Doctoral Thesis

Stable finite element boundary element Galerkin schemes for acoustic and electromagnetic scattering

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Stable Finite Element Boundary Element Galerkin Schemes for Acoustic and Electromagnetic Scattering

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Doctor of Sciences

presented by
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Zusammenfassung


lers. Standard Interpolationfehlerabschätzungen ermöglichen die Herleitung theoretischer Konvergenzraten selbst im Falle resonanter Frequenzen. Numerische Experiment für des Helmholtz Transmissionsproblem bestätigen diese Konvergenzresultate und zeigen die Überlegenheit der stabilisierten Kopplungsstrategie gegenüber der symmetrischen Methode auf. Abschliessend kann gesagt werden, dass die regularisierten variationellen Formulierungen selbst im resonanten Fall eindeutige diskrete Lösungen liefern, welche mit der optimalen Rate gegen die exakte Lösung konvergieren.
Abstract

This thesis is concerned with the stabilisation of Ritz-Galerkin schemes for time-harmonic acoustic and electromagnetic scattering problems. The main focus lies on the numerical treatment of transmission problems, where we are interested in the effect an inhomogeneous region of bounded support has on an incident field. The design and implementation of numerical algorithms for transmission problems are very challenging because one has to take into account the computation of the scattered wave on the unbound exterior region.

Because of the infinite exterior domain, boundary element methods (BEM) are best suited for the discretisation of the scattered field, whereas on the interior region due to the inhomogeneous medium and the possible presence of sources or sinks we opt for a finite element (FEM) discretisation of the total field. The coupling of boundary integral equations with variational formulations on the medium is based on Dirichlet-to-Neumann maps (DtN), which map the Dirichlet data of scattered waves onto their Neumann data. Such DtN maps can be constructed from the Calderón projector for the exterior region, which can be used to identify Cauchy data of radiating Helmholtz or Maxwell solutions. According to the properties of the DtN map we distinguish between the symmetric and the non-symmetric coupling approach. A severe disadvantage of all known strategies is that they all suffer from so-called spurious resonances. This refers to a countable set of critical frequencies, for which the coupled variational formulations fail to possess unique solutions. These frequencies are closely related to the eigenvalues of an interior Dirichlet eigenvalue problem. Note that this is an artificial difficulty, since the transmission problem is in general uniquely solvable. Non-uniqueness of solutions now causes the discrete systems, which arise from a conforming discretisation of the coupled variational formulations, to be extremely ill-conditioned whenever the wave number is close to one of the resonant frequencies. Close to the limit this effect becomes so strong that any direct or iterative solver applied to the linear system will suffer a breakdown.

One possible solution to this dilemma is the use of complex Robin-type trace mappings at the interface boundary of the inhomogeneous medium, which can be obtained from the standard Dirichlet and Neumann traces by means of a trace transformation operator. For the construction of these transformations regularising operators will be needed, similar to the ones that are already in use for the stabilisation of boundary element methods in acoustic and electromagnetic scattering. Regularised versions of the exterior Calderón projector can be obtained by applying the transformed traces to the representation formula. From the regularised versions of the exterior Calderón projector stable DtN maps can be constructed, which lead to resonance-free coupled variational formulations for the transmission problem. The occurrence of operator products in the variational formulations forces us to introduce additional non-physical variables on the interface boundary. A closer look reveals that these artificial unknowns do not pose any further problem, since it can be shown that they always evaluate to zero, independently of the wave number, the material parameters, the shape of the obstacle, or the incident field.

On one hand, we can use compactness properties of the regularising operators to establish so-called Gårding inequalities for the underlying sesquilinear forms. On the other, the transformed traces enforce uniqueness of weak solutions. Hence, we can employ standard theory about conforming discretizations of coercive variational formulations and establish asymptotic inf-sup estimates for the underlying sesquilinear forms, which allows us to derive existence, uniqueness, and stability of discrete solutions and an asymptotic quasi-optimality estimate for the discretization error. Furthermore, standard interpolation error estimates for finite element and boundary element spaces allow us to establish theoretical convergence rates even in the resonant case. Numerical experiments for the Helmholtz transmission problem confirm these results and clearly indicate superiority of regularised coupling strategies compared to symmetric ones in the presence of resonant frequencies. As a concluding remark we can say that, the mixed
regularised variational formulations provide unique discrete solutions, which converge with the optimal rate towards the exact solution of the transmission problem.
Part I

The Helmholtz Case
Chapter 1

Introduction

Scattering theory plays an important role in modern physical practice, with applications ranging from acoustic remote sensing systems such as SONAR (Sound Detection and Ranging), a hydroacoustic instrument used to navigate or detect other vessels, or SODAR (Sonic Detection and Ranging), a meteorological device used to measure the thermodynamic structure of the lower atmosphere, up to medical sonography, which denotes an ultrasound-based diagnostic imaging method, employed by health care professionals to visualize muscles, tendons, internal organs, or fetuses in prenatal care.

Acoustic scattering problems are focused on the effect, which inhomogeneous media have on incident sound waves. Hence, if we adopt the splitting of the total field $U$ into a prescribed incident part $U^i$ and a resulting scattered field $U^s$, we obtain the direct scattering problem, which aims at finding $U^s$ from the knowledge of $U^i$ and the physical laws that determine wave motion. The direct scattering problem can be further subdivided into the scattering of incident acoustic waves from impenetrable, homogeneous objects, which we refer to as Helmholtz scattering problems, and scattering from penetrable, inhomogeneous media, for which we adopt the name Helmholtz transmission problems.

1.1 Acoustic Waves

First of all, we start with a brief overview of the physical background of the acoustic scattering problem, as presented in [28, Sect. 8.1]. We consider propagation of acoustic waves of small amplitude in an inhomogeneous medium consisting of various types of material having different properties, which are assumed to be contained inside the volume $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$). The physical model describing the wave motion is considered as a problem in fluid dynamics assuming homogeneous material parameters in the exterior air region $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega}$. For any $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, let $p(x, t)$ denote the pressure, $\mathbf{v}(x, t)$ the velocity vector of a fluid particle, $\rho(x, t)$ the density, and $S(x, t)$ the specific entropy of the fluid. The propagation of acoustic waves inside the fluid is then governed by the following set of equations:

\begin{align}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho} \nabla p &= 0, \\
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) &= 0, \\
p &= f(\rho, S), \\
\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S &= 0,
\end{align}

(1.1)
where \( f \) is a function depending on the nature of the fluid. Euler’s equation describes the momentum conservation, which holds true in a frictionless fluid with no volume forces, and the equation of continuity enforces mass conservation inside the fluid if neither sources nor sinks are present. For the rest of this section we assume \( p, \mathbf{v}, \rho, \) and \( S \) to be small perturbations of the steady state solution \( \mathbf{v}(x,t) = \mathbf{v}_0 = 0, \rho(x,t) = \rho_0(x), S(x,t) = S_0(x), \) and \( p_0 = f(\rho_0, S_0) = \text{const.} \) A straightforward linearisation of (1.1) as described in [28, Sect. 8.1] yields the following equations for the perturbed solution

\[
\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0} \mathbf{grad} p = 0, \\
\frac{\partial p}{\partial t} + \rho_0 \mathbf{div} \mathbf{v} = 0, \\
\frac{\partial p}{\partial t} = c^2(x) \left( \frac{\partial p}{\partial t} + \mathbf{v} \cdot \mathbf{grad} \rho_0 \right),
\]

where \( c \) denotes the speed of sound, which is defined by the following expression

\[
c^2(x) := \frac{\partial f}{\partial \rho}(\rho_0(x), S_0(x)).
\]

