} L^{-\gamma} \log P_0 \left[ \hat{T}_{bL}^{\ell'} < T_L^{\ell'} \right] < 0.
\end{equation}

Condition $(T')$ relative to $\ell$ is then the requirement that

\begin{equation}
(3.1.10) \quad \text{holds for all } \gamma \in (0, 1),
\end{equation}

and condition $(T)$ relative to $\ell$ refers to the case where

\begin{equation}
(3.1.12) \quad \text{holds for } \gamma = 1.
\end{equation}

It is clear that $(T)$ implies $(T')$ and we show in Theorem 3.2.6 that $(T')$ is equivalent to $(T)_\gamma$ when $\gamma \in (\frac{1}{2}, 1)$. Moreover, it is conjectured that the conditions $(T)_\gamma, \gamma \in (0, 1]$ are all equivalent.
Let us also mention that for $\gamma \in (0,1]$, condition $(T)_\gamma$ relative to $\ell$ is in essence equivalent to almost sure transience in direction $\ell$ together with finiteness of a stretched exponential moment of the size of the trajectory up to a certain regeneration time, see [46], Theorem 3.1 therein or [55] for a similar result in the discrete setting. The latter formulation of condition $(T')$ is especially appropriate to study the asymptotic properties of the diffusion. Indeed Schmitz showed in [46], Theorem 4.5, (see also [54]) that when $d \geq 2$, it enables us to verify the sufficient conditions of [48] for a ballistic law of large numbers and a central limit theorem. However, the more geometrical expression (3.1.10) is better suited for our present purpose.

Despite the interest of the two above mentioned formulations of condition $(T')$, they are not “effective conditions” that can be checked by local inspection of the environment. Concrete examples where $(T')$ holds besides the easy case where the projection of the drift on some unit vector is uniformly bounded away from 0, see [46] Proposition 5.1, originate from a stronger condition going back to Kalikow, see [23], [57]. For instance, it is shown in [46], Theorem 5.2, and [47], Theorem 2.1 that there exists a constant $c_\epsilon > 0$ depending only on $K, \nu, R, d$ [see (3.1.2)-(3.1.5)], such that condition $(T)$ holds when

\[ \mathbb{E}[(b(0, \omega) \cdot \ell)_+ - c_\epsilon \mathbb{E}[(b(0, \omega) \cdot \ell)_-]]. \]

In the first part of this work we derive an effective criterion in the above sense. We show [see Theorem 3.2.6] that when $d \geq 2$ for any direction $\ell$, $(T')|\ell$ is in essence equivalent to

\[ \inf_{B, \alpha \in (0,1]} \{c(d)\bar{L}^{d-1}L^{3(d-1)+1}\mathbb{E}[^{\alpha}B] \} < 1, \text{ with } \]

\[ \rho_B = \frac{P_{0,\omega}[X_{T_B} \notin \partial_B]}{P_{0,\omega}[X_{T_B} \in \partial_B]}, \]

provided in the above infimum, $B$ runs over all large boxes transversal to $\ell$ consisting of the points $x$ with $x \cdot \ell \in (-L + R + 2, L + 2)$ and other coordinates in an orthonormal basis with first vector $\ell$, smaller in absolute value than $\bar{L}$, for $L \geq c'(d), R + 2 \leq \bar{L} < L^3$. In the above formula for $\rho_B$, $T_B$ denotes the exit time from $B$ and $\partial_B$ is the part of the boundary of $B$ where $x \cdot \ell = L + 2$. The proof of Theorem 3.2.6 follows the strategy of Sznitman [55] and the sufficiency of the effective criterion is obtained.
by an induction argument along a growing sequence of boxes $B_k$ that tend to look like infinite slabs and in which suitable moments of $\rho_{B_k}$ are used to control moments of $\rho_{B_{k+1}}$. This allows us to deduce the asymptotic exit behaviour (3.1.10) from slabs. As a first application of the effective criterion, we show the equivalence between $(T')$ and $(T)_{\gamma}$ when $\gamma \in \left(\frac{1}{2}, 1\right)$. Note also that $\rho_B$ in (3.1.15) reminds us of the decisive quantity appearing in the one-dimensional theorem of Solomon, [50]. We will see in Section 3.2.1 that when $d = 1$, the box $B$ is replaced with an interval $(-L, L)$ and the existence of an $a \in (0, 1)$ and a $L > R$ such that $\mathbb{E}[\rho^0_{(-L, L)}] < 1$ is equivalent to $(T')$ and $(T)$ as well as to almost sure convergence to $+\infty$. Hence in opposition to the multidimensional case, condition $(T)$ does not imply ballistic behaviour when $d = 1$.

In the second part of this article we use the effective criterion to construct a new class of ballistic diffusions. We show [see Theorem 3.3.1] that when $d \geq 4$, for any $\eta > 0$, Brownian motion perturbed with a random drift $b(\cdot, \omega)$ such that

$$\sup_{x \in \mathbb{R}^d, \omega \in \Omega} |b(x, \omega)| \leq \epsilon \text{ and } \mathbb{E}[b(0, \omega) \cdot e_1] \geq \epsilon^{2-\eta}, \text{ for } \epsilon > 0, \tag{3.1.16}$$

satisfies the effective criterion with $\ell = \epsilon_1$ if $\epsilon$ is small enough. The conditions (3.1.16) allow for laws $\mathbb{P}$ of the environment such that (3.1.13) does not apply. Indeed, since the constant $c_\epsilon$ is larger than 1 as one can see from an inspection of the proof of [46] Theorem 2.5., (3.1.13) requires that $\mathbb{E}[b(0, \omega) \cdot e_1]$ is larger than $(c_\epsilon - 1)\mathbb{E}[(b(0, \omega) \cdot e_1)_-]$ which can be chosen to be of order $\epsilon$ under (3.1.16). Note that in the discrete setting, Sznitman (see [56]) obtained similar results under conditions significantly weaker than (3.1.16). Indeed, he showed that a discrete version of the effective criterion is satisfied by randomly perturbed simple random walk on $\mathbb{Z}^d$ with a drift $d(0, \omega) \overset{\text{def}}{=} E_0,\omega[X_1 - X_0]$ of size $\epsilon$ such that $\mathbb{E}[d(0, \omega) \cdot e_1]$ is larger than $\epsilon^{5/2-\eta}$ when $d = 3$ respectively larger than $\epsilon^{3-\eta}$ when $d \geq 4$. The strength of this result in contrast to ours is that it includes expected drifts of an order not larger than $\epsilon^2$, which enabled him to construct examples for condition $(T')$ where Kalikow's condition, see for instance [56], (5.3) therein, fails. Considering condition (5.23) of [46] as a continuous analogue of Kalikow's condition, we believe that such examples also exist in our setting. Since however the continuous setup with the finite range dependence tends to complicate the arguments, we did not attempt to retrieve the full strength of Sznitman's
result.

Let us now briefly describe the proof leading to the new example. In order to verify the effective criterion (3.1.14) under (3.1.16), we slice a large box $B$ (as defined below (3.1.14)) into thinner slabs transversal to $e_1$ and propagate good controls on the exit behaviour out of these slabs to the box $B$ using a refinement of the estimate (see Lemma 3.2.3 and Proposition 3.3.3) that was instrumental in the induction argument leading to the effective criterion. The heart of the matter is then to prove these good controls for the thinner slabs. To this end, we express the probability that the trajectory exits through the right side of a slab with the help of the Green operator of the diffusion killed when exiting the slab, see (3.3.23). This quantity is linked to the Green operator of killed Brownian motion via a certain perturbation equality, see (3.3.40). For Brownian motion, however, an explicit formula obtained by the well known “method of the images” from electrostatics [see (3.3.30)], allows us to compute all necessary estimates.

Let us finally explain how this article is organized. In Section 3.2 we first introduce some notation and then we show the equivalence between the effective criterion and condition $(T')$ when $d \geq 2$, see Theorem 3.2.6. The key estimate for the induction step is given by Proposition 3.2.2. In Section 3.2.1, we discuss the one-dimensional case. In Section 3.3, we use the effective criterion to show that a certain perturbed Brownian motion satisfies condition $(T')$ when $d \geq 4$. In Section 3.3.1, we state the main Theorem 3.3.1 and a refinement of Lemma 3.2.3, see Proposition 3.3.3. In Section 3.3.2, we define the Green operators and Green’s functions for which we provide certain deterministic estimates in the case of Brownian motion, see Lemmata 3.3.7 and 3.3.9. We also prove a perturbation equality, see Proposition 3.3.8. In Section 3.3.3, we use the results from the previous sections to prove the main Theorem 3.3.1. In Appendix A, we give the proof of Lemma 3.2.3 which is similar to that of [55] Proposition 1.2. In Appendix B, we prove Lemma 3.3.9 using a technique similar to [56] Lemma 2.1.

Convention on constants. Unless otherwise stated, constants only depend on the quantities $d$, $K$, $\nu$, $R$. We denote with $c$ positive constants with values changing from place to place and with $c_0, c_1, \ldots$ positive constants with values fixed at their first appearance. Dependence on additional parameters appears in the notation.
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3.2 An effective criterion when $d \geq 2$

In this section we show that condition $(T')$, see (3.1.11), is equivalent to the effective criterion, see (3.2.53), which controls the exit probability from some finite box. By an induction argument we propagate this control to larger boxes that tend to look like infinite slabs. Then one can infer the fast decay of exit probabilities from slabs through “the left” side as required by condition $(T')$.

We first need some notation. For $A, B \subset \mathbb{R}^d$ an open and a closed set, we denote with $T_A = \inf\{t \geq 0; X_t \notin A\}$ the exit time from $A$ and with $H_B = \inf\{t \geq 0; X_t \in B\}$ the entrance time into $B$. For any stopping time $S$, we call $S_0 = 0, S_{k+1} = S \circ \theta_{S_k} + S_k, k \geq 0$ the iterates of $S$. Here, $\theta_t$ denotes the canonical time shift. We consider a direction $\ell \in S^{d-1}$ and a rotation $\mathcal{R}$ of $\mathbb{R}^d$ such that $R(e_1) = \ell$. The vectors $e_i, i = 1, \ldots, d$ constitute the canonical basis. As a shortcut notation for the stopping times in (3.1.9), we write $T_u = T_u^\ell$ and $\bar{T}_u = \bar{T}_u^\ell, u \in \mathbb{R}$. Moreover, we introduce

$$|z|_L = \max_{j \geq 2} |z \cdot \mathcal{R}(e_j)|, \text{ for } x \in \mathbb{R}^d.$$  

For positive numbers $L, L', \tilde{L}$, we introduce the box

$$B = B(\mathcal{R}, L, L', \tilde{L}) \overset{\text{def}}{=} \mathcal{R}((-L, L') \times (-\tilde{L}, \tilde{L})^{d-1}),$$

and the positive respectively negative part of its boundary

$$\partial_+ B = \partial B \cap \{x \in \mathbb{R}^d : \ell \cdot x = L'\}, \quad \partial_- B = \partial B \setminus \partial_+ B.$$ 

We also define the following random variables: for $\omega \in \Omega$, 

$$p_B(\omega) = P_0,\omega[X_{T_B} \in \partial_+ B] = 1 - q_B(\omega), \quad \text{and}$$

$$\rho_B(\omega) = \frac{q_B(\omega)}{p_B(\omega)} \in [0, \infty].$$
In the sequel we will use different length scales \( L_k, \tilde{L}_k \geq 0, k = 0, 1 \ldots \) and
the following shortcut notations (cf. (3.1.5) for the definition of \( R \)):

\[
B_k = B(R, L_k - R - 1, L_k + 1, \tilde{L}_k), \quad \text{for } k \geq 0, \quad \text{and}
\]
\[
p_k = p_{B_k}, \quad q_k = q_{B_k}, \quad \rho_k = \rho_{B_k}.
\]

Finally let us set for \( k \geq 0 \),

\[
N_k = \frac{L_{k+1}}{L_k}, \quad n_k = \lfloor N_k \rfloor, \quad \tilde{N}_k = \frac{\tilde{L}_{k+1}}{\tilde{L}_k}.
\]

We start with an easy Lemma, introducing the counterpart of a discrete ellipticity constant.

**Lemma 3.2.1.** Let \( C_L \) be the tube \( \{ z \in \mathbb{R}^d : -\frac{1}{4} < z \cdot e_1 < L, \sup_{2 \leq j \leq d} |z \cdot e_j| < \frac{L}{4} \} \). There exists a constant \( 0 < \kappa \leq \frac{1}{2} \), such that for any \( L \geq 1, \omega \in \Omega, \) and any rotation \( R \),

\[
P_{0,\omega}[T_L^{R(e_1)}] < T_R(C_L)] \geq \kappa^{L+1}, \quad \text{and} \quad P_{0,\omega}[\tilde{T}_{-L}^{R(e_1)}] < T_R(-C_L)] \geq \kappa^{L+1}.
\]

**Proof.** We define the function \( \psi(s) = \frac{5}{4} R(e_1) s \), for \( 0 \leq s \leq 1 \). With the support theorem, see [3] p. 25, we obtain that there is a constant \( c > 0 \) such that for all \( x \in \mathbb{R}^d, \omega \in \Omega, P_{x,\omega}[\sup_{0 \leq s \leq 1} |X_s - X_0 - \psi(s)| < \frac{1}{4}] \) and \( P_{x,\omega}[\sup_{0 \leq s \leq 1} |X_s - X_0 + \psi(s)| < \frac{1}{4}] \) are both larger than \( c \). Then we set \( \kappa = \min\{c, \frac{1}{2}\} \). The claim follows by applying the Markov property \([L] \) times.

We are now ready to prove the main induction step which in essence bounds moments of \( \rho_1 \) in terms of moments of \( \rho_0 \).

**Proposition 3.2.2.** \((d \geq 2)\) There exist \( c_1 > R + 2, c_2, c_3 > 1, \) such that when \( N_0 \geq 3, L_0 \geq c_1, N_0 \geq 150 N_0, \tilde{L}_0 \geq R + 2, \) for any \( a \in (0, 1) \):

\[
\mathbb{E}[\rho_1^a] \leq c_2 \left\{ \kappa^{-10L_1} \left( c_3 L_1^{d-2} \frac{L_1^3}{L_0^2} \tilde{L}_0 \mathbb{E}[q_0] \right)^{\frac{N_0}{12N_0}} \sum_{0 \leq m \leq n_0 + 1} (c_3 \tilde{L}_1^{d-1} \mathbb{E}[(\rho_0^2)^a])^{\frac{1}{2}(n_0 + m - 1)} \right\}.
\]
3. Effective criterion and new example

**Proof.** For \( i \in \mathbb{Z} \) and \( L_0 > R + 2 \), we introduce the slabs of width \( R \)

\begin{equation}
S_i = \{ x \in \mathbb{R}^d : iL_0 - \frac{R}{2} \leq x \cdot \ell \leq iL_0 + \frac{R}{2} \}
\end{equation}

and denote by \( I(\cdot) \) the function on \( \mathbb{R}^d \) such that \( I(x) = i \) if \( x \cdot \ell - iL_0 \in \left[ -\frac{L_0}{2}, \frac{L_0}{2} \right) \), \( i \in \mathbb{Z} \). In particular, \( I \) takes the value \( i \) on \( S_i \), for all \( i \in \mathbb{Z} \). We define the successive times of visit to the different slabs \( S_i \) as the iterates \( V_k, k \geq 0 \) of the stopping time

\begin{equation}
V = \inf \{ t \geq 0 : X_t \in S_{I(X_0) - 1} \cup S_{I(X_0) + 1} \}.
\end{equation}

We also need the stopping time

\begin{equation}
\hat{T} = \inf \{ t > 0 : |X_t|_{\perp} \geq \tilde{L}_1 \}.
\end{equation}

In a first step we obtain a control on \( \mathbb{E}[\rho_1^a] \) using the following quantities: for \( \omega \in \Omega, i \in \mathbb{Z} \),

\begin{equation}
\hat{\rho}(i, \omega) = \sup \left\{ \frac{\hat{q}(x, \omega)}{\hat{p}(x, \omega)} : x \in S_i, |x|_{\perp} < \tilde{L}_1 \right\}, \quad \text{where}
\end{equation}

\begin{equation}
\hat{q}(x, \omega) = P_{x, \omega} [X_{V_1} \in S_{I(X_1) - 1}] = 1 - \hat{p}(x, \omega).
\end{equation}

The first step then comes with the following lemma.

