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 Calibration and Interference Removal for a 61 kHz Resolution Instrument, and Analysis of Decimetric Solar Radio Spikes

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1 Abstract

A calibration has been carried out for the Phoenix 3 instrument for frequencies below 1 GHz which provides solar data within 10% uncertainty. The method developed maps directly to the calibration of other frequency bands for future Phoenix 3 observations in the (1-5) GHz range.

An effective new algorithm has been created to detect and remove Radio Frequency Interference (RFI) from the high spectral resolution data of Phoenix 3. This algorithm detects RFI using spectral gradient, with identification criteria specific to each channel. The algorithm has a learning facility which ensures that newly introduced carrier signals and pulsing RFI are treated appropriately.

1478 radio spikes have been analysed in the frequency range (600-850) MHz from a solar event on 23rd August 2005 with 61 kHz resolution. The instrument, Phoenix 3, provides the most accurate spectral study of solar Spikes to date by an order of magnitude. Spikes were observed with bandwidth 0.3 MHz; smaller than the narrowest to date of 2 MHz[1]. The spikes' skewness had a mean of -0.5 and was negative for 75% of cases. This study indicates the energy emitted in spikes follows an exponential law and the law's parameters are estimated to within 20% uncertainty.

2 Introduction

Decimetric Radio Spikes are solar emissions of decimetric wavelength with narrow band-width (tens of MHz) and time duration (tens of ms)[1]. The Power Spectral Density (PSD) of the spikes is in the order of hundreds of Solar Flux Units (sfu), where 1 sfu = 10^{-22} J s^{-1} m^{-2} Hz^{-1}. As terrestrial Radio Frequency Interference (RFI) affects the data, cleaning is required before analysis.

Spikes are the finest spectral solar features currently studied using broadband instruments. The most accurate observations of spikes to date were carried out with a resolution of 1 MHz[4]. The narrowest spike bandwidth observed was only double the resolution, suggesting the instrument was limiting observations. This motivated the 61 kHz resolution analysis documented in this report.

The report is separated into three parts: the first details the instrument; its calibration and an assessment of the uncertainty of data it produces. The second describes the development and effectiveness of an algorithm designed to remove RFI from this and future data sets. In the third section, the data from the solar spikes is analysed.
3 Theoretical Background

It has been suggested\cite{3,8} that decimetric spikes are caused by the MASER gyroresonance of pockets of electrons which are excited in the presence of a strong magnetic field. The electrons are forced into a distribution of quantum rotation states with greater energies than would correspond to a Boltzmann distribution at the local temperature by an unknown effect (possibly mirroring in converging magnetic fields).

The $z$-component of magnetic moment ($\mu_z$) of an electron with a magnetic quantum number ($m_j$) is given by $\mu_z = m_j \mu_b$, where $\mu_b = e\hbar/2m_e$ is the Bohr magneton. The potential energy of a moment $\mu_j$ in a magnetic field $B$ in the $z$-direction is given by $U = -\mu_z B$. Thus the stable probability distribution of the energy of an electron with total quantum number $j$ at temperature $T$ is:

$$p(U_i) = p(-\mu_b i B) = \frac{\exp\left(-\mu_b (i) B/kT\right)}{\sum_{i=-j}^{+j} \exp\left(-\mu_b (i) B/kT\right)}$$

(1)

Where $i$ can take any value in the range: $-j, -j + 1, -j + 2, ..., j - 2, j - 1, j$ and $j$ can be either an integer, or an integer plus one half. The excited energy states can only be $U_{i'} = -\mu_b i' B$ for the same possible values of $i'$ as those for $i$, except the angular momentum of the electron changes from $j$ to $j'$. When the electrons fall from the excited energy states to equilibrium states, the total angular momentum and energy must be conserved: thus a circularly polarised photon is emitted with angular momentum $J = (j' - j)\hbar$ and energy $E = \mu_b (i - i') B = \mu_b (n/2) B$ for integer $n$.

Hence the possible frequencies ($\nu_n = E/h$) emitted from a region with a magnetic flux $B$ are given in equation 2-A. The bandwidth of a single spike therefore corresponds to the variation in the local magnetic field in the region where electrons are MASing, as in equation 2-B.

$$\text{(A)} \quad \nu_n = \frac{\pi e B n}{2m_e}$$

$$\text{(B)} \quad \Delta \nu_n = \frac{\pi e n}{2m_e} \Delta B$$

(2)

MASing occurs when excited electrons are induced to emit by similar frequency photons already emitted by neighboring electrons. This is a chain reaction, but occurs in finite regions because:

- the region of excited electrons is finite
- as the field strength varies with position, (already emitted) inducing photons have different frequency than those they induce. Emission is less probable for greater a frequency difference; narrow bandwidth spikes are thought to originate from areas of high local field variation.
The photons released pass through plasma, which is expected to attenuate lower frequencies more as the refractive index of the plasma becomes negative for frequencies below the plasma frequency \( \nu_p \propto \text{electron density} \). Therefore the frequency profile of the spikes is expected to be weighted toward higher frequencies, corresponding to a negative skew (see appendix C).

**Part I**

**The Instrument - Phoenix 3**

The instrument used in this study is Phoenix 3, a spectrometer which uses Fast Fourier Transform (FFT) technology as opposed to frequency agile scanning, and can provide higher frequency resolution: in this study 61 kHz. The extra noise expected from narrower frequency resolution (see § 5) is avoided since each frequency can be integrated over the full time step between each scan of the spectrum, in this study 100 ms, as opposed to sharing the time-step with many channels in an agile scan.

The spectrometer is fed by a 5 m dish with a linear antennae at Bleien Observatory, 50 km West of Zürich. The gain of the antennae is approximately 7 dB. For the data set studied in this report, polarisation was ignored, but in future the instrument will measure left and right circular polarisations separately. The frequency range studied here is from \( (0 - 1000) \) MHz, although the instrument gain is only significant in the range \( (300 - 850) \) MHz, due to depleted effective area of the dish for lower frequencies, and the effect of low pass filters for higher frequencies.

The instrument’s calibration is described in § 4, and the uncertainty associated with the data it produces is discussed in § 5.

**4 Calibration**

In order to analyse the data, it is essential to calibrate the apparatus. The response of the system is expected to depend on frequency, so each frequency channel must be calibrated individually. At a particular time \( t \), the digital output \( y(\nu) \) from the instrument at a frequency \( \nu \) is given by some function of the true solar flux \( F(\nu) \). Writing the function as a Taylor Expansion:

\[
y(\nu) = \sum_{n=0}^{\infty} a_n(\nu) (F(\nu))^n = a_0(\nu) + a_1(\nu)F(\nu) + a_2(\nu)(F(\nu))^2 + \ldots
\]  

(3)
4.1 Validating and Defining a Linear Calibration of the Instrument

For sufficiently low flux, $n \geq 2$ terms in equation 3 can be neglected and the system is in the linear limit. The instrument has been designed to operate in this limit over the frequency range (300 - 850) MHz. To estimate how significant the non-linear components are, sinusoidal input signals such as $F = A \cos(2\pi \nu t)$ can be used, and their harmonics observed. Retaining the first non-linear term, for constants evaluated at $\nu$:

$$y \approx a_0 + Aa_1 \cos(2\pi \nu t) + A^2 a_2 \cos^2(2\pi \nu t) = \left[a_0 + \frac{a_2 A^2}{2}\right] + a_1 A \cos(2\pi \nu t) - \frac{a_2 A^2}{2} \cos(2\pi [2\nu] t)$$

Thus a peak occurs at $2\nu$, caused by the non-linear response of the system with a magnitude $a_2 A^2/2$. Neglecting contributions from higher order terms, $a_2$ can be estimated from the Instrument’s output for the main peak and its first harmonic; $y_p = (a_1 A)$ and $y_h = (a_2 A^2/2)$ respectively.