Hence, we conclude that the pressure field \( p \) satisfies

\[
\frac{1}{c^2(x)} \frac{\partial p}{\partial t} - \rho_0(x) \mathbf{div} \left( \frac{1}{\rho_0(x)} \mathbf{grad} p \right) = 0.
\]

Furthermore, if we assume all terms containing \( \mathbf{grad} \rho_0 \) to be negligible and that \( p \) is a harmonically oscillating function with a single angular frequency \( \omega > 0 \), that is

\[
p(x,t) = \Re \{ U(x) \exp(-i\omega t) \},
\]

then \( U \) is a solution to the following equation

\[
\Delta U + \frac{\omega^2}{c^2(x)} U = 0 \quad \text{in } \mathbb{R}^d.
\] (1.2)

Based on our assumptions, we conclude that the speed of sound assumes a constant value \( c(x) = c_0 \) for all \( x \in \Omega^+ \) and by definition of the wave number \( \kappa := \omega/c_0 > 0 \) and the refractive index \( n(x) := c_0^2/c^2(x) \), (1.2) can be cast into the following form

\[
\Delta U + \kappa^2 n(x) U = 0 \quad \text{in } \mathbb{R}^d.
\]

Note that the refractive index might display some spatial variation inside \( \Omega \) but assumes the constant value \( n = 1 \) in the exterior air region \( \Omega^+ \).

It turns out that an additional condition is needed to obtain a well-posed problem. To do so, we need to distinguish between prescribed incident fields \( U^i \), which satisfy the homogeneous Helmholtz equation

\[
\Delta U^i + \kappa^2 U^i = 0 \quad \text{in } \mathbb{R}^d,
\]

in the background medium, and resulting scattered waves \( U^s \). Prominent examples for \( U^i \) are so-called plane waves, which are given by

\[
U^i(x) := A \exp(i \kappa \mathbf{d} \cdot \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,
\]

where \( \mathbf{d} \in \mathbb{R}^d \) (\( d = 2, 3 \)), \( |\mathbf{d}| = 1 \), denotes the direction of propagation and \( A \in \mathbb{R} \) the amplitude of the incident field. Hence the total field \( U \) can be split into an incident part \( U^i \) and a scattered wave \( U^s \), satisfying \( U = U^i + U^s \). For the two spherically symmetric waves

\[
U_\pm(x) := \frac{\exp(\pm i \kappa |x|)}{|x|},
\]
only the one labelled with + satisfies the Sommerfeld radiation condition \cite{80}

\[
\frac{\partial U^s}{\partial r} - i k U^s = o \left( r^{(1-d)/2} \right) \quad \text{uniformly for } r := |x| \to \infty,
\]

in three dimensions. Moreover, the following identity

\[
\text{Re} \left\{ U_\pm(x) \exp(-i\omega t) \right\} = \frac{\cos(k \cdot d \cdot x - \omega t)}{|x|},
\]

confirms that \( U_+ \) corresponds to an outgoing spherical wave, whereas \( U_- \) denotes an incident field or particle. Hence, the asymptotic boundary condition (1.3) distinguishes between incoming and outgoing time-harmonic sound waves.

We finish the introduction with the following important convention. For the rest of this thesis \( \Omega \subset \mathbb{R}^d \) (\( d = 2,3 \)) denotes either the volume occupied by

1. a bounded, impenetrable object, or
2. by an inhomogeneous medium of bounded support.

In practical applications even combinations of both cases can occur. However, any numerical method, which is flexible enough to compute approximate solutions, for 2. can be modified to deal with a combination of both cases. Thus 1. and 2. should merely serve as model problems to derive numerical methods for time-harmonic acoustic scattering.

\subsection*{1.2 Sobolev Spaces, Traces, and Differential Operators}

In this section we define the relevant Sobolev spaces on domains and boundaries and review some of their properties. The main reference for this section are the textbooks \cite{76,64}.

In what follows we assume \( \Omega \) to be a curvilinear Lipschitz polyhedron in the parlance of \cite[Sect. 1]{31}, e.g. a simply connected Lipschitz domain, whose boundary \( \Gamma := \partial \Omega \) consists of piecewise smooth and open faces \( \Gamma_j, j = 1, \ldots, N_\Gamma \). For simplicity reasons we assume also \( \Gamma \) to be connected. The exterior unit normal vector field \( \mathbf{n} \) belongs to \( L^\infty(\Gamma) \) and is directed from \( \Omega \) into \( \Omega^+ \).

Without any further explanations we will make use of Sobolev spaces of arbitrary order on domains, in particular \( H^s(\Omega) \) and \( H^{-s}(\Omega) \) for \( s \geq 0 \), cf. \cite{2} and \cite[Chap. 2]{64}. The corresponding Fréchet spaces on unbounded domains will be labelled by a subscript \( \text{loc} \), e.g. \( H^s_{\text{loc}}(\mathbb{R}^+) \) for \( s \geq 0 \). Their dual spaces will be tagged by the subscript \( \text{comp} \) to indicate that they contain compactly supported distributions, e.g. \( H^{-s}_{\text{comp}}(\mathbb{R}^+) \) for \( s \geq 0 \). Furthermore, for any \( D \subset \mathbb{R}^d \) (\( d = 2,3 \)),

\[
H^1_{\text{loc}}(\Delta, D) := \{ U \in H^1_{\text{loc}}(D); \Delta U \in L^2_{\text{loc}}(D) \},
\]

denotes the domain of definition of the Laplacian, where the subscript \( \text{loc} \) will be dropped if \( D \) is bounded. The Sobolev spaces and related functionals on boundaries, \( H^s(\Gamma) \) and \( H^{-s}(\Gamma) \), can be defined invariantly for \( 0 \leq s \leq 1 \), cf. \cite[Thm. 1.3.3]{42}, and especially for \( s > 1 \) we set

\[
H^s(\Gamma) := \left\{ u \in H^1(\Gamma); u_{|\Gamma_j} \in H^s(\Gamma_j), j = 1, \ldots, N_\Gamma \right\}.
\]

Labelling interior and exterior traces from \( \Omega^+ \) and \( \Omega \) by + and −, we obtain the following point-wise restrictions of smooth functions \( W \in C^\infty(\Omega^+) \) and \( V \in C^\infty(\bar{\Omega}) \) onto \( \Gamma \):

\[
(\gamma^+_D V)(x) := V(x), \quad (\gamma^+_N V)(x) := \text{grad} \, V(x) \cdot \mathbf{n}(x), \quad x \in \Gamma,
\]
\[
(\gamma^-_D W)(x) := W(x), \quad (\gamma^-_N W)(x) := \text{grad} \, W(x) \cdot \mathbf{n}(x), \quad x \in \Gamma.
\]

Their extension to Sobolev spaces is detailed in \cite[Lem. 3.1, Lem. 3.2]{30}.
Lemma 1.1. The Dirichlet traces \( \gamma_D^+ : H^1(\Omega) \mapsto H^{1/2}(\Gamma) \) and \( \gamma_D^- : H^{1}_{\text{loc}}(\Omega^+) \mapsto H^{1/2}(\Gamma) \) are continuous, surjective and possess continuous right inverse. Furthermore, the Neumann traces \( \gamma_N^+ : H(\Delta, \Omega) \mapsto H^{-1/2}(\Gamma) \) and \( \gamma_N^- : H^{1}_{\text{loc}}(\Delta, \Omega^+) \mapsto H^{-1/2}(\Gamma) \) provide continuous mappings.