**Lemma 3.2.3.** Under the assumptions of Proposition 3.2.2,

\begin{equation}
\mathbb{E}[\rho_1^a] \leq \kappa^{-a(L_1 + 1)} \mathbb{P}[G^c] + 2 \sum_{0 \leq m \leq n_0 + 1} \prod_{-n_0 + 1 < i \leq m} \mathbb{E}[\hat{\rho}(i, \omega)^{2a}]^{1/2},
\end{equation}

\begin{equation}
\text{where } G = \left\{ \omega \in \Omega : P_{0, \omega} [\hat{T} \leq \tilde{T}_{-L_1 + R + 1} \wedge T_{L_1 + 1}] \leq \kappa^{9L_1} \right\}.
\end{equation}

The proof of this lemma is similar to the proof of [55], (2.39) in Proposition 2.1., or [56], Lemma 1.2. For the reader’s convenience, we include the argument in Appendix A.

We now complete the proof of Proposition 3.2.2. Except for a few modifications due to the continuous setup, we follow the steps in the proof of [55],
Proposition 2.1. We first bound \( \mathbb{P}[G^c] \) in terms of \( \mathbb{E}[q_0] \). In the next chapter, we infer a different bound on this probability, see (3.3.13). By Chebychev’s inequality, we find that
\[
(3.2.17) \quad \mathbb{P}[G^c] \leq \kappa^{-9L_1} P_0[\tilde{T} \leq \tilde{T}_{-L_1+R+1} \land T_{L_1+1}],
\]
and our task is to derive an upper bound on the right-hand side. We introduce for \( u \in \mathbb{R}, j \geq 2 \), the stopping times
\[
(3.2.18) \quad \sigma_{u,j}^\pm = \inf\{t \geq 0 : \pm X_t \cdot \mathcal{R}(e_j) \geq u\}, \quad \text{and}
\]
\[
(3.2.19) \quad \bar{L} = 2(n_0 + 2)(\bar{L}_0 + 1) + R, \quad J = \left\lfloor \frac{\bar{L}_1}{\bar{L}} \right\rfloor.
\]
Since \( \bar{L}_1 = \tilde{N}_0 \tilde{L}_0 \geq 150n_0 \tilde{L}_0, n_0 \geq 3 \) and \( \tilde{L}_0 \geq 2 + R \), it follows that \( J \geq 15 \). On the event \{\( T \leq \tilde{T}_{-L_1+R+1} \land T_{L_1+1} \}\}, \( P_0 \)-a.s., at least one of the projections \( |X_t \cdot \mathcal{R}(e_j)|, j \geq 2 \), reaches the value \( J \bar{L} \) before \( X_t \) exits the box \( B_1 \). Hence:
\[
(3.2.20) \quad P_0[\tilde{T} \leq \tilde{T}_{-L_1+R+1} \land T_{L_1+1}] \leq \sum_{j \geq 2} P_0[\sigma_{J \bar{L}}^+ \leq T_{B_1}] + P_0[\sigma_{J \bar{L}}^- \leq T_{B_1}].
\]
Let us write \( \sigma_u \) in place of \( \sigma_{u,2}^+ \) and bound the term \( P_0[\sigma_{J \bar{L}} \leq T_{B_1}] \), the other terms being treated similarly. The strong Markov property yields that
\[
(3.2.21) \quad P_0[\sigma_{J \bar{L}} \leq T_{B_1}] \leq \mathbb{E} E_{0,\omega} [\sigma_{(J-1) \bar{L}} \leq T_{B_1}, P_{X_{\sigma_{(J-1) \bar{L}} \omega}}[\sigma_{J \bar{L}} \leq T_{B_1}]].
\]
We define the auxiliary box
\[
(3.2.22) \quad B' = B(\mathcal{R}, L_0 - R, L_0, \tilde{L}_0 + 1),
\]
cf. (3.2.2) for the notation and let \( H^i, i \geq 0 \) denote the iterates of the stopping time \( H^1 = T_{B_1} \land T_{X_0 + B'} \). Then for any \( \omega \in \Omega, x \in B_1 \) with \( x \cdot \mathcal{R}(e_2) = (J - 1) \bar{L} \), we have:
\[
(3.2.23) \quad P_{x,\omega}[\sigma_{J \bar{L}} > T_{B_1}] \geq P_{x,\omega} \left[ \bigcap_{k=0}^{2(n_0+1)-1} \theta_{H^k}^{-1}\{H^1 < T_{\partial_- B'+X_0}\} \right],
\]
because on the event in the right-hand side, the trajectory either exits \( B_1 \) before \( \sigma_{J \bar{L}} \) right away on \( \{H^1 < T_{\partial_- B'+X_0}\} \) or it exits the box \( B_1 \) through “the right”, since for every \( k \geq 0 \), on \( \theta_{H^k}^{-1}\{H^1 < T_{\partial_- B'+X_0}\} \) the trajectory
$P_{x, \omega}$-a.s. moves between time $H^k$ and $H^{k+1}$ at most a distance $\tilde{L}_0 + 1$ into direction $\mathcal{R}(e_2)$ and at least a distance $L_0$ into direction $\ell$ until it leaves $B_1$, and since

\begin{equation}
2(n_0 + 1)(\tilde{L}_0 + 1) = \tilde{L} - 2(\tilde{L}_0 + 1) - R < \tilde{L}
\end{equation}

and $2(n_0 + 1)L_0 > 2L_1 - R$, the width of $B_1$ in direction $\ell$. In order to obtain a lower bound on the right-hand side of (3.2.23) with the help of the strong Markov property, we cover the set

$$G(J - 1) \deq \{ x \in B_1 : |x \cdot \mathcal{R}(e_2) - (J - 1)\tilde{L}| \leq 2(n_0 + 1)(\tilde{L}_0 + 1) \},$$

which contains the trajectories up to $T_{B_1}$ described by the event in the right-hand side of (3.2.23), with a collection of disjoint and rotated unit cubes $C_m$ with centers $x_m$. The cardinality of this collection is proportional to the volume of $G(J - 1)$.

For any $k, m \geq 0$ and any $\omega \in \Omega$, we have that on $\{X_{H^k} \in C_m\}$, $P_{0, \omega}$-a.s.,

\begin{equation}
\begin{aligned}
(3.2.25) \quad & P_{X_{H^k}, \omega}[X_{T_{B'} + X_0} \in \partial_+ B' + X_0] \geq P_{X_{H^k}, \omega}[X_{T_{n_0 + x_m}} \in \partial_+ B_0 + x_m],
\end{aligned}
\end{equation}

as for any $x \in C_m$, it follows from the definitions of $B'$, see (3.2.22) and $B_0$, see (3.2.6), that $\partial_+ B_0 + x_m \subset (\overline{B'} + x)^c$, $\partial_- B' + x \subset \overline{B}_0 + x_m$ and $x \in B_0 + x_m$, see Figure 1. Here $\overline{U}$ denotes the closure of $U \subset \mathbb{R}^d$. Therefore any piece of
trajectory contained in $B_0 + x_m$, connecting $x \in C_m$ to $\partial_+ B_0 + x_m$ has to exit $B' + x$, but cannot touch $\partial_- B' + x$.

As a consequence, we deduce from (3.2.23) using the strong Markov property that for any $\omega \in \Omega$, $x \in B_1$ with $x \cdot \mathcal{R}(e_2) = (J - 1)L$,

\begin{equation}
P_{x, \omega}[\sigma_{JL} > T_{B_1}] \geq \left( \inf_{m} \inf_{x \in C_m} P_{x, \omega}[X_{T_{B_0 + x_m}} \in \partial_+ B_0 + x_m] \right)^{2(n_0 + 1)} \overset{\text{def}}{=} 1 - \phi(J - 1, \omega),
\end{equation}

and thus, in view of (3.2.21), we find:

\begin{equation}
P_0[\sigma_{J \bar{L}} \leq T_{B_1}] \leq \mathbb{E}[P_{0, \omega}[\sigma_{(J-2)\bar{L}} \leq T_{B_1}] \phi(J - 1, \omega)].
\end{equation}

From (3.2.24), we see that $G(J - 1) \subset \{x \in B_1 : x \cdot \mathcal{R}(e_2) \geq (J - 2)\bar{L} + 2(\tilde{L}_0 + 1) + R \}$, and therefore the random variable $\phi(J - 1, \cdot)$ is $\mathcal{H}_{\{z \in \mathcal{R}(e_2) \geq (J - 2)\bar{L} + R\}}$-measurable whereas $P_0[\sigma_{(J-2)\bar{L}} \leq T_{B_1}]$ is $\mathcal{H}_{\{z \in \mathcal{R}(e_2) \leq (J - 2)\bar{L}\}}$-measurable. Thus the finite range dependence property implies that

\begin{equation}
P_0[\sigma_{J \bar{L}} \leq T_{B_1}] \leq P_0[\sigma_{(J-2)\bar{L}} \leq T_{B_1}] \mathbb{E}[\phi(J - 1, \omega)].
\end{equation}

Using the notation (3.2.26) and observing that $1 - p^k \leq k(1 - p)$ for $k \geq 1, p \geq 0$, we obtain:

\begin{equation}
\mathbb{E}[\phi(J - 1, \omega)] \leq 2(n_0 + 1)\mathbb{E}\left[ \sup_{m} \sup_{x \in C_m} P_{x, \omega}[X_{T_{B_0 + x_m}} \in \partial_- B_0 + x_m] \right]
\end{equation}

We now observe that the cardinality of the collection of cubes $C_m$ is proportional to $2L_1 \cdot 4(n_0 + 1)(\tilde{L}_0 + 1) \cdot (2\tilde{L}_1)^{d-2} \leq c \tilde{L}_1^{d-2} \frac{L_1^3}{\tilde{L}_0} \tilde{L}_0$. Then translation invariance and an application of Harnack’s inequality to the harmonic function $x \mapsto P_{x, \omega}[X_{T_{B_0}} \in \partial_- B_0]$ yield that

\begin{equation}
\mathbb{E}[\phi(J - 1, \omega)] \leq c' \tilde{L}_1^{d-2} \frac{L_1^3}{\tilde{L}_0} \tilde{L}_0 \mathbb{E}[q_0],
\end{equation}

where we used the notation (3.2.4). Coming back to (3.2.28), we see that:

\begin{equation}
P_0[\sigma_{J \bar{L}} \leq T_{B_1}] \leq P_0[\sigma_{(J-2)\bar{L}} \leq T_{B_1}] c \tilde{L}_1^{d-2} \frac{L_1^3}{\tilde{L}_0} \tilde{L}_0 \mathbb{E}[q_0],
\end{equation}

and by induction

\begin{equation}
\leq \left\{ c \tilde{L}_1^{d-2} \frac{L_1^3}{\tilde{L}_0} \tilde{L}_0 \mathbb{E}[q_0] \right\}^m, \quad \text{for all } 0 \leq m \leq \left\lfloor \frac{J}{2} \right\rfloor.
\end{equation}
Similar bounds hold for each term in the right-hand side of (3.2.20) and since \( \left\lfloor \frac{J}{2} \right\rfloor \geq \frac{\tilde{N}_0}{12N_0} \) from our assumptions on \( N_0, \tilde{N}_0 \) and \( \tilde{L}_0 \) we conclude from (3.2.17), (3.2.20) and (3.2.31) that:

\[
\mathbb{P}[G^c] \leq \kappa^{-gL_1} 2(d-1) \left\{ c \tilde{L}_1^{d-2} \frac{L_1^3}{L_0^2} \tilde{L}_0 \mathbb{E}[q_0] \right\}^{\frac{\tilde{N}_0}{12N_0}}.
\]

So far we found an upper bound for the first term of the right-hand side of (3.2.15). To complete the proof of (3.2.9), we are now going to bound the second term.

For any \( i \in \mathbb{Z} \), we cover the set \( \{ x \in S_i : \|x\|_\perp < \tilde{L}_1 \} \) (appearing in the definition of \( \tilde{\rho}(i, \omega) \), see (3.2.13)) with a collection of disjoint and rotated unit cubes \( \tilde{C}_k \) with cardinalinity at most \( (R+1)(2\tilde{L}_1 + 1)^{d-1} \). As a result, for \( 0 < a < 1 \),

\[
\mathbb{E}[\tilde{\rho}(i, \omega)^{2a}] \leq \sum_k \mathbb{E} \left[ \sup_{x \in \tilde{C}_k} \tilde{q}(x, \omega)^{2a} \right].
\]

By Harnack’s inequality, there is a constant \( c \geq 1 \) such that \( \frac{\sup_{x \in \tilde{C}_k} \tilde{q}(x, \omega)}{\inf_{x \in \tilde{C}_k} \tilde{p}(x, \omega)} \leq c^2 \frac{\tilde{q}(x_k, \omega)}{\tilde{p}(x_k, \omega)} \), for every \( 1 \leq k, \omega \in \Omega \). Moreover, observe that \( \tilde{q}(x_k, \omega) \leq q_0 \circ t_{x_k} \omega \), cf. (3.2.6) for the notation. Using translation invariance, we see that the second term on the right-hand side of (3.2.15) is less than or equal to

\[
2 \sum_{0 \leq m \leq n_0 + 1} \left( (R+1)(2\tilde{L}_1 + 1)^{d-1} c^{4a} \mathbb{E}[\rho_0^{2a}] \right)^{\frac{m+n_0-1}{2}}.
\]

Choosing \( c_3 \geq (R+1)^{3d-1}c^4 \) sufficiently large completes the proof of Proposition 3.2.2.

Similarly to [55], we are going to iterate (3.2.9) along an increasing sequence of boxes \( B_k \), which tend to look like infinite slabs transversal to the direction \( \ell \). For the definition of these boxes, we consider:

\[
(3.2.35) \quad u_0 \in (0, 1], \quad v = 8, \quad \alpha = 240,
\]

\[
\frac{1}{4} \left\lfloor \frac{J}{2} \right\rfloor \geq \frac{J}{2} - \frac{1}{2} = \frac{\tilde{N}_0 \tilde{L}_0}{4(n_0+2)(L_0+1)+2R} - 1 \geq \frac{\tilde{N}_0 \tilde{L}_0}{4\left(\frac{\tilde{N}_0}{11}n_0\right) \frac{3}{2}L_0 + n_0 L_0} - 1 \geq \frac{\tilde{N}_0}{11N_0} - 1 \geq \frac{\tilde{N}_0}{N_0 \left( \frac{11}{4} - \frac{1}{150} \right)} \geq \frac{\tilde{N}_0}{12N_0}.
\]
and choose two sequences $L_k, \tilde{L}_k, k \geq 0$, such that

\begin{equation}
L_0 \geq c_1, R + 2 \leq \tilde{L}_0 \leq L_0^3, \quad \text{and} \quad k \geq 0, \quad L_{k+1} = N_k L_k, \quad \text{with} \quad N_k = \frac{\alpha}{u_0} v^k \quad \text{and} \quad \tilde{L}_{k+1} = N_k^3 \tilde{L}_k.
\end{equation}

As a consequence we see that for $k \geq 0$:

\begin{equation}
L_k = \left( \frac{\alpha}{u_0} \right)^k v^{\frac{k(k-1)}{2}} L_0, \quad \text{and}
\end{equation}

\begin{equation}
\tilde{L}_k = \left( \frac{L_k}{L_0} \right)^3 \tilde{L}_0.
\end{equation}

**Lemma 3.2.4.** There exists $c_4 > c_1$, such that when for some $L_0 \geq c_4, R + 2 \leq \tilde{L}_0 \leq L_0^3, a_0 \in (0,1], u_0 \in [\kappa \frac{L_0}{\alpha}, 1],$

\begin{equation}
\varphi_0 \overset{\text{def}}{=} c_3 \tilde{L}_1^{(d-1)} L_0 \mathbb{E}[\rho_0^{a_0}] \leq \kappa u_0 L_0,
\end{equation}

then for all $k \geq 0$,

\begin{equation}
\varphi_k \overset{\text{def}}{=} c_3 \tilde{L}_{k+1}^{(d-1)} L_k \mathbb{E}[\rho_k^{a_k}] \leq \kappa u_k L_k, \quad \text{with} \quad a_k = a_0 2^{-k}, u_k = u_0 v^{-k}.
\end{equation}

As the proof is purely algebraic and hence identical to the proof of [55], Lemma 2.2., we omit it here. We now use the induction result to control the exit behaviour from a slab.