A large, narrow RFI peak at 391 MHz and its harmonic were measured as $y_p = 75000 \pm 10000$ units and $y_h = 50 \pm 10$ units above background, where the uncertainty was obtained statistically using five time-slices. Using the calibration from § 4.4, the parent peak’s flux $A \approx y_p/A = 15000 \pm 2000$, and $a_1 \leq 5$ therefore:

$$a_2 = \frac{2 a_1 y_h}{A y_p} \quad \text{and} \quad A \approx y_p/A \quad \Rightarrow \quad a_2 = (4 \pm 1) \times 10^{-7} \text{ units sfu}^{-2}$$

The data set to be studied has solar flux no greater than 1000 sfu, which corresponds to a maximum non-linearity of 0.1 %. Assuming similar non-linearity for all frequencies, a linear calibration was sufficient. Re-labelling $a_0(\nu) = c(\nu)$ and $a_1(\nu) = m(\nu)$, the calibration takes the linear form:

$$y(\nu) = m(\nu) \times F(\nu) + c(\nu) \quad \Rightarrow \quad y[i] = m[i] \times F[i] + c[i]$$

Where $[i]$ labels the channel number, and the $i$th channel has $\nu = (i/16.384)$ MHz. To determine two unknown constants for each $i$, two unique equations must be supplied. Therefore the spectra with two different known Solar Flux distributions must be measured using Phoenix 3. The two cases chosen are negligible flux (§ 4.2), and the quiet sun (§ 4.3).
4.2 Calibration Set 1 - Negligible Flux

For zero Flux, equation 6 reduces to $y[i] = c[i]$. The telescope was pointed below the horizon after sunset, thus picking up both interference and thermal radiation from the hills (figures 1-A and 1-B), but nothing from the sun. Assuming a plank distribution, for hills with a surface temperature ($T$), the thermal energy density per unit frequency is given by:

$$u(\nu, T) = \frac{8\pi \hbar \nu^3}{c^3} \frac{1}{\exp(\frac{\hbar \nu}{kT}) - 1} \quad \text{as } \frac{\hbar \nu}{kT} \rightarrow 0 \quad \frac{8\pi \nu^2 kT}{c^3}$$

Since $\frac{\hbar \nu}{kT} \ll 1$ (7)

The Flux of power into the telescope from the hills is given by equation 8. In the valid range of $\nu$, $F_{neg}$ is below the minimum statistical noise of 2 sfu (see §5), and can therefore be neglected.

$$F_{neg} = cu = \frac{8\pi \nu^2 kT}{c^2} = 1.118 \times 10^{-8} \nu^2 \ll 2 \text{ sfu} \quad \text{For } 0 \text{ MHz} \leq \nu \leq 1000 \text{ MHz}$$

Figure 1: A diagram depicting the Output of Phoenix 3 for Negligible Input Flux

Figure 1-B shows a close-up of a region with limited interference. The small amplitude fluctuations are random noise but there is also a systematic sinusoidal oscillation with a period $\approx 2$ MHz. This is probably caused by a bad impedance match of a component of length in the order of 50 m, for the reasons described in appendix B. The strong interference peaks and random noise in figures 1-A and 1-B prevent the use of a single, untreated spectra to be used to estimate $c[i]$.

As much RFI in each time slice as possible was removed and interpolated linearly using the algorithm described in §9, but with the inputs: $m[i] = 1$, $c[i] = 0$ and $his[i] = 0$ for all $i$. Then $n = 300$ time slices were averaged; reducing the random noise from 1.3% (described in §5) to 0.07%. A running mean over 7 channels smoothed any remaining fluctuations caused by very low amplitude RFI, and rose the uncertainty by an approximate factor $g = 10$. $g$ was estimated from the change in $m$ divided by 0.07 $m$ resulting from the running mean.
4.3 Calibration Set 2 - Quiet Sun Flux

The quiet sun flux \( F_q \) is the radio emission from the sun when no significant solar flares are active. \( F_q(\nu) \) has been parameterised as a function of sunspot number \( N_s \) to within 10% uncertainty\(^1\) by linearly interpolating quiet sun data\(^2,6\) on a log-log scale. The parameterisation is given in equation 9, where the values of \( k_0, k_1, p_0 \) and \( p_1 \) depend on the frequency range:

\[
F_q(\nu) = k_0 \nu^{p_0} + N_s k_1 \nu^{p_1}
\] (9)

A period of quiet sun was isolated a few minutes before the spikes began (reducing the affect of condition changes on the calibration). \( N_s \) was measured\(^7\) as 36 ± 5, and the parameterised quiet sun for that day is shown in figure 2. The same processes of RFI cleaning, averaging and running mean calculation as in § 4.2 was carried out, producing: \( y_q[i] \).

![Figure 2: The Parameterised Flux of the Quiet Sun](image)

4.4 Implemented Calibration

From \( c[i] \) and \( y_q[i] \), a smooth estimate for \( m[i] \) was obtained via: \( m[i] = (y_q[i] - c[i])/F_q[i] \). Despite the precautions taken to reduce the affect of RFI on the calibration, an RFI peak remained in both \( m \) and \( c \) at 390 MHz. These peaks were removed by fitting a fifth order polynomial to the surrounding region by minimising Chi-squared as is described in Appendix A. The polynomials were used to maintain the shape of \( m \) and \( c \) as much as possible. A close-up of \( m \) in the vicinity of the interference peak is shown in figure 3.

\(^1\)The data points used for the parameterisation has uncertainty of 7% the interpolation is expected to have slightly larger uncertainty: close to 10%

6
The calibration resulted in estimated values of $c[i]$ and $m[i]$ for frequencies 250 MHz to 900 MHz, enclosing the valid range 300 MHz to 850 MHz. From figures 4-A and 4-B, it can be seen that both $m[i]$ and $c[i]$ vary considerably with frequency; the gain drops significantly outside the valid range. The small scale fluctuations are not noise: they are permanent and are probably caused by poor impedance matching of components as described in Appendix B.

5 Statistical Uncertainty Analysis

The uncertainty in the readings of Phoenix 3 are required for scientific integrity, and also for the RFI removal algorithm (see § 9). The Radiometer equation can help predict the statistical uncertainty,
and is verified to apply for this instrument in § 5.1. The post calibration uncertainty for channels free of RFI is calculated in § 5.2.