Furthermore, we introduce jumps and averages of traces across the boundary \( \Gamma \) according to

\[
\begin{align*}
[\gamma_D^+ V]_\Gamma &= \gamma_D^+ V - \gamma_D^- V, & [\gamma_D^- V]_\Gamma &= \gamma_N^+ V - \gamma_N^- V, \\
\{\gamma_D^+ V\}_\Gamma &= \frac{1}{2} (\gamma_D^+ V + \gamma_D^- V), & \{\gamma_D^- V\}_\Gamma &= \frac{1}{2} (\gamma_N^+ V + \gamma_N^- V).
\end{align*}
\]

In the sequel \( (\cdot, \cdot)_\Gamma \) will stand for the inner product on the space of square integrable, scalar functions on the interface boundary \( \Gamma \)

\[
(u, v)_\Gamma := \int_\Gamma u v \, dS, \quad u, v \in L^2(\Gamma),
\]

which can be extended to a duality pairing on \( H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \). Adjoints of operators with respect to \( (\cdot, \cdot)_\Gamma \) will be tagged by \( * \). In addition, we denote by

\[
L_\ell^2(\Gamma) := \left\{ u \in L^2(\Gamma); \ u \cdot n = 0 \text{ a.e. on } \Gamma \right\},
\]

the space of square integrable, tangential vector fields endowed with the following inner product

\[
(u, v)_\ell := \int_\Gamma u \cdot \nabla v \, dS, \quad u, v \in L_\ell^2(\Gamma).
\]

This allows us to introduce the following subspaces

\[
H^{1/2}_s(\Gamma) := \left\{ u \in H^{1/2}(\Gamma); \ (1, u)_\Gamma = 0 \right\}, \quad H^{-1/2}_s(\Gamma) := \left\{ \vartheta \in H^{-1/2}(\Gamma); \ (\vartheta, 1)_\Gamma = 0 \right\},
\]

which provide crucial tools for establishing ellipticity results for boundary integral operators in two and three dimensions.

On Lipschitz polyhedra surface differential operators can be defined in a face-by-face fashion in the following way, cf. [76, Sect. 4.3]: For \( j = 1, \ldots, N_\Gamma \) and any \( q \in H^1(\Gamma_j) \), we introduce an extension \( q_\varepsilon \) of \( q \) onto

\[
U_\varepsilon := \left\{ z \in \mathbb{R}^d; \ \exists (x, \alpha) \in \Gamma_j \times (-\varepsilon, \varepsilon) : z = x + \alpha n(x) \right\}, \quad \varepsilon > 0,
\]
along the normal direction \( n \), by
\[
q_*(x + \alpha n(x)) := q(x), \quad (x, \alpha) \in \Gamma \times (-\varepsilon, \varepsilon).
\]

For any \( x \in \Gamma \) the surface gradient is now given by \( \text{grad}_\Gamma q(x) := \text{grad} q_*(x) \). A definition of the surface gradient on curvilinear Lipschitz polyhedra is based on locally Lipschitz-continuous charts, see section 4.2 for details. It is well known that \( \text{grad}_\Gamma \) provides a continuous linear map from \( H^1(\Gamma) \) onto \( L^2(\Gamma) \), cf. [18, Prop. 3.3]. The surface divergence \( \text{div}_\Gamma : L^2(\Gamma) \mapsto H^{-1}(\Gamma)/\mathbb{C} \) is given by the adjoint of the surface gradient w.r.t. the sesquilinear pairing \( (\cdot, \cdot)_\Gamma \), cf. [16, Def. 2.3]. In particular, the following identity holds
\[
(\text{div}_\Gamma u, q)_\Gamma = -(u, \text{grad}_\Gamma q)_\Gamma \quad \forall u \in L^2(\Gamma), q \in H^1(\Gamma).
\]

Obviously, \( \text{div}_\Gamma \) is linear and continuous as a mapping from \( L^2(\Gamma) \) to \( H^{-1}(\Gamma) \). Finally, we introduce the Laplace-Beltrami operator by \( \Delta_\Gamma := \text{div}_\Gamma \circ \text{grad}_\Gamma : H^1(\Gamma) \mapsto H^{-1}(\Gamma) \).

1.3 Potentials and Boundary Integral Operators

This section introduces potentials and boundary integral operators for the Helmholtz and Laplace equation and provides a brief review of the most important properties, as they are used in the numerical analysis of boundary element methods. The main reference for this section is the pioneering work of Costabel [30] and the textbooks [76, 64].

We start with the definition of the fundamental solution \( G_\kappa(z) \) to the Helmholtz and Laplace equation in two and three dimensions
\[
G_\kappa(z) := \begin{cases} 
\frac{i}{4} H^{(1)}_0(k|z|) & \text{for } k \neq 0 \text{ and } d = 2, \\
-\frac{1}{2\pi} \log(|z|) & \text{for } k = 0 \text{ and } d = 2, \\
\frac{1}{4\pi} \exp(ik|z|)/|z| & \text{for } k \in \mathbb{R} \text{ and } d = 3,
\end{cases}
\]

which allow us to introduce the Newton potential
\[
N_\kappa(U)(x) := \int_{\mathbb{R}^d} G_\kappa(x-y)U(y)\,dy.
\]

The Newton potential satisfies the following mapping properties, cf. [69, Chap. 6.1] and [81, Chap. 3].

**Theorem 1.2.** The Newton potential \( N_\kappa : H^s_{\text{comp}}(\mathbb{R}^d) \mapsto H^{s+2}_{\text{loc}}(\mathbb{R}^d) \) and the operator \( N_\kappa - N_0 : H^s_{\text{comp}}(\mathbb{R}^d) \mapsto H^{s+4}_{\text{loc}}(\mathbb{R}^d) \) are continuous for all \( s \in \mathbb{R} \).

Following the ideas of [30] and [76, Def. 3.1.5] we introduce the potentials
\[
\begin{align*}
\Psi_{SL}^\kappa &:= N_\kappa \circ \gamma_D^*, \\
\Psi_{DL}^\kappa &:= N_\kappa \circ \gamma_N^*.
\end{align*}
\]

The single-layer and double-layer potentials provide continuous mappings, cf. [76, Thm. 3.1.16]
\[
\begin{align*}
\Psi_{SL}^\kappa : H^{-1/2}(\Gamma) &\mapsto H^1_{\text{loc}}(\mathbb{R}^d), \\
\Psi_{DL}^\kappa : H^{1/2}(\Gamma) &\mapsto H^1_{\text{loc}}(\mathbb{R}^d \setminus \Gamma) \cap H_{\text{loc}}(\Delta, \Omega \cup \Omega^+).
\end{align*}
\]
Furthermore, for any given function $\varphi \in L^1(\Gamma)$ we obtain the following representations for (1.5) on $\mathbb{R}^d$ off the boundary

$$
\Psi_{SL}^\kappa(\varphi)(x) = \int_{\Gamma} G_\kappa(x - y) \varphi(y) \, dS(y), \quad x \in \mathbb{R}^d \setminus \Gamma,
$$

$$
\Psi_{DL}^\kappa(\varphi)(x) = \int_{\Gamma} \frac{\partial}{\partial n_y} G_\kappa(x - y) \varphi(y) \, dS(y), \quad x \in \mathbb{R}^d \setminus \Gamma,
$$
in a point-wise sense, see [76, Thm. 3.1.6] for details. For a fixed wave number $\kappa > 0$, a distribution $U$ on $\mathbb{R}^d$ is called a radiating Helmholtz solution, if

$$
\Delta U + \kappa^2 U = 0 \quad \text{in } \Omega \cup \Omega^\perp,
$$

$$
\lim_{r \to \infty} r^{(d-1)/2} \left( \frac{\partial U}{\partial r} - i\kappa U \right) = 0, \quad (1.6)
$$

where the limit is assumed to hold uniformly in all directions. It can be shown that for any $\vartheta \in H^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$ the potentials $\Psi_{SL}^\kappa(\vartheta)$ and $\Psi_{DL}^\kappa(v)$ provide solutions to (1.6), cf. [76, Prop. 3.1.7] and [76, Cor. 3.1.17]. Furthermore, any radiating Helmholtz solution $U$ can be expressed via jumps of its Cauchy data across the interface boundary $\Gamma$ using the representation formula, cf. [76, Thm. 3.1.6]

$$
U = -\Psi_{SL}^\kappa(\gamma_N U|_\Gamma) + \Psi_{DL}^\kappa(\gamma_D U|_\Gamma). \quad (1.7)
$$