**Proposition 3.2.5.** There exists $c_5 \geq c_4, c_6 > 1$, such that when for some $L_0 \geq c_5, R + 2 \leq \tilde{L}_0 \leq L_0^3$,

\begin{equation}
c_6 \left( \log \frac{1}{\kappa} \right)^{3(d-1)} \tilde{L}_0^{(d-1)} L_0^{3(d-1)+1} \inf_{a \in (0,1]} \mathbb{E}[\rho_0^a] < 1,
\end{equation}

with $B_0, \rho_0$ as in (3.2.6), then for some $c > 0$,

\begin{equation}
\limsup_{L \to \infty} L^{-1} \exp\{c(\log L)^{\frac{1}{2}}\} \log P_0[\tilde{T}_{-b L} < T_L] < 0, \quad \text{for all} \ b > 0.
\end{equation}

**Proof.** In view of (3.2.37),(3.2.38), we see that (3.2.39) is equivalent to

\begin{equation}u_0^{-3(d-1)} \kappa^{-u_0 L_0} c_3 \alpha^{3(d-1)} \tilde{L}_0^{(d-1)} L_0 \mathbb{E}[\rho_0^{a_0}] \leq 1.
\end{equation}
The minimum of the function \( [\kappa \frac{L_0}{d}, 1] \ni u_0 \mapsto u_0^{-3(d-1)}k^{-u_0L_0} \) is 
\( c'(L_0 \log \frac{1}{\kappa})^{3(d-1)} \), provided \( L_0 \geq c_5 \). Hence choosing \( c_6 = 2c'c_3\alpha^{3(d-1)} \), we can make sure that whenever (3.2.41) holds, for some \( L_0 \geq c_5, R + 2 \leq \tilde{L}_0 \leq L_0^3 \), then (3.2.39) holds for some \( a_0 \in (0, 1], u_0 \in [\kappa \frac{L_0}{d}, 1] \). By Lemma 3.2.4, (3.2.40) holds for all \( k \geq 0 \). For any \( b > 0 \), we are now looking for a bound on \( P_0[\hat{T}_{-bL} < T_L] \) when \( L \) is large. For every large enough \( L \), we can find a unique \( k \) with:

\[
L_k < bL \leq L_{k+1}.
\]

We then introduce the auxiliary box \( B'_k = B(\mathcal{R}, L_k - R, L_k, \tilde{L}_k + 1) \), and use an argument similar to (3.2.23)-(3.2.26) to find a lower bound for \( P_{0,\omega}[\hat{T}_{-bL} > T_L] \), that is we require in essence that the trajectory successively exits certain translates of the box \( B'_k \) through the ”right” side, \( \lfloor \frac{L_k}{e} \rfloor + 1 \) times. We therefore cover the set

\[
G' = \{ x \in \mathbb{R}^d, |x|_1 \leq (\frac{L}{L_k} + 1)(\tilde{L}_k + 1), x \cdot \ell \in (-bL, L) \},
\]

playing the role of former set \( G(J - 1) \), with disjoint and rotated unit cubes \( C'_j \) with centers \( x'_j \). The cardinality of this collection is at most \( m_k = ((b + 1)L + 1)(2(\frac{L}{L_k} + 1)(\tilde{L}_k + 1) + 1)^{(d-1)} \). For \( L \) large, we introduce the event

\[
(3.2.45) \quad \Gamma = \{ \omega \in \Omega : \sup_j x \in C'_j P_{x,\omega}[X_{T_{B'k+x'_j}} \in \partial_- B_k + x'_j] \geq \kappa^{\frac{1}{2}u_kL_k} \}.
\]

Then, for any \( \omega \in \Gamma^c \), we obtain by arguments as before that

\[
(3.2.46) \quad P_{0,\omega}[\hat{T}_{-bL} > T_L] \geq (1 - \kappa^{\frac{1}{2}u_kL_k}) \left(\frac{L_k}{e} + 1\right).
\]

On the other hand, using translation invariance, Harnack’s and Chebychev’s inequality, we find that there is a \( c > 0 \) such that

\[
(3.2.47) \quad P[\Gamma] \leq cm_k \kappa^{-\frac{1}{2}u_kL_k} \mathbb{E}[q_k].
\]

Since \( q_k \leq \rho_k^{a_k} \) and because of (3.2.40), we obtain that

\[
(3.2.48) \quad P[\Gamma] \leq c' \frac{m_k}{c_3 \tilde{L}_k^{(d-1)} L_k} \kappa^{\frac{1}{2}u_kL_k},
\]
and a simple computation using (3.2.44) and (3.2.36) shows that for $L$ large enough,

$$
(3.2.49) \quad \frac{m_k}{\bar{L}_{k+1}^{(d-1)} L_k} \leq \frac{(b + 1)L + 1) (2(\frac{L}{L_k} + 1) (\tilde{L}_k + 1) + 1)^{d-1}}{N_k^{3(d-1)} L_{k}^{d-1} L_k} \\
\leq \frac{(\frac{1}{b} + 2) L_{k+1} \delta^{(d-1)} (L_{k+1}^{(d-1)} L_k + 1)^{d-1} \tilde{L}_k^{d-1}}{N_k^{3(d-1)} L_{k}^{d-1} L_k} \\
\leq c(b) \left( \frac{1}{b} + \frac{1}{N_k} \right)^{d-1} N_k^d N_{k}^{-3(d-1)} \leq c'(b),
$$

since $d \geq 2$. As a consequence, we obtain from (3.2.48) that

$$
(3.2.50) \quad \mathbb{P}[\Gamma] \leq c(b) \kappa^{\frac{1}{2} u_k L_k}.
$$

Assembling (3.2.46),(3.2.50) and using $1 - p^m \leq m(1 - p)$, $p, m \geq 0$, we see that for large $L$:

$$
(3.2.51) \quad P_0[\tilde{T}_{-bL} < T_L] \leq \left( c(b) + \frac{L}{L_k} + 1 \right) \kappa^{\frac{1}{2} u_k L_k}.
$$

From (3.2.40), (3.2.36), we obtain $u_k L_k = \frac{u^2}{\alpha} v^{-2k} L_{k+1} \geq \frac{u^2}{\alpha} v^{-2kbL}$, and so the right-hand side of (3.2.51) is less than $c'(b) N_k \kappa^{\frac{1}{2} u_k v^{-2kbL}}$. From the inequality $L_k \leq bL$ and (3.2.37), we deduce that if $L$ is large, then $k \leq c(\log bL)^{1/2}$, and we obtain:

$$
(3.2.52) \quad P_0[\tilde{T}_{-bL} < T_L] \leq c'(b) \exp \left\{ - \frac{1}{4} \frac{u^2}{\alpha} \left( \log \frac{1}{k} \right) bL \exp \left( - c(\log(bL))^{1/2} \right) \right\},
$$

for large $L$. This implies the claim (3.2.42).
We are now ready to prove the main result of this section.

**Theorem 3.2.6.** There exists a constant $c_7(d) > 1$, such that for $\ell \in S^{d-1}$ the following conditions are equivalent:

(i) There exist $a \in (0,1]$ and a box $B = B(\mathcal{R}, L - R - 2, L + 2, \bar{L})$ with $\mathcal{R}(e_1) = \ell$, $L \geq c_5$, $R + 2 \leq \bar{L} < L^3$ with

\[
c_7(\log \frac{1}{\kappa})^3(d-1) \bar{L}^{d-1} L^{3(d-1) + 1} \mathbb{E}[\rho_B^a] < 1,
\]

(ii) $(T')$ holds with respect to $\ell$ [see (3.1.11)],

(iii) $(T)_{\gamma}$ holds with respect to $\ell$ for some $\gamma \in (\frac{1}{2}, 1)$ [see (3.1.10)].

**Proof.** The implication (i) implies (ii) is proved in the same way as the corresponding statement in [55], Theorem 2.4. Indeed, we define $c_7 = 2(d-1)c_6$ and observe that as a result of (3.2.53),

\[
c_6(\log \frac{1}{\kappa})^3(d-1) \bar{L}'^{d-1} L^{3(d-1) + 1} \mathbb{E}[\rho_B^a] < 1,
\]

with $\bar{L}' = (\bar{L} + 2) \wedge L^3 \in (\bar{L}, 2\bar{L})$. If $B'$ denotes the box $B(\mathcal{R}', L - R - 1, L + 1, \bar{L}')$ and if the rotation $\mathcal{R}'$ is close enough to $\mathcal{R}$,

\[
p_B \leq p_{B'} \text{ and hence } \rho_{B'} \leq \rho_B.
\]

As a result, whenever $\mathcal{R}'$ is sufficiently close to $\mathcal{R}$, we can apply Proposition 3.2.5 to the box $B'$, and find that

\[
\limsup L^{-\gamma} \log P_0[T_{bL}^{\ell'} < T_{L}^{\ell'}] < 0, \text{ for any } \gamma \in (0,1), b > 0 \text{ with } \ell' = \mathcal{R}(e_1).
\]

This proves (ii). It is plain that (iii) follows from (ii).

We now show that (iii) implies (3.2.53). The neighbourhood appearing in the definition of $(T)_{\gamma}$ contains for some small $\alpha > 0$ and all $j = 2, \ldots, d$, the vectors $\ell'_j = \cos(\alpha)\ell + \sin(\alpha)\mathcal{R}(e_j)$, $\ell''_j = \cos(\alpha)\ell - \sin(\alpha)\mathcal{R}(e_j)$. For large $L'$ and $0 < b < 1$, we choose $L + 2 = L'\frac{1-b}{2\cos(\alpha)}$ and $\bar{L} = L'\frac{1+b}{2\sin(\alpha)}$. (In particular $\bar{L} \leq L^3$ if $L'$ is large enough depending on $\alpha$ and $b$.) As a consequence, if we set $B = B(\mathcal{R}, L - 2 - R, L + 2, \bar{L})$ then $\partial_- B$ is included in the region where
3.2. An effective criterion when $d \geq 2$

$x \cdot \ell_j' \leq -bL'$ or $x \cdot \ell_j'' \leq -bL'$ for some $2 \leq j \leq d$ (see also Figure 2). In other words,

$$(3.2.57) \quad \mathbb{E}[q_B] \leq P_0 \left[ \text{there exists } \ell' \in \bigcup_{j=2}^{d} \{ \ell_j', \ell_j'' \} : T_{-bL'} < T_{\ell'} \right],$$

and from (3.1.10), we see that for some $c > 0$:

$$(3.2.58) \quad \mathbb{E}[q_B] \leq 2(d-1)e^{-cL^\gamma}, \quad \text{if } L \text{ is large enough.}$$

Hence for large $L$, for $a \in (0,1)$ and $c' > 0$:

$$(3.2.59) \quad \mathbb{E}[\rho_B^a] \leq \mathbb{E}[\rho_B^a, p_B \geq e^{-c'L^\gamma}] + \mathbb{E}[\rho_B^a, p_B < e^{-c'L^\gamma}],$$

so that using the definition (3.2.5) and Jensen’s inequality to bound the first term and $\rho_B \leq \kappa^{-(L+3)}$ because of Lemma 3.2.1 to control the second term, we find for large $L$, $a \in (0,1)$ and $c' > 0$:

$$(3.2.60) \quad \mathbb{E}[\rho_B^a] \leq e^{c'aL^\gamma} \mathbb{E}[q_B]^a + \kappa^{-a(L+3)} \mathbb{P}[q_B \geq 1 - e^{-c'L^\gamma}]$$

$$(3.2.58) \quad \leq (2(d-1))^a e^{(c'-c)aL^\gamma} + 2\kappa^{-a(L+3)} 2(d-1)e^{-c'L^\gamma}.$$
3. Effective criterion and new example

If we choose $a = L^{-\frac{1}{2}}$ and $c' > 0$ sufficiently small, we obtain:

\[
\limsup_{L \to \infty} L^{-\left(\gamma - \frac{1}{2}\right)} \log \mathbb{E} \left[ \rho_B^{L^{-1/2}} \right] < 0.
\]

This implies (3.2.53) and thus finishes the proof of Theorem 3.2.6.

Remark 3.2.1 As mentioned in the Introduction, it is conjectured that the conditions $(T)$, $(T')$ and $(T)_\gamma$ for a $\gamma \in (0, 1)$ are all equivalent and Theorem 3.2.6 proves part of it. An improvement of the rather crude bound on the second term on the right-hand side of (3.2.59) is likely to yield the equivalence of $(T')$ and $(T)_\gamma$ also for $\gamma$ smaller than 1/2. Moreover, the latter theorem together with Proposition 3.2.5 strengthen the belief that $(T)$ and $(T')$ are equivalent. Indeed, we have in fact obtained the equivalence of $(T')|_{\ell}$ and

\[
\limsup_{L \to \infty} L^{-1} \exp \left\{ c(\log L)^{\frac{1}{2}} \right\} \log \mathbb{P}_0 \left[ T_{-bL}^{T_{\ell}} < T^\ell_L \right] < 0,
\]

and $\ell'$ close to $\ell$, cf. (3.2.42), which is just slightly weaker than $(T)|_{\ell}$, since $\exp \left\{ c(\log L)^{\frac{1}{2}} \right\}$ grows slower than any polynomial. Also note that using [46], (3.36) therein, $(\ast)$ actually holds for $\ell' \in S^{d-1}$ satisfying $\ell' \cdot v > 0$, where $v = \lim_{t \to \infty} \frac{X_t}{t}$ denotes the limiting velocity which has been shown to be deterministic and non zero (ballistic behaviour) under $(T')|_{\ell}$ when $d \geq 2$, see [46], [48] and also [55] in the discrete setting.

3.2.1 The one-dimensional case.

We introduce here the one-dimensional counterpart of the effective criterion and show that condition $(T)$ is equivalent to $(T')$ and to $P_0$-a.s. transience, see Proposition 3.2.7. Unlike the multidimensional case, condition $(T')$ does not imply ballistic behaviour when $d = 1$, since one can construct one-dimensional diffusions in random environments that tend to infinity, hence satisfy $(T')$, and have zero limiting velocity. A natural question to ask is then whether directional transience, i.e. convergence to $\infty$ into some direction, or at least ballistic behaviour implies $(T')$ also in higher dimensions.

We first adapt the definitions (3.2.4), (3.2.5), (3.2.10), (3.2.13) to the one-dimensional setting. Instead of boxes or slabs, we now consider intervals.
For any $L > 0$, $\rho_B$, see (3.2.5), is replaced by

\begin{equation}
(3.2.62) \quad \rho_L = \frac{P_{0,\omega}[\tilde{T}_L < T_L]}{P_{0,\omega}[\tilde{T}_L > T_L]}.
\end{equation}

For $L_0 \geq 1$, $i \in \mathbb{Z}$, we redefine $S_i$, see (3.2.10), as $S_i = iL_0$. The definition of the stopping-times $V_k$, $k \geq 0$, see (3.2.11), remains unchanged. Then we set for $\omega \in \Omega, i \in \mathbb{Z}$:

\begin{equation}
(3.2.63) \quad \hat{\rho}(i, \omega) = \frac{\hat{q}(iL_0, \omega)}{\hat{p}(iL_0, \omega)}, \text{ where}
\end{equation}

\begin{equation}
(3.2.64) \quad \hat{q}(x, \omega) = P_{x,\omega}[X_{V_1} \in S_{I(x)-1}] = P_{x,\omega}[\tilde{T}_{L_0+x} < T_{L_0+x}] = 1 - \hat{p}(x, \omega).
\end{equation}

**Proposition 3.2.7.** ($d = 1$). The following conditions are equivalent:

(i) There exists $a \in (0, 1]$, $L > R$, such that $E[\rho_L^a] < 1$.

(ii) There exists $L > R$, such that $E[\log \rho_L] < 0$.

(iii) Condition $(T)$ holds relative to $e_1$.

(iv) Condition $(T')$ holds relative to $e_1$.

(v) $\lim_{t \to \infty} X_t \cdot e_1 = \infty$, $P_0$-a.s.

