Pricing Credit Derivatives

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Abstract

The size of global credit derivatives markets rose from USD 586 billion at the beginning of this millennium to more than USD 20 trillion of outstanding notional by the end of 2006 as recently reported by the British Bankers’ Association. This outstanding growth and the fact that the mathematics of credit derivatives are in some respects quite different from classical equity and fixed income derivatives made the valuation of credit derivatives products an active research area of mathematical finance. In this thesis we address the challenge of pricing credit derivatives in presence of different sorts of default dependencies: Dependence across the underlying obligors of a credit portfolio and dependence between these underlying credits and other related financial factors such as interest-rates and foreign exchange rates.

In part I we present a modelling framework for portfolio credit risk which incorporates the dependence between risk-free interest-rates and the aggregated portfolio loss process. The innovation in this approach is that –besides the traditional diffusion-based covariation between the portfolio loss intensity and interest-rates– a direct dependence between interest-rates and the portfolio loss process is allowed. The model is set up using a set of loss-contingent forward interest-rates $f_n(t, T)$ and loss-contingent forward credit protection rates $F_n(t, T)$ to parameterize the market of default-free bonds and credit portfolio-sensitive assets such as single-tranche collateralized debt obligations. We show that existence of such a parameterization is equivalent to absence of static arbitrage opportunities in the underlying assets and we give necessary and sufficient conditions on the stochastic evolution of the parameterization to ensure absence of dynamic arbitrage strategies. Moreover we identify the particular form of the model-implied portfolio spot loss intensity under an equivalent martingale measure.

We then turn away from the issue of loss-dependent interest rates and address the question of what type of generic spot model for the portfolio loss intensity produces flexible term structures of loss-contingent forward credit
Abstract

In single-obligor default risk modelling, using a background filtration in conjunction with a suitable embedding ($\mathbb{H}$) hypothesis has proven a very successful tool to separate the actual default event from the default intensity model. We analyze the conditions under which this approach can be extended to the credit portfolio case with focus on the so-called top-down view. We introduce the natural $\mathbb{H}$-hypothesis of this setup—the successive $\mathbb{H}$-hypothesis—and we show that it is equivalent to a seemingly weaker one-step $\mathbb{H}$-hypothesis. Furthermore, we provide a canonical construction of loss processes in this setup and we show that under little regularity every loss process satisfying the successive $\mathbb{H}$-property actually stems from a canonical construction. In a special case, we provide closed-form solutions for some pricing problems.

As will be made obvious in the analysis of part I on the example of loss-dependent interest-rates, in credit risk modelling one has to take into account that related financial factors may show dependence with the underlying credits not only through diffusion-type covariation, but also via joint jumps at the default times of the underlying obligors.

In part II this observation is taken up in the case of a related foreign exchange rate (FX). We analyze the connections between the credit spreads that the same single-name credit risk commands in different currencies. We show that the empirically observed differences in these credit spreads are mostly driven by the dependence between the default risk of the underlying obligor and the exchange rate. In our model there are two different channels to capture this dependence: First, the diffusions driving FX rate and default intensities may be correlated, and second, an additional jump in the exchange rate may occur at the time of default. The differences between the default intensities under the domestic and foreign pricing measures are analyzed and closed-form prices for a variety of securities affected by default risk and FX risk—including credit default swaps (CDS) are given. In the empirical analysis we find that a purely diffusion-based correlation between exchange rate and default intensity is not able to explain the observed differences between JPY and USD CDS rates for a set of large Japanese corporate obligors. The data implies a significant additional jump in the FX rate at default.

The contents of this thesis are summarized in Ehlers and Schönbucher [2004, 2006b, 2006a].

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Part I

Credit Portfolio Derivatives
Chapter 1

Pricing Interest-Sensitive Credit Portfolio Derivatives

1.1 Introduction to Chapter 1

In this chapter we present a modelling framework for portfolio credit risk which incorporates a new methodology to model the dependence between risk-free interest-rates and the default loss process, allowing direct dependence between interest-rates and the loss process.

We provide a stochastic and arbitrage-free framework for the evolution of the prices of a set of contingent-claims on the credit portfolio's loss distribution. This set of contingent claims is complete in the sense that it spans all European contingent claims on the loss process $L(t)$.\footnote{1} In particular, the prices of index credit default swaps (CDS) and single-tranche collateralized debt obligations (STCDO) of all maturities and attachment points can be easily constructed from these contingent claims. This allows a straightforward calibration of the model to the so-called “correlation smile”. In contrast to Schönbucher (2006), though, these prices can not be interpreted as probabilities under the spot martingale measure as we allow dependence between the loss process and risk-free interest-rates.

The prices of the basic contingent claims are parameterized using a set of loss-contingent forward interest-rates $f_n(t, T)$ and loss-contingent forward credit protection rates $F_n(t, T)$. The forward interest-rates $f_n(t, T)$ must be

\footnote{1}: any contingent claim with payoff $X$ at time $T$, where $X$ is $\sigma (L(T))$-measurable can be replicated using a static portfolio of these contingent claims.
loss contingent in order to allow us to capture its credit dependence. These rates can be viewed as the interest-rates of forward-rate agreements that are contingent on a certain number of losses $L(T) = n$. Clearly, if there is dependence between the loss process and the default-free interest-rates, the loss-contingent forward rates $f_n(t, T)$ must differ over different values of $n$. We show that (up to weak regularity conditions), existence of such a parameterization is necessary and sufficient for the absence of static arbitrage opportunities in the underlying assets, i.e. the parameterization fully describes the set of arbitrage-free price systems in this model.

Next, we analyze the possible dynamics of the thus defined market for portfolio credit derivatives and interest-rate derivatives. We give necessary conditions and sufficient conditions on the dynamics of the parameterization which ensure absence of dynamic arbitrage opportunities in the model. Similar to the HJM drift restrictions for default-free interest-rates, these conditions take the form of restrictions on the drifts of $f_n(t, T)$ and $F_n(t, T)$, together with a set of regularity conditions on the stochastic characteristics of the parameterization.

The modelling framework presented here can be applied very efficiently to the pricing of exotic portfolio credit-derivatives such as options on index CDS, leveraged super-senior tranches, options on STCDOs and in particular also hybrid derivatives on credit portfolios and interest-rates. Given the increased liquidity of the markets for index CDS and STCDOs, a particular advantage of this model over most competing models is that it can be calibrated very easily to a given set of prices for index CDS and STCDOs. Such a calibration only requires changing the initial conditions of the loss-contingent forward interest- and protection-rates $f_n(t, T)$ and $F_n(t, T)$ but it does not require any changes to the dynamics of the model.

Empirical studies of the dependence between default rates and risk-free interest-rates usually find a negative dependence between interest-rates and defaults. Such results are found investigating the interest-rate correlations of credit spreads of corporate bonds (e.g. Duffee (1998), Duffee (1995), Collin-Dufresne et al. (2001)), and also when actual default event arrivals are investigated (see e.g. Duffie et al. (2006)). The most common explanation of this negative dependence is that high default rates and low interest-rates tend to coincide in recessions, while booms are usually characterized by low default rates and high interest-rates.

Traditionally, interest-rate risk is incorporated in intensity-based models of credit risk by specifying diffusion-based dynamics for the interest-rates and correlating them with the dynamics of the default intensities of the obligor(s) in question. Most importantly, a direct influence from the credit event itself to
the level or dynamics of the default-free interest-rates is usually (implicitly) ruled out. This approach is followed most frequently in models of single-obligor default risk (e.g. Duffie and Singleton (1997), Madan and Unal (1998), Jarrow et al. (1997) and the literature following it) where indeed it may be argued that the individual obligor under consideration is “small” compared to the macroeconomic forces determining the level of default-free interest-rates and where the influence of firm-specific factors determining the default-risk dominates. Thus it is plausible that the dependence between the default of any given individual firm and interest-rates should be indirect and weak. Nevertheless, even in this situation some difficulties arise: e.g. negative covariation between interest-rates and intensities is essentially impossible to achieve in an exponentially affine framework with positive interest-rates and positive intensities (the most common specification of intensity-based default risk).

The assumption of weak and indirect dependence cannot be sustained when default risk is considered on a portfolio level. Almost by definition, if a representative portfolio suffers a large number of defaults, a recession or a similar macroeconomic crisis must be present: if not as the cause, then certainly as the effect of the defaults. This crisis in turn will directly influence the level of default-free interest-rates through the interest-rate policies of central banks and through effects on investor preferences and inflation expectations in fixed-income markets. Thus, for economic reasons, there should be a stronger and more direct link between portfolio defaults and interest-rates.

Moreover, diversification effects also shift the focus towards macro-variables when moving from a single-obligor point of view to portfolio credit risk modelling as idiosyncratic influences average out in a portfolio context. Thus, we find it very plausible that the dependence between interest-rates and large losses in credit portfolios should be stronger and more direct than the dependence between interest-rates and the default of one specific individual obligor.

Also from a mathematical viewpoint it is natural to allow non-defaultable quantities in the model to be driven by the portfolio loss process: If the model’s setup admits a martingale representation theorem, the resulting representation must include integrals with respect to the compensated loss process \( \tilde{L}(t) \) (or another martingale which generates the jumps of \( L(t) \)). After all, the loss process itself is adapted to the model’s filtration. Thus, if one were to specify the generic dynamics of any other martingale (e.g. a discounted bond price process) in the model, it would exhibit joint jumps with \( L \) — and if it does not have such joint jumps this amounts to a modelling assumption which needs to be justified.

As argued above, we believe that a credit portfolio’s loss process can
Chapter 1. Pricing Interest-Sensitive Credit Portfolio Derivatives

convey important macroeconomic information which should affect the prices of many different financial instruments. With the same justification we also argue to allow influences between L and other variables such as equity indices, exchange rates etc. Here we focus on the dependence between losses and default-free interest-rates. For practical implementations this is the most important case because interest-rates enter the pricing formulae of all securities, in particular also the pricing formulae of elementary calibration securities such as defaultable bonds, CDS or STCDOs.

Unfortunately, almost invariably the literature on portfolio credit risk assumes independence between defaults and interest-rates. For multivariate intensity models some dependence can be introduced by correlating default intensities and interest-rates in the same way as for single-obligor intensity models, yet – as mentioned above – this approach is not without difficulties and problems. In other types of portfolio credit risk models the incorporation of interest-rate dependence is not so straightforward, this applies in particular to portfolio credit risk models based upon static latent factors to drive the dependence like the very widely-used Gauss copula models (Li (2000)).

The model presented in this chapter follows the top-down approach to credit risk modelling. Models of this class do not focus on the defaults or survival of individual obligors but rather attempt to capture the behavior of the aggregate portfolio loss process. Representatives of this approach are e.g. Davis and Lo (2001), Giesecke and Goldberg (2005) or Frey and Backhaus (2004), and also mixture models used in credit risk management are developed from a top-down view (see e.g. McNeil et al. (2005), chapter 8.4). More concretely, the parameterization of the prices of the basic securities and the idea of modelling the price dynamics of a whole term- and strike structure of STCDOs in this work goes back to the forward loss surface modelling approach proposed in Schönbucher (2006), which in turn is closely related to the model proposed by Sidenius et al. (2005). We show later that the model by Bennani (2005) can be viewed as a projection of the model presented here, removing the loss-dependence from the state variables. Furthermore, the model used in Brigo et al. (2006) and the empirical study of CDO prices in Longstaff and Rajan (2006) can be viewed as simplified versions of these loss surface models.

The rest of this chapter is structured as follows: In the following part we introduce the basic securities for portfolio loss modelling: loss contingent zero coupon bonds \( P_n(t, T) \) and up-front prices \( U_n(t, T) \) for protection against the \( n \)-th default. After showing how liquid portfolio credit derivatives like index CDS and STCDOs can be priced using solely these instruments, we analyze the conditions under which a system of these prices can be represented using loss-contingent forward interest-rates \( f_n(t, T) \) and loss-contingent forward
1.2. Basic Portfolio Credit Derivatives

We discuss the economic interpretation of these rates and show that existence of such a representation is essentially equivalent to absence of static arbitrage opportunities in the basic securities.

Next we address the problem which dynamics of the parameterization \( \{f_n(t, T), F_n(t, T)\} \) are consistent with absence of dynamic arbitrage opportunities. We characterize dynamics that \( \{f_n(t, T), F_n(t, T)\} \), the loss process \( L(t) \) and the short rate process \( r(t) \) have to follow under any spot martingale measure and we also give sufficient conditions for the dynamics to be arbitrage-free. As these conditions entail very specific forms for the drift parameters of the \( f_n(t, T) \) and \( F_n(t, T) \), we also have to address the problem of existence of solutions to these restricted stochastic differential equations.²

1.2 Basic Portfolio Credit Derivatives

In this section we show that index CDS and STCDOs on a portfolio of \( N \) obligors can be priced by setting up simple static portfolios of the building block securities \( P_0, P_1, \ldots, P_N \) and \( U_1, \ldots, U_N \). Index CDS and STCDOs are by far the most liquid portfolio credit derivatives in the market and the most popular assets used to calibrate portfolio credit risk models. The calibration of our model to these instruments is facilitated by the relatively simple form of the pricing formulae.

Furthermore, we show how the payoff functions of many more exotic portfolio credit derivatives can be represented in our setup. Thus, these derivatives can be priced by simulation in our model. For an excellent overview of credit derivatives products the reader is referred to Bruyère et al. (2006).

1.2.1 Setup

We are on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). \( T^* < \infty \) is a finite investment horizon, \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T^*]} \) satisfies the usual conditions, all processes are \( \mathbb{F} \)-adapted and càdlàg and \( \mathbb{P} \) is understood as the physical probability measure if nothing else is specified. \( \mathcal{P} \) denotes the predictable and \( \mathcal{O} \) the optional \( \sigma \)-field, \( \mathcal{B}(\{0, T^*\}) \) is the Borel \( \sigma \)-field on \([0, T^*]\). \( \mathcal{L} \) denotes the Lebesgue measure and \( M(\mathbb{P}) \) is the space of uniformly integrable \( \mathbb{P} \)-martingales. When we make a statement about a term structure, that is a family of processes \( X(t, T)_{t \in [0, T]} \) indexed by \( T \in [0, T^*] \), we will omit the phrase “for all \( 0 \leq t \leq T \leq T^* \)” if no ambiguities can arise, and we always assume \( X(t, \omega, T) \) to be \( \mathcal{O} \times \mathcal{B}([0, T^*]) \)-measurable.

²The contents of this chapter are summarized in Ehlers and Schönbucher (2006b).
Chapter 1. Pricing Interest-Sensitive Credit Portfolio Derivatives

We consider a given credit portfolio consisting of $N \in \mathbb{N}$ obligors whose cumulative credit losses are described by the loss process $L_t$. We assume $L_t$ is a piecewise constant, bounded process with jumps of size one and $L_0 = 0$ a.s. The credit exposures of the obligors are normalized to unity so that we can identify the number of losses with the amount lost in the portfolio. The time of the $n^{th}$ default (loss) in the portfolio is denoted by $\tau_n := \inf \{ t > 0 : L_t \geq n \}$, $n = 1, \ldots, N$.\footnote{We use the convention $\inf \emptyset = \infty$. Because $L_t$ only has one-step increments, no two jump times will coincide if one of them is finite, i.e. $\tau_i < \tau_{i+1}$ for all $i \leq N$ with $\tau_i < \infty$.}

We assume the following financial instruments are traded at all times $t \in [0, T^*]$ for all $T \geq t$ and $n = 0, \ldots, N$:

- $P_n(t, T)$, is the price at time $t$ of an asset paying 1$ at time $T$ if and only if $L_T = n$.
- $U_n(t, T)$, is the up-front price for protection against the $n^{th}$ loss, i.e. the price at time $t$ of an asset paying 1$ at $\tau_n$ if and only if $\tau_n \in (t, T]$.

It is natural to set $P_{m-1}(t, T) = U_m(t, T) = 0$ for all $m \leq L_t$. Further, money can always be invested in the money market account $b_t$, which is nondecreasing and satisfies $b_0 = 1$ a.s.

### 1.2.2 Index CDS and Single-Tranche CDOs

In an index credit default swap (index CDS) with maturity $T$, the protection seller promises to pay all credit losses that occur in the portfolio up to time $T$, i.e. he promises to pay 1 at $\tau_n$ if $\tau_n \leq T$ for all $n \leq N$. Expressed in terms of the building blocks, the value of such a protection at time $t$ is

$$\text{Prot}_{0,N}(t, T) := \sum_{n=1}^{N} U_n(t, T).$$

The protection buyer on the other hand promises to pay at the payment dates $T_k$, $k = 1, \ldots, K$ a protection fee $\overline{\pi}$ times the remaining notional of the index and times a daycount fraction. It is important to note that at a default, the notional on which the fee is paid is reduced: At any time $t$, the remaining notional of the index is $N - L_t = \sum_{n=1}^{N} 1_{[\tau_n > t]}$. If we approximate the discrete payments with a continuous payment stream the value of the fee stream at time $t$ is

$$\overline{\pi} \cdot \text{Fee}_{0,N}(t, T) := \overline{\pi} \int_{t}^{T} \sum_{n=0}^{N} (N - n) P_n(t, u) du.$$
1.2. Basic Portfolio Credit Derivatives

The difference between a STCDO and an index CDS is that the protection of the STCDO only covers a certain range \([a, b]\) of the possible losses. In a STCDO with attachment point \(a\) and detachment point \(b > a\) and maturity \(T\), the protection seller promises to pay 1 at \(\tau_n\) if \(\tau_n \leq T\) for all \(n \in [a + 1, b]\). (An index CDS is reached by setting \(a = 0\) and \(b = N\).) The value of this protection is

\[
\text{Prot}_{a,b}(t, T) := \sum_{n=a+1}^{b} U_n(t, T).
\]

Similar to the case of the index CDS, the protection fee on the STCDO is also paid on a variable notional amount which is reduced by all losses that fall in the range \([a, b]\) between attachment point and detachment point. Overall the value of the fee stream is

\[
\overline{s} \cdot \text{Fee}_{a,b}(t, T) := \overline{s} \int_t^T (b - a) \sum_{n=0}^{a} P_n(t, u) du + \overline{s} \int_t^T \sum_{n=a+1}^{b} (b - n) P_n(t, u) du,
\]

which is composed of the value of the fee payment on the full notional which is paid as long as \(L_t \leq a\) (the first term), and the value of the fees paid on a notional amount which is reduced by the losses already incurred.

Given these formulae, the \textit{fair fee} quoted in the market is the fee that makes the value of the fee leg equal to the value of the protection leg:

\[
\overline{s}_{a,b}(t, T) := \frac{\text{Prot}_{a,b}(t, T)}{\text{Fee}_{a,b}(t, T)}.
\]

Our assumption of unit losses given default allowed us to slightly simplify the payoffs by identifying the losses with the default count. In a real-world STCDO, payoffs are based solely upon the cumulative loss process, i.e. the protection seller pays the increments in the tranche loss process \((b - L_t)_+ - (a - L_t)_+\), where \(a, b \in \mathbb{R}\). For the fee payment, the notional of a STCDO is reduced by the tranche’s loss process, while the notional of a real-world index CDS is reduced by the total notional of the defaulted obligor (and not just the loss in default). If a constant recovery rate is assumed, it is straightforward to adapt the pricing formulae given above to these modifications.

Recently, \textit{tranchelets}, i.e. very thin tranches have gained popularity in the market. In our setup, the thinnest possible tranche would cover one default,
i.e. $b = a + 1$. Its protection leg has simply the value $U_b(t, T)$, while the fee leg has the value $\int_t^T \sum_{n=0}^\infty P_n(t, u)du$.

1.2.3 Forward-Starting STCDOs and Tranche Options

A $T_1$-into-$(T_2 - T_1)$ forward-starting STCDO with attachment point $a$ and detachment point $b > a$ is a contract which at time $T_1$ turns into a STCDO over $[T_1, T_2]$, where the protection fee has already been fixed at the initial time $t < T_1$ at the level $\tilde{\pi}'$. The twist is that the attachment and detachment points are shifted by any loss amounts that occur before $T_1$, i.e. it turns into a STCDO with attachment point $(a + L_{T_1})$ and detachment point $(b + L_{T_1})$, and not simply into a STCDO with attachment point $a$ and detachment point $b$. (The latter case would have been straightforward to price.)

While it is not possible to directly give the price of a forward-starting STCDO, we can at least give its payoff at $T_1$. At $T_1$, the value of the position to the protection seller is

$$\left(\tilde{\pi}' - \tilde{\pi}(a + L_{T_1}), (b + L_{T_1})(T_1, T_2)\right) \cdot \text{Fee}_{(a + L_{T_1}), (b + L_{T_1})}(T_1, T_2).$$

A Put (Payer) is usually defined as the right to by protection (i.e. sell risk) and a Call (Receiver) as the right to sell protection (i.e. buy risk). Tranche options come in various shapes and forms and we can only give the simplest variations: The payoff at time $T_1$ of a European Put on a forward-starting STCDO is

$$\left(\tilde{\pi}(a + L_{T_1}), (b + L_{T_1})(T_1, T_2) - \tilde{\pi}'\right)^+ \cdot \text{Fee}_{(a + L_{T_1}), (b + L_{T_1})}(T_1, T_2),$$

while a European Put on a plain STCDO without adjustment of attachment-and detachment points has the payoff

$$\left(\tilde{\pi}_{a,b}(T_1, T_2) - \tilde{\pi}'\right)^+ \cdot \text{Fee}_{a,b}(T_1, T_2).$$

Leveraged Super-Senior tranches are a form of barrier option on a STCDO, usually a "super-senior" tranche, i.e. a tranche with a high attachment point. In these transactions, a trigger event $\tau$ is specified, usually in terms of a hitting time of the loss process $L_t$ or a hitting time of the portfolio spread $\tilde{\pi}_{0,N}(t, T)$ or a combination of these. Upon the trigger event, the protection seller makes a payment to the protection buyer which is linked to the value of a given STCDO-tranche. The precise specification of such transactions varies but it is clear that the trigger events and the payoffs of these structures can be expressed in terms of our basic securities.
1.3 Forward Model and Absence of Arbitrage

1.3.1 Static No-Arbitrage Conditions

In our setup, the default-free zero coupon can be expressed as 
\[ B(t, T) := \sum_{n=0}^{N} P_n(t, T), \]
and it is well-known that 
\[ B(t, T) \in (0, 1] \] 
and nonincreasing in \( T \) and \( B(t, t) = 1. \) \((\ast B)\)

is a necessary condition in order to avoid static arbitrage opportunities if cash can be stored at zero cost. In particular, if \( L_t = N, \) then \( P_N(t, T) = B(t, T) \)
will be the only remaining traded asset apart from the money market account \( b_t. \) When \( L_t < N, \) the situation more involved. Obviously, absence of (static) arbitrage requires
\[ P_n(t, T) \geq 0 \text{ and } P_n(t, t) = 1_{\{L_t=n\}}, \ n = 0, \ldots, N. \] \((\ast P)\)
\[ U_n(t, T) \] 
nondecreasing in \( T \) and \( U_n(t, t) = 0, \ n = 1, \ldots, N. \) \((\ast U)\)
because \( P_n \) and \( U_n \) yield nonnegative payoffs, and the larger \( T, \) the more protection provides \( U_n. \) Additionally we define the portfolios\(^4\)
\[ \Pi_n(t, T) := P_n(t, t) + U_n(t, T) - U_{n+1}(t, T) - P_n(t, T). \]
By the definition of \( P_n \) and \( U_n, \) holding \( \Pi_n(t, T) \) pays off as follows:
- \(+1\) $ at \( t \) and \(-1\) $ at \( \tau_{n+1} \wedge T \) for \( n = L_t \)
- \(+1\) $ at \( \tau_n \wedge T \) and \(-1\) $ at \( \tau_{n+1} \wedge T \) for \( n > L_t \)
Consequently, for any \( \delta > 0, \) \( \Pi_n(t, T + \delta) - \Pi_n(t, T) \) pays nothing if the loss level \( n \) is not reached in \([T, T + \delta)\) (i.e. \( L_u \neq n \) for all \( u \in [T, T + \delta) \)), and otherwise \(+1\) $ at \( \max\{T, \tau_n\} \) and \(-1\) $ at \( \min(T + \delta, \tau_{n+1}) \) (i.e. later). Hence in an arbitrage-free market, \( \Pi_n(t, T) \) must at least satisfy
\[ \Pi_n(t, T) \text{ nondecreasing in } T, \ n = 0, \ldots, N. \] \((\ast \Pi)\)
We call \((\ast B), \ (\ast P), \ (\ast U)\) and \((\ast \Pi)\) the static no-arbitrage conditions. If they hold, then it is easily checked that also the following statements are true: \( P_n(t, T) \leq 1 \) and for each \( T \) there is always at least one \( n \geq L_t \) with \( P_n(t, T) > 0. \) \( U_n(t, T) \in [0, 1] \) and is nonincreasing in \( n. \) \( \Pi_n(t, T) \in [0, 1] \)
and \( \Pi_n(t, t) = 0. \) And the defaultable zero bond \( S_n(t, T) := \sum_{j=L_t}^{j=n} P_j(t, T), \)
which pays \$1 at \( T \) if at most \( n \) losses have occurred by time \( T, \) satisfies \( S_n(t, T) \in [0, 1] \)
and is nonincreasing in \( T \) and \( S_n(t, t) = 1 \) for all \( n = L_t, \ldots, N. \)

\(^4\)If \( L_t = n, \) then \( P_j(t, T) \) and \( U_{j+1}(t, T), \ j = 0, \ldots, n-1 \) are not traded anymore, and we are free to set them equal to zero. Further we make the convention \( U_{N+1}(t, T) = 0. \)
Remark 1 (πₙ as Loss Contingent Forward Lending). If the law of each τₙ is diffuse, then a full term structure of πₙ(t, T) allows to approximate the nᵗʰ loss-contingent forward interest rate arbitrarily well in the following sense. Suppose n ≥ L₁. Then for every T ≥ t and ε > 0, there exists a.s. an Fₜₙ-measurable δ = δ(t, ω) > 0 such that \( P \left[ \tauₙ, \tauₙ+₁ ∈ (T, T + δ) \mid Fₜ \right] > 1 - ε \) (on the set \( \{L₂ ≤ n\} \)). And hence with probability greater than 1 − ε, \( πₙ(t, T + δ) - πₙ(t, T) \) will pay⁵

\[
+1$ at T and $-1$ at T + δ if \( Lₜ = n \),
nothing otherwise,
\]

i.e. holding \( πₙ(t, T + δ) - πₙ(t, T) \) approximately replicates forward lending of 1$ over \( [T, T + Δ] \), contingent on \( Lₜ = n \).

We denote the space of rightcontinuous functions in \( T \) with \( C^{0,r}(T) \), and \( C^{1,r}(T) \) is the space of continuous functions that admit a right derivative which is in \( C^{0,r}(T) \). We write \( ∂ₜ \) for the right derivative operator. Suppose now \( (•B) \), \( (•P) \), \( (•U) \) and \( (•Π) \) hold and \( Pₙ(t, T) \) and \( Uₙ(t, T) \) are in \( C^{1,r}(T) \). Then

\[
Fₙ(t, T) := 1_{\{Pₙ(t, T) ≠ 0\}} \frac{1}{Fₙ(t, T)} ∂ₜ Uₙ₊₁(t, T) \quad \text{and} \quad fₙ(t, T) := 1_{\{Pₙ(t, T) ≠ 0\}} \frac{1}{Fₙ(t, T)} ∂ₜ πₙ(t, T)
\]

(1.3)

are well-defined and nonnegative. Assuming in addition that \( \int_{t}^{T} fₙ(t, u) + Fₙ(t, u)du < ∞ \) for all \( n = 0, \ldots, N \) and defining \( P⁻¹ := 0 \) ensures that we have by construction

\[
Pₙ(t, T) = e^{t, T}_n \left( 1_{\{Lₜ = n\}} + \int_{t}^{T} (e^{t, u})⁻¹ P⁻¹(t, u)F⁻¹(t, u)du \right) \quad \text{and} \quad Uₙ(t, T) = \int_{t}^{T} P⁻¹(t, u)F⁻¹(t, u)du,
\]

(1.4)

where \( e^{t, T}_n := e^{- \int_{t}^{T} fₙ(t, u) + Fₙ(t, u)du} \). This way, \( P₀ = \cdots = P_{L₁−¹} = 0 \) and \( U₁ = \cdots = U_{L₁} = 0 \) is automatically satisfied, and

\[
∂ₜ Pₙ(t, T) = -Pₙ(t, T) \left( fₙ(t, T) + Fₙ(t, T) \right) + P⁻¹(t, T)F⁻¹(t, T)
\]

(1.5)

And with probability less than ε it pays +1$ and then −1$ within \( [T, T + δ] \) as described above.

⁵with the convention \( Fₙ(t, T) = 0 \).
1.3. Forward Model and Absence of Arbitrage

holds with initial condition \( P_n(t, t) = 1_{[L_t = n]} \).

The classical forward rate curve \( f(t, T) \in C^{0,r} \) for default-free interest-rates is characterized by \( f(t, T) := -\partial_T \log B(t, T) \), which is equivalent to \( \int_t^T B(t, u) f(t, u)du = 1 - B(t, T) \) for all \( T \in [t, T^*] \). Similarly, we can uniquely define a curve \( F(t, T) \in C^{0,r} \) such that

\[
\text{Prot}_{0,N}(t, T) = \sum_{n=1}^N U_n(t, T) = \int_t^T B(t, u) F(t, u)du
\]

for all \( T \in [t, T^*] \). The integral \( \int_t^T B(t, u) F(t, u)du \) is the value of a continuous non-constant payment stream of \( F(t, u)du \) over the interval \( u \in [t, T] \) which by construction is equal to the value \( \text{Prot}_{0,N}(t, T) \) of protection against all losses over \([t, T]\) (see also equation (1.1)). Hence, \( F(t, T) \) can be interpreted as a type of non-standard protection fee which is not subject to the usual reduction of a standard index CDS (cp. the discussion preceding (1.2)).

Using this notation, we obtain

**Proposition 1.** \( f(t, T) \) and \( F(t, T) \) decompose into

\[
f(t, T) = \sum_{n=0}^N \frac{P_n(t, T)}{B(t, T)} f_n(t, T) \quad \text{and} \quad F(t, T) = \sum_{n=0}^{N-1} \frac{P_n(t, T)}{B(t, T)} F_n(t, T),
\]

i.e. \( f(t, T) \) and \( F(t, T) \) are weighted averages over \( f_n(t, T) \) and \( F_n(t, T) \), respectively.

**Proof.** Note that \( \sum_{n=0}^N \Pi_n(t, T) = 1 - B(t, T) \) and by definition we have \( B(t, T) f(t, T) = -\partial_T B(t, T) \). Hence

\[
B(t, T) f(t, T) = \sum_{n=0}^N \partial_T \Pi_n(t, T) = \sum_{n=0}^N P_n(t, T) f_n(t, T).
\]

Analogously, \( B(t, T) F(t, T) = \partial_T \text{Prot}_{0,N}(t, T) = \sum_{n=1}^N \partial_T U_n(t, T) = \sum_{n=1}^N P_n(t, T) F_n(t, T) \). 

Thus, our model refines the notion of forward interest-rates and forward protection rates by conditioning these rates on the realization of \( L(t) = n \). The corresponding unconditional rates are reached by projection. A model which only uses unconditional forward loss rates \( F(t, T) \) and does not allow a dependence on \( L_t \) is the forward loss model of Bennani (2005). This modelling approach amounts to an implicit assumption of \( F(t, T) = F_n(t, T) \) (and also
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\( f(t, T) = f_n(t, T) \) for all \( n \). This has implausible consequences, e.g. the price of the next-to-default zero bond (which defaults at the next loss event)

\[
P_{L_1}(t, T) = e^{-\int_t^T f_n(t, u) + F_n(t, u) du}
\]
does not depend on the current loss level \( L_1 \).

We have seen that if the no-arbitrage requirements (\( \bullet B \)), (\( \bullet P \)), (\( \bullet U \)) and (\( \bullet \Pi \)) hold, then \( f_n \) and \( F_n \) as defined in (1.3) are nonnegative. Next, we turn to the converse case.

**Definition 1.** If there exists nonnegative term structures \( f_n(t, T) \), \( F_n(t, T), \) \( n = 0, \ldots, N \), with \( \int_t^T f_n(t, u) + F_n(t, u) du < \infty \), \( n = L_1, \ldots, N \) for all \( t \leq T^* \), such that \( P_n \) and \( U_n \) satisfy (1.4) a.s., then we say \( P_n \) and \( U_n \) have the representation property with respect to \( f_n \) and \( F_n \). (For short, they have the representation property.)

In definition 1, we do not require that \( P_n \) and \( U_n \) are in \( C^{1,r}(T) \), and \( f_n \) and \( F_n \) need not be in \( C^{0,r}(T) \). Also \( f_0, \ldots, f_{L_1-1} \) and \( F_0, \ldots, F_{L_1-1} \) can take arbitrary values without affecting \( P_n \) and \( U_n \) in (1.4). Therefore, \( f_n F_n \) in definition 1 are not unique.

The results below are understood \( t \)-wise a.s.

**Theorem 2.** Let \( P_n \) and \( U_n \) have the representation property. Then (\( \bullet B \)), (\( \bullet P \)), (\( \bullet U \)) and (\( \bullet \Pi \)) hold.

The proof can be found at the end of this section. With a slight abuse of notation we obtain\(^7\)

**Corollary 3.** Let \( f_n \) and \( F_n \) be in \( C^{0,r}(T) \) with \( \int_t^T |f_n(t, u)| + |F_n(t, u)| du < \infty \) for \( n = 0, \ldots, N \). Then the following are equivalent.

(i) \( P_n \) and \( U_n \) are in \( C^{1,r}(T) \) and satisfy (\( \bullet B \)), (\( \bullet P \)), (\( \bullet U \)) and (\( \bullet \Pi \)).

(ii) \( P_n \) and \( U_n \) have the representation property with respect to \( f_n \) and \( F_n \), in particular \( f_1, \ldots, f_N \) and \( F_1, \ldots, F_{N-1} \) are nonnegative.

Informally, we can state that under weak regularity, for absence of static arbitrage in the market consisting of \( P_n \) and \( U_n \) (and \( b_t \)) it is necessary that \( P_n \) and \( U_n \) have the representation (1.4) with \( F_n \) and \( f_n \) nonnegative.

---

\(^7\)By abuse of notation we mean that \( f_n \) and \( F_n \) can be defined likewise by (1.3) or definition 1. But as we showed, under sufficient regularity the two notions coincide.
1.3. Forward Model and Absence of Arbitrage

Remark 4. In practical applications, it would be possible to fit all traded tranches with piecewise constant (in $T$) term structures $f_n(t, T)$ and $F_n(t, T)$. Therefore we think that requiring $P_n(t, T)$ and $U_n(t, T)$ to be continuously differentiable in $T$ (or almost equivalently $f_n(t, T)$ and $F_n(t, T)$ be continuous in $T$) might be too restrictive.

Strictly Positive Interest Rates, Default- and Survival Probabilities

At the beginning of this section, we required only minimal properties for the evolution of the bank account $b_t$ and the distribution of the loss times $\tau_n$, $n = 1, \ldots, N$ under $P$. It is realistic and removes implausible investment opportunities\textsuperscript{8} to assume in addition that $b_t$ is a.s. increasing and that $t$-wise a.s. for all $n = L_t + 1, \ldots, N$,

$$P[ \tau_n > T \mid \mathcal{F}_t ], T \geq t$$

is (strictly) positive and decreasing in $T$.

Intuitively, there always remains a minimum of uncertainty about the occurrence of losses in the future (unless the entire portfolio has already defaulted) and the money market account pays positive interest. In this modified setup,

$$U_n(t, T) \text{ increasing in } T \text{ and } U_n(t, t) = 0, n = L_t + 1, \ldots, N. \quad (\ast U')$$

$$\Pi_n(t, T) \text{ increasing in } T, n = L_t, \ldots, N. \quad (\ast \Pi')$$

are necessary conditions for absence of arbitrage. In case $(\ast B)$, $(\ast P)$, $(\ast U')$ and $(\ast \Pi')$ hold, $f_n$ and $F_n$ as defined in (1.3) will be positive for $n \geq L_t$, and it holds in addition: $B(t, T)$ is decreasing in $T$. $P_n(t, T) \in (0, 1)$ for all $T > t$. $U_n(t, T) \in (0, 1)$ and is decreasing in $n$. $\Pi_n(t, T) \in (0, 1)$ for all $T > t$ and $\Pi_n(t, t) = 0$. And the defaultable zero bond $S_n(t, T)$ is decreasing in $T$ and satisfies $S_n(t, t) = 1$ for all $n = L_t, \ldots, N$. With a little more work along the proof of theorem 2, we then obtain the counterpart of theorem 2 and its corollary.

Theorem 5. Let $P_n$ and $U_n$ have the representation property with respect to $f_n$ and $F_n$, and let $f_n$, $n = L_t, \ldots, N$ and $F_n$, $n = L_t, \ldots, N - 1$ be positive a.s. Then $(\ast B)$, $(\ast P)$, $(\ast U')$ and $(\ast \Pi')$ hold.

Corollary 6. Let $f_n$ and $F_n$ be in $C^{0,r}(T)$ with $\int_T^T |f_n(t, u)| + |F_n(t, u)| \, du < \infty$ for $n = 0, \ldots, N$. Then the following are equivalent.

(i) $P_n$ and $U_n$ are in $C^{1,r}(T)$ and satisfy $(\ast B)$, $(\ast P)$, $(\ast U')$ and $(\ast \Pi')$.

\textsuperscript{8}E.g. that $P_n(t, T) = 1$ for $T > t$ or that $U_n(t, T)$ trades at zero even though $L_t < n$. 
(ii) $P_n$ and $U_n$ have the representation property with respect to $f_n$ and $F_n$, and $f_n, n = L_1, \ldots, N$ and $F_n, n = L_1, \ldots, N - 1$ are positive.

Proof of Theorem 2. It is clear that $P_0 = \ldots = P_{L_t-1} = U_1 = \ldots = U_{L_t} = 0$ and $P_{L_t}(t, T) = e_{L_t}^{T,T} \in (0, 1]$. Hence if $L_t < N$, then $(\star B)$, $(\star P)$ and $(\star U)$ hold. On the other hand, if $L_t = N$, we may assume that for some $n_0 \geq L_t$, we have $P_n(t, T), U_n(t, T) \in [0, 1]$, $U_n(t, T)$ is nondecreasing in $T$, and $P_n$ and $U_n$ satisfy $U_n(t, T) = \int_t^T P_{n-1}(t, u) F_{n-1}(t, u) du$ and $P_n(t, T) = 1_{[L_t = n_0]} - \int_t^T P_n(t, u) \left( f_n(t, u) + F_n(t, u) \right) du + U_n(t, T)$ for all $n \leq n_0$. This implies in particular $U_n(t, T) = 0$ and $P_n(t, T) = 1_{[L_t = n_0]}$ for all $n \leq n_0$.

Then $U_{n_0+1}(t, T) := \int_t^T P_{n_0}(t, u) F_{n_0}(t, u) du$ as well as $P_{n_0+1}(t, T) := \int_t^T e^{t,u,T} P_{n_0}(t, u) F_{n_0}(t, u) du$ are well-defined and nonnegative. $U_{n_0+1}(t, T)$ is nondecreasing in $T$, $U_{n_0+1}(t, t) = 0$ and $P_{n_0+1}(t, t) = 1_{[L_t = n_0+1]}$ by construction. Further

$$P_{n_0+1}(t, T) \leq U_{n_0+1}(t, T) \leq \int_t^T P_{n_0}(t, u) \left( f_{n_0}(t, u) + F_{n_0}(t, u) \right) du = P_{n_0}(t, t) - P_{n_0}(t, T) + U_{n_0}(t, T) \leq 1$$

and, using the abbreviation $\overline{f}_n(t, T) := f_n(t, T) + F_n(t, T)$, we obtain

$$P_{n_0+1}(t, T) = \int_t^T \left( e_{n_0+1}^{t,T} - e_{n_0+1}^{t,u} \right) \left( e_{n_0+1}^{t,v} \right)^{-1} P_{n_0}(t, v) F_{n_0}(t, v) dv + \int_t^T P_{n_0}(t, v) F_{n_0}(t, v) dv$$

$$= \int_t^T \left( e_{n_0+1}^{t,u} \int_t^T \overline{f}_{n_0+1}(t, u) du \right) \left( e_{n_0+1}^{t,v} \right)^{-1} P_{n_0}(t, v) F_{n_0}(t, v) dv + U_{n_0+1}(t, T)$$

$$= \int_t^T \overline{f}_{n_0+1} \int_t^T \left( e_{n_0+1}^{t,v} \right)^{-1} P_{n_0}(t, v) F_{n_0}(t, v) dv \overline{f}_{n_0+1}(t, u) du + U_{n_0+1}(t, T)$$

$$= 1_{[L_t = n_0+1]} - \int_t^T P_{n_0+1}(t, u) \overline{f}_{n_0+1}(t, u) du + U_{n_0+1}(t, T).$$

This way $(\star P)$, $(\star U)$, $B(t, T) > 0$ (since $P_L(t, T) > 0$) and $B(t, t) = 1$ follow by induction in $n_0$. By definition we have $B(t, T) = 1 - \sum_{n=0}^N \Pi_n(t, T)$. Hence it remains to show $(\star U)$. But by the above, we have for all $n \leq N$

$$\Pi_n(t, T) = \int_t^T P_n(t, u) f_n(t, u) du.$$
1.3. Forward Model and Absence of Arbitrage

which is nondecreasing in $T$.

1.3.2 Representation with Auxiliary Markov Chain

If $P_n$ and $U_n$ have the representation property and moreover $f_n$ and $F_n$ are in $C^0(T)$, then the building blocks $P_n(t, T)$ and $U_n(t, T)$ can also be interpreted as follows. For fixed $t$ and $\omega$, consider the auxiliary $\{L_t, \ldots, N\}$-valued Markov chain $\tilde{L} = (\tilde{L}_T)_{T \in [t, T]}$ with law $\tilde{P}$, defined by $\tilde{P}[L_t = L_T] = 1$ and the time-inhomogeneous generator matrix $\tilde{F}(t, T)_{T \in [t, T]}$ with (omitting the $(t, T)$-argument)

$$
\tilde{F} := \begin{pmatrix}
-F_0 & F_0 & 0 & \cdots \\
0 & -F_1 & F_1 & 0 \\
\vdots & 0 & \ddots & \ddots \\
0 & \cdots & 0 & -F_N
\end{pmatrix}
$$

This way the law $\tilde{P}$ is uniquely characterized. Further, we define an auxiliary short rate process $\tilde{r} = (\tilde{r}_T)_{T \in [t, T]}$ by

$$
\tilde{r}_T := \int_{L_t} (t, T)
$$

and auxiliary stopping times $\tilde{t}_{L_n} := t$ and $\tilde{t}_n := \inf\{T > t; \tilde{L}_T \geq n\}$ for $n = L_t + 1, \ldots, N$. $\tilde{\mathbb{E}}[\cdot]$ denotes expectation wrt $\tilde{P}$. Then we have

**Proposition 2.** For every $n = L_t, \ldots, N$ and $T \geq t$ it holds

$$
P_n(t, T) = \tilde{\mathbb{E}}\left[1_{\{L_T = n\}} e^{-\int_{L_t}^{\tilde{r}_T} \tilde{r}_s ds}\right] \quad \text{and}
$$

$$
U_n(t, T) = \tilde{\mathbb{E}}\left[1_{\{\tilde{t}_n \leq T\}} e^{-\int_{L_t}^{\tilde{t}_n} \tilde{r}_s ds}\right].
$$

I.e. under weak regularity, there always (i.e. for every $t$) exists a unique auxiliary Markov chain model $(\tilde{L}, \tilde{P})$ together with an auxiliary short rate $\tilde{r} = \tilde{r}(\tilde{L})$ that allow to represent $P_n(t, T)$ and $U_n(t, T)$ as their expected discounted final payoffs. The proof of proposition 2 can be found at the end of this subsection.

**Remark 7.** This statement also informally confirms that there exist no static arbitrage opportunities if $f_n$ and $F_n$ in (1.4) are nonnegative (cf. corollary 3 where we show necessity). Indeed, if the auxiliary Markov chain model were the true model, then the prices $U_n$ and $P_n$ as defined in proposition 2 would be arbitrage-free, there would not exist any admissible trading strategy offering
arbitrage opportunities. In particular, this holds for the static trading strategies\(^9\). Whether a static trading strategy admits arbitrage depends only on the initial asset prices and the final asset payoffs (and whether these payoffs occur with positive probability), which are identical in the true and the auxiliary model. Formally, to complete the argument, we would need to check here, that asset payoffs indeed have the same null sets in both the true and the auxiliary model, e.g. that \(P \left[ \tau_n > T \mid F_t \right] > 0 \) if and only if \(Q \left[ \tau_n > T \right] > 0\). We will not go further into this but refer to corollary 9 and the remark following it, which show that under weak regularity the auxiliary model is a certain projection of the true model. Intuitively, this makes clear that asset payoffs have the same null sets in both models.

Omitting the argument \((t, T)\), we additionally define the matrices \(P := (P_0, \ldots, P_n), \quad f = -\text{diag}(f_0, \ldots, f_N)\) and the vectors \(e_0 := (1, 0, \ldots, 0)', e_1 := (0, 1, 0, \ldots, 0)', \ldots, e_N := (0, \ldots, 0, 1)'\). Then the term structure ODE (1.5) reads more compactly

\[
\partial_T P(t, T) = P(t, T) \left( f(t, T) + F(t, T) \right), \quad P(t, t) = e'_L, \]

This ODE looks similar to the Kolmogorov equation in Schönbucher (2006). However, as we have noted in proposition 2, in general \(P(t, T)\) is not a probability distribution (its elements do not sum up to one), and the matrix \(f(t, T) + F(t, T)\) is not a generator matrix (since \(F(t, T)\) is a generator matrix, but \(f(t, T)\) is not). Consider \(D(t, T) := \frac{1}{nT} P(t, T)\) instead, which is a probability distribution for all \(T \geq t\). By differentiation we get

\[
\partial_T D(t, T) = D(t, T) \left( f^{\text{ex}}(t, T) + F(t, T) \right), \quad D(t, t) = e'_L, \]

where \(f^{\text{ex}} := -\text{diag}(f_0 - f, \ldots, f_N - f)\).\(^{10}\) This distinguishes our results from those in Schönbucher (2006) as follows. For every \(t, D(t, T)\) is a Markov chain in \(T \in [t, T^{+}]\), if and only if

\[
f_n(t, T) = f(t, T) \quad (1.6)
\]

for all \(T \geq t\) and \(n = L_1, \ldots, N\), i.e. the term structure of interest rates coincides with the term structure of loss-contingent interest rates for all possible loss levels \(n\). See also remark 11 below for further comments on this.

\(^9\)To be precise, with a static trading strategy, we mean that one invests at \(t\) in a number of traded assets (i.e. \(P_n(t, T), U_n(t, T), b_n\) and holds each of these positions until its respective maturity.