### 5.1 Verifying the Radiometer Equation

The Radiometer equation for constant flux $F$ measured by an instrument with bandwidth $\Delta \nu$, and integrated over a time $\Delta t$, is shown in equation 10-A. The constant is caused by a D.C. offset added to signals prior to amplification. However, the relation between $F$ and $y$ is linear, and the output of Phoenix 3 includes the D.C. offset, hence equation 10-B applies for the uncalibrated data.

(A) \[ \frac{\sigma_F}{F + \text{const.}} = \frac{1}{\sqrt{\Delta \nu \Delta t}} \]

(B) \[ \beta = \frac{\sigma_y}{y} = \frac{1}{\sqrt{\Delta \nu \Delta t}} \] (10)

For the instrument’s configuration on 23rd August 2005, $\Delta \nu = 61.035$ kHz and $\Delta t = 100$ ms; the expected value of $\beta = 0.013$, for all flux and frequency channels. To verify this, the two data sets of negligible flux and quiet sun (see § 4) have been used. They are suitable since they detect roughly constant signals in time, but with different amplitudes relative to one another.

The average $\bar{y}$ and standard deviation $\sigma_y$ for each frequency channel was calculated from three hundred spectra in both cases; spanning 30 seconds. $\beta$ was estimated from $\sigma_y/\bar{y}$ and is plotted against frequency for the negligible and quiet sun input in figures 5-A and 5-B respectively. In both cases, $\beta$ was approximately constant over all channels (ignoring channels with RFI), with a value of $\beta = 0.014 \pm 0.001$. This suggests the radiometer equation estimates $\beta$ to within 7 % uncertainty.

Figure 5: $\beta$ for negligible and Quiet Sun Inputs

### 5.2 Determining the Uncertainty of the Calibrated Output

An expression for the statistical uncertainty ($\sigma_F$) is desired for all frequencies and any flux. To use quadrature, one requires $F(\nu)$ as a function of constants and variables with statistically independent
error. The standard form \( y[i] = m[i] \times F[i] + c[i] \) contains both \( m[i] \) and \( c[i] \) which do not have independent errors. The form in equation 11 is mathematically equivalent, but expresses \( F \) as a function of \( y, F_q, y_q \) and \( c \) which are statistically independent:

\[
F = F_q \left( \frac{y - c}{y_q - c} \right) \quad \Rightarrow \quad \sigma_F^2 = \left( \frac{\partial F}{\partial y} \right)^2 \sigma_y^2 + \left( \frac{\partial F}{\partial c} \right)^2 \sigma_c^2 + \left( \frac{\partial F}{\partial y_q} \right)^2 \sigma_{y_q}^2 + \left( \frac{\partial F}{\partial F_q} \right)^2 \sigma_{F_q}^2
\] (11)

Hence, for \( n \) time slices averaged to give \( c \) and \( y_q \) with a fractional increase in uncertainty of \( g \):

\[
\sigma_F = \sqrt{A(\nu)F^2 + B(\nu)F + C(\nu)} \approx \sqrt{A} F + \sqrt{C} \approx 0.1 F + 2 \text{ sfu} \quad (12)
\]

\( A(\nu) = g^2 \beta^2 / n + g^2 \beta^2 c^2 / (m^2 n^2 F_q^2) + \beta^2 y_q^2 / (m^2 F_q^2) + \beta'^2 \) \quad \text{For } \beta' = \sigma_{Fq} / F_q

\( B(\nu) = 2 g^2 \beta^2 c / (m n) - 2 g^2 \beta^2 c^2 / (m^2 n^2 F_q^2) \)

\( C(\nu) = 2 g^2 \beta^2 c^2 / (m^2 n) \)

The latter is a conservative, linear approximation of the error with constants determined using typical values of \( m, c \) and \( F_q \) in the valid frequency range from 300 MHz to 850 MHz.

**Part II**

**Removing Radio Frequency Interference**

To analyse the radio spikes recorded on 23rd August 2005, it is desirable to first remove as much RFI as possible. In future, the Phoenix 3 instrument will operate with the same high spectral resolution\(^2\), but with a variety of integration times. An algorithm is desired which can clean RFI from both the recorded data, and real-time data in future, with various integration times. The form of the RFI is studied to determine how it may be best detected in \( \S \) 6, and a number of methods of doing so are briefly examined in \( \S \) 7. Then the gradient method implemented is examined in detail in \( \S \) 8, and the final algorithm developed is described fully in \( \S \) 9.

\(^2\)Although a number of channels will be integrated over before saving the data
6 An examination of RFI

Here, features of RFI in the (300 - 850) MHz range are examined. The data is from a period of quiet sun; therefore any features significantly above noise can be attributed to RFI or the instrument itself. It is desirable to remove both types of features, so they are herein both regarded as RFI. Figure 6 is a broadband logarithmic spectra depicting a variety of RFI peaks. In § 6.1, some general features of the RFI peaks is uncovered before looking at some specific cases in § 6.2.

Figure 6: A Broadband Spectra Showing many types of RFI Peak

6.1 General Features of RFI

A sample of 300 time-slices was studied to identify the distribution of RFI bandwidth (figure 7) and maximum local percentage jump (figure 8). The jump is particularly relevant for the algorithm developed (see § 9 for its definition), and is essentially the step in Phoenix 3 output from one channel to the next. A zoom for bandwidths greater than 0.5 MHz overlays figure 7-A for clarity. The cumulative probability is the probability of any given value being less than that on the x-axis.

The method involved subtracting the quiet sun model described in § 4 from each spectra, leaving random fluctuations and RFI peaks as the only deviation from zero flux. First the RFI peak with the largest amplitude was identified and its edges located by finding the nearest minima on either side below half power. The peak’s bandwidth and the largest percentage jump between its edges were recorded, then the peak was removed from the spectra. The process was repeated on each spectra until the maximum flux was 3σ above zero, therefore recording all significant RFI peaks.
The bandwidth of RFI is most commonly below the frequency of a single channel. This information can be used to indicate which type of windowing function should be used for Phoenix 3 in future.

All RFI above $3\sigma$ in this sample has a jump greater than 5%. However, the analysis in § 8 indicates jumps of 10% occur due to legitimate solar features. Figure 8-B can be used to estimate how much interference would be missed if only features with jumps larger than (e.g.) 10% were excluded.

6.2 Detailed Features of Specific Types of RFI

Carrier bands dominate the spectrum, and tend not to vary significantly with time (no useful information will ever be obtained from channels in such regions). Two of these are shown more clearly in frequency space in figure 9. However, many smaller, narrow peaks exist, one of which is examined in figure 10 in both frequency and time.
Figure 9: Logarithmic Spectra of Large Amplitude RFI peaks

Figure 9-A shows large RFI can be composed of multiple peaks, and sometimes the edges decay slowly with no clear boundary. Figure 9-B shows small subsidiary peaks can surround large peaks.

Figure 10: Linear Spectra and Time Analysis of a Small Amplitude RFI peak

Figure 10-A, depicts an RFI peak spanning a single frequency channel. The amplitude is small relative to figure 9, but dwarfs the noise, therefore corrupting data if not removed. Figure 10-B follows the 375 MHz band over time, showing the RFI signal only occurs at specific times. Ideally, an algorithm would only reject this frequency channel when the RFI signal was present.