*Proof.* The fact that (i) implies (ii) follows from Jensen’s inequality since by Lemma 3.2.1 $E[\rho_L^a] \leq \kappa^{-a(L+1)} < \infty$. We now show that (ii) implies (iii). We have from (ii) that $-\mu \overset{\text{def}}{=} E[\log \rho_{L_0}] < 0$ for some $L_0 > R$. We are going to use a similar argument as in Appendix A or as in [54], Proposition 2.6 therein. For any $b > 0$ and any real $L > 4L_0/b$, we define

\begin{equation}
(3.2.65) \quad n' = \left\lfloor \frac{bL}{L_0} \right\rfloor \text{ and set } n_0 = \left\lfloor \frac{L'}{L_0} \right\rfloor.
\end{equation}

(In the spirit of Appendix A, $-n'$ plays the role of $-n_0 + 1$, see e.g. (3.4.2) or (3.4.3).) We define the function $\hat{f}$ on $\{-n', -n' + 1, \ldots, n_0 + 2\}$ by (3.4.1) and modify the definition of $\tau$, see (3.4.3), as follows:

\begin{equation}
(3.2.66) \quad \tau = \inf\{k \geq 0; X_{V_k} \in S_{n_0+2} \cup S_{-n'}\}.
\end{equation}
Since $-\hat{p}(X_{\nu_m}) + \hat{q}(X_{\nu_m})\rho(I(X_{\nu_m}))^{-1}$ vanishes $P_{0,\omega}$-a.s. for all $m \geq 0$, we obtain by an argument similar to (3.4.4)-(3.4.7), that for all $\omega \in \Omega$ and $L > \frac{4L_0}{b}$:

\begin{equation}
(3.2.67) \quad P_{0,\omega}[\overline{T}_{-bL} < T_L] \leq \frac{f(0)}{f(-n')} \frac{\prod_{n'=-n'+n_0+1}^{-1}}{\prod_{n'=-n'+n_0+1}^{-1}} \leq 1.
\end{equation}

We then take the expectation with respect to $\mathbb{P}$ of the left-hand side and split it according to the sets where $\sup_{0 \leq k \leq n_0+1} \prod_{n',-n'+1}^{-1}Y_{n',k}$ is smaller respectively larger than $\frac{1}{n_0+2}e^{-c_\mu L}$ with $c_\mu \overset{\text{def}}{=} \frac{\mu b}{8L_0}$. As a consequence:

\begin{equation}
(3.2.68) \quad P_0[\overline{T}_{-bL} < T_L] \leq e^{-c_\mu L} + \left(\frac{L}{L_0} + 2\right) \sup_{0 \leq k \leq n_0+1} \mathbb{P} \left[ \sum_{j=-n'+1}^{k} \log \hat{\rho}(j) \geq -c_\mu L - \log(n_0 + 2) \right].
\end{equation}

Then we decompose the sum appearing in the second term into three sums of independent random variables $\hat{\rho}(j), j = i \mod 3$ where $i = 0, 1$ or 2. Moreover, since $n_0 \leq \frac{n'+1}{b}$ and by the choice of $c_\mu$, we observe for all $n'$ large enough that for any $0 \leq k \leq n_0 + 1$, we have $\frac{1}{n'+k} (c_\mu L + \log(n_0 + 2)) \leq \frac{\mu}{4}$. Hence the probability on the right-hand side of (3.2.68) is less than

\begin{equation}
(3.2.69) \quad \sum_{i=0}^{3} \mathbb{P} \left[ \frac{3}{n'+k} \sum_{\substack{j=-n'+1 \leq j \leq k \mod 3\ \text{mod}\ 3}} (-\log \hat{\rho}(j) - \mu) \leq -\frac{\mu}{4} \right].
\end{equation}

As for any $j \in \mathbb{Z}, \omega \in \Omega, |\log \hat{\rho}(j)| \leq (L_0 + 1) \log \left(\frac{1}{n'}\right)$, by (3.2.8), it follows from an Azuma-type inequality [see e.g. [1]] that for any $0 \leq k \leq n_0 + 1$, 

\begin{equation}
\text{(3.2.65)} \quad \sum_{i=0}^{3} \mathbb{P} \left[ \frac{3}{n'+k} \sum_{\substack{j=-n'+1 \leq j \leq k \mod 3\ \text{mod}\ 3}} (-\log \hat{\rho}(j) - \mu) \leq -\frac{\mu}{4} \right].
\end{equation}
(3.2.69) is less than

\[ (3.2.70) \]

\[
\leq \sum_{i=0}^{3} \exp \left\{ -\frac{1}{2} \left( \frac{\mu' n' + k}{3} \right)^2 \right\} \left| \{ j \in [-n' + 1, k] \mid j = i \mod 3 \} \right|^{-1} \times \\
((L_0 + 1) \log \kappa^{-1} + \mu)^{-2} \leq 3 \exp \left\{ -c(\mu, L_0) \left( \frac{bL}{L_0} - 1 \right) \right\}.
\]

In view of (3.2.68), this implies condition \((T)\), see (3.1.12).

The implication \((\text{iii}) \Rightarrow (\text{iv})\) is clear. To show \((\text{iv}) \Rightarrow (\text{i})\), we follow the argument of the corresponding multidimensional statement, (see Theorem 3.2.6, \((\text{iii}) \Rightarrow (\text{i})\)), that is in place of (3.2.58) and (3.2.60), we have \(P_0[\tilde{T}_{-L} < T_L] \leq e^{-cL\gamma} \) and \(E[\rho_q^L] \leq e^{(c' - c)\alpha L\gamma} + 2\kappa^{-\alpha(L+3)}e^{-cL\gamma} \), for \(L\) large.

We now come to the implication \((\text{v}) \Rightarrow (\text{ii})\). We follow the arguments in [7], Theorem 2, point b) in the case of a line. This theorem applied to the discrete Markov chain \(X_{V_k}, k \geq 0\) under \(P_{0,\omega}\) for an \(L_0 > R\) in fact shows the equivalence of (ii) and (v). For the readers convenience, we extract and present here the ideas which are relevant for the implication \((\text{v}) \Rightarrow (\text{ii})\). For \(\omega \in \Omega, L_0 > R, n \in \mathbb{Z}\), we introduce the shortcut notations \(p_n = \hat{p}(nL_0, \omega) = 1 - q_n\) and \(\delta_n \overset{\text{def}}{=} P_{nL_0,\omega}[\tilde{T}_{(n-1)L_0} = \infty] \overset{\text{def}}{=} 1 - \eta_n\). We claim that

\[ (3.2.71) \]

\[
P[\delta_0 > 0] = 1.
\]

Indeed, let us assume by contradiction that there exists some \(\omega'\) in the set of full measure \(\{\omega \in \Omega : P_{-L_0,\omega}[X_t \to \infty] = 1\}\) such that \(\delta_0(\omega') = 0\). Then repeated use of the strong Markov property shows that \(P_{0,\omega'}[\liminf_t X_t \leq -L_0] = 1\), a contradiction.

Next, we see that for \(n \in \mathbb{Z}\),

\[ (3.2.72) \]

\[
\eta_n = P_{nL_0,\omega}[\tilde{T}_{(n-1)L_0} < \infty] = q_n + p_n \eta_{n+1} \eta_n, \quad \text{and thus} \quad \eta_n = \frac{q_n}{1 - p_n \eta_{n+1}}.
\]
As a consequence, for all \( \omega \in \Omega, n \leq -1 \),
\[
\delta_n = 1 - \eta_n \overset{(3.2.72)}{=} 1 - \frac{1 - p_n}{1 - p_n \eta_{n+1}} = \hat{\rho}(n)^{-1} \eta_n \delta_{n+1}, \quad \text{and by induction}
\]
(3.2.73)
\[
= (\hat{\rho}(n) \hat{\rho}(n+1) \ldots \hat{\rho}(-1))^{-1} \eta_n \eta_{n+1} \ldots \eta_{-1} \delta_0.
\]
Taking the logarithm of the latter expression and splitting the resulting sum into sums of i.i.d. random variables (similarly as below (3.2.68)), we obtain from the law of large numbers:
\[
\lim_{n \to \infty} \frac{1}{n} \log \delta_n = \mathbb{E}[-\log \hat{\rho}(0)] + \mathbb{E}[\log \eta_0], \quad \mathbb{P}\text{-a.s.,}
\]
(3.2.74)
\[
\lim_{n \to \infty} \frac{1}{n} \log \delta_n = \mathbb{E}[-\log \hat{\rho}(0)] + \mathbb{E}[\log \eta_0], \quad \mathbb{P}\text{-a.s.,}
\]
since \( \lim_n \frac{1}{n} \log \delta_0 = 0, \mathbb{P}\text{-a.s.} \) by (3.2.71).

On the other hand, by translation invariance of \( \mathbb{P} \), we see that for any \( \epsilon > 0 \),
\[
\mathbb{P}[|\log \delta_n| > \epsilon n] = \mathbb{P}[|\log \delta_0| > \epsilon n] \overset{n \to \infty}{\longrightarrow} \mathbb{P}[\delta_0 = 0] \overset{(3.2.71)}{=} 0.
\]
(3.2.75)
\[
\mathbb{P}[|\log \delta_n| > \epsilon n] = \mathbb{P}[|\log \delta_0| > \epsilon n] \overset{n \to \infty}{\longrightarrow} \mathbb{P}[\delta_0 = 0] \overset{(3.2.71)}{=} 0.
\]
In other words, \( \frac{1}{n} \log \delta_n \) converges to 0 in probability, so the right-hand side of (3.2.74) vanishes and \( \mathbb{E}[\log \hat{\rho}(0)] = \mathbb{E}[\log \eta_0] \) which is strictly negative because of (3.2.71). This proves the implication (v) \( \Rightarrow \) (ii).

To show the converse implication, we use the fact that (ii) implies condition (T). Following [46], see the proof of (3.1) \( \Rightarrow \) (3.2) therein, we observe that \( P_0[T_L = \infty] \leq P_0[T_{-L} < T_L], \) since \( P_0[T_{-L} = T_L = \infty] = 0 \) as in every time unit, the trajectory can escape from the interval \([-L, L]\) with a probability bounded away from 0. Observe that the left-hand side increases with \( L \) while the right-hand side tends to 0 by condition \((T)|_{\epsilon_1}\). Hence \( P_0\text{-a.s.}, \limsup_{t \to \infty} X_t = \infty. \) From the strong Markov property and translation invariance of \( \mathbb{P}, \) we obtain for any \( L > 0 \):
\[
(3.2.76) \quad P_0[T_{L} < T_L \circ \theta_{T_L}] = P_0[T_{L} < T_L].
\]
Under condition \((T)|_{\epsilon_1}, \) the left-hand side decreases exponentially and hence and application of Borel-Cantelli's lemma yields that \( P_0\text{-a.s.} \) for large integer \( L, T_{L} < T_L \circ \theta_{T_L} + T_L. \) As a result, we can \( P_0\text{-a.s.} \) construct an integer-valued sequence \( L_k \uparrow \infty, \) with \( L_{k+1} = \lceil \frac{4}{3} L_k \rceil \) and \( T_{L_{k+1}} < \widetilde{T}_{L_k} \circ \theta_{T_{L_k}} + T_{L_k}, k \geq 0. \) This shows (v). \( \square \)
Remark 3.2.2 Let us mention that for any $L > 0$,

\[ \mathbb{E}[\log \rho_L] = -2L \mathbb{E}[b(0)/a(0)], \]

and as a consequence, if conditions (i) or (ii) above are satisfied for some $L \geq R$, they are in fact satisfied for all $L > 0$. Indeed, using the scale function $s(x, \omega) = \int_0^x \exp \left\{ - \int_0^y 2b(u, \omega)/a(u, \omega) du \right\} dy$, for $x \in \mathbb{R}, \omega \in \Omega$, see for instance [3] p.78 and p.88, we can write $\rho_L = \frac{s(L)}{-s(-L)}$. It follows that for $L > 0$, $\mathbb{E}[\log \rho_L]$ equals

\[ \mathbb{E}\left[ \log \int_0^L e^{-\int_0^y \frac{2b(u, \omega)}{a(u, \omega)} du} dy \right] - \mathbb{E}\left[ \log \int_{-L}^0 e^{-\int_{-L}^y \frac{2b(u, \omega)}{a(u, \omega)} du} dy \right]. \]

Because of translation invariance of $\mathbb{P}$, the second term becomes

\[ \mathbb{E}\left[ \log \int_0^L e^{-\int_0^y \frac{2b(u, \omega)}{a(u, \omega)} du} dy \right] + \mathbb{E}\left[ \int_{-L}^0 \frac{2b(u, \omega)}{a(u, \omega)} du \right], \]

so that the first term of (3.2.78) is canceled out. Fubini’s theorem then yields $\mathbb{E}[\log \rho_L] = - \int_{-L}^0 \mathbb{E}\left[ \frac{2b(u, \omega)}{a(u, \omega)} \right] du$, and the claim follows from translation invariance of $\mathbb{P}$. □

### 3.3 An example of a ballistic diffusion

#### 3.3.1 Main result and preliminaries

In this section, we use the effective criterion to show that a Brownian motion perturbed by a small random drift which is bounded by $\epsilon > 0$ and whose expectation in direction $\ell = e_1$ is of order $\epsilon^{2-\eta}$ with $\eta > 0$ satisfies condition $(T')|e_1$. The interest of this class of diffusions stems from the fact that it contains new examples of ballistic diffusions which in particular do not fulfill the criterion of [46], Theorem 5.2 therein, which states that there exists a constant $c_\epsilon > 1$ such that if

\[ \mathbb{E}[(b(0, \omega) \cdot e_1)_+] > c_\epsilon \mathbb{E}[(b(0, \omega) \cdot e_1)_-], \]

\[ \mathbb{E}[(b(0, \omega) \cdot e_1)_+] > c_\epsilon \mathbb{E}[(b(0, \omega) \cdot e_1)_-], \]
then \((T')|e_1\) holds. Before we give further explanations on this matter, see Remark 3.1 below, we introduce the family of perturbed Brownian motions studied in this section. For any \(\epsilon \in (0, \bar{K}], \eta > 0\) and \(\omega \in \Omega\), we consider the class of diffusions attached to an operator of the form

\[
\mathcal{L} = \frac{1}{2} \Delta + b(x, \omega) \cdot \nabla,
\]

where we require that for all \(x \in \mathbb{R}, \omega \in \Omega\),

\[
|b(x, \omega)| \leq \epsilon, \quad \lambda \overset{\text{def}}{=} \mathbb{E}[b(0, \omega) \cdot e_1] \geq \epsilon^{2-\eta}.
\]

Note that the constant \(\bar{K}\), the ellipticity constant \(\nu\) and the dependence range \(R\), see (3.1.2)-(3.1.5), do not depend on \(\epsilon\). We keep the convention concerning constants stated at the end of the Introduction. Moreover, when we write that an expression holds “for large enough \(L\)” we mean that the expression holds for all \(L\) larger than some \(c(\eta)\).

The main result of the section is

**Theorem 3.3.1.** When \(d \geq 4\), for any \(\eta \in (0, 1)\) there is \(e_0(\eta, d) > 0\) such that whenever (3.3.3) holds for \(0 < \epsilon < e_0\) then condition \((T')|e_1\) is satisfied.

**Remark 3.3.1.** Clearly, (3.3.1) is equivalent to

\[
\mathbb{E}[b(0, \omega) \cdot e_1] > (c_\epsilon - 1)\mathbb{E}[(b(0, \omega) \cdot e_1)_-].
\]

An inspection of the proof of [46], Theorem 5.2., reveals that \(c_\epsilon > 1\), and hence (3.3.4) fails when \(\epsilon > 0\) is small, if \(\mathbb{E}[b(0, \omega) \cdot e_1]\) is of order \(\epsilon^{2-\eta}\) with \(0 < \eta < 1\) and \(\sup_{\omega \in \Omega} (b(0, \omega) \cdot e_1)_-\) is of order \(\epsilon\) under an adequate choice of \(\mathbb{P}\). With this observation one can rather straightforwardly produce examples where (3.3.3) holds with \(\epsilon < e_0(\eta, d)\), but (3.3.1) or (3.3.4) fail. \(\square\)

The rest of the section is devoted to the proof of Theorem 3.3.1. We will verify the effective criterion (3.2.53) when \(\epsilon\) is smaller than some \(e_0(\eta, d)\) for \(a = 1/2\) and a box \(B = B(\text{Id}, NL' - R - 2, NL' + 2, \frac{1}{4}(NL')^3)\), see (3.2.2), where

\[
N = L^3 \text{ and } L = L' - \frac{R}{2} \text{ is an integer such that } L = \left\lfloor \frac{1}{4\epsilon} \right\rfloor.
\]
3.3.1. Main result and preliminaries

The starting point to estimate \( \mathbb{E}[\rho_B^{1/2}] \) is (3.2.15). Here we set, cf. (3.2.6), \( L_1 = NL', \tilde{L}_1 = \frac{1}{4}(NL')^3, L_0 = L', n_0 = N \) and \( a = 1/2 \). With these choices, the box \( B \) defined above, on which we want to check (3.2.53), equals \( B_1 + e_1 \). In order to apply (3.2.15) we use the following

**Lemma 3.3.2.** For \( a \in (0, 1] \) and \( B_1 \) a box as in (3.2.6) with \( \ell = e_1, L_1 \geq R + 3 \) and \( R = \text{Id} \),

\[
\mathbb{E}[\rho_{B_1 + e_1}^a] \leq c^a \mathbb{E}[\rho_{B_1}^a].
\]

**Proof.** Since for every \( \omega \in \Omega, P_{x,\omega}[X_{TB_1} \in \partial B_1] \) is harmonic on \((-2, 2)^d \), Harnack’s inequality implies that \( \frac{\mathbb{P}_{-e_1,\omega}[X_{TB_1} \in \partial B_1]}{\mathbb{P}_{-e_1,\omega}[X_{TB_1} \in \partial B_1]} \leq c \rho_{B_1}(\omega) \). The claim then follows from translation invariance of \( \mathbb{P} \). \( \square \)

For the purpose of this section, we need a bound on \( \mathbb{P}[\mathcal{G}^c] \) appearing in (3.2.15) which differs from (3.2.32) and which is essentially the same as the estimate in [56], Theorem 1.1. We now follow [56] to introduce the notation used for this bound. Let \( h, H, M \) be positive integers with

\[
2h \leq H \leq \frac{(NL')^3}{32} \quad \text{and} \quad M = \left\lfloor \frac{(NL')^3}{32H} \right\rfloor.
\]