\(^{10}\)\(f^{\text{ex}}\) can be termed “loss-contingent excess forward rate matrix”.
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Proof of proposition 2. Let $(\tilde{\mathcal{F}}_T)_{T \in [t, T^*]}$ be the natural filtration of $\tilde{L}$. We will first show that $P_n(t, T)$ as defined above satisfies (1.5). Clearly $\mathbb{E}\left[1_{\{\tilde{L}_n = n\}}|\tilde{\mathcal{F}}_T\right] = 1_{\{\tilde{L}_n = n\}} = P_n(t, t)$. We use Landau’s symbol $o(\Delta)$ for $g$ with $g(\Delta)/\Delta \to 0$ as $\Delta \downarrow 0$. By construction, we have $\mathbb{E}\left[1_{\{\tilde{L}_{n+\Delta} - \tilde{L}_T > 1\}}\right] = o(\Delta)$ and on the set $\{\tilde{L}_T = \tilde{L}_{T+\Delta} = n\}$ it holds $e^{-\int_T^{T+\Delta} \tilde{r}_s ds} = 1 - \Delta f_n(t, T) + o(\Delta)$ $\tilde{P}$-a.s. Thus

$$P_n(t, T + \Delta) = \sum_{k=0}^{n-L_t} \mathbb{E}\left[e^{-\int_T^{T+\Delta} \tilde{r}_s du} 1_{\{\tilde{L}_T = n-k\}} \mathbb{E}\left[1_{\{\tilde{L}_{T+\Delta} - \tilde{L}_T = k\}} e^{-\int_T^{T+\Delta} \tilde{r}_s du} | \tilde{\mathcal{F}}_T\right]\right]$$

$$= P_n(t, T)(1 - \Delta F_n(t, T))(1 - \Delta f_n(t, T)) + o(\Delta)$$

$$+ \mathbb{E}\left[e^{-\int_T^{T+\Delta} \tilde{r}_s du} 1_{\{\tilde{L}_T = n-1\}} \mathbb{E}\left[1_{\{\tilde{L}_{T+\Delta} - \tilde{L}_T = 1\}} | \tilde{\mathcal{F}}_T\right]\right]$$

$$- \mathbb{E}\left[e^{-\int_T^{T+\Delta} \tilde{r}_s du} 1_{\{\tilde{L}_T = n-1\}} \mathbb{E}\left[1_{\{\tilde{L}_{T+\Delta} - \tilde{L}_T = 1\}}(1 - e^{-\int_T^{T+\Delta} \tilde{r}_s du}) | \tilde{\mathcal{F}}_T\right]\right].$$

Notice that $1_{\{\tilde{L}_T = n-1\}} \mathbb{E}\left[1_{\{\tilde{L}_{T+\Delta} - \tilde{L}_T = 1\}} | \tilde{\mathcal{F}}_T\right] = 1_{\{\tilde{L}_T = n-1\}} \Delta F_{n-1}(t, T) + o(\Delta)$ $\tilde{P}$-a.s. Using this and $\tilde{r}_u \leq f_{n-1}(t, u) + f_n(t, u)$ on $\{n-1 \leq \tilde{L}_u \leq n\}$, we deduce

$$\frac{1}{\Delta} \mathbb{E}\left[e^{-\int_T^{T+\Delta} \tilde{r}_s du} 1_{\{\tilde{L}_T = n-1\}} \mathbb{E}\left[1_{\{\tilde{L}_{T+\Delta} - \tilde{L}_T = 1\}}(1 - e^{-\int_T^{T+\Delta} \tilde{r}_s du}) | \tilde{\mathcal{F}}_T\right]\right]$$

$$\leq \left(1 - e^{-\int_T^{T+\Delta} f_{n-1}(t, u) + f_n(t, u) du}\right) \frac{1}{\Delta} \mathbb{E}\left[1_{\{\tilde{L}_{T+\Delta} - \tilde{L}_T = 1\}} \right] \xrightarrow{\Delta \downarrow 0} 0.$$ 

Combining the steps above we obtain that $\mathbb{E}\left[1_{\{\tilde{L}_T = n\}} e^{-\int_T^{T+\Delta} \tilde{r}_s du}\right]$ solves the forward ODE (1.5). Second, we show that $U_n(t, T)$ as defined above, satisfies (1.4). Since $\int_T^{T+\Delta} F_{\tilde{L}_n}(t, u) du$ is the predictable compensator of $\tilde{L}$ under $\tilde{P}$,

$$\mathbb{E}\left[1_{\{t < \tilde{L}_n \leq T\}} e^{-\int_T^{\tilde{L}_n} \tilde{r}_s du}\right] = \mathbb{E}\left[\int_t^{\tilde{L}_n} 1_{\{\tilde{L}_n = n\}} e^{-\int_{\tilde{L}_n}^{t} \tilde{r}_s du} d\tilde{L}_n\right]$$

$$= \int_t^T \mathbb{E}\left[1_{\{\tilde{L}_n = n\}} e^{-\int_{\tilde{L}_n}^{t} \tilde{r}_s du}\right] F_{n-1}(t, u) du$$

$$= \int_t^T P_{n-1}(t, u) F_{n-1}(t, u) du = U_n(t, T).$$
1.3.3 Dynamic No-Arbitrage Conditions

So far we have only excluded static arbitrage opportunities by restricting the term structures \( f_n \) and \( F_n \) in the representation (1.4) separately for every \( t \). In this section, we are concerned with the dynamic evolution of \( f_n \) and \( F_n \). For derivatives pricing, we mostly like to set up a model directly under an equivalent martingale measure (EMM) and the link to the real-world probability measure is of less importance. We therefore give necessary and sufficient martingale/drift restrictions for the dynamics of \( f_n \) and \( F_n \) under an EMM, but we do not question the existence of an EMM. Once the model is set up under an EMM, every candidate for a real-world probability measure can be constructed using appropriate measure transformations. Consequently, \( P \) does not necessarily denote the real-world probability measure here.

**Assumption 1.**

(i) \( P_n \) and \( U_n \) have the representation property with respect to \( f_n \) and \( F_n \).

(ii) The money market account is absolutely continuous.

(iii) A floating rate note with a continuous dividend rate \( r_{fl}^f \), ensuring that the value of the floating rate note is constant, is traded.

By (ii) there exists a short rate process \( r_t \) with \( b_t = e^{\int_0^t r_s \, ds} \) and hence the discount factor \( \beta_t := 1/b_t \) satisfies \( \beta_t = 1 - \int_0^t \beta_s r_s \, ds \). Note that \( P_n(t, T) \) is an asset without dividends and terminal value \( 1_{[L=0]} \) and \( U_n(t, T) \) is a dividend-paying asset with terminal value zero \( (U_n(T, T) = 0) \) and dividend \( 1_{[L_{t-}>0]} dL_t \). It is well-known that absence of arbitrage is equivalent to the existence of an equivalent (local) martingale measure.

**Definition 2 (EMM).** We say \( Q \sim P \) is an equivalent martingale measure (EMM), if every asset (i.e. \( b_t, P_n, U_n \) and the floating rate note) with price process \( S_t \) and accumulated dividend process \( D_t \) (i.e. with dividend \( dD_t \)), satisfies

\[
\beta_t S_t + \int_0^t \beta_s dD_s \in \mathcal{M}^\text{loc}(Q). \tag{1.7}
\]

For the floating rate note, this means \( \beta_t + \int_0^t \beta_s r_{fl}^f \, ds \) must be a \( Q \)-local martingale. But since it is continuous and of bounded variation, it must be constant, which is equivalent to \( r_{fl}^f = r_t \, P \times 1(dt) \)-a.e. This allows to characterize the short rate (if it exists) in an arbitrage-free market as the unique\(^\text{11} \) process \( r_t \).

\(^\text{11} \) Unique in \( L_1(P \times 1) \).
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such that \( \beta_t + \int_0^t \beta_s r_s \, ds \) is a \( P \)-martingale (cf. Hughston and Rafailidis (2005)). Similarly, the predictable compensator of \( L \) under \( P \) is the unique predictable nondecreasing process \( A_t \), with \( A_0 = 0 \) such that \( L_t = A_t \) is a \( P \)-martingale. If \( A \) is absolutely continuous, then we may write \( A_t = \int_0^t \lambda_s \, ds \), and we say \( L \) admits the loss intensity \( \lambda \) under \( P \).

Clearly, \( P_n \) and \( U_n \) are bounded with bounded dividends. Hence if \( P \) itself is an EMM, then it holds \( P_n(t, T) = E \left[ 1_{\{L_T = T \}} \beta_T / \beta_t \mid F_t \right] \) and

\[
U_n(t, T) = E \left[ \int_t^T 1_{\{L_s = n-1 \}} \beta_s / \beta_t \, dL_s \bigg| F_t \right].
\]

Now we are ready to formulate the main result of this subsection.

**Theorem 8.** Let \( \beta_t P_n(t, T) f_n(t, T) \) and \( \beta_t P_n(t, T) F_n(t, T) \) be an EMM, then it holds \( P_n(t, T) = E \left[ 1_{\{L_T = n \}} \beta_T / \beta_t \mid F_t \right] \) and \( U_n(t, T) = \int_t^T 1_{\{L_s = n-1 \}} \beta_s / \beta_t \, dL_s \mid F_t \).

Conversely, let \( P \) be an EMM, and assume that \( f_n(t, T_0) \) and \( F_n(t, T_0) \) are integrable and \( \lim_{t \to T_0} E \left[ |x_n(t, T) - x_n(t, T_0)| \right] = 0 \) for all \( t \leq T_0 \leq T^* \), \( n = 0, \ldots, N \) and \( x = f, F \). Then \( \beta_t P_n(t, T) f_n(t, T) \) and \( \beta_t P_n(t, T) F_n(t, T) \) are versions of the short rate and the loss intensity, respectively.

We immediately derive the following corollary, which illustrates best the relation between \( r \) and the “forward rates” \( f_n, F_n \).

**Corollary 9.** Let \( L \) admit an intensity \( \lambda \) under \( P \), and assume that \( f_n(t, T_0) \) and \( F_n(t, T_0) \) are integrable and \( \lim_{t \to T_0} E \left[ |x_n(t, T) - x_n(t, T_0)| \right] = 0 \) for all \( t \leq T_0 \leq T^* \), \( n = 0, \ldots, N \) and \( x = f, F \). Then \( P \) is an EMM if and only if

\[
\begin{align*}
f_n(t, T) &= 1_{\{P_n(t, T) \neq 0\}} \frac{E \left[ \beta_T 1_{\{L_T = n\}} / \beta_t P_n(t, T) \mid F_t \right]}{\beta_t P_n(t, T)} \text{ and } \\
F_n(t, T) &= 1_{\{P_n(t, T) \neq 0\}} \frac{E \left[ \beta_T 1_{\{L_T = n\}} / \beta_t P_n(t, T) \mid F_t \right]}{\beta_t P_n(t, T)}. \tag{1.9}
\end{align*}
\]

Proof. If \( \lambda_t = \beta_T 1_{\{L_T = n\}} / \beta_t P_n(t, T) \), then \( P_n(T, T) F_n(T, T) = 1_{\{L_T = n\}} F_n(T, T) = 1_{\{L_T = n\}} \beta_T 1_{\{L_T = n\}} / \beta_t P_n(t, T) \). Analogously, \( P_n(T, T) f_n(T, T) = 1_{\{L_T = n\}} r_T \). Then the claim follows from theorem 8. \( \square \)
Remark 10. The possibility that the loss intensity $\lambda_j$ of a credit portfolio jumps (up) at the occurrence of defaults is usually referred to as contagion. Due to theorem 8, $\lambda_j = F_{L_j}(t, t)$ must hold up to regularity under every EMM. This implies in particular that modelling $F_n(t, T)$ as continuous processes (for every $T$) does not rule out contagion because by construction, $\lambda_j = F_{L_j}(t, t)$ automatically admits discontinuities at each loss event; contagion is a natural feature of our model. However, we will give an argument in chapter 2 that if the rates $F_n(t, T)$ are stochastic, then they should at least exhibit jumps at the loss times $t_1, \ldots, t_N$. We will present a jump diffusion-based Heath-Jarrow-Morton-type (HJM) setup of the model in section 1.4.

For the rest of this section we suppose the conditions of corollary 9 are satisfied and $P$ is an EMM. We denote with $P^T$ the $T$-forward measure as introduced in Jamshidian (1987).

Remark 11. If $P \left[ L_T = n \right] > 0$, then $P_n(0, T) > 0$ and –similar to the $T$-forward measure– we may define the the $n^{th}$ loss-contingent $T$-forward measure $P^T_n \ll P$ by

$$\frac{dP^T_n}{dP} \bigg|_{F_t} := \frac{\beta_t P_n(t, T)}{P_n(0, T)}$$

Then (1.9) reads to $f_n(t, T) = E^{P^T_n} \left[ r_T | F_t \right]$ and $F_n(t, T) = E^{P^T_n} \left[ \lambda_T | F_t \right]$. It is well-known that $f(t, T) = E^{P^T} \left[ r_T | F_t \right]$ and using proposition 1 we also obtain $F(t, T) = E^{P^T} \left[ \lambda_T | F_t \right]$. This shows that both standard and loss-contingent forward interest rates and forward loss rates are projections of the true future interest rates and default rates, with respect to an appropriate probability measure. It also shows that $f_n(t, T) \geq 0$ and $F_n(t, T) \geq 0$ must be the case if $P$ is an EMM.

We mentioned in section 1.3.2 that the representation (1.4) discussed in this chapter coincides with that obtained in Schönbucher (2006) if and only if (1.6) holds, i.e. $f_n(t, T) = f(t, T)$ for all $n$. This is in turn is equivalent to

$$E^{P^T} \left[ 1_{\{L_T = n\}} r_T | F_t \right] = E^{P^T} \left[ 1_{\{L_T = n\}} | F_t \right] E^{P^T} \left[ r_T | F_t \right],$$

i.e. conditional on $F_t$, $L_T$ and $r_T$ are uncorrelated with respect to the $T$-forward measure $P^T$. (To see this, note $L_T = \sum_{n=0}^{N} n 1_{\{L_T = n\}}$.) Empirical studies as e.g. Duffie et al. (2006) indicate that correlation between interest rates and defaults is typically negative.

$P^T$ is induced by the $P$-martingale $\beta_t B(t, T)$ via $\frac{dP^T}{dP} \bigg|_{F_t} = \frac{\beta_t B(t, T)}{B(0, T)}$. 

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12 $P^T$ is induced by the $P$-martingale $\beta_t B(t, T)$ via $\frac{dP^T}{dP} \bigg|_{F_t} = \frac{\beta_t B(t, T)}{B(0, T)}$. 

Proof of theorem 8. To show the first result, we only need to check that all traded assets satisfy (1.7). The floating rate note satisfies (1.7) by construction. \( \int_0^t F_{L_x}(s, s) ds \) being the predictable compensator of \( L \) under \( \mathbb{P} \), we have

\[
\mathbb{E} \left[ \int_t^T \beta_t 1_{[t, s]}dL_s \middle| \mathcal{F}_t \right] = \mathbb{E} \left[ \int_t^T \beta_t 1_{[L_{s-} = n-1]} F_{L_x}(s, s) ds \middle| \mathcal{F}_t \right] = \mathbb{E} \left[ \int_t^T \beta_t P_{n-1} F_{n-1}(s, s) ds \middle| \mathcal{F}_t \right] = \int_t^T \beta_t P_{n-1}(t, s) F_{n-1}(t, s) ds = \beta_t U_n(t, T) \tag{1.10}
\]

for all \( n = 1, \ldots, N \) and \( t \leq T \leq T^* \) (in the second equality we used that \( L_{x-} = L_x \mathbb{P} \times t(ds) \)-a.e. and \( P_{n-1}(s, s) = 1_{[L_{s-} = n-1]} \)). Hence \( U_n \) satisfies (1.7). As concerns \( P_n \), we note that, since \( r_t = f_{L_x}(t, t) \),

\[
\beta_T 1_{[L_T = n]} = \beta_t 1_{[L_t = n]} - \int_t^T \beta_s 1_{[L_s = n]} f_{L_x}(s, s) ds + \int_t^T \beta_s (1_{[L_s = n-1]} - 1_{[L_s = n]}) dL_s.
\]

Then, the latter together with (1.10) and the definition of \( \Pi_n \) and \( P_n(T, T) = 1_{[L_T = n]} \) yields

\[
\mathbb{E} \left[ \beta_T P_n(T, T) \middle| \mathcal{F}_t \right] = \beta_t \Pi_n(t, T) + \beta_t P_n(t, T) - \int_t^T \mathbb{E} \left[ \beta_s 1_{[L_s = n]} f_{L_x}(s, s) \middle| \mathcal{F}_t \right] ds = \beta_t \Pi_n(t, T) + \beta_t P_n(t, T) - \int_t^T \beta_s P_n(t, s) f_{n}(t, s) ds = \beta_t P_n(t, T)
\]

for all \( n = 0, \ldots, N \) and \( t \leq T \leq T^* \), i.e. \( P_n \) satisfies condition (1.7).

For the converse direction, we will first prove that the martingale property holds and then show that both \( L_T = \int_0^T F_{L_x}(s, s) ds \) and \( \beta_T + \int_0^T \beta_s f_{L_x}(s, s) ds \) are in \( \mathcal{M}(\mathbb{P}) \). Let \( 0 \leq t_1 < t_2 \leq T^* \) and \( F \in \mathcal{F}_{t_1} \). First of all we note that
$U_n(t_1, T) = U_n(t_1, t_2) + \int_{t_2}^{T} P_{n-1}(t_1, u) F_{n-1}(t_1, u) du$ for $T \geq t_2$. Hence, if $P$ is an EMM, then, using the abbreviation $P_n F_n(t, u) := P_n(t, u) F_n(t, u)$,

\[
0 = E \left[ 1_F \left( \beta_{t_2} U_n(t_2, T) - \beta_{t_1} U_n(t_1, T) + \int_{t_1}^{t_2} \beta_1 1_{\{L_s = n-1\}} dL_s \right) \right]
\]

\[
= E \left[ 1_F \int_{t_2}^{T} \beta_{t_2} P_{n-1} F_{n-1}(t_2, u) - \beta_{t_1} P_{n-1} F_{n-1}(t_1, u) du \right]
\]

\[
- E \left[ 1_F \beta_{t_1} U_n(t_1, t_2) - \int_{t_1}^{t_2} \beta_1 1_{\{L_s = n-1\}} dL_s \right]
\]

\[
= \int_{t_2}^{T} E \left[ 1_F \left( \beta_{t_2} P_{n-1} F_{n-1}(t_2, u) - \beta_{t_1} P_{n-1} F_{n-1}(t_1, u) \right) \right] du \quad (1.11)
\]

for all $T \in [t_2, T^*]$ and $n = 1, \ldots, N$. In the last equality we used (1.8). This implies that the integrand in (1.11) vanishes for $t$-a.e. $u \in [t_2, T]$. For all $t \leq u$ and $n$, $E \left[ 1_F \beta_{t_1} P_{n-1}(t, u) F_{n-1}(t, u) \right]$ exists. It is right-continuous in $u$ for all $t$ and $n$ because

\[
E \left[ 1_F \beta_{t_1} |P_n F_n(t, u) - P_n F_n(t, u_0)| \right] 
\]

\[
\leq E \left[ |P_n F_n(t, u) - P_n F_n(t, u_0)| \right] 
\]

\[
\leq E \left[ P_n(t, u) |F_n(t, u) - F_n(t, u_0)| \right] + E \left[ P_n(t, u_0) |P_n(t, u) - P_n(t, u_0)| \right] 
\]

\[
\leq E \left[ |F_n(t, u) - F_n(t, u_0)| \right] 
\]

\[
\rightarrow 0 
\]

\[
\lim_{u \downarrow u_0} E \left[ F_n(t, u_0) |P_n(t, u) - P_n(t, u_0)| \right] = 0 
\]

followed by dominated convergence. Hence the integrand in (1.11) vanishes for every $u \in [t_2, T]$ and it follows $\beta_{t_1} P_n(t, T) F_n(t, T) \in \mathcal{M}(P)$ for all $n = 1, \ldots, N$ and $T \leq T^*$.

\[
\Pi_n(t, T) = 1_{\{L_s = n\}} - P_n(t, T) + E \left[ \int_{t}^{T} \frac{\beta_s}{\beta_t} (1_{\{L_s = n-1\}} - 1_{\{L_s = n\}}) dL_s \left| \mathcal{F}_t \right. \right] 
\]

is a consequence of (1.8) and the definition of $\Pi_n(t, T)$. We proceed similar as above. $\Pi_n(t_1, T) = \Pi_n(t_1, t_2) + \int_{t_2}^{T} P_n(t_1, u) f_n(t_1, u) du$ for $T \geq t_2$. Hence, if $P$ is an EMM, then

\[
0 = E \left[ 1_F \left( \beta_{t_2} \Pi_n(t_2, T) - \beta_{t_1} \Pi_n(t_1, T) - \beta_{t_2} 1_{\{L_s = n\}} + \beta_{t_1} 1_{\{L_s = n\}} \right) \right]
\]

\[
+ E \left[ \int_{t_1}^{t_2} \beta_1 (1_{\{L_s = n-1\}} - 1_{\{L_s = n\}}) dL_s \right]
\]

\[
= \int_{t_2}^{T} E \left[ 1_F \left( \beta_{t_2} P_n(t_2, u) f_n(t_2, u) - \beta_{t_1} P_n(t_1, u) f_n(t_1, u) \right) \right] du 
\]
for all \( n = 0, \ldots, N \) and \( T \leq T^* \). With the same arguments as above this shows that \( \beta_t P_n(t, T) f_n(t, T) \in \mathcal{M}(P) \) for all \( n = 0, \ldots, N \) and \( T \leq T^* \).

From the martingale property of \( \beta_t P_n(t, T) F_n(t, T) \) for all \( n \) we deduce

\[
\sum_{n=1}^{N} \beta_t P_{n-1}(t, s) F_{n-1}(t, s) = \mathbb{E}\left[ \beta_t F_{L_t}(s, s) \mid \mathcal{F}_s \right] \text{ for every } s \geq t.
\]

Note that \( dL_t = 1_{[L_t < N]} dL_t \). Using (1.8), we observe

\[
\mathbb{E}\left[ \int_t^T \beta_s dL_s \mid \mathcal{F}_t \right] = \sum_{n=1}^{N} \beta_t U_n(t, T) = \mathbb{E}\left[ \int_t^T \beta_s f_{L_t}(s, s) ds \mid \mathcal{F}_t \right]
\]

for every \( t \leq T \leq T^* \). This proves that \( \int_t^T \beta_s dL_s \) admits the intensity \( \beta_t F_{L_t}(t, t) \) and, since \( \beta_t \) is positive a.s., it follows that \( L \) admits the intensity \( \lambda_t = F_{L_t}(t, t) \).

Analogously, \( \sum_{n=0}^{N} \beta_t P_n(t, s) f_n(t, s) = \mathbb{E}\left[ \beta_t f_{L_t}(s, s) \mid \mathcal{F}_t \right] \) for every \( s \geq t \). We note that \( 1 - B(t, T) = \sum_{n=0}^{N} \Pi_n(t, T) \) and therefore

\[
\mathbb{E}\left[ \beta_t - \beta_T \mid \mathcal{F}_t \right] = \beta_t (1 - B(t, T))
\]

\[
= \sum_{n=0}^{N} \beta_t \Pi_n(t, T) = \mathbb{E}\left[ \int_t^T \beta_s f_{L_t}(s, s) ds \mid \mathcal{F}_t \right]
\]

for every \( t \leq T \leq T^* \). As pointed out above, this uniquely characterises \( f_{L_t}(t, t) \) as the short rate \( r_t \).

\[ \Box \]

**Remark 12.** Recall that \( 1 - B(t, T) = \sum_{n=0}^{N} \Pi_n(t, T) \). If \( P_n \) and \( U_n \) are in \( \mathcal{C}^{1/2}(T) \), then so are \( \Pi_n \), and the marginal (short-maturity) bond return rate can be defined as usual as

\[
\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \left( \frac{1}{B(t, t + \Delta)} - 1 \right) = f_{L_t}(t, t).
\]

If moreover \( P \) is an EMM and \( f_n \) and \( F_n \) are sufficiently regular (cf. the converse part of theorem 8), then the marginal bond return rate is also equal to the short rate. However, this would not yet prove that \( f_{L_t}(t, t) \) satisfies \( \beta_t + \int_t^T \beta_s f_{L_t}(s, s) ds = \text{const. a.s.} \) because we did not use (1.12) as the definition of the short rate.

### 1.4 A HJM-Type Forward Model

In this section, we consider a HJM-type specification of the *loss-contingent* forward interest and loss rates \( f_n \) and \( F_n \) with jumps. As we mentioned in
remark 10 (and we will further discuss this matter in chapter 2), assuming that 
f_n and F_n are continuous processes seems at first sight not too restrictive but in a realistic setting, these rates should exhibit jumps, at least at the loss times \( \tau_1, \ldots, \tau_N \). Here we will allow finitely many jumps in these rates.

We maintain the general setup and notation of section 1.3, but here, \( \mathbb{P} \) is not necessarily the physical probability measure. \( W_t \) is a \( d \)-dimensional Brownian motion \((d \in \mathbb{N})\) under \( \mathbb{P} \), \((Z, Z)\) is a measurable space and \( \mu(dz, dt) \) is a random measure on \( Z \times [0, T^*] \) with finite activity, i.e. \( \mu(Z \times [0, T^*]) < \infty \) a.s. We assume there exists a predictable function\(^{13}\) \( \phi^L(z, t) \) with

\[
L_t = \int_0^t \int_Z \phi^L(z, s) \mu(dz, ds)
\]

and the predictable compensator measure \( \nu(dz, dt) \) of \( \mu(dz, dt) \) under \( \mathbb{P} \) is of the form \( \nu(dz, dt) = dF_t(z)\lambda_t^\mu dt \). Then \( L_t \) admits the intensity

\[
\lambda_t^\mu := \int_Z \phi^L(z, t) dF_t(z) \cdot \lambda_t^\mu
\]

and it holds \( \int_0^T \lambda_t^\mu dt \leq \int_0^T \lambda_t^\mu dt < \infty \) a.s. We define \( \pi := \nu \) and sometimes we abbreviate \( \nu := \mu(dz, dt) \) etc.

**Assumption 2 (HJM-type Forward Rates).** For every \( T \leq T^* \), there exist nonnegative processes \( f_n(t, T), n = 0, \ldots, N \), and \( F_n(t, T), n = 0, \ldots, N - 1 \), satisfying

\[
df_n(t, T) = \alpha_n^f(t, T)dt + \sigma_n^f(t, T)dW_t + \int_Z \phi_n^f(z, t, T)\pi(dz, dt),
\]

\[
dF_n(t, T) = \alpha_n^F(t, T)dt + \sigma_n^F(t, T)dW_t + \int_Z \phi_n^F(z, t, T)\pi(dz, dt),
\]

\( \sigma_n^f(t, \omega, u) \) and \( \sigma_n^f(t, \omega, u) \) are \( \mathcal{P} \otimes \mathcal{B}([0, \infty)) \)-measurable and \( \phi_n^f(z, t, \omega, u) \) is \( \mathcal{Z} \otimes \mathcal{P} \otimes \mathcal{B}([0, \infty)) \)-measurable for \( n = 0, \ldots, N \) and \( x = f, F \) and

1. \( \int_0^T \alpha_n^f(0, u) + F_n^2(0, u)du < \infty \) a.s. for all \( n \).
2. \( \int_0^T \left( \int_0^u \left( \sigma_n^f(s, u) \right)^2 ds \right) du < \infty \) a.s. for all \( n \) and \( x = f, F \).
3. \( \int_0^T \left( \int_0^u \left( \sigma_n^F(s, u) \right)^2 ds \right) du < \infty \) a.s. for all \( n \) and \( x = f, F \).
4. \( \int_0^T \int_Z \left( \phi_n^f(z, s, u) \right)^2 dz \) a.e. to ensure that \( \Delta L_t \in [0, 1] \) for all \( t \) a.s.

\( \phi^L(z, t) \) must be \( [0, 1] \)-valued \( \mathbb{P} \times \nu(dz, dt) \)-a.e. to ensure that \( \Delta L_t \in [0, 1] \) for all \( t \) a.s.

\( \text{(iv) is equivalent to } \int_0^T \int_Z \left( \phi_n^f(z, s, u) \right)^2 du \mu(dz, ds) < \infty \) a.s. for all \( n \) and \( x = f, F \).
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(v) \( P_n \) and \( U_n \) have the representation property with respect to \( f_n \) and \( F_n \).

Assumption 2 (i) to (v) guarantee that the integrals \( \int_T f_n(t, u)du \) and \( \int_T F_n(t, u)du \) are finite for all \( t \leq T \leq T^* \) a.s. (cf. Heath et al. (1992) if \( \phi^*_n = 0 \) or lemma 14 below for the general case). In the sequel we will use the abbreviations \( \mathcal{F}_n := f_n + F_n, \alpha_n := \alpha^f_n + \alpha^F_n, \sigma_n := \sigma^f_n + \sigma^F_n \) Further, for \( t \leq u \leq T \), we denote \( \gamma^{t,T} := \int_t^T \gamma(t, u)du \) and \( \gamma^{t,u,T} := \int_t^u \gamma(t, v)dv \) for \( \gamma = \alpha, \alpha_n, \sigma, \sigma_n, \sigma_n, \sigma_n \), respectively. 

\[ e_n^{t,u,T} := e^{-\int_t^u \mathcal{F}_n(t, v)dv}. \]

If a product of functions which share the same argument is considered, we put that argument for brevity only at the end of respective product, e.g. \( P_n f_n(T) \) stands for \( P_n f_n(T) \). Also, when integrating functions, we will sometimes completely omit a term structure argument \((t, T), (s, u)\) etc. if it can be guessed/recovered from the integration bounds, e.g. \( \int_T^{t,u,T} P_n du \) is a short-form of \( \int_T^{t,u,T} P_n(t, u)du \). \( P_n^-, f_n^- \) and \( F_n^- \) denote the left \( t \)-limits of \( P_n \), \( f_n \) and \( F_n \), respectively. We set \( v_{-1} := 0, u_{-1} := 0 \) and \( \psi_{-1} \) := 0 and then we iteratively define

\[ v_n(t, T) := 1_{[t_{-1} = n]} P_n^- \sigma_n^{t,T} - 1_{[t_{-1} < n]} \int_T^{t_{-1},u,T} \left( P_n^- \sigma_n - P_{n-1}^- \sigma_n^{u,T} - F_{n-1}^- v_{n-1} \right) du \]

\[ \psi_n^P(z, t, T) := 1_{[t_{-1} = n]} (1 - \phi^P_n) P_n^- e^{-z \phi^P_n} + 1_{[t_{-1} < n]} \phi^P_n e^{t_{-1,T} - z \phi^P_n} \]

\[ + \int_T^{t_{-1},u,T} e^{-z \phi^P_n} \left( P_{n-1}^- + \psi_{n-1}^P \right) (F_{n-1}^- + \phi^P_{n-1}) du - P_{n-1}^- \]

\[ u_n(t, T) := 1_{[t_{-1} = n]} P_n^- \left( \int_T^{t_{-1},u,T} - \alpha_n^{t,T} + \frac{1}{2} \left| \sigma_n^{t,T} \right|^2 + \int_Z \phi^P_n F_i(z) \cdot \lambda_i^{P_n} \right) \]

\[ - 1_{[t_{-1} < n]} \int_T^{t_{-1},u,T} \left( P_n^- \alpha_n + \sigma_n^{t,T} v_n - P_{n-1}^- \alpha_n^{u,T} \right. \]

\[ \left. - F_{n-1}^- u_{n-1} - v_{n-1} \sigma_n^{u,T} \right) du \]

\[ + 1_{[t_{-1} < n]} \int_T^{t_{-1},u,T} \left( P_n^- \phi_n \left. \right|_t^{u,T} \right. \]

\[ \left. - P_{n-1}^- \phi_n^{P_n} - F_{n-1}^- \psi_{n-1}^P \right) F_i(z) du \cdot \lambda_i^{P_n} \]
for \( n = 0, \ldots, N \), and for \( n = 1, \ldots, N \) we define

\[
b_n(t, T) := \int_t^T \left( P_{n-1}^- \sigma_{n-1}^F + F_{n-1}^- v_{n-1} \right) du
\]

\[
c_n(z, t, T) := \int_t^T \left( P_{n-1}^- \phi_{n-1}^F + F_{n-1}^- \psi_{n-1}^P + \phi_{n-1}^F \psi_{n-1}^P \right) du
\]

\[
a_n(t, T) := \int_t^T \left( P_{n-1}^- \sigma_{n-1}^F + F_{n-1}^- u_{n-1} + v_{n-1}' \sigma_{n-1}^F \right) du
\]

\[
+ \int_t^T \int_Z \phi_{n-1}^F \psi_{n-1}^P dF_t(z) du \cdot \lambda_t^u = 1_{\{L_t=n-1\}} F_{n-1}(t, t)
\]

**Theorem 13** (Representation). \( v_n, u_n, a_n, b_n \) are \( \mathcal{P} \otimes \mathcal{B}([0, T^+]) \)-measurable processes with \( \int_0^T \|v_n(s, T)\|^2 + \|u_n(s, T)\|^2 + \|b_n(s, T)\|^2 + \|a_n(s, T)\| ds < \infty \) a.s. and the functions \( \psi_n^P, c_n \) are \( S \otimes \mathcal{P} \otimes \mathcal{B}([0, T^+]) \)-measurable with \( \int_0^T \int_Z |\psi_n^P(z, s, T)| ds < \infty \) a.s. for all \( n \) and \( T \leq T^+ \) and it holds

\[
dP_n(t, T) = u_n(t, T) dt + v_n(t, T) dW_t + \int_Z \psi_n^P(z, t, T) \tilde{\pi}(dz, dt),
\]

\[
dU_n(t, T) = a_n(t, T) dt + b_n(t, T) dW_t + \int_Z c_n(z, t, T) \tilde{\pi}(dz, dt).
\]

**1.4.1 The Proof of Theorem 13**

To begin with, we briefly review a few properties of the HJM-type term structures \( f_\gamma(t, T) \) and \( F_\gamma(t, T) \) that we will use in subsequent proofs. Modelling the term structure of forward interest rates \( f(t, T) \), Heath et al. (1992) showed that in the pure diffusion setup,\(^\text{15}\) one has \( \int_0^T f(t, T) dt < \infty \) a.s. and

\[
d\int_T f(t, u) du = \alpha^i t dt + \sigma^i T dW_t - f(t, t) dt \quad \text{for all } T \leq T^+.
\]

Björk et al. (1997) extended these integration rules to the case where the forward rates \( f(t, T) \) are driven by a jump diffusion \( (W_t, \tilde{\pi}) \), which is our case, but (for the ease of exposition) they did not provide the proper technical conditions. We give them in

**Lemma 14.** Let \( \sigma(t, \omega, T), \alpha(t, \omega, T) \) be \( \mathcal{P} \otimes \mathcal{B}([0, T^+]) \)-measurable with \( \int_0^T \int_Z |\sigma(s, u)|^2 du ds < \infty \) a.s., let \( \phi(z, t, \omega, u) \) be a \( Z \otimes \mathcal{P} \otimes \mathcal{B}([0, T^+]) \)-measurable

\(^\text{15} \)I.e. \( \phi_n^u = 0 \) for all \( n, T \) and \( x = f, F \), in our case.
for all $T \leq T^*$. Then $\int_0^T X(u, u) du < \infty$ a.s. and for all $T \leq T^*$, $\alpha^{t, T}, \sigma^{t, T}$ are $\mathcal{P}$-measurable with $\int_0^T |\alpha^{1, T}| + \|\sigma^{t, T}\|^2 dt < \infty$ a.s., $\phi^{t, T}(z)$ is $Z \otimes \mathcal{P}$-measurable with $\int_0^T |\phi^{1, T}(z)|v(dz, dt) < \infty$ a.s. and
\[
d\int_t^T X(t, u) du = \alpha^{t, u, T} dt + \sigma^{t, u, T} dW_t + \int_Z \phi^{t, u, T}(z)\mathbb{P}(dz, dt) - X(t, t) dt.
\]

### Corollary 15
Let $u \leq T$ be fixed. Under the conditions of lemma 14, it holds for all $t \leq u$
\[
d\int_u^T X(t, v) dv = \alpha^{1, u, T} dt + \sigma^{1, u, T} dW_t + \int_Z \phi^{1, u, T}(z)\mathbb{P}(dz, dt).
\]

### Proof of lemma 14
Heath et al. (1992) showed that for $\phi = 0$ the result follows from the stochastic Fubini theorem (see e.g. Protter (2004), chapter IV, theorem 65). We can thus assume without restriction that $\alpha = 0$, $\sigma = 0$ and $X(0, \cdot) = 0$. It can be checked easily that $\phi^{t, T}(z)$ is a predictable function with $\int_0^T \int_Z |\phi(z, s, u)| v(dz, ds) < \infty$ a.s. and by the property of the predictable compensator, the integrals $\int_0^T \int_Z |\phi(z, s, u)| d\mu(dz, ds)$ and $\int_0^T \int_Z |\phi(z, s, u)| \mu(dz, ds)$ are finite for all $t \leq T^*$ a.s. Then, by the ordinary Fubini theorem,
\[
\int_0^T X(u, u) du \leq \int_0^T \int_0^u |\phi(t, u)| d\mu du + \int_0^T \int_0^u |\phi(t, u)| dv du = \int_0^T \int_0^T |\phi(t, u)| dv du + \int_0^T \int_0^T |\phi(t, u)| du dv < \infty
\]
lemma yield low easily with Itô’s lemma (see (1.14) below). Recalling (1.4), we notice that compute iteratively the dynamics of integrals like in order to derive the dynamics of can be done via lemma 14. The difficulty, however, is to show that the inte-

\[ \int_t^T X(t, u) du = \int_t^T \int_0^u \int_Z \phi(z, s, u) \bar{P}(dz, ds) du \]

\[ = \int_t^T \int_Z \int_s^T \phi(z, s, u) du \bar{P}(dz, ds) \]

\[ - \int_t^T \int_{Z_s} \int_s^T \phi(z, s, u) du \bar{P}(dz, ds) \]

\[ = \int_t^T \phi^{s,T} \bar{P}(dz, ds) - \int_0^u \int_Z \phi(z, s, u) \bar{P}(dz, ds) du \]

\[ = \int_0^T \phi^{s,T} (z) \bar{P}(dz, ds) - \int_0^T X(u, u) du. \]

Application of lemma 14 and its corollary to \( X := T_n \) together with Itô’s lemma yield

\[ \frac{d e_n^{i,u,T}}{e_n^{i,u,T}} = \left( \mathbb{1}_{[i=0]} f_n(t, t) - \alpha_n^{i,u,T} + \frac{1}{2} \left\| \sigma_n^{i,u,T} \right\|^2 \right) dt - \sigma_n^{i,u,T} dW_t \]

\[ + \int_Z \left( e^{-\phi_n^{i,u,T}} - 1 + \phi_n^{i,u,T} \right) dv + \int_Z \left( e^{-\phi_n^{i,u,T}} - 1 \right) d\bar{P}. \] (1.13)

Then the dynamics of \( \mathbb{1}_{[i=0]} e_n^{i,u,T} \), in particular of \( P_0(t, T) = \mathbb{1}_{[i=0]} e_n^{i,u,T} \), follow easily with Itô’s lemma (see (1.14) below). Recalling (1.4), we notice that in order to derive the dynamics of \( P_n(t, T) \) and \( U_n(t, T) \) for \( n \geq 1 \), we need to compute iteratively the dynamics of integrals like \( \int_0^T P_{n-1} F_{n-1}(t, u) du \). This can be done via lemma 14. The difficulty, however, is to show that the inte-

grands as e.g. \( X := P_{n-1} F_{n-1} \) satisfy indeed the conditions of lemma 14.

To shortly illustrate this, suppose we have shown that theorem 13 holds for \( n - 1 \). This means in particular that \( \int_0^T \| v_{n-1}(s, T) \|^2 ds < \infty \) a.s. for all \( T \leq T^* \). Then \( d(P_{n-1} F_{n-1})(t, T) = F_{n-1} v_{n-1}(t, T) dW_t + \ldots \) and in order to apply lemma 14 to \( X = P_{n-1} F_{n-1} \) we must verify that indeed \( \int_0^T \int_S \| F_{n-1} v_{n-1}(s, u) \|^2 du ds < \infty \) a.s. Obviously, this does not simply follow from Hölder-type inequalities.

A key observation to solve these problems and to prove a number of technical lemmata which we will state below is given in

**Proposition 3.** The processes \( \int_0^T v_{n-1}^2(t, u) du \) are locally bounded for all \( n \).

The proofs of this proposition and all technical lemmata used below are postponed to the end of this section. Now we are ready for the
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Proof of theorem 13. From the dynamics of \( e_{n,T}^{i} = e_{n}^{i,T} \) in (1.13) and Itô’s lemma we deduce

\[
\frac{d(1_{\{L_{i}=n\}}e_{n,T}^{i})}{e_{n-T}^{i}} = 1_{\{L_{i}=n\}} \left\{ \left( T_{n}(t,t) - \alpha_{n,T}^{i} + \frac{1}{2} \left\| \sigma_{n,T}^{i} \right\|^{2} \right) dt - \sigma_{n,T}^{i} dW_{t} \right. \\
+ \int_{Z} \left( (1 - \phi_{n}^{i}) e^{-\phi_{n}^{i}T} - 1 + \phi_{n}^{i}T \right) d\nu \\
+ \int_{Z} \left( (1 - \phi_{n}^{i}) e^{-\phi_{n}^{i}T} - 1 \right) d\Pi \\
\left. + 1_{\{L_{i}=n+1\}} \left\{ \int_{Z} \phi_{n}^{i} e^{-\phi_{n}^{i}T} d\nu + \int_{Z} \phi_{n}^{i} e^{-\phi_{n}^{i}T} d\Pi \right\} \right) \\
(1.14)
\]

Since \( \nu(dz, dt) = dF_{t}(z)\lambda_{t}^{p} dt \) is absolutely continuous, the integrals above with respect to \( \nu \) contribute solely to the drift of \( 1_{\{L_{i}=n\}}e_{n,T}^{i} \). Also note that \( T_{n}(t,T) \) is nonnegative processes, and thus we have \( e_{n,T}^{i,u} \leq 1 \) and it holds \( \phi_{n}(z,t,T) \geq -T_{n}(t,T) \) \( P \times \nu(dz, dt) \)-a.e. for all \( T \leq T^{*} \), which implies

\[
e^{-\phi_{n}^{i,T}(z)} \leq e^{\int_{T}^{T_{n}(t,T)} d\nu} = (e_{n,T}^{1,u})^{-1} \quad (1.15)
\]

\( P \times \nu(dz, dt) \)-a.e. for all \( t \leq u \leq T \leq T^{*} \). This, together with \( P_{0}(t,T) = 1_{\{L_{i}=0\}}e_{n,T}^{i} \) proves the theorem for \( n = 0 \).

We may hence continue by induction assuming the theorem holds for \( n-1 \). We first treat \( U_{n} \). Note that

\[
d(P_{n-1}F_{n-1}) = y_{n-1} dt + x_{n-1} dW_{t} + \int_{Z} \phi_{n-1}^{p} \psi_{n-1}^{p} d\nu + \int_{Z} z_{n-1} d\Pi
\]

where we used the abbreviations

\[
x_{n} := P_{n-1}^{*} \sigma_{n}^{F} + F_{n}^{*} \psi_{n}, \\
y_{n} := P_{n-1}^{*} \alpha_{n}^{F} + F_{n}^{*} u_{n} + v_{n}^{*} \sigma_{n}^{F}, \\
z_{n} := P_{n-1}^{*} \phi_{n}^{F} + F_{n}^{*} \psi_{n}^{F} + \phi_{n}^{F} \psi_{n}^{p}
\]

for \( n \leq N-1 \). We will now show that lemma 14 applies to \( X := P_{n-1}F_{n-1} \). Note that \( \int_{Z} \phi_{n-1}^{p} \psi_{n-1}^{p} d\nu = \left( \int_{Z} \phi_{n-1}^{p} \psi_{n-1}^{p} dF_{t}(z) \lambda_{t}^{p} \right) dt \) makes part of the drift of \( P_{n-1}F_{n-1} \). Since \( 0 \leq P_{n-1} \leq 1 \) and the theorem holds for \( n-1 \), it is ensured that \( |\psi_{n-1}^{p}(z,t,T)| \leq 1 \) \( P \times \nu(dz, dt) \)-a.e. for all \( T \leq T^{*} \). First, this

\[\text{In particular} \int_{0}^{T} f_{z}(\sigma_{n}(z,t),T) \nu(dz, ds) < \infty \text{ a.s. follows from (1.15).}\]
implies that the additional drift part satisfies

\[
\int_0^T \int_{Z_s} \phi_{n-1}(z, s, u) dF_s(z) \cdot \lambda_s^P \, du \, ds \leq \int_0^T \int_Z \int_{Z_s} \phi_{n-1}(z, s, u) \left| v(dz, ds) \right| < \infty \quad \text{a.s.}
\]

(1.16)

by assumption 2 (iv). Second, it shows \(|z_{n-1}| \leq 2|\phi_{n-1}^F| + F_{n-1}^P \times v(dz, dt)\) a.e. for all \(T \leq T^*\). By localisation we may assume \(\int_0^T F_{n-1}^{-1}(t, u) \, du \leq C < \infty\) (cf. proposition 3), which implies

\[
\int_0^T \int_Z \int_Z F_{n-1}^{-1}(s, u) \, du \, v(dz, ds) \leq C \cdot v\left(Z \times [0, T^*]\right) < \infty \quad \text{a.s.}
\]

Together with the second inequality in (1.16), this shows that \(z_{n-1}\) satisfies

\[
\int_0^T \int_Z \int_Z z_{n-1}(s, u) \, du \, v(dz, ds) < \infty \quad \text{a.s.}
\]

We continue with a technical

**Lemma 16.** \(\int_0^T \int_Z \|x_n(s, u)\|^2 + |y_n(s, u)| \, du \, ds < \infty\) a.s. for all \(n \leq N - 1\).

Hence \(\alpha := y_{n-1} + \int_Z \phi_{n-1}^F \psi_{n-1}^P dF_t(z) \lambda_t^P, \sigma := x_{n-1} \) and \(\phi := z_{n-1}\) satisfy the conditions of lemma 14. Further it holds \(\int_0^T P_{n-1} F_{n-1}(0, T) \, dT \leq \int_0^T F_{n-1}(0, T) \, dT < \infty\) a.s. Thus lemma 14 applies to \(X = P_{n-1} F_{n-1}\):

\[
dU_n(t, T) = d\left(\int_0^T P_{n-1} F_{n-1}(t, u) \, du\right)
\]

\[
= y_{n-1}^F t \, dt + \int_0^T \phi_{n-1}^F \psi_{n-1}^P dF_t(z) \cdot \lambda_t^P \, dt + x_{n-1}^T \, dW_t
\]

\[
+ \int_Z z_{n-1}(z, t) \, d\mu(dz, dt) - 1_{[L_i = n-1]} F_{n-1}(t, t) \, dt
\]

\[
= a_n(t, T) \, dt + b_n(t, T) \, dW_t + c_n(z, t, T) \, d\mu(dz, dt).
\]

We continue with \(P_n\). Its characteristics restricted to the set \(\{L_i = n\}\) follow
already from (1.14). On \([L_t < n]\), we need to evaluate \(d(e_n^{t,u,T} P_{n-1} F_{n-1})\).

\[
\frac{d(e_n^{t,u,T} P_{n-1} F_{n-1})}{e_n^{t,u,T} F_{n-1}} = \left( P_{n-1}^{-} F_{n-1}^{-} \left( -\alpha_n^{t,u,T} + \frac{1}{2} \|\sigma_n^{t,u,T}\|^2 \right) \right) \cdots (1.17)
\]

\[
-\sigma_n^{t,u,T} x_{n-1} + y_{n-1} \right) dt + \left( -P_{n-1}^{-} F_{n-1}^{-} \sigma_n^{t,u,T} + x_{n-1} \right) dW_t
\]

\[
+ \int_Z \left( P_{n-1}^{-} F_{n-1}^{-} (e^{-\phi_n^{t,u,T}} - 1 + \phi_n^{t,u,T}) \right) \cdots + z_{n-1}(e^{-\phi_n^{t,u,T}} - 1) + \phi_n^{F} \psi_{n-1}^P d\nu
\]

\[
+ \int_Z \left( P_{n-1}^{-} F_{n-1}^{-} (e^{-\phi_n^{t,u,T}} - 1) + z_{n-1}(e^{-\phi_n^{t,u,T}}) \right) d\mu
\]

**Lemma 17.** \(\int_0^T \int_Z F_{n-1}^{-} (s-u) \|\sigma_n^{t,u,T}\|^2 du ds < \infty a.s.\) for all \(n = 1, \ldots, N\) and \(T \leq T^*\).