6.3 An Algorithm is Desired to:

1. remove long term carrier bands, and react to newly introduced carrier bands
2. remove small non-permanent RFI as possible
3. remove less significant subsidiary peaks surrounding large peaks
4. remove tails of large peaks significantly above noise
7 Possible Methods of RFI Exclusion

A spectrograph can be analysed for RFI: spectrally, temporally or in a combination of both. Spectral analysis is preferred for use online, as temporal analysis requires the comparison of new spectra with old, consuming significant memory, and increasing processing time.

7.1 Spectral Analysis

Some characteristic spectral features of RFI peaks determined in § 6 include their amplitude, bandwidth, and gradient:

- The amplitude of RFI peaks varies enormously, and many are below the flux that can occur in large solar flares, preventing the use of amplitude as an RFI indicator.
- The bandwidth is normally less than 0.2 MHz, which is significantly narrower than Solar features, although this only applies for less significant small RFI peaks.
- Common to all RFI is the large flux jump between contaminated channels, making gradient an ideal criteria to identify RFI.

However, the gradient of solar spikes can also be large, and occasionally it is larger than the minimum jump required to identify all RFI peaks. Thus some additional criteria must be used to distinguish RFI peaks from solar spikes.

7.2 Temporal Analysis

The minimum temporal resolution of Phoenix 3 is 10 ms, which is too large to over-sample radio spikes which have a duration of a few milliseconds\textsuperscript{[1]}. The Spectral Kurtosis suggested by Gelu et al\textsuperscript{[5]} is only a valid method of removing RFI if the most quickly varying solar features are over-sampled. Thus a spectral kurtosis method should not be used to clean Phoenix 3 data.

Many frequency channels contain RFI at nearly all times, particularly large carrier bands such as that in figure 9. Information about a channel’s RFI history could help determine if it currently contains RFI. Ideally this information would be condensed to a single, numerical ‘score’ for each channel to reduce memory use, and it should be dynamic to cope with newly introduced RFI.
8 The method selected: Gradient in Frequency Space

The simplest gradient algorithm detects large changes in raw data from one channel compared with the previous: \(\Delta y[i] = y[i] - y[i-1]\). If \(\Delta y[i]\) is above a local tolerance \((tol[i])\), it’s assumed to be caused by an RFI channel, which is normally a member of a local RFI band, consisting of one or more channels. The edges of the band are estimated, and all channels between are flagged so they can be neglected during analysis.

However, non RFI contributions to \(\Delta y[i]\) must be considered to ensure \(tol[i]\) is optimal. In this section we look at contributions from random fluctuations (§8.1), solar variation (§8.2) and the instrument’s frequency dependent modulation of incoming flux (§8.3).

8.1 Statistical Fluctuations Contribution to \(\Delta y\)

For the raw data \(y[i]\) output by the \(i^{th}\) frequency channel of Phoenix 3, it has been shown in §5 that the uncertainty \((\sigma_y[i])\) on \(y[i]\) is \(\beta \bar{y}[i]\). The radiometer equation is used to estimate \(\beta\), and \(\bar{y}[i]\) is the mean of \(y[i]\) over an infinite number of measurements. Acknowledging the central limit theorem, the probability density function \(P_i(y)\) is assumed to be Gaussian, centered on \(\bar{y}[i]\) with a standard deviation \(\sigma_y[i]\).

Here, the probability density function \(\varphi_i(\Delta y[i])\) of the jump \(\Delta y[i]\) is also shown to be Gaussian, centered on zero with a standard deviation \(\sqrt{2} \sigma[i]\). This assumes that \(\bar{y}[i] = \bar{y}[i-1]\), which ensures \(\sigma_y[i] = \sigma_y[i-1]\) and therefore \(P_i(y) = P_{i-1}(y)\). For algebraic clarity, we shift \(\bar{y}[i] = \bar{y}[i-1] \rightarrow 0\), and rename \(\sigma_y[i] = \sigma_y[i-1] \equiv \sigma_y; \Delta y[i] = \Delta y;\) and \(y[i] \equiv y_i\). Due to their statistical independence, the probability of obtaining specific values of \(y_i\) and \(y_{i-1}\) is the product of the probabilities of obtaining those values in isolation. Integrating over all specific values which provide \(y_i - y_{i-1} = \Delta y\):

\[
\varphi_i(\Delta y) = \int_{-\infty}^{\infty} P_{i-1}(\Delta y - y_i) P_i(y_i) \, dy_i = \int_{-\infty}^{\infty} P_i(\Delta y - y_i) P_i(y_i) \, dy_i \propto [P_i * P_i](\Delta y)
\]

Where \((*)\) denotes the convolution operator. From Fourier transform theory; the Fourier transform of a convolution is the product of the Fourier transforms of the operators being convolved. The Fourier transform of a Gaussian is simply another Gaussian, with the standard deviation inverted:

\[
\tilde{\varphi}_i(\omega) \propto [\tilde{P}_i(\omega)]^2 \propto \exp\left(-\omega^2 \sigma_y^2\right) = \exp\left(-\frac{\omega^2 (\sqrt{2} \sigma_y)^2}{2}\right)
\]

Since \(\tilde{P}_i(\omega) \propto \exp\left(-\frac{\omega^2 \sigma_y^2}{2}\right)\)

Taking the reverse transform:

\[
\varphi_i(\Delta y) \propto \exp\left(-\frac{(\Delta y)^2}{2(\sqrt{2} \sigma_y)^2}\right) = \exp\left(-\frac{(\Delta y)^2}{2 \sigma_y^2}\right) \propto \exp\left(-\frac{(\Delta y)^2}{2 \sigma_y^2}\right)
\]

For \(\sigma_y = \sqrt{2} \sigma_y\)
8.2 Solar Contribution to $\Delta y$

Solar spikes have the largest spectral gradient of any known solar radio feature, and should provide an upper limit on the solar contribution to $\Delta y/y$. A Gaussian Function\(^3\) provides an approximate model of each spike at a given time (see § 11.1). Relating the standard deviation of that Gaussian to its half power bandwidth ($\Delta \nu_{1/2}$), and shifting the frequency $\nu$ to center the Gaussian on $\nu = 0$:

\[
F(\nu) = F_{\text{max}} \exp\left(\frac{-\nu^2}{2\sigma^2_{\nu}}\right) \quad \text{With} \quad \sigma_{\nu} = \frac{\Delta \nu_{1/2}}{\sqrt{8 \ln2}}
\]

The magnitude of the gradient $|F'(\nu)|$ has a maximum value when $\nu = \sigma_{\nu}$, and is given by:

\[
|F'_{\text{max}}| = \frac{F_{\text{max}} \exp(-1/2)}{\sigma_{\nu}} = \frac{F_{\text{max}} \sqrt{8 \ln2} \exp(-1/2)}{\Delta \nu_{1/2}} \approx \frac{1.43 \, F_{\text{max}}}{\Delta \nu_{1/2}}
\]