Later on, see (3.3.51), we will choose \( H \) and \( h \) to be of order \( (NL')^2 \) and \( L^2 \), respectively. We introduce the exit time \( S \) from a tube:

\[
S = \inf \{ t > 0; |(X_t - X_0) \cdot e_1| \geq L \text{ or } \sup_{j \geq 2} |(X_n - X_0) \cdot e_j| \geq h \}.
\]

and the expected displacement

\[
\Delta(x, \omega) = E_{x,\omega}[X_S] - x, \quad x \in \mathbb{R}^d, \omega \in \Omega.
\]

Moreover, for \( 0 < \gamma \leq 1 \), later chosen to be of order \( \epsilon^{1-\eta} \), see (3.3.51), we define

\[
p_L = \inf_{j \geq 2} \mathbb{P}[\text{for all } z \in \tilde{B}^j, \Delta(z, \omega) \cdot e_1 \geq \gamma L],
\]

where for \( 2 \leq j \leq d \),

\[
\tilde{B}^j = \{ y \in B, |y \cdot e_j| < H \}.
\]

Let us now state the analog of Theorem 1.1 in [56].
Proposition 3.3.3. There exists a constant $c_8 > R + 3$ such that when $L \geq c_8$ and
\begin{equation}
\delta^{-1} \overset{\text{def}}{=} \exp \left\{ -\frac{\gamma N}{128} + \frac{10N}{\gamma} \exp \left\{ -\frac{\gamma N}{32} \left( \frac{H}{2hN} - \frac{4}{\gamma} \right)^2 \right\} \right\} < 1,
\end{equation}
than for any $0 < a \leq 1$
\begin{equation}
\mathbb{E}[\rho_B^a] \leq c^a \kappa^{-aNL'}2d \exp \left\{ -\frac{M}{2} \left( p_L - \frac{10NL \log \kappa^{-1}}{M \log \delta} \right)^2 \right\} + c^a \frac{2\mathbb{E}[\hat{\rho}(0,\omega)^{2a}]^{N/2}}{(1 - \mathbb{E}[\hat{\rho}(0,\omega)^{2a}]^{1/2})_+}.
\end{equation}

Since the proof is very similar to the one of Theorem 1.1 in [56], we only make a few comments here. Because of (3.3.6), we can estimate $\mathbb{E}[\rho_B^a]$ with the help of (3.2.15). We bound the second term on the right-hand side in the latter expression using translation invariance of $\mathbb{P}$ and obtain the second term on the right-hand side of (3.3.13). The intuitive idea behind the estimate on $\mathbb{P}[G^c]$ in the first term on the right-hand side of (3.2.15), leading to the first term in the right-hand side of (3.3.13), is to consider nested boxes $B_k = (-NL' + R + 2, NL' + 2) \times (-k4H, k4H)^d - 1$ for $0 \leq k \leq M$ contained in the big box $B$. Then in order not to exit through "the left or right" of $B$, the trajectory has to reach the boundary of box $B_k$ before exiting $B$ and then move from box $B_k$ to box $B_{k+1}$ without exiting $B$. The probability of this last step is related to the quantity $1 - p_L$.

Note that the coefficient in the first term of $\delta^{-1}$ differs from the result in [56] as the width of $B$ is a multiple of $L'$ while the definition of the time $S$ uses the quantity $L = L' - \frac{R}{2}$. This affects the right-hand side of the expression below (1.24) in [56].

Despite the finite range dependence, the remark in [56] below (1.29) still holds since [in the notation of [56]] the random variables $Z_k(\epsilon)$ and $Z_{k-1}(\epsilon)$ are respectively $H_{\{z \in \mathbb{R}^d : z \geq 4kH - H - h\}}$ and $H_{\{z \in \mathbb{R}^d : z \leq 4(k-1)H + H + h\}}$ - measurable. The involved half-spaces are separated by a distance $2(H - h)$ which is larger than $H$ by (3.3.7). Hence $(Z_k)_{0 \leq k \leq M}$ are independent if $H$ and thus $L$ are large enough.
3.3.2 Bounds on the Green operator

The main Theorem 3.3.1 will follow after choosing \( h \) and \( H \) as in (3.3.51) once we show exponential decay in \( L \propto \epsilon^{-1} \) of both terms on the right-hand side of (3.3.13) for \( a = 1/2 \). Therefore the goals of this section are to find a tractable expression for \( \bar{\rho}(0, \omega) \), see Lemma 3.3.5, that involves the Green operator of the diffusion killed when exiting the open slab \( S \defeq \{ x \in \mathbb{R}^d : |x \cdot e_1| < L \} \), and then investigate its relation with the Green operator of killed Brownian motion, see Proposition 3.3.8. Certain deterministic estimates on the latter operator and its kernel, see Lemmas 3.3.7 and 3.3.9, will then be instrumental in the proof of the desired exponential decay of \( \mathbb{E}[\rho_{1/2}^s] \), see Proposition 3.3.10.

Throughout this section, we use the shortcut notation \( b_i \defeq b \cdot e_i \) and we set \( \|f\|_\infty = \sup_{x \in S} |f(x)| \), for any function \( f \) on \( S \). For any bounded measurable function \( f \) on \( S \) and any \( x \in S, \omega \in \Omega \), let us denote with

\[
G_{S}^{\omega} f(x) \defeq E_{x,\omega} \left[ \int_0^{T_S} f(X_s) ds \right],
\]

respectively

\[
G_{S} f(x) \defeq E \left[ \int_0^{T_S} f(x + W_s) ds \right],
\]

the Green operator of the diffusion respectively Brownian motion killed when exiting the slab \( S \). (Here \( E \) denotes the expectation with respect to some measure under which \( W_s \) is a Brownian motion.) Note that by (3.3.16) below, these operators acting on \( L^\infty \) have norm bounded by \( 2L^2 \). Moreover, the semi-group \( P_t^{\omega} \) of the diffusion in environment \( \omega \) killed when exiting \( S \) is defined as

\[
P_t^{\omega} f(x) = E_{x,\omega}[f(X_t), t < T_S], \quad \text{for } x \in S, t \geq 0.
\]

In a similar fashion, we denote with \( P_t \) the semi-group of a Brownian motion killed when exiting \( S \).

The following lemma states basic bounds on the expected exit time from the slab \( S \) and on the supremum-norm of the operator \( P_t^{\omega} \).

**Lemma 3.3.4.** For \( \omega \in \Omega, \epsilon \in (0, 1/4), x \in S \), under the assumption (3.3.3) and with the definition (3.3.5),

\[
\frac{2}{3} (L^2 - (x \cdot e_1)^2) \leq E_{x,\omega}[T_S] \leq 2(L^2 - (x \cdot e_1)^2).
\]
For any bounded measurable function $f$ and any $\omega \in \Omega$, 
\begin{equation}
\|P_t^\omega f\|_\infty \leq c_{10}\|f\|_\infty \exp\left(-c_{11}t/L^2\right), \quad \text{for } t > 0.
\end{equation}

**Proof.** To show (3.3.16), we consider for $x \in S, \omega \in \Omega$ the $P_{x,\omega}$-martingale
\begin{equation}
(X_t \wedge T_S - \epsilon_1)^2 - (X_0 - \epsilon_1)^2 - \int_0^{t \wedge T_S} 2b_1(X_s, \omega)(X_s - \epsilon_1)ds - t \wedge T_S.
\end{equation}

After taking expectations and using the monotone convergence theorem, we obtain (3.3.16) from our assumption $|b(\cdot, \cdot)| \leq \epsilon$, see (3.3.3), and the choice $L \leq \frac{1}{4\epsilon}$, see (3.3.5).

We now turn to (3.3.17). By the support theorem [see [3]] applied to the rescaled diffusion $\frac{1}{L}X_L^{2t}$ and the fact that $|Lb_1| \leq \frac{1}{4}$, the probability under $P_{x,\omega}$ that the trajectories leave the slab within time $L^2$ when starting in $x \in S$ is bounded away from 0 by some constant $c_{11}$. Hence the strong Markov property yields for any $t > 0, x \in S, \omega \in \Omega$, that $P_{x,\omega}[t \leq T_S] \leq c_{10} \exp(-c_{11}t/L^2)$, and (3.3.17) follows from the definition (3.3.15). \(\square\)

**Remark 3.3.2:**

1. For Brownian motion starting at $x \in S$, the expected exit time from the slab $S$ equals $L^2 - (x \cdot e_1)^2$. The analogue of (3.3.17) for Brownian motion is also valid.

2. We point out that since $T_S$ has a finite moment under $P_{x,\omega}$ by (3.3.16), Fubini’s Theorem applied to (3.3.14) yields for any bounded measurable function $f$ and any $\omega \in \Omega, x \in S$ that
\begin{equation}
G_S^{\omega} f(x) = \int_0^\infty P_t^{\omega} f(x)dt.
\end{equation}

Of course, the same relation holds for the killed Brownian motion. \(\square\)

Let us now introduce the following shortcut notation for the set appearing in the definition of $\hat{\rho}(0, \omega)$, see (3.2.13):
\begin{equation}
\mathcal{V} \overset{\text{def}}{=} \{x \in \mathbb{R}^d; |x \cdot e_1| \leq \frac{R}{2}, |x|_\perp \leq \frac{1}{4}(NL')^3\}.
\end{equation}
For later purposes, we observe that (3.3.16) and our assumption (3.3.3) on \( \lambda \) imply, that there are constants \( c_{12} > 0 \) and \( L_{1}(c_{12}, \eta) \) such that when \( L \geq L_{1} \) then for any \( x \in \mathcal{V}, \omega \in \Omega \),

\[
G_{S}^{\omega} \lambda(x) = \lambda E_{x, \omega}[T_{S}] \geq c_{12} L^{\eta}.
\]

The next lemma provides a tractable expression of \( \rho(0, \omega) \) in terms of the Green operator \( G_{S}^{\omega} \).

**Lemma 3.3.5.** For \( L \geq 3R, \omega \in \Omega \), with (3.3.3) and (3.3.5),

\[
\hat{\rho}(0, \omega) = \sup_{x \in \mathcal{V}} \frac{L - x \cdot e_{1} - G_{S}^{\omega}(b_{1}(\cdot, \omega))(x)}{L + x \cdot e_{1} + G_{S}^{\omega}(b_{1}(\cdot, \omega))(x)} \leq 5.
\]

(See (3.2.13), (3.3.20) for the notation.)

**Proof.** For any \( x \in \mathcal{S}, \omega \in \Omega \), \( X_{t \wedge T_{S}} \cdot e_{1} - X_{0} \cdot e_{1} - \int_{0}^{t \wedge T_{S}} b_{1}(X_{s}, \omega) ds \) is a \( P_{x, \omega} \)-martingale. Hence, after taking expectations, we obtain from the dominated convergence theorem that (see (3.2.14) for the notation)

\[
\hat{\rho}(x, \omega) = \frac{x \cdot e_{1} + L + G_{S}^{\omega}(b_{1}(\cdot, \omega))(x)}{2L}.
\]

Inserting this expression into the definition (3.2.13) of \( \rho(0, \omega) \) yields the claimed equality. Using (3.3.16), (3.3.5), we see that for all \( L > 0 \),

\[
|G_{S}^{\omega}(b_{1}(\cdot, \omega))(x)| \leq \frac{L}{2},
\]

and thus the inequality in (3.3.22) follows when \( L \geq 3R \).

In order to explore the relationship between the Green operators of the diffusion and Brownian motion, see (3.3.40), we need to collect a few facts about the semi-group of Brownian motion. From [51], Theorem 8.1.18 we have that whenever \( f \) is a continuous and bounded function, then \( (t, x) \mapsto P_{t}f(x) \) is bounded and in \( C^{1,2}([0, \infty) \times \mathcal{S}, \mathbb{R}) \). Moreover,

\[
\frac{\partial}{\partial t} P_{t}f = \frac{1}{2} \Delta P_{t}f \text{ in } (0, \infty) \times \mathcal{S},
\]

\[
\lim_{t \to 0} P_{t}f(x) = f(x), \quad x \in \mathcal{S}.
\]
Since every point on the boundary of $S$ is regular according to [51], (8.1.16) therein, we have the following continuity property at the boundary (see [51], Theorem 8.1.18):

\[
\lim_{(t,x)\to(s,a)} P_t f(x) = 0, \text{ for } (s,a) \in (0, \infty) \times \partial S.
\]

Our next step is to express $P_t$ and $G_S$ in terms of kernels using "the method of images" from electrostatics.

**Proposition 3.3.6.** Let $f$ be a bounded measurable function on $\mathbb{R}^d$. If we define for $t > 0; x, y \in S$

\[
p(t, x, y) = \sum_{k=-\infty}^{\infty} p_d(t, x, y + 2k2L\varepsilon_1) - p_d(t, x, y^* + (2k + 1)2L\varepsilon_1),
\]

where $p_d(t, x, y) \overset{\text{def}}{=} (2\pi t)^{-d/2} \exp \left\{ |x - y|^2 / 2t \right\}$ is the $d$-dimensional heat kernel and $y^*$ is the image of $y$ under reflection with respect to $\{z \in \mathbb{R}^d : z \cdot \varepsilon_1 = 0\}$, then

\[
P_t f(x) = \int_S p(t, x, y)f(y)dy.
\]

Moreover, when $d \geq 4$, if we define the Green's function for distinct $x, y \in S$ by

\[
g(x, y) = \sum_{k=-\infty}^{\infty} g_d(x, y + 2k2L\varepsilon_1) - g_d(x, y^* + (2k + 1)2L\varepsilon_1),
\]

where $g_d(x, y) \overset{\text{def}}{=} \int_0^{\infty} p_d(t, x, y)dt = \gamma_d|x - y|^{2-d}$ for $x \neq y$ and an appropriate constant $\gamma_d$, then

\[
G_S f(x) = \int_S g(x, y)f(y)dy.
\]

**Proof.** The fact that $p(t, x, y)$ in (3.3.28) satisfies the equality in (3.3.29), follows from [24], Proposition 8.10 after mapping the interval $[0, a]$ to $[-L, L]$.
and after multiplying with $p_{d-1}$. It is well known that $g_d(x,y)$ equals $\gamma_d|x - y|^{2-d}$ for an appropriate constant $\gamma_d$ when $d \geq 3$ and $x \neq y$, see e.g. [51], (8.4.10). To see that the expression in (3.3.30) is indeed the kernel of $G_S$, we observe that $p(t,x,y)$ is integrable over $t$ for $x \neq y$, since by the monotone convergence theorem, we have $\int_0^\infty p(t,x,y) \, dt \leq \sum_{k=-\infty}^{\infty} g_d(x,y + 2k2L\epsilon_1) + g_d(x,y^* + (2k + 1)2L\epsilon_1)$, and since the latter series converges absolutely when $d \geq 4$. Moreover, with dominated convergence,

$$g(x,y) = \int_0^\infty p(t,x,y) \, dt \text{ for } x \neq y,$$

Then we insert (3.3.29) into (3.3.19) and since $(t,y) \mapsto p(t,x,y)f(y)$ is product integrable by Tonelli’s theorem and (3.3.17), we obtain (3.3.31) from Fubini’s theorem and (3.3.32).

The next lemma provides gradient estimates on the semi-group and the Green operator of killed Brownian motion which play an important role in the derivation of the perturbation equality (3.3.40) and in the proof of Proposition 3.3.10.