Since \(\| - P_{n-1}^{-} F_{n-1}^{-} \sigma_n^{t,u,T} + x_{n-1} \|^2 \leq 4 \left( F_{n-1}^{-} \|\sigma_n^{t,u,T}\|^2 + \|x_{n-1}\|^2 \right)\), lemmata 16 and 17 make sure that the volatility of \(e_n^{t,u,T} P_{n-1} F_{n-1}\) satisfies the assumptions of lemma 14. Next, we notice that by (1.15) the jump part of \(e_n^{t,u,T} P_{n-1} F_{n-1}\) is bounded by \(P_{n-1}^{-} F_{n-1}^{-} + |z_{n-1}|\). That \(z_{n-1}\) is sufficiently integrable was already shown above and

\[
\int_0^T \int_Z P_{n-1}^{-} F_{n-1}^{-} (s-u) du ds = \int_0^T U_n(s-\cdot, T^*) ds \leq T^*
\]

holds by construction, thus the jump part of \(e_n^{t,u,T} P_{n-1} F_{n-1}\) satisfies the conditions of lemma 14. The integral in (1.17) with respect to \(\nu(dz, dt)\) is a drift component of \(e_n^{t,u,T} P_{n-1} F_{n-1}\). By the latter remark and since \(|\psi_{n-1}^P| \leq 1,\) its only critical term is \(e_n^{t-u,T} P_{n-1}^{-} F_{n-1}^{-} \psi_n^{t,u,T}\).

**Lemma 18.** \(\int_0^T \int_Z F_{n-1}^{-} (s-u) \|\phi_n^{t,u,T}\| \nu dz, ds < \infty a.s.\) for all \(n \leq N\) and \(T \leq T^*\).

Thus the “\(d\nu\)”-part of the drift satisfies the conditions of lemma 14. We turn to the “\(dt\)”-part of the drift

**Lemma 19.** For every \(n = 1, \ldots, N\) and \(T \leq T^*\) it holds a.s.

\[
\int_0^T \int_0^T \left( \sigma_n^{t,u,T} \left( -\alpha_n^{t,u,T} + \frac{1}{2} \|\sigma_n^{t,u,T}\|^2 \right) + \sigma_n^{t,u,T} x_{n-1} \right) (s, u) du ds < \infty.
\]
Together with lemma 16 this shows that the drift of \( e_n^{t,u,T} P_{n-1} F_{n-1} \) satisfies the conditions of lemma 14. We further note that \( \int_0^T e_n^{t,u,T} P_{n-1} F_{n-1} du \leq \int_0^T F_{n-1} du < \infty \text{ a.s. for all } T \leq T^* \). Hence lemma 14 applies to \( X := e_n^{t,u,T} P_{n-1} F_{n-1} \). In an intermediate step, we notice that lemmata 17 and 19 also ensure that

\[
\begin{align*}
\frac{d}{dt} & \left( e_n^{t,u,T} P_{n-1} F_{n-1} \right) \\
& = \int_0^T e_n^{t,u,T} \left( \frac{d}{du} \left( P_{n-1} F_{n-1} \right) \right) du \left( \alpha_n \sigma_n + \sigma_n \partial \psi_n \right) dt
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} & \left( e_n^{t,u,T} P_{n-1} F_{n-1} \right) \\
& = \int_0^T e_n^{t,u,T} \left( \frac{d}{du} \left( P_{n-1} F_{n-1} \right) \right) du \left( \alpha_n \sigma_n + \sigma_n \partial \psi_n \right) dt
\end{align*}
\]
1.4. A HJM-Type Forward Model

Note that \( e_n^{i,T} = e_n^T, P_n^{-1}F_{n-1}^+ + z_{n-1} = (P_n^{-1} + \psi_P^{n-1})(F_n^{-1} + \phi_P^{n-1}) \) and \( \int_T^T e_n^{i,T} P_n^{-1}F_{n-1}^+ du = 1_{|t_i - \infty|} P_n^{-1} \). Together with (1.14) this completes the proof.

Proofs of Lemmata used in Proof of Theorem 13

Proof of proposition 3. Without restriction we may assume \( F_n(0, \cdot) = 0 \). We then first notice that

\[
F_n(t, u) \leq \int_0^t |\alpha_n^F(s, u)| ds + \int_0^t \sigma_n^F(s, u) dW_s \\
+ \int_0^t \int_Z |\phi_n^F(z, s, u)| v(dz, ds) + \int_0^t \int_Z |\phi_n^F(z, s, u)| \mu(dz, ds)
\]

and thus by Hölders inequality and taking left limits

\[
F_n^2(t-, u) \leq 4^2 \left( \left( \int_0^t |\alpha_n^F(s, u)|^2 ds \right)^2 + \left( \int_0^t \sigma_n^F(s, u) dW_s \right)^2 \right)
+ v(Z \times [0, t]) \cdot \int_0^t \int_Z |\phi_n^F(z, s, u)|^2 v(dz, ds)
+ \mu(Z \times [0, t]) \cdot \int_0^t \int_Z |\phi_n^F(z, s, u)|^2 \mu(dz, ds)
\]

for all \( t \leq T^* \). Note that \( v(Z \times [0, t]) \) and \( \mu(Z \times [0, t]) \) are locally bounded processes which do not depend on \( u \) and

\[
\int_0^t \int_Z |\phi_n^F(z, s, u)|^2 v(dz, ds) du = \int_0^T \int_0^T |\phi_n^F(z, s, u)|^2 du v(dz, ds)
\]

is continuous by construction, hence locally bounded. Similarly we estimate

\[
\int_0^T \int_0^T |\phi_n^F(z, s, u)|^2 \mu(dz, ds) du \leq \int_0^T \int_0^T |\phi_n^F(z, s, u)|^2 du \mu(dz, ds)
\]

The righthand side is the left limit of a nondecreasing càdlàg process, and is thus locally bounded. Therefore, to prove that \( \int_0^T F_n^2(t-, u) du \) is locally
bounded, it remains to show that the processes below are locally bounded:

\[ A_t := \int_0^T \left( \int_0^t |\alpha_n^F(s,u)| ds \right)^2 du \]

\[ B_t := \int_0^T \left( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \right)^2 du \geq \int_t^T \left( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \right)^2 du \]

First, we will show that \( A_t \) is continuous, and hence locally bounded. Note

\[ A_t = \int_0^T \left( \int_0^t |\alpha_n^F(s,u)| ds \right)^2 du \]

\[ \leq \int_0^T \sup_{t \leq T^*} \left( \int_0^t |\alpha_n^F(s,u)| ds \right)^2 du \]

\[ = \int_0^T \left( \int_0^u |\alpha_n^F(s,u)| ds \right)^2 du < \infty \text{ a.s.} \]

Thus \( A_t \) is continuous by dominated convergence. Concerning \( B_t \) we notice that

\[ \int_0^{T^*} \left( \sup_{t \leq T^*} \left( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \right)^2 \right) du \]

By localisation, we may assume \( \int_0^{T^*} \left( \sup_{t \leq T^*} \left( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \right)^2 \right) du < \infty \) a.s. Then, due to Doob’s inequality,

\[ \mathbb{E} \left[ \int_0^{T^*} \left( \sup_{t \leq T^*} \left( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \right)^2 \right) du \right] \leq 4 \int_0^{T^*} \mathbb{E} \left[ \int_0^{u} \| \sigma_n^F(s,u) \|^2 ds \right] du < \infty, \]

which shows that \( \int_0^{T^*} \sup_{t \leq T^*} \left( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \right)^2 du < \infty \text{ a.s.} \). Then the dominated convergence theorem applies and since \( \int_0^{t \wedge u} \sigma_n^F(s,u) dW_s \) is continuous for all \( u, B_t \) is continuous and thus locally bounded.

**Proof of lemma 16.** We show the lemma in three steps. By localisation, we can always assume without restriction \( \int_0^T F_n(t-u)+F_n^2(t-u)du \leq C < \infty \) for all \( t \leq T^* \) a.s. (cf. proposition 3).

**Step 1:** \( \int_0^{T^*} \int_s^T \|x_n(s,u)\|^2 du \) ds < \( \infty \) a.s. for \( n=0, \ldots, N-1 \).

---

17 The space of locally bounded processes is closed under multiplication.
Clearly, $\|x_n(s, u)\|^2 \leq 4(\|\sigma_n F(s, u)\|^2 + F_n^2(s, u) \|v_n(s, u)\|^2)$. It thus suffices to show $\int_0^{T^*} \int_s^{T^*} F_n^2(s, u) \|v_n(s, u)\|^2 \, du \, ds < \infty$. With Hölders inequality, we obtain $\|\sigma_n u\|^2 \leq u \int_s^u \|\sigma_n (s, v)\|^2 \, dv$, and setting $x_{-1}(s, u) := 0$, we notice that
\[
\|v_n(s, u)\|^2 \leq u \int_s^u \|\sigma_n (s, v)\|^2 \, dv + 4u \int_s^u \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \\
\leq 5u \int_s^u \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \\
(1.18)
\]
for $n = 0, \ldots, N$. Using (1.18) and the localisation argument above, we find
\[
\int_0^{T^*} \int_s^{T^*} F_n^2(s, u) \|v_n(s, u)\|^2 \, du \, ds \\
\leq 5 \int_0^{T^*} \int_s^{T^*} F_n^2(s, u) \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \, du \\
\leq 5T^* \int_0^{T^*} \int_s^{T^*} F_n^2(s, u) \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \\
\leq 5CT^* \int_0^{T^*} \int_s^{T^*} \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \, ds.
\]
For $n = 0$, the righthandside is a.s. finite, hence $\int_0^{T^*} \int_s^{T^*} \|x_0(s, u)\|^2 \, du \, ds < \infty$ a.s. Then Step 1 follows by induction in $n$. \hfill \square \text{Step 1}

Step 2: $\int_0^{T^*} \int_s^{T^*} \|v_n(s, u)\|^2 \, du \, ds < \infty$ a.s. for $n = 0, \ldots, N$.

We set $x_{-1} := 0$. With (1.18) and the last estimate in Step 1, we immediately see that
\[
\int_0^{T^*} \int_s^{T^*} \|v_n(s, u)\|^2 \, du \, ds \\
\leq 5 \int_0^{T^*} \int_s^{T^*} u \int_s^u \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \, du \\
\leq 5T^* \int_0^{T^*} \int_s^{T^*} \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \\
\leq 5T^{n+2} \int_0^{T^*} \int_s^{T^*} \|\sigma_n (s, v)\|^2 + \|x_{n-1}(s, v)\|^2 \, dv \, ds
\]
for $n = 0, \ldots, N$. Then Step 2 follows from Step 1. \hfill \square \text{Step 2}
Step 3: \( \int_0^T \int_s^T |y_n(s, u)| \, du \, ds < \infty \) a.s. for \( n = 0, \ldots, N - 1 \).

First of all \( |y_n(s, u)| \leq |\sigma_n^F (s, u)| + F_n(s, u)|u_n(s, u)| + |v_n^F \sigma_n^R (s, u)|. \) By
Step 2 and Hölder’s inequality, \( \int_0^T \int_s^T |v_n^F \sigma_n^R (s, u)| du \, ds < \infty \) a.s. It is thus
sufficient to show \( \int_0^T \int_s^T F_n(s, u)|u_n(s, u)| \, du \, ds < \infty \) a.s. for all \( n = 0, \ldots, N - 1 \). We continue with a result on the bounds of \( \psi_n^P \).

**Lemma 21.** \(-P_n^- \leq \psi_n^P \leq 1 - 1\{\mu_{n-} = 0\} \phi^L - P_n^- \ P \times \nu(dz, dt)\) a.e. for all \( n \leq N \) and \( T \leq T^* \).

Therefore we have indeed \( |\psi_n^P| \leq 1 \ P \times \nu(dz, dt)\) a.e.\(^{18}\) Using this,
(1.15) and \( \int_0^T |\phi_n^{r,v}(z)| \, dv \leq u \int_0^T \phi_n(z, s, v) \, dv, \) and with the convention \( y_{-1} := 0, \) we estimate

\[
|u_n(s, u)| \leq \mathcal{T}_n(s, s) + \int_s^u \left( |\alpha_n| + \frac{u}{2} |\sigma_n|^2 + |\sigma_n^R v_n| + |y_{n-1}| \right)(s, v) \, dv
\]

\[
+ \int_{s/2}^u \left( 2 + \left| (1 + u) |\phi_n| + |\phi_n^{r-1}| + F_{n-1}^{-} \right| (z, s, v) \right) \, dv \frac{dy}{ds}
\]

for \( n = 0, \ldots, N. \) By localisation we can assume \( \int_T^T F_n(s, u) du \leq C. \) Then

\[
\int_0^T \int_s^T F_n^- |u_n| (s, u) \, du \, ds
\]

\[
\leq \mathcal{T}_n(s, s) \int_s^u \int_0^T F_n^{-} (s, u) \, du \, dv \, ds + 2 \mathcal{T}_n(s, u) \int_s^u \int_0^T F_n^-(s, u) \, du \, dv \, ds
\]

\[
+ \int_0^T \int_s^T \left( |\alpha_n| + \frac{T^*}{2} |\sigma_n|^2 + |\sigma_n^R v_n| + |y_{n-1}| \right) \, dv \, du \, ds
\]

\[
+ \int_0^T \int_s^T \left( 1 + T^* \right) |\phi_n| + |\phi_n^{r-1}| + F_{n-1}^- \right| \, dv \, du \, dv
\]

\[
\leq C \mathcal{T}_n(s, s) ds + (2C + C^2) \mathcal{V}(Z \times [0, T^*))
\]

\[
+C \int_s^u \left( |\alpha_n| + \frac{T^*}{2} |\sigma_n|^2 + |\sigma_n^R v_n| + |y_{n-1}| \right) \, dv \, ds
\]

\[
+C(1 + T^*) \int_0^T \int_s^T \left( |\phi_n| + |\phi_n^{r-1}| \right) \, dv \, du \, dv\).\]

\(^{18}\)Note that without the above lemma, it would be inaccurate to presume that \( |\psi_n^P| \leq 1, \) e.g. be-
cause \( P_n \) is \( [0, 1] \)-valued. This reasoning is only correct, once we have proven that theorem 13
holds for \( n, \) which is what we are doing.
\[ \int_0^T \int_s^T |\sigma_n^{s,u,T}| (s, v) dv ds < \infty \ a.s. \] is again due to Hölder’s inequality and Step 2. Thus for \( n = 0 \) the righthandside above is a.s. finite and then Step 3 follows by induction in \( n \). □

Now, the proof of lemma 16 is direct from Step 1 and Step 3.

**Proof of Lemma 17.** By Hölder’s inequality \( \|\sigma_n^{s,u,T}\| ^2 \leq T \int_u^T \|\sigma_n(s, v)\| ^2 dv \) for all \( T \leq T^* \) and by localisation we may assume \( \int_s^T F_{n-1}^2 (s, v) du \leq C < \infty \) for all \( s \leq T \) a.s. Then

\[
\int_0^T \int_s^T F_{n-1}^2 (s, u) \|\sigma_n^{s,u,T}\|^2 dv ds
\]

\[
\leq T \int_0^T \int_s^T \|\sigma_n(s, v)\|^2 dv du ds
\]

\[
\leq T^* \int_0^T \int_s^T \|\sigma_n(s, v)\|^2 dv ds
\]

\[
\leq CT^* \int_0^T \|\sigma_n(s, v)\|^2 dv ds < \infty \ a.s. \quad \square
\]

**Proof of Lemma 18.** We may assume \( \int_s^T F_{n-1}^2 (t, u) du \leq C < \infty \) a.s. Then

\[
\int_0^T \int_s^T F_{n-1}^2 (s, u) |\phi_n^{s,u,T}| dv v(dz, ds)
\]

\[
\leq T \int_0^T \int_s^T |\phi_n(z, s, v)| dv v(dz, ds)
\]

\[
\leq C T^* \int_0^T \int_s^T |\phi_n(z, s, v)| dv v(dz, ds) < \infty \ a.s. \quad \square
\]

**Proof of Lemma 19.** By localisation, we may assume \( \int_s^T F_{n-1}^2 (t, u) du \leq C < \infty \). We will treat every summand of the integrand separately. First

\[
\int_0^T \int_s^T |\sigma_n^{s,u,T}| dv ds
\]

\[
\leq T \int_0^T \int_s^T |\sigma_n(s, v)| dv du ds
\]

\[
\leq C T^* \int_0^T \int_s^T |\sigma_n(s, v)| dv ds < \infty \ a.s.
\]
Second, since \( \|\sigma_{n}^{u,T}\|^{2} \leq T \int_{u}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \), we have

\[
\int_{0}^{T} \int_{s}^{T} F_{n-1}^{-} \left[ \sigma_{n}^{u,T} \right]^{2} \, du \, ds \leq T \int_{0}^{T} \int_{s}^{T} F_{n-1}^{-}(s, u) \int_{u}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \, du \, ds \\
\leq CT^{*} \int_{0}^{T} \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \, ds < \infty
\]

Third, by the definition of \( x_{n-1} \), we have \( \|x_{n-1}\| \leq F_{n-1}^{-} \|v_{n-1}\| + \|\sigma_{n-1}^{F}\| \). With (1.18) we find

\[
\int_{0}^{T} \int_{s}^{T} F_{n-1}^{-} \left[ \sigma_{n}^{x,u,T} \right] \left[ \|v_{n-1}\| \right] \, du \, ds \\
\leq \int_{0}^{T} \int_{s}^{T} F_{n-1}^{-} \left( T \int_{u}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \, \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \right)^{\frac{1}{2}} \, du \, ds \\
\leq \sqrt{ST} \int_{0}^{T} \int_{s}^{T} F_{n-1}^{-} \left( \|\sigma_{n}(s, v)\|^{2} \, \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \right)^{\frac{1}{2}} \, du \, ds \\
\leq C \sqrt{ST} \int_{0}^{T} \int_{s}^{T} \left( \|\sigma_{n}(s, v)\|^{2} + \|x_{n-1}\|^{2} \right) (s, v) \, dv \, ds \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \, ds \right)^{\frac{1}{2}} \, ds \\
\leq C \sqrt{ST} \left( \int_{0}^{T} \int_{s}^{T} \left( \|\sigma_{n}(s, v)\|^{2} + \|x_{n-1}\|^{2} \right) (s, v) \, dv \, ds \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \, ds \right)^{\frac{1}{2}} < \infty
\]

a.s. On the other hand, remember \( \|\sigma_{n}^{x,u,T}\|^{2} \leq T \int_{u}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \), we have

\[
\int_{0}^{T} \int_{s}^{T} \left[ \sigma_{n}^{F,-1} \right] \left[ \sigma_{n}^{x,u,T} \right] \, du \, ds \\
\leq \int_{0}^{T} \int_{s}^{T} \left[ \sigma_{n}^{F,-1}(s, u) \right] \left( T \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \right)^{\frac{1}{2}} \, du \, ds \\
\leq \left( \int_{0}^{T} \int_{s}^{T} \left[ \sigma_{n}^{F,-1}(s, u) \right] \, du \right) \left( \int_{0}^{T} \int_{s}^{T} \|\sigma_{n}(s, v)\|^{2} \, dv \, ds \right)^{\frac{1}{2}} \\
\leq T^{*} \left( \int_{0}^{T} \int_{s}^{T} \left[ \sigma_{n}^{F,-1} \right]^{2} \, du \, ds \int_{s}^{T} \|\sigma_{n}\|^{2} \, dv \, ds \right)^{\frac{1}{2}} < \infty \quad \text{a.s.} \quad \Box
\]

Proof of lemma 20. Recall \( e_{n}^{t,x,T} = e_{n}^{t,u,v} e_{n}^{t,v,T} \) for all \( t \leq u \leq v \leq T \leq T^{*} \).
Then
\[
\int_t^T e_n^{t-u,T} p_{n-1}^- F_n^{-1}(t, u) \sigma_n^{t,u,T} \, du \\
= \int_t^T e_n^{t-u,T} p_{n-1}^- F_n^{-1}(t, u) \int_u^T \sigma_n(t, v) \, dv \, du \\
= \int_t^T \left( \int_u^T e_n^{t-u,v} p_{n-1}^- F_n^{-1}(t, u) \, du \right) e_n^{t,v,T} \sigma_n(t, v) \, dv \\
= 1_{\{ L_n < n \}} \int_t^T e_n^{t-u,T} p_n^- \sigma_n(t, v) \, dv.
\]

The integral involving \( \phi_n^{t,u,T} \) transforms analogous. For the second result, we will carry out three Fubini-type transformations separately. Admissibility is guaranteed by lemma 19.

\[
\int_t^T e_n^{t-u,T} p_{n-1}^- F_n^{-1}(t, u) \sigma_n^{t,u,T} \, du \\
= \int_t^T \int_u^T e_n^{t-u,T} p_{n-1}^- F_n^{-1}(t, u) \sigma_n(t, v) \, dv \, du \\
= \int_t^T \int_u^T e_n^{t-u,v} p_{n-1}^- F_n^{-1}(t, u) \, du e_n^{t,v,T} \sigma_n(t, v) \, dv \\
= 1_{\{ L_n < n \}} \int_t^T e_n^{t-v,T} p_n^- \sigma_n(t, v) \, dv.
\]

It is easy to verify that \( \frac{1}{2} \| \sigma_n^{t,u,T} \|^2 = \int_u^T \left( \int_v^T \sigma_n(t, y) \, dy \right) \sigma_n(t, v) \, dv \), and thus

\[
\int_t^T e_n^{t-u,T} p_{n-1}^- F_n^{-1} \frac{1}{2} \| \sigma_n^{t,u,T} \|^2 \, du \\
= \int_t^T \int_u^T e_n^{t-u,T} p_{n-1}^- F_n^{-1}(t, u) \left( \int_v^T \sigma_n(t, y) \, dy \right) \sigma_n(t, v) \, dv \, du \\
= \int_t^T \int_u^T e_n^{t-u,v} p_{n-1}^- F_n^{-1}(t, u) \, du e_n^{t,v,T} \left( \int_v^T \sigma_n(t, y) \, dy \right) \sigma_n(t, v) \, dv \\
= 1_{\{ L_n < n \}} \int_t^T e_n^{t-v,T} p_n^- \sigma_n(t, y) \left( \int_v^T e_n^{t-v,y} p_n^- \sigma_n(t, v) \, dv \right) \, dy.
\]
\[ n \equiv \mathbf{1}_{\{L_i < n\}} \int_T \mathbf{e}_n^{t,v \cdot T} \sigma_n(t, v) \left( \int_T \mathbf{e}_n^{t,u \cdot T} \mathbf{p}_n^+ \sigma_n(t, u) \, du \right) \, dv \]

Third, we check that
\[
\begin{aligned}
\int_T \mathbf{e}_n^{t,u \cdot T} \mathbf{c}_n^{t,v \cdot T} \, dx &= \int_T \int_T \mathbf{e}_n^{t,u \cdot T} \sigma_n(t, v) \, x_{n-1}(t, u) \, dv \, du \\
&= \int_T \int_T \mathbf{e}_n^{t,u \cdot T} \sigma_n(t, v) \, x_{n-1}(t, u) \, dv \, du \\
&= \int_T \mathbf{e}_n^{t,v \cdot T} \sigma_n(t, v) \int_T \mathbf{e}_n^{t,u \cdot T} x_{n-1}(t, u) \, du \, dv.
\end{aligned}
\]

Combining these three results and the definition of \( v_n \) yields the claim.

**Proof of Lemma 21.** For \( n = 0 \) the claim is trivially satisfied. If the lower bound rule holds for \( n \), i.e. \( \psi_n^P \geq -P_n^+ \mathbf{P} \times \mathbf{v} \cdot \mathbf{a.e.} \), then \( (P_n^- + \psi_n^P)(F_n^- + \phi_n^F) \geq 0 \mathbf{P} \times \mathbf{v} \cdot \mathbf{a.e.} \) because \( F_n^- \) is nonnegative, thus the lower bound holds for \( n + 1 \).

For the upper bound we need an explicit formula for \( \psi_n^P \), which is given in the following lemma.

**Lemma 22.** Define \( \hat{F}_n := F_n^- + \phi_n^F \), \( \mathbf{e}_n^{t,T} := \mathbf{e}_n^{t,-T} e^{-\phi_n^T} \) and \( \mathbf{e}_n^{t,u \cdot T} := \mathbf{e}_n^{t,u \cdot T} e^{-\phi_n^{u \cdot T}} \). Then \( \psi_n^P = -P_n^- + (1 - \phi_L^T) \mathbf{e}_n^{t,T} \) and for \( n \geq 2 \) it holds
\[
\psi_n^P = \int_T \mathbf{e}_n^{t,v \cdot T} \sigma_n(t, v) \left( \int_T \mathbf{e}_n^{t,u \cdot T} \mathbf{p}_n^+ \sigma_n(t, u) \, du \right) \, dv
\]

\[
\begin{aligned}
&= \int_T \mathbf{e}_n^{t,v \cdot T} \sigma_n(t, v) \int_T \mathbf{e}_n^{t,u \cdot T} x_{n-1}(t, u) \, du \, dv
\end{aligned}
\]
With lemma 22, the upper bound is easily checked for \( n = 0, 1 \). Further, define \( \hat{f}_n := f_n + \phi_n \). Then, for each \( k = 0, \ldots, N - 1 \), we have
\[
\int_0^T \phi_k(t, v)du \leq \int_0^T \phi_k(t, v)(\hat{f}_k + \hat{F}_k)(t, v)dv = 1 - \epsilon_k^T \leq 1 \text{ for all } t \leq T,
\]
and similarly \( \int_0^T \phi_k(t, v)dv = 1 - \epsilon_k^T \leq 1 \) for all \( t \leq u \leq T \). Using these estimates iteratively, we observe that every multi-integral in (1.19) is bounded from above by one. Then the claim follows.

**Proof of lemma 22.** The formulae for \( \psi_0^P \) and \( \psi_1^P \) follow directly from the definition. Using \( \psi_1^P \) as given in the lemma, we find
\[
\psi_1^P = -P_2^- + 1_{\{L_{n-2}\}}(1 - \phi^L)P_2^- e^{\phi_n^T} + 1_{\{L_{n-1}\}}\phi^L e_n^T e_n^T \\
+ \int_t^T e_n^T \left(1_{\{L_{n-1}\}}(1 - \phi^L)e_1^T + 1_{\{L_{n-2}\}}(\phi^L e_n^T + \cdots)
+ (1 - \phi^L) \int_0^T \phi_0^n \hat{F}_1(t, u) e_1^T du \right)F_1 \left( t, u \right)du
\]
\[
= -P_2^- + 1_{\{L_{n-2}\}}(1 - \phi^L)e_2^T \\
+ 1_{\{L_{n-1}\}} \left\{ \phi^L e_2^T + (1 - \phi^L) \int t^T e_1^T \hat{F}_1(t, u) e_1^T du \right\} \\
+ 1_{\{L_{n-2}\}} \phi^L \int t^T e_1^T \hat{F}_1(t, u) e_1^T du + (1 - \phi^L) 1_{\{L_{n-2}\}} \int t^T e_1^T \hat{F}_1(t, u) e_1^T du du_0.
\]
i.e. the lemma holds for \( n = 2 \). We continue by induction assuming the lemma holds for some \( n \geq 2 \). Then, on the set \( \{L_{n-1} \geq n - 1 \} \), we have
\[
\psi_{n+1}^P = -P_{n+1}^- + 1_{\{L_{n-1} \geq n - 1 \}}(1 - \phi^L)P_{n+1}^- e^{\phi_n^{n+1}} + 1_{\{L_{n-1} = n\}} \phi^L e_n^{n+1} e^{\phi_n^{n+1}} \\
+ 1_{\{L_{n-2} \geq n-1 \}} \int t^T e_n^{n+1} T e^{\phi_n^{n+1}} (P_{n+1}^- + \phi_n^P) (t, u)du \]
\[
= -P_{n+1}^- + 1_{\{L_{n-1} \geq n - 1 \}}(1 - \phi^L)e_n^{n+1} + 1_{\{L_{n-1} = n\}} \phi^L e_n^{n+1} \\
+ \int t^T e_n^{n+1} \hat{F}_n(t, u) e_n^{n+1} (1_{\{L_{n-1} = n\}} (1 - \phi^L) e_n^{n+1})du \]
\[
+ \int t^T e_n^{n+1} \hat{F}_n(t, u) 1_{\{L_{n-1} = n-1\}} \left( \phi^L e_n^{n-1} + \cdots \right) \\
+ (1 - \phi^L) \int_0^T e_n^{n+1} \hat{F}_{n-1}(t, u) e_n^{n-1} du_{n-1} \right)du_n
\]
\[
= -P_{n+1}^- + 1_{\{L_{n-1} \geq n - 1 \}}(1 - \phi^L)e_n^{n+1}
\]
It is desirable to have a rule telling how to directly set up the model such as already sufficient in the HJM-type model. Is necessary in order for \( \mathbb{P} \) and on the set \( \{ L_t < n - 1 \} \) it holds

\[
\psi^P_{n+1} = 1_{\{ L_t < n - 1 \}} \int_T \cdots \int_T \cdots 1_{\{ L_t < n - 1 \}} \int_T \cdots \int_T \cdots \left( \left( 1 - \phi^L_i \sum_{j=0}^{n-2} 1_{\{ L_t = -j \}} \int_T \cdots \int_T \cdots \left( \left( 1 - \phi^L_i \sum_{j=0}^{n-2} 1_{\{ L_t = -j \}} \int_T \cdots \int_T \cdots \right) \right) \right) \right)
\]

1.4.2 EMM Relationship

It is desirable to have a rule telling how to directly set up the model such that \( \mathbb{P} \) itself is already an EMM. Such a rule is partly given by theorem 8 and its corollary. Given the dynamics in theorem 13, we can only restrict \( \alpha_i^a \) such that \( \beta_i\mathbb{P}_x(t, T) \in \mathcal{M}^{lo}(\mathbb{P}) \) for all \( T, n \) and \( x = f, F \), which is necessary in order for \( \mathbb{P} \) to be an EMM. In fact, it will turn out that this is already sufficient in the HJM-type model.
1.4. A HJM-Type Forward Model

As concerns the converse direction of theorem 8, we note that in the HJM-type forward model, \( \lim_{T \to T_0} E \left[ |x_n(t, T) - x_n(t, T_0)| \right] = 0 \) for all \( t \leq T_0 \leq T^* \), \( n = 0, \ldots, N \) and \( x = f \), \( F \) is in general not satisfied (nor are \( f_n(t, T) \) and \( F_n(t, T) \) integrable). We will first prove a general result and then come back to this point in lemma 26 and corollary 27.

**Theorem 23** (EMM HJM model). \( \mathbf{P} \) is an EMM if and only if \( \mathbf{P} \times \ell(dt, dT) \)-a.e.

\[
P_n^{-\sigma_n} + v_n^* \sigma_n + \int_Z \psi_n^P \phi_n^P \frac{dv}{dt} = 0 \quad \text{and} \quad P_n^{-\sigma_n} + c_n^* \sigma_n + \int_Z \psi_n^P \phi_n^P \frac{dv}{dt} = 0.
\]

for each \( n = 0, \ldots, N \), and \( f_L, (t, \cdot) = r_t \) and \( F_L, (t, \cdot) = \lambda_t^L \) are càdlàg versions of the short rate and the loss intensity, respectively.

**Proof of Theorem 23.** It is clear that \( \mathbf{P} \) is an EMM if and only if it holds for every \( T \leq T^* \) (note that \( L_{t-} = L, \mathbf{P} \times \ell(dt) \)-a.e.)

\[
\begin{align*}
\alpha_n & = P_n^{-r_t}, \\
\sigma_n & = U_n^{-r_t} - 1 \text{[for] } n = 0, \ldots, N, \\
\mathbf{P} \times \ell(dt) \text{-a.e. Restricted to } & \{L_{t-} = n\}, \quad P_n^- \text{ is strictly positive for all } T \leq T^*, \\
\text{and we have} & \|\sigma_n^{*T}\| = 2 \int_0^T \sigma_n \sigma_n( u, T) du = -2 \int_0^T \frac{1}{P_n} \psi_n^P \sigma_n( t, u) du \\
\psi_n^P (t, T) & = P_n^- \left( 1 - \phi_t \right) e^{-\phi_t T} - 1 \\
& = P_n^- \left( 1 - \phi_t \right) \left( 1 - \int_0^T e^{-\phi_t u} \phi_n( t, u) du \right) - 1 \\
& = -P_n^- \left( \phi_t + \int_0^T \frac{1}{P_n} (\psi_n^P + P_n^-) \phi_n( t, u) du \right).
\end{align*}
\]

Putting these steps together and using the definition \( \lambda_t^L = \int_Z \phi_t \frac{dv}{dt} \), we find

\[
\begin{align*}
1_{\{L_{t-} = n\}} u_n( t, T) &= 1_{\{L_{t-} = n\}} \int_0^T \frac{1}{P_n} \left( \psi_n^P \phi_n + \frac{dv}{dt} \right) du \\
&= 1_{\{L_{t-} = n\}} \int_0^T \frac{1}{P_n} \left( \psi_n^P \phi_n + \frac{dv}{dt} \right) du.
\end{align*}
\]

With \( n = 0 \), this shows that \( u_0 = P_0^{-r_t} \) for all \( T \leq T^* \) if and only if \( 1_{\{L_{t-} = 0\}} (\psi_0^P \phi_0 - \lambda_t) = 0 \mathbf{P} \times \ell(dt) \)-a.e. and

\[
P_0^{-\sigma_0} + c_0^* \sigma_0 + \int_Z \psi_0^P \phi_0 \frac{dv}{dt} = 0
\]
We may hence continue by induction assuming (1.21) holds for some \( n \geq 0 \). Then we have

\[
P_{n+1} = U_t^- \, 1_{\{L_t^- < n+1\}} + \int_t^T (P_n^- \sigma_n^F + \psi_n^P \phi_n^F \, du) \quad (1.22)
\]

\( P \times \ell(dt, dT) \)-a.e. We suppose the latter two conditions hold. In an intermediate step we notice that on the set \( \{L_t^- < n+1\} \)

\[
\psi_{n+1}^P = 1_{\{L_t^- = n\}} \phi^L e^{\int_{t+1}^T -e^{-\phi^T_n}} - P_{n+1}^- \\
+ \int_t^T e^{t-u} \, (P_n^- + \psi_n^P \phi_n^F) (F_n^- + \psi_n^F) \, du
\]

\[
= 1_{\{L_t^- = n\}} \phi^L e^{\int_{n+1}^T -e^{-\phi_{n+1}^T}} \left( 1 - \int_t^T e^{-\phi_{n+1}^T} \phi_{n+1}(t, v) \, dv \right) - P_{n+1}^- \\
+ \int_t^T e^{t-u} \, (P_n^- + \psi_n^P \phi_n^F) (F_n^- + \psi_n^F) \, du
\]

\[
= 1_{\{L_t^- = n\}} \phi^L e^{\int_{n+1}^T -e^{-\phi_{n+1}^T}} + \int_t^T e^{t-u} \, (P_n^- + \psi_n^P \phi_n^F) (F_n^- + \psi_n^F) \, du - P_{n+1}^- \\
- \int_t^T e^{t-u} \, 1_{\{L_t^- = n\}} \phi^L e^{\int_{n+1}^T -e^{-\phi_{n+1}^T}} \phi_{n+1}(t, v) \, dv
\]

\footnote{In the first condition, we used that \( L_t^- = L_t \, P \times \ell(dt) \)-a.e.}
If we insert this in the formula for \( u_{n+1}(t, T) \) on \( \{ L - < n + 1 \} \) and use (1.22) and \( u_n = P_n^{-1} r_t \) (which is a consequence of (1.21)), then we get

\[
I_{\{ L - < n + 1 \}} u_{n+1}(t, T) = I_{\{ L - < n + 1 \}} \left( P_{n+1}^{-1} r_t \cdots \right.
\]

\[
\cdots - \int_t^T e^\nu_uT \left( \left( P_{n+1}^{-1} a_{n+1} + \sigma_{n+1} \right) P_{n+1}^{-1} \phi_{n+1} + \int_Z \psi^p_{n+1} \phi_{n+1} du \right) \bigg) dt.
\]

Together with (1.20), this shows that (under (1.21)) \( u_{n+1} = P_n^{-1} r_t \) \( P \times \ell (dt) \)-a.e. if and only if \( I_{\{ L - < n + 1 \}} \left( \int_{L - } (t, t) - r_t - \lambda_n \right) = 0 \) \( P \times \ell (dt) \)-a.e. and

\[
P_{n+1}^{-1} a_{n+1} + \sigma_{n+1} \psi_{n+1} + \int_Z \psi^p_{n+1} \phi_{n+1} \frac{dv}{dt} = 0
\]

\( P \times \ell (dt, dT) \)-a.e. Repeating this step until \( n + 1 = N \), this proves that \( P \) is an EMM if and only if (1.21) holds for \( n = N \). Then the claim follows, since \( \mathcal{F}_N = f_N (F_N = 0) \). \( \square \)

In order to set up the model directly under an EMM \( Q \), theorem 23 suggest to define \( r_t := f_{L_t} (t, t) \) and \( \lambda^Q_{L_t} := f_{L_t} (t, t) \) and

\[
\alpha_n^x := \alpha_n^x Q := -\nu^x_n a_n^x + \frac{1}{P_n^-} \int_Z \psi^p_n \phi^x_n d\nu
\]

for each \( n = 0, \ldots, N \) and \( x = f, F \). However, in general, for \( \sigma_n^x \) and \( \phi_n^x \) satisfying (iii) and (iv) of assumption 2, \( \alpha_n^x Q \) does not automatically satisfy (ii) of assumption 2. Nevertheless, if the forward rates \( f_n, F_n \) have no jumps, we can exclude these cases by imposing a growth condition on the volatilities \( \sigma_n^x \). Interestingly, in this case we need not to further restrict \( \sigma_n^f \) beyond assumption (iii) of 2.
Lemma 24. Assume \(\phi_0^x = 0 \times v\cdot a.e.\) for all \(T, n = 0, \ldots, N\) and \(x = f, F.\) If there exists a predictable process \(\xi\) with \(\int_0^T \xi^2\,dt < \infty\) a.s. and such that 
\[
\|\sigma_n^F(t, T)\| \leq \xi\, F_n^-(t, T) \quad \text{for all } t \leq T \leq T^* \quad \text{and all } n = 0, \ldots, N - 1,
\]
then 
\[
\int_0^{T^*} \left( \int_0^u |\alpha_{nQ}^x(s, u)|\,ds \right)^2 \,du < \infty \quad \text{a.s.}
\]
for each \(n = 0, \ldots, N\) and \(x = f, F.\)

Proof of lemma 24. We start with a preliminary result.

Lemma 25. If there exists a process \(\xi\) with 
\[
\|v_n(t, T)\| \leq P_n^-(t, T) \left( \sum_{j=0}^n \left( T \int_t^T \|\sigma_j(t, v)\|^2 \,dv \right)^{1/2} + (n-1)\xi_t \right)
\]
for all \(T \leq T^*\) and \(n = 0, \ldots, N.\)

By lemma 25 and using that \(\phi_0^x = 0 \times v\cdot a.e.,\) we can first estimate \(\alpha_{nQ}^x\) as follows.

\[
|\alpha_{nQ}^x(s, u)| = \frac{|v_n(s, u)|\sigma_n^x(s, u)}{P_n(s, u)} \leq \frac{|v_n(s, u)| \|\sigma_n^x(s, u)\|}{P_n(s, u)} \leq \left( \sum_{j=0}^n \left( u \int_s^u \|\sigma_j(t, v)\|^2 \,dv \right)^{1/2} + (n-1)\xi_t \right) \|\sigma_n^x(s, u)\|
\]

Using this estimation and Hölder’s inequality, we observe

\[
\int_0^{T^*} \left( \int_0^u |\alpha_{nQ}^x(s, u)|\,ds \right)^2 \,du \leq 4 \int_0^{T^*} \left( \int_0^u \sum_{j=0}^n \left( u \int_s^u \|\sigma_j(s, v)\|^2 \,dv \right)^{1/2} \|\sigma_n^x(s, u)\| \,ds \right)^2 \,du
\]

\[
+ 4\int_0^{T^*} \left( \int_0^u (n-1)\xi_t \|\sigma_n^x(s, u)\| \,ds \right)^2 \,du
\]

\[
\leq 4n^2 \sum_{j=0}^n \int_0^{T^*} \left( \int_0^u \|\sigma_j(s, v)\|^2 \,dv \,ds \right) \left( \int_0^u \|\sigma_n^x(s, u)\|^2 \,ds \right) \,du
\]
We continue by induction assuming the lemma holds for \( n \in \mathbb{N} \).

**Proof of Lemma 25.** The lemma holds for \( n = 0 \) because for all \( t \leq T \leq T^* \) we have

\[
\|v_0(t, T)\| = P_0^{-}(t, T) \|\sigma_{0,T}^n\| \leq P_0(t_-, T) \left( t \int_0^T \|\sigma_0(t, v)\|^2 dv \right)^{1/2}.
\]

We continue by induction assuming the lemma holds for \( n \). Then, by the definition of \( v_{n+1} \) and Lemma 20,

\[
\|v_{n+1}(t, T)\| \\
\leq 1_{\{L_{n+1} = n+1\}} P_{n+1}^{-}(t, T) \|\sigma_{n+1}^n\| \\
+ 1_{\{L_n = n\}} \int_0^T e_{n+1}^{t,u,T} \left( P_n^{-} \left( F_n^{-} \left( \|\sigma_{n+1}^n\|^2 + \|\sigma_{n+1}^n(u)\|^2 \right) \right) + F_n^{-} \|v_n\| \right) du \\
\leq 1_{\{L_{n+1} = n+1\}} P_{n+1}^{-}(t, T) \left( t \int_0^T \|\sigma_{n+1}(t, v)\|^2 dv \right)^{1/2} \\
+ 1_{\{L_n = n\}} \int_0^T e_{n+1}^{t,u,T} P_n^{-} \left( \left( t \int_0^T \|\sigma_{n+1}(t, v)\|^2 dv \right) + \xi_t \right) du \\
+ 1_{\{L_n = n\}} \int_0^T e_{n+1}^{t,u,T} P_n^{-} \left( \sum_{j=0}^n \left( u \int_0^T \|\sigma_j(t, v)\|^2 dv \right) + (n-1)\xi_t \right) du \\
\leq 1_{\{L_{n+1} = n+1\}} P_{n+1}^{-}(t, T) \left( t \int_0^T \|\sigma_{n+1}(t, v)\|^2 dv \right)^{1/2} \\
+ 1_{\{L_n = n\}} \left( \sum_{j=0}^{n+1} \left( \int_0^T \|\sigma_j(t, v)\|^2 dv \right)^{1/2} + n\xi_t \right) \int_0^T e_{n+1}^{t,u,T} P_n^{-} F_n^{-} du \\
\leq P_{n+1}^{-}(t, T) \left( \sum_{j=0}^{n+1} \left( \int_0^T \|\sigma_j(t, v)\|^2 dv \right)^{1/2} + n\xi_t \right). \quad \square
The Link to Theorem 8

Finally, we come back to the question when \( f_n \) and \( F_n \) as considered in assumption 2 satisfy the conditions of theorem 8.

**Lemma 26.** Let \( f_n(0, T) \) and \( F_n(0, T) \) be in \( C^{0,r}(T) \) for all \( n \) and \( \sigma_n^x, \alpha_n^x \) and \( \phi_n^x \) be in \( C^{0,r}(T) \) for all \( t \in [0, T^*] \) a.s. for all \( n \) and \( x = f, F \). If there exist predictable processes \( \alpha_t \) and \( \sigma_t \) and a predictable function \( \phi_t(z) \) with \( E \left[ \int_0^T |\alpha_t| + \sigma_t^2 dt + \int_0^T \phi_t(z) v(dz, dt) \right] < \infty \) such that \( |\alpha_n^x| \leq \alpha_t \), \( \|\sigma_n^x\| \leq \sigma_t \) P \( \times \) \( P(dz, dt) \)-a.e. and \( |\phi_n^x| \leq \phi_t P \times v(dz, dt) \)-a.e. for all \( n \), \( T \geq t \) and \( x = f, F \) and \( f_n(t, T) \) and \( F_n(t, T) \) are integrable and

\[
\lim_{T \uparrow T_0} E \left[ |x_n(t, T) - x_n(t, T_0)| \right] = 0
\]

for all \( t \leq T_0 \leq T^*, \ n = 0, \ldots, N \) and \( x = f, F \).

Under the conditions of lemma 26, the EMM drift restrictions obtained in theorem 23 follow straightforward from applying Ito’s lemma to the products \( \beta_t \beta_n f_n(t, T) \) and \( \beta_t \beta_n F_n(t, T) \), which must be martingales under every EMM by theorem 8.

**Proof of lemma 26.** From the Itô isometry it is clear that \( f_n(t, T) \) and \( F_n(t, T) \) are integrable. Further

\[
E \left[ |x_n(t, T) - x_n(t, T_0)| \right] \leq E \left[ \int_0^t |\alpha_n^x(s, T) - \alpha_n^x(s, T_0)| \, ds \right] + \left[ \int_0^t \sigma_n^x(s, T) - \sigma_n^x(s, T_0) dW_t \right] + 2 \left[ \int_0^t |\phi_n^x(s, T) - \phi_n^x(s, T_0)| v(dz, ds) \right] \leq E \left[ \int_0^t |\alpha_n^x(s, T) - \alpha_n^x(s, T_0)| \, ds \right] + \left[ \int_0^t \|\sigma_n^x(s, T) - \sigma_n^x(s, T_0)\|^2 \, ds \right]^{1/2} + 2 \left[ \int_0^t |\phi_n^x(s, T) - \phi_n^x(s, T_0)| v(dz, ds) \right] .
\]

Then the claim follows with dominated convergence letting \( T \downarrow T_0 \). \( \square \)

Finally, as a concatenation of lemma 26 and theorem 8 we obtain the following corollary which acts as a good summary of the EMM relationship of the HJM-type forward model.
Corollary 27. Under the conditions of lemma 26 the following assertions are equivalent.

(i) \( P \) is an EMM.

(ii) \( P_n^{-\alpha_n^1} + v_n \sigma_n^1 + \int_{2}^{\psi_n \phi_n^1 \psi_n} \frac{dt}{dT} \times 0 \times dt \times dT \) - a.e. for each \( n \) and \( x = f, F \) and \( f_{L_i}(t, \tau) = r_i \) and \( F_{L_i}(t, \tau) = \lambda_i \) are versions of the short rate and the loss intensity.

(iii) \( \beta_n P_n f_n(t, T) \) and \( \beta_n P_n F_n(t, T) \) are in \( M(\mathcal{P}) \) for all \( n \) and \( T \leq T^* \), and \( f_{L_i}(t, \tau) = r_i \) and \( F_{L_i}(t, \tau) = \lambda_i \) are versions of the short rate and the loss intensity.
Chapter 2

Canonical Loss Processes & the Successive $H$ Hypothesis

2.1 Introduction to Chapter 2

In single-obligor default risk modelling, using a background filtration in conjunction with an embedding hypothesis ($H$-hypothesis) has proven a very successful tool to separate the actual default event from the model for the default arrival intensity (see e.g. Lando (1998), or Elliott et al. (2000)). In particular, this approach allows to express the prices of all credit sensitive securities in terms of the default intensity alone, without reference to the actual default arrival process. This approach has become the de-facto standard modelling approach for single-obligor intensity models.

For portfolio credit risk, there are two alternative approaches to set up intensity-based models. In the top-down approach, only the aggregate loss process of a given credit portfolio is modelled and the individual obligors are not referenced or identified. In the bottom-up approach on the other hand, the default events of every individual obligor are modelled.

Top-down models have advantages in certain situations when the individual obligor in the portfolio “does not matter much”: Either – as in the case of the standard credit index reference portfolios\(^1\) – in cases when there is a liquid and important market for CDOs on the portfolio whose importance dwarfs the individual obligors, or – as in the case of retail portfolios or portfolios of small

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\(^1\)The most liquid standard credit index portfolios are the iTraxx portfolio for European obligors and the CDX.NA portfolio of North-American obligors.
and medium-sized enterprises – in cases where the individual obligor is very small compared to the entire reference portfolio. In these cases, the simplification of the problem that is incurred in the top-down approach is outweighed by the gain in flexibility and tractability.

The strong growth of the markets for credit derivatives on standard index reference portfolios has increased the interest in the top-down credit models, recent papers include Schönbucher (2006), Sidenius et al. (2005), and Ehlers and Schönbucher (2006b) who use a forward-modelling approach similar to Heath et al. (1992), and also Giesecke and Goldberg (2005), Errais et al. (2006) or Frey and Backhaus (2004) who do not use a forward-modelling approach but directly model the intensity of the portfolio’s cumulative loss process. In the context of the theme of this chapter it is particularly interesting that Sidenius et al. (2005) explicitly build their model around a setup with a background filtration, albeit without considering the details of the corresponding $\mathbb{H}$-hypothesis and its implications. Later on, we will comment on the approach taken in Sidenius et al. (2005) in more detail.