Applying the trace mappings yields the following four continuous boundary integral operators

$$
V_\kappa : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \quad V_\kappa := \{\gamma_D\}_\Gamma \circ \Psi_{SL}^\kappa,
$$

$$
K_\kappa : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma), \quad K_\kappa := \{\gamma_N\}_\Gamma \circ \Psi_{DL}^\kappa,
$$

$$
K'_\kappa : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \quad K'_\kappa := \{\gamma_N\}_\Gamma \circ \Psi_{SL}^\kappa,
$$

$$
W_\kappa : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \quad W_\kappa := -\{\gamma_N\}_\Gamma \circ \Psi_{DL}^\kappa,
$$

for a scale of Sobolev spaces with $|s| < \frac{1}{2}$, see [30, Thm. 1]. For the sake of implementations, more concrete boundary integral representations of the operators $V_\kappa, K_\kappa, K'_\kappa,$ and $W_\kappa$ are indispensable. A comprehensive treatment for second order elliptic operators can be found in [64, Sect. 7.2]. Thus for any $\varphi \in L^\infty(\Gamma)$ the following identities hold

$$
V_\kappa(\varphi)(x) = \int_{\Gamma} G_\kappa(x - y) \varphi(y) \, dS(x), \quad x \in \Gamma,
$$

$$
K_\kappa(\varphi)(x) = \int_{\Gamma} \frac{\partial}{\partial n_y} G_\kappa(x - y) \varphi(y) \, dS(x), \quad x \in \Gamma,
$$

$$
K'_\kappa(\varphi)(x) = \int_{\Gamma} \frac{\partial}{\partial n_x} G_\kappa(x - y) \varphi(y) \, dS(x), \quad x \in \Gamma,
$$

$$
W_\kappa(\varphi)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} G_\kappa(x - y) \varphi(y) \, dS(x), \quad x \in \Gamma,
$$
on the boundary in a point-wise sense. From the jump relations [76, Thm. 3.3.1]

$$
[\gamma_D \Psi_{SL}^\kappa(\vartheta)]_{\Gamma} = 0, \quad [\gamma_N \Psi_{SL}^\kappa(\vartheta)]_{\Gamma} = -\vartheta, \quad \forall \vartheta \in H^{-1/2}(\Gamma),
$$

$$
[\gamma_D \Psi_{DL}^\kappa(v)]_{\Gamma} = v, \quad [\gamma_N \Psi_{DL}^\kappa(v)]_{\Gamma} = 0, \quad \forall v \in H^{1/2}(\Gamma),
$$

we can directly deduce the following four identities

$$
\gamma_D^\pm \Psi_{SL}^\kappa = V_\kappa, \quad \gamma_N^\pm \Psi_{SL}^\kappa = K'_\kappa \mp \frac{1}{2} \text{id},
$$

$$
\gamma_D^\pm \Psi_{DL}^\kappa = K_\kappa \pm \frac{1}{2} \text{id}, \quad \gamma_N^\pm \Psi_{DL}^\kappa = -W_\kappa.
$$

Crucial for any variational formulation based on boundary integral operators will be the following lemma, see [76, Lem. 3.9.8], [30, Thm. 2].
Lemma 1.3. The following operators are compact
\[
\begin{align*}
\delta V_\kappa &:= V_\kappa - V_0 : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma), \\
\delta K_\kappa &:= K_\kappa - K_0 : H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma), \\
\delta K'_\kappa &:= K'_\kappa - K'_0 : H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma), \\
\delta W_\kappa &:= W_\kappa - W_0 : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma).
\end{align*}
\]

The proof relies on the fact that both \( V_\kappa - V_0 \) and \( K_\kappa - K_0 \) turn out to be integral operators with continuous and bounded kernels, which ensures that they map into \( H^1(\Gamma) \), which is compactly embedded in \( H^{1/2}(\Gamma) \). Details can be found in [21, Sect. 2].

Lemma 1.4. In three dimensions, the single-layer and the hypersingular operator, \( V_0 \) and \( W_0 \), are self-adjoint and elliptic on \( H^{-1/2}(\Gamma) \) and \( H^1(\Gamma) \) in the sense that there exists a constant \( C > 0 \) such that
\[
(\vartheta, V_0(\vartheta))_\Gamma \geq C\|\vartheta\|_{H^{-1/2}(\Gamma)}^2, \quad (W_0(v), v)_\Gamma \geq C\|v\|_{H^{1/2}(\Gamma)}^2, \tag{1.11}
\]
holds for all \( \vartheta \in H^{-1/2}(\Gamma) \) and \( v \in H^{1/2}(\Gamma) \). The same result holds true for the hypersingular operator in two dimensions. For the single-layer operator, (1.11) carries over to two dimensions only under the additional constraints \( \vartheta \in H^{-1/2}(\Gamma) \), with \( \text{diam}(\Omega) < 1 \), or \( \vartheta \in H^{-1/2}(\Gamma) \).

Proof. For a proof see [81, Sect. 6.6]. \( \square \)

Combining lemma 1.3 and 1.4, we finally arrive at the following estimates for the Helmholtz single-layer and the hypersingular operator \( W_\kappa \), cf. [76, Prop. 3.5.5] and [30, Thm. 2].

Lemma 1.5. The operators \( V_\kappa \) and \( W_\kappa \) satisfy a generalised Gårding inequality in the sense that there exists a constant \( C > 0 \) and compact operators
\[
\begin{align*}
T_V : H^{-1/2}(\Gamma) &\rightarrow H^{1/2}(\Gamma), \\
T_W : H^{1/2}(\Gamma) &\rightarrow H^{-1/2}(\Gamma),
\end{align*}
\]
such that
\[
\begin{align*}
\text{Re} \left\{ (\vartheta, (V_\kappa + T_V)(\vartheta))_\Gamma \right\} &\geq C\|\vartheta\|_{H^{-1/2}(\Gamma)}^2, \\
\text{Re} \left\{ ((W_\kappa + T_W)(v), v)_\Gamma \right\} &\geq C\|v\|_{H^{1/2}(\Gamma)}^2,
\end{align*}
\]
holds true for all \( \vartheta \in H^{-1/2}(\Gamma) \) and \( v \in H^{1/2}(\Gamma) \).

Furthermore, \( K'_\kappa \) is the \((\cdot, \cdot)_\Gamma\)-adjoint of \( K_\kappa \) up to a compact perturbation:

Lemma 1.6. There exists a compact operator \( T_\kappa : H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma) \) such that
\[
(\vartheta, (K_\kappa + T_\kappa)(\vartheta))_\Gamma = ((K'_\kappa + T_\kappa)(\vartheta), \vartheta)_\Gamma
\]
holds true for all \( \vartheta \in H^{-1/2}(\Gamma) \) and \( v \in H^{1/2}(\Gamma) \), where \( K_\kappa^* \) denotes the \( L^2(\Gamma) \)-adjoint of \( K_\kappa \).