**Lemma 3.3.7.** $(d \geq 4)$ For any bounded, continuous function $f$, there exists $c_{13}, c_{14} > 0$ such that for all $x \in S, t > 0$ and $L > 0$,

$$|\nabla P_t f(x)| \leq \left( \frac{c_{13}}{L} + \frac{c_{14}}{\sqrt{t}} \right) \exp\left( -\frac{\epsilon_1}{2} t/L^2 \right) ||f||_\infty, \quad \text{and}$$

$$|\nabla G_S f(x)| \leq c_{15} ||f||_\infty L.$$

**Proof.** We first show (3.3.33). Let $(x^{(1)}, \ldots, x^{(d)})$ denote the coordinates of a point $x$ in $\mathbb{R}^d$. We estimate the partial derivatives $\partial_i$, $i = 1, \ldots d$ of $P_t f(x)$ separately. As a consequence of the semi-group property, we have that for $t > 0, x \in S$,

$$P_t f(x) = \int_S p(t/2, x,z) P_{t/2} f(z) \, dz.$$
We let $U \subset S$ be a neighbourhood of $x$. To compute $\partial_t P_t(x)$ by interchanging derivation and integration, we need to show that $|\partial_t p(t/2, x, z)P_{t/2}f(z)|$ is $dx \times dz$ integrable over $U \times S$. After an application of (3.3.17), we see that

\[
(3.3.36) \sup_{x \in U} \int_S |\partial_t p(t/2, x, z)P_{t/2}f(z)| \, dz \leq \exp(-c_{11}t/L^2)\|f\|_\infty \sup_{x \in U} \int_S |\partial_t p(t/2, x, z)| \, dz.
\]

For $i = 1$, according to (3.3.28), $|\partial_1 p(t/2, x, z)|$ is smaller than

\[
(3.3.37) \prod_{j=2}^d p_1(t/2, x^{(j)}, z^{(j)}) \times \\
\sum_{k=-\infty}^\infty |\partial_1 p_1(t/2, x^{(1)}, z^{(1)} + 4kL)| + |\partial_1 p_1(t/2, x^{(1)}, -z^{(1)} + (2k + 1)2L)|.
\]

The integral over $\mathbb{R}^d$ of the first $d - 1$ factors in the latter expression equals 1 and using monotone convergence, we find that for any $x \in U, t > 0$, the integral on the right-hand side of (3.3.36) is smaller than

\[
(3.3.38) \sum_{k \neq 0} p_1(t/2, x^{(1)}, L + 4kL) + p_1(t/2, x^{(1)}, -L + 4kL) + \\
\sum_{k \neq -1} p_1(t/2, x^{(1)}, L + (2k + 1)2L) + \sum_{k \neq 0} p_1(t/2, x^{(1)}, -L + (2k + 1)2L) + \\
\int_{-L}^L \frac{1}{\sqrt{\pi t}} e^{-\frac{4(x^{(1)} - z)^2}{t}} \left| \frac{x^{(1)} - z}{t} \right| \, dz + p_1(t/2, x^{(1)}, -L) + p_1(t/2, x^{(1)}, L).
\]

For any $x \in U$, the function $z \mapsto p_1(t/2, x^{(1)}, z)$ is monotone on $(-\infty, -2L]$ and on $[2L, \infty)$. Therefore the first sum in (3.3.38) is less than $\frac{4}{L} \int_{-\infty}^\infty p_1(t/2, x^{(1)}, z) \, dz = \frac{4}{L}$. A similar argument yields that the second and third sum in (3.3.38) are less than $\frac{c}{L}$. The integral in (3.3.38) is less than $2 \int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-u^2} u \, du = \frac{1}{\sqrt{\pi t}}$ and the last two terms can also be bounded by $\frac{c}{\sqrt{t}}$. Collecting our estimates, we obtain for $i = 1$ that the left-hand side of (3.3.36) is less than

\[
(3.3.39) \exp(-c_{11}t/L^2)\|f\|_\infty \left( \frac{c}{L} + \frac{c'}{\sqrt{t}} \right).
\]
Hence we can interchange the derivative $\partial_1$ with the integral in (3.3.35), and for any $x \in S$, $|\partial_1 P_t(x)|$ is bounded by (3.3.39). Similar bounds on $|\partial_i P_t(x)|$, for $2 \leq i \leq d$, follow from an easier version of the above arguments. Indeed, in an expression corresponding to (3.3.37), the last factor containing the sum, which was more delicate to treat, will not be affected by the derivative $\partial_i$ and thus its integral over $[-L, L]$ equals one. This proves (3.3.33). Since the latter estimate shows that $\nabla P_t f(x)$ is integrable with respect to $t > 0$, (3.3.34) is an immediate consequence of (3.3.19).

The link between the Green operators of the killed diffusion and the killed Brownian motion is expressed by the following perturbation equality:

**Proposition 3.3.8.** Let $f$ be a bounded, continuous function on $\mathbb{R}^d$. Then we have for all $x \in S, \omega \in \Omega$ that

\[
G^\omega_S f(x) = G_S f(x) - G^\omega_S (b(\cdot, \omega) \cdot \nabla)G_S f(x).
\]

**Proof.** The classical idea of the proof is to take the derivative of $P_t^\omega P_{u-t} f(x)$ with respect to $t$, which yields $P_t^\omega ((\mathcal{L} - \Delta)P_{u-t} f)(x)$. Then one integrates both sides with respect to $t$ from 0 to $u$ and with respect to $u$ from 0 to infinity. The result then follows from Fubini’s theorem. Let us now present the details of the proof. For $\omega \in \Omega, u > 0, x \in S$, we claim that

\[
P_u^\omega f(x) - P_u f(x) = \int_0^u P_t^\omega (b(\cdot, \omega) \cdot \nabla P_{u-t} f)(x) dt.
\]

To prove the claim, we define for $h > 0$ the function

\[
\epsilon(t, x) \overset{\text{def}}{=} P_{u+h-t} f(x), \quad \text{with } 0 \leq t \leq u, x \in S.
\]

According to [51], Theorem 8.1.18, $\epsilon$ is in $C^{1,2}((0, u) \times S)$. Hence we can apply Ito’s formula to a function $\epsilon_n \in C^{1,2}((0, u) \times \mathbb{R}^d)$ such that $\epsilon_n(t, \cdot) = \epsilon(t, \cdot)$ on $D_n \overset{\text{def}}{=} \{x \in S, \text{dist}(x, \partial S) \geq 1/n\}$ and $\epsilon_n(t, \cdot) = 0$ on $S^c$. Because of (3.3.25), we obtain for all $\omega \in \Omega, x \in D_n$ after taking expectations:

\[
E_{x, \omega} \left[ \epsilon(u \wedge T_{D_n}, X_{u \wedge T_{D_n}}) - \epsilon(h \wedge T_{D_n}, X_{h \wedge T_{D_n}}) - \int_{h \wedge T_{D_n}}^{u \wedge T_{D_n}} b(X_s, \omega) \cdot \nabla \epsilon(s, X_s) ds \right] = 0.
\]
When $n$ tends to $\infty$, $t \wedge T_{D_n}$ increases to $t \wedge T_S$, and it follows from the dominated convergence theorem and (3.3.27) that for any $\omega \in \Omega, x \in S$,

$$E_{x,\omega}[e(u \wedge T_{D_n}, X_{u \wedge T_{D_n}}), u \geq T_S] \xrightarrow{n \to \infty} 0.$$ 

The same result holds for $h$ in place of $u$. From (3.3.33), we have that $\sup_{0 \leq t \leq u, x \in S} |\nabla e(v, x)|$ is finite. Thus coming back to (3.3.43) and letting $n \to \infty$, we obtain with dominated convergence that for any $x \in S$,

$$E_{x,\omega}[e(u, X_u), u < T_S] - E_{x,\omega}[e(h, X_h), h < T_S]$$

$$= E_{x,\omega}\left[ \int_h^{u \wedge T_S} b(X_s, \omega) \cdot \nabla e(s, X_s) ds, h < T_S \right].$$

We now insert the definition (3.3.42) into the above expression and let $h$ tend to 0 using dominated convergence. This concludes the proof of (3.3.41).

The integral with respect to $u > 0$ of the left-hand side of (3.3.41) equals $G^\omega_S f(x) - G_S f(x)$, see (3.3.19). On the right-hand side, (3.3.33) and (3.3.17) imply that the iterated integral:

$$\int_0^\infty \int_0^\infty \left| P^\omega_t (b(\cdot, \omega) \cdot \nabla P_{u-t} f)(x) \right| 1_{\{t < u\}} \, du \, dt$$

$$\leq \int_0^\infty \int_t^\infty c \|f\|_\infty \left( \frac{C_{13}}{L} + \frac{C_{14}}{\sqrt{u-t}} \right) e^{-\frac{C_{14}}{2L^2}(t+u)} \, du \, dt$$

is finite. Hence we can integrate the right-hand side of (3.3.41) with respect to $u$, use Fubini's theorem and then substitute $u-t$ with $u$. It follows for $\omega \in \Omega, x \in S$,

$$G^\omega_S f(x) - G_S f(x) = \int_0^\infty \int_0^\infty P^\omega_u (b(\cdot, \omega) \cdot \nabla P_t f)(x) \, du \, dt.$$

The same argument as before allows us to interchange the integrals once more. Finally with (3.3.33) and a further application of Fubini's theorem we can move the $dt$-integral inside $P^\omega_u (\cdot)$ and interchange it with the gradient. This finishes the proof of Proposition 3.3.8. □

We close this section with estimates on the Green's function (3.3.30) of killed Brownian motion and on its gradient. They are at the heart of the proof of Proposition 3.3.10.
Lemma 3.3.9. \((d \geq 4)\) For all \(x, y \in S\) and \(L > 0\) we have

\[
\begin{align*}
(3.3.48) \quad & g(x, y) \leq c_{16}|x - y|^{2-d} \exp\left(-c_{17}|x - y|_\perp/L\right), \\
(3.3.49) \quad & \nabla g(x, y) \leq (c_{18}|x - y|^{1-d} + c_{19}L^{1-d}) \exp\left(-c_{17}|x - y|_\perp/L\right).
\end{align*}
\]

Moreover, for any bounded H"older continuous function \(f\), \(G_Sf\) is twice continuously differentiable on \(S\) and

\[
(3.3.50) \quad \frac{1}{2} \Delta G_Sf(x) = -f(x), \quad \text{for } x \in S.
\]

The proof is included in the Appendix B and the arguments showing (3.3.48) and (3.3.49) are similar to the proof of [56], (2.11), (2.13) therein.

### 3.3.3 Proof of Theorem 3.3.1

The starting point for the proof is (3.3.13) with \(a = \frac{1}{2}\). We first specify the quantities \(h, H, \gamma\) involved in the first term on the right-hand side of (3.3.13), see (3.3.7), (3.3.10):

\[
(3.3.51) \quad h \overset{\text{def}}{=} L'^2, \quad H \overset{\text{def}}{=} [(NL')^2], \quad \gamma \overset{\text{def}}{=} \frac{1}{4} c_{12} L^{\eta-1}.
\]

It is clear that the main Theorem 3.3.1 follows from the effective criterion once we show exponential decay in \(L \propto \epsilon^{-1}\) of both terms on the right-hand side of (3.3.13). We first examine the second term. It suffices to show that for large enough \(L\)

\[
(3.3.52) \quad \mathbb{E}[\hat{\rho}(0, \omega)] \leq \exp\left(-\frac{c_{12}}{2} L^{-1}\right),
\]

where \(c_{12}\) is defined in (3.3.21). Indeed, since we assumed \(N = L^3\), see (3.3.5), the second term of (3.3.13) then becomes smaller than \(cL \exp\left(-\frac{c_{12}}{4} L^2\right)\), which will be more than sufficient for the application of the effective criterion (3.2.53).
To prove (3.3.52), we use (3.3.22) and write $E[\hat{\rho}(0, \omega)]$ as

\[(3.3.53)\]

\[
\mathbb{E} \left[ \sup_{x \in \mathcal{V}} \frac{L - x \cdot e_1 - G_{S}^{\omega}(b_1(\cdot, \omega))(x)}{L + x \cdot e_1 + G_{S}^{\omega}(b_1(\cdot, \omega))(x)} \right] \inf_{x \in \mathcal{V}} G_{S}^{\omega}(b_1(\cdot, \omega))(x) \geq \frac{c_{12}}{2} L^{\eta} \]

\[+ 5 \mathbb{P} \left[ \inf_{x \in \mathcal{V}} G_{S}^{\omega}(b_1(\cdot, \omega))(x) < \frac{c_{12}}{2} L^{\eta} \right].\]

When $L$ is larger than some $c(\eta)$, the first term becomes smaller than $1 - \frac{c_{12}}{2} L^{\eta-1} \leq \exp \left( -\frac{c_{12}}{2} L^{\eta-1} \right)$. Hence (3.3.52) follows from the next proposition which estimates the second term of (3.3.53).

**Proposition 3.3.10.** $(d \geq 4)$ For any $\eta \in (0, 1)$, under the assumption (3.3.3) and with (3.3.5), we have that

\[(3.3.54)\]

\[
\limsup_{L \to \infty} L^{-\frac{3}{2} \eta} \log \mathbb{P} \left[ \inf_{x \in \mathcal{V}} G_{S}^{\omega}(b_1(\cdot, \omega))(x) < \frac{c_{12}}{2} L^{\eta} \right] < 0,
\]

where $\mathcal{V}$ and $c_{12}$ are defined in (3.3.20) and (3.3.21).

Before proving the proposition, we show that (3.3.54) together with our choices in (3.3.51) also yield exponential decay of the first term on the right-hand side of (3.3.13), which then finishes the proof of the main theorem. Using (3.3.5), we find that

\[(3.3.55)\]

\[
\delta^{-1} \leq \exp \left( -c L^{2+\eta} \right) + c' L^{4-\eta} \exp \left\{ -c'' L^{2+\eta} \left( L^3 - c''' L^{1-\eta} \right)^2 \right\},
\]

which tends to 0 as $L$ goes to $\infty$, so that (3.3.12) holds when $L$ is large. If in addition, we know that [see (3.3.10) for the notation]

\[(3.3.56)\]

\[
\liminf_{L \to \infty} p_L = 1,
\]

an easy calculation using (3.3.51) and $M \geq c_{13} NL$ (see (3.3.7) for the definition) shows that for $L$ large enough, the first term on the right-hand side of (3.3.13) is less than $c \exp(-cNL)$, and the effective criterion (3.2.53) is satisfied for large $L$.

We now prove that Proposition 3.3.10 implies (3.3.56). First we cover the sets $\tilde{B}_j$, $2 \leq j \leq d$ [see (3.3.11)] with a collection of disjoint cubes of side
length $\frac{R}{2}$. The cardinality of this collection is for large $L$ at most $L''$ where $\nu$ only depends on $d$. Translation invariance then yields:

$$p_L \geq 1 - \sup_{2 \leq j \leq d} c' L'' \mathbb{P}[\inf_{x \in [-\frac{R}{2}, \frac{R}{2}]} \Delta(x, \omega) \cdot e_1 < \gamma L].$$

In this expression we will in essence replace $\Delta(x, \omega) \cdot e_1$ with $G_S^\omega(b_1(\cdot, \omega))(x)$. More precisely we claim that for large $L$ and for all $\omega \in \Omega, x \in [-\frac{R}{2}, \frac{R}{2}]^d$,

$$|\Delta(x, \omega) \cdot e_1 - G_S^\omega(b_1(\cdot, \omega))(x)| \leq c_{20}. \tag{3.3.58}$$

Then with our choice of $\gamma$, see (3.3.51), and with (3.3.57), Proposition 3.3.10 implies (3.3.56) since $[-\frac{R}{2}, \frac{R}{2}]^d \subset \mathcal{V}$. We now prove (3.3.58). The martingale argument leading to (3.3.23) also shows that for any $x \in S, \omega \in \Omega$

$$G_S^\omega(b_1(\cdot, \omega))(x) = E_{x,\omega}[X_{T_S} \cdot e_1] - x \cdot e_1. \tag{3.3.59}$$

The support theorem, see [3], applied to the rescaled diffusion $\frac{1}{T}X_{L^2t}$ yields a lower bound $c > 0$ (uniform in $x \in \mathbb{R}^d, \omega \in \Omega$) for the probability under $P_{x,\omega}$ that $X$ exits a cube of side length $L$ centered at $x$ through the “left or right”. Hence with the strong Markov property, for all $\omega \in \Omega, x \in \mathbb{R}^d$,

$$P_{x,\omega}[S < T_{-L+x \cdot e_1} \wedge T_{L+x \cdot e_1}] \leq 2(d - 1)(1 - c)^L, \tag{3.3.60}$$

which becomes smaller than $L^{-1}$ for large enough $L$. Since $|X_S \cdot e_1| \leq L + |x \cdot e_1|$, $P_{x,\omega}$-a.s. we obtain from (3.3.59) and (3.3.60) that for large enough $L$ and for all $\omega \in \Omega, x \in [-\frac{R}{2}, \frac{R}{2}]^d$, the left-hand side of (3.3.58) is less than

$$|E_{x,\omega}[(X_S - X_{T_S}) \cdot e_1, S = \tilde{T}_{-L+x \cdot e_1} \wedge T_{L+x \cdot e_1}]| + c. \tag{3.3.61}$$

On the event $\{S = \tilde{T}_{-L+x \cdot e_1} \wedge T_{L+x \cdot e_1}\} \cap \{(X_S \cdot e_1)(X_{T_S} \cdot e_1) > 0\}$, the trajectory $P_{x,\omega}$-a.s. leaves the slab $S$ and the box $[-L, L] \times [-h, h]^{d-1} + x$ “through the same side”. Hence on this event, $|(X_S - X_{T_S}) \cdot e_1| \leq \frac{R}{2}, P_{x,\omega}$-a.s. for $x \in [-\frac{R}{2}, \frac{R}{2}]^d$. It remains to show that for all $\omega \in \Omega, x \in [-\frac{R}{2}, \frac{R}{2}]^d$,

$$|E_{x,\omega}[(X_S - X_{T_S}) \cdot e_1, S = \tilde{T}_{-L+x \cdot e_1} \wedge T_{L+x \cdot e_1}, (X_S \cdot e_1)(X_{T_S} \cdot e_1) < 0]| \leq c. \tag{3.3.62}$$