Even more than in the single-obligor case, it is of great advantage for portfolio credit risk models if one can simplify the pricing problem to a problem which does not reference the loss process itself any more through an extension of the background-filtration/$\mathbb{H}$-embedding approach. We only mention the aspects of the numerical implementation of these models: Monte-Carlo based methods will converge significantly faster if it is not necessary to simulate extremely low-probability events (like the occurrence of large losses), and in Markov-diffusion setups, the absence of jumps (i.e. the absence of the loss process) allows the use of standard solvers for the associated partial differential equations. For some pricing problems, even closed-form solutions may become available.

Here we investigate the possible extensions of the modelling approach with background filtrations and an embedding via a $\mathbb{H}$-hypothesis to the problem of portfolio credit risk modelling, with a specific focus on the top-down approach. It is well-known that the implications of hypothesis $\mathbb{H}$ on the modelling of the credit risk of multiple obligors are stronger than the implications for single-obligor default models, for example as observed by Kusuoka (1999), the most common setup for bottom-up models precludes the possibility of default contagion, i.e. the empirically observed phenomenon that at the default of an obligor, the default intensities of the other obligors jump up. We show in this chapter that – as opposed to the bottom-up approach – in a top-down approach default contagion is compatible with the background-filtration modelling approach.

The rest of the chapter is structured as follows: In section 2.2, the math-
2.1. Introduction to Chapter 2

The mathematical framework of this chapter is set up and we give a further illustration of the advantages of a \( \mathbb{H} \)-based pricing approach. Furthermore, we are also able to give some general Bayes’ type pricing formulae for defaultable claims which do not explicitly reference the loss process any more.

Next, we introduce two different \( \mathbb{H} \)-type embedding assumptions: a successive \( \mathbb{H} \) property and a (seemingly weaker) one-step \( \mathbb{H} \) property, and show that in the setup of top-down portfolio credit risk modelling, the two assumptions are actually equivalent. This and related results on \( \mathbb{H} \)-embedding can be found in section 2.3.

In section 2.4, we construct a canonical loss process such that the successive \( \mathbb{H} \) property holds. This can be regarded as the portfolio-analogue of the well-known construction of default times with exponential random variables in the single-obligor case. Conversely, we show in 2.4.2 that (under weak regularity conditions) every loss process satisfying the successive \( \mathbb{H} \) property is actually a canonical loss process; thus the canonical construction indeed carries its name with some justification.

In section 2.5 we show that in a special case of the canonical loss model (including e.g. the Sidenius et al. (2005) model) the complete conditional transition probabilities can be computed in closed form. We call this special case the conditional Markov model, as in this situation the canonical loss process is a Markov chain when conditioned on the full background information. We take the \( \mathbb{H} \)-based pricing pricing up again and comment on some problems that one inevitably faces when trying to price credit portfolio derivatives using the SPA framework.

Opposed to the spot modelling approach of the canonical construction, in section 2.6 we prove existence of a suitable forward representation within the conditional Markov loss framework using a set of loss-contingent forward loss rates which are adapted to the background filtration. Unfortunately, it turns out that setting directly up a model for these forward loss rates involves almost unsurmountable consistency (in a sense made precise below) and marking-to-market problems.

Nevertheless we can relate these background forward loss rates to the forward loss rates \( F_m(t, T) \) of the model presented in chapter 1 and in Schönbucher (2006). This way we obtain an important warning concerning too restrictive modelling assumptions for the rates \( F_m(t, T) \).\(^2\)

\(^2\)The contents of this chapter are summarized in Ehlers and Schönbucher (2006a).
2.2 Set-Up and Motivation

We work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with an initial (or background) filtration \(\mathbb{F}^0 := (\mathcal{F}^0_t)_{t \geq 0}\). Our main object of investigation is a given reference credit portfolio of \(N\) obligors. The random times \(\tau_1 < \ldots < \tau_N\) denote the ordered times of defaults in the portfolio, i.e. \(\tau_n\) is the time of the \(n^{th}\) default, and not necessarily the time of default of the \(n\)-th obligor. The loss process \(L_t\) associated with the \(\tau_n\) counts the number of defaults in the reference portfolio. Normalizing all losses in default to one, it is defined as

\[
L_t := \sum_{n=1}^{N} 1[\tau_n \leq t].
\]

In pricing applications, the probability measure \(\mathbb{P}\) should be viewed as spot martingale measure.

2.2.1 Motivation of the Pricing Approach

Let us consider two filtrations: a market filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\), and a background filtration \(\mathbb{F}^0 = (\mathcal{F}^0_t)_{t \geq 0}\), with the inclusion \(\mathbb{F}^0 \subset \mathbb{F}\) that the background filtration contains less information than the market filtration. In particular, we assume the loss process \(L_t\) is \(\mathbb{F}\)-adapted, but not \(\mathbb{F}^0\)-adapted (i.e. the default times \(\tau_n\) are \(\mathbb{F}\)-stopping times, but not \(\mathbb{F}^0\)-stopping times).

As an example of a generic exotic portfolio credit derivative, we shall consider a contingent claim with maturity \(t > 0\) and payoff \(X\). We are particularly interested in claims whose payoffs are a function of (i) losses \(L_t\) up to time \(t\), (ii) the prices of synthetic CDOs at time \(t\), and possibly (iii) other market variables \(Z\) such as interest-rates or exchange rates. A number of claims with this type of payoff function has recently gained increasing attention, for example options on portfolio CDS, options on CDOs, forward-starting CDOs and in particular leveraged super-senior tranches (cf. section 1.2).

Synthetic CDOs are the most important portfolio credit derivatives on any given reference portfolio and frequently serve as underlying assets for more exotic portfolio credit derivatives. As shown in detail in chapter 1, if interest rates are assumed independent of default risk, then knowledge of the loss distribution

\[
\{P_n(t, T) := \mathbb{P}[ L_T = n \mid \mathcal{F}_t ] \mid n = 0, \ldots, N; T > t \}
\]

We regard portfolio CDS as synthetic CDOs with attachment point 0% and detachment point 100%.
2.2. Set-Up and Motivation

is sufficient in order to determine the prices of all synthetic CDOs at time $t$, i.e. to price CDOs for all attachment points and maturity dates. Thus, we may replace the dependence on synthetic CDO prices in the payoff function of our contingent claim with a dependence on the loss distribution $\{P_n(t, T) \mid n = 0, \ldots, N; T > t\}$. To shorten notation, we will in fact reduce the dependence from the whole loss distribution $\{P_n(t, T) \mid n = 0, \ldots, N; T > t\}$ to just one $P_m(t, T)$ alone, the argument would remain valid for any number of loss probabilities $P_n(t_k, T_k), k = 1, \ldots, K$ in the payoff function. Finally, we assume that $Z$ is in the background information, i.e. $\mathcal{F}^0_t$-measurable.

Under these assumptions, we can write the payoff $X$ as a function $G(\cdot)$ of loss process, loss distribution, and background variables:

$$X = G(L, P_m(t, T), Z) = \sum_{n=0}^{N} 1_{[L_n = n]} G(n, P_m(t, T), Z) = \sum_{n=0}^{N} 1_{[L_n = n]} G(n, 1_{[L_n = n]} P_m(t, T), Z).$$

We will show in detail later on that, under suitable assumptions, we will have

$$1_{[L_n = n]} P_m(t, T) = 1_{[L_n = n]} \hat{P}_{n,m}(t, T), \quad (2.1)$$

where the variables $\hat{P}_{n,m}(t, T)$ are only $\mathcal{F}^0_0$-adapted, i.e. they do not directly reference the loss process any more. The payoff decomposition becomes:

$$X = \sum_{n=0}^{N} 1_{[L_n = n]} G(n, 1_{[L_n = n]} \hat{P}_{n,m}(t, T), Z) = \sum_{n=0}^{N} 1_{[L_n = n]} G(n, \hat{P}_{n,m}(t, T), Z),$$

and the problem of pricing the generic contingent claim reduces to the evaluation of

$$\mathbb{E}[X] = \sum_{n=0}^{N} \mathbb{E}
\left[
1_{[L_n = n]} G(n, \hat{P}_{n,m}(t, T), Z)
\right] = \sum_{n=0}^{N} \mathbb{E}
\left[
\mathbb{E}
\left[
1_{[L_n = n]} G(n, \hat{P}_{n,m}(t, T), Z) \mid \mathcal{F}^0_t
\right]
\right] = \sum_{n=0}^{N} \mathbb{P}
\left[
L_n = n \mid \mathcal{F}^0_t
\right] G(n, \hat{P}_{n,m}(t, T), Z).$$
In the last equation all variables are $\mathcal{F}_t^0$-measurable, thus all reference to the loss process has been removed and the pricing problem has been reduced to a problem under the background filtration $\mathbb{F}^0$.

The following questions were left open in the argument above and will be answered in this thesis:

- What assumptions are necessary (and/or sufficient) for (2.1) to hold?
- Under these assumptions, can we give closed-form expressions for the building blocks $P_{n,m}(t, T)$ and $P\left[ L_t = n \mid \mathcal{F}_T^0 \right]$?

### 2.2.2 Filtrations Set-Up

Filtrations are abbreviated with math blackboard letters and the $\sigma$-fields that make up the filtration with math calligraphic letters (e.g. $\mathcal{G}$ stands for $(\mathcal{G}_t)_{t \geq 0}$). “$\vee$” is the $\sigma$-field product operator and when $\vee$ and $\subseteq$ are used between filtrations, they are understood $t$-wise for all $t \geq 0$.\(^4\) All processes are assumed to be càdlàg (if nothing else is specified) and $\mathcal{M}^2(\mathcal{G})$ denotes the space of square-integrable $\mathcal{G}$-martingales. For convenience we set $\tau_0 := 0$ and we assume $L_0 = 0$ a.s.

**Definition 3 (Filtrations).**

(i) For each $n$, we define $\mathbb{F}^n$ as the smallest filtration which contains $\mathbb{F}^0$, satisfies the usual hypotheses, and makes $\tau_1, \ldots, \tau_n$ stopping times. I.e.

$$\mathbb{F}^n_t = \mathbb{F}_t^0 \vee T_{\tau_n+} \vee \cdots \vee T_{\tau_1+},$$

where $\mathbb{T}^n$ is given by $T_{\tau_n} = \sigma(\{\tau_n > u \mid u \leq t\}).$\(^5\)

(ii) The market filtration $\mathbb{F}$ is defined as the filtration $\mathbb{F} := \mathbb{F}^N$, which admits all loss times in the portfolio as stopping times.

Clearly, $L$ is $\mathbb{F}^N$-adapted and $\mathbb{F}^0 \subseteq \mathbb{F}^1 \subseteq \cdots \subseteq \mathbb{F}^N$. $\mathbb{F}^0$ can be thought of as the flow of default-free market information (to which all assets that are insensitive to credit portfolio losses are adapted).

In order to avoid trivialities we assume that $\mathbb{F}^n$ does not contain “too much” information about the next loss time $\tau_{n+1}$.

**Assumption 3.** It holds $E\left[ I_{[\tau_{n+1} > 1]} \mid \mathbb{F}^n_t \right] > 0$ a.s. for every $n$ and $t$.

Then $\tau_{n+1}$ is in particular not an $\mathbb{F}^n$-stopping time.\(^6\)

---

\(^4\)I.e. $g^1 \vee g^2$ is the smallest $\sigma$-field containing both $g^1$ and $g^2$ and $G^1 \vee G^2 := (g_1^1 \vee g_2^1)_{t \geq 0}$.

\(^5\)I.e. it is the smallest filtration making $\tau_n$ a stopping time.

\(^6\)Since also $\tau_{n+1} < \infty$ a.s.
Remark 28. \( \mathcal{G}^n := \mathbb{F}^0 \vee \mathbb{T}^1 \vee \cdots \vee \mathbb{T}^n \), the minimal enlargement of \( \mathbb{F}^0 \) which makes \( \tau_1, \ldots, \tau_n \) stopping times, is in general \textit{not} right-continuous, as remarked e.g. in Protter (2004), p. 370.\(^7\) We find it more convenient to work directly with the minimal enlargements satisfying the usual conditions. Nevertheless almost all results we obtain in this chapter, remain valid if \( \mathbb{F}^n \) is replaced with \( \mathcal{G}^n \) for all \( n \).

Also note that \( \mathbb{F}^{n+1} \) can be characterized equivalently in recursive fashion as the smallest filtration containing \( \mathbb{F}^n \) and making \( \tau_{n+1} \) a stopping time, which satisfies the usual conditions, i.e.

\[
\mathbb{F}^{n+1} = \bigcap_{\epsilon > 0} \left( \mathbb{F}^{n}_t \vee \mathbb{F}^{1}_t \vee \cdots \vee \mathbb{F}^{n+1}_t \right) = \bigcap_{\epsilon > 0} \left( \mathbb{F}^{n}_t \vee \mathbb{F}^{1}_t \vee \cdots \vee \mathbb{F}^{n+1}_t \right).
\]

It will be convenient to use the following notation throughout this chapter. For a vector \( \mathbf{x} := (x_1, \ldots, x_N) \in \mathbb{R}^N_+ \), we denote with \( \mathbf{x}^k \in \mathbb{R}^k \) the sub-vector of its first \( k \) components, i.e. \( \mathbf{x}^k := (x_1, \ldots, x_k) \). Comparisons between vectors are understood componentwise. Then the explicit generator \( \pi^n \) of the \( \sigma \)-field \( \mathbb{F}^0 \vee \mathbb{F}^1 \vee \cdots \vee \mathbb{F}^n \) reads

\[
\pi^n := \left\{ F^0 \cap \left[ \mathbf{x}^n > \mathbf{u}^n \right] : F^0 \in \mathbb{F}^0, \mathbf{u} \in [0, t]^N \right\}
\]

and the \( \sigma \)-field \( \mathbb{F}^{n-1} \vee \mathbb{F}^n \supseteq \mathbb{F}^0 \vee \mathbb{F}^1 \vee \cdots \vee \mathbb{F}^n \) is generated by the \( \pi \)-system

\[
\pi^n := \left\{ F \cap \left[ \mathbf{x}^n > \mathbf{u}^n \right] : F \in \mathbb{F}^{n-1}, \mathbf{u} \in [0, t]^N \right\} \supseteq \pi^n.
\]

2.2.3 Some Conditional Loss Transition Probabilities

Given the particular nature of the filtrations \( \mathbb{F}^0 \subseteq \mathbb{F}^1 \subseteq \cdots \subseteq \mathbb{F}^N = \mathbb{F} \) we are now able to say more about the question raised in section 2.2.1, as to how it may be possible to decompose the loss probabilities \( P_m(t, T) \) on the sets \( \{ L_t = 0 \}, \ldots, \{ L_t = m \} \). It should be noticed that \( 1_{\{L_t = 0\}} = 1_{\{L_t \leq 0\}}1_{\{L_t \leq 0\}} \) whereas \( 1_{\{L_t \geq 0\}} = 1_{\{L_t \leq 0\}}1_{\{L_t \geq 0\}} \) is \( \mathbb{F}^n \)-measurable but \( 1_{\{L_t \geq 0\}} = 1_{\{L_t \geq 0\}}1_{\{L_t \geq 0\}} \) is \textit{not}.

Hence, by assumption 3, the \( \mathbb{F}^n \)-\textit{conditional} transition probabilities

\[
\tilde{P}_{n,m}(t, T) := 1_{\{L_t \geq 0\}} \mathbb{E} \left[ 1_{\{L_t = m\}} \bigg| \mathbb{F}^n_t \right]
\]

are well-defined for all \( 0 \leq n \leq m \leq N \) and \( t \leq T \). A useful Bayes’ type rule is stated in

\[^7\] \( \mathbb{F}^0 \vee \mathbb{T}^1 \vee \cdots \vee \mathbb{T}^n \) is obviously complete (\( \mathbb{F}^0 \) already contains all null sets). In contrast, \( \mathbb{T}^n \) is automatically right-continuous but not complete, as noticed e.g. in Dellacherie (1968/69).
Chapter 2. Canonical Loss Processes & the Successive $\mathbb{H}$ Hypothesis

Lemma 29. \(1_{\{L_i=n\}} \mathbf{E}\left[ 1_{\{L_T=m\}} \mid \mathcal{F}_t^{m+1} \right] = 1_{\{L_i=n\}} \hat{P}_{n,m}(t, T) \) for all \(T \geq t\).

The proof of this simple lemma, despite its technical form, is based solely on the monotonicity of the random times \(\tau_1, \ldots, \tau_N\) and is postponed to the end of this section. A similar result is given in Lemma 30.

Lemma 30. Let \(Y \in L^1(\mathcal{F}_t^n)\). Then it holds

\[
1_{\{L_i=n\}} \mathbf{E}\left[ Y 1_{\{L_T=m\}} \mid \mathcal{F}_t^{m+1} \right] = 1_{\{L_i=n\}} \mathbf{E}\left[ Y \hat{P}_{n,m}(T; t, T) \mid \mathcal{F}_t^n \right]
\]

where we use the slightly different conditional transition probabilities

\[
\hat{P}_{n,m}(S; t, T) := 1_{\{L_i\geq n\}} \frac{\mathbf{E}\left[ 1_{\{L_i=n\}} 1_{\{L_T=m\}} \mid \mathcal{F}_t^n \right]}{\mathbf{E}\left[ 1_{\{L_i=n\}} \mid \mathcal{F}_t^n \right]} \tag{2.3}
\]

Since \(\mathbf{E}\left[ \hat{P}_{n,m}(T; t, T) \mid \mathcal{F}_t^n \right] = \hat{P}_{n,m}(t, T)\), lemma 29 is obviously a special case of lemma 30. For the proof see the end of this section.

Remark 31. For \(m = n = 1\), lemma 30 can be found e.g. in Blanchet-Scalliet and Jeanblanc (2004), lemma 1 (p. 147).

So far, lemmata 29 and 30 are only of theoretical interest because first, they do not tell how to explicitly compute \(\hat{P}_{n,m}(t, T)\) and \(\hat{P}_{n,m}(T; t, T)\) nor if we can further decompose these quantities into \(\mathbb{P}^0\)-adapted components, separately on each set \(\{L_i = k\}\) as claimed in (2.1). Second, for derivatives pricing, one needs to “know” the \(P_m(t, T)\) and expressions like \(\mathbf{E}\left[ Y 1_{\{L_T=m\}} \mid \mathcal{F}_t \right]\) which are conditional expectations with respect to the full market filtration \(\mathbb{F} = \mathbb{F}^N\), and not only with respect to \(\mathbb{P}^{m+1}\) (with possibly \(m + 1 < N\)) as on the respective lefthandsides in the lemmata above. Therefore we have to answer the following questions.

- (When) can one switch from \(\mathbb{P}^{m+1}\)- to \(\mathbb{F}^N\)-conditional expectations?
- Under which assumptions is this simultaneously possible for all \(m\)?

We treat these issues in the next section by studying successive $\mathbb{H}$-type filtration embeddings.

Proofs of Lemmata in this Section

Proof of lemma 29. Since the \(\sigma\)-fields \(\mathcal{F}_t^n\) do not have an explicit generator, we consider filtrations which are a bit larger than \(\mathbb{P}^n\). For each \(n\) and each \(m \geq n\) let \(\mathbb{P}^{m,\tau_m} := (\mathcal{F}_t^{m,\tau_m})_{t \geq 0}\) where \(\mathcal{F}_t^{m,\tau_m} := \mathcal{F}_t^n\) and

\[
\mathcal{F}_t^{m,\tau_{m+1}} := \left\{ F \in \mathcal{F}_\infty^{m+1} \mid \exists F_i \in \mathcal{F}_t^{m,\tau_m}: F \cap \{\tau_{m+1} > t\} = F_i \cap \{\tau_{m+1} > t\} \right\}
\]
2.2. Set-Up and Motivation

for \( m \geq n \), \( \mathcal{F}_t^{n, \tau_m} \) automatically satisfies the usual conditions and by construction \( \mathcal{F}_t^{m+1} \subseteq \mathcal{F}_t^{m, \tau_{m+1}} \subseteq \mathcal{F}_t^{n, \tau_m} \) for every \( m > n \). (For \( m = n \), these filtrations were introduced in Jeulin and Yor (1978), p. 81).\(^8\) It is thus sufficient to show \( \mathbb{1}_{\{L_t = n\}} \mathbb{E} \left[ \mathbb{1}_{\{L_T = m\}} \mid \mathcal{F}_t^{n, \tau_m} \right] = \mathbb{1}_{\{L_t = n\}} \tilde{P}_{n,m}(t, T) \), then the result follows with the law of iterated expectations. Let \( F \in \mathcal{F}_t^{m, \tau_m} \). We first note that for \( m \geq n \)

\[
F \cap \{L_t = n\} = F \cap \{\tau_{n+1} > t\} \cap \cdots \cap \{\tau_{m+1} > t\} \cap \{L_t = n\}.
\]

By definition of \( \mathcal{F}_t^{m, \tau_m}, \ldots, \mathcal{F}_t^{n, \tau_n} \), there exist \( F_t^{m} \in \mathcal{F}_t^{m, \tau_m}, \ldots, F_t^{n} \in \mathcal{F}_t^{n, \tau_n} = \mathcal{F}_t^{n} \) with \( F \cap \{L_t = n\} = F_t^{m} \cap \{L_t = n\} = \cdots = F_t^{n} \cap \{L_t = n\} \). Therefore

\[
\mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \mathcal{F}_t^{n} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \mathcal{F}_t^{n} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \mathcal{F}_t^{n} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \mathcal{F}_t^{n} \right].
\]

\[\text{Proof of lemma 30.}\] We use the same \( F \) as above. Recall \( F \cap \mathbb{1}_{\{L_t = n\}} = F_t^{n} \cap \mathbb{1}_{\{L_t = n\}} \). This yields

\[
\mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathcal{F}_t^{n} \right].
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\]

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\[
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\]

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\[
\mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathcal{F}_t^{n} \right].
\]

\[\text{Proof of lemma 30.}\] We use the same \( F \) as above. Recall \( F \cap \mathbb{1}_{\{L_t = n\}} = F_t^{n} \cap \mathbb{1}_{\{L_t = n\}} \). This yields

\[
\mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathcal{F}_t^{n} \right].
\]

\[\text{Proof of lemma 30.}\] We use the same \( F \) as above. Recall \( F \cap \mathbb{1}_{\{L_t = n\}} = F_t^{n} \cap \mathbb{1}_{\{L_t = n\}} \). This yields

\[
\mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathcal{F}_t^{n} \right].
\]

\[\text{Proof of lemma 30.}\] We use the same \( F \) as above. Recall \( F \cap \mathbb{1}_{\{L_t = n\}} = F_t^{n} \cap \mathbb{1}_{\{L_t = n\}} \). This yields

\[
\mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathbb{1}_{\{L_T = m\}} \right] = \mathbb{E} \left[ F \mathbb{1}_{\{L_t = n\}} \mathcal{F}_t^{n} \right].
\]
2.3 The Successive $\mathbb{H}$ Property

Definition 4 (i) below is the standard definition of the martingale preserving property / embedding property ($\mathbb{H}$ hypothesis) in the case when the enlargement of the filtration is done in one step, i.e. if only two filtrations are involved. In the top-down credit loss setup on the other hand, a succession of $N$ filtration enlargements $F^0 \subseteq F^1 \subseteq \cdots \subseteq F^N$ exists. We define two extensions of $\mathbb{H}$ hypothesis:

**Definition 4 ($\mathbb{H}$ Hypotheses).**

(i) Let $F$ be a subfiltration of $G$. The enlargement $G \supseteq F$ satisfies the $\mathbb{H}$ property and we write $F \xrightarrow{H} G$ iff

$$\mathcal{M}^2(F) \subseteq \mathcal{M}^2(G).$$

(ii) $F^0 \subseteq F^1 \subseteq \cdots \subseteq F^N$ satisfies the *successive $\mathbb{H}$ property* iff

$$F^n \xrightarrow{H} F^{n+1} \quad \text{for every } n = 0, \ldots, N - 1.$$  \hfill (\star)

(iii) $F^0 \subseteq F^1 \subseteq \cdots \subseteq F^N$ satisfies the *one-step $\mathbb{H}$ property* iff

$$F^0 \xrightarrow{H} F^N.$$  \hfill (\star\star)

**Remark 32.** In general, the successive $\mathbb{H}$ property (\star) is a stronger assumption than the one-step $\mathbb{H}$ property (\star\star), i.e. in an arbitrary successive enlargement $G^I \subseteq G^{II} \subseteq G^{III}$, $G^I \xrightarrow{H} G^{III}$ does in general not imply $G^{II} \xrightarrow{H} G^{III}$ although it does of course imply $G^I \xrightarrow{H} G^{II}$. As a simple counterexample take an enlargement $G^{II} \subseteq G^{III}$ which does not satisfy $\mathbb{H}$ hypothesis 9 and let $G^I$ be trivial. Then $G^I \xrightarrow{H} G^{III}$ holds, but of course by construction we do not have $G^{II} \xrightarrow{H} G^{III}$.

The following theorem is the main result of this section. It states that in the case of a successive enlargement by an increasing sequence of stopping times, the one-step $\mathbb{H}$-property is actually equivalent to the successive $\mathbb{H}$-property:

9Enlargements by random times which do not satisfy $\mathbb{H}$ hypothesis are e.g. enlargements by $G^{II}$-honest times which are not $G^{II}$-stopping times (see e.g. Azéma et al. (1993), chapter 2, p. 137).
2.3. The Successive \( \mathbb{H} \) Property

Theorem 33. The following assertions are equivalent (in our enlargement setup):

(i) \( F^0 \overset{H}{\rightarrow} F^N \): The one-step \( \mathbb{H} \) property (**) holds.

(ii) \( F^n \overset{H}{\rightarrow} F^{n+1} \) for every \( n = 0, \ldots, N - 1 \): The successive \( \mathbb{H} \) property (***) holds.

(iii) \( F^n \overset{H}{\rightarrow} F^N \) for every \( n = 0, \ldots, N - 1 \).

Proof. See below.

Remark 34. Theorem 33 can be extended straightforward to the case \( N = \infty \). Precisely, for an increasing sequence of random times \( \tau_n, n \in \mathbb{N} \), let \( \mathcal{F}_t^\infty := \bigcap_{\varepsilon > 0} (\mathcal{F}_t^0 \vee \bigvee_{n=0}^{\infty} \tau_t^{\varepsilon +} ) \). Then the following assertions are equivalent.

(i') \( F^0 \overset{H}{\rightarrow} F^\infty \)

(ii') \( F^n \overset{H}{\rightarrow} F^{n+1} \) for all \( n \in \mathbb{N} \)

(iii') \( F^n \overset{H}{\rightarrow} F^\infty \) for all \( n \in \mathbb{N} \)

Noting that \( \mathcal{F}_t^0 \vee \bigvee_{n=0}^{\infty} \tau_t^n \) is generated by sets of the form \( F^0 \cap \{ N > U_t^n \} \) with \( F^0 \in \mathcal{F}_t^0, U_t^n \in [0, \infty) \) and \( N \in \mathbb{N} \), the proof goes analogous to that of theorem 33.

Theorem 33 states that if adding the full running loss information \( \vee_{n=1}^{\infty} u^n \) to the default-free market information \( F^0 \) "at once" preserves the martingale property, then "nothing can go wrong" any more in the intermediate steps, the successive \( \mathbb{H} \) assumption already holds.

It is hard to argue directly why the successive \( \mathbb{H} \) assumption (**) would hold in real markets. This would require several arguments for each of the successive default times \( \tau_n \) and their corresponding enlargements of filtrations \( \mathbb{F}_n \overset{H}{\rightarrow} \mathbb{F}^{n+1} \), moving from one hypothetical situation ("if we could only observe the first \( n \) defaults. . .") to another hypothetical situation ("if we could only observe the first \( n + 1 \) defaults. . ."). Both of these hypothetical situations are rather contrived. (Why do we only observe the first \( n \) defaults? Why not more? Why not less? Why any defaults at all? Which \( N - n \) obligors do we not observe - what are their names?) Thus, forming an opinion about the differences between them (i.e. validating \( \mathbb{F}_n \overset{H}{\rightarrow} \mathbb{F}^{n+1} \) with an economic argument) seems almost impossible.
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Fortunately, theorem 33, relieves us from this task and tells us that it is sufficient to motivate (\ref{eq:one-step-hypothesis}), i.e. $\mathbb{F}^0 \overset{\mathbb{H}}{\rightarrow} \mathbb{F}^N$. This is much easier: First, there is only one hypothesis to support, and not $N$. Second, that hypothesis is about the difference between the situation when we observe no defaults (i.e. $\mathbb{F}^0$), and the situation when we observe all defaults (i.e. $\mathbb{F}^N$). The situation $\mathbb{F}^N$ is the market filtration, it is the information that is available in reality, it is not a hypothetical situation. And information $\mathbb{F}^0$, i.e. not observing any defaults is also a clearly defined situation that one can imagine without difficulty in a thought experiment.

Nevertheless, theorem 33 does not do all the work for us. It does not in itself provide a “proof” of the one-step $\mathbb{H}$ hypothesis (\ref{eq:one-step-hypothesis}). (\ref{eq:one-step-hypothesis}) is still an assumption which has to be justified from economic arguments outside the mathematical model.

Furthermore, it was noticed by Kusuoka (1999) that $\mathbb{H}$ hypothesis is in general not invariant under an equivalent change of probability measure. Hence, for derivatives pricing, assumption $\mathbb{F}^0 \overset{\mathbb{H}}{\rightarrow} \mathbb{F}^N$ has to be made directly under the/an equivalent martingale (pricing) measure. There are some interesting related results, e.g. it can be shown that if the defaultable/loss $\mathbb{F}^N$-market is arbitrage-free\footnote{Then $\mathbb{F}^0$ is arbitrage-free too. Precisely, let $T^*$ be a finite time horizon. If there exists an $\mathbb{F}^N$-equivalent martingale measure (EMM) $\mathbb{Q} \sim \mathbb{P}$, on $\mathcal{F}^N_T$, then $\mathbb{Q}|_{\mathcal{F}^0_{T^*}}$, the restriction of $\mathbb{Q}$ to the $\mathbb{F}^0$-market, is obviously an $\mathbb{F}^0$-EMM.} and the default-free/initial $\mathbb{F}^0$-market is complete, then $\mathbb{F}^0 \overset{\mathbb{H}}{\rightarrow} \mathbb{F}^N$ is satisfied under every $\mathbb{F}^N$-equivalent martingale measure (see Blanchet-Scalliet and Jeanblanc (2004), p. 150).

While the one-step $\mathbb{H}$ property (\ref{eq:one-step-hypothesis}) is much easier to justify, it is unfortunately less convenient to work with (otherwise we would not need theorem 33): For concrete calculations, the successive $\mathbb{H}$ property has the advantage of allowing a reduction of the pricing problems to an iterative sequence of problems, where the $n$-th problem only involves one stopping time $\tau_n$.

Problems involving one stopping time under $\mathbb{H}$ have already been treated extensively in the literature. The main workhorse for such problems is the following well-known lemma (see e.g. Brémaud and Yor (1978) or Elliott et al. (2000)) which summarizes the key properties of expectations under filtration enlargements satisfying $\mathbb{H}$:

**Lemma 35.** The following statements are equivalent to $\mathbb{F} \overset{\mathbb{H}}{\rightarrow} \mathbb{G}$.

(i) $\mathbb{E}[F | \mathcal{F}_t] = \mathbb{E}[F | \mathcal{G}_t]$ for every $F \in L^2(\mathcal{F}_\infty)$.

(ii) $\mathbb{E}[G | \mathcal{F}_\infty] = \mathbb{E}[G | \mathcal{F}_t]$ for every $G \in L^2(\mathcal{G}_t)$. 

\[\]
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(iii) $E[F | \mathcal{F}_t] = E[F | \mathcal{F}_t] E[G | \mathcal{F}_t]$ for every $F \in L^2(\mathcal{F}_\infty)$ and $G \in L^2(\mathcal{g}_t)$, i.e. $\mathcal{F}_\infty$ and $\mathcal{g}_t$ are conditionally independent given on $\mathcal{F}_t$.

Note that by virtue of (i) and theorem 33, as soon as the one-step $H$ property (**) holds, we may replace $\mathcal{F}^{m+1}_t$ in lemmata 29 and 30 with $\mathcal{F}_t$, i.e. it holds

$$1_{[L_t = n]} P_m (t, T) = 1_{[L_t = n]} \hat{P}_{n,m} (t, T).$$

This is the first step on the way to finding a decomposition as claimed in (2.1).

Remark 36. $L^2$ can be replaced with $L^\infty$ in lemma 35.

2.3.1 Proof of Theorem 33

In this subsection we give a detailed proof of theorem 33. The reader who is only interested in the main results may skip the rest of this section at first and recommence reading with section 2.4.

The proof, apart from some technical details, is based on the monotonicity of the random times $\tau_1, \ldots, \tau_N$ and the explicit generator $\mathcal{G}_n$ of $\mathcal{F}^{n-1}_t \vee \mathcal{T}^n_t$ as given in (2.2). We need an auxiliary lemma concerning general enlargements (starting from filtrations which satisfy the usual hypotheses).

Lemma 37. Let $\mathcal{G}$ be any filtration and $\mathcal{G}^+$ be the minimal right-continuous filtration containing $\mathcal{G}$.

(i) Let an $\mathcal{F}$ satisfy the usual conditions and $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F} \overset{\text{H}}{\rightarrow} \mathcal{G}$ implies $\mathcal{F} \overset{\text{H}}{\rightarrow} \mathcal{G}^+$.

(ii) Every bounded $\mathcal{G}$-martingale with a càdlàg version is also a $\mathcal{G}^+$-martingale.

Proof. (i): Due to lemma 35 it is sufficient to show that it holds $E[F | \mathcal{F}_t] = E[F | \mathcal{G}_t]$ for all $F \in L^\infty(\mathcal{F}_\infty)$. Let $F \in L^\infty(\mathcal{F}_\infty)$ and let $M$ be the càdlàg version of the (bounded) $\mathcal{F}$-martingale $E[F | \mathcal{F}_t]$. By assumption we have a.s. $E[F | \mathcal{G}_t] = M_{t+1/n}$ for all $n \in \mathbb{N}$. This implies, a.s.

$$E[F | \mathcal{G}_t] = E[E[F | \mathcal{G}_{t+1/n}] | \mathcal{G}_t] = E[M_{t+1/n} | \mathcal{G}_t]$$

and hence, with dominated convergence,

$$E[F | \mathcal{G}_t] = \lim_{n \to \infty} E[M_{t+1/n} | \mathcal{G}_t] = E[M_t | \mathcal{G}_t] = M_t.$$
(ii): Let $M$ be the càdlàg version of a bounded $\mathbb{G}$-martingale and let $s < t$. Then a.s. $E\left[M_t \mid \mathcal{G}_{s+1/n}\right] = M_{s+1/n}$ for all $n \in \mathbb{N}$ (sufficiently large). Therefore a.s.

$$E\left[M_t \mid \mathcal{G}_{s+}\right] = \lim_{n \to \infty} E\left[E\left[M_t \mid \mathcal{G}_{s+1/n}\right] \mid \mathcal{G}_{s+}\right] = \lim_{n \to \infty} E\left[M_{s+1/n} \mid \mathcal{G}_{s+}\right] = E\left[\lim_{n \to \infty} M_{s+1/n} \mid \mathcal{G}_{s+}\right] = E\left[M_t \mid \mathcal{G}_{s+}\right] = M_s. \quad \Box$$

**Remark 38.** The idea to pass from $\mathcal{G}_t$ to $\mathcal{G}_{t+}$ by the right-continuity of $\mathbb{F}$, respectively of the $\mathbb{F}$-martingale $M$, stems from Dellacherie and Meyer (1978), p. 71.

**Proof of theorem 33.** It is clear that “(iii) $\Rightarrow$ (ii)” and “(ii) $\Rightarrow$ (i)”. To show “(i) $\Rightarrow$ (iii)” we will prove that $\mathbb{F}^0 \overset{H}{\longrightarrow} \ldots \overset{H}{\longrightarrow} \mathbb{F}^{k-1} \overset{H}{\longrightarrow} \mathbb{F}^N$ implies $\mathbb{P}^{k-1} \overset{H}{\longrightarrow} \mathbb{P}^k \overset{H}{\longrightarrow} \mathbb{P}^N$ for every $k = 1, \ldots, N-1$. Then the claim will follow by induction in $k$.

So let $\mathbb{F}^0 \overset{H}{\longrightarrow} \ldots \overset{H}{\longrightarrow} \mathbb{F}^{k-1} \overset{H}{\longrightarrow} \mathbb{F}^N$ hold for some $0 < k < N$. Since $\mathbb{P}^k \subseteq \mathbb{P}^N$, it remains to show $\mathbb{P}^k \overset{H}{\longrightarrow} \mathbb{P}^N$. By virtue of lemma 35 and lemma 37, (i) we must only prove

$$E\left[X^k \mid \mathcal{F}^k_t \lor \mathcal{T}^{k+1}_t \lor \ldots \lor \mathcal{T}^N_t\right] = E\left[X^k \mid \mathcal{F}^k_t\right] \text{ for all } X^k \in L^2(\mathcal{F}^k_\infty),$$

and hence by Dynkin’s lemma, it is sufficient to show that for any $u^N \in [0, t]^N$ and $F^k \in \mathcal{F}^k_t$

$$E\left[X^k 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right] = E\left[E\left[X^k \mid \mathcal{F}^k_t\right] 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right].$$

We first take $X^k$ of the form $X^k = X^{k-1} 1_{\{\tau_k > T\}}$ where $X^{k-1} \in L^2(\mathcal{F}^{k-1}_\infty)$ and $T \geq 0$. By assumption, it holds $E\left[X^{k-1} \mid \mathcal{F}^N_t\right] = E\left[X^{k-1} \mid \mathcal{F}^{k-1}_t\right] = E\left[X^{k-1} \mid \mathcal{F}^k_t\right]$. Thus, if $T < t$, we have

$$E\left[X^k 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right] = E\left[E\left[X^{k-1} \mid \mathcal{F}^N_t\right] 1_{\{\tau_k > T\}} 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right] \overset{H}{=} E\left[E\left[X^{k-1} \mid \mathcal{F}^k_t\right] 1_{\{\tau_k > T\}} 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right] = E\left[E\left[X^{k-1} 1_{\{\tau_k > T\}} \mid \mathcal{F}^k_t\right] 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right] = E\left[X^k \mid \mathcal{F}^k_t\right] 1_{\mathcal{L}^N \supseteq u^N} \mid F^k\right].$$
On the other hand, if \( T \geq t \), we notice that by the monotonicity of \( \tau_1, \ldots, \tau_N \) we have

\[
1_{\{\tau_1 > T\}} = 1_{\{\tau_1 > t\}} 1_{\{\tau_2 > t\}} \text{ and } 1_{\{\tau_1 > T\}} 1_{\{\tau_N > \tau_N^N\}} = 1_{\{\tau_1 > t\}} 1_{\{\tau_N > \tau_N^N\}}.
\]

Hence

\[
E \left[ X^k 1_{\{\tau_N > \tau_N^N\}} 1_{F^k} \right] = E \left[ X^{k-1} 1_{\{\tau_1 > T\}} 1_{\{\tau_1 > t\}} 1_{\{\tau_2 > \tau_2\}} 1_{F^k} \right]
\]

\[
= E \left[ E \left[ X^{k-1} 1_{\{\tau_1 > T\}} 1_{\{\tau_1 > t\}} 1_{\{\tau_2 > \tau_2\}} 1_{F^k} \right] \right]
\]

\[
= E \left[ E \left[ X^{k-1} 1_{\{\tau_1 > T\}} 1_{\{\tau_1 > t\}} 1_{\{\tau_N > \tau_N^N\}} 1_{F^k} \right] \right].
\]

The claim follows because \( \{X^k = X^{k-1} 1_{\{\tau_1 > T\}} | X^{k-1} \in L^2(\mathcal{F}_{\infty}^{k-1}), \ T \geq 0\} \) is dense in \( L^2(\mathcal{F}_{\infty}^{k}) \).  

\[\square\]

### 2.4 The Canonical Loss Process

The aim of this section is to provide a concrete situation in which the successive\( \mathbb{H} \) property holds: the canonical construction of a loss process. Proposition 44 will show that this construction is indeed the stochastic representation of the successive\( \mathbb{H} \) property: Up to regularity, assuming the successive\( \mathbb{H} \) property (\( * \)) is equivalent to assuming the construction below.

#### 2.4.1 The Canonical Construction

As we want to construct random times \( \tau_1, \ldots, \tau_N \) satisfying (\( * \)) (they are not given a priori), we need some "construction material":

**Assumption 4.** On \( (\Omega, \mathcal{F}, P) \) there are \( N \) random variables \( E_1, \ldots, E_N \) which are i.i.d. unit exponentially distributed and independent of \( \mathcal{F}_0^\infty \).

**Definition 5** (Canonical Construction). Let assumption 4 hold.

(A) For each \( n = 0, \ldots, N - 1 \), successively:

1. Choose \( \lambda_n^N \geq 0 \), \( \mathbb{F}^n \)-adapted with \( \int_0^t \lambda_n^N ds < \infty \) a.s. for all \( t < \infty \) and \( \int_0^\infty \lambda_n^N ds = \infty \) a.s.
2. Define \( \tau_{n+1} := \inf \{ t > \tau_n \mid \int_{\tau_n}^t \lambda_n^N ds \geq E_{n+1} \} \)
3. Let \( \mathbb{P}^{n+1} \) be as in definition 3.

---

I thank Freddy Delbaen for pointing this out to me.

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(B) Define the loss process $L_t := \sum_{n=1}^{N} 1_{[\tau_n \leq t]}$ and let $F := \mathbb{F}^N$ be the market filtration.

(C) We call $L_t$ the canonical loss process with respect to $\lambda^n_t$; $A_t$ denotes the predictable $\mathbb{F}^N$-compensator of $L_t$.

For the one-obligor case ($N = 1$) this is the Cox process construction by Lando (1998). Here, $L_t$ can be interpreted as a generalization of the Cox process construction because $\lambda^n_t$ may vary across different values of $n$, and furthermore we allow $\lambda^n_t$ to depend on (a part of) the history of $L$: the “truncated” loss history $\sigma \left( \{ L_u \wedge n, u \leq t \} \right)$. Nevertheless, as in Lando (1998), also here the “intensity” $\lambda^n_t$ is well-defined even after $\tau_n$.

Here are some elementary properties of the canonically constructed loss process.

**Proposition 4.** The canonical loss process has the following properties.

1. \( 0 < t_0 < t_1 < \ldots < t_N < \infty \) a.s.

2. \( P \left[ \tau_{n+1} > t \mid \mathbb{F}^n_{\infty} \right] = e^{-\int_0^t \lambda^n_s \, ds} \)

3. The successive $H$ property (•) holds.

4. \( A_t = \sum_{n=0}^{N-1} \int_0^t 1_{[L_u = n]} \lambda^n_s \, ds \).

5. For every $\mathbb{F}^n$-stopping time, it holds $P \left[ \Theta = \tau_{n+1} \right] = 0$.

Properties P1 and P2 ensure that assumption 3 always holds for the canonical construction and by P4 the canonical loss process admits the $\mathbb{F}^N$-intensity

\[
\lambda_t := \sum_{n=0}^{N-1} 1_{[L_u = n]} \lambda^n_t.
\]

This shows that $\tau_n$ is $\mathbb{F}^N$- (and hence $\mathbb{F}^n$-) totally inaccessible for each $n$ and that contagion, usually referred to as the ability of the loss intensity to jump (up) at the occurrence of defaults, is a natural feature of the canonical model. The size of the jump of the loss intensity at the $n$-th default is

\[
\Delta \lambda_{\tau_n} = \lambda^n_{\tau_n} - \lambda^{n-1}_{\tau_n} \quad \text{for all } n.
\]  

**Remark 39.** It is quite obvious that e.g. for credit-interest rate or credit-FX hybrids pricing, the canonical construction can be extended to include e.g. loss-dependent interest or FX rates or relevant fundamental processes. In order to construct a loss-dependent short rate process $r_t$, simply insert...
(n1.1) Choose \( r_t^n \geq 0, \mathbb{F}^n \)-adapted with \( \int_0^t r_s^n ds < \infty \) a.s. for all \( t < \infty \).

after (n1) and after (B) add

(B.1) Choose \( r_t^N \geq 0, \mathbb{F}^N \)-adapted with \( \int_0^t r_s^N ds < \infty \) a.s. for all \( t < \infty \) and define \( r_t := \sum_{n=0}^{N} \mathbf{1}_{[L_t=n]} r_t^n \).

Proof of proposition 4. P1: Since \( \tau_0 = 0 \) a.s., we may assume we have shown \( \tau_n < \infty \) a.s. Then

\[
P \left[ \tau_n < \tau_{n+1} < \infty \right] = \lim_{m \to \infty} P \left[ \tau_n + 1/m < \tau_{n+1} \leq m \right] = \lim_{m \to \infty} P \left[ \int_{\tau_n}^{\tau_n+1/m} \lambda^n_s ds < E_{n+1} \leq \int_{\tau_n}^{1/m} \lambda^n_s ds \right] = P \left[ 0 < E_{n+1} < \infty \right] = 1
\]

which implies P1. We continue with an auxiliary

Lemma 40. \( E_n, \ldots, E_N \) are independent of \( \mathcal{F}_\infty^n \) for all \( n = 0, \ldots, N-1 \).

Its proof can be found below. P2: Using this lemma, we find first

\[
P \left[ \tau_{n+1} > t \mid \mathcal{F}_\infty^n \right] = P \left[ E_{n+1} > \int_t^{\tau_{n+1}} \lambda^n_s ds \mid \mathcal{F}_\infty^n \right] = e^{-\int_t^{\tau_{n+1}} \lambda^n_s ds}.
\]

P3: Second, since \( P \left[ \tau_{n+1} > t \mid \mathcal{F}_\infty^n \right] \) is \( \mathcal{F}_\infty^n \)-measurable, it must be equal to

\[
P \left[ \tau_{n+1} > t \mid \mathcal{F}_\infty^n \right] \text{ and it is well-known that in our setup this is equivalent to}
\]

\[
\mathbb{F}^n \overset{\text{st}}{\rightarrow} \mathbb{F}^{n+1} \text{ (see e.g. Dellacherie and Meyer (1978) (p. 71), or Bielecki and Rutkowski (2002), section 6.1.1).}
\]

P4: By property P3 it is sufficient to show that \( A_t^{n+1} := \int_0^t \mathbf{1}_{[L_t=n]} \lambda^n_s ds \) is the predictable \( \mathbb{F}^{n+1} \)-compensator of \( \mathbf{1}_{[\tau_{n+1} \leq t]} \). Further, due to lemma 37, (ii) and since \( \mathbf{1}_{[\tau_{n+1} \leq t]} - A_t^{n+1} \) is càdlàg (and locally bounded), it is sufficient to show that \( A_t^{n+1} \) is the predictable \( \mathbb{F}^n \)-compensator of \( \mathbf{1}_{[\tau_{n+1} \leq t]} \). Obviously, \( A_t^{n+1} \) is nondecreasing and \( \mathbb{F}^{n+1} \)-predictable (continuous and \( \mathbb{F}^{n+1} \)-adapted). It remains to show the martingale property. Take \( s < t \) and \( F := F^n \cap \{ \tau_{n+1} > u \} \) for some \( F^n \in \mathcal{F}_s^n \) and \( u \leq s \). Then

12Bielecki and Rutkowski (2002) show only equivalence to \( \mathbb{F}^n \overset{\text{st}}{\rightarrow} \mathbb{F}^n \). One can then pass from \( \mathbb{F}^n \) to \( \mathbb{F}^{n+1} \) by the rightcontinuity of \( \mathbb{F}^n \) as in lemma 31, (ii).
13Then \( \mathbf{1}_{[\tau_{n+1} \leq t]} - A_t^{n+1} \) is a locally bounded \( \mathbb{F}^{n+1} \)-martingale, hence an \( \mathbb{F}^N \)-martingale.
\(1_F 1_{[\tau_{n+1} > t]} = 1_F 1_{[\tau_{n+1} > s]}\). Using repeatedly P2, we deduce

\[
\mathbb{E}[1_F (1_{[\tau_{n+1} \leq t]} - 1_{[\tau_{n+1} \leq s]})] = \mathbb{E}[1_F (1_{[\tau_{n+1} > s]}) - 1_{[\tau_{n+1} > t]})]
= \mathbb{E}[1_F \int_0^t e^{-\int_y^{\tau_{n+1}} \lambda_s ds} - e^{-\int_y^{\tau_{n+1}} \lambda_s ds}] dy
= \int_0^t \mathbb{E}[1_F 1_{[\tau_{n+1} > s]} 1_{[\tau_{n+1} \leq y]} \lambda_s dy]
= \mathbb{E}[1_F 1_{[\tau_{n+1} > t]} \int_0^t \lambda_s dy]
= \mathbb{E}[1_F (A_t + 1 - A_{t+1})]
\]

By Dynkin’s lemma, this holds for every \(F \in \mathcal{F}_s \cup \mathcal{F}_{s+1}\) because \(\mathcal{F}_s \cup \mathcal{F}_{s+1}\) is generated by sets of the form \(F^m \cap \{\tau_{n+1} > u\}\) with \(F^m \in \mathcal{F}_s\) and \(u \leq s\).