Proof. From (1.5) and (1.8), we recall the representations
\[
K_\kappa = \{\gamma_D\}_\Gamma \circ N_\kappa \circ \gamma_N^*, \quad K'_\kappa = \{\gamma_N\}_\Gamma \circ N_\kappa \circ \gamma_D^*,
\]
where \( N_\kappa : \mathcal{H}_{\text{comp}}(\mathbb{R}^d) \rightarrow \mathcal{H}_{\text{loc}}(\mathbb{R}^d) \) is the Newton potential for the Helmholtz kernel
\[
K_\kappa^* - K'_\kappa = \{\gamma_N\}_\Gamma \circ (N_\kappa - N_\kappa^*) \circ \gamma_D^*.
\]
Observe that
\[(\mathcal{N}_\kappa - \mathcal{N}_\kappa^+) (V)(x) = 2i \int_{\mathbb{R}^d} \text{Im} \{G_\kappa(x - y)\} V(y) \, dy,\]
is an integral operator with an analytic kernel in both two and three dimensions, which maps continuously \(H^{-1}_\text{comp}(\mathbb{R}^d) \hookrightarrow H^s_\text{loc}(\mathbb{R}^d)\) for any \(s \in \mathbb{R}\). Thus, \(K_\kappa^* - K_\kappa : H^{-1/2}(\Gamma) \hookrightarrow H^{1}(\Gamma)\) is continuous and the compact embedding \(H^{1}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)\) finishes the proof. \(\square\)

Finally, it can be shown that the imaginary part of the sesquilinear form corresponding to the single-layer operator \(\mathcal{V}_\kappa\) is positive, see [82, Lem. 3.1].

**Lemma 1.7.** For all \(\vartheta \in H^{-1/2}(\Gamma)\) we have
\[\text{Im} \{\langle \mathcal{V}_\kappa(\vartheta), \vartheta \rangle_\Gamma \} \geq 0.\]

### 1.4 Calderón Projectors

A crucial tool for the coupling of the variational equations on \(\Omega\) and boundary integral equations on \(\Gamma\) are the two Calderón projectors [76, Sect. 3.6]
\[P_{\pm} := \begin{pmatrix} \frac{1}{2} \text{Id} \pm K_\kappa & \mp \mathcal{V}_\kappa \\ \mp W_\kappa & \frac{1}{2} \text{Id} \mp K'_\kappa \end{pmatrix} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma). \quad (1.12)\]

They arise from applying the trace operators \(\gamma_D^\pm\) and \(\gamma_N^\pm\) to (1.7) and using (1.10). The operators \(P_+\) and \(P_-\) obviously satisfy the identity
\[P_+ + P_- = \text{Id}. \quad (1.13)\]

The Calderón projectors can be used to characterise pairs of functions in \(H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\) that are eligible as traces of Helmholtz solutions, see [85].

**Theorem 1.8.** If and only if \((v, \vartheta) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)\) belongs to the range of \(P_{\pm}\), there is a Helmholtz solution \(U\) such that \(v = \gamma_D^+ U\) and \(\vartheta = \gamma_N^+ U\).

This theorem paves the way for establishing formal expressions for the exterior Dirichlet-to-Neumann map for the Helmholtz problem in \(\Omega^+\). This is the operator \(\text{DtN}_\kappa^+ : H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)\) returning the Neumann traces of an exterior Helmholtz solution matching prescribed Dirichlet boundary conditions on \(\Gamma\). Three different formulae can instantly be obtained from (1.12), at least formally, because the inverses of operators might not exist:
\[\text{DtN}_\kappa^+ := V_\kappa^{-1} \circ (K_\kappa - \frac{1}{2} \text{Id}), \quad (1.14)\]
\[\text{DtN}_\kappa^+ := - (\frac{1}{2} \text{Id} + K'_\kappa)^{-1} \circ W_\kappa, \quad (1.15)\]
\[\text{DtN}_\kappa^+ := - W_\kappa + (\frac{1}{2} \text{Id} - K'_\kappa) \circ V_\kappa^{-1} \circ (K_\kappa - \frac{1}{2} \text{Id}). \quad (1.16)\]

Only the third formula reflects the essential symmetry of the boundary value problem in the case \(\kappa = 0\). It will be the starting point for symmetric coupling.

**Remark 1.9.** If the incident wave \(U^i\) can be extended to an interior Helmholtz solution, which is evidently the case, when \(U^i\) is a plane wave or generated by a sound source compactly supported in \(\Omega^+\), then, by (1.7) and (1.9), its traces on \(\Gamma\) will fulfill
\[\begin{bmatrix} \gamma_D U^i \\ \gamma_N U^i \end{bmatrix} = P_- \begin{bmatrix} \gamma_D U^i \\ \gamma_N U^i \end{bmatrix} \Leftrightarrow P_+ \begin{bmatrix} \gamma_D U^i \\ \gamma_N U^i \end{bmatrix} = 0. \quad (1.16)\]
For the same reasons, the scattered field \( U^s \) satisfies
\[
\begin{bmatrix}
\gamma_D^+ U^s \\
\gamma_N^+ U^s 
\end{bmatrix} = P_+ \begin{bmatrix}
\gamma_D^+ U^s \\
\gamma_N^+ U^s 
\end{bmatrix}.
\tag{1.17}
\]

Using that the total field in \( \Omega^+ \) is given by \( U = U^s + U^i \), we can eliminate \( U^s \) from (1.16) and (1.17) and end up with
\[
\begin{bmatrix}
\gamma_D^+ U \\
\gamma_N^+ U 
\end{bmatrix} = P_+ \begin{bmatrix}
\gamma_D^+ U \\
\gamma_N^+ U 
\end{bmatrix} - \begin{bmatrix}
g_D \\
g_N 
\end{bmatrix}.
\tag{1.18}
\]

As above Dirichlet-to-Neumann maps for the total field can be constructed from this relationship.

### 1.5 Functional Analytic Framework

In this section we will give a short review of the functional analytic framework, which is used to establish existence and uniqueness of solutions and quasi-optimality error estimates to variational formulations based on elliptic boundary value problems and boundary integral equations. The main references here are the articles [77, 89] and the textbook [64].

Let \( V \) be a Hilbert space with induced norm \( \| \cdot \|_V \) and let \( a : V \times V \to \mathbb{C} \) be a bounded, sesquilinear form, which satisfies the continuity estimate
\[
|a(u,v)| \leq C_S \|u\|_V \|v\|_V \quad \text{for all } u, v \in V,
\]
with a constant \( C_S > 0 \). Furthermore, let \( V' \) denote the dual of \( V \) and for any \( g \in V' \), \( v \in V \) we introduce the following continuous bilinear and sesquilinear pairings
\[
\langle g,v \rangle_{V' \times V} := g(v), \quad (g,v)_{V' \times V} := \langle g,v \rangle_{V' \times V}
\]
on the product space \( V' \times V \). Key requirements when establishing existence of solutions to variational equations on Hilbert spaces using the Fredholm alternative are stated in the following definition, cf. [64].

**Definition 1.10.** A sesquilinear form \( a : V \times V \to \mathbb{C} \) is said to
1. satisfy a Gårding inequality, if there exists a compact operator \( T : V \to V' \) and a constant \( C > 0 \), such that
\[
\Re \left\{ a(u,u) + (T(u), u)_{V' \times V} \right\} \geq C \|u\|_V^2,
\]
for all \( u \in V \).
2. be injective, if \( a(u,v) = 0 \) for all \( v \in V \), implies \( u = 0 \).