When $x \cdot e_1 = 0$ the above quantity vanishes. We now consider the case where $0 < x \cdot e_1 \leq \frac{R}{2}$. The remaining case is treated analogously. We find
that for $0 < x \cdot e_1 \leq \frac{R}{2}$,

\[ (3.3.63) \quad P_{x,\omega}[S = \hat{T}_{-L+xe_1} \land T_{L+x \cdot e_1}, (X_S \cdot e_1)(X_{T_S} \cdot e_1) < 0] \leq P_{x,\omega}[T_L < \hat{T}_{-L+xe_1} < T_{L+x \cdot e_1}] + P_{x,\omega}[\hat{T}_{-L+xe_1} < T_L < T_{-L}]. \]

We estimate the first term on the right-hand side. The strong Markov property implies that for all $\omega \in \Omega, 0 < x \cdot e_1 \leq \frac{R}{2}$,

\[ (3.3.64) \quad P_{x,\omega}[T_L < \hat{T}_{-L+xe_1} < T_{L+x \cdot e_1}] \leq E_{x,\omega}[T_L < \hat{T}_{-L+xe_1}, P_{\hat{T}_L,\omega}[\hat{T}_{-L+\frac{R}{2}} < T_{L+\frac{R}{2}}]]. \]

The function $e(x) \overset{\text{def}}{=} -e^{4e(x \cdot e_1)} + e^{4e(L+\frac{R}{2})}$ satisfies $Le(x) < 0$ since $|b(\cdot, \cdot)| \leq \epsilon$. Hence $e(X_t)$ is a supermartingale under $P_{x,\omega}$ for any $x \in \mathbb{R}^d, \omega \in \Omega$. Since $e(x)$ is non negative when $x \cdot e_1 \leq L + \frac{R}{2}$, Chebychev’s inequality and the stopping theorem yield for any $y \in \mathbb{R}^d$ with $y \cdot e_1 = L$,

\[ (3.3.65) \quad P_{y,\omega}[\hat{T}_{-L+\frac{R}{2}} < T_{L+\frac{R}{2}}] \leq \frac{E_{y,\omega}[e(X_{\hat{T}_{-L+\frac{R}{2}} \land T_{L+\frac{R}{2}}})]}{e^{4e(L+\frac{R}{2})} - e^{4e(-L+\frac{R}{2})}} \leq \frac{1 - e^{-4e\frac{R}{2}}}{1 - e^{-8\epsilon L}} \leq c \epsilon \leq c'L^{-1}, \]

for large enough $L$. Inserting this bound into (3.3.64) and repeating the same type of argument for the second term on the right-hand side of (3.3.63), we obtain that its left-hand side is of order $L^{-1}$. This finishes the proof of (3.3.62) since $(X_S - X_{T_S}) \cdot e_1$ is of order $L$, $P_{x,\omega}$-a.s. for $x \in [-\frac{R}{2}, \frac{R}{2}]^d$. Thus (3.3.58) follows in view of (3.3.61). As a consequence, Proposition 3.3.10 implies (3.3.56) and the main theorem follows as we explained below (3.3.56).

**Proof of Proposition 3.3.10.** The idea of the proof is to decompose the $e_1$ projection of the drift $b_1(x, \omega)$ into its expectation $\mathbb{E}[b_1 \cdot e_1] = \lambda$ and a mean-zero term $\tilde{b}(x, \omega)$. As a consequence, the Green operator applied to $b_1$ splits into two terms: a leading term $G_S^* \lambda$ which is larger than twice the bound imposed on the Green operator in the event of interest in (3.3.54) by our choice of constants and by (3.3.21); an error term $G_S^* \tilde{b}$ that we decompose using the perturbation equality (3.3.40) and which turns out to make no
substantial contribution to the leading term with high probability. Hence the event of interest in (3.3.54) is very unlikely. We now give the details of the proof. Let us introduce the box

\[ \mathcal{U} \overset{\text{def}}{=} \{ x \in \mathbb{R}^d; |x \cdot e_1| \leq L - 1, |x|_\perp \leq \frac{1}{4}(NL')^2 + L^2 \} \]

which will be useful later in a discretisation step where we need to restrict ourselves to points located at a constant distance of \( \partial S \). As mentioned above we define [see (3.3.3)]

\[ \tilde{b} \overset{\text{def}}{=} b_1 - \lambda. \]

[For the sake of simplicity we drop the \( \omega \) dependence of \( b_1, \tilde{b} \) from the notation.] Then the perturbation equality (3.3.40) applied to \( G_{S \tilde{b}} \) together with (3.3.21) yields that for large enough \( L \),

\[ \begin{align*}
\mathbb{P} \left[ \inf_{x \in \mathcal{V}} G_{S \tilde{b}}(x) < \frac{c_{12}}{2} L^n \right] \\
\leq \mathbb{P} \left[ \inf_{x \in \mathcal{V}} G_S b(x) - G_S b_{\omega}(b \cdot \nabla) G_S \tilde{b}(x) \leq -\frac{c_{12}}{2} L^n, \sup_{y \in \mathcal{U}} |\nabla G_S \tilde{b}(y)| \leq L^{-1 + \eta/3} \right] \\
+ \mathbb{P} \left[ \sup_{y \in \mathcal{U}} |\nabla G_S \tilde{b}(y)| > L^{-1 + \eta/3} \right].
\end{align*} \]

The proposition obviously follows once we prove the following three claims: there exist \( \nu', \nu'' \geq 1 \) depending only on \( d \) such that for large enough \( L \),

\[ \begin{align*}
\text{(3.3.69)} & \quad \{ \omega \in \Omega; \sup_{y \in \mathcal{U}} |\nabla G_S \tilde{b}(y)| \leq L^{-1 + \eta/3} \}, \\
\text{on the set } & \quad \sup_{x \in \mathcal{V}} |G_{S \tilde{b}}(b \cdot \nabla) G_S \tilde{b}(x)| \leq cL^{\eta/3}, \\
\text{(3.3.70)} & \quad \mathbb{P} \left[ \sup_{y \in \mathcal{U}} |\nabla G_S \tilde{b}(y)| > L^{-1 + \eta/3} \right] \leq L^{\nu'} \exp \left( -c'L^{2/3} \right), \\
\text{(3.3.71)} & \quad \mathbb{P} \left[ \inf_{x \in \mathcal{V}} G_S \tilde{b}(x) \leq -\frac{c_{12}}{4} L^n \right] \leq L^{\nu''} \exp \left( -c'L^{c_{21} + 2\eta} \right),
\end{align*} \]

where \( c_{21} = 1 \) when \( d = 4 \) and \( c_{21} = 2 \) when \( d \geq 5 \).
We now show (3.3.69). In view of (3.3.34) and (3.3.5), we have that
\[ \sup_{x \in S} |\nabla G_S b(x)| \leq c_{15} 2eL \leq c_{15}/2. \]
Therefore for any \( \omega \in \Omega \) satisfying \( \sup_{y \in U} |\nabla G_S b(y)| \leq L^{-1+\eta/3} \) and any \( x \in \mathcal{V} \) we find that
\[
(3.3.72) \quad G_S^\omega (b \cdot \nabla) G_S b(x) \leq \epsilon L^{-1+\eta/3} G_S^\omega 1_U(x) + \epsilon c_{15} G_S^\omega 1_{\{z \in S; \text{dist}(z, \partial S) \leq 1\}}(x) + \epsilon c_{15} G_S^\omega 1_{\{z \in S; |z|_\perp \geq 1/4(NL')^3 + L^2\}}(x).
\]
The first term on the right-hand side is smaller than \( \frac{1}{4} L^{-2+\eta/3} E_{x,\omega}[T_S] \leq \frac{1}{2} L^{\eta/3} \) by (3.3.16).

To bound the second term on the right-hand side of (3.3.72), we define for \( L \geq 4(1 + R) \) the auxiliary set \( \hat{S} = \{x \in S; \text{dist}(x, \partial S) < 2\} \). With a martingale argument similar to (3.3.18), (3.3.16), we obtain that for any \( \omega \in \Omega \) and \( x \in S \), \( E_{x,\omega}[T_{\hat{S}}] \leq (1 - 2\epsilon)^{-1} \leq 2 \). Then we introduce the successive times of entrance in \( \{x \in \mathbb{R}^d; |x \cdot e_1| \geq L - 1\} \) and departure from \( \{x \in \mathbb{R}^d; |x \cdot e_1| \geq L - 2\} \):
\[
R_1 = T_{L-1} \wedge T_{L+1}, \quad D_1 = T_{\{x \in \mathbb{R}^d; |x \cdot e_1| > L - 2\}} \circ \theta_{R_1} + R_1,
\]
\[
(3.3.73) \quad R_k = R_1 \circ \theta_{D_{k-1}} + D_{k-1}, \quad D_k = T_{\{x \in \mathbb{R}^d; |x \cdot e_1| > L - 2\}} \circ \theta_{R_k} + R_k.
\]
With the help of these definitions we now express the Green operator appearing in the second term on the right-hand side of (3.3.72): for any \( \omega \in \Omega, x \in \mathcal{V} \), we have
\[
(3.3.74) \quad G_S^\omega (x) 1_{\{z \in S; \text{dist}(z, \partial S) \leq 1\}} = \sum_{k \geq 1} E_{x,\omega} \left[ E_{X_{R_k}} 1_{\{z \in S; \text{dist}(z, \partial S) \leq 1\}}(X_s) ds, R_k < T_S \right]
\]
\[
\quad \leq \sum_{k \geq 1} E_{x,\omega} \left[ E_{X_{R_k}} 1_{\{R_k < T_S\}} \right] \leq 2 \sum_{k \geq 1} P_{x,\omega}[R_k < T_S].
\]
The sum is bounded by a constant since the strong Markov property and the support theorem imply that for \( k \geq 1, x \in \mathcal{V} \), \( P_{x,\omega}[R_k < T_S] \leq (1 - c)^{k-1} \). Hence the second term on the right-hand side of (3.3.72) is less than \( c'L^{-1} \).

We now examine the last term on the right-hand side of (3.3.72). We call \( \hat{U} \) the set \( \{z \in S; |z|_\perp \geq 1/4(NL')^3 + L^2\} \) appearing in that term. For any
\( \omega \in \Omega, x \in \mathcal{V} \), the Markov property yields:

\[
G_{S}^{\omega}1_{\tilde{U}}(x) = \mathbb{E}_{x,\omega}\left[\mathbb{E}_{H_{\tilde{U}},\omega}\left[\int_{0}^{T_{S}}1_{\tilde{U}}(X_{s})\,ds\right], H_{\tilde{U}} < T_{S}\right] \\
\leq \sup_{z \in \mathcal{S}} \mathbb{E}_{z,\omega}\left[T_{S}\right] P_{x,\omega}\left[H_{\tilde{U}} < T_{S}\right].
\]

Using (3.3.16) and a scaling argument similar to the one leading to (3.3.60), we find that the latter expression is smaller than \( cL^{2}e^{-c' L} \). As a consequence, the last term on the right-hand side of (3.3.72) is smaller than \( L^{-1} \) for large enough \( L \). This proves (3.3.69).

Next we turn to the proof of (3.3.70). In order to deal with the supremum over the set \( \mathcal{U} \), we cover \( \mathcal{U} \) with disjoint cubes of side-length \( e^{3} \) and centers \( y_{i}, i \in \mathcal{I} \) where \( |\mathcal{I}| \leq cL^{12d-8} \). If \( Q \) is such a cube with center \( y_{i} \), then according to Lemma 3.3.9, \(-\frac{1}{2}G_{S}\tilde{b}(y)\) is twice continuously differentiable on \( Q' = y_{i} + (-\frac{1}{2}, \frac{1}{2})^{d} \subset \mathcal{S} \) and satisfies the equation \( \Delta u = \tilde{b} \) on \( Q' \). Therefore [17], (3.20), p.41, applies and we find that for any \( y \in Q \)

\[
\frac{\nabla G_{S}\tilde{b}(y) - \nabla G_{S}\tilde{b}(y_{i})}{|y - y_{i}|} \leq c|y - y_{i}|\left(\sup_{z \in Q'} |G_{S}\tilde{b}(z)| + \sup_{z \in Q'} |\tilde{b}(z)|\right) \left(\log\left(\frac{c'}{|y - y_{i}|}\right) + 1\right).
\]

Since the bounds in (3.3.16) also hold for Brownian motion, we have that \( \sup_{z \in Q'} |G_{S}\tilde{b}(z)| \leq 2L^{2}2e < L \). Thus the right-hand side of (3.3.76) is less than \( cL^{-2+\eta} \) for large enough \( L \). With this discretization step we obtain for large enough \( L \):

\[
P\left[\sup_{y \in \mathcal{U}} |\nabla G_{S}\tilde{b}(y)| > L^{-1+\eta/3}\right] \leq \sum_{i \in \mathcal{I}} P\left[|\nabla G_{S}\tilde{b}(y_{i})| > \frac{1}{2}L^{-1+\eta/3}\right].
\]

To bound the terms of the sum on the right-hand side of (3.3.77), we estimate \( P\left[\partial_{j}G_{S}\tilde{b}(y_{i}) > \frac{1}{2d}L^{-1+\eta/3}\right] \) and \( P\left[\partial_{j}G_{S}\tilde{b}(y_{i}) < -\frac{1}{2d}L^{-1+\eta/3}\right] \) separately for \( j = 1, \ldots, d \) with the help of an Azuma-type inequality. Therefore we cover the slab \( \mathcal{S} \) with disjoint cubes of side-length \( R \) and assign these cubes to \( 2^{d} \) disjoint families of cubes that are spaced by a distance \( R \). We denote with \( Q_{k}^{m} = x_{m,k} + [-\frac{R}{2}, \frac{R}{2})^{d} \), \( 1 \leq m \leq 2^{d}, k \geq 1 \) the cubes associated to the
mth family and define for $i \in \mathcal{I}, 1 \leq j \leq d, \omega \in \Omega$,

\begin{equation}
Y_{i,k}^m(\omega) = \int_{Q_k^m \cap S} \partial_j g(y_i, z)b(z, \omega)dz, \ k \geq 1.
\end{equation}

For fixed $m \in \{1, \ldots, 2^d\}$ and $i \in \mathcal{I}, 1 \leq j \leq d$, these random variables are $\mathbb{P}$-independent (as $k$ varies) and have mean 0 by Fubini’s theorem. Moreover, it follows from (3.3.49) that for all $\omega \in \Omega; m, i, j, k \geq 1$,

\begin{equation}
|Y_{i,k}^m(\omega)| \leq cL^{-1}(|x_{m,k} - y_i|^{1-d+1} + L^{1-d}) \exp \left(-c_1|y_{m,k} - y_i| L\right) \overset{\text{def}}{=} \gamma_{m,k}.
\end{equation}

Indeed, either $|y_i - x_{m,k}| \leq \sqrt{d}R$ and using polar coordinates we obtain that $|Y_{i,k}^m| \leq c\int_{B_{2\sqrt{d}R}(y_i)} (r^{1-d} + L^{1-d}) r^{d-1} dr \leq c' L^{-1} (1 + L^{1-d})$. Or $|y_i - x_{m,k}| \geq \sqrt{d}R$ and we can bound the integral by the supremum of the integrand times the constant volume of $Q_k^m$. Using a slight variation of the proof of Azuma’s inequality, we find for $1 \leq j \leq d, i \in \mathcal{I}$,

\begin{equation}
P \left[ \partial_j G S \tilde{b}(y_i) > \frac{1}{2d}L^{-1+\eta/3} \right] \leq \sum_{m=1}^{2^d} \sum_{k \geq 1} \mathbb{P} \left[ Y_{k}^m(\omega) > \frac{1}{d^{2d+1}}L^{-1+\eta/3} \right] \\
\leq \sum_{m=1}^{2^d} \exp \left(- \frac{d^{-2}2^{-2(d+1)}L^{-2+2/3}\eta}{\sum_{k \geq 1}(\gamma_{m,k})^2} \right) \leq 2^d \exp \left(- \frac{cL^{2/3}\eta}{d} \right),
\end{equation}

since the following easy computation and (3.3.79) show that $\sum_{k \geq 1}(\gamma_{m,k})^2$ is of order $L^{-2}$ for all $m \geq 1$:

\begin{equation}
L^2 \sum_{k \geq 1}(\gamma_{m,k})^2 \leq c \sum_{|x_{m,k} - y_i| \leq 4L} (|x_{m,k} - y_i|^{-d+1} + 1 + L^{-d+1})^2 + \\
\sum_{|x_{m,k} - y_i| \geq 2L} L^{-2d+2} \exp \left(- c'|x_{m,k} - y_i| \right) L = c \int_1^{4L} (r^{-2d+2} + L^{-2d+2}) r^{d-1} dr + L^{-2d+3} \int_L^\infty e^{-c' r/L} r^{d-2} dr \\
\leq c + L^{-d+2} \int_1^\infty e^{-c'u} du \leq c''.
\end{equation}
The same bound as in (3.3.80) holds for the terms $\mathbb{P} \left[ \partial_j G_S \bar{b}(y_i) < -\frac{1}{2d} L^{-1+\frac{\eta}{2}} \right]$, $1 \leq j \leq d$. Collecting the estimates (3.3.77), (3.3.80) and recalling that the cardinality of $\mathcal{I}$ is polynomial in $L$, we have proved the claim (3.3.70).