**P5:** Let \(\Theta\) be an \(\mathcal{F}\)-stopping time. \(\Theta = \tau_{n+1}\) implies \(\int_0^{\Theta} \lambda_s ds = E_{n+1}\).

Hence

\[
\mathbb{P}[\Theta = \tau_{n+1}] \leq \mathbb{P}\left[\int_0^{\Theta} \lambda_s ds = E_{n+1}\right]
= \mathbb{E}\left[\mathbb{P}\left[\int_0^{\Theta} \lambda_s ds = E_{n+1} \mid \mathcal{F}_{\Theta}\right]\right] = 0
\]

because \(E_{n+1}\) has diffuse law and is independent of \(\mathcal{F}_{\Theta}\) (again by lemma 40).

**Proof of lemma 40.** For \(n = 0\) the claim is trivially satisfied. We (may) continue assuming the claim holds for \(n < N - 1\), i.e.

\[
\mathbb{P}\left[\bigcap_{k=n+1}^N \{E_k > x_k\} \cap F\right] = \mathbb{P}[F] \prod_{k=n+1}^N e^{-x_k}
\]

for all \(x_{n+1}, \ldots, x_N \geq 0\) and \(F \in \mathcal{F}_{\infty}\). We will show that the claim holds for \(n + 1\). Therefor let \(F := \bigcup_{k=n+1}^{n+1} \{F^m \cap F^0\} \cap \{\tau_{n+1} > u\}\) with \(u \in [0, \infty)\) and \(F^0 \in \mathcal{F}_{\infty}\) and define \(S_y := \int_y^{\tau_{n+1}} \lambda_s ds\), \(y \geq 0\). Then \(\{\tau_{n+1} > u\} = \{E_{n+1} > S_{u+1}\}\) and, since \(S_{u+1}\) is nonnegative and \(\mathcal{F}_{\infty}\)-measurable, there exist \(x_1, \ldots, x_m \geq 0\) and \(F_1, \ldots, F_m \in \mathcal{F}_{\infty}\) disjoint, for each \(m \in \mathbb{N}\) such
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that \( \sum_{j=1}^{m} x_j^m \mathbb{1}_{F_j^m} \uparrow S_{\mu+1} \).\footnote{See e.g. the proof of theorem 4.13 in Wheeden and Zygmund (1977).} Note that \( F^n := \{ \mathbb{1}^n \geq \mathbb{u}^n \} \cap F^0 \in \mathcal{F}_\infty^n \). Thus

\[
P \left[ \bigcap_{k=n+2}^{N} \{ E_k > x_k \} \cap F \right] \]

\[
= \lim_{m \to \infty} \sum_{j=1}^{m} P \left[ \bigcap_{k=n+2}^{N} \{ E_k > x_k \} \cap \left( \bigcap_{j=1}^{m} F_j^m \right) \cap F^n \right] e^{-x_j^m} \prod_{k=n+2}^{N} e^{-x_k}
\]

\[
= \lim_{m \to \infty} \sum_{j=1}^{m} P \left[ \left( \bigcap_{k=n+2}^{N} \{ E_k > x_k \} \right) \cap F^n \right] \prod_{k=n+2}^{N} e^{-x_k} = \prod_{k=n+2}^{N} e^{-x_k} ,
\]

i.e. the claim holds for \( n+1 \) since \( \mathcal{F}_\infty^{n+1} \) is generated by sets of the form \( F \).

\( \square \)

**Remark 41.** Enlargements by a sequence of ordered random times were also studied in Jeanblanc and Rutkowski (2000), chapter, 5.1. They let \( 9_t \) be nonincreasing \( F^0 \)-adapted processes and define

\[
\tau_n := \inf \left\{ t \geq 0; \ e^{-\Psi^n_t} \geq E \right\} \quad n = 1, 2,
\]

where \( E \) is one unit exponential random variable independent of \( \mathcal{F}_\infty^0 \). This construction is evidently different from ours. In particular in their setup \( \mathcal{F}_\infty^2 \) is generated by \( \mathcal{F}_\infty^0 \) and \( E \) whereas in our case \( \mathcal{F}_\infty^2 \) contains \( \mathcal{F}_\infty^0, E_1 \) and \( E_2 \). Also, their \( \tau_2 \) does not avoid the \( \mathbb{V}^1 \)-stopping times (cf. proposition 4).

2.4.2 Property (\( \ast \)) and the Canonical Model

Here we put ourselves back in the general setup of section 2.2. In 2.4.1 we saw that the canonical construction yields a loss process \( L_t \) such that the successive
\[ \mathbb{H} \] property (\(*\)) holds and \( \tau_1, \ldots, \tau_N \) are \( \mathbb{F}^N \)-totally inaccessible. In this part we shall prove a converse result: if \( \tau_1, \ldots, \tau_N \) are \( \mathbb{F}^N \)-totally inaccessible and (\(*\)) holds, then, under little additional regularity, \( L_t \) must be a canonical loss process.

We begin with a comment on the form of the predictable compensator \( A_t \) in our setup.

**Proposition 5.** There exist \( \mathbb{F}^n \)-predictable processes \( \Lambda^n \) for \( n = 0, \ldots, N-1 \), such that

\[ A_t = \sum_{n=0}^{N-1} \Lambda^n_{t \wedge \tau_n} - \Lambda^n_{t \wedge \tau_n}. \]  

Moreover, under assumption 3, the \( \Lambda^n \) above are unique.

The proof of this proposition follows inductively from a remark in Dellacherie et al. (1992), p. 186, that on the set \( \{ \tau_{n+1} \geq t \} \), every \( \mathbb{F}^{n+1} \)-predictable process is equal to an \( \mathbb{F}^n \)-predictable process, and this process is unique if and only if \( \mathbb{P} \left[ \tau_{n+1} \leq t \mid \mathcal{F}^n \right] < 1 \) for all \( t \geq 0 \). We provide the full proof at the end of this section.

If \( A_t \) is absolutely continuous, i.e. \( A_t = \int_0^t \lambda_s \, ds \), then the loss times \( \tau_1, \ldots, \tau_N \) are indeed \( \mathbb{F}^N \)-totally inaccessible stopping times, and for each \( n = 0, \ldots, N-1 \), there exists a unique\(^{15}\) \( \mathbb{F}^n \)-adapted \( \lambda^n \) such that

\[ A_t = \sum_{n=0}^{N-1} \int_0^t \mathbf{1}_{[L_s = n]} \lambda^n_s \, ds. \]  

I.e. \( A_t \) is of the same form as in the canonical model. Note that this follows solely from the absolute continuity of \( A_t \). Now we are ready to formulate the standing assumption for this section.

**Assumption 5.** It holds

(i) \( \mathbb{F}^0 \overset{\mathbb{H}}{\longrightarrow} \mathbb{F}^N \),

(ii) \( \tau_{n+1} \) avoids the \( \mathbb{F}^n \)-stopping times for all \( n = 0, \ldots, N-1 \).

(iii) \( A_t \) is absolutely continuous and \( \lambda^n \) in (2.7) satisfies \( \int_0^t \lambda^n_s \, ds < \infty \) a.s. for all \( t < \infty \) and \( \int_0^\infty \lambda^n_s \, ds = \infty \) a.s. for all \( n = 0, \ldots, N-1 \).

Intuitively, (ii) means that when passing from \( \mathbb{F}^n \) to \( \mathbb{F}^{n+1} \), the added random time \( \tau_{n+1} \) is “something completely new”. Also note that \( \int_0^t \lambda^n_s \, ds < \infty \)

\(^{15}\)More precisely, a unique càdlàg version.
a.s. does not follow from $E[A_t] = E[L_t] \leq N < \infty$, it is indeed an additional condition on $\lambda^n$, which “lives” beyond $\tau_{n+1}$, unlike the predictable compensator $A^{n+1}_t$ of $I_{[\tau_{n+1} \leq t]}$ which remains constant after $\tau_{n+1}$ (and satisfies $A^{n+1}_t = \int_0^t I_{[L_s = \omega_n]} \lambda^n ds$). A useful equivalence it given in

**Lemma 42.** Let $\mathbb{F}$ be any filtration satisfying the usual conditions and $\tau$ be random time and let $\mathbb{G}$ be the minimal right-continuous filtration containing $\mathbb{F}$ and making $\tau$ a stopping time. If $\tau$ is $\mathbb{G}$-totally inaccessible, then the following assertions are equivalent.

(i) $\tau$ avoids the $\mathbb{F}$-stopping times, i.e. $P[\tau = \Theta] = 0$ for all $\mathbb{F}$-stopping times $\Theta$.

(ii) The $\mathbb{F}$-martingales do not jump at $\tau$, i.e. $\Delta M_\tau = 0$ a.s. for all $\mathbb{F}$-martingales $M$.

**Proof.** (i) $\Rightarrow$ (ii) : Let (i) hold and $M$ be an $\mathbb{F}$-martingale. We recall that $M$ is càdlàg (has a càdlàg version), hence the stopping times $\Theta^k_\epsilon$ denoting the $k$th jump time of $M$ larger than $\epsilon$ satisfy $\lim_{k \to \infty} \Theta^k_\epsilon = \infty$ a.s. for all $\epsilon > 0$. Thus

$$P[\Delta M_\tau \neq 0] = \lim_{m \to \infty} P[|\Delta M_\tau| > 1/m] = \lim_{m \to \infty} \sum_{k=1}^\infty P[\tau = \Theta^1_k] = 0.$$  

(ii) $\Rightarrow$ (i) : Conversely, assume there exists an $\mathbb{F}$-stopping time $\Theta$ with $P[\Theta = \tau] > 0$. Without restriction we may assume that $\Theta$ is $\mathbb{F}$-totally inaccessible. Then there exists an $\mathbb{F}$-martingale $M$ with exactly one jump, of size one and occurring at $\Theta$ (see Protter (2004), Chapter III, Theorem 22, p. 124) and hence

$$P[\Delta M_\tau \neq 0] = P[\tau = \Theta] > 0. \quad \Box$$

**Remark 43.** The lemma above can in general not be applied to the enlargement from $\mathbb{G}^n = \mathbb{F}^0 \vee \mathbb{T}^1 \vee \cdots \vee \mathbb{T}^n$ to $\mathbb{G}^{n+1} = \mathbb{F}^0 \vee \mathbb{T}^1 \vee \cdots \vee \mathbb{T}^{n+1}$ because, as we mentioned, $\mathbb{G}^n$ does in general not satisfy the usual conditions (unless $n = 0$ or $\mathbb{F}^0$ is trivial). This is one of the reasons why we prefer to work with $\mathbb{G}^n$ instead of $\mathbb{G}^n$.

Now, we will show in three steps that every loss process satisfying assumption 5 is a canonical loss process. The end result is

---

16By Protter (2004) (Chapter III, Theorem 3, p. 104) there exist unique $\mathbb{F}$-stopping times $\Theta_a$ and $\Theta_i$ such that $\Theta_a$ is $\mathbb{F}$-accessible and $\Theta_i$ is $\mathbb{F}$-totally inaccessible, $P[\Theta_a < \infty, \Theta_i < \infty] = 0$ and $\Theta = \Theta_a \wedge \Theta_i$, $\tau < \infty$ a.s. implies $P[\Theta = \tau] = P[\Theta_a = \tau] + P[\Theta_i = \tau]$. Since $\tau$ is $\mathbb{G}$-totally inaccessible, it avoids the $\mathbb{G}$- and hence the $\mathbb{F}$-accessible stopping times. Therefore $P[\Theta_a = \tau] = 0$ and $P[\Theta_i = \tau] > 0$. 

---
Theorem 44. Under assumption 5, \( L \) is a canonical loss process with respect to \( \lambda^n \).

To prove this theorem, we will explicitly construct a sequence of random variables \( E_1, \ldots, E_N \) such that \( \tau_1, \ldots, \tau_N \) can be reconstructed using step (n2) of the canonical construction in definition 5 and then we will show that these \( E_1, \ldots, E_N \) are indeed i.i.d. unit exponentially distributed and independent of \( F^0_{\infty} \). Therefore, we carry out three preliminary steps in the following lemmata and proposition and their corollaries.

We first establish a result similar to that in lemma 30, i.e. we show how to "pass" from \( F^N_t \) to \( F^n_t \) in conditional expectations for certain payoffs.

Lemma 45. For fixed \( n \in \{0, \ldots, N-1\} \), let \( f \in L^1(F^N_t) \). Then

\[
E \left[ I_{[L \leq n]} f \mid F^n_t \right] = E \left[ e^{-\int_{\tau_n}^{T_n} \lambda^*_n dU} f \mid F^n_t \right].
\]

Proof. By dominated convergence, it is sufficient to prove the claim for \( f \) bounded.\(^{17}\) Then it is clear that \( F^N_t \) above can be replaced with \( F^{n+1}_t \) by assumption 5. We notice that \( Y^n \) is an \( F^n_t \)-local martingale of locally finite variation and with exactly one jump at \( \tau_{n+1} \). Next, we define the bounded \( F^n_t \)-, and hence \( F^{n+1}_t \)-martingale

\[
M_t := E \left[ e^{-\int_{\tau_n}^{T_n} \lambda^*_n dU} f \mid F^n_t \right] \equiv E \left[ e^{\int_{\tau_n}^{T_n} \lambda^*_n dU} f \mid F^{n+1}_t \right].
\]

\( Y^n M \) is an \( F^n_t \)-local martingale since \( [Y^n, M]^c = 0 \) and \( \Delta Y^n \Delta M = 0 \) a.s.,\(^{19}\) and \( Y^n M \) is bounded because \( |Y^n M_t| = E \left[ |f| \mid F^n_t \right] \), which is bounded. Thus \( Y^n M \) is an \( F^n_t \)-martingale which implies

\[
E \left[ I_{[L \leq n]} f \mid F^{n+1}_t \right] = E \left[ Y^n M_T \mid F^{n+1}_t \right] = Y^n M_t = 1_{[L \leq n]} E \left[ e^{-\int_{\tau_n}^{T_n} \lambda^*_n dU} f \mid F^n_t \right].
\]

\(^{17}\)Take \( f_k := (f \wedge k) \vee (-k) \in [-k, k] \) which satisfies \( |\delta f_k| \leq |f| \in L^1 \) and \( \lim_{k \to \infty} f_k = f \) a.s.

\(^{18}\)Because \( dY^0_t = -Y^0_t I_{[\tau_{n-1}, \tau_n]} dt \).

\(^{19}\)\( \Delta Y^n \Delta M_t = 1_{[L = \tau_{n+1}]} \Delta Y^n \Delta M_t = 0. \)
2.4. The Canonical Loss Process

Corollary 46. Let \( \phi_t \) be \( \mathbb{F}^n \)-predictable and \( \mathbb{P} \times \lambda^n d t \)-integrable. Then

\[
\mathbb{E} \left[ \int_t^T 1_{\{L_s = n\}} \phi_s d L_s \bigg| \mathcal{F}_t^N \right]
\]

\[
= 1_{\{L_t = n\}} \int_t^T \mathbb{E} \left[ e^{-\int_t^s \lambda^n d u} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
+ 1_{\{L_t < n\}} \int_t^T \mathbb{E} \left[ 1_{\{L_s \geq n\}} \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

Proof. It is sufficient to show the result for \( \phi \) bounded.\(^{20}\) Then, by the property of the predictable compensator, \( \mathbb{F}^{n+1} \rightarrow \mathbb{F}^N \) and the lemma above, we obtain

\[
\mathbb{E} \left[ \int_t^T 1_{\{L_s = n\}} \phi_s d L_s \bigg| \mathcal{F}_t^N \right]
\]

\[
= \int_t^T \mathbb{E} \left[ 1_{\{L_s = n\}} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
\overset{\mathbb{H}}{=} \int_t^T \mathbb{E} \left[ 1_{\{L_s \leq n\}} \phi_s \lambda^n_s - 1_{\{L_s < n\}} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
= 1_{\{L_s \leq n\}} \int_t^T \mathbb{E} \left[ e^{-\int_t^s \lambda^n d u} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
- \int_t^T \mathbb{E} \left[ 1_{\{L_s < n\}} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

Then the claim follows because

\[
1_{\{L_s \leq n\}} \int_t^T \mathbb{E} \left[ e^{-\int_t^s \lambda^n d u} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
= 1_{\{L_t = n\}} \int_t^T \mathbb{E} \left[ e^{-\int_t^s \lambda^n d u} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
+ \int_t^T \mathbb{E} \left[ 1_{\{L_s < n\}} e^{-\int_t^s \lambda^n d u} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds
\]

\[
+ \int_t^T \mathbb{E} \left[ 1_{\{L_s \leq n\}} \phi_s \lambda^n_s \bigg| \mathcal{F}_t^N \right] ds. \quad \square
\]

\(^{20}\)Note \( \mathbb{E} \left[ \int_t^T 1_{\{L_s = n\}} \phi_s d L_s \right] = \mathbb{E} \left[ \int_t^T 1_{\{L_s = n\}} \phi_s \lambda^n_s ds \right] \leq \mathbb{E} \left[ \int_0^\infty |\phi_s| \lambda^n_s ds \right] < \infty \) and apply the dominated convergence theorem.
Second, we show that under assumption 5, \( L_t \) has property P2 of the canonical model.

**Proposition 6.** For all \( T \geq 0 \), it holds

\[
E \left[ 1_{\{L_T \leq n\}} \big| \mathcal{F}_\infty \right] = E \left[ 1_{\{L_T \leq n\}} \big| \mathcal{F}_T \right] = e^{-\int_T^T \lambda_n \, dt}.
\]

**Proof.** \( E \left[ 1_{\{L_T \leq n\}} 1_F \right] = E \left[ e^{-\int_T^T \lambda_n \, dt} 1_F \right] \) for all \( F \in \mathcal{F}_T \) follows directly from setting \( t = 0 \) in lemma 45.

And an aside this shows \( \tau_N < \infty \) a.s. (cf. P1 in proposition 4) because assuming \( \tau_n < \infty \) a.s. implies (recall \( r_0 = 0 \))

\[
P \left[ \tau_{n+1} < \infty \right] = 1 - \lim_{T \to \infty} E \left[ 1_{\{L_T \leq n\}} \right] = 1 - E \left[ \lim_{T \to \infty} e^{-\int_T^T \lambda_n \, dt} \right] = 1.
\]

This yields inductively \( \tau_N < \infty \) a.s. Using \( 1_{\{L_T = n\}} = 1_{\{\tau_n \geq T\}} 1_{\{L_T \leq n\}} \), we further observe

\[
E \left[ 1_{\{L_T = n\}} \big| \mathcal{F}_\infty \right] = E \left[ 1_{\{L_T = n\}} \big| \mathcal{F}_T \right] = 1_{\{\tau_n \geq T\}} T \leq 1_{\{L_T \leq n\}} e^{-\int_T^T \lambda_n \, dt}.
\]

**Corollary 47.** Let \( \phi \) be \( \mathbb{P}^n \)-predictable and \( \mathbb{P} \times \lambda_n \, dt \)-integrable. Then

\[
E \left[ \int_0^T 1_{\{L_t = n\}} \phi_t dL_t \big| \mathcal{F}_\infty \right] = E \left[ \int_0^T e^{-\int_T^T \lambda_n \, dt} \phi_t \lambda_n \, dt \big| \mathcal{F}_\infty \right].
\]

**Proof.** It is sufficient to the prove the claim for \( \phi \) bounded.\(^{21}\) Let \( M \) be a bounded \( \mathbb{P}^n \)-martingale. Then \( \int_0^T \left( \int_0^T 1_{\{L_t = n\}} \phi_t dL_t \right) dM_t \) is an \( \mathbb{P}^{n+1} \)-martingale\(^{22}\) and it holds \( [M, \int_0^T 1_{\{L_t = n\}} \phi_t dL_t] = 0 \).\(^{23}\) Hence Itô’s rule yields

\[
E \left[ M_{\infty} \int_0^T 1_{\{L_t = n\}} \phi_t dL_t \right] = E \left[ M_T \int_0^T 1_{\{L_t = n\}} \phi_t dL_t \right] = E \left[ \int_0^T M_{\infty} 1_{\{L_t = n\}} \phi_t dL_t \right] = E \left[ \int_0^T M_T 1_{\{L_t = n\}} \phi_t dL_t \right] = E \left[ \int_0^T M_{\infty} 1_{\{L_t = n\}} \phi_t dL_t \right].
\]

\(^{21}\) By assumption.

\(^{22}\) By assumption.

\(^{23}\) By assumption.
The last equality is due to assumption 5, (ii) and lemma 42. Using the property of the predictable compensator, \( L_\cdot = L \mathbf{P} \times \mathbf{I} (dt) \)-a.e. and equation (2.8), we observe

\[
\mathbb{E} \left[ \int_0^T M_t 1_{\{L_t = n\}} \phi_t dL_t \right] = \int_0^T \mathbb{E} \left[ M_t 1_{\{L_t = n\}} \phi_t \lambda^n_t \right] dt \\
= \int_0^T \mathbb{E} \left[ M_t \mathbb{E} \left[ 1_{\{L_t = n\}} \mid \mathcal{F}_t^n \right] \phi_t \lambda^n_t \right] dt \\
= \int_0^T \mathbb{E} \left[ M_\infty 1_{\{\tau_n \leq t\}} e^{-\int_0^t \lambda^n_u du} \phi_t \lambda^n_t \right] dt \\
= \mathbb{E} \left[ M_\infty 1_{\{\tau_n < T\}} e^{-\int_{\tau_n}^T \lambda^n_u du} \phi_t \lambda^n_t dt \right].
\]

Then the claim follows because for every \( F \in \mathcal{F}_\infty^n \), \( M_t := \mathbb{E} \left[ 1_F \mid \mathcal{F}_t^n \right] \) is a bounded \( \mathbb{P}^n \)-martingale with \( M_\infty = 1_F \) a.s. \( \square \)

Third, we note that in the canonical setup of section 2.4, the exponential random variables \( E_1, \ldots, E_N \) satisfied \( E_{n+1} = \int_{\tau_n}^{\tau_{n+1}} \lambda^n_s ds \) for every \( n = 0, \ldots, N-1 \), which is hence the only reasonable choice to define \( E_1, \ldots, E_N \) here. Now we formulate the main result of this section.

**Lemma 48.** For every \( n = 0, \ldots, N-1 \), the random variable \( E_{n+1} := \int_{\tau_n}^{\tau_{n+1}} \lambda^n_s ds \) is unit exponentially distributed and independent of \( \mathcal{F}_\infty^n \).

**Proof.** Let \( x \geq 0 \) and recall that \( \tau_n < \tau_{n+1} < \infty \) a.s. Therefore we may write

\[
e^{-x \int_{\tau_n}^{\tau_{n+1}} \lambda^n_s du} = \lim_{T \to \infty} \int_0^T 1_{\{L_t = n\}} e^{-x \int_0^t \lambda^n_u du} dL_t.
\]

\( \phi_t := e^{-x \int_0^{\tau_{n+1}} \lambda^n_u du} \) is \( \mathbb{F}^n \)-predictable and by localization we may assume that \( \int_0^T \lambda^n_t dt \) is bounded (for fixed \( T \)). Then \( 1_{\{t \leq T\}} \phi_t \) is \( \mathbb{P} \times \lambda^n_t dt \)-integrable.\(^{25}\)

---

\(^{24}\)(M_0 - M_{\tau_0})1_{\{L_{\tau_0} = n\}} \phi dL_t = 1_{\{t = \tau_{n+1} < \infty\}} \Delta M_{\tau_{n+1}} \phi_{\tau_{n+1}} = 0 \) a.s.

\(^{25}\)\( e^{-x \int_0^{\tau_{n+1}} \lambda^n_u du} \lambda^n_t dt \leq \int_0^T \lambda^n_t dt. \)
Hence corollary 47 applies.

\[
E\left[e^{-x \int_{T_{n+1}}^{T_n} \lambda_t^n dt}\mid \mathcal{F}_\infty^n\right] = \lim_{T \to \infty} E\left[\int_0^T \mathbf{1}_{[L_{n+1} = n]} e^{-x \int_{T_{n+1}}^{T_n} \lambda_t^n dt} dL_t \mid \mathcal{F}_\infty^n\right]
\]
\[
= \lim_{T \to \infty} \mathbf{1}_{(\tau_n < T)} \int_{T_n}^T e^{-(x+1) \int_{T_n}^t \lambda_s^n ds} \lambda_t^n dt
\]
\[
= \lim_{T \to \infty} \mathbf{1}_{(\tau_n < T)} \frac{1}{1+x} \left(1 - e^{-(x+1) \int_{T_n}^T \lambda_s^n ds}\right)
\]
\[
= \frac{1}{1+x} \ a.s.
\]

This is the Laplace transform of a unit exponential random variable. \(\Box\)

Now it is clear that by the nonnegativity of \(\lambda_t^n\) and since \(\tau_n < \tau_{n+1}\) a.s., it holds indeed

\[
\tau_{n+1} = \inf\left\{t > \tau_n : \int_{T_n}^t \lambda_s^n ds \geq \int_{T_n}^{T_{n+1}} \lambda_s^n ds\right\} = \inf\left\{t > \tau_n : \int_{T_n}^t \lambda_s^n ds \geq E_{n+1}\right\}
\]

for all \(n = 0, \ldots, N-1\), i.e. given \(\tau_1, \ldots, \tau_n\), the next-loss time \(\tau_{n+1}\) can be reconstructed via (n2) in the canonical construction using \(E_{n+1} = \int_{T_n}^{T_{n+1}} \lambda_s^n ds\).

Further, as \(E_n\) is \(\mathcal{F}_\infty^n\)-measurable for every \(n = 1, \ldots, N\), the variables \(E_1, \ldots, E_N\) are also i.i.d.

This concludes the proof of theorem 44. \(\Box\)

**Proof of Proposition 5**

Proof of proposition 5. By Dellacherie et al. (1992), p. 186, there exists an \(\mathbb{P}^{N-1}\)-predictable process \(\Lambda^{N-1}\) with

\[
\mathbf{1}_{[t \leq T_N]} \left(\Lambda^{N-1}_t - \Lambda_t\right) = 0.
\]

In particular, this shows also that \(\Lambda^{N-1}_{T_N} = \Lambda_{T_N}\) on \([\tau_N < \infty]\). Clearly, \(\Lambda_t = \Lambda_{T_N \wedge t}\) (because \(L_t = L_{T_N \wedge t}\)) and thus

\[
\Lambda_t = \mathbf{1}_{[t \leq T_N]} \Lambda_t + \mathbf{1}_{[t > T_N]} \Lambda_{T_N} = \mathbf{1}_{[t \leq T_N]} \Lambda^{N-1}_t + \mathbf{1}_{[t > T_N]} \Lambda^{N-1}_{T_N} = \Lambda^{N-1}_{T_N \wedge t},
\]

which proves the claim for \(N = 1\). For \(N > 1\) we (may) continue by induction, assuming we have shown that there exist \(\mathbb{P}^n\)-predictable \(\Lambda^n_t\) for \(n = k, \ldots, N-1\) such that

\[
\Lambda_t = \Lambda^{k}_{T_N \wedge t} + \sum_{n=k+1}^{N-1} \Lambda^n_{T_N \wedge t} - \Lambda^n_{T_{n+1} \wedge t}.
\]
Then the $\mathbb{F}^N$-compensator of $L_{t\wedge\tau}$ is given by $\Lambda_{t\wedge\tau} = \Lambda_{k}^{t\wedge\tau}$. Again, by the argument stated above, there exists an $\mathbb{F}^{k-1}$-predictable process $\Lambda_{k}^{t\wedge\tau}$ with $1_{[t \leq \tau_k]}(\Lambda_{t}^{k-1} - \Lambda_{t}^{k}) = 0$ and hence $\Lambda_{t_k}^{k-1} = \Lambda_{t_k}^{k}$ on $[t_k < \infty]$, which implies

$$
\Lambda_{t} = \Lambda_{t_{k+1}\wedge t}^{k} + \sum_{n=k+1}^{N-1} \Lambda_{t_{n+1}\wedge t}^{n} - \Lambda_{t_{n}\wedge t}^{n} \quad \text{for all } t \geq 0.
$$

Repeating this step until $k = 1$ proves existence\(^{26}\). Further each $\Lambda_{k}^{t\wedge\tau}$ is unique if (and only if) $P \left[ \tau_k \leq t \mid \mathcal{F}_{t}^{k-1} \right] < 1$ for all $t \geq 0$, as shown in Dellacherie et al. (1992), p. 186.

\[\square\]

### 2.5 The Conditional Markov Loss Model

So far we have shown that the canonical construction of $L$ yields a concrete and vastly general stochastic representation of the loss process. Throughout this section we work under

**Assumption 6.** $L$ is a canonical loss process with respect to $\lambda^n$.

Next we will tackle some of the pricing problems that were motivated in section 2.2.1. The key step there was in equation (2.1) which looked to express $1_{[L_{t} = n]} P_{m}(t, T) = 1_{[L_{t} = n]} \tilde{P}_{n,m}(t, T)$ for some suitably defined $\tilde{P}_{n,m}(t, T)$. Recall that the canonical construction of $L$ has the successive $\mathbb{H}$ property (\(\ast\)) and decomposition (2.4). In order to find a decomposition (2.1), we will explicitly compute the quantities $\tilde{P}_{n,m}(t, T) \in L(\mathcal{F}^n_{t})$, but it will come in handy to study first the $\mathcal{F}^n_{\infty}$-conditional transition probabilities

$$
\tilde{P}_{n,m}^\infty(t, T) := 1_{[L_{t} \geq n]} \frac{E \left[ 1_{[L_{t} = n]} 1_{[L_{T} = m]} \mid \mathcal{F}^n_{\infty} \right]}{E \left[ 1_{[L_{t} = n]} \mid \mathcal{F}^n_{\infty} \right]},
$$

\(^{26}\)Recall $\tau_0 = 0$ a.s.
With proposition 4, P2 and noting that \( I_{\{L_t=n\}} I_{\{L_T=m\}} = I_{\{L_t\geq n\}} I_{\{L_{t+1}\geq T\}} \) for all \( T \geq t \), it is easy to see that

\[
\tilde{P}_{n,m}^\infty(t, T) = I_{\{L_t\geq n\}} e^{-\int_t^T \lambda_n^m ds},
\]

(2.9)

hence \( I_{\{L_t=m\}} P_m(t, T) = I_{\{L_t=m\}} \tilde{P}_{m,m}(t, T) = I_{\{L_t=m\}} E \left[ e^{-\int_t^T \lambda_m^m ds} \mid F_t^m \right] \).

Using (**) we observe that if \( \lambda^m \) were \( \mathbb{P}^0 \)-adapted, then on the set \( \{ L_t = m \} \), \( P_m(t, T) \) would indeed be equal to

\[
E \left[ e^{-\int_t^T \lambda_n^m ds} \mid F_t^m \right] \mathbb{H} E \left[ e^{-\int_t^T \lambda_n^m ds} \mid F_t^0 \right] \in L(F_t^0),
\]

which is what we are after. Therefore we make the following additional assumption in the canonical construction of \( L \).

**Assumption 7.** The processes \( \lambda^n \) in (n1) are \( \mathbb{P}^0 \)-adapted for each \( n \).

In this case, \( L \) can be interpreted as a conditional finite-state Markov chain, conditional on \( F^0_\infty \). This assumption, although more restrictive, does still not rule out contagion (cf. (2.5)). It is useful to define the \( F^0_\infty \) - and \( F_t^0 \)-conditional transition probabilities

\[
\tilde{P}_{n,m}^\infty(t, T) := \frac{E \left[ I_{\{L_t=m\}} I_{\{L_T=m\}} \mid F^0_\infty \right]}{E \left[ I_{\{L_t=m\}} \mid F^0_\infty \right]} \quad \text{and}
\]

(2.10)

\[
\tilde{P}_{n,m}(t, T) := \frac{E \left[ I_{\{L_t=m\}} I_{\{L_T=m\}} \mid F_t^0 \right]}{E \left[ I_{\{L_t=m\}} \mid F_t^0 \right]}
\]

and loss probabilities \( \tilde{P}_n(t) := E \left[ I_{\{L_t=n\}} \mid F^0_t \right] \mathbb{H} E \left[ I_{\{L_t=n\}} \mid F^0_\infty \right] \). We begin with an intuitive result, which is due to the interpretation of \( L \) as an \( F^0_\infty \)-conditional Markov chain.

**Lemma 49.** Define \( e_n^\infty.t, T := e^{-\int_t^T \lambda_n^m ds} \) for each \( n \). Then it holds that

\[
\tilde{P}_{n,m}^\infty(t, T) = I_{\{m=n\}} e_n^\infty.t, T + \int_t^T e_n^\infty.n, T \tilde{P}_{n,m-1}^\infty(t, u) \lambda_u^{m-1} du
\]

and \( \tilde{P}_{n,m}^\infty(t, T) = I_{\{L_t\geq n\}} \tilde{P}_{n,m}^\infty(t, T) \).

A precise proof can be found at the end of this section. The formulae for \( \tilde{P}_{n,m}(t, T) \) are given in
2.5. The Conditional Markov Loss Model

Corollary 50. It holds

\[ \tilde{P}_{n,m}(t, T) = \mathbb{E} \left[ \prod_{i=0}^{\infty} \mathbf{1}_{[L_i = n]} \mathbf{1}_{[L_T = m]} \, | \mathcal{F}_T^0 \right] + \int_t^T \mathbb{E} \left[ \mathbf{1}_{[L_t = n]} \mathbf{1}_{[L_T = m]} \mathcal{F}_T^0 \right] du \]

and we have the intuitive relationship \( \tilde{P}_{n,m}(t, T) = 1_{[L_t \geq n]} \tilde{P}_{n,m}(t, T) \).

Lemma 49 and its corollary also show that \( \tilde{P}_{n,m}(t, T) \) is continuous and \( \tilde{P}_{n,m}(t, T) \) is right-continuous in \( T \) a.s.

Proof of corollary 50. The formula for \( \tilde{P}_{n,m}(t, T) \) follows immediately from the lemma and

\[ \tilde{P}_{n,m}(t, T) = \mathbb{E} \left[ \mathbf{1}_{[L_t = n]} \, | \mathcal{F}_T^0 \right] \]

and then the relationship \( \tilde{P}_{n,m}(t, T) = 1_{[L_t \geq n]} \tilde{P}_{n,m}(t, T) \) is due to

\[ \tilde{P}_{n,m}(t, T) \equiv \mathbb{E} \left[ \tilde{P}_{n,m}(t, T) \, | \mathcal{F}_T^0 \right] = 1_{[L_t \geq n]} \tilde{P}_{n,m}(t, T) \]

The main consequence of lemma 49 and its corollary is that in the conditional Markov set-up the CDO building blocks \( P_m \) satisfy

\[ P_m(t, T) = \sum_{n=0}^{N} 1_{[L_t = n]} \tilde{P}_{n,m}(t, T), \quad (2.11) \]

i.e. restricted to any set \( [L_t = n] \) they depend only on the “default-free” information \( \mathcal{F}_T^0 \). This is the desired relationship that was shown to be very useful to price exotic credit portfolio derivatives in section 2.2.1.

Derivatives Pricing in the Conditional Markov Model

Coming back to the example discussed in 2.2.1, we may conclude that the price at \( t = 0 \) of the derivative with payoff \( X \) is

\[ \mathbb{E} [X] = \sum_{n=0}^{N} \mathbb{E} \left[ \tilde{P}_n(t) G(n, \tilde{P}_{n,m}(t, T), Z) \right]. \]
Since $L_0 = 0$ a.s., we have of course that $\hat{P}_n(t) = \hat{P}_0^\infty(0, t)$. Pricing at a general time $s < t$ however, must be done using lemma 30. Recall definition (2.3) and note that in the conditional Markov model, we have

$$\hat{P}_{n,m}(T; t, T) = 1_{\{L_s \geq n\}} \mathbb{E}\left[ \frac{1_{\{L_s = n\}} 1_{\{L_T = m\}}}{\mathbb{E}\left[ 1_{\{L_s = n\}} \big| \mathcal{F}_T^m \right]} \bigg| \mathcal{F}_t^n \right] \mathbb{E}\left[ \frac{1_{\{L_s = n\}} 1_{\{L_T = m\}}}{\mathbb{E}\left[ 1_{\{L_s = n\}} \big| \mathcal{F}_T^m \right]} \bigg| \mathcal{F}_t^n \right] \equiv \hat{P}_{n,m}^\infty(t, T).$$

In the conditional Markov loss set-up lemma 30 reads

**Proposition 7.** For every $Y \in L^1(\mathcal{F}_T^0)$ it holds

$$1_{\{L_s = n\}} \mathbb{E}\left[ Y 1_{\{L_T = m\}} \bigg| \mathcal{F}_T^n \right] = 1_{\{L_s = n\}} \mathbb{E}\left[ Y \hat{P}_{n,m}^\infty(t, T) \bigg| \mathcal{F}_T^0 \right].$$

**Proof.** Using the particular form of $\hat{P}_{n,m}(T; t, T)$ and lemma 30, we deduce

$$1_{\{L_s = n\}} \mathbb{E}\left[ Y 1_{\{L_T = m\}} \bigg| \mathcal{F}_T^n \right] \equiv 1_{\{L_s = n\}} \mathbb{E}\left[ Y \hat{P}_{n,m}(T; t, T) \bigg| \mathcal{F}_T^m+1 \right]$$

$$= 1_{\{L_s = n\}} \mathbb{E}\left[ Y \hat{P}_{n,m}^\infty(t, T) \bigg| \mathcal{F}_T^{m+1} \right]$$

$$\equiv 1_{\{L_s = n\}} \mathbb{E}\left[ Y \hat{P}_{n,m}^\infty(t, T) \bigg| \mathcal{F}_T^0 \right].$$

In order to price the claim $X$ due at $t$ at time $s < t$, we first decompose the payoff $X$ as follows

$$X = \sum_{k=0}^N \sum_{n=0}^N 1_{\{L_s = k\}} 1_{\{L_n = n\}} G(n, \hat{P}_{n,m}(t, T), Z).$$

Since $G(n, \hat{P}_{n,m}(t, T), Z) \in L(\mathcal{F}_T^0)$, proposition 7 applies and the price at time $s < t$ of the claim $X$ due at $t$ is given by

$$\mathbb{E}\left[ X \bigg| \mathcal{F}_s \right] = \sum_{k=0}^N \sum_{n=0}^N 1_{\{L_s = k\}} \mathbb{E}\left[ 1_{\{L_n = n\}} G(n, \hat{P}_{n,m}(t, T), Z) \bigg| \mathcal{F}_s \right] \quad (2.12)$$

$$= \sum_{k=0}^N \sum_{n=0}^N 1_{\{L_s = k\}} \mathbb{E}\left[ \hat{P}_{s,k}^\infty(s, t) G(n, \hat{P}_{n,m}(t, T), Z) \bigg| \mathcal{F}_s^0 \right].$$

Apart from the current loss level $L_s$, all variables needed to price this derivative are $\mathcal{F}_T^0$-measurable. And again, all reference to the loss process inside the conditional expectations brackets above has been removed.
A Remark on the SPA Model

In a different approach, Sidenius et al. (2005) consider the quantities
\[ p_x(t, T) \ := \ E \left[ 1_{\{L_T \leq x\}} \mid \mathcal{F}_T^{SPA} \right], \quad x \geq 0 \]  
(2.13)
for a suitable subfiltration \( \mathbb{F}^{SPA} \subseteq \mathbb{F}^0 \). They suggest that \( p_x(t, \cdot) \), being positive and nonincreasing in \( T \) (and \( \mathbb{F}^{SPA} \)-adapted), is parameterized
\[ p_x(t, T) := e^{-\int_t^T \lambda_x(s, t) ds} \]
with nonnegative \( \mathbb{F}^{SPA} \)-adapted curves \( \lambda_x(t, \cdot) \), and they argue that under regularity there exists a consistent loss process \( L \) such that (2.13) indeed holds.

When \( L \) is \([0, \ldots, N]\)-valued as in our setup and \( \mathbb{F}^{SPA} = \mathbb{F}^0 \), then \( p_x(t, \cdot) \) obviously depends on \( P_m(t, \cdot) \), and hence on \( \tilde{P}_{n,m}(t, \cdot) \) via
\[
\begin{align*}
\ p_x(t, T) &= \sum_{m=0}^{[x]} \sum_{n=0}^{\lfloor x \rfloor} \ E \left[ P_m(t, T) \mid \mathcal{F}_T^0 \right] \\
&= \sum_{m=0}^{[x]} \sum_{n=0}^{\lfloor x \rfloor} \ E \left[ 1_{\{L_n = n\}} \mid \mathcal{F}_T^0 \right] \tilde{P}_{n,m}(t, T) \\
&= \sum_{m=0}^{[x]} \sum_{n=0}^{\lfloor x \rfloor} \tilde{P}_n(t) \tilde{P}_{n,m}(t, T). \quad (2.14)
\end{align*}
\]
But the \( P_m(t, \cdot) \) and \( \tilde{P}_{n,m}(t, \cdot) \) can in general not be recovered from the \( p_x(t, \cdot) \) and the loss history \( \sigma(L_s, s \leq t) \),\(^{27}\) which is evident because they basically project our \( N(N + 1)/2 \) building blocks \( \tilde{P}_{n,m}, 0 \leq n \leq m \leq N \), down to \( N \) building blocks \( p_x, x = 1, \ldots, N \). It should be noticed that for a derivative with payoff \( X = G(L_t, P_m(t, T), Z) \) as discussed above and in (2.2.1), in general neither the payoff
\[
X = \sum_{n=0}^{N} 1_{\{L_n = n\}} G(n, \tilde{P}_{n,m}(t, T), Z)
\]

nor its \( \mathcal{F}_T^0 \)-projection
\[
E \left[ X \mid \mathcal{F}_T^0 \right] = \sum_{n=0}^{N} \tilde{P}_n(t) G(n, \tilde{P}_{n,m}(t, T), Z)
\]

\(^{27}\)Unless at time \( t = 0 \) since \( L_0 = 0 \) a.s.
Finally, we show that derivatives. The Conditional Markov Property

because the predictable compensator

Forward ODE

Proof of lemma 49. As an aside of lemma 49 we also observe the \( F^0_\infty \)-conditional Kolmogorov forward ODE

As an aside of lemma 49 we also observe the \( F^0_\infty \)-conditional Kolmogorov forward ODE

Proofs of Lemmata in this Section

Proof of lemma 49. Note \( \hat{P}^\infty_n(t) := E \left[ \mathbf{1}_{\{L_T=n\}} \mid F^n_\infty \right] = 1_{\{L_T=n\}} e^{-\int_0^T \lambda_n ds} \) follows from P2 in proposition 4. Let \( n \) be fixed. For \( m = n \) the result follows from (2.9) and

For \( m > n \), we notice that \( E \left[ \mathbf{1}_{\{L_T=m\}} \mathbf{1}_{\{L_T \leq m-1\}} \mid F^m_\infty \right] \) is continuous in \( T \). We may assume the theorem holds for \( m = n, \ldots, M - 28 \)

Right-continuous by the right-continuity of \( L \), and, left-continuous because \( L_- \) exists and

because the predictable compensator \( A^m \) of \( 1_{\{t_m \leq T\}} \) is continuous a.s.
1. Then the random variable $\mathbf{1}_{\{L_t=n\}} \tau_M$ admits a right-continuous density because the right derivative

$$
\partial_T E \left[ \mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{\tau_M \leq T\}} \mid \mathcal{F}_n^\infty \right] = -\partial_T E \left[ \mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{L_T \leq M-1\}} \mid \mathcal{F}_n^\infty \right] \\
= -\tilde{P}_n^\infty (t) \sum_{m=n}^{M-1} \partial_T \tilde{P}_n^{\infty, m} (t, T) \\
= -\tilde{P}_n^\infty (t) \sum_{m=n}^{M-1} \partial_T \tilde{P}_n^{\infty, m} (t, T) \\
= \tilde{P}_n^\infty (t) \tilde{P}_n^{\infty, M-1} (t, T) \lambda_T^{M-1}
$$

exists and is rightcontinuous (simply apply the right derivative operator $\partial_T$ to $\tilde{P}_n^\infty (t, T)$ for $m = n, \ldots, M-1$). Together with the fact that $L_t = n$ implies $\tau_M > t$, we obtain

$$
E \left[ \mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{L_T = M\}} \mid \mathcal{F}_n^\infty \right] = E \left[ \mathbf{1}_{\{L_t=n\}} E \left[ \mathbf{1}_{\{L_T = M\}} \mid \mathcal{F}_n^\infty \right] \mid \mathcal{F}_n^\infty \right] \\
= E \left[ \mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{L_T \geq M\}} e^{-\int_t^T \lambda_u^M du} \mid \mathcal{F}_n^\infty \right] \\
= E \left[ \mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{L_T \geq M\}} \mid \mathcal{F}_n^\infty \right] \\
= \int_t^T e^{-\int_t^T \lambda_u^M du} \partial_u E \left[ \mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{L_T \geq u\}} \mid \mathcal{F}_n^\infty \right] du \\
= \tilde{P}_n^\infty (t) \int_t^T e^{-\int_t^u \lambda_s^M ds} \tilde{P}_n^{\infty, M-1} (t, u) \lambda_u^{M-1} du.
$$

Hence on the set $\{L_t \geq n\}$, \( \tilde{P}_n^\infty (t, T) \) is equal to the $\mathcal{F}_T^0$-measurable random variable $Z_M := \int_t^T e^{-\int_t^u \lambda_s^M ds} \tilde{P}_n^{\infty, M-1} (t, u) \lambda_u^{M-1} du$. Finally, we must show that indeed $Z_M = \tilde{P}_n^\infty (t, T)$. By definition, we have

$$
\tilde{P}_n^\infty (t, T) = \frac{E \left[ \tilde{P}_n^\infty (t) \tilde{P}_n^{\infty, M} (t, T) \mid \mathcal{F}_n^0 \right]}{E \left[ \tilde{P}_n^\infty (t) \mid \mathcal{F}_n^0 \right]} = E \left[ \tilde{P}_n^\infty (t) Z_M \mid \mathcal{F}_n^0 \right] = Z_M.
$$

Then the claim follows by induction in $M$. 

\footnote{Note that for all $T \geq 0$ we have $\mathbf{1}_{\{L_t=n\}} \tau_M > T$ if and only if $\mathbf{1}_{\{L_t=n\}} \mathbf{1}_{\{L_T \leq M-1\}} = 1$.}

\footnote{Note that $I_n (t) = \mathbf{1}_{\{L_t=n\}} \tau_n (t)$.}
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Proof of lemma 51. We distinguish the cases \( n_k < n \) and \( n_k = n \). For \( n_k < n \), the numerator on the lefthand side simplifies as follows

\[
E \left[ I_{\{L_T=m\}} I_{\{L_i=n\}} I_{\{L_{u_k}=n_{k}\}} \cdots I_{\{L_{u_1}=n_1\}} \mid \mathcal{F}_\infty^0 \right]
\]

\[
= E \left[ E \left[ I_{\{L_T=m\}} I_{\{L_i=n\}} \mid \mathcal{F}_\infty^n \right] I_{\{L_{u_k}=n_{k}\}} \cdots I_{\{L_{u_1}=n_1\}} \mid \mathcal{F}_\infty^0 \right]
\]

\[
= \tilde{P}_{n,m}(t, T) E \left[ E \left[ I_{\{L_T=n\}} \mid \mathcal{F}_\infty^n \right] I_{\{L_{u_k}=n_{k}\}} \cdots I_{\{L_{u_1}=n_1\}} \mid \mathcal{F}_\infty^0 \right].
\]

and the denominator satisfies

\[
E \left[ I_{\{L_T=n\}} I_{\{L_{u_k}=n_{k}\}} \cdots I_{\{L_{u_1}=n_1\}} \mid \mathcal{F}_\infty^0 \right]
\]

\[
= E \left[ E \left[ I_{\{L_T=n\}} \mid \mathcal{F}_\infty^n \right] I_{\{L_{u_k}=n_{k}\}} \cdots I_{\{L_{u_1}=n_1\}} \mid \mathcal{F}_\infty^0 \right].
\]

In the case \( n_{k-1} < n_k = n \), we must notice that on the set \( \{L_T = n\} \), we have that \( I_{\{L_{u_k}=n_{k}\}} = I_{\{L_{u_1}=n_1\}} \) which is \( \mathcal{F}_\infty^n \)-measurable. Then the proof is analogous.

2.6 The Conditional Markov Forward Loss Model

As we argued, the aim of the frameworks presented here and in chapter 1 is to explain (match) a large set of STCDO spreads in the market, rather than to price these. A fitted version of the model can then be used to price more exotic derivatives with the tranches as underlyings. In this section we work again under

Assumption 8. \( L \) is a conditional Markov loss process with respect to \( \lambda^n \).

Unlike the forward modelling approach taken in chapter 1, here the difficulty in applications will be to find a concrete spot model \( \lambda^0, \ldots, \lambda^{N-1} \) such that the full term structure of STCDOs can indeed be fitted with the model-implied \( \tilde{P}_{n,m}(t, T) \) and the resulting \( P_{n,m}(t, T) \). We also recall that the evaluation of \( \tilde{P}_{n,m}(t, T) \) in the conditional Markov model involves a conditional expectation operation, conditional on \( \mathcal{F}_t^n \) (cf. corollary 50). That means in particular given a model for \( \lambda^0, \ldots, \lambda^{N-1} \), in order to price a derivative with formula (2.12) using MC, one effectively has to simulate the model \( (\lambda^0, \ldots, \lambda^{N-1}) \) until the maturity \( T \) of the underlying (not only until the maturity \( t < T \) of the derivative).

We do not need to consider \( n_{k-1} = n_k = n \) since \( I_{\{L_{u_k-1}=n\}} I_{\{L_{u_k}=n\}} I_{\{L_T=n\}} = I_{\{L_{u_k-1}=n\}} I_{\{L_T=n\}} \).

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By contrast, if one could directly model $P_{n,m}(t, T)$ at time $t$, e.g. with a suitable forward model, then the MC-simulation must be run only until the option maturity $t < T$, and furthermore the model could be automatically fitted to the full term structure of market STCDO prices at any point in time.

In this section we will prove existence of a forward representation for $P_{n,m}(t, T)$ within the conditional Markov loss framework using a set of $\mathbb{F}^0$-adapted loss-contingent forward loss rates $\tilde{\lambda}_{n,m}(t, T)$. These rates can be viewed as $T$-forward loss rates, conditional on $L_t = n$ and $L_T = m$ and given $\mathbb{F}_t^0$.

Unfortunately, it turns out that setting directly up a model for the rates $\tilde{\lambda}_{n,m}(t, T)$ involves two major almost unsurmountable problems. First, in general it is not possible to find a conditional Markov loss process $L$ which is consistent with $\tilde{\lambda}_{n,m}(t, T)$ (in a sense made precise below), and second, the initial term structures $\tilde{\lambda}_{n,m}(0, T)$ needed to calibrate such a model cannot be directly implied from market STCDO prices.

Nevertheless we can relate the rates $\tilde{\lambda}_{n,m}(t, T)$ to the forward loss rates $F_m(t, T)$ of the model presented in chapter 1 and in Schönbucher (2006). This way we obtain an important warning concerning too restrictive (and nonsensical) modelling assumptions for the rates $F_m(t, T)$.

### 2.6.1 $\mathbb{F}^0$-Conditional Forward Loss Rates

First we impose an additional integrability condition on the intensities $\lambda^n$ in

**Assumption 9.** For all $n$, $\lambda^n$ are (strictly) positive and it holds for all $t < \infty$

$$\lim_{\delta \downarrow 0} \mathbb{E} \left[ \sup_{n \in [t, t+\delta]} \{ \lambda^n \} \right] < \infty.$$ 

Then it is ensured that $\tilde{P}_{n,m}(t, T) > 0$, and hence $P_{n,m}(t, T) > 0$ a.s. for all $n \leq m$ and $T > t$, and in particular, $\lambda^n_t$ and $\int_0^t \lambda^n_s ds$ are integrable for every $n$ and $t$. We may, hence, make the following

**Definition 6.** For every $n \leq m$ and every $t < T$, we let

$$\tilde{\lambda}_{n,m}(t, T) := \frac{\mathbb{E} \left[ \tilde{P}_{n,m}(t, T) \lambda^m_T \mid \mathbb{F}_t^0 \right]}{\mathbb{E} \left[ \tilde{P}_{n,m}(t, T) \mid \mathbb{F}_t^0 \right]}.$$ 

For $m = n$, we naturally extend the definition to $T = t$, i.e. $\tilde{\lambda}_{n,n}(t, t) = \lambda^n_t$. 
\( \tilde{\lambda}_{n,m}(t, T) \) can be seen as \( \mathcal{F}_T^0 \)-conditional \( T \)-forward loss rates given \( L_t = n \) and \( L_T = m \) because one can easily check that

\[
\tilde{\lambda}_{n,m}(t, T) = \mathbf{E} \left[ \mathbf{1}_{[L_T = m]} \mathbf{1}_{[L_T = m]} \tilde{\lambda}_{n,m}^m \bigg| \mathcal{F}_T^0 \right] / \mathbf{E} \left[ \mathbf{1}_{[L_T = m]} \bigg| \mathcal{F}_T^0 \right].
\] (2.16)

**Remark 52.** It is important to notice that the rates \( \tilde{\lambda}_{n,m}(t, T) \), being projections of (the true conditional loss rate) \( \lambda_{T}^{n,m} \) with respect to different \( T \)-forward loss transition measures (depending on \( n \)), do i.g. differ over different values of \( n \) (for fixed \( T > t \) and \( m < N \)).

Also note that \( \tilde{\lambda}_{n,m}(t, T) \) is right-continuous in \( T \) by construction for \( T > t \) (see remark 53 for the case \( T = t \)). \( \tilde{\partial}_T \) denotes the right-derivative operator. The proposition below holds irrespective of how the rates \( \tilde{\lambda}_{n,m}(t, T) \) are defined for \( m > n \).\(^32\)

**Proposition 8.**

\[
\tilde{\partial}_T \tilde{P}_{n,m}(t, T) = -\tilde{P}_{n,m}(t, T) \tilde{\lambda}_{n,m}(t, T) + \tilde{P}_{n,m-1}(t, T) \tilde{\lambda}_{n,m-1}(t, T)
\]

The proof of this proposition follows from the conditional Kolmogorov ODE (2.15) and the integrability condition in assumption 9 and can be found at the end of this subsection. If \( \int_t^T \tilde{\lambda}_{n,m}(t, u) \, du < \infty \) a.s. for all \( n, m \) and \( T < \infty \), then the solution of the ODE in proposition 8 is given by

\[
\tilde{P}_{n,m}(t, T) = \mathbf{1}_{[n=m]} e^{\tilde{\tilde{\lambda}}_{n,m}^T T} + \mathbf{1}_{[n<m]} \int_t^T e^{\tilde{\tilde{\lambda}}_{n,m}^u T} \tilde{P}_{n,m-1}(t, u) \tilde{\lambda}_{n,m-1}(t, u) \, du
\] (2.17)

where \( \tilde{\tilde{\lambda}}_{n,m}^T := e^{-\int_t^T \tilde{\tilde{\lambda}}_{n,m}(t, v) \, dv} \) and \( \tilde{\tilde{\lambda}}_{n,m}^u^T := e^{-\int_t^u \tilde{\tilde{\lambda}}_{n,m}(t, v) \, dv} \). Coming from a general spot model for \( \tilde{\tilde{\lambda}}_{0}^n, \ldots, \tilde{\tilde{\lambda}}_{n}^{N-1} \), it is difficult to give a rule telling when \( \int_t^T \tilde{\tilde{\lambda}}_{n,m}(t, u) \, du < \infty \) a.s. But at least, if the \( \tilde{\lambda}_n \) are bounded, then so are \( \tilde{\tilde{\lambda}}_{n,m}(t, T) \), and hence \( \int_t^T \tilde{\tilde{\lambda}}_{n,m}(t, u) \, du \) remains indeed finite for all \( n \leq m \leq N \) and \( T < \infty \).\(^33\) Now we come back to the right-continuity of \( \tilde{\lambda}_{n,m}(t, T) \) in \( T \) at the short end \( T = t \).

---

\(^{32}\)With the convention \( 0 \cdot \infty = 0 \).

\(^{33}\)Let \( \overline{\lambda} \) be the bound. Then \( \tilde{\tilde{\lambda}}_{n,m}(t, T) = \mathbf{E} \left[ \tilde{P}_{n,m}^m(t, T) \bigg| \mathcal{F}_T^0 \right] / \mathbf{E} \left[ \mathbf{1}_{[L_T = m]} \bigg| \mathcal{F}_T^0 \right] \leq \frac{\mathbf{E} \left[ \tilde{P}_{n,m}^m(t, T) \bigg| \mathcal{F}_T^0 \right]}{\mathbf{E} \left[ \mathbf{1}_{[L_T = m]} \bigg| \mathcal{F}_T^0 \right]} = \overline{\lambda} \).
2.6. The Conditional Markov Forward Loss Model

Remark 53. Let $\lambda^n$ be bounded for all $n$ and assume that for all $n$, $m$ and $t$ we have $\lim_{t \to 0} \mathbb{E} \left[ \sup_{u \in (t,t+\delta)} \frac{\tilde{p}^\infty_{n,m}(t,u)}{P_{n,m}(t,u)} \lambda^n_T \right] < \infty$. This holds e.g. if $\lambda^n$ is bounded and bounded away from zero for all $n$.\(^{34}\) Then we have indeed

$$
\lim_{T \to 0} \mathbb{E} \left[ \frac{\tilde{p}^\infty_{n,m}(t,T)}{P_{n,m}(t,T)} \lambda^n_T \right] = \mathbb{E} \left[ \lim_{T \to 0} \frac{\tilde{p}^\infty_{n,m}(t,T)}{P_{n,m}(t,T)} \lambda^n_T \right]
$$

for all $n$, $m$ and $t$. In this case $\tilde{\lambda}_{n,m}(t,t) := \lim_{T \to 0} \tilde{\lambda}_{n,m}(t,T) = \lambda^n_T$ can be used as definition of $\tilde{\lambda}_{n,m}(t,t)$ for $m > n$. As we mentioned, this does not affect the validity of the ODE in proposition 8.

Relation to Schönbucher (2006) and Chapter 1

At this point, it is useful to recall that the $\mathbb{P}^N$-conditional forward loss probabilities $P_m(t,T)$ in the Schönbucher (2006) model and the model presented in chapter 1\(^{35}\) are parameterized

$$
P_m(t,T) = \mathbb{1}_{(L_i = m)} e^{-\int_t^T F_m(t,u) du} + \int_t^T e^{-\int_u^T F_m(t,u) du} P_{m-1} F_{m-1}(t,u) du
$$

using $\mathbb{P}^N$-conditional forward loss rates $F_m(t,T)$. Thus (2.11) implies that in the conditional Markov loss model $F_m(t,T)$ satisfies\(^{36}\)

$$
F_m(t,T) = \sum_{n=0}^m \mathbb{1}_{(L_i = n)} \tilde{\lambda}_{n,m}(t,T)
\tag{2.18}
$$

\(^{34}\)Let $\tilde{\lambda}$ be these bounds. Then $\tilde{p}^\infty_{n,m}(t,t+\delta) \in \left[ e^{-\tilde{\lambda} \delta} \frac{(\tilde{\lambda} T)^{m-n}}{(m-n)!}, \frac{(\tilde{\lambda} T)^{m-n}}{(m-n)!} \right]$, which implies $0 < e^{-\tilde{\lambda} \delta} (\tilde{\lambda} T)^{m-n} \leq \tilde{p}^\infty_{n,m}(t,t+\delta) \leq e^{-\tilde{\lambda} \delta} (\tilde{\lambda} T)^{m-n} < \infty$.

\(^{35}\)Assuming zero interest rates or by normalizing $P_m(t,T) := P_m(t,T) / B(t,T)$, cf. remark 7.

\(^{36}\)We mean the specification of $F_m$ with $F_m(t,T) = \mathbb{1}_{(L_i \leq m)} F_m(t,T)$. 

Recalling remark 52, we notice that \( F_m(t, T) \) is hence in general not continuous at the loss times \( t = \tau_n \). In Schönbucher (2006) and in the forward loss model of chapter 1 we would feel tempted to directly model \( F_m(t, T) \) as continuous processes satisfying \( dF_m(t, T) = \mu_m(t, T)dt + \sigma_m(t, T)dW_t \). Given (2.18), this seems too restrictive because these dynamics can hardly stem from a canonical (or a conditional Markov) loss model, unless \( \lambda^n \) are deterministic functions. In contrast, if \( \mathbb{F}^0 \) is e.g. a Brownian filtration, assuming \( \tilde{\lambda}_{m,n}(t, T) \) continuous is reasonable (cf. definition 6).

**Proof of Proposition 8**

**Proof of proposition 8.** Recall the formula for \( \tilde{P}_{m,n}^\infty \) in lemma 49. Since \( 1 - e^{-x} \leq x \) for \( x \geq 0 \), we first estimate

\[
\left| \tilde{P}_{m,n}^\infty(t, T + \delta) - \tilde{P}_{n,m}^\infty(t, T) \right| \leq \int_T^{T+\delta} e^{-\int_t^T \lambda_u^m ds} \tilde{P}_{n,m-1}^\infty(t, u) \lambda_u^{m-1} du
\]

and second, using in particular the triangle inequality, we find that for \( m > n \)

\[
\frac{1}{\delta} \left| \tilde{P}_{n,m}^\infty(t, T + \delta) - \tilde{P}_{n,m}^\infty(t, T) \right| \leq \frac{1}{\delta} e^{-\int_T^{T+\delta} \lambda_u^m ds} \int_T^{T+\delta} e^{-\int_t^T \lambda_u^m ds} \tilde{P}_{n,m-1}^\infty(t, u) \lambda_u^{m-1} du + \frac{1}{\delta} \left( 1 - e^{-\int_T^{T+\delta} \lambda_u^m ds} \right) \tilde{P}_{n,m-1}^\infty(t, T)
\]

the latter of which is integrable for \( \delta \) sufficiently small by assumption. This implies that \( \frac{1}{\delta} \left| \tilde{P}_{n,m}^\infty(t, T + \delta) - \tilde{P}_{n,m}^\infty(t, T) \right|, \delta > 0, \) is uniformly bounded by an integrable random variable (separately for every \( T \)). Using dominated convergence, we may now compute the right-derivative

\[
\partial_T \tilde{P}_{n,m}(t, T) = \partial_T \mathbb{E} \left[ \tilde{P}_{n,m}^\infty(t, T) \mid \mathcal{F}_t^0 \right]
\]

\[
= \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{E} \left[ \tilde{P}_{n,m}^\infty(t, T + \delta) - \tilde{P}_{n,m}^\infty(t, T) \mid \mathcal{F}_t^0 \right]
\]
2.6. The Conditional Markov Forward Loss Model

\[
\begin{align*}
&= \mathbb{E} \left[ \lim_{\delta \downarrow 0} \frac{1}{\delta} \left( \hat{P}_{n,m}^\infty (t, T + \delta) - \hat{P}_{n,m}^\infty (t, T) \right) \bigg| \mathcal{F}_t^0 \right] \\
&= \mathbb{E} \left[ \partial_T \hat{P}_{n,m}^\infty (t, T) \bigg| \mathcal{F}_t^0 \right]
\end{align*}
\]

For \( T > t \), the conditional Kolmogorov ODE (2.15) and the definition of \( \tilde{\lambda}_{n,m}(t, T) \) imply
\[
\begin{align*}
\mathbb{E} \left[ \partial_T \hat{P}_{n,m}^\infty (t, T) \bigg| \mathcal{F}_t^0 \right] &= \mathbb{E} \left[ -\hat{P}_{n,m}^\infty (t, T) \lambda_t^m + \hat{P}_{n,m-1}^\infty (t, T) \lambda_t^{m-1} \bigg| \mathcal{F}_t^0 \right] \\
&= -\hat{P}_{n,m}^\infty (t, T) \tilde{\lambda}_{n,m}(t, T) + \hat{P}_{n,m-1}^\infty (t, T) \tilde{\lambda}_{n,m-1}(t, T)
\end{align*}
\]

and for \( T = t \), we notice that irrespective of how \( \tilde{\lambda}_{n,m}(t, t) \) is defined for \( m > n \)
\[
\begin{align*}
\partial_T \bigg|_{T=t} \hat{P}_{n,m}^\infty (t, T) &= -\hat{P}_{n,m}^\infty (t, t) \lambda_t^m + \hat{P}_{n,m-1}^\infty (t, t) \lambda_t^{m-1} \\
&= -1_{[m=n]} \lambda_t^m + 1_{[m-1=n]} \lambda_t^{m-1} \\
&= -1_{[m=n]} \lambda_{n,n}(t, t) + 1_{[m-1=n]} \lambda_{n,n}(t, t) \\
&= -1_{[m=n]} \lambda_{n,m}(t, t) + 1_{[m-1=n]} \lambda_{n,m-1}(t, t) \\
&= -\hat{P}_{n,m}(t, t) \tilde{\lambda}_{n,m}(t, t) + \hat{P}_{n,m-1}(t, t) \tilde{\lambda}_{n,m-1}(t, t).
\end{align*}
\]

2.6.2 Forward Modelling and Consistency

To summarize, in the subsection above we showed that every sufficiently regular \( \mathbb{F}^0 \)-conditional Markov loss model \( (\lambda^n; L) \) induces a term structure of \( \mathbb{F}^0 \)-adapted forward loss rates \( \tilde{\lambda}_{n,m}(t, T) \) such that \( \hat{P}_{n,m}(t, T) \) are given by (2.17).

A reverse approach is to directly model \( \mathbb{F}^0 \)-adapted forward loss rates \( \tilde{\lambda}_{n,m}(t, T) \) and, using these, to define \( \hat{P}_{n,m}(t, T) \) by (2.17) without yet specifying a conditional Markov loss model \( (\lambda^n; L) \). As we have shown, these are the quantities needed to price a large number of credit portfolio derivatives.

After that, in order to give a concrete meaning to the so-constructed conditional "transition probabilities" \( \hat{P}_{n,m}(t, T) \), we wish to construct a canonical loss process \( L \) which is consistent with \( \tilde{\lambda}_{n,m}(t, T) \) in the sense below.

**Definition 7** (Consistency). Let \( \tilde{\lambda}_{n,m}(t, T) \) be \( \mathbb{F}^0 \)-adapted forward loss rates and let \( L \) be a canonical loss process with respect to some \( \mathbb{F}^0 \)-adapted \( \lambda^n \). Then \( \tilde{\lambda}_{n,m}(t, T) \) are said to be consistent with \( (\lambda^n; L) \) if \( \hat{P}_{n,m}(t, T) \) defined
as the functional in (2.17) are equal with $\tilde{P}_{n,m}(t, T)$ as originally defined in (2.10).

Remembering that in the conditional Markov set-up the canonical construction of $L$ relies on $\lambda_0, \ldots, \lambda^{N-1}$ $\mathbb{F}^0$-adapted and $E_1, \ldots, E_N$ independent of $\mathbb{F}^0$, it becomes obvious from definition 6 that consistency can be achieved only if

$$\lambda^n_t = \tilde{\lambda}_{n,n}(t, t)$$

(2.19)

for all $n$ and $t$. However, note that (2.19) does not guarantee consistency because setting $\lambda^n_t := \tilde{\lambda}_{n,n}(t, t)$ already fully specifies the canonical construction—and thus the $\mathcal{F}^0_t$-conditional distribution—of the canonical loss process. Hence (2.19) also determines $\tilde{\lambda}_{n,m}(t, T)$ for all $m > n$ through definition 6.

That means an arbitrarily chosen forward model for $\tilde{\lambda}_{n,m}(t, T)$ only and construct a canonical loss process using (2.19). This seems to solve the problem. There are however a few warnings.

1. **Marking-to-Market.** When the full term structure of STCDOs is being traded, the forward loss distribution $P_{m}(t, T)$, $m \geq L_t$, can be inferred from market data at any time $t$.\(^{37}\) Regarding (2.11) and (2.18), respectively, we notice however that $\tilde{\lambda}_{n,m}(t, T)$ can be implied from the market for only one single $n$, namely $n = L_t$. Hence marking $\tilde{\lambda}_{k,k}(t, T)$ with $k > L_t$ to market is in general not possible.

2. **Forward Modelling.** For $m > n$, $\tilde{\lambda}_{n,m}$ loses the forward modelling feature because its initial values $\tilde{\lambda}_{n,m}(0, T)$ can only be obtained via definition 6, which involves the joint distribution of $\lambda^n_0, \ldots, \lambda^m_t$ for all(!) $t \leq T$.

3. **Drift Restriction.** The “diagonal” rates $\tilde{\lambda}_{n,n}(t, T)$ might be subject to a martingale (drift) restriction that depends on other $\tilde{\lambda}_{k,k}(t, T)$ with $k < m$, which are not directly observable in the market.

Below, we let $\mathbb{F}^0_t$ be a Brownian filtration and we will postulate HJM dynamics

$$d\tilde{\lambda}_{n,m}(t, T) = \tilde{\alpha}_{n,m}(t, T)dt + \tilde{\sigma}_{n,m}(t, T)dW_t.$$  

This will enable us to identify further conditions, which are necessary for the existence of a consistent canonical loss process for a given set of forward loss rates $\tilde{\lambda}_{n,m}(t, T)$, in particular we obtain a restriction on the drifts $\tilde{\alpha}_{n,m}(t, T)$ and we show that the drift restriction for the “diagonal” elements $\tilde{\alpha}_{n,n}(t, T)$ depends on the volatilities $\tilde{\sigma}_{n,n}(t, T)$ only.

\(^{37}\)Interest rates must be uncorrelated with $L$ in a suitable sense. See chapter 1 for details.
2.6.3 The Conditional Markov Loss-HJM Model

Here we assume \( \mathbb{F}^0 \) is the natural augmented filtration generated by a \( d \)-dimensional Brownian Motion \( W_t \) and we work with a finite time horizon \( T^* \). The Lebesgue measure is denoted with \( \mathcal{L} \).

**Assumption 10.** \( \tilde{\lambda}_{n,m}(\cdot, T), \ T \leq T^* \) are nonnegative processes which satisfy

\[
d\tilde{\lambda}_{n,m}(T, T) = \tilde{\sigma}_{n,m}(T, T)dt + \tilde{\sigma}_{n,m}(T, T)dW_t
\]

where \( \tilde{\sigma}_{n,m}(t, \omega, u) \) and \( \tilde{\sigma}_{n,m}(t, \omega, u) \) are \( \mathcal{F} \otimes \mathcal{B}([0, T^*]) \)-measurable and

(i) \[ \int_0^T \tilde{\lambda}_{n,m}(0, u)du < \infty \text{ for all } n, m. \]
(ii) \[ \int_0^T \int_u^T \| \tilde{\sigma}_{n,m}(s, u) \|^2 ds du < \infty \text{ a.s. for all } n, m. \]
(iii) \[ \int_0^T \left( \int_0^T |\tilde{\sigma}_{n,m}(s, u)| ds \right)^2 du < \infty \text{ a.s. for all } n, m. \]
(iv) \( \tilde{P}_{n,m}(t, T) \) are defined by (2.17).

We define \( \tilde{\nu}_{n,m} := 0 \) and \( \tilde{\nu}_{n,m} := 0 \) for \( m < n \) and then iteratively

\[
\tilde{\nu}_{n,m}(t, T) := -1_{[n=m]} \tilde{P}_{n,n} \tilde{\sigma}_{n,n}^{t,T} - 1_{[n<m]} \int_t^T \tilde{\sigma}_{n,m}^{t,u} \left( \tilde{P}_{n,m} \tilde{\sigma}_{n,m} + \cdots \right) du,
\]
\[
\tilde{\nu}_{n,m}(t, T) := 1_{[n=m]} \tilde{P}_{n,n} \left( \tilde{\lambda}_{n,n}(t, t) + \frac{1}{2} \left\| \tilde{\sigma}_{n,n}^{t,T} \right\|^2 \right) - 1_{[n=m]} \int_t^T \tilde{\sigma}_{n,m}^{t,u} \left( \tilde{P}_{n,m} \tilde{\sigma}_{n,m} + \cdots \right) du,
\]

for \( n = 0, \ldots, N \) and \( m = n, \ldots, N \). The dynamics of \( \tilde{P}_{n,m} \) are given in

**Theorem 54** (Representation). Then the functions \( \tilde{\nu}_{n,m} \) and \( \tilde{\nu}_{n,m} \) are \( \mathcal{F} \otimes \mathcal{B}([0, T^*]) \)-measurable with \( \int_0^T \| \tilde{\nu}_{n,m}(s, T) \|^2 + |\tilde{\nu}_{n,m}(s, T)| ds < \infty \) a.s. for every \( 0 \leq n \leq m \leq N \) and \( T \leq T^* \) and

\[
d\tilde{P}_{n,m}(t, T) = \tilde{\nu}_{n,m}(t, T)dt + \tilde{\nu}_{n,m}(t, T)dW_t
\]

**Proof.** The proof goes analogous to that of theorem 13. \( \square \)

\footnote{Here, \( \mathcal{F} \) is the predictable \( \sigma \)-field with respect to the initial filtration \( \mathbb{F}^0 \).}
So far we have not specified the loss process itself. We continue with

**Assumption 11.**

(i) $L$ is a canonical loss process with respect to $\lambda^n_t := \tilde{\lambda}_n(t, t)$.

(ii) $L$ is consistent with $\tilde{\lambda}_{n,m}(t, T)$.

(iii) $\tilde{\lambda}_{n,m}(t, t) = \tilde{\lambda}_{m,m}(t, t) \mathbb{P} \times \mathcal{L}(dt)$-a.e. for all $n \leq m$ (cf. remark 53).

(iv) $\tilde{\lambda}_{n,m}(t, T)$ is right-continuous in $T$ for all $n \leq m$.

Then $P_m$ satisfies (2.11), $F_m$ is given by (2.18) and is, hence, continuous (until $\tau_m + 1$) if and only if $\tilde{\lambda}_{m,m}(t, T) = \tilde{\lambda}_{m,m}(t, T) \mathbb{P} \times \mathcal{L}(dt)$-a.e. for all $n \leq m$. We define $\overline{L}$ as the compensated loss process with $d\overline{L}_t = dL_t - \sum_{n=0}^{m-1} 1_{\{L_{n-1} = n\}} \lambda^n_{n,t} dt$. The dynamics of $P_m(t, T)$ and $F_m(t, T)$ are given in

**Proposition 9.** $P_m(t, T)$ satisfies the SDE

\[
dP_m(t, T) = u_m(t, T) dt + v_m(t, T) dW_t + \psi^P_m(t, T) d\overline{L}_t
\]

where

\[
v_m(t, T) = \sum_{n=0}^{m} 1_{\{L_{n-1} = n\}} \tilde{u}_{n,m}(t, T)
\]

\[
\psi^P_m(t, T) = \sum_{n=0}^{m} 1_{\{L_{n-1} = n\}} \left( 1_{\{n < m\}} \tilde{\psi}_{n+1,m}(t, T) - \tilde{\psi}_{n,m}(t, T) \right)
\]

\[
u_m(t, T) = \sum_{n=0}^{m} 1_{\{L_{n-1} = n\}} \tilde{u}_{n,m}(t, T) + \psi^P_m(t, T) \lambda^n_{n}
\]

and $F_m(t, T)$ solves\(^{39}\)

\[
dF_m(t, T) = \sigma^F_m(t, T) dt + \sigma^F_m(t, T) dW_t + \phi^F_m(t, T) d\overline{L}_t
\]

where

\[
\sigma^F_m(t, T) = \sum_{n=0}^{m} 1_{\{L_{n-1} = n\}} \tilde{\sigma}_{n,m}(t, T)
\]

\[
\phi^F_m(t, T) = \sum_{n=0}^{m} 1_{\{L_{n-1} = n\}} \left( 1_{\{n < m\}} \tilde{\phi}_{n+1,m}(t, T) - \tilde{\phi}_{n,m}(t, T) \right)
\]

\[
\alpha^F_m(t, T) = \sum_{n=0}^{m} 1_{\{L_{n-1} = n\}} \tilde{\alpha}_{n,m}(t, T) + \phi^F_m(t, T) \lambda^n_{n}
\]

\(^{39}\)We mean the specification of $F_m$ with $F_m(t, T) = 1_{\{L \leq m\}} F_m(t, T)$.
2.6. The Conditional Markov Forward Loss Model

Proof. Note that $\tilde{P}_{n,m}(t, T)$ and $\tilde{\lambda}_{n,m}(t, T)$ are continuous processes and recall that $d\mathbf{T}_t = dL_t - \sum_{n=0}^{N-1} 1_{\{L_t = n\}}\lambda^n_t dt$. Then the claim follows from Itô’s lemma because by consistency we may decompose $P_m$ and $F_m$ as in (2.11) and (2.18).

Recall that $P_m(t, T)$ are $\mathbb{F}$-martingales and by consistency, $P_m(t, T)$ can be defined via (2.11). As an application of theorem 23 and proposition 9, this immediately yields

**Proposition 10.** $P_m(t, T) = \sum_{n=0}^{m} 1_{\{L_t = n\}} P_{n,m}(t, T)$ are $\mathbb{F}$-martingales for all $m \leq N$ and $T \leq T^*$ if and only if for all $n \leq m \leq N$ and $T \leq T^*$ it holds $P \times \mathbf{L}^{(dt, dT)}$-a.e.

$$\tilde{P}_{n,m} \delta_{n,m} + \tilde{\sigma}_{n,m} = 1_{\{n < m\}} \lambda^n_t \tilde{P}_{n+1,m} (\tilde{\lambda}_{n,m} = \tilde{\lambda}_{n+1,m}).$$

Proof. Set $r_t = 0$ in theorem 23 and then note that the drift restriction for $F_m(t, T) = \sum_{n=m}^{N} 1_{\{L_t = n\}} \tilde{\lambda}_{n,m}(t, T)$ given there must be satisfied separately on every set $\{L_t = n\}$. The “short-end” martingale-condition $\lambda_t = F_{L_t}(t, t)$ of theorem 23 is automatically satisfied under consistency because

$$F_{L_t}(t, t) = \sum_{n=0}^{N-1} 1_{\{L_t = n\}} F_m(t, t) = \sum_{n=0}^{N-1} 1_{\{L_t = n\}} \sum_{m=0}^{m} 1_{\{L_t = m\}} \tilde{\lambda}_{n,m}(t, t)$$

$$= \sum_{n=0}^{N-1} 1_{\{L_t = n\}} \tilde{\lambda}_{n,n}(t, t) = \sum_{n=0}^{N-1} 1_{\{L_t = n\}} \lambda^n_t = \lambda_t. \Box$$

Consistent Forward Modelling for the “Diagonal” Rates $\tilde{\lambda}_{n,n}(t, T)$

Proposition 10 shows that under consistency, the “diagonal” drift $\tilde{\alpha}_{n,n}(t, T)$ depends only on $\tilde{\sigma}_{n,n}(t, T)$, precisely it satisfies the well-known HJM drift condition

$$\tilde{\alpha}_{n,n}(t, T) = \tilde{\delta}_{n,n}^{T} \tilde{\sigma}_{n,n}(t, T). \tag{2.21}$$

That means, if we specify a forward model for $\tilde{\lambda}_{n,n}(t, T)$ for all $n$ and $T$, i.e. an initial term structure $\tilde{\lambda}_{n,n}(0, T)$ for all $n$ and a volatility model $\tilde{\sigma}_{n,n}(t, T)$ for all $n$ and $T$, and we define $\tilde{\lambda}_{n,m}(T, T)$ for $m > n$ and $T > t$ via definition 6, then (2.21) ensures consistency.

**Remark 55.** Further, assuming that $\tilde{\alpha}_{n,m}$ and $\tilde{\sigma}_{n,m}$ are sufficiently smooth in $T$, e.g. $\tilde{\alpha}_{n,m}(t, T) = \tilde{\sigma}_{n,m}(t, t) + \int_t^T \nabla \tilde{\sigma}_{n,m}(t, u) du$ with uniformly bounded
derivatives $\partial_T \tilde{\sigma}_{n,m}(t,T)$ (cf. Brace et al. (1997)) allows to apply the stochastic Fubini theorem (see e.g. Protter (2004), chapter IV, theorem 65), which yields

$$
\tilde{\lambda}_{n,m}(t,t) = \tilde{\lambda}_{n,m}(0,t) + \int_0^t \tilde{\alpha}_{n,m}(s,t)ds + \int_0^t \tilde{\sigma}_{n,m}(s,t)dW_s
$$

$$
= \tilde{\lambda}_{n,m}(0,t) + \int_0^t \left( \tilde{\alpha}_{n,m}(s,s) + \int_s^t \partial_u \tilde{\alpha}_{n,m}(s,u)du \right)ds
$$

$$
+ \int_0^t \left( \tilde{\sigma}_{n,m}(s,s) + \int_s^t \partial_u \tilde{\sigma}_{n,m}(s,u)du \right)dW_s
$$

$$
= \tilde{\lambda}_{n,m}(0,t) + \int_0^t \tilde{\alpha}_{n,m}(s,s)ds + \int_0^t \int_0^u \partial_u \tilde{\alpha}_{n,m}(s,u)ds du
$$

$$
+ \int_0^t \tilde{\sigma}_{n,m}(s,s)dW_s + \int_0^t \int_0^u \partial_u \tilde{\sigma}_{n,m}(s,u)dW_s du
$$

Together with assumption 11, (iii) this implies then

$$
d\tilde{\lambda}_{n,m}(t,t) = d\tilde{\lambda}_{n,m}(t,t) = \ldots dt + \tilde{\sigma}_{n,m}(t,t)dW_t.
$$

Hence under consistency and sufficient regularity, for fixed $m$ the volatilities $\tilde{\sigma}_{n,m}$ “meet” at the short end of the term structure, i.e. for all $m$ it holds

$$
\tilde{\sigma}_{n,m}(t,t) = \tilde{\sigma}_{m,m}(t,t)
$$

for all $n \leq m$ and $t \leq T^*$. 

Part II

The Influence of FX Risk on Credit Spreads
Chapter 3

Introduction to Part II

In modern debt markets, many large debtors issue debt in more than one currency, e.g. a large Japanese obligor may find it advantageous to issue debt in US dollar (USD), or a European obligor in Japanese Yen (JPY). Furthermore, since the advent of liquid markets for credit default swaps (CDS) there are markets for credit protection in currencies different from the obligors “home” currency, even if the obligor has not issued bonds in that currency. (There is demand for this in order to hedge loan exposures or OTC derivatives transactions.) In particular, CDS protection on many international corporations is now available in their home currency and Euro (EUR) and USD.

Given this situation, it is natural to ask about the correct relative pricing of the credit risk in the different currencies: How should the credit spread (either on bonds or on CDS) be adjusted if a different currency is used? And: What information about foreign exchange (FX) risk, e.g. likely FX movements in crisis events, can we imply from the relative difference of CDS spreads in different currencies?

We aim to answer this question in two steps. First, we analyse the connections between local and foreign currency credit spreads on a theoretical basis in an intensity-based framework and highlight the effects that we can expect to encounter. We find that the essential feature driving differences between credit spreads in different currencies is the dependency between default risk and FX risk. If default risk and FX risk are independent (in a sense which will be made precise later on), credit spreads in different currencies should not differ.

In order to capture empirically observed differences, we model this dependency in two different ways. First, there may be correlation between the
diffusions driving default intensities and FX rate (before default), and second, an additional jump in the exchange rate may occur at the time of default, i.e. the default “causes” a devaluation of the currency. We give closed-form solutions for CDS rates and defaultable bond prices in a model which encompasses both cases using an affine jump-diffusion (AJD) model.

In chapter 6 these models are estimated using a historical database of CDS rates in JPY and USD on a set of major Japanese corporate obligors. Besides being relatively clean, liquid and standardised, CDS data has the additional advantage that the recovery rates on USD-denominated and JPY-denominated CDS will be identical by definition (by documentation, to be more precise). Thus, credit spread differences in CDS rates cannot be caused by the effects of different legal regimes and bond specification which may affect corporate bond data.

In the pure diffusion hypothesis (i.e. without jumps in the FX rate), we first estimate the model parameters using USD CDS spreads and the JPY/USD rate, without using the JPY CDS spreads. In the second step, we then calculate the JPY CDS spread which would hold if the model were correct and compare it to the empirically observed JPY CDS spreads. In all cases, we can strongly reject the hypothesis that the empirically observed JPY CDS spreads are noisy observations of the model predictions, the predicted spread difference is only a small fraction of the observed spread difference. Consequently, we reject the pure diffusion model, there must be jumps in the exchange rate at default.

In many cases, the implied jump in FX rates at default of the obligors seems quite large and is difficult to explain unless we assume that the obligor’s default was caused by a major macroeconomic crisis. The techniques used in this part are also useful for a number of other applications like the pricing of counterparty risk and the pricing of sovereign default risk. Some of these applications are also pointed out in section 4.5, where we discuss and price some more exotic FX-related credit derivatives.

Sparked by the Asian crisis in the late 1990’s (and the Peso crisis earlier on) there is a large literature on sovereign default risk, banking crises and currency crises in emerging markets\(^1\). While some of the techniques used in this part can also be applied to these situations, this article has a different focus from the domestic debt of a sovereign obligor which is special because (at least in theory) the sovereign could always repay that debt by simply printing more domestic currency. Similarly, this part also differs in focus from papers which empirically investigate sovereign credit spreads, like e.g. Duffie et al. (2003) and Singh (2003).

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\(^{1}\) See e.g. Kaminsky and Reinhart (1999), Reinhart (2002), Bulow and Rogoff (1989) or other papers listed on N. Roubini’s [http://www.stern.nyu.edu/globalmacro/](http://www.stern.nyu.edu/globalmacro/)
Here we focus on multinational corporations which face a quite different situation from sovereigns. Here, the foreign sovereign often has negligible default risk compared to the corporate (unless it is an emerging market), and the exchange rate floats freely. In addition, the available data is significantly different: CDS on large corporations are routinely quoted in the major currencies (USD, EUR, and also JPY, GBP, CHF). Despite this focus on corporate risk, the techniques of this thesis can be used to back out market implied information about the sovereign, in particular values for the expected currency devaluation upon a default of the corporate. Furthermore, it is not difficult to extend the model to cover corporates which are based in a country with non-negligible default risk.

A related paper is Jankowitsch and Pichler (2005), there the authors address the question of the construction of corporate credit spread curves from corporate bond prices in different currencies. In their sample, the authors find strong evidence against the assumption of independence of corporate bond credit spreads and exchange rates. This article differs from Jankowitsch and Pichler (2005) in several respects: First, we provide a theoretical model which is able to capture stochastic dependency between default intensities and exchange rates, and we empirically estimate these models. Second, we base our analysis on CDS spreads which are probably better suited to the empirical and theoretical analysis of these questions than bond spreads as that avoids the issues caused by differences in recovery rates in different currencies. Another related paper is Warnes and Acosta (2002) who extend the classical Merton (1974) firm’s value approach to incorporate debt in a foreign currency and provide closed-form solutions for debt prices under the assumption of constant interest-rates in both countries.

The rest of this part is structured as follows: We first recapitulate the payoff mechanics of credit default swaps and show that the delivery option in the protection leg of the CDS makes the effective recovery rate currency-independent. In chapter 4 we then set up a general mathematical model of FX and default risk and show how the distributional properties of default risk change when moving from the domestic- to a foreign-currency martingale measure. We provide general pricing rules for credit-risky claims and apply these to the valuation bonds and CDSs in different currencies in section 4.3. As our focus is on the difference between domestic and foreign CDS spreads, we discuss in section 4.4 the effects which we expect to influence this quantity. Furthermore, in section 4.5 it is shown how the techniques used for the CDS analysis can be applied to other default-sensitive FX derivatives.

To provide a concrete specification of the model we specify an affine jump-diffusion (AJD) version of the model in chapter 5 and chapter 6 contains
an empirical analysis of a two-factor AJD-model using historical CDS data on a number of large Japanese obligors. We show that there is a persistent, significant and rather large difference between CDS rates in JPY and USD which cannot be explained by a purely diffusion-based dependency between default intensity and FX rate alone. Thus, we conclude in section 6.3 that the market must be pricing an implicit devaluation at default into these CDS spreads.²

### 3.1 CDS in Multiple Currencies

Credit default swaps (CDS) are derivative instruments which allow the trading of payoffs contingent on the occurrence of a credit event. Single-name CDS are still the most important class of credit derivatives transactions. In the latest BBA credit derivatives report (Barrett and Ewan (2006)), it is estimated that by the end of 2006 CDS account for 33.3% of the entire credit derivatives market (in terms of notional outstanding), this is equivalent to a notional of around 6.65 trillion USD; compared to 51% and 2.56 trillion USD at the end of 2004. Despite fall in market share, this growth of absolute market size is expected to continue.

In many cases, the liquidity of the CDS market has surpassed the liquidity of the market for the bonds of the underlying obligor. This trading volume and liquidity has been made possible by the standardisation of the documentation for CDS transactions which has been proposed by the International Swap Dealers Association (ISDA), see [http://www.isda.org](http://www.isda.org) and (ISDA) (1999). In particular when it comes to the analysis of the default risk of any given obligor in two or more different currencies, and thus in two different jurisdictions, this standardisation is essential: Bonds in domestic and foreign currency are typically issued in different jurisdictions and are therefore governed by different legal rules which has a significant impact on the resulting recovery rates of the bonds (see e.g. Davydenko and Franks (2004)). CDS referencing the obligor on the other hand will be governed by the same standardised ISDA documentation even if they are denominated in different currencies, in particular they will have the same recovery rates. Thus, for the purposes of this part we consider CDS to be more standardised and more easily comparable than the underlying corporate bonds.

We now present a quick summary of the payoff mechanics of an ISDA-standard CDS with physical settlement in order to explain why the recovery rate of a CDS is typically independent from the currency of its denomination:

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²The contents of this part are summarized in Ehlers and Schönbucher (2004).
3.1. CDS in Multiple Currencies

Being an over-the-counter (OTC) traded derivative, a CDS is a contract between two counterparties: the protection buyer and the protection seller. The protection buyer makes the payments of the fee leg of the CDS, the protection seller pays the protection leg. In order to define these payment streams, the following data is specified in every CDS:

- the notional amount \( N \), and the currency \( c \) of the notional amount,
- the maturity date \( T \),
- the CDS rate \( \tau \),
- the reference credit (i.e. the obligor whose credit risk is traded)
- the applicable (precise) definition of the credit event, and
- the set of deliverable obligations.

It is important to note that both the definition of the credit event and the list of the deliverable obligations do usually not depend on the currency of the CDS. The ISDA-definition of a credit event includes bankruptcy, failure to make due payments on bonds or loans ("failure to pay"), repudiation or moratorium, cross-acceleration, obligation default, distressed restructuring and credit events upon mergers. These events apply globally to the reference obligor and are in most cases objectively verifiable and independent from local legal rules. The set of deliverable obligations contains most bonds and loans issued by the reference credit irrespective of their currency, excluding special cases such as subordinated bonds or bonds with unusual maturity dates but including all major bond issues.

**The Fee Leg.** The protection buyer makes fee payments to the protection seller at regular intervals until the CDS matures or until a credit event occurs. The fee payments are made in the currency of the CDS and are calculated as:

\[ \text{Fee Payment} = \text{daycount fraction} \times \text{CDS rate} \times \text{Notional} \times \Delta t. \]

**The Protection Leg.** At the credit event, the protection buyer chooses a portfolio from the set of deliverable obligations such that the total notional amount of the portfolio is \( N \). If any obligations in the portfolio are denominated in other currencies than the CDS reference currency \( c \), then the notional amount of these obligations is converted into the reference currency using the actual exchange rate of the day. The protection buyer then delivers this portfolio to the protection seller who has to pay the full notional \( N \) for it.

Clearly, although they will trade at a significant discount to par, not all deliverable obligations will trade at the same price. The protection buyer has a delivery option: The protection buyer will choose to deliver those bonds which trade at the highest discount to their par value, the cheapest-to-deliver bonds. Interestingly, the choice of the cheapest-to-deliver bond is independent of the

\footnote{Usually fees are paid quarterly, \( \Delta t = 1/4 \).}
currency in which that bond is denominated, it only depends on the relative
discount of the bond to its par value.

To illustrate this let us assume the bond that we want to deliver trades
at a discount of \( q \), i.e. at a price of \((1 - q)\) per unit 1.00 of notional in its
currency \( \tilde{c} \neq c \). If one unit of \( \tilde{c} \) is worth \( X \) units of \( c \) at the time of the credit
event, then in order to reach a portfolio of notional \( N \) in currency \( c \), we have
to buy a portfolio of notional \( N/X \) in the bond’s currency \( \tilde{c} \). Thus the delivery
portfolio costs \((1 - q) \cdot X \cdot N/X = (1 - q) \cdot N \) in the CDS’s currency. This
portfolio is put to the protection seller for a payment of \( N \) in \( c \) which yields a
net value of the protection payment of \( N \cdot q \) in \( c \).

This value does not depend on the exchange rate \( X \) any more. Thus, in
order to maximise the value of the protection payment, the protection buyer
will choose to deliver a portfolio of those bonds which have the lowest \((1 - q)\),
irrespective of the exchange rate for the currency of denomination of this bond.
In particular, the effective recovery rate for the CDS will be independent from
the currency in which the CDS is denominated.

This does not mean that CDS which are denominated in different curren-
cies are identical. Consider two CDS on the same reference credit with the
same deliverable obligations but denominated in different currencies \( c \) (with
notional \( N \)), and \( \tilde{c} \) (with notional \( \tilde{N} \)). At default, the first CDS will pay off
\( N \cdot q \) in currency \( c \), and the other will pay off \( \tilde{N} \cdot q \) in currency \( \tilde{c} \). Thus,
the amount of protection that the \( \tilde{c} \)-CDS provides in currency \( c \) depends on
the exchange rate at the time of default. This is the relationship which we are
going to explore herein.
Chapter 4

A Joint Model for FX and Single-Name Default Risk

Preliminaries

Our model is set in a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, Q)\) with finite time horizon \(T^* (< \infty)\). The filtration \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T^*]}\) satisfies the usual conditions and is rich enough to carry an \(N\)-dimensional Brownian motion (BM) \(W_t\) and a jump measure \(\mu(dz, dt)\) on \([0, T^*] \times Z\) with jumps in \(Z := (0, 1]^K\) for some \(K \in \mathbb{N}\). All processes are càdlàg and \(\mathbb{F}\)-adapted and, for any process \(X\), \(\mathbb{F}_X\) denotes the natural augmented filtration of \(X\). The transpose of a matrix \(M\) is denoted by \(M^0\), and if \(x\) is a vector, then diag\( (x)\) is a diagonal matrix with the elements of \(x\) on its diagonal. Standard arithmetical functions, integrals and comparisons of vectors are meant componentwise, except in the case of multiplications we will use matrix multiplications. \(I\) denotes the Lebesgue measure. Regarding time points \((t\) and term structure arguments \((t, T))\) we always assume \(0 \leq t \leq T \leq T^*\).

\(v(dz, dt)\) denotes the predictable compensator measure of \(\mu(dz, dt)\) under \(Q\) and we assume \(\int_0^T \int_Z v(dz, dt) < \infty\) a.s. Then \(N_t := \int_0^t \int_Z \mu(dz, ds)\) is a counting process, \(J_t := \int_0^t \int_Z z \cdot \mu(dz, ds)\) is a purely discontinuous \(\mathbb{R}_+^K\)-valued process, \(W_t + J_t\) is a jump diffusion and \(v(dz, dt)\) can be decomposed

\[v(dz, dt) = dF(t; z) dA_t,\]

where \(A_t\) is the predictable compensator of \(N_t\) and \(F(t; z)\) is a predictable
4.1 FX and Default Risk Set-Up

We are in an arbitrage-free market with a domestic currency $c_d$ and a foreign currency $c_f$ and a risky obligor which underlies a number of traded securities. The probability $Q$ is the domestic martingale measure (DSMM)$^2$ under which all payoffs in domestic currency (or denominated in domestic currency) are priced. $Q$ does not need to be unique, we only assume it is the domestic pricing measure chosen by the market and we sometimes write $Q_d$ instead of $Q$ when we find it necessary to emphasize the fact that $Q$ is the domestic SMM. The risky obligor defaults at the first jump time of $N_t$, i.e. at the stopping time

$$\tau := \inf \{t; \, N_t > 0\}.$$  

(with the usual convention $\inf 0 = \infty$) and given default, i.e. $\tau \leq T^*$, the severity of default$^3$ is characterized by the realization of the marker

$$z_\tau := J_\tau = \Delta J_\tau.$$  

$r_i(t)$ denote the short-term interest rates, which satisfy $\int_0^T r_i(t)dt < \infty$ a.s., and $b_i(t) := e^{\int_0^t r_i(s)ds}$ are money market accounts in the respective currencies $c_i, i = d, f$. The exchange (FX) rate between $c_f$ and $c_d$ is denoted with the nonnegative process $X$, i.e. $X_t$ is the value at time $t$ of one unit of $c_f$, expressed in units of $c_d$.

**Assumption 12.**

(i) There exists a process $\lambda_t$ with $A_{t,x} = \int_0^t \lambda_s ds$ and $\lambda_t$ solves the SDE

$$d\lambda_t = \mu_\lambda(t)dt + \sigma_\lambda(t)dW_t$$

where $\mu_\lambda, \sigma_\lambda$ are predictable with $\int_0^T |\mu_\lambda(t)| + \|\sigma_\lambda(t)\|^2 dt < \infty$ a.s.

(ii) The FX rate process satisfies an SDE of the form

$$\frac{dX_t}{X_{t-}} = (r_d - r_f)(t)dt + \sigma_X(t)dW_t - \int_Z \delta(z, t)(\mu - \nu)(dz, dt) \quad (4.1)$$

---

1. See e.g. Jacod and Shiryaev (1988) for the definition of predictable functions.
2. I.e. the discounted price of every traded asset (denominated in domestic currency) is a $Q$-local martingale.
3. One can think of the loss given default rate, we will make this more precise below.
where \( \sigma_X \) is a predictable process and \( \delta(z, t) \leq 1 \) is a predictable function with \( \int_0^T \| \sigma_X(t) \|^2 \, dt + \int_0^T \int_Z |\delta(z, t)| \, v(\,dz, \, dt) < \infty \) a.s.

By (i) \( \tau \) is a totally inaccessible stopping time and \( \int_0^{\tau \wedge t} \lambda_s \, ds \) is the predictable compensator of the default indicator \( 1_{[\tau \leq t]} \), hence \( 1_{[\tau \leq t]} \lambda_t \) can be termed default intensity under \( Q \). Sometimes we call \( \lambda_t \) itself default intensity, omitting the indicator function in the mathematically correct expression.

(ii) has two implications. First, FX rate and default intensity may be conditionally correlated if \( d \, T \); \( X \rightarrow Z \); \( X \rightarrow D \) for example we have positive local correlation, an increasing (decreasing) FX rate will indicate a rise (lowering) in the default intensity. Second, there is a direct jump-influence from the default event itself on the exchange rate which is captured in the function \( \delta \) via \( d[N, X]_t = -X_t - \int Z \delta(z, t) \mu(\,dz, \, dt) \) or

\[
\Delta X_t = -X_t - \delta(z, t).
\]

At default \( \tau \), the foreign currency \( c_f \) is devaluated (relative to \( c_d \)) in a jump of a fraction \( \delta(z, \tau) \) of the pre-default value of \( X \).

Regarding the drift term in (4.1) we say \( X \) satisfies the FX drift restriction under \( Q \) because

\[
L_t^X := \frac{X_t b_f(t)}{X_0 b_d(t)},
\tag{4.2}
\]

the discounted value in \( c_d \) of a foreign \( (c_f) \) bank account (normalized by \( X_0 \)), is a \( Q \)-local martingale if and only if the drift of \( X \) under \( Q \) equals indeed \( X_t - (r_d - r_f)(t) \) as in (4.1).

Of course not only \( X \) and \( \lambda \) but also \( r_d, r_f, \lambda, X \) may show mutual correlation and \( r_d, r_f \) might also jump at \( \tau \). However, in this part we mainly focus on the dependence of FX and credit risk, and we will not treat these additional features in detail.

**Remark 56.** In standard FX derivatives pricing models (see e.g. Musiela and Rutkowski (2005), chapter 4), the martingale term \( \int Z \delta(z, t) (\mu - v)(\,dz, \, dt) \) in the dynamics of \( X \) does not appear. In an arbitrage-free market however, \( L_t^X \) as defined in (4.2) must only be a (nonnegative) local martingale under any DSM. Thus, from a mathematical point of view, and given the particular nature of our underlying probability space which carries both continuous and purely discontinuous martingales, it is natural to include a jump term.
\[ \int_{\mathbb{R}} \delta(z, t) (\mu - \nu) (dz, dt) \] in the dynamics of \( X \). Setting \( \delta(z, t) = 0 \) would imply that default cannot cause any immediate FX movements (or vice versa). Such an additional model assumption would need to be economically justified.

**Remark 57 (Extension)**. The setup presented here can be generalized in order to allow discontinuities in the FX rate if we define

\[ \tau := \inf \left\{ t > 0 : \int_0^t \int_{\mathbb{R}} \phi^{def}(z, s) \mu(dz, ds) \geq 1 \right\} \]

where \( \phi^{def}(z, t) \) is a \((\mathcal{Q} \times \nu(dz, dt))-\text{a.e.}\) \([0, 1]\)-valued predictable function. Then default does not necessarily occur at the first jump event. This way the setup can be made compatible with the portfolio setup of part I.

### 4.2 Default Risk under the Foreign SMM

The dependency between defaults and the movements of the exchange rate \( X \) has important consequences for the dynamics of the model under the pricing measures that we will introduce in the following. For technical reasons, we make an additional

**Assumption 13**. The discounted value in \( c_d \) of a foreign bank account \( L_t^X \) is a positive \( \mathcal{Q} \)-martingale.

Starting from a DSMM \( \mathcal{Q} \), the following pricing measure is commonly used to analyze FX-related instruments when assumption 13 is satisfied (see Musiela and Rutkowski (2005), chapter 4).

**Definition 8** (Foreign Spot Martingale Measure). The probability \( \mathcal{Q}_f \sim \mathcal{Q}_d \) defined by

\[ \frac{d\mathcal{Q}_f}{d\mathcal{Q}_d} := L_T^X, \]

is called the foreign spot martingale measure (FSMM) induced by \( \mathcal{Q}_d \) and \( X \).

Let \( E_i \) denote the expectation operator wrt. \( \mathcal{Q}_i \) for \( i = d, f \). The FSMM is useful for pricing foreign currency payoffs because the price in \( c_d \) at time \( t \) of an instrument that pays \( Z \) units of \( c_f \) at time \( T \) (\( X_T Z \) units of \( c_d \)) is

\[ p(t) = E_d \left[ e^{-\int_0^t r_f(s) ds} X_T Z \mid \mathcal{F}_t \right] = X_t E_f \left[ e^{-\int_t^T r_f(s) ds} Z \mid \mathcal{F}_t \right], \quad (4.3) \]

\( ^4 \)We emphasize that every DSMM induces another FSM. If \( \mathcal{Q}_d \) is unique, then so is \( \mathcal{Q}_f \).
4.2. Default Risk under the Foreign SMM

i.e. \( E[T e^{\int_t^T r(s)ds} \mid F_t] \) is the price of this instrument in \( c_f \). Note that (4.3) remains valid if \( T \) is a \([t, T^*] \)-valued stopping time. Thus, in order to price a foreign currency contingent claim, only the distributional properties of the considered claim (and foreign interest rates) under the FSMM are needed and, once an FSMM is determined/chosen, any foreign currency pricing problem is reduced to a related “domestic” pricing equation (with \( c_f \) acting as domestic currency).

4.2.1 Jump Diffusions under Change of Measure

We first give a general form of Girsanov’s theorem which applies to the probability space we consider. For the proof see e.g. Jacod and Shiryaev (1988).

**Theorem 58** (Girsanov’s Theorem). Let \( L \) be an \( F \)-martingale under \( Q \) with

\[
\frac{dL_t}{L_t} = \sigma(t)dW_t + \int_Z (\Phi(z, t) - 1)(\mu - v)(dz, dt), \quad L_0 = 1
\]

where \( \sigma \) is a predictable process and \( \Phi \) is a nonnegative predictable function with \( \int_T^\infty |\sigma(t)|^2 dt + \int_0^T \int_Z |\Phi(z, t)| v(dz, dt) < \infty \) a.s. Then, the probability measure \( \tilde{Q} \) on \((\Omega, \mathcal{F})\), defined by

\[
\frac{d\tilde{Q}}{dQ} = L_T,
\]

is absolutely continuous with respect to \( Q \) and it holds that:

(i) The process \( \tilde{W}_t = W_t - \int_0^t \sigma(s)ds \) is a BM under \( \tilde{Q} \).

(ii) The predictable compensator measure \( \tilde{v}(dz, dt) \) of \( \mu(dz, dt) \) under \( \tilde{Q} \) satisfies \( 1_{[t \leq z]} \tilde{v}(dz, dt) = 1_{[t \leq z]} d\tilde{F}(t; z) \tilde{\lambda}_t dt \), where

\[
\tilde{\lambda}(t) = \lambda(t) \int_Z \Phi(z, t) dF(t; z)
\]

and \( \tilde{F} \) is a distribution function on \( Z \) for all \( t \in [0, T^*] \) with

\[
d\tilde{F}(t; z) = \begin{cases} \frac{\Phi(z, t)}{\int_Z \Phi(z, t) dF(t; z)} dF(t; z) & \text{if } \int_Z \Phi(z, t) dF(t; z) > 0, \\ \text{otherwise} \end{cases}
\]

Note that by (ii), \( \tau \) remains totally inaccessible under \( \tilde{Q} \) and \( \int_0^{T^*} \tilde{\lambda}_s ds \) is the predictable \( \tilde{Q} \)-compensator of \( 1_{[t \leq \tau]} \), i.e. \( 1_{[t \leq \tau]} \tilde{\lambda}_t \) is the default intensity under \( \tilde{Q} \). The following corollary is a straightforward implication of Girsanov’s theorem.
Corollary 59. Let \( L_t \) be as in theorem 58 and \( \tilde{Q} \sim Q \). Then \( \langle L, N \rangle_{1 \wedge \tau} = 0 \) a.s. for all \( t \), if and only if

\[ \mathbf{1}_{[t \leq T]} \tilde{\lambda}(t) = \mathbf{1}_{[t \leq T]} \lambda(t) \quad Q \times \mathcal{L}(dt) - \text{a.e.} \]

i.e. default arrives with the same intensity under two equivalent measures \( \tilde{Q} \) and \( Q \) if and only if \( \langle d\tilde{Q}/dQ, N \rangle_{1 \wedge \tau} = 0 \) a.s. However, this does not imply that \( \tilde{v} = v \).

Proof of Corollary 59. If \( \tilde{Q} \sim Q \), then \( L_t \) is a.s. positive, and for all \( t \). By theorem 58 the predictable covariation of \( L \) and \( 1_{[t \leq T]} \) under \( Q \) is given by

\[
\langle L, 1_{[t \leq T]} \rangle_t = \langle L, N \rangle_{1 \wedge \tau} = \int_0^{\mathbf{1}_{t \leq T}} \int_Z L_s \left( \Phi(z, s) - 1 \right) q(dz, ds)
\]

\[
= \int_0^{\mathbf{1}_{t \leq T}} \int_Z L_s \left( \Phi(z, s) - 1 \right) dF_\tau(z) \lambda_s ds
\]

\[
= \int_0^{\mathbf{1}_{t \leq T}} L_s \left( \tilde{\lambda}_s - \lambda_s \right) ds.
\]

Since \( L \) is a.s. positive, this shows that \( \mathbf{1}_{[t \leq T]} (\tilde{\lambda} - \lambda) = 0 \) \( Q \times \mathcal{L}(dt) - \text{a.e.} \) if and only if \( \langle L, N \rangle_{1 \wedge \tau} = 0 \) a.s. for all \( t \).

4.2.2 Domestic and Foreign Default Intensities

We define \( \lambda_d := \lambda \) to emphasize that \( \lambda_d \) is the default intensity under the domestic SMM and, if \( L = L^X \) or equivalently \( \tilde{Q} = Q^X \), then we call \( \lambda_f := \tilde{\lambda} \) the default intensity under the foreign SMM (for short, the domestic and the foreign default intensity).

Corollary 59 implies that if the FX rate \( X \), and so the Radon-Nikodym density process \( L^X_t \), jumps at default, then the domestic and the foreign default intensity do in general not coincide, and thus default occurs with different probabilities under the FSMM and under the DSMM. As we showed, \( \lambda_f = \lambda_d \) if and only if \( \langle N, L^X \rangle_{1 \wedge \tau} = 0 \). Here, given the particular nature of our numeraire asset, more can be said about the link between the devaluation fraction of the currency and the two default intensities.

Proposition 11 (FSMM Default Intensity). Define the locally expected devaluation fraction \( \widehat{\delta}_t \) at time \( t \) (under \( Q_d \), conditional on default occurring at \( \tau = t \)) by

\[
\widehat{\delta}_t := \int_Z 1_{[t \leq T]} \delta(z, t) dF(t; z).
\]

5Clearly, \( \tau \) may also have different distributions under \( Q_d \) and \( Q_f \) if \( \lambda_d = \lambda_f \) but e.g. \( W \neq \bar{W} \).
Then the domestic and the foreign default intensity satisfy the relationship

\[ \lambda_f(t) = (1 - \delta_t) \lambda_d(t). \]

**Proof.** Substitute (4.1) in theorem 58.

Intuitively, the adjustment factor between domestic and foreign default intensity is equal to the locally expected devaluation of the FX rate \( X \), if a default were to happen at time \( t \).

### 4.3 Pricing Credit Securities

We turn to the pricing of credit-risky claims. Single-name credit derivatives usually involve two types of cash flows: payments upon survival at prespecified (deterministic) dates and claims payable at default. We first provide general pricing rules for these types of claims. Then we apply these rules to price default-sensitive fixed income securities such as bonds and CDSs in different currencies.

#### 4.3.1 General Pricing Rules

Note that by assuming \( W \) is a Brownian motion not only with respect to its natural filtration \( \mathbb{F}^W \) but also with respect to the larger model filtration \( \mathbb{F} \), we implicitly assumed that the following hypothesis holds under \( Q \).

\[ H: \text{Every square-integrable } \mathbb{F}^W\text{-martingale is an } \mathbb{F}\text{-martingale.} \]

Equivalent characterizations of hypothesis \( H \) are in Elliott et al. (2000) or lemma 35, chapter 2. The financial interpretation of \( H \) hypothesis is that if the (default-free) \( \mathbb{F}^W \)-market is arbitrage-free, then default does not introduce arbitrage in this market. Also note that \( t \), being \( \mathbb{F} \)-totally inaccessible, automatically avoids the \( \mathbb{F}^W \)-stopping times (cf. assumption 5 in chapter 2). In presence of these properties and an additional technical assumption (see below) the valuation of a “defaultable claim” of the form \( G \mathbb{1}_{T > T} \) with \( G \in L^1(\mathbb{F}^W_T) \) can be reduced to the pricing of a “default-free” claim, i.e. a conditional expectation of a related \( \mathbb{F}^W_T \)-measurable random variable (see e.g. Elliott et al. (2000)). The extension of this method to the portfolio case was treated in lemma 45 of chapter 2.

---

6By the predictable representation property of Brownian motion (see e.g. Karatzas and Shreve (1991), theorem 4.15, page 182), every square-integrable \( \mathbb{F}^W \)-martingale is of the form \( M_t = M_0 + \int_0^t \sigma_s dW_s \) where \( \sigma_t \) is \( \mathbb{F}^W \)-predictable with \( \mathbb{E} \left[ \int_0^T \| \sigma_t \|^2 dt \right] < \infty \).
Assumption 14. $\mu, \sigma$ are $\mathbb{F}^W$-predictable processes and, restricted to \( \{ t \leq \tau \} \), \( F(t; z) \) is an $\mathbb{F}^W$-predictable function.

Then $\lambda_t$ is $\mathbb{F}^W$ adapted and the following lemma gives the rule for pricing defaultable claims.

**Lemma 60.** Let $G \in L^1(\mathcal{F}^W_T)$. Then

$$
\mathbb{E} \left[ 1_{[\tau < T]} G \mid \mathcal{F}_T \right] = 1_{[\tau < T]} \mathbb{E} \left[ e^{-\int_{\tau}^T \lambda_s ds} G \mid \mathcal{F}_T \right].
$$

Note that on the lefthandside of the equation above, $\mathcal{F}_T$ can be replaced by $\mathcal{F}^W_T$ due to hypothesis. With Fubini’s theorem and since $\tau$ is totally inaccessible, this also implies that for any $\mathbb{F}^W$-predictable $g_t$ with $\int_0^T g_t dt \in L^1$, it holds

$$
\mathbb{E} \left[ \int_{t}^{T} 1_{[t \leq \tau]} g_s ds \mid \mathcal{F}_t \right] = 1_{[\tau < T]} \int_t^T \mathbb{E} \left[ e^{-\int_{t}^{s} \lambda_u du} g_s \mid \mathcal{F}_t \right] ds \quad (4.5)
$$

the lefthandside of which can be interpreted as (is) the value of a continuous fee payable until default. Using the property of the predictable compensator we derive a rule for pricing cash flows due at default.

**Corollary 61.** Let $f(z, t)$ be an $\mathbb{F}^W$-predictable $Q \times \nu(dz, dt)$-integrable function. Define $G_t := \int_Z f(z, t) dF(t; z)$. Then

$$
\mathbb{E} \left[ 1_{[t \leq \tau]} f(\Delta J_t, \tau) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \int_{t}^{T} \int_Z 1_{[t \leq \tau]} f(z, s) \mu(dz, ds) \mid \mathcal{F}_t \right] = 1_{[\tau < T]} \int_t^T \mathbb{E} \left[ e^{-\int_{t}^{s} \lambda_u du} G_s \mid \mathcal{F}_t \right] ds.
$$

**Proof of Lemma 60.** By the dominated convergence theorem, it is sufficient to prove the claim for $G$ bounded. Then

$$
M_t := \mathbb{E} \left[ e^{-\int_{t}^{T} \lambda_s ds} G \mid \mathcal{F}_t^W \right]
$$

is a bounded $\mathbb{F}^W$-martingale, hence a.s. continuous. Next, we define the $\mathcal{F}^W$-local martingale $Y_t := 1_{[t \leq \tau]} \int_0^T \lambda_s ds$. By hypothesis (see (i) in lemma 35 in chapter 2), $M$ is also an $\mathbb{F}^W$-martingale and thus a.s. continuous. Since $Y_t$ is

---

7This implies $1_{[t \leq \tau]} = 1_{[t \leq \tau]} Q \times l(dt)$-a.e.

8$\mathbb{E} \left[ \int_0^T 1_{[t \leq \tau]} f(\cdot, t) dF(t; z) \lambda_s dt \right] < \infty$ is sufficient.
purely discontinuous, $MY$ is also an $\mathbb{F}$-local martingale.\footnote{Note that this follows only from hypothesis $H$.} Further $|M_t Y_t| \leq \mathbb{E}[|G| | \mathcal{F}_t]$, hence $MY$ is bounded and thus an $\mathbb{F}$-martingale which implies

$$
1_{\{t < T\}} \mathbb{E}\left[ e^{-\int_t^T \lambda(s)ds} G_T \middle| \mathcal{F}_t \right] = M_t Y_t = \mathbb{E}[M_T Y_T | \mathcal{F}_T] = \mathbb{E}\left[ 1_{\{t < T\}} G_T \middle| \mathcal{F}_T \right].
$$

**Proof of corollary 6.1.** By theorem 1.8 in Jacod and Shiryaev (1988), chapter II, condition $\mathbb{E}\left[ \int_0^T \int_Z 1_{\{s \leq t\}} f(z, s)dF(t; z)\lambda_t dt \right] < \infty$ (or $f(z, t) Q \times \nu(dz, dt)$-integrable) ensures that $\int_0^T \int_Z 1_{\{s \leq t\}} f(z, s)(\mu - \nu)(dz, ds)$ is a martingale. Thus

$$
\mathbb{E}\left[ 1_{\{t \in (0, T]\}} f(\Delta J_t, \tau) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[ \int_t^T \int_Z 1_{\{s \leq t\}} f(z, s)\mu(dz, ds) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[ \int_t^T 1_{\{s \leq t\}} G_s \lambda_s ds \middle| \mathcal{F}_t \right]
$$

Then the claim follows from (4.5).

Hence, for defaultable claims as well as for payoffs at default, it remains to evaluate conditional expectations of the form

$$
\mathbb{E}\left[ e^{-\int_t^T \lambda_s ds} G \middle| \mathcal{F}_t \right].
$$

A popular class of models where these quantities can be expressed in closed-form are the affine models. We give an affine jump diffusion (AJD) version of the model in chapter 5 but we first discuss the valuation of bonds and CDSs in different currencies in the general framework.

### 4.3.2 Bonds and CDSs in Different Currencies

Note that in the abstract pricing formulas above, the stochastic discount factors $\beta_i(t), i = d, f$ are implicitly included in the $\mathcal{F}_t^W$-measurable random variable $G$. Therefor and in order to have the “same conditions” under the FSMM as under the DSMM, we need an additional assumption.

**Assumption 15.** The FX characteristics $\sigma_X$ and $\delta(z, t)$ and the interest rates $r_i(t), i = d, f$ are $\mathcal{F}_t^W$-predictable.
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The $\mathbb{P}^W$-predictability of $\sigma_X$ implies that $\tilde{W}$ is $\mathbb{P}^W_\tau$-adapted and hence $\mathbb{P}^W = \mathbb{P}^\tilde{W}$, and $\delta(t, z) \cdot \mathbb{P}^W_\tau$-predictable ensures that $\tilde{F}(t; z)$ is also a $\mathbb{P}^W$-predictable function and thus $\lambda_f(t)$ is $\mathbb{P}^W$-predictable.

Now we are ready to apply the rules of section 4.3.1 to the valuation of bonds and CDSs in both currencies. First note that according to (4.3) with $Z = 1$, domestic and foreign default-free zero coupon bonds (ZCB), which pay one unit of the respective currency at their maturity $T$ with certainty, trade at

\[ B_i(t, T) = E_i \left[ e^{-\int_t^T r(s)ds} \bigg| \mathcal{F}_t \right] \]

units of $c_i$, $i = d, f$.

**Assumption 16 (Recoveries).** We model the loss given default (of a $c_i$-bond) with a predictable $[0, 1]$-valued function $q_i(z, t)$, $i = d, f$, which captures the dependency of recovery on the default severity marker $z_t = \Delta J_t$.

This way, conditional dependence between recovery and FX devaluation (given default occurs) becomes a natural feature of the model through their joint dependence on the default severity marker $z_t$.

**Defaultable Zero Coupon Bonds**

Similar as in (4.4), we define the $c_i$-locally expected loss given default rates

\[ \tilde{q}_i(t) := \int_Z q_i(z, t)dF_i(t; z), \quad i = d, f. \]

We present the prices of ZCB under a variety of recovery assumptions that have been proposed in the literature:

**Zero Recovery.** A $c_i$-ZCB with zero recovery (ZR) pays $1_{[\tau > T]}$ units of $c_i$ at its maturity $T$, $i = d, f$. By equation (4.3) with $Z = 1_{[\tau > T]}$ and lemma 60, the price of a $c_i$-ZCB with ZR is

\[ \overline{B}_i(t, T) := E_i \left[ e^{-\int_t^T r(s)ds} 1_{[\tau > T]} \bigg| \mathcal{F}_t \right] = 1_{[\tau > T]}E_i \left[ e^{-\int_t^T (r(s) + \lambda_i s)ds} \bigg| \mathcal{F}_t \right] \]

in the respective payoff currency $c_i$. Note that not only are the payoffs in different currencies discounted with different interest rates, but also with different default intensities.

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10The recovery rate $1 - q_i(z, t)$ of a foreign-issued bond is in general not equal to that of a domestic bond because the respective bankruptcy courts use different legal rules.
4.3. Pricing Credit Securities

Recovery of Par. (See e.g. Duffie (1998).) In addition to the survival payoff of \(1_{\{t > T\}}\) units of \(c_i\) at maturity \(T\), in the recovery of par setting (RP) a defaultable ZCB pays

\[
1 - q_i(z, \tau) = \int_0^T \int \lambda(z, t) \mu(dz, dt)
\]

units of \(c_i\) at the default time \(\tau\) if \(\tau \leq T\). Using corollary 61, the no-arbitrage prices of these bonds before default (i.e. on the set \(\{t < \tau\}\)) are thus

\[
B_i^{RP}(t, T) = B_i(t, T) + \int_t^T E_i \left[ e^{-\int_t^\tau (\mu + \lambda(s)) ds} (1 - \tilde{q}_i(s)) \lambda_i(s) \right] d\tau
\]

\(i = d, f\).

Multiple Defaults. (See e.g. Schönbucher (1998) or Duffie et al. (2003).) As pointed out in section 3.1, the ISDA definition of default includes many scenarios (restructuring, temporary insolvency etc.) in which a company may continue running its business after a default; usually the lenders accept a reduction of the notional (or/and the coupons) on their outstanding bonds in this case. Reflecting this, a multiple default (MD) ZCB can default more than once and, at any default, the promised payoff is reduced by a fraction \(q_i\) of its pre-default-value. We therefore write

\[
\overline{B}_i^{MD}(t, T) = E_i \left[ e^{-\int_t^\tau (\mu + \lambda(s)) ds} p_i^{MD}(T) \right] \mid \mathcal{F}_t
\]

for the price of an MD-ZCB in \(c_i\) with maturity \(T\), where the payoff \(p_i^{MD}(T)\) solves the SDE

\[
\frac{dp_i^{MD}(t)}{p_i^{MD}(t)} = - \int \lambda_i(z, t) \mu(dz, dt), \quad p_i^{MD}(0) = 1,
\]

\(i = d, f\). The value \(p_i^{MD}(t)\) is the remaining promised payoff at time \(t\). In order to derive closed form solutions of ZCB prices in the MD setting, knowledge of the compensator \(\nu(dz, dt)\) after \(\tau\) is needed. That is the distributional properties of the stopping times \(\tau_j := \inf\{t > \tau_{j-1}; N_t > N_{\tau_{j-1}}\}\) denoting the time of the \(j\)th default and the conditional distribution functions \(F(\tau_j; z)\) of the \(j\)th default severity given the \(j\)th default occurs. Here we make the additional

Assumption 17 (MD). \(A_t = \int_0^t \lambda_s ds\) and \(F(t; z)\) is \(\mathbb{F}^W\)-predictable for all(!) \(t\) and the loss fractions \(q_i(z, t)\) are \(\mathbb{F}^W\)-predictable for \(i = d, f\).
Under this assumption, $N_t$ is a Cox process (combine theorem 44 and remark 34, chapter 2) and we have the following

**Proposition 12.**

$$\overline{B}_i^{MD}(t, T) = p_i^{MD}(t) E_i \left[ e^{-\int_t^T (r_i + \tilde{q}_i(t)) ds} \mid \mathcal{F}_t \right]$$

This shows that at any default, the price of an MD ZCB is reduced by same fraction as is its promised payoff.\(^{11}\)

**Proof.** Note that $\tilde{q}_i(t) := \int_Z q_i(z, t)dF_i(t; z)$ is $\mathbb{F}^W$-predictable for all $t$ and let $G_i$ be bounded $\mathcal{F}_t^W$-measurable random variables for $i = d, f$. Then the proof goes much in line with that of lemma 60. We define the $\mathcal{Q}_t$-martingales $M_i(t) := E_i \left[ e^{-\int_t^T \tilde{q}_i(t)\lambda_i(s) ds} G_i \mid \mathcal{F}_t \right]$ and the $\mathcal{Q}_t$-local martingales $Y_i(t) := p_i^{MD}(t)e^{\int_t^T \tilde{q}_i(s)\lambda_i(s) ds}$ for $i = d, f$.\(^{12}\) Due to $\mathbb{H}$ hypothesis $M_i$ is also an $\mathbb{F}^W$-martingale under $\mathcal{Q}_t$, hence continuous, and since $Y_i$ is purely discontinuous, $M_i Y_i$ is also a $\mathcal{Q}_t$-local martingale for $i = d, f$. Further $|M_i(t)Y_i(t)| \leq E_i \left[ |G_i| \mid \mathcal{F}_t \right]$, which is bounded, hence $M_i Y_i$ is a $\mathcal{Q}_t$-martingale which shows

$$E_i \left[ p_i^{MD}(T) G_i \mid \mathcal{F}_t \right] = E_i \left[ M_i(T) Y_i(T) \mid \mathcal{F}_t \right] = M_i(t) Y_i(t)$$

for $i = d, f$. Then setting $G_i := e^{-\int_t^T r_i(s)ds}$ yields the claim.\(\square\)

**Recovery of Treasury.** (See e.g. Jarrow and Turnbull (1995).) Under recovery of treasury (RT), if default occurs before maturity, a ZCB holder receives

$$1 - q_i(z, \tau) = \int_0^T \int_Z (1-q_i(z, t)) 1_{\{\tau \leq t\}}(dz, dt)$$

default-free ZCBs with the same maturity $T$ and par value 1 at default. Using the domestic and foreign $T$-forward measures $P_d^T$ and $P_f^T$, as introduced in

\(^{11}\) By $\mathbb{H}$ hypothesis, $E_i \left[ e^{-\int_t^T (r_i + \tilde{q}_i(t))ds} \mid \mathcal{F}_t \right] = E_i \left[ e^{-\int_t^T (r_i + \tilde{q}_i(t))ds} \mid \mathbb{F}_t^W \right]$ is continuous a.s.

\(^{12}\) $Y_i(t)$ is a $\mathcal{Q}_t$-local martingale because

$$\frac{dY_i(t)}{Y_i(t-)} = \tilde{q}_i(t)\lambda_i(t)dt - \int_Z q_i(z, t)\mu(dz, dt) = -\int_Z q_i(z, \tau)(\mu - \nu_i)(dz, dt)$$
4.3. Pricing Credit Securities

Jamshidian (1987) and defined by

\[
\frac{dP^T_i}{dQ_i} |_{\mathcal{F}_t} := L^T_i(t) := \frac{B_i(t, T)}{B_i(0, T) h_i(t)}
\]

for \(i = d, f\), we can always reduce the RT case to a related RP case. Namely, on the set \(\{ t < \tau \}\), the value of the RT-recovery satisfies

\[
E^T_i \left[ \int_t^T e^{-\int_t^s r_i(v) dv} \left( 1 - q_i(z, s) \right) B_i(s, T) 1_{[s \leq \tau]} \mu(dz, ds) \bigg| \mathcal{F}_t \right] = \int_t^T E^T_i \left[ B_i(s, T) e^{-\int_t^s r_i(v) dv} \left( 1 - \tilde{q}_i(s) \right) \lambda_i(s) \bigg| \mathcal{F}_t \right] ds = B_i(t, T) \int_t^T E^T_i \left[ (1 - \tilde{q}_i(s)) \lambda_i(s) \bigg| \mathcal{F}_t \right] ds \quad (4.6)
\]

where \(E^T_i\) denotes expectation wrt. \(P^T_i\) for \(i = d, f\).

Credit Default Swaps

The cash flows involved in a CDS contract were already described in section 3.1. After default a CDS contract is unwound, thus we always assume that default has not yet occurred (i.e. we always assume we are on the set \(\{ \tau > t \}\)) in order to avoid trivialities. We also assume the amount of notional insured is always 1 unit of \(c_i, i = d, f\).

For a contract with maturity \(T\) entered at \(t\), the fee leg then consists of payments \(\bar{s}_i(t, T) \times (T_j - T_{j-1})\) in \(c_i\) at quarterly dates \(T_j\). We approximate this payment stream by an integral. The Value of the Fee Stream (in its payoff currency) is thus given by

\[
V^f_{i, ee}(t, T) = \bar{s}_i(t, T) \int_t^T B_i(t, s) ds
\]

The fee stream of a CDS can be interpreted as a defaultable coupon bond with continuously paid coupon \(\bar{s}_i(t, T) ds\) and par value zero.

Value of Protection Leg. The protection payment of a CDS takes place if and only if \(\tau \leq T\). In this case it is made at the time of default \(\tau\), and its size is \(1 - \bar{B}_{ctd}(\tau)\) units of \(c_i\), where \(\bar{B}_{ctd}\) is the price of the cheapest-to-deliver bond. As argued in section 3.1, the cheapest-to-deliver bond will usually be a
coupon-bearing bond and the only relevant quantity is its relative discount to par value in the currency in which it was issued. We model this bond using the RP setup which is the most appropriate choice for CDS recovery modelling (see e.g. Houweling and Vorst (2005)). Then the value of the protection leg is

\[
V_{i}^{\text{prot}}(t, T) = \int_{t}^{T} \mathbb{E}_{t}\left[ e^{-\int_{t}^{s}(r_{i}+\lambda_{i})(v)dv}\left(1-\tilde{q}_{i}(s)\right)\lambda_{i}(s) \bigg| \mathcal{F}_{t} \right] ds. \tag{4.7}
\]

The fair CDS rate \( \tilde{s}_{i}(t, T) \) is obtained when the value of fee and protection leg are equal, i.e.

\[
\tilde{s}_{i}(t, T) = \frac{V_{i}^{\text{prot}}(t, T)}{\int_{t}^{T} B_{i}(t, s) ds}. \tag{4.8}
\]

And at the short end of the term structure, CDSs spreads satisfy the intuitive relationship given in

**Lemma 62.** Let \( \mathbb{E}_{t}\left[ \sup_{t \in [0, T^{*}]} \lambda_{i}(t) \right] < \infty \) hold. Then, for every \( t \), we have

\[
\lim_{T \downarrow t} \tilde{s}_{i}(t, T) = (1 - \tilde{q}_{i}(t))\lambda_{i}(t) \quad a.s.
\]

**Proof.** By dominated convergence and since \( \lambda_{i} \) and \( \tilde{q}_{i} \) are càdlàg, for fixed \( t \), both \( \mathbb{E}_{t}\left[ e^{-\int_{t}^{T}(r_{i}+\lambda_{i})(v)dv}\left(1-\tilde{q}_{i}(T)\right)\lambda_{i}(T) \bigg| \mathcal{F}_{t} \right] \) and \( B_{i}(t, T) \) are a.s. right-continuous in \( T \). Thus, on \( \{ \tau > t \} \)

\[
\lim_{T \downarrow t} \tilde{s}_{i}(t, T) = \lim_{T \downarrow t} \frac{\int_{t}^{T} \mathbb{E}_{t}\left[ e^{-\int_{t}^{s}(r_{i}+\lambda_{i})(v)dv}\left(1-\tilde{q}_{i}(s)\right)\lambda_{i}(s) \bigg| \mathcal{F}_{t} \right] ds}{\int_{t}^{T} B_{i}(t, s) ds}
\]

\[
= \lim_{T \downarrow t} \frac{\lim_{T \downarrow t} \mathbb{E}_{t}\left[ e^{-\int_{t}^{T}(r_{i}+\lambda_{i})(v)dv}\left(1-\tilde{q}_{i}(T)\right)\lambda_{i}(T) \bigg| \mathcal{F}_{t} \right]}{\lim_{T \downarrow t} B_{i}(t, T)}
\]

\[
= \mathbb{E}_{t}\left[ \lim_{T \downarrow t} e^{-\int_{t}^{T}(r_{i}+\lambda_{i})(v)dv}\left(1-\tilde{q}_{i}(T)\right)\lambda_{i}(T) \bigg| \mathcal{F}_{t} \right]
\]

\[
= (1 - \tilde{q}_{i}(t))\lambda_{i}(t) \quad a.s. \quad \square
\]

**Remark 63.** A sufficient condition for \( \mathbb{E}_{t}\left[ \sup_{t \in [0, T^{*}]} \lambda_{i}(t) \right] < \infty \) is e.g. that

\[
\mathbb{E}_{t}\left[ \int_{0}^{T} \left| \mu_{2i}(t) \right| + \|\sigma_{2i}(t)\|^{2} dt \right] < \infty.
\]
4.4 Relationship Between Domestic and Foreign CDS Spreads

In this section we want to achieve some intuition regarding the relationship between domestic and foreign CDS spreads $\bar{s}_d$ and $\bar{s}_f$. We will identify the three components of credit risk (default intensity risk, default event risk and default severity risk) and the default-free term structures of interest rates as the driving factors of this relation.

Note that if default occurs, there will be one unique recovery rate to both CDS due to the protection buyer’s delivery option discussed on section 3.1. The quantities $1 - \tilde{q}_d(t)$ and $1 - \tilde{q}_f(t)$ however, may differ from each other (even if $\tilde{q}_d(t)$ and $\tilde{q}_f(t)$ are deterministic) because they are the expectations of this recovery rate under different measures.

Mathematically this becomes clear from the definition of $\tilde{F}$ in (ii) of Girsanov’s theorem 58. From an economic viewpoint this is explained by the possible dependence of recovery and devaluation fraction of the foreign currency at default. E.g. if recovery and devaluation show negative correlation, then the protection buyer of a foreign CDS is unprivileged (compared to a domestic protection buyer): In a light default scenario, i.e. when recovery is large, the LGD paid to the protection buyer will be small. In the case of a severe default, i.e. when recovery is small, the LGD amount in foreign currency will be large, but its value in domestic currency will typically be reduced by a high devaluation fraction. Clearly, when the devaluation fraction $\delta(z, t)$ is constant, then $\tilde{q}_d(t) = \tilde{q}_f(t)$, and when the recovery rate function is constant, i.e. $q(z, t) = q$, then $\tilde{q}_d(t) = \tilde{q}_f(t) = q$ is also constant. In the sequel we assume such a $q$ exists but one could easily include the considerations concerning recovery risk (=default severity risk) in the subsequent discussion.

The value of the protection leg in (4.7) was expressed as an integral over securities with a payoff in units of defaultable ZCBs. The following probability measures are often used in this situation (see Schönbucher (1999) and Schönbucher (2004)). $\bar{P}^T_d$ and $\bar{P}^T_f$, defined by the Radon-Nikodym derivatives

$$\frac{d\bar{P}^T_i}{d\bar{Q}^T_i} := \bar{L}^T_i(t) := \frac{B_i(t, T)}{\bar{B}_i(0, T) b_i(t)}.$$  

$i = d, f$, are called the domestic and the foreign $T$-survival measure.\(^{13}\) Let $\bar{E}_i$ be the expectation operator wrt $\bar{P}^T_i$ for $i = d, f$. Further we define the

\(^{13}\)We assume $\bar{L}^T_i$ is a $\bar{Q}_i$-martingale.
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term structures of default intensities

\[ \lambda_i(t, T) := E^d_0 \left[ \lambda_i(T) \mid \mathcal{F}_t \right] \]

and defaultable weights \( \overline{w}_i(s; t, T) := \overline{B}_i(t, s) \int_t^T \overline{B}_i(t, s) ds \) for \( t \leq s \leq T \) and \( i = d, f \). Then (4.8) simplifies to

\[ \overline{s}_i(t, T) = q \int_t^T \overline{w}_i(s; t, T) \lambda_i(t, s) ds. \] (4.9)

The weights \( \overline{w}_i \) are proportional to the defaultable ZCB price for the corresponding maturities. Overall, the price curve of defaultable ZCBs will be downward-sloping. The currency with the higher level of defaultable interest rates will have a stronger downward slope in the defaultable ZCB price \( \overline{B}_i(t, s) \) and thus it will have a higher weight on early (small \( s \)) values of \( \lambda_i(t, s) \), compared to the currency with a lower level of defaultable interest rates.

In many cases the fact that we have different weights \( \overline{w}_d \) and \( \overline{w}_f \) will however have only a small influence on the differences in CDS rates, because the weights \( \overline{w}_i \) will not differ by much, and the term-structure of default intensities will be quite flat (i.e. \( \lambda_i(t, s) \) is close to a constant function of \( s \)). For flat term structures of \( \lambda_i(t, s) \) the weighting has no influence at all; and if the slope is small, the influence of slightly varying weights will also be small. In these cases we may argue that \( \overline{s}_i(t, T) \approx q \lambda_i(t) \).

It remains to analyze the difference between the domestic and foreign term structure of default intensities. Chaining densities we can define the \( P_d^T \)-martingale

\[ T_{df}^T(t) := \frac{dP^T_0}{dP^T_d} \bigm| \mathcal{F}_t = L^X(t) \frac{L^T_f(t)}{L^T_d(t)} = X_0 \frac{\overline{B}_f(t, T)/\overline{B}_d(t, T)}{\overline{B}_d(0, T)/\overline{B}_d(0, T)}. \]

It should be noted that when domestic and foreign ZCB price do not differ by much, then \( T_{df}^T \) is almost proportional to the FX rate \( X \). In any case

\[ \lambda_f(t, T) = E^d_0 \left[ \frac{T_{df}^T(T)}{L_{df}^T(t)} (1 - \tilde{\delta}_T) \lambda_d(T) \mid \mathcal{F}_t \right] \]

\[ = E^d_0 \left[ (1 - \tilde{\delta}_T) \lambda_d(T) \mid \mathcal{F}_t \right] + \text{cov}_d^T \left[ \frac{T_{df}^T(T)}{L_{df}^T(t)}, (1 - \tilde{\delta}_T) \lambda_d(T) \mid \mathcal{F}_t \right]. \]
Thus, different term structures of default intensities may arise for two reasons. First, when domestic and foreign default intensities are not equal, i.e. when the locally expected devaluation fraction $\delta_t$ is not identically zero. Second, when $L_{df}^T$ and the domestic default intensity $\lambda_d(t)$, respectively $(1 - \delta_t)\lambda_d(t)$, are correlated under the domestic $T$-survival measure.

Overall we have identified four drivers of the difference between CDS rates in domestic and foreign currency. First, foreign and domestic default intensities are not equal, when the FX rate is subject to default event risk, i.e. when $X$ jumps at default (unless the expected devaluation fraction is equal to zero). Second, foreign and domestic term structure of default intensities do not coincide when there is nonzero covariance between $\lambda_d$ (respectively $\delta\lambda_d$) and the $P_{d}^T$-martingale $L_{df}^T$ under the domestic survival measure $P_{d}^T$ for any $T$. Conversely, it is easy to see that these term structures are equal, up to the multiplicative constant $(1 - \delta)$, when $\delta_t$ is constant, the default-free interest rates $r_d, r_f$ are independent of $\lambda$ and $[\lambda, X]=0$, i.e. when $X$ is not subject to default intensity risk.

Third, the locally expected LGD rates $\tilde{q}_d(t)$ and $\tilde{q}_f(t)$ differ if the LGD function $q(z, t)$ and devaluation fraction function $\delta(z, t)$ are correlated under the DSMM $Q_d$, which implies that devaluation is not independent of default severity risk. And fourth, the slopes of domestic and foreign term structure of default free interest rates determine (via their impact on the weights $w_t$) how the respective term structures of default intensities must be weighted in order to derive the CDS rate in each currency.

### 4.5 Extensions: Some other Default-Sensitive FX Derivatives

We used CDS spreads with denomination in different currencies to illustrate the key ideas and methods which must be used to analyse credit risk in multiple currencies, but there are also a number of other situations in which the methods of this thesis can be used, for example some exotic credit derivatives and the analysis of counterparty credit risk in OTC derivatives transactions. Here we present some of these connections.

As a first application, we define default-sensitive equivalents for the most common FX derivatives such as FX swaps and FX forwards. These derivatives behave like their default-free equivalents except that they are cancelled at default. It will turn out that some of these instruments have a natural connection to the problem of pricing CDSs in different currencies.
We define the defaultable FX forward rate $\overline{X}$ as

$$\overline{X}(t, T) = X(t) \frac{B_f(t, T)}{B_d(t, T)}.$$  

This is the forward exchange rate to be used in a FX forward contract which is cancelled at default (i.e. if a default occurs before the settlement date $T$). Using the defaultable zero coupon bond prices with zero recovery, the defaultable FX forward rates can be given in closed-form.

In a defaultable currency swap one side of the swap pays a stream of the defaultable currency swap rate $\overline{x}(t, T)$ in $c_f$, and the other side of the swap pays a stream of 1 in $c_d$, and both payment streams stop at default, or — if no default occurs before $T$ — at the maturity date $T$. (In contrast to a classical currency swap we assume no exchange of principal at maturity.) As both legs of the currency swap must have the same value, we reach the following representation for $\overline{x}(t, T)$:

$$\overline{x}(t, T) = \frac{\int_t^T \frac{B_f(t, s)}{B_d(t, s)} ds}{\int_t^T \frac{B_d(t, s)}{B_d(t, s)} ds}.$$ (4.10)

Hence the defaultable FX swap rate $\overline{x}(t, T)$ satisfies

$$\overline{x}(t, T) = X(t) \frac{\int_t^T \frac{B_f(t, s)}{B_d(t, s)} ds}{\int_t^T \frac{B_d(t, s)}{B_d(t, s)} ds} = \int_t^T \frac{\overline{X}(t, s)B_d(t, s)}{B_d(t, s)} ds.$$  

Thus, we can view the rate $\overline{x}$ as a weighted average of the defaultable FX-forward rates $\overline{X}(t, s)$ over the life of the swap, or as a survival-contingent exchange rate for payment streams. Alternatively, it can be viewed as the relative price of a unit payoff stream (an annuity) in foreign-currency $\int_t^T \frac{B_f(t, s)}{B_d(t, s)} ds$, measured in units of the domestic defaultable annuity $\int_t^T \frac{B_d(t, s)}{B_d(t, s)} ds$.

The defaultable currency swap in (4.10) allows us to transform any CDS fee stream in $c_f$ into a corresponding fee stream in $c_d$. This leads naturally to the following instrument:

A Quanto CDS is a credit-default swap, which has a protection payment in one currency (e.g. $c_f$), but the fee payment is made in another currency (e.g. $c_d$) at the rate $\overline{x}$. Using the defaultable currency swap to transform the fee streams we reach immediately: $\overline{x}(t, T) = \frac{\int_t^T \frac{B_f(t, s)}{B_d(t, s)} ds}{\int_t^T \frac{B_d(t, s)}{B_d(t, s)} ds} \overline{x}(t, T)$.

A Default-Contingent FX Forward is a contract to exchange 1 unit of foreign currency for $\overline{x}(t, T)$ units of domestic currency at the time of default $\tau$, if and only if default occurs before its maturity $T$. 

This instrument may seem a bit artificial but we believe there should be interest in it. Often, investors in foreign companies have secured strong collateralisation of their loans. Upon a credit event, these investors will have to liquidate the collateral and convert a significant amount of cash back into their domestic currency. Such investors might be interested in a default-contingent FX forward, i.e. a FX forward contract which allows the investor to exchange if (and only if) a credit event has occurred.

Assuming constant recovery rates $1 - q$, we can model this contract as a portfolio of two CDS contracts, one of which is short protection in foreign currency with notional $1/q$ (thus paying one unit of $c_f$ at default), and one that is long protection in domestic currency with notional $X/q$ (which pays $X/q$ units of $c_d$ at default). We can convert the fees of the $c_f$ CDS into domestic currency using the quanto CDS introduced above to reach a net value in $c_d$ for the default-contingent FX forward of

$$\frac{1}{q} \left( \bar{x}_f(t, T) \bar{x}(t, T) - \bar{X}_d(t, T) \bar{x}_d(t, T) \right) \int_t^T B_d(t, s) ds,$$

which implies that the fee in domestic currency to be paid for this contract is $\frac{1}{q} (\bar{x}_f \bar{x} - \bar{X}_d \bar{x}_d)$. Finally, for the contract to be fair (i.e. for the fee to be zero), we need to set the default-contingent FX forward rate to

$$\bar{X}_d(t, T) = \bar{x}(t, T) \frac{\bar{x}_f(t, T)}{\bar{x}_d(t, T)}$$

(4.11)

The Ratio of Domestic to Foreign CDS Spreads: From (4.11) we reach immediately

$$\frac{\bar{x}_f(t, T)}{\bar{x}_d(t, T)} = \frac{\bar{X}_d(t, T)}{\bar{x}(t, T)}.$$  

(4.12)

The ratio of the foreign to the domestic CDS spreads is the ratio of two exchange rates: The exchange rate $\bar{X}_d(t, T)$ that applies at the time of default only if a default occurs over $[t, T]$ to the exchange rate $\bar{x}(t, T)$ that applies over $[t, T]$ only for the time the obligor survives.
Chapter 5

AJD-Version of the Model

One way to specify a tractable version of the model, which can also be statistically estimated, is to work with an affine jump diffusion (AJD) process, which drives all relevant processes and variables in the market. When taking conditional expectations (and when $\tau$ is the first jump event as in our setup), one can always get rid of the jump part of an AJD as in 4.3.1 and is then left with an affine diffusion (AD) part of an AJD. An AD is a continuous time-homogeneous $N$-dimensional (for some $N \in \mathbb{N}$) Markov\(^1\) process $Y_t$, characterized as the unique strong solution of an SDE

$$dY_t = \gamma(Y_t) dt + \sigma(Y_t) dW_t, \quad Y(0) = Y_0,$$

(5.1)

where $\gamma(\cdot)$ and $\sigma \sigma(\cdot)$ are affine functions.\(^2\) The main feature of these processes, which makes them so appealing for applications in fixed income derivatives pricing, is that under technical conditions, for affine functions $f$ and $g$,

$$\mathbb{E}\left[ e^{\int_t^T f(Y_s) ds} g(Y_T) \left| \mathcal{F}_t \right. \right] = (A(T-t) + B(T-t)' Y_t) e^{A(T-t)+B(T-t)' Y_t},$$

where $A, B, A, B$ are real functions determined by a set of ordinary differential equations (ODEs) involving $\gamma, \sigma \sigma'$ and $f, g$.

There is a large literature on AD in the context of fixed income derivatives pricing. Particularly well-known examples of (one-dimensional) AD models in finance are the CIR (see Cox et al. (1985)) and the Vasicek (see Va-

\(^1\)Under $\mathbb{H}$ hypothesis, if $Y_t$ is Markov with respect to $\mathbb{F}^W$, then it is Markov with respect to $\mathbb{F}$.

\(^2\)Indeed as shown by Duffie and Kan (1996), under light technical assumptions given in section 5.1, SDE (5.1) admits a unique strong solution on $(\Omega, \mathcal{F}, \mathbb{P})$ because the drift $\gamma$ is Lipschitz-continuous and the volatility matrix $\sigma$ satisfies a multivariate extension of the Yamada condition.
sick (1977) model of the term structure of interest rates. The general (multi-dimensional) case was first studied in Duffie and Kan (1996) and a succinct mathematical foundation of affine processes is given in Filipovic et al. (2003).

In this chapter, we first briefly review the properties and classification of ADs. We then specify an AD version of the FX-credit risk model and derive closed-form expressions for the bond and CDS prices as given in section 4.3.2 in the AJD setup. All notation from chapter 4 is carried over to this chapter.

### 5.1 Classification and Properties of the ADs

When one wants to parameterize the functions $\gamma$ and $\sigma \sigma'$ in (5.1), it has to be taken into account that $\sigma$ is a square root of the affine matrix function $\sigma \sigma'(Y)$, hence it may not be well-defined for all values of $Y$. In the light of this problem Dai and Singleton (2000) introduced a parameterization under which the admissibility of the matrix $\sigma \sigma'$ can be checked easily. They call a pair $(\gamma, \sigma \sigma')$ admissible if (5.1) admits a unique strong solution.\(^3\) Furthermore their parameterization allows for a simple classification of ADs. We present a (very slight) extension of their parameterization.\(^4\)

For $m \in \{0, 1, \ldots, N\}$ fixed, $\mathcal{A}_m(N)$ is the class of admissible $N$-dimensional ADs with $\sigma$ depending on exactly $m$ components of $Y$. Consider the following parameterized version of the SDE (5.1)

$$dY = (\Theta - K) dt + \sqrt{S} \, dW, \quad Y(0) = Y_0, \quad (5.2)$$

where $S$ is a diagonal matrix with $S_{ii} = a_i + \sum_{j=1}^{m} b_{ij} Y_j$ for $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^{N \times N}$, $Y_0, \Theta \in \mathbb{R}^N$ and $K \in \mathbb{R}^{N \times N}$. Now, if the following conditions are satisfied then $(\gamma, \sigma)$ is admissible and the solution $Y$ of (5.2) belongs to $\mathcal{A}_m(N)$:

- $b \geq 0$, and for all $1 \leq i \neq j \leq m$ we have:
  $$Y_{0i} \geq 0, \quad \Theta_i \geq 0, \quad a_i = 0, \quad K_{ij} \leq 0, \quad b_{ii} = 1, \quad b_{ij} = 0.$$

- and for all $m < k \leq N$ we have:
  $$\Theta_k = 0, \quad a_k \in [0, 1], \quad b_{ik} = 0 \text{ for all } 1 \leq i \leq N, \quad K_{ik} = 0 \text{ for all } 1 \leq i \leq m.$$

\(^3\)In particular, a real matrix $\sigma_t$ with $\sigma_t \sigma'_t = \sigma \sigma'(Y_t)$ is $Q$-a.s. definable for all $t \in [0, T^\tau]$.

\(^4\)A comment in Ait-Sahalia and Kimmel (2002) shows that the Dai and Singleton (2000) parameterization does not include all ADs.
Then the first $m$ components $Y_{t1}, \ldots, Y_{tm}$ of $Y_t$ are nonnegative and $\sigma(Y_t)$ depends only on these components. $Y$ is called a \textit{canonical representative} of the class $C_m(N)$ which is formed by all regular affine transforms of $Y$ (i.e. all processes $Z = \eta + \theta Y$, where $\eta \in \mathbb{R}^N$, $\theta \in \mathbb{R}^{N \times N}$ invertible). If furthermore:

- $\mathcal{K}_{ii} > 0$ for all $1 \leq i \leq N$,
- $\Theta_i > 0$ for all $1 \leq i \leq m$,
- $\alpha_i + \sum_{j=1}^{N} b_{ij} > 0$ for all $1 \leq i \leq N$,

then $(Y_{t1}, \ldots, Y_{tm})'$ remains strictly positive, $Y$ is a.s. non-explosive $[0, T^*]$ and it holds $E^W = E^Y$. In the sequel of the chapter we will always consider this case.

**Proposition 13** (Quadratic Variation). \textit{Let $Y$ be a canonical representative of $C_m(N)$ and $\beta \in \mathbb{R}^N$. Then the quadratic variation of $\beta' Y$ satisfies}

\[
\frac{d}{dt} [\beta' Y]_t = v(\beta) + w(\beta)' Y_t
\]

\textit{where $v$ and $w$ in matrix notation are given by}

\[
v(\beta) = a' \text{diag}(\beta) \beta = \beta' \text{diag}(a) \beta \quad \text{and} \quad w(\beta) = b' \text{diag}(\beta) \beta.
\]

**Proof.** By matrix multiplication and Itô’s rule we have

\[
\frac{d}{dt} [\beta' Y] = \beta' S \beta = \sum_{i=1}^{N} \beta_i^2 \left( a_i + \sum_{j=1}^{N} b_{ij} Y_j \right) = \sum_{i=1}^{N} \beta_i^2 a_i + \sum_{i,j=1}^{N} \beta_i^2 b_{ij} Y_j.
\]

We observe that $v(\cdot) \geq 0$ and $w(\cdot)$ is $\mathbb{R}_+^m \times \{0\}^{N-m}$-valued. Further note the rules

\[
v(\beta_1 + \beta_2) = v(\beta_1) + v(\beta_2) + 2 \beta_1 \text{diag}(a) \beta_2
\]
\[
w(\beta_1 + \beta_2) = w(\beta_1) + w(\beta_2) + 2 \beta_1 \text{diag}(b) \beta_2.
\]

Hence the partial derivatives of $v$ and $w$ wrt. $\beta$ in matrix notation are given by

\[
\frac{\partial v(\beta)}{\partial \beta} = 2a' \text{diag}(\beta) = 2\beta' \text{diag}(a) \quad \text{and} \quad \frac{\partial w(\beta)}{\partial \beta} = 2b' \text{diag}(\beta).
\]

The following lemmata for the calculation of \textit{extended transforms} are well-known (see e.g. Duffie and Kan (1996) or Dai and Singleton (2000)), in our parameterization they are:
Lemma 64. Let \( Y \) be the canonical representative of \( h_m(N) \) satisfying (5.2), \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R}^N \), and \( A : \mathbb{R} \to \mathbb{R} \) and \( B : \mathbb{R} \to \mathbb{R}^N \) solve the Riccati ODEs

\[
\frac{\partial}{\partial x} A(x) = \alpha + \Theta' B(x) + \frac{1}{2} v(B(x)), \quad \frac{\partial}{\partial x} B(x) = \beta - \mathcal{K}' B(x) + \frac{1}{2} w(B(x)) \tag{5.3}
\]

with initial conditions \( A(0) = 0 \) and \( B(0) = 0 \). If there exists \( \mathcal{B}^* < \infty \) with \( |\mathcal{B}| \leq \mathcal{B}^* \) on \([0, T]\) and \( E \left[ e^{\int_0^T w(B^*)Y_{t}dt} \right] < \infty \), then

\[
E \left[ e^{\int_0^T (\alpha + \beta' Y_{t})dt} \mid \mathcal{F}_t \right] = e^{A(T-t)+\mathcal{B}(T-t)'Y_{t}}.
\]

And the following lemma generalizes the result of lemma 64.

Lemma 65. Let the assumptions of lemma 64 be satisfied, \( \zeta \geq 0 \) and \( \xi \in \mathbb{R}_+^m \times [0]^{N-m} \) and \( A : \mathbb{R} \to \mathbb{R} \) and \( B : \mathbb{R} \to \mathbb{R}^N \) solve the ODEs

\[
\frac{\partial A(x)}{\partial x} = \Theta' B(x) + \mathcal{B}(x)' \text{diag}(\alpha) B(x), \quad \frac{\partial B(x)}{\partial x} = -\mathcal{K}' B(x) + b' \text{diag}(B(x)) B(x). \tag{5.4}
\]

with \( A(0) = \zeta \) and \( B(0) = \xi \). If \( E \left[ \exp \left\{ \frac{1}{2} \int_0^T w(B^* + \frac{B^*}{\alpha + \mathcal{B}^*})' Y_{t}dt \right\} \right] < \infty \), where \( \alpha_* = \min_{0 \leq t \leq T} A(t) \), \( \mathcal{B}_* = \min_{0 \leq t \leq T} B(t) \), then

\[
E \left[ e^{\int_0^T (\alpha + \beta' Y_{t})dt} (\zeta + \xi' Y_{T}) \mid \mathcal{F}_t \right] = (A(T-t)+B(T-t)'Y_{t}) e^{A(T-t)+\mathcal{B}(T-t)'Y_{t}}.
\]

For completeness, we give a proof which is adapted to our parameterization at the end of this section. We say \((\alpha, \beta)\) is \(Q\)-regular if \(\alpha, \beta\) and the parameters \((\Theta, \mathcal{K}, \alpha, \beta)\) governing the dynamics of \(Y\) under \(Q\) satisfy the conditions of lemma 64, and if the conditions of lemma 65 are satisfied, we say \((\alpha, \beta, \zeta, \xi)\) is \(Q\)-regular.

Remark 66. (i) The following relationship is useful for applications and follows immediately from lemma 65. If \((\alpha, \beta, \zeta = \alpha, \xi = \beta)\) is \(Q\)-regular, then

\[
E \left[ e^{\int_0^T (\alpha + \beta' Y_{t})dt} (\alpha + \beta' Y_{T}) \mid \mathcal{F}_t \right] = \frac{\partial}{\partial t} e^{A(T-t)+\mathcal{B}(T-t)'Y_{t}},
\]

i.e. \(A\) and \(B\) are the derivatives of \(A\) and \(B\) with respect to the time argument.
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(ii) The result of lemma 65 can be generalized to values of \( \zeta \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \), but then one has to impose a condition which is more difficult to check than \( \mathbb{E} \left[ \exp \left( \frac{1}{2} \int_0^T w \left( \mathcal{B}^* + \frac{\beta^*}{\Lambda^* + \mathcal{B}^*} \right) Y_t dt \right) \right] < \infty \).

Duffie and Singleton (1999) directly calculate conditional expectations of the form

\[
\mathbb{E} \left[ e^{\int_0^T \alpha + \beta' Y_t ds + \gamma' Y_T} (\zeta + \xi' Y_T) \bigg| \mathcal{F}_T \right]
\]

using "differentiation through the integral". We believe we have found a more natural way to look at this, namely we choose \( \tilde{D}p \) and \( \tilde{S} \) in Girsanov's theorem 58. Then

\[
L_t = e^{\psi(Y_t-Y_0)-\int_0^t \theta - \psi Y_s ds - \frac{1}{2} \int_0^t \psi'(\gamma) + w(\gamma) Y_s ds}
\]

and (see also proposition 14) \( Y \) is also an AD under \( \tilde{Q} \). Thus we can always find a measure \( \tilde{Q} \sim Q \) under which \( Y \) remains an AD and \( \tilde{Q} ; Q \) such that

\[
\mathbb{E} \left[ e^{\int_0^T \alpha + \beta' Y_t ds + \gamma' Y_T} (\zeta + \xi' Y_T) \bigg| \mathcal{F}_T \right] = L_t \tilde{E} \left[ e^{\int_0^T \tilde{\alpha} + \tilde{\beta}' Y_t ds} (\zeta + \xi' Y_T) \bigg| \mathcal{F}_T \right],
\]

hence lemma 64 or 65 apply again. For our applications, the circuitous detour described above to find \( \tilde{Q} ; Q \) in order to derive the respective expectations will never be necessary. The martingale \( L_t \), rather than the factor \( e^{\gamma Y_T} \), has a financial interpretation, e.g. \( L_t = L_t^X \).

Proof of lemma 64. The boundedness of \( \mathcal{B} \) (and hence \( \mathcal{A} \)) guarantees that \( \mathcal{A} \) and \( \mathcal{B} \) are \( C^1 \)-functions and thus it can be checked easily via Itô’s lemma that \( M(t) = e^{\int_0^t \alpha + \beta' Y_s ds + A(T-t) + B(T-t)' Y_t} , t \in [0,T] \) solves the SDE

\[
\frac{dM_t}{M_t} = \mathcal{B}(T-t) \sqrt{S} dW_t
\]

if \( \mathcal{A} \) and \( \mathcal{B} \) solve the ODE (5.3), hence \( M \) is a local martingale. By Novikov’s criterion, \( \mathbb{E} \left[ e^{\int_0^T \alpha + \beta' Y_s ds} \bigg| \mathcal{F}_T \right] < \infty \), \( M \) is a true martingale and the boundary conditions \( A(0) = 0 \) and \( \mathcal{B}(0) = 0 \) ensure that the terminal value is indeed \( M_T = e^{\int_0^T \alpha + \beta' Y_s ds} \).

Proof of lemma 65. If \( \xi = 0 \), then lemma 64 applies. Hence we consider only the case where \( \xi \) has at least one positive component. Note that (5.4) is linear in \( B \) with bounded coefficients as long as \( \mathcal{B} \) is bounded, i.e. \( B \) is also bounded. Moreover \( B \geq 0 \), for at least one component \( \min_{0 \leq t \leq T} B_t(t) > 0 \) and \( A \geq 0 \). Thus

\[
\mathcal{M}(t) := e^{\int_0^T \alpha + \beta' Y_s ds + A(T-t) + B(T-t)' Y_t} \left( A(T-t) + B(T-t)' Y_t \right)
\]
is $Q$-a.s. positive for all $t \in [0, T]$ and it follows with Itô’s lemma that $\mathcal{M}$ solves
\[
\frac{d\mathcal{M}(t)}{\mathcal{M}(t)} = \left( B(T-t) + \frac{B(T-t)}{A(T-t) + B(T-t)\xi_t} \right) \sqrt{S_t} dW_t,
\]
i.e. $\mathcal{M}$ is a local martingale. $E \left[ \exp \left\{ \frac{1}{2} \int_0^T \left[ (\frac{B}{A} + \frac{B^*}{A^*}) Y_t \right] dW_t \right\} \right] < \infty$ (Novikov’s condition) guarantees that $\mathcal{M}$ is a martingale and the terminal value $\mathcal{M}_T = e^{\int_0^T a + \beta Y_t dt} (\xi + \xi' Y_T)$ is due to $A(0) = \xi$ and $B(0) = \xi$. 

\section{5.2 Credit and FX Risk in AJD Framework}

In this section, we specify an AJD version of the FX-credit risk model of section 4.1 and we apply lemmata 64 and 65 to the general bond and CDS formulas given in section 4.3.2.

**Assumption 18 (AJD Setup).** Let $Y$ be a canonical representative of $\mathbb{L}_m(N)$.

(i) Default intensity and interest rates are affine in $Y$, i.e. there exist $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}^N$, $i = d, f$ and $\alpha_\lambda \geq 0$, $\beta_\lambda \in \mathbb{R}_+^{m} \times \{0\}^{N-m}$ (This ensures nonnegativity of $\lambda$) such that
\[
\begin{align*}
    r_i & = \alpha_i + \beta_i' Y, \quad i = d, f \quad \text{and} \\
    \lambda & = \alpha_\lambda + \beta_\lambda' Y.
\end{align*}
\]

(ii) $F(t; z) = F(z)$ is a deterministic distribution function.

(iii) There exist $\gamma \in \mathbb{R}^N$ and $\delta(z)$ with $\int_z \delta(z) dF(z) < \infty$ such that in the FX dynamics (4.1)
\[
\sigma_X(t) = \sqrt{S_t} \gamma \quad \text{and} \quad \delta(z, t) = \delta(z).
\]

(iv) The LGD rates are deterministic functions $q_i(z, t) := q_i(z) \leq 1$.

Occasionally, we prefer to write $\alpha_{\lambda, i}$ and $\beta_{\lambda, i}$ instead of $\alpha_\lambda$ and $\beta_\lambda$. Note that under assumption 18 the covariation between default intensity and exchange rate $[\lambda, X]$ is completely determined by the inner product $\gamma' \beta_\lambda$ because $d[\lambda, X] = X_- \gamma' \beta_\lambda dt$. The assumption on the functional form of $\delta(z, t)$ and $q_i(z, t)$ is restrictive but it preserves the affine structure. Despite this, using a higher-dimensional marker space $Z$ we still have a large degree of flexibility in the modelling of the correlation between loss given default $q_i(z)$ and FX devaluation $\delta(z)$. We write $\hat{\delta} := \int_Z \delta(z) dF(z)$ and
5.2. Credit and FX Risk in AJD Framework

\( \bar{q}_i := \int_\mathcal{Z} q_i(z) dF(z), \ i = d, f \) for the expected devaluation fraction and LGD rates. Also note that the \( \frac{dQ_f}{dQ_\mathcal{A}} \)-Radon Nikodym derivative

\[
L_t^X = e^{r'(Y_t - Y_0) - \frac{1}{2} \int_0^t [r'(Y_s - Y_0)]^2 ds} \cdot (1 - 1_{\{\tau \leq t\}} \delta(z_\tau, \tau))
\]

is exponentially affine. The following proposition provides the relations between the domestic and the foreign spot martingale measure in an AJD framework.

**Proposition 14.**

(i) \( Y \) is also an AD under \( Q_f \) and satisfies the SDE

\[
dY = (\Theta_f - \mathcal{K}_f Y) dt + \sqrt{K_f} dW_f, \quad Y(0) = Y_0,
\]

where \( \Theta_f = \Theta + \text{diag}(\gamma)a, \mathcal{K}_f = \mathcal{K} - \text{diag}(\gamma)b \) and \( W_f \) is a \( Q_f \)-BM.

(ii) The default intensity under \( Q_f \) is given by

\[
\lambda_f(t) = \hat{\lambda}(t).
\]

(iii) The conditional distribution function of the severity of default variable \( z_\tau \) under \( Q_f \) satisfies

\[
dF_f(z) = \frac{\delta(z)}{\delta} dF(z).
\]

**Proof.** Apply Girsanov’s theorem 58 to our AJD framework.

Mean reversion speed and level of \( Y \) (and thus of \( \lambda, r_d, r_f \)) transform under the changes of measure considered in proposition 14, whereas the parameters \( a \) and \( b \) governing the volatility of \( Y \) are clearly invariant. Note that the parameter restrictions for canonical ADs given in section 5.1 are automatically satisfied for \( \mathcal{K}_f \) and the \( m \) first components of \( \Theta_f \), but in general not for the \( N - m \) last components of \( \Theta_f \). Of course \( \Theta_f, \mathcal{K}_f, a, b \) is an admissible parameter set, but it does not necessarily belong to a canonical representative of \( \mathcal{A}_mN \). Further the inequalities \( \mathcal{K}_{fii} > 0, i = 1, \ldots, N \) may be violated, i.e. we possibly lose mean reversion under the FSMM, which might not be economically meaningful. In applications one should check that these inequalities hold.
Remark 67.

(i) In general it is difficult to check the validity of assumption 13. In the AJD case however, the martingale property of $L^X$ is ensured if the following condition holds (see e.g. Lépingle and Mémin (1978)). Let $X$ be as in proposition 14, and

$$
E \left[ \exp \left( \frac{1}{2} \int_0^T w(y)Y_t dt + \int_0^T g(z) dF(\lambda(t)) dt \right) \right] < \infty,
$$

where $g(z) := \log (1 - \delta(z)) \cdot (1 - \delta(z)) - 1$. If the foreign currency can only be devaluated at default, i.e. $\delta(z) = 0$ for all $z \in \mathbb{Z}$, it suffices to check the Novikov condition $E \left[ \exp \left( \frac{1}{2} \int_0^T w(y)Y_t dt \right) \right] < \infty$.

(ii) Proposition 14 remains true for all sufficiently regular short rates (which are not affine in $Y$) because the short rates $r_i$ enter only the FX rate $X$ but not the Radon-Nikodym process $L^X$.

5.2.1 Bonds and CDSs in AJD Framework

We first define the (constant) coefficients $\alpha_{\lambda_f} := \tilde{\alpha}_\lambda$ and $\beta_{\lambda_f} := \tilde{\beta}_\lambda$, then it holds

$$
\dot{\lambda}_f(t) = \alpha_{\lambda_f} + \beta_{\lambda_f} Y_t.
$$

Let $(-\alpha_i, -\beta_i)$ be $\mathbb{Q}$-regular parameters for $i = d, f$. Then domestic and foreign default-free ZCB prices exist for all $0 \leq t \leq T \leq T^*$ and satisfy

$$
B_i(t, T) = e^{A_i(T-t) + B_i(T-t)'Y_t},
$$

where $A_i, B_i$ solve (5.3) with $(-\alpha_i, -\beta_i, \Theta_i, \mathcal{K}_i, a, b)$ for $i = d, f$. Clearly, if we do not allow for negative interest rates, i.e. $\alpha_i \geq 0$ and $\beta_i \in \mathbb{R}_+^m \times \mathbb{R}^N$ for $i = d, f$, then the existence of ZCB prices is immediate. However, note that, due to the affine structure, in this case $\lambda$ can only have nonnegative correlation with $r_1$ (because then $\beta'_2 S \beta_i \geq 0$), which is not always a desirable property (see e.g. Duffee (1998) for empirical evidence of negative correlation).

Next, we provide the defaultable ZCB formulas of section 4.3.2 in our AJD framework. The proofs of the formulae below follow immediately from lemmata 64 and 65.

**Zero Recovery.** Let $(-\overline{\alpha}_i, -\overline{\beta}_i)$ with $\overline{\alpha}_i := \alpha_i + \alpha_{\lambda_i}$ and $\overline{\beta}_i := \beta_i + \beta_{\lambda_i}$, be $\mathbb{Q}$-regular. Then by lemma 64,

$$
\overline{B}_i(t, T) = 1_{\{t \geq \tau\}} e^{A_i(T-t) + B_i(T-t)'Y_t},
$$

(5.5)
where \(A_i\) and \(B_i\) solve (5.3) with \((-\overline{\alpha}_i, -\overline{\beta}_i, \Theta_i, K_i, a, b)\), \(i = d, f\).

**Recovery of Par.** Let \((-\overline{\alpha}_i, -\overline{\beta}_i, \xi_i := \alpha_{\beta_i}, \xi_i := \beta_{\alpha_i}\) be \(Q_i\)-regular. Then by lemma 65, on \([t < \tau]\) we have

\[
\overline{B}_i^R(t, T) = \overline{B}_i(t, T) + \hat{q}_i \int_0^{T-t} (A_i(s) + B_i(s)'Y_i) e^{A_i(s) + B_i(s)'Y_i} ds
\]

where \(A_i, B_i\) are as in (5.5) and \(A_i, B_i\) solve (5.4) with \((-\overline{\alpha}_i, -\overline{\beta}_i, \xi_i := \alpha_{\beta_i}, \xi_i := \beta_{\alpha_i}, \Theta_i, K_i, a, b)\). Note that by remark 66 the existence of

\[
E_i \left[ e^{-\int_0^T (r_t + \lambda_t) dt} dV_i(s) \bigg| \mathcal{F}_t \right]
\]

is immediate if \(\tau_i \geq 0\).

**Multiple Defaults.** As we pointed out, to value an MD bond, we have to specify the predictable compensator of \(\mu\) after \(\tau\). Here, in addition to assumption 17, we require \(\hat{F}(t; z) = F(z)\) and \(q_i(z, t) = q_i(z), i = d, f\), for all \(t\). Now let \((-\overline{\alpha}_i^{MD}, -\overline{\beta}_i^{MD})\) be \(Q_i\)-regular, where \(\overline{\alpha}_i^{MD} := \alpha_i + \gamma_i \alpha_i\) and \(\overline{\beta}_i^{MD} := \beta_i + \gamma_i \beta_i\), for \(i = d, f\). Then

\[
\overline{B}_i^{MD}(t, T) = \overline{B}_i^{MD}(t) e^{A_i^{MD}(T-t) + B_i^{MD}(T-t)'Y_i},
\]

where \(A_i^{MD}\) and \(B_i^{MD}\) solve (5.3) with \((-\overline{\alpha}_i^{MD}, -\overline{\beta}_i^{MD}, \Theta_i, K_i, a, b)\) for \(i = d, f\).

**Remark 68.** If we relax the assumption on \(\hat{q}_i(t)\) by letting it be exponentially affine in \(Y_t\), i.e. \(\hat{q}_i(t) = e^{\alpha_{\beta_i} + \beta_{\alpha_i} Y_t}\) with \(\alpha_{\beta_i} < 0\) and \(\beta_{\alpha_i} \in \mathbb{R}^m \times [0, 1)^{N-m}\), then \(\hat{q}_i(t)\) remains \((0, 1)\)-valued and) lemma 64 can still be used to price the claim

\[
1 - q_i(z, \tau).
\]

This way, conditional dependence between the locally expected LGD and default intensity (e.g. over the credit cycle) can be captured. However, if \(\hat{\delta}_t\) is exponentially affine, i.e. \(\hat{\delta}_t = e^{\alpha_{\beta} + \beta_{\alpha} Y_t}\), then \(\lambda_f\) fails to be affine in \(Y_t\) (see proposition 14).

**Recovery of Treasury.** As argued above, the pricing of an RT recovery can be reduced to the valuation of a related RP recovery under the \(T\)-forward measure \(P_i^T\). In the AJD framework the Radon-Nikodym densities

\[
\frac{dP_i^T}{dQ_i} \bigg| \mathcal{F}_t = e^{B_i(T-t)'Y_t - B_i(T)'Y_t - \int_0^T d[B_i(T-)'Y_t]}
\]
Chapter 5. AJD-Version of the Model

$i = d, f$, are exponentially affine, but in general time-inhomogeneous. GirSANOV’s theorem 58 yields that $Y$ has mean reversion level $\Theta_i^{T} (t) := \Theta_i + \text{diag}(\mathbf{B}_i(T - t))a$ and speed $\mathcal{K}_i^{T} (t) := \mathcal{K}_i - \text{diag}(\mathbf{B}_i(T - t))b$ under the respective measure $P_i^{T}$. This leads to a AD pricing problem with deterministic but time-dependent coefficients $(\Theta_i(t), \mathcal{K}_i(t), a, b)$. We will not further treat the time-inhomogeneous case here. See for e.g. appendix B of Duffie et al. (2000) for details.

Credit Default Swaps

Finally, we translate the CDS formulae (4.8) in our AJD framework. It follows directly from ZR and the RP case that

$$
\mathcal{T}_i(t, T) = \frac{\int_{0}^{T - t} (A_i(s) + B_i(s)') Y_i e^{A_i(s) + B_i(s)'} dS}{\int_{0}^{T - t} e^{A_i(s) + B_i(s)'} Y_i dS}, \quad (5.6)
$$

where $A_i$ and $B_i$ solve (5.3) with $(\bar{\mathbf{w}}_i, \bar{\mathbf{p}}_i, \Theta_i, \mathcal{K}_i, a, b)$ and $A_i$ and $B_i$ solve (5.4) with $(\bar{\mathbf{w}}_i, \bar{\mathbf{p}}_i, \zeta_i := \alpha_{i}, \xi_i := \beta_{i}, \Theta_i, \mathcal{K}_i, a, b)$.

---

5Unless $\beta_i = (0, \ldots, 0)'$, then $\mathbf{B}_i(\cdot)$ is constant due to (5.3).
Chapter 6

Empirical Results

In section 4.4 we identified four drivers of the difference between domestic and foreign CDS spreads. Here we focus on the first two drivers, a possible jump of the FX rate at default and continuous correlation of $\lambda_d$ and $X$ (before default). There is theoretical and empirical evidence that the correlation between default-free interest-rates and default intensities has only a very small effect on CDS rates (see e.g. Houweling and Vorst (2005) or Schönbucher (2002)). Regarding the relative prices of CDS, this effect is likely to be even smaller here, so we feel justified in ignoring this effect. Furthermore, we will also assume that the correlation between recovery rate and devaluation fraction is not significant.

Assumption 19 (Empirical Estimation Setup).

(i) Domestic and foreign interest-rates have zero covariation with exchange rate and default intensity, i.e.

$$[r_d, X] = 0 = [r_d, \lambda] \quad \text{and} \quad [r_f, X] = 0 = [r_f, \lambda].$$

(ii) The cheapest-to-deliver bond of a CDS has a constant LGD rate $q$ and, at default, the FX rate is devaluated by a constant fraction $\delta$.

By virtue of (i) we do not need an affine model for the default-free interest-rates any more. We directly use the current 1M Libor rates as approximation for the short rate in the FX drift (4.1) and the current term-structure of interest-rates to discount future cash-flows.

As data source for CDS quotes we use the ValuSpread CDS database by Lombard Risk Systems Ltd., a provider of a data pooling service for the CDS
market. The data starts in the third quarter of 1999 and ends in June 2005. At the beginning of the period the set of CDSs on the same reference entity which are available in more than one currency is relatively sparse and the data frequency is only monthly but from 2002 onwards the data quality improves significantly both in frequency (weekly, then daily from 2003) and in the number of obligors with CDS rates in both JPY and USD. From the available set of Japanese reference names we selected the 25 obligors with the largest number of simultaneous data points in both JPY and USD. For default-free interest-rates we used JPY and USD term-structures of interest-rates based upon Bloomberg swap and money-market data. The JPY/USD exchange rate data was also taken from Bloomberg. The snapshots in figure 6 act as a good example for the magnitude of the spread between domestic and foreign CDS rates in our dataset. JPY CDS rates on many large Japanese reference entities trade typically around 20 percent lower than their USD equivalent.

6.1 A Simple Devaluation Fraction Estimator

As a first step, we tested for the presence of significant differences between JPY and USD CDS rates using a simple statistic based upon the limiting property in lemma 62, which states that CDS rates for short maturities are approximately equal to LGD times default intensity. Remembering that $\lambda_f = (1 - \delta)\lambda_d$ (see proposition 11), the following (approximate) estimator for $\delta$ is straightforward.

$$
\hat{\delta}_k := \frac{1}{n} \sum_{i=1}^{n} \left\{ 1 - \frac{\tau_y(t_i, t_i + k)}{\tau_y(t_i, t_i + k)} \right\}, \quad k > 0.
$$

Apart from the limit argument in lemma 62, we saw another argument why this estimator can be interpreted as an implied devaluation fraction in equation (4.12). We give the estimates $\hat{\delta}_k$ for the maturities $k = 1$ years and $k = 5$ years in table 6–1. The 1-year maturity is the shortest maturity available to us, it is reported in the first pair of columns. We also added the 5-year maturity because market liquidity is usually concentrated at this point of the CDS term

---

1 At regular (daily) intervals, Lombard Risk Systems collects CDS quotes on a large number of reference names from a set of major market makers and dealers. The submitted data is cleaned and averaged and then added to the ValuSpread CDS data base and distributed back to the original data providers and to other subscribers of the service who use this data to mark their books and for various risk management purposes.

2 For three companies (NEC, Sharp and Tokio Marine and Fire Insurance), the data ends in the first quarter of 2004 and for Bank of Tokyo-Mitsubishi it ends already in the third quarter of 2003.
6.1. A Simple Devaluation Fraction Estimator

Fig. 6–1. 5y CDS rates on Sumitomo Mitsui Bank (top) and Sony in USD and JPY. The mean CDS spread difference amounts 4.1 bp for Sony and 7.6 bp for Sumitomo Mitsui Bank.
structure. Using a $t$-test on $1 - \bar{s}_Y / \bar{s}_S$ we tested the hypothesis whether the deviations of the CDS ratio $\bar{s}_Y / \bar{s}_S$ from 1 is just noise in the data. In all cases this hypothesis could be clearly rejected: We are facing a systematic feature of the data. Interestingly the devaluation fractions implied by the data are typically higher for the 1-year CDS rates than for the 5-year CDS rates.

### 6.2 A Correlation Model

The differences between JPY and USD CDS rates reported in table 6–1 need not necessarily be caused by an implied devaluation of the JPY at default (i.e. a $\delta > 0$). As seen above, a difference between CDS rates in different currencies can also arise when default intensities and FX rate are correlated via their diffusion components. Thus, we now want to investigate whether it is possible to reproduce the observed differences in CDS rates without assuming a devaluation at default, but only using the dependency between defaults and exchange rate $X$ that is generated in a purely diffusion-based setup.

We build up a concrete model for the observable background driving process $Y$ using the classification of ADs into the families $\tilde{h}(N)$. Because assumption 19 relieves us from modelling default-free interest-rates, we are left with the task of modelling the correlation structure of two variables: $\lambda$ and the diffusion part of $X$, hence we need a dimension of at least $N \geq 2$. Second we
have to choose the number \( m \leq N \) of components of \( Y \) that we want to remain positive. In our case \( \lambda > 0 \) is desirable, hence we choose \( m \geq 1 \). Thus, an \( \mathcal{A}_1(2) \) or an \( \mathcal{A}_2(2) \) model is appropriate. We chose \( Y \in \mathcal{A}_1(2) \) because for fixed \( N \), the number of parameter restrictions is increasing in \( m \) (see 5.1), and if \( Y \in \mathcal{A}_2(2) \), then the two components driving the stochastic volatility of \( Y \), and thus of \( \lambda \) and \( X \), can only have nonnegative correlation (see Dai and Singleton (2000)).

Then \( \lambda \) is a CIR process up to a positive additive constant. We restrict this constant and the lower left part of the matrix \( b \) defined in (5.2) to zero. This yields the following model under the DSMM \( \mathbb{Q}_d \).

\[
\begin{align*}
\frac{d\lambda}{\lambda} &= \kappa(\theta - \lambda)dt + \sigma \sqrt{\lambda} \, dW^1 \\
\frac{dX}{X} &= (r_d - r_f)dt + \gamma_1 \sqrt{\lambda} \, dW^1 + \gamma_2 \, dW^2 - \delta(dN - \lambda dt) \quad (6.2)
\end{align*}
\]

with \( \kappa, \theta, \sigma, \gamma_2 > 0, \gamma_1 \in \mathbb{R} \) and \( \delta < 1 \) and \( (W^1, W^2)' \) a standard BM under \( \mathbb{Q}_d \). The instantaneous correlation of default intensity and log FX rate,

\[
\rho(\log X^c, \lambda) := \frac{\frac{d}{dt}[\log X^c, \lambda]}{\sqrt{\frac{d}{dt}[\log X^c] \frac{d}{dt}[\lambda]}} = \text{sgn}(\gamma_1) \left( 1 + \frac{(\gamma_2/\gamma_1)^2}{\lambda} \right)^{-1/2},
\]

is essentially controlled by the ratio \( \gamma_1/\gamma_2 \), but also depends on the current level of the stochastic process \( \lambda \). Importantly its sign depends only on the sign of \( \gamma_1 \). Further we have to link the DSMM to the physical measure \( \mathbb{P} \), under which the data was generated. For tractability we assume that the state price density is of the form

\[
\frac{d\mathbb{P}}{d\mathbb{Q}_d} := L_T \quad \text{where} \quad \frac{dL}{L} = \phi_1 \sqrt{\lambda} \, dW^1 + \phi_2 \, dW^2 - \Phi(dN - \lambda dt)
\]

for \( \phi_1, \phi_2 \in \mathbb{R} \) and \( \Phi < 1 \). Then \( Y \) is also an AD \( \in \mathcal{A}_1(2) \) under \( \mathbb{P} \).

In order to estimate the parameters of an \( \mathcal{A}_m(N) \) model a large number of relatively demanding estimation methodologies have been proposed for non-Gaussian AD models (Singleton (2001), Ait-Sahalia (2002), Gallant and Tauchen (1996)). Given the simplicity of our model and the high frequency of our data (daily for most of the dataset) we used a simple quasi maximum likelihood (QML) estimator by approximating (6.2) with its Euler discretisation. Then the parameter estimation reduces to a linear regression problem in transformed variables. Note that for every volatility estimator of the above model the value of \( \delta \) plays no role as long as default has not yet occurred.
As the intensity $\lambda$ is not directly quoted in the markets, we need to find a proxy for it. Here, the limiting property in lemma 62 suggests that CDS rates with a short maturity are approximately proportional to $\lambda$, and equation (4.9) show that also for longer times to maturity, CDS rates are proportional to a weighted average of forward hazard rates. Therefore, we decided to use CDS rates as approximation for $q \lambda(t)$. Ideally, we would have liked to use 1-year CDS rates, but the data for the 5-year maturity turned out to be cleaner and more liquid, so we used 5-year USD CDS rates, $\pi_q(t, t + 5)$.

If $q$ is unknown, then $\gamma_1$ can only be estimated up to a positive constant. Therefore, we give QML estimates and 95%-confidence intervals for $\frac{\gamma_1}{q\sigma}$ in table 6–2. However, the level of correlation seems to be rather low: For only

<table>
<thead>
<tr>
<th>Ticker (Company)</th>
<th>$\hat{\gamma}_1$ (mean, [%])</th>
<th>95%-CI</th>
<th>$\hat{\rho}(\log X^c, \lambda)$ (mean, [%])</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASAGLA (Asahi Glass Company, Ltd.)</td>
<td>-1.07</td>
<td>[-8.08, 5.95]</td>
<td>-1.47</td>
</tr>
<tr>
<td>BOT1 (Bank of Tokyo-Mitsubishi, Ltd.)</td>
<td>-2.80</td>
<td>[-5.49, 0.01]</td>
<td>-16.58</td>
</tr>
<tr>
<td>EJRAIL (East Japan Railway Company)</td>
<td>-2.97</td>
<td>[-10.22, 4.29]</td>
<td>-3.43</td>
</tr>
<tr>
<td>FUJITSU (Fujitsu Ltd)</td>
<td>-0.17</td>
<td>[-1.54, 1.21]</td>
<td>-0.88</td>
</tr>
<tr>
<td>HITACH (Hitachi, Ltd.)</td>
<td>-1.36</td>
<td>[-5.31, 1.79]</td>
<td>-3.93</td>
</tr>
<tr>
<td>HONDA (Honda Motor Co., Ltd.)</td>
<td>-2.33</td>
<td>[-7.28, 2.52]</td>
<td>-4.37</td>
</tr>
<tr>
<td>MATSEL (Matsushita Electric Industrial Co., Ltd.)</td>
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<td>[-4.03, 1.93]</td>
<td>-2.93</td>
</tr>
<tr>
<td>MITSCO (Mitsubishi Corp)</td>
<td>-3.97</td>
<td>[-8.18, 0.23]</td>
<td>-7.15</td>
</tr>
<tr>
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<td>[-8.29, -0.12]</td>
<td>-7.32</td>
</tr>
<tr>
<td>NECORP (NEC Corporation)</td>
<td>-1.51</td>
<td>[-3.60, 0.38]</td>
<td>-7.23</td>
</tr>
<tr>
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<td>-3.88</td>
<td>[-9.65, 1.88]</td>
<td>-5.52</td>
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<tr>
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<td>[-9.19, -2.03]</td>
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</tr>
<tr>
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<td>[-2.98, 0.99]</td>
<td>-3.61</td>
</tr>
<tr>
<td>TOYOTA1 (Toyota Motor Corporation)</td>
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<td>[-11.02, 7.66]</td>
<td>-1.69</td>
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<tr>
<td>YAMAHA (Yamaha Motor Co., Ltd.)</td>
<td>0.26</td>
<td>[-1.76, 2.28]</td>
<td>0.87</td>
</tr>
</tbody>
</table>

Table 6–2. QLM estimates of the Correlation Parameter and Averaged Correlation.

four companies, MITSCO, NIPSTL, SHARP and SUMT, the null hypothesis $H_0 : \gamma_1 = 0$ can be rejected on the 95% level. To provide a quantity which is more intuitively understandable, we also computed the average instantaneous correlation function $\rho(\log X^c, \lambda)$ by combining the estimates for $\frac{\gamma_1}{q\sigma}$ with the estimates of $\gamma_2$ and $\sigma$. This quantity can be viewed as the average correlation between default intensity and exchange rate over our sample period.
6.3 Evidence for Devaluation at Default

As we outlined in section 4.4, \( y_1 \neq 0 \) if and only if \([X, \lambda] \neq 0\), which implies different term structures of default intensities in USD and JPY. We now attempt to answer the question whether this purely correlation based dependence is able to explain the whole difference between USD and JPY CDS rates, i.e. the question whether we can we can make our life much easier and set the jump interaction parameter \( \delta \) to zero? Formally, our null hypothesis is: “The difference between JPY and USD CDS spreads is caused by a model like (6.2) with no devaluation at default, i.e. \( \delta = 0 \).”

In order to evaluate the Null, we must first completely specify the model (6.2). Given \( \delta = 0 \), and for a given value of the loss given default rate \( q \) (which is also provided in our CDS database), we are able to compute QML-estimates for all parameters in (6.2) including the market prices of risk \( \phi_1, \phi_2 \). In the estimates of these parameters only USD CDS rates (and not JPY CDS rates) are used besides the exchange rate and interest-rate data.

Next, we try to find out whether there are any combinations of \( \lambda \) and the default-free term structures for which the resulting ratio of JPY CDS spreads to USD CDS spreads is of a similar order of magnitude as the one observed in table 6–1. We do not need to consider the current value of \( X \) as it only enters the CDS spreads through its dependency with \( \lambda \) under the different measures (see e.g. equation (4.9)). For this, we took the mean term structures of default-free interest rates over the sample period in USD and JPY. We also considered the term structure which turned out to be most favorable (least favorable) to our null hypothesis: the steepest (flattest) default-free USD term structure of interest-rates of this period, together with the JPY term structure of the same day. We first used \( \lambda(t_i) \approx \frac{1}{q} \bar{\gamma}_5(t_i, t_i + 5) \) as an approximation for the default intensity at \( t_i \) and then calculated at each \( t_i \) the corresponding 5Y USD and JPY CDS rates (4.9).

The resulting difference between USD and JPY CDS rates was in no case even close to the difference that we observed in the data: Even with the steepest term structure of USD-interest-rates we typically found a theoretical relative difference of less than 7%, (with the mean term structure one of less than 4% and with the flattest less than 3%): We applied the estimator (6.1) to the resulting theoretical 5Y CDS rates and also calculated

\[
\delta_{5,\text{max}} := \max_i \left\{ 1 - \frac{\bar{\gamma}_Y(t_i, t_i + 5)}{\bar{\gamma}_5(t_i, t_i + 5)} \right\}.
\]

3In the case of Yamaha we observed the largest values of both, \( \delta_5 \) and \( \delta_{5,\text{max}} \). This is probably caused by the fact that its CDS spreads fell from nearly 250bp in 2002 (where the data is quite sparse) to around 25bp from 2004 onwards.
With the mean and the steepest term structures of interest rates in this period we reached the values reported in table 6–3. (We considered only the tickers which showed positive mean reversion \( \kappa \) under \( Q \).) Comparing these results to the relative differences reported in table 6–1 we reject the null hypothesis \( \delta = 0 \).

On the other hand, for any given relative difference between USD and JPY CDS rates (and for any given term structures of default-free interest-rates) we can find a nonzero value of \( \delta \neq 0 \) such that this relative difference is reproduced. So we cannot reject the hypothesis that the devaluation fraction \( \delta \) is not zero.

### 6.4 Discussion of Empirical Results

The difference between domestic and foreign CDS rates implies a currency devaluation of between 9 and 26% at a default of one of the larger obligors in our sample. This may seem rather large, and it may be (at least partially) caused by market imperfections. Nevertheless, the size and persistence of the effect (it did not change significantly although the market liquidity has multiplied in our sample) indicates that there is – at least to some extent – a real foundation to it. For example, a default of a large Japanese firm will most likely happen in a serious recession scenario, or it will be an indicator of severe fundamental problems in the economy (e.g. a bank default would be a strong indicator that the Japanese bad-loans problem is more serious than expected).

Furthermore, we would like to point out that we excluded some sources of dependency in our setup, in particular the possibility of joint jumps of FX rate
and default intensity before default, and the dependency across obligors. In particular the latter effect should be investigated further: A macro-economic effect of a default (e.g. FX devaluation) should be particularly large if the default itself was caused by a systematic, macro-economic reason. But then the default is also more likely to affect other obligors. Thus, the implied FX devaluation at default may even give us information on the dependency between the obligors and the macro economy if all variables (defaults and FX rate) are driven by the same macro-variables.

We would have expected to see significantly different results for firms that are active in foreign trade as exporters, as importers, and firms that mostly service the domestic market. Making this distinction for large Japanese corporations is rather difficult and from a qualitative inspection of our results we did not see any such systematic connection.

To the methodology of our study it was irrelevant that the reference obligors were Japanese companies. What we needed was that $X$ was the exchange rate between the CDS-currencies, but not that $X$ had any connection to the reference credit. Thus, a similar study can also be performed using reference credits that are not incorporated in the “foreign” country. A particularly interesting case arises if the reference credit of the CDS is a sovereign itself. Usually, this sovereign will not be a G7 country, so (unless this is explicitly stated in the term sheet) the CDS will not be denominated in its currency. Nevertheless, many developing countries have issued debt in multiple currencies, e.g. USD, EUR, and JPY, and CDS on these sovereigns can also be traded in these currencies. The relative difference of these CDS will then allow us to make statements about the implied effect of a sovereign default of e.g. Brazil on the EUR/USD exchange rate in much the same way as we made statements about the JPY/USD exchange rate upon default of a corporate reference credit.\(^4\) In this case, it will not be clear which currency we would expect to be devalued, i.e. $\delta$ may also be negative.

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\(^4\)In the case of a sovereign obligor, one has to watch out for the delivery specifications in the CDS contract. They might exclude local currency debt in foreign currency denominated CDS.
Bibliography