Now for \( f \in V' \) consider the following weak variational problem on \( V \):
Find \( u \in V \) such that for all \( v \in V \) there holds
\[
a(u,v) = \langle f,v \rangle_{V' \times V}.
\tag{1.19}
\]

The following theorem establishes existence of solutions to the variational equation (1.19) from their uniqueness, based on the Fredholm alternative, cf. [64, Thm. 2.33].

**Theorem 1.11.** Let \( a : V \times V \to \mathbb{C} \) be an injective sesquilinear form satisfying a Gårding inequality. Then, for any \( f \in V' \), the variational problem (1.19) has a unique solution \( u \in V \).
Let $V_h \subset V$, $h > 0$ be a family of finite dimensional subspaces, which possess the approximation property
\[
\lim_{h \to 0} \inf_{v_h \in V_h} \|u - v_h\|_V = 0 \quad \text{for all } u \in V. \tag{1.20}
\]

Using this family of finite dimensional vector spaces $V_h$ for both ansatz and test functions, we arrive at the following Ritz-Galerkin discretization of problem (1.19):

Find $u_h \in V_h$ such that for all $v_h \in V_h$ there holds
\[
a(u_h, v_h) = (f, v_h)_{V', V}. \tag{1.21}
\]

Existence and uniqueness of discrete solutions $u_h \in V_h$ and the convergence $u_h \to u$ can only hold if the sesquilinear form $a$ satisfies a discrete inf-sup condition, cf. [89, Eq. 2.3.6]: There exists a constant $\gamma > 0$ such that for all $v_h \in V_h$ and the whole family $V_h$ with $h \to 0$, there holds
\[
\sup_{0 \neq w_h \in V_h} \frac{|a(v_h, w_h)|}{\|w_h\|_V} \geq \gamma \|v_h\|_V. \tag{1.22}
\]

The following lemma establishes the discrete inf-sup conditions for all sesquilinear forms satisfying items 1. and 2. from definition 1.10, cf. [89, Thm. 2.9].

**Theorem 1.12.** Assume the variational formulation (1.19) to be uniquely solvable and that the underlying sesquilinear form $a$ satisfies a Gårding inequality. Then there exists $h_0 > 0$ such that the Galerkin method (1.21) is stable satisfying the discrete inf-sup condition (1.22) for all $0 < h < h_0$.

The following lemma is a straightforward consequence of theorem 1.12 and establishes asymptotic quasi-optimality of the discretization error, when conforming discretizations are used for an approximation of the solution $u$.

**Lemma 1.13.** If the discrete inf-sup condition (1.22) holds, then the Galerkin equation (1.21) is uniquely solvable and we have the quasi-optimality error estimate
\[
\|u - u_h\|_V \leq \left( 1 + \frac{CS}{\gamma} \right) \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{1.23}
\]
Chapter 2

The Helmholtz Scattering Problem

2.1 Problem Formulation

Direct acoustic scattering from an impenetrable obstacle boils down to the following exterior Dirichlet problem for the scattered field $U^s$ on the unbound exterior air region

\[ \Delta U^s + \kappa^2 U^s = 0 \quad \text{in } \Omega^+, \]

\[ \gamma_D^+ U^s = g_D \quad \text{on } \Gamma, \]

\[ \frac{\partial U^s}{\partial r} - i\kappa U^s = o \left( r^{(1-d)/2} \right) \quad \text{uniformly for } r := |x| \to \infty, \]

where the wave number $\kappa > 0$ and the Dirichlet data is given by $g_D := -\gamma_D^+ U^i$. The boundary condition $\gamma_D^+ U^s = -\gamma_D^+ U^i$ corresponds to a sound-soft obstacle with vanishing total field on the surface. Based on Rellich’s theorem [74], it can be shown that problem (2.1) has unique solutions $U^s \in H^1_{\text{loc}}(\Delta, \Omega^+)$.}

Another important case is direct scattering from so-called sound-hard obstacles where the normal velocity of the total field vanishes on $\Gamma$. Mathematically speaking, this boils down to the following exterior Neumann problem for the scattered field

\[ \Delta U^s + \kappa^2 U^s = 0 \quad \text{in } \Omega^+, \]

\[ \gamma_N^+ U^s = g_N \quad \text{on } \Gamma, \]

\[ \frac{\partial U^s}{\partial r} - i\kappa U^s = o \left( r^{(1-d)/2} \right) \quad \text{uniformly for } r := |x| \to \infty, \]
where the wave number $\kappa > 0$ and the Neumann data is given by $g_N := -\gamma_N^+ U^1$. Again Rellich’s theorem can be employed to establish uniqueness of solutions $U^g \in H^1_{\text{loc}}(\Delta, \Omega^+)$.  

## 2.2 Boundary Element Methods

In this section we introduce variational formulations based on boundary integral equations for the computation of approximate solutions to the sound-soft scattering problem. The main reference in this section are the textbook [76, Chap. 3] and the articles [20, 82] for the regularised equations.

### 2.2.1 Indirect Methods

Indirect methods are based on a solution ansatz in terms of the single-layer or the double-layer potentials, or a complex linear combination of both. Since both potentials provide radiating Helmholtz solutions, in the sense of equation (1.6), we conclude that the Helmholtz equation and the Sommerfeld radiation condition are already satisfied by the trial expression. Thus, Galerkin discretizations of boundary integral equations for indirect methods are only concerned with the approximation of boundary integral operators and the Dirichlet or Neumann data on the boundary $\Gamma$. For sound-soft scattering problems, a large variety of different strategies have been proposed, see [9, 61, 72, 23] and [21] among others. In this section we consider some of the most popular indirect methods and review their properties.

### Single-Layer Ansatz

The *single-layer ansatz* (SLA) is based on the following trial expression for the solution to problem (2.1)

$$ U := \Psi_{\text{SL}}(\vartheta), \quad \vartheta \in H^{-1/2}(\Gamma). \quad (2.2) $$

Clearly, $U$ provides a radiating Helmholtz solution on $\Omega \cup \Omega^+$ in the sense of (1.6). An application of the exterior Dirichlet trace together with the trace relations (1.10) yields the following variational formulation:

Find $\vartheta \in H^{-1/2}(\Gamma)$ such that for all $\varphi \in H^{-1/2}(\Gamma)$ there holds

$$ (\varphi, \nabla \vartheta \varphi)^\Gamma = (\varphi, g_D)^\Gamma. \quad (2.3) $$

Now lemma 1.5 immediately provides us with a Gårding inequality for the corresponding sesquilinear form, in the sense of item 1. of definition 1.10. However, uniqueness of solutions for the variational formulation remains elusive. If we assume the resonant case, we can find $W \in H^1(\Omega) \setminus \{0\}$ such that

$$ \Delta W + \kappa^2 W = 0 \quad \text{in} \quad \Omega, \quad \gamma_D^- W = 0 \quad \text{on} \quad \Gamma. $$

Due to the fact that $\gamma_D^- W = 0$ we can employ theorem 1.8 and conclude

$$ \begin{bmatrix} 0 \\ \gamma_N^- W \end{bmatrix} = \mathcal{P} \begin{bmatrix} 0 \\ \gamma_N^- W \end{bmatrix} = \begin{bmatrix} \nabla \kappa(\gamma_N^- W) \\ (\frac{1}{2} \mathbb{I} + K'_N)(\gamma_N^- W) \end{bmatrix}, $$

which means that $(0, \gamma_N^- W)$ provides a solution to (2.3) in the case of $g_D = 0$. This destroys injectivity and prohibits us from applying the Fredholm alternative. Moreover, whenever $\kappa$ is close to one of the resonant frequencies the linear system arising from a Galerkin discretization of (2.2) will be extremely ill-conditioned. A profound analysis of the impact of spurious modes for electromagnetic scattering can be found in [26].
Combined Field Integral Equations

In 1965 Brakhage and Werner [9], Leis [61], and Panich [72] independently introduced the \textit{combined field integral equation}, whose trial expression for the solution is based on the following complex linear combination of the single-layer and double-layer potentials

\[ U := \Psi_{SL}^e(u) + i\eta \Psi_{DL}^e(u), \quad u \in H^{1/2}(\Gamma), \]

for some \( 0 \neq \eta \in \mathbb{R} \). Obviously, \( U \) is a radiating Helmholtz solution in the sense of (1.6). The combined field integral equation was proposed to obtain boundary integral formulations, which produce unique solutions for either sound-soft or sound-hard scattering problems and all positive wave numbers.