From this, and using the calibration $y[i] = m[i] \times F[i] + c[i]$, an upper estimate of $\Delta y/y$ using the most extreme observed values of constants for the instrument and for spikes (bandwidth\(^1\) and peak flux from § 11.2) is given in equation 15. Where $\Delta \nu$ is the frequency separation of channels:

\[
\Delta y[i] \leq m[i] \times |F'_{\text{max}}| \times \Delta \nu \quad \text{and} \quad y[i] \approx m[i] \times (0.6 \, F_{\text{max}}) + c[i] \quad \text{For the maximum case}
\]

\[
\Rightarrow \quad \frac{\Delta y}{y} \leq \frac{1.43 \, m[i] \times F_{\text{max}} \, \Delta \nu}{(m[i] \times (0.6 \, F_{\text{max}}) + c[i]) \Delta \nu_{1/2}} \approx 0.1
\]

This tolerance provides an indication of the minimum that tolerance should be to permit spikes to pass through. However the model of the spikes as perfect Gaussian functions may be too crude, so $\Delta y/y \leq 0.5$ should be allowed to pass in perfect channels (see § 9.5).

8.3 Calibration Contribution to $\Delta y$

Over a range of approximately two hundred frequency channels, the modulation that Phoenix 3 has on the incoming PSD flux $F$ is given by:

\[
\Delta y[i] = \left\{ m^{(0)} + m^{(1)} \sin(\omega \, i + \phi) \right\} \times F[i] + \left\{ c^{(0)} + c^{(1)} \sin(\omega \, i + \phi) \right\}
\]

\(^3\)Some spikes contain internal structure, and cannot be accurately modeled using a Gaussian
The values of \( m^{(0)} \) and \( c^{(0)} \) are the average of \( m[i] \) and \( c[i] \) respectively (see § 4) over the local 200 channels. The sinusoidal fluctuations are caused by the impedance mismatching described in appendix B. The ‘frequencies’ with which \( m \) and \( c \) oscillate are the same and there is no relative phase shift because the origin of the oscillation in \( m \) and \( c \) is common.

The maximum \( \Delta y \) caused by this effect is given by \( \Delta y = (m^{(1)} F + c^{(1)}) \Delta \nu \omega \). For approximate values of the constants, \( \Delta y_{\text{calibration}} \approx 0.01 y \). Although this is relatively small (compared to the estimate of 0.1 for solar spikes) its contribution to \( \Delta y \) is calculable, and it is therefore possible to remove its affect altogether. It is important to determine the statistical contribution \( \Delta y[i] \) as accurately as possible in the algorithm, (see the \( g \)-vector in § 9.2). The choice the algorithm makes on where interference bands begin and end will be wildly inaccurate if no correction is made for this contribution to \( \Delta y \).

9 Description of the Cleansing Algorithm

The algorithm analyses individual time-slices in frequency space. It scans through each frequency channel, and its basic structure is as follows:

1. The jump in magnitude to one channel from the previous is determined
2. If the jump is greater than a certain size, that channel is flagged as containing RFI
3. Suspicious neighbouring channels with less significant jumps are identified and are also flagged

This description is split into sections. First the inputs are given. Secondly, key parameters used are defined and a short description is given. Then the method used to identify characteristic RFI channels is explained, and the criteria for removing suspicious neighbouring channels are also given.

9.1 Inputs

For \( 0 \leq i \leq (L - 1) \), where \( L = 2^{14} \) is the total number of channels:

1. \( y[i] \): The raw data output from the \( i^{th} \) frequency channel
2. \( m[i] \) and \( c[i] \): The multiplicative and additive calibration constants (see § 4 for the \( i^{th} \) frequency channel
3. \( h[i] \): The ‘history’ constant with value \( 0 \leq h[i] \leq 1 \), defined for each channel, which is refined every time the algorithm is run.
4. \( \Delta t \) and \( \Delta \nu \): The integration time and channel-bandwidth of the instrument
9.2 Parameters Defined from the Inputs

For all $0 \leq i \leq (L - 1)$, and all $0 \leq j \leq (N - 1)$, where $N$ has a value $0 \leq N \leq L$:

1. $F[i]$: The Calibrated Flux, before RFI removal, calculated from:

   $$F[i] = \frac{y[i] - c[i]}{m[i]}$$

2. $\tilde{F}[i]$: A first estimate of what the Flux would be without interference. The estimate is made using a sample; which is large enough to ignore most interference peaks (see § 6), but narrow enough to retain most solar variations.

   $$\tilde{F}[i] = \text{average of the lowest 10 values in the range: } F[i-50], F[i-19], \ldots, F[i+48], F[i+49].$$

3. $\tilde{y}[i]$: A first estimate of the raw data without RFI: obtained from back-calibrating $\tilde{F}$:

   $$\tilde{y}[i] = m[i] \times \tilde{F}[i] + c[i]$$

4. $\delta[i]$: The jump from the $(i - 1)^{th}$ to the $i^{th}$ channel, removing any contribution caused by the uneven calibration. It is calculated from:

   $$\delta[i] = m[i] \times (F[i] - F[i-1])$$

5. $\beta$: This constant provides an estimate of the standard deviation ($\sigma_\delta[i]$) of $\delta[i]$, via: $\sigma_\delta[i] = \beta \times \tilde{y}[i]$. It assumes that the true flux into the $(i - 1)^{th}$ and the $i^{th}$ channels is equal, and incorporates theory from § 8.1. It is calculated from:

   $$\beta = \sigma_\delta[i] \times \frac{\sqrt{2}}{\sqrt{\Delta t \Delta \nu}}$$

6. $tol[i]$: A tolerance is defined for each channel which determines how large a jump must be before it’s considered to be caused by RFI. It takes values of $4 \sigma_\delta[i] \leq tol[i] \leq 0.5 \tilde{y}[i]$, and has the following form, which is explained in § 9.5:

   $$tol[i] = (4 \beta + (0.5 - 4 \beta) h^{1/3}) \times \tilde{y}[i]$$

7. $s[j]$: A vector containing the index of all channels which we suspect may contain RFI. In fact, many represent valid channels, mis-selected because solar features make them appear to contain RFI. The elements of $s$ will naturally form groups of consecutive numbers. If one channel in a group is deemed to contain RFI, the other group members are marked too.