Finally we come to (3.3.71). The argument is similar to the previous one. First we handle the infimum over $\mathcal{V}$ by covering $\mathcal{V}$ with disjoint cubes of the form $x_i + [-\frac{R}{2}, \frac{R}{2}]^d$; for some adequate points $x_i, i \in \mathcal{I}'$ where $x_i \cdot e_1 = 0$ and $|\mathcal{I}'| \leq cL^{12(d-1)}$. Then it follows from (3.3.34) that for all $\omega \in \Omega$ and $|x - x_i| \leq \frac{R}{2}$,

$$\tag{3.3.82} |G_S \bar{b}(x) - G_S \bar{b}(x_i)| \leq c_{15} 2\epsilon L \frac{R}{2} \sqrt{d} \leq c. $$

Hence the discretization step implies that the left-hand side of (3.3.71) is less than

$$\tag{3.3.83} \sum_{x_i} \mathbb{P} \left[ G_S \bar{b}(x_i) \leq -\frac{c_{15}^2}{8} L^{\eta} \right]. $$

Then we use the same $2^d$ $R$-disjoint families of boxes $Q^m_k$ as before to cover the slab $S$ and we define for $i \in \mathcal{I}'$, $m \geq 1$ and all $\omega \in \Omega$,

$$\tag{3.3.84} \check{Y}^m_{i,k}(\omega) = \int_{Q^m_k \cap S} g(x_i, z) \bar{b}(z, \omega) dz, \quad k \geq 1. $$

Again we observe that for fixed $m \in \{1, \ldots, 2^d\}$ and $i \in \mathcal{I}'$, these random variables are $\mathbb{P}$-independent and have mean 0. Moreover, it follows from (3.3.48) that for all $\omega \in \Omega; m, i, k \geq 1$

$$\tag{3.3.85} |\check{Y}^m_{i,k}(\omega)| \leq cL^{-1}(|x_{m,k} - x_i|^{2-d} \wedge 1) \exp \left( -c_{17} |x_{m,k} - x_i|_L / L \right) \overset{\text{def}}{=} \tilde{\gamma}_{m,k}. $$

A computation like in (3.3.81) shows that for large enough $L$ and for all $1 \leq m \leq 2^d$:

$$\tag{3.3.86} \sum_{k \geq 1} (\tilde{\gamma}_{m,k})^2 \leq L^{-2} \left\{ \begin{array}{ll} c \log L, & d = 4, \\ c, & d \geq 5. \end{array} \right. $$

Then the same Azuma-type argument as before yields for large enough $L$ that each term in (3.3.83) is less than

$$\tag{3.3.87} \exp \left( -cL^{2+2\eta} / \log(L) \right) \text{ when } d = 4 \text{ respectively } \exp \left( -cL^{2+2\eta} \right) \text{ when } d \geq 5. $$

This finishes the proof (3.3.71) and thus of Proposition 3.3.10.
3.4 Appendix A

We now give the proof of Lemma 3.2.3. In order to bound \( \rho_1(\omega) \) on \( \mathcal{G} \) [see (2.3.17)], we first construct a function which -after appropriate normalization- dominates \( P_{x,\omega}[\tilde{T}_{L_1+R+1} < \tilde{T} \wedge T_{L_1+1}] \). For the construction, we divide the box \( B_1 \) into slabs of width \( L_0 \) and consider an expression inspired from the solution of a discrete one-dimensional Dirichlet problem for the exit probability of a Markov chain whose states correspond in essence to the slabs \( S_i, i \in \mathbb{Z} \).

Indeed, we recall (3.2.13) and for integers \( a < b \), we consider the products 
\[
\prod_{a,b} = \prod_{j=a+1}^{b} \hat{\rho}(j,\omega)^{-1}
\]
and set \( \prod_{a,a} = 1 \). Then we define the function \( f \) on \( \{-n_0 + 1, -n_0 + 2, \ldots, n_0 + 2\} \times \Omega \) via
\[
(3.4.1) \quad f(n_0 + 2, \omega) = 0, \quad f(n_0 + 1, \omega) = 1,
\]
\[
f(i, \omega) = \sum_{i \leq m \leq n_0 + 1} \prod_{m,n_0 + 1} \text{ for } i \leq n_0.
\]

For simplicity we drop the \( \omega \)-dependence from the notation. We now show that for \( \omega \in \Omega \),
\[
(3.4.2) \quad P_{0,\omega}[\tilde{T}_{-L_1+R+1} < \tilde{T} \wedge T_{L_1+1}] \leq \frac{f(0)}{f(1 - n_0)}.
\]

Let us introduce the \((\mathcal{F}_{V_m})_{m \geq 0}\) -stopping time
\[
(3.4.3) \quad \tau = \inf\{m \geq 0 : X_{V_m} \in S_{n_0+2} \cup S_{1-n_0}\}.
\]

Observe that \( P_{0,\omega}\text{-a.s. on the event which appears in (3.4.2), } X_{V_\tau} \in S_{1-n_0} \) and \( V_\tau < \tilde{T} \), and thus for \( \omega \in \Omega \),
\[
(3.4.4) \quad P_{0,\omega}[\tilde{T}_{-L_1+R+1} < \tilde{T} \wedge T_{L_1+1}] \leq \frac{E_{0,\omega}[f(I(X_{V_\tau^+})), V_\tau < \tilde{T}]}{f(1 - n_0)}.
\]

As we will see now, the numerator on the right-hand side is less than \( f(0) \): for \( \omega \in \Omega, m \geq 0 \),
\[
(3.4.5) \quad E_{0,\omega} \left[ f(I(X_{V_{(m+1)^\wedge \tau}})), V_{(m+1)^\wedge \tau} \leq \tilde{T} \right] 
\leq E_{0,\omega} \left[ f(I(X_{V_{m^\wedge \tau}})), V_m^\wedge \tau \leq \tilde{T}, \tau \leq m \right] + E_{0,\omega} \left[ f(I(X_{V_{m+1}})), V_m \leq \tilde{T}, \tau > m \right]
\]
and by the strong Markov property, the second term on the right-hand side equals

\[(3.4.6) \quad E_{0,\omega} \left[ V_m \leq \tilde{T}, \tau > m, E_{X_{V_m},\omega} [f(I(X_{V_1}))] \right]. \]

However on \( \{ V_m \leq \tilde{T}, \tau > m \} \), \( P_{0,\omega}\)-a.s.:

\[(3.4.7) \quad E_{X_{V_m},\omega} [f(I(X_{V_1}))] = f(I(X_{V_m}))
+ \hat{\rho}(X_{V_m}) \{ f(I(X_{V_m}) + 1) - f(I(X_{V_m})) \} + \hat{q}(X_{V_m}) \{ f(I(X_{V_m}) - 1) - f(I(X_{V_m})) \}
= f(I(X_{V_m})) + \prod_{I(X_{V_m}), n_0 + 1} \left[ -\hat{\rho}(X_{V_m}) + \hat{q}(X_{V_m}) \rho(I(X_{V_m}))^{-1} \right]. \]

Note that \( P_{0,\omega}\)-a.s., \( X_{V_m} \in S_{I(X_{V_m})} \), for \( m \geq 0 \). Hence the expression inside the square brackets is non-positive, see (3.2.13). As a result, we obtain that the left-hand side of (3.4.5) is smaller than or equal to

\[ E_{0,\omega} \left[ f(I(X_{V_m} \wedge \tau)), V_m \wedge \tau \leq \tilde{T} \right]. \]

The latter expression is hence non-increasing with \( m \). Since \( \tau \) is \( P_{0,\omega}\)-a.s. finite, it follows from Fatou’s inequality that for \( \omega \in \Omega \),

\[(3.4.8) \quad E_{0,\omega} \left[ f(I(X_{\tau})), V_\tau \leq \tilde{T} \right] \leq f(0). \]

Together with (3.4.4), this implies (3.4.2).

We now derive a bound on \( \rho_1 \). Let us define for \( \omega \in \Omega \),

\[(3.4.9) \quad A = P_{0,\omega} [I_{\tilde{T}-L_1 + R+1} < \tilde{T} \wedge T_{L_1 + 1}] + P_{0,\omega} [\tilde{T} < \tilde{T}_L - L_1 + R + 1 \wedge T_{L_1 + 1}]. \]

Observe that \( q(0, \omega) \leq A \) and since \( \frac{q}{1-q} \) is non-decreasing in \( q \), we obtain for \( \omega \in \Omega \) that \( \rho_1(\omega) \leq \frac{A}{(1-A)^+} \). Using (3.4.2) and (2.3.17), it follows for \( \omega \in \mathcal{G} \) that

\[(3.4.10) \quad \rho_1(\omega) \leq \frac{f(0) + f(1 - n_0) \kappa^{gL_1}}{(f(1 - n_0) - f(0) - f(1 - n_0) \kappa^{gL_1})_+}. \]

Let us for the time being assume that there is a \( c_1 > R + 2 \) such that for \( L_0 \geq c_1 \) and \( \omega \in \Omega \),

\[(3.4.11) \quad f(0) + f(1 - n_0) \kappa^{gL_1} \leq 2f(0), \quad \text{and} \]
\[(3.4.12) \quad f(1 - n_0) - f(0) - f(1 - n_0) \kappa^{gL_1} \geq \prod_{n_0 + 1, n_0 + 1}. \]
Then in view of (3.4.10) and the definition of $f(0)$, for $L_0 \geq c_1$, $\omega \in \mathcal{G}$,

\begin{equation}
(3.4.13) \quad \rho_1(\omega) \leq 2 \sum_{0 \leq m \leq n_0 + 1} \prod_{-n_0 + 1 < j \leq m} \hat{\rho}(j, \omega).
\end{equation}

Observe that by the definition (3.2.13), \{\hat{\rho}(j, \omega), j \text{ even}\} and \{\hat{\rho}(j, \omega), j \text{ odd}\} are two collections of independent random variables, as $\rho(j, \omega)$ and $\rho(j+2, \omega)$ depend on regions separated by a distance $R$. With the help of Cauchy-Schwarz’s inequality and $(u + v)^a \leq u^a + v^a$, for $u, v \geq 0$ and $a \in (0, 1]$, we find that for $L_0 \geq c_1$,

\begin{equation}
(3.4.14) \quad \mathbb{E}[\rho_1^a, \mathcal{G}] \leq 2 \sum_{0 \leq m \leq n_0 + 1} \prod_{-n_0 + 1 < j \leq m} \mathbb{E}[\hat{\rho}(j, \omega)^{2a}]^{1/2}.
\end{equation}

From Lemma 3.2.1, we have that for all $\omega \in \Omega$, $\rho_1(\omega) \leq \kappa^{-L_1-1}$. This inequality and (3.4.14) immediately implie the claim (3.2.15).

Let us now show (3.4.11). Using again Lemma 3.2.1, we have for all $\omega \in \Omega$, $-n_0 + 1 \leq j \leq n_0 + 1$:

\begin{equation}
(3.4.15) \quad \kappa^{L_0+1} \leq \hat{\rho}(j, \omega) \leq \kappa^{-(L_0+1)}.
\end{equation}

In view of (3.4.1) and since $L_0 + 1 \leq 2L_0$, we find that

\begin{equation}
\kappa^{9L_1} \leq (2n_0 + 1)\kappa^{-(L_0+1)2n_0} \kappa^{9L_1} \leq (2n_0 + 1)\kappa^{5n_0L_0}.
\end{equation}

If $L_0 \geq c_1 \geq R + 2$ large enough, it follows that for all $\omega \in \Omega$ and all $n_0 \geq 3$,

\begin{equation}
(3.4.16) \quad f(1 - n_0)\kappa^{9L_1} \leq \kappa^{4n_0L_0} < 1.
\end{equation}

Clearly $f(0) \geq 1$ and we obtain (3.4.11). To see (3.4.12), we note that:

\begin{equation}
(3.4.17) \quad f(1 - n_0) - f(0) \geq \prod_{-n_0+1, n_0+1} + \prod_{-1, n_0+1} \geq \prod_{-n_0+1, n_0+1} + \kappa^{(L_0+1)(n_0+2)}.
\end{equation}

Since $(L_0 + 1)(n_0 + 2) \leq 4L_0n_0$ and because of (3.4.16), the claim (3.4.12) follows, provided that $L_0 \geq c_1$. This finishes the proof of Lemma 3.2.3.
3.5 Appendix B

We now prove Lemma 3.3.9. We start with the proof of (3.3.49). A similar and easier argument also shows (3.3.48). Since for $d \geq 4$, we have

$$|\partial_t g_d(x, y)| \leq c|x - y|^{-d} \quad \text{and} \quad |\partial_t \partial_j g_d(x, y)| \leq c'|x - y|^{-d},$$

the sum of the first and second derivatives of the terms with $k \geq 2$ appearing in (3.3.30) converges uniformly for all $x, y \in S$. Hence $g(x, y)$ is twice continuously differentiable for $x, y \in S, x \neq y$, and interchanging differentiation and summation yields for all $x, y \in S$

$$|\nabla g(x, y)| \leq 3|\nabla g_d(x, y)| + c(2L)^{-d+1} \quad \text{and}$$
$$|\partial_t \partial_j g(x, y)| \leq 3|\partial_t \partial_j g_d(x, y)| + c'(2L)^{-d}, \quad \text{as well as}$$
$$\Delta g(x, y) = 0, \quad \text{for } x \neq y.$$

For any $x \in S$, we consider a small vector $h$ with $x + h \in S$ and an point $y \in S$ with $|x - y|_\perp \geq L$. Moreover we denote with $W$ a $d$-dimensional Brownian motion starting at $y$ under some measure $P$ and with $T$ the stopping time $\inf\{t \geq 0; |W_t - y|_\perp \geq \frac{1}{2}|x - y|\}$. Since $g(x, y)$ is symmetric in $x$ and $y$, it is also harmonic in $y$ and thus $g(x, W_{T \wedge T_S})$ is a bounded martingale under $P$. The stopping theorem thus implies that

$$\frac{1}{|h|} |g(x + h, y) - g(x, y)| = \frac{1}{|h|} \left| E_P \left[ g(x + h, W_{T \wedge T_S}) - g(x, W_{T \wedge T_S}) \right] \right|.$$

Direct inspection of $g(x, y)$ shows that it vanishes on the boundary of $S$. Hence using the mean value theorem, the latter expression is smaller than

$$\sup\{|\nabla g(x', y')|; x' \in B(x, h),|y' - y|_\perp = \frac{1}{2}|x - y|\} P[T < T_S].$$

Because of (3.5.19) the first factor above is less than $c|x - y|^{-d} + c'L^{1-d}$ and a scaling argument similar to the one leading to (3.3.60) yields that $P[T < T_S] \leq \exp\left(-c\frac{|x - y|_\perp}{L}\right)$. Letting $h$ tend to 0 in (3.5.22),(3.5.23) and treating the cases $|x - y| < L$ and $|x - y| > L$ separately we obtain the claimed result (3.3.49). The same martingale argument also leads to (3.3.48).

We now prove (3.3.50). For any $x_0 \in S$, we define the auxiliary set $U = \{x \in S; |x - x_0|_\perp < 1\}$. From (3.3.31) we can write the Green operator for a
bounded Hölder continuous function $f$ as follows: we define $\tilde{g} = g - g_d$ and for any $x \in U$, we find:

\[(3.5.24)\]

$$G_S f(x) = \int_{U} g_d(x, y) f(y) dy + \int_{U} \tilde{g}(x, y) f(y) dy + \int_{S \setminus U} g(x, y) f(y) dy.$$ 

According to [17], Lemma 4.2, the first term on the right-hand side is twice continuously differentiable on $U$, and its Laplacian equals $-2f(x)$. With the same argument as below (3.5.18), we see that $\tilde{g}(\cdot, y)$ is harmonic on $U$ for any $y \in S$. Hence Fubini's theorem together with the mean value theorem, see [17], Theorem 2.7, yield that the second term on the right-hand side of (3.5.24) is harmonic on $U$. The same is valid for the last term, since from (3.5.21), $g(\cdot, y)$ is harmonic on $U$ for any $y \in S \setminus U$. As $x_0 \in S$ is arbitrary, we obtain (3.3.50). This finishes the proof of Lemma 3.3.9.
Bibliography


Curriculum vitae

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