In the sound-soft case an application of the exterior Dirichlet trace together with the trace relations (1.10) yields the following equation

\[ \nabla \kappa(u) + i\eta (\kappa + \frac{1}{2} \text{id})(u) = g_D. \quad (2.4) \]

Note that this equation is set up in \( H^{1/2}(\Gamma) \) and thus needs to be tested by functions from \( H^{-1/2}(\Gamma) \). Unfortunately, since the density \( u \) needs to be chosen from \( H^{1/2}(\Gamma) \), matching trial and test spaces cannot be obtained and hence any variational formulation based on (2.4) is not suited for a straightforward discretization by means of boundary elements.

On the other hand, we can abandon the idea of working with the natural Dirichlet and Neumann trace spaces and consider the equation (2.4) in \( L^2(\Gamma) \) with a density function \( u \in L^2(\Gamma) \) as well. Thus we are able to retain matching trial and test spaces and arrive at a variational formulation which is amenable to a Ritz-Galerkin discretization. However, this does not guarantee coercivity of the underlying variational formulation, since the double-layer operator \( \kappa \) fails to be compact as a mapping from \( L^2(\Gamma) \) to \( L^2(\Gamma) \) on non-smooth boundaries in three dimensions, which prevents us from establishing a Gårding inequality for the sesquilinear form associated with (2.4). Nevertheless, it is well known that variational formulations based on combined field integral equations have unique solutions, cf. [61, 9].

A solution to the problem of non-matching test and trial spaces has been discovered by Panich [72]. Based on an additional operator that lifts the argument of the double-layer potential into the right trace space, a regularised combined field integral equation can be derived, for which all boundary integral operators related to the double-layer potential become compact perturbations of the single-layer operator.

We start from an abstract perspective and call \( M : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma) \) a \textit{regularising operator}, if it satisfies the following assumptions, cf. [21, Sect. 3.2].

\textbf{Assumption 2.1.} We suppose that

1. \( M : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma) \) is compact, and
2. \( \text{Re} \{ \langle \vartheta, M(\vartheta) \rangle_\Gamma \} > 0 \) for all \( \vartheta \in H^{-1/2}(\Gamma) \setminus \{0\} \).

The regularised combined field integral equation for problem (2.1) is based on the following trial expression

\[ U := \Psi_{SL}^e(\vartheta) + i\eta (\Psi_{DL}^e \circ M)(\vartheta), \quad \vartheta \in H^{-1/2}(\Gamma), \quad (2.5) \]

for fixed \( 0 \neq \eta \in \mathbb{R} \). As above, \( U \) provides a radiating Helmholtz solution on \( \Omega \cup \Omega^+ \) in the sense of (1.6). An application of the exterior Dirichlet trace together with the trace relations (1.10) leaves us with the boundary integral equation

\[ \nabla \kappa(\vartheta) + i\eta (\frac{1}{2} \text{id} + \kappa) \circ M(\vartheta) = g_D. \]
**Lemma 2.2.** The boundary integral operator

\[ \mathcal{V}_\kappa + i\eta \left( \frac{1}{2} \text{Id} + \mathcal{K}_\kappa \right) \circ \mathcal{M} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \]

is injective for all \( \eta \neq \eta \in \mathbb{R} \) and \( \mathcal{M} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) satisfying item 1. and 2. from assumption 2.1.

*Proof.* For a proof see [21, Lem. 3.1]. \qed

Furthermore, item 1. of assumption 2.1 confirms the boundary integral operator \( \left( \frac{1}{2} \text{Id} + \mathcal{K}_\kappa \right) \circ \mathcal{M} \) to be a compact mapping from \( H^{-1/2}(\Gamma) \) onto \( H^{1/2}(\Gamma) \). This implies a Gårding inequality on \( H^{-1/2}(\Gamma) \) for the sesquilinear form underlying the following variational formulation:

Find \( \varphi \in H^{-1/2}(\Gamma) \) such that for all \( \varphi \in H^{-1/2}(\Gamma) \) there holds

\[ (\varphi, \mathcal{V}_\kappa(\varphi))_\Gamma - i\eta (\varphi, ((\frac{1}{2} \text{Id} + \mathcal{K}_\kappa) \circ \mathcal{M})(\varphi))_\Gamma = (\varphi, g_D)_\Gamma. \] (2.6)

However, due to the operator product \( \mathcal{K}_\kappa \circ \mathcal{M} \) problem (2.6) is not amenable to a straightforward discretization by means of conforming boundary element spaces. In principle, this problem can be solved by introducing an additional variable and switching to a mixed variational formulation. A simple eligible operator for regularisation has been introduced by Buffa and Hiptmair in [21]:

For each \( \varphi \in H^{-1}(\Gamma) \), find \( \mathcal{M}(\varphi) \in H^1(\Sigma) \) such that for all \( q \in H^1(\Gamma) \) there holds

\[ (\varphi, \mathcal{M}(\varphi), \text{grad}_\Gamma q)_\Gamma + (\mathcal{M}(\varphi), q)_\Gamma = (\varphi, q)_\Gamma. \] (2.7)

Compactness for this operator immediately follows from the embedding \( H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \) and \( H^{-1/2}(\Gamma) \hookrightarrow H^{-1}(\Gamma) \). Obviously, \( \mathcal{M} \) satisfies the identity

\[ (\varphi, \mathcal{M}(\varphi))_\Gamma = \|\mathcal{M}(\varphi)\|^2_{L^2(\Gamma)} + \|\text{grad}_\Gamma \mathcal{M}(\varphi)\|^2_{L^2(\Gamma)}, \quad \forall \varphi \in H^{-1}(\Gamma). \]

Furthermore, if we assume \( \mathcal{M}(\varphi) = 0 \), we conclude \( \text{grad}_\Gamma \mathcal{M}(\varphi) = 0 \) and equation (2.7) confirms \( \varphi = 0 \), which renders \( \mathcal{M} \) an injective operator. For later use we define the sesquilinear form

\[ b(p, q) := (\text{grad}_\Gamma p, \text{grad}_\Gamma q)_\Gamma + (p, q)_\Gamma, \quad p, q \in H^1(\Gamma), \]

which allows us to restate definition 2.7 in the following way

\[ b(\mathcal{M}(p), q) = (p, q)_\Gamma, \quad \forall q \in H^1(\Gamma). \] (2.8)

Hence, by introducing the auxiliary variable \( p := \mathcal{M}(\varphi) \in H^1(\Omega) \), we can get rid of all operator products in (2.6) and we arrive at the mixed variational formulation:

Find \( \varphi \in H^{-1/2}(\Gamma), p \in H^1(\Gamma) \) such that for all \( \varphi \in H^{-1/2}(\Gamma), q \in H^1(\Gamma) \) there holds

\[ (\varphi, \mathcal{V}_\kappa(\varphi))_\Gamma - i\eta (\varphi, ((\frac{1}{2} \text{Id} + \mathcal{K}_\kappa)(p))_\Gamma = (\varphi, g_D)_\Gamma, \]

\[ - (\varphi, q)_\Gamma + b(p, q) = 0. \] (2.9)

Obviously, the first component of any solution \( (\varphi, p) \) of (2.9), will provide a solution to (2.6). Furthermore, all off-diagonal blocks of the underlying sesquilinear form are compact due to the embedding \( H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \). Thus a straightforward application of the Fredholm alternative will provide us with existence, uniqueness, and stability of solutions. Moreover, the following remark tells us how to reconstruct the scattered field \( U^\kappa \) from solutions \( (\varphi, p) \in H^{-1/2}(\Gamma) \times H^1(\Gamma) \) to the regularised combined field integral equation.
Remark 2.3. If $(\theta, p) \in H^{-1/2}(\Gamma) \times H^{1}(\Gamma)$ are solutions to (2.9), then the trial expression

$$U := \Psi_{SL}^\varepsilon(\theta) + \Psi_{DL}^\varepsilon(p)$$

provides a solution to problem (2.1).

This is not the only possible operator that fits into the framework of regularising operators. Another possible choice is based on the single-layer potential of the pseudo-differential operator $(\mathbf{1} - \Delta)^{1+\varepsilon}$ with $0 < \varepsilon \leq 1$, see [23]. The fundamental solution (cf. [71, Ex. 2.2]) is given by

$$G_{\varepsilon}(x) := \frac{2^{-\varepsilon}}{(2\pi)^{3/2} \Gamma(1+\varepsilon)} |x|^{\varepsilon-1/2} K_{\varepsilon-1/2}(|x|),$$

where $K_{\nu}$ is the modified Bessel function (cf. [1, Sect. 9.6]) and $\Gamma(\cdot)$ denotes the Gamma function (cf. [1, Sect. 6.1]). The corresponding single-layer potential is given by

$$B_{\varepsilon}(\varphi)(x) := \int_{\Gamma} G_{\varepsilon}(x-y) \varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^3.$$

**Theorem 2.4.** Let $1/2 < l < 2$, $l \neq 3/2$. Then, for any $\varepsilon \in (0,1]$, the potential

$$B_{\varepsilon} : H^{1/2-l}(\Gamma) \rightarrow H^{2+2\varepsilon-l}_{\text{loc}}(\mathbb{R}^3)$$

is continuous. For any $\varepsilon \in (0,1/2) \cup (1/2,1]$, the operator

$$\gamma_{D} B_{\varepsilon} : H^{1/2-\varepsilon}(\Gamma) \rightarrow H^{1/2+\varepsilon}(\Gamma)$$

is continuous. If the surface is smooth this holds for $\varepsilon = 1/2$ as well.

**Proof.** For a proof see [23, Thm. 2.5].

**Theorem 2.5.** For any $\varepsilon \in (0,1/2) \cup (1/2,1]$, the sesquilinear form

$$(\cdot, \gamma_{D} B_{\varepsilon}(\cdot))_{\Gamma} : H^{-1/2-\varepsilon}(\Gamma) \times H^{-1/2-\varepsilon}(\Gamma) \rightarrow \mathbb{C}$$

is hermitian and $H^{-1/2-\varepsilon}(\Gamma)$-elliptic. For smooth surfaces, this holds true for $\varepsilon = 1/2$ as well.

**Proof.** For a proof see [23, Thm. 2.6].

These theorems together with the compact embedding $H^{-1/2}(\Gamma) \hookrightarrow H^{-1/2-\varepsilon}(\Gamma)$, for $0 < \varepsilon < 1/2$, ensure item 1. and 2. of assumption 2.1 for the regularising operator defined by

$$M := \gamma_{D} B_{\varepsilon} : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma). \quad (2.10)$$

Recently, Engleder and Steinbach [82] introduced a novel type of a regularising operator based on a stabilised version of $W_0$ given by

$$W_\ast(v) := W_0(v) + (v, 1)_{\Gamma} 1, \quad \forall v \in H^{1/2}(\Gamma).$$

Note that $W_\ast$ is self-adjoint and $H^{-1/2}(\Gamma)$-elliptic, cf. [70, 81], which gives rise to the following bounded, linear operator

$$M := W_\ast^{-1} \circ (\frac{1}{2} \mathbf{1} + K_{\varepsilon-n}) : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma). \quad (2.11)$$

In contrast to the regularising operators introduced in (2.7) and (2.10), $M$ defined by (2.11) is no longer compact as mapping from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$, but merely continuous. However, due
to its specific structure it can still be used to stabilise trial expressions. The solution ansatz (2.5) for the sound-soft scattering problem now yields the following boundary integral equation

\[(V_\kappa + i\eta ((\frac{1}{2}I + K_\kappa) \circ M))(\varphi) = g_D.\]

This equation is set in \(H^{1/2}(\Gamma)\) and can be tested by functions from \(H^{-1/2}(\Gamma)\), which yields the following variational formulation:

Find \(\varphi \in H^{-1/2}(\Gamma)\) such that for all \(\varphi \in H^{-1/2}(\Gamma)\) there holds

\[\left(\varphi, (V_\kappa(\varphi))_\Gamma + i\eta (\varphi, ((\frac{1}{2}I + K_\kappa) \circ M)(\varphi))_\Gamma\right) = (\varphi, g_D)_\Gamma.\]

(2.12)

Although \(M\) defined by (2.11) does not satisfy item 2. of assumption 2.1, which ensures uniqueness of solutions to stabilised variational formulations, injectivity of the sesquilinear form underlying (2.12) can still be established, cf. [82, Thm. 3.1].

Lemma 2.6. The boundary integral operator

\[V_\kappa + i\eta ((\frac{1}{2}I + K_\kappa) \circ M) : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma),\]

with \(M\) given by (2.11) and \(\eta > 0\), is injective.

Moreover, a Gårding inequality for the underlying sesquilinear form directly follows from lemma 1.5 and (2.11), cf. [82, Sect. 3]. Hence, we can rely on a Fredholm argument to ensure existence, uniqueness, and stability of solutions.

Due to the operator product \(KK \circ M\) (2.12) is not amenable to a straightforward Ritz-Galerkin discretization and we have to introduce the auxiliary variable

\[\text{and switch to a mixed variational formulation:}\]

Find \(\varphi \in H^{-1/2}(\Gamma), u \in H^{1/2}(\Gamma)\) such that for all \(\varphi \in H^{-1/2}(\Gamma), v \in H^{1/2}(\Gamma)\) there holds

\[\left(\varphi, (V_\kappa(\varphi))_\Gamma - i\eta (\varphi, ((\frac{1}{2}I + K_\kappa)(u))_\Gamma\right) = (\varphi, g_D)_\Gamma,\]

\[i\eta (\varphi, ((\frac{1}{2}I + K_\kappa)(u))_\Gamma - i\eta (W_\kappa(u), v)_\Gamma = 0.\]

(2.13)

Now, a direct evaluation of the associated sesquilinear form for \(\varphi := \varphi, v := u\), together with lemma 1.4 and lemma 1.3 yields a Gårding inequality for the sesquilinear form underlying problem (2.13) on \(H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)\). Hence, applying the Fredholm alternative makes