All channels are initially in $s$. The $(i - 1)^{th}$, $i^{th}$ and $(i + 1)^{th}$ are removed if the following conditions are true:

   For $j = i - 2, i - 1, i, i + 1, i + 2$:

   (a) $|\delta[j]| \leq 3 \sigma_\delta[j]$ for any of the $j$ defined
   (b) $\text{SIGN}(\delta[j])$ is not the same for all the $j$ defined
   (c) $y[i] \leq 1.5 \tilde{y}[i]$
9.3 Locating and Removing Bad Channels

A channel is considered to contain RFI if:

\[ |\delta[i]| \geq tol[i] \]

This large jump in frequency-space indicates the rise or fall at the edges of an interference peak. However RFI peaks often have crests with jumps that are not quite as high in frequency space, and subsidiary peaks which also have less significant peaks. In order to remove as much of the RFI next to the marked jump as possible, the entire suspicious region next to the peak is removed, as shown in figure 11. The channels to be removed are recorded in the inclusion vector \( I[i] \), defined in equation 17:

\[
I[i] = \begin{cases} 
0 & \text{If } F[i] \text{ is to be removed} \\
1 & \text{If } F[i] \text{ is not to be removed} 
\end{cases}
\]  

(17)

Figure 11: A diagram depicting how a time-slice is split up into two types of section

9.4 Resizing the Calibrated Data

If averaging occurs to resize the data, marked channels are excluded from the average as shown in equation 18. If \( Norm[i] = 0 \), \( F_{out}[i] \) is simply flagged as total RFI. For resizing by a factor of \( 2^p \):

\[
F_{out}[i] = \frac{1}{Norm[i]} \sum_{j=i-2^{(p-1)}}^{i+2^{(p-1)}-1} I[j] F[j] \quad \text{Where} \quad Norm[i] = \sum_{j=i-2^{(p-1)}}^{i+2^{(p-1)}-1} I[j]
\]  

(18)
9.5 The History and Tolerance Constants: $h[i]$ and $tol[i]$

$h[i]$ is a ‘quality’ factor, and dictates the tolerance ($tol[i]$) of jump permitted in the $i^{th}$ channel before it is flagged as RFI. High $h[i]$ (close to 1) occurs in channels which rarely experience large jumps, thus more variation is tolerated there. Low $h[i]$ occurs in the converse situation. $h[i]$ is updated after each run of the program as follows:

$$h_{\text{new}}[i] = \frac{h_{\text{old}}[i] \times (n - 1) + \tau[i]}{n}$$

Where

$$\tau[i] = 0 \quad \text{if} \quad |\delta[i]| \geq \lambda \tilde{y}[i]$$

$$= 1 \quad \text{otherwise}$$

$(n)$ effects how long the memory of the system is, and provides a more accurate $h$ for large $n$, however response time is shorter for small $n$: a value $n \approx 300$ was used for this report. ($\lambda$) marks the magnitude of $\delta[i]/\tilde{y}[i]$ assumed to be caused by RFI. $\lambda = 0.10$ should register 99% of RFI (see § 6), and is double $\beta_{\text{max}} \approx 0.05$, preventing excessive incorrect flagging. There remains a probability ($P_{\beta}(\beta)$) that a channel will be incorrectly marked as RFI by $h[i]$ due to statistical fluctuations. Thus, $h'[i]$ is the history due to RFI alone: $h'[i] = h[i] + P_{\beta}(\beta)$ with the additional restriction: $h'[i] \leq 1$.

$$P_{\beta}(\beta) = \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \, dx$$

Only channels with $h'[i] \geq 0.95$ experience a significant increase in tolerance, as shown in figure 12. This is the purpose of the high power in the form: $tol = (4\beta + (0.5 - 4\beta) h'[i]^{13}) \tilde{y}[i]$

Figure 12: The variation of Tolerance of a Channel with its History
10 Effectiveness of the Algorithm

The calibrated spectra shown in figure 13-A is from period of particularly high activity on 23rd August 2005. The many solar spikes present have large spectral gradients, which could be mistaken as RFI if the algorithm was not designed appropriately.

Figure 13: A Cleaned and an Uncleaned Spectra containing Solar Spikes

![Figure 13: A Cleaned and an Uncleaned Spectra containing Solar Spikes](image)

Figure 13-B shows the interpolated, cleaned spectra. The history vector $h[i]$ was given initial values of 1 for all channels, and refined by running on 300 spectra before being used in the algorithm for this spectra. The solar spikes remain, but all visible RFI has been removed except for the strong band around 390 MHz, which is significantly attenuated. The zoom in figure 14 is of a section chosen at random, and shows the cleaning in more detail. The region removed around each peak seems to be chosen well; each RFI peak is removed to below noise, without excessive information loss.

Figure 14: A zoom of a section of the Uncleaned Spectra with a Vertically Shifted, Cleaned Overlay

![Figure 14: A zoom of a section of the Uncleaned Spectra with a Vertically Shifted, Cleaned Overlay](image)
Part III

Data Analysis

It is of scientific interest to estimate the bandwidth, skewness and net Flux associated with decimetric spikes. These characteristics are determined for 1478 spikes in §11, using statistical techniques and a modified Gaussian to model each spike. The modified Gaussian failed to model some spikes accurately, revealing significant internal structure which is documented in §12.

11 Features of Spikes

Spikes are so numerous in the data sample that they often overlap one-another. This prevents numerical statistical analysis, as individual spikes cannot be easily isolated. For example, figure 15 shows the sum of two Gaussians which represent spikes. The total flux was separated at the minima between the two peaks, and the mean and standard deviation of each section calculated. The values for the smaller peak deviated from that of the original Gaussian by more than 25%.

Figure 15: A diagram showing the sum of two different Gaussian Curves

A solution to this problem is to minimise Chi-squared for the sum of a modelling function for each spike, which can then be analysed independently. In this example, one would recover the original Gaussian curves. A good model of the spikes should be simple enough to allow optimisation of Chi-squared, but must allow enough variation to fit the spikes accurately. It must also contain some antisymmetric contributions to model non-zero skewness.
11.1 Modelling the spikes

An initial model of a spike is Gaussian, with a peak \( F_0 \), centered about frequency \( \nu_0 \), where \( F_0 = F(\nu_0) \) is the maximum local flux. The standard deviation \( \sigma \) can be estimated from measuring the width \( (\Delta \nu_A) \) at a given constant flux \( (A < F_0) \), making the Gaussian unique:

\[
F_{\text{model}}(\nu) = F_0 \exp \left( -\frac{(\nu - \nu_0)^2}{2 \sigma^2} \right)
\]

Where
\[
\sigma^2 = \frac{(\Delta \nu_A)^2}{8 \ln(F_0/A)}
\]  

(19)

Each spectra was first cleaned of RFI, calibrated and the background (the average of the lowest 5% of flux values for each channel) was subtracted. A Gaussian model was then fit to the largest spike, with \( A = 40 \) sfu, and the estimated values of \( F_0, \nu_0 \) and \( \sigma \) were recorded before subtracting the model from the spectra: deleting that spike. The process was repeated on the adjusted spectra until a Gaussian approximation of all spikes with peaks above 40 sfu in the spectra was made.

The more complex model given in equation 20 reduces to the Gaussian in equation 19 when \( M, A, B \) and \( C \) are all zero. The sum of the models for each spike with \( F_0, \nu_0 \) and \( \sigma \) estimated as above, and \( M = A = B = C = 0 \) provided an initial approximation of \( F(\nu) \) for the original spectra. The optimal values of all seven constants were then found for each spike in the spectra using Chi-square minimisation\(^4\).

\[
F_{\text{model}}(\nu) = F_0 \left[ M z + 1 \right] \exp \left( -\frac{z^2}{2} \right) + \frac{F_0 \left[ A z^2 + B z + C \right]}{1 + \left[ z/\alpha \right]^8}
\]

For: \( z = \frac{\nu - \nu_0}{\sigma} \)

(20)

Equation 20 is simple enough to provide statistical information analytically (see appendix C), but allows enough variation to fit most spikes accurately. By formulating \( F(\nu) \) as a function of \( z \), each term is either symmetric or anti-symmetric in frequency about \( \nu_0 \).

The denominator of \( Y(\nu) \) has a value equal to unity within 2% for \( z \leq 0.5 \alpha \), and becomes very large for \( z \geq 2 \alpha \). Taking \( \alpha = 0.5 \) and keeping the magnitude of \( M, A, B \) and \( C \) below 0.8 ensures that the polynomial has significant freedom to contribute to \( F(\nu) \) in the region surrounding \( \nu_0 \), but not more than \( \sigma \) from it. It also ensures the moments of \( F(\nu) \) required for analysis, are finite.

11.2 Results

The distributions of the total observed energy per area \( (E) \) for each spike, their skewness, and bandwidth are depicted in figures 16-A, 16-B and 17 respectively.

\(^4\)The error on each data point is the full form of equation 12 in § 5, and is infinite for interpolated values where interference had to be removed
Figure 16: Probability Density of the Energy per Area and Skewness of the spikes detected

Lines of best fit on the logarithmic energy per area probability density \((P_E(E))\) in figure 16 provide the exponential relationship (equation 21). The lowest \(E\) reading was discounted due to the selection effect of choosing only features larger than 40 sfu as candidates for spike analysis. The uncertainty for \(P_E(E)\) is the Poisson error, and does not account for inaccuracies in the modeling method.

\[
P_E(E) = P_0 \exp\left(-\frac{E}{E_0}\right) \quad \text{For } P_0 = 0.020 \pm 0.004 \text{ and } E_0 = (50 \pm 5) \times 10^{-15} \text{ J m}^{-2} \quad (21)
\]

The skewness is clearly negative for most spikes, as expected from theory (see § 3). The positive skew found could be due to inaccuracies in the modeling process.

Figure 17: Probability Density of the Bandwidth of the spikes detected
The lower limit of bandwidth of the spikes is narrower than in other studies\cite{1}, possibly due to strict selection criteria don’t identify smaller spikes. It is also possible that the better frequency resolution of the Phoenix 3 instrument simply allows narrower spikes to be identified. Figure 18 depicts a particularly narrow peak, with a bandwidth of 0.3 MHz.

Figure 18: Close up of narrow band-width (0.3 MHz) Solar Spike and adjacent spectra

This puts a new lower limit on the bandwidth of spikes, this spike’s bandwidth spans 5 frequency channels suggesting that the instrument’s resolution is not limiting the observation. The spectra before and after the spike was measured are shown to indicate the background. No RFI was found in this frequency region when searched for.

12 Internal Structure

The model described in §11.1 could not accurately fit some spikes, because of significant internal structure. Figure 19 has had the background subtracted, and the sigma-error\textsuperscript{5} is included on the plot to indicate how significant the structure is. The regions of missing error indicate that RFI has been removed and interpolated.

The uncertainty of about 4 sfu cannot account for the structure shown in the zoom in figure 19. The largest gradient is 50 sfu MHz\textsuperscript{-1}. The internal structure observed varies significantly over frequencies of less than one MHz, and are thus the finest solar spike observed to date. Relating back to theory, the structure could be due to separated pockets of unstable electrons MASEing, and inducing one-another to MASE. These pockets would have to be very close together to explain the very similar emission frequency (which is proportional to the local magnetic field strength $B$).

\textsuperscript{5}The sigma error is that described in § 5, but with the contribution from the quiet sun model removed, which is systematic and should only scale the intensity of the entire spike over this small frequency range.
13 Conclusions

The principle conclusions of this report are:

1. Solar Radio Spikes have a minimum bandwidth of at most 0.3 MHz

2. Some solar spikes have internal structure: variations significantly above noise within the bandwidth of the most accurately fitting Gaussian.

3. The RFI removal algorithm described in this report removed all identified RFI peaks with bandwidth under 5 MHz. For a fuller verification of the algorithm, it should be tested with lower integration times, which would produce more noise.

4. The Phoenix 3 Instrument should not be used to observe frequencies below 350 MHz due to reduced effective dish area in that domain which produces very low gain.

Some spikes may have been excluded from the analysis because their peak flux was below the selection threshold of 40 sfu. Other spikes which would have fit the model accurately may have been excluded because a spike with significant internal structure was present in the same spectra; increasing Chi-squared and causing that spectra to be ignored in analysis. The results observed from spikes in this study are not necessarily typical of all solar spikes, and could form a subset. Further work could involve analysis of other data sets, and comparing them with this one.
A Chi Squared Method for Polynomial Fitting

For a function \( f(x) \) modelling \( y(x) \), Chi Squared can provide an estimate of how well the function fits a data set \( y_i \) and \( x_i \) where there are \( M \) data points, and uncertainties \( \sigma_{x_i} \) and \( \sigma_{y_i} \) respectively. However, Chi Squared can also be used to determine parameters of the function \( f(x) \).

In this report, it was desired to find a function \( f \) which related the frequency \( x \) to the calibration curve of Phoenix 3 \( y \), i.e. \( y = f(x) \).

\[
\chi^2 = \sum_{i=0}^{M} \left( \frac{y_i - f(x_i)}{\sigma_i} \right)^2
\]

(22)

Where \( \sigma_i^2 = \sigma_{y_i}^2 + \frac{dy}{dx} \sigma_{x_i}^2 \), and \( \frac{dy}{dx} \) can be estimated by first establishing an approximate \( f(x) \) using \( \sigma_i = \text{constant} \) and differentiating it such that \( \frac{dy}{dx} \approx \frac{df}{dx} \).

Smaller \( \chi^2 \) corresponds to a better model: parameters within the model can be estimated by minimising \( \chi^2 \) with respect to each parameter. Hence if we approximate \( f \) using a finite series expansion:

\[
f(x) = \sum_{n=0}^{N} a_n x^n \quad \Rightarrow \quad \chi^2 = \sum_{i=0}^{M} \left( \frac{y_i - \sum_{n=0}^{N} a_n x_i^n}{\sigma_i} \right)^2
\]

(23)

Stationary values of \( \chi^2 \) occur for each co-efficient \( a_m \) when \( \frac{\partial \chi^2}{\partial a_m} = 0 \), hence:

\[
0 = \sum_{i=0}^{M} \left( \frac{y_i - \sum_{n=0}^{N} a_n x_i^n}{\sigma_i} \right) a_i^m \quad \Rightarrow \quad \sum_{i=0}^{M} y_i x_i^m = \sum_{i=0}^{M} \sum_{n=0}^{N} a_n x_i^{n+m} \sigma_i^2 = \sum_{n=0}^{N} \left( \sum_{i=0}^{M} a_n x_i^{n+m} \sigma_i^2 \right)
\]

(24)

Calling \( A_{mn} = \sum_{i=0}^{M} \frac{a_n x_i^{n+m}}{\sigma_i^2} \) and \( b_m = \sum_{i=0}^{M} \frac{y_i x_i^m}{\sigma_i^2} \), and using the summation convention, linear algebra shows:

\[
A_{mn} a_n = b_n \quad \Rightarrow \quad a_m = A_{mn}^{-1} b_n
\]

(25)

Where \( A_{mn}^{-1} \) is the inverse matrix of \( A_{mn} \). Hence for the data set for a given frequency, IDL was set to compute the \( N \) by \( N \) matrix \( A_{mn} \) and the vector \( b_n \), and the inverse was calculated to double precision using the Gaussian Elimination method by a standard IDL program, and the \( N^{\text{th}} \) order polynomial which best fit the data was determined.
B Sinusoidal variation in the instrument gain due to an Impedance miss-match

Most impedances of the Phoenix 3 system between the receiver and the ADC are close to the standard 50 Ω. However, small deviations from this value are inevitable, and they result in reflections at the interface. The magnitude of reflected signal is a function of frequency. Here it is shown that for small impedance changes, the proportion of reflected power is approximately sinusoidal in frequency space, with a peak-to-peak frequency period related to the length of the badly matched impedance device.

Figure 20 depicts waves in a section of transmission line of impedance $Z_2$ between two sections each of impedance $Z_1$ resulting from a unit-amplitude sinusoidal input. Four simultaneous equations are formed from imposing the standard boundary conditions of continuity of signal $V$, and continuity of the gradient of $V$.

Figure 20: A Diagram Showing the Waves Resulting from an impedance Miss-Match

Solving to isolate $t$, the amplitude of the wave which is transmitted to the ADC:

$$t = \frac{4k_1k_2}{(k_1 + k_2)^2 \exp(-ik_2a) - (k_1 - k_2)^2 \exp(i k_2a)} \quad (26)$$

For similar impedances $k_1 \approx k_2$. Calling $k_2 - k_1 = \delta$ and $k_1 + k_2 = X$:

$$|t|^2 = \frac{16k_1^2(k_1 + \delta)^2}{X^4 + \delta^4 - 2\delta^2 X^2 \cos(2k_2a)} \quad \text{as} \quad \delta/k_2 \to 0 \quad 1 - \frac{\delta^2}{2k_2^2} [1 - \cos(2k_2a)] \quad (27)$$

The power transmitted is $|t|^2$, and is unity for perfect transmission. The small amplitude power transmission variation is directly proportional to $\cos(2k_2a)$, which is sinusoidal in $k_2$ as expected. The period in $k_2$ is $\Delta k_2 = \pi/a$.

The speed of electromagnetic radiation in most transmission lines is approximately $0.6 \, c$, from the wave equation, we can relate $k_2$ to the frequency of the incoming waves:
\[ \nu = \frac{\text{speed}}{\text{wavelength}} = 0.6 \frac{k_2 c}{2\pi} \Rightarrow \Delta \nu = 0.3 \frac{\Delta k_2 c}{\pi} = \frac{0.3 c}{a} \] (28)

The sinusoidal oscillation has a ‘period’ \( \Delta \nu = (0.3c/a) \), and given that the sinusoidal variation discovered during calibration had a period of approximately 2 MHz, \( a \) must be approximately 50 m, which is in fact the length of the cable which takes the signal from the telescope to the observatory building. This cable is difficult to replace (as it would require digging up all the underground cabling), so in future, efforts should be taken to reduce the impact of this inevitable impedance miss-match (which with lower resolution instruments would have gone undetected!)

C  Calculating Statistical Features of a spike from its Modelling Function

\[ F(\nu) = F_0 [M z + 1] \exp \left( -\frac{z^2}{2} \right) + \frac{F_0 [A z^2 + B z + C]}{1 + [z/\alpha]^8} \quad \text{For} \quad z = \frac{\nu - \nu_0}{\sigma} \] (29)

The modelling function is given by equation 29. The algebra can be simplified by defining the following integrals, and shifting the origin temporarily to \( \nu_0 \):

\[ I_n^{(1)} = \sigma^{n+1} \int_{-\infty}^{\infty} z^n \exp \left( -\frac{z^2}{2} \right) \, dz \quad \text{and} \quad I_n^{(2)} = \sigma^{n+1} \alpha^8 \int_{-\infty}^{\infty} \frac{z^n}{\alpha^8 + z^8} \, dz \]

\[ I_0^{(1)} = \sigma \sqrt{2\pi} \quad I_2^{(1)} = \sigma^3 \sqrt{2\pi} \quad I_4^{(1)} = 3 \sigma^5 \sqrt{2\pi} \]

\[ I_0^{(2)} = \frac{\pi \alpha \sigma}{2} \sqrt{1 + 1/\sqrt{2}} \quad I_2^{(2)} = \frac{\pi \alpha^3 \sigma}{2} \sqrt{1 - 1/\sqrt{2}} \quad I_4^{(2)} = \frac{\pi \alpha^5 \sigma}{2} \sqrt{1 - 1/\sqrt{2}} \]

Where \( \alpha = 1/2 \). The integral set \( I_n^{(1)} \) has been calculated using integration by parts, and integral set \( I_n^{(2)} \) using contour integration and residue calculus. The total flux \( J \) associated with the entire spike is given by:

\[ J = \int_{-\infty}^{\infty} F(\nu) \, d\nu = F_0 \left( I_0^{(1)} + A I_2^{(2)} + C I_0^{(2)} \right) \] (30)

\(^6\)The period is here taken to be the distance in frequency space between two adjacent peaks of the sinusoidal curve. The period is not in time space.

28
The mean, and the second and third moments with $\nu$ shifted are given by equations 31, 32 and 33 respectively:

$$\bar{\nu} = \frac{1}{J} \int_{-\infty}^{\infty} \nu F(\nu) \, d\nu = \frac{F_0 \left( MI_2^{(1)} + BI_2^{(2)} \right)}{J}$$ (31)

$$Mom(2) = \frac{1}{J} \int_{-\infty}^{\infty} (\nu - \bar{\nu})^2 F(\nu) \, d\nu = \bar{\nu}^2 - \nu^2$$ (32)

$$Mom(3) = \frac{1}{J} \int_{-\infty}^{\infty} (\nu - \bar{\nu})^3 F(\nu) \, d\nu = \frac{F_0 \left( MI_4^{(1)} + BI_4^{(2)} \right)}{J} - 3 \bar{\nu}^2 \nu + 2 \nu^3$$ (33)

Where $$\nu^2 = \frac{1}{J} \int_{-\infty}^{\infty} \nu^2 F(\nu) \, d\nu = \frac{F_0 \left( I_2^{(1)} + AI_4^{(2)} + CI_2^{(2)} \right)}{J}$$

The true mean ($\nu_{\text{mean}}$), standard deviation ($\nu_{\sigma}$) and skew ($Skew$) of the spike’s model are:

$$\nu_{\text{mean}} = \bar{\nu} + \nu_0 \quad \nu_{\sigma} = \sqrt{Mom(2)} \quad Skew = \frac{Mom(3)}{[Mom(2)]^{3/2}}$$ (34)

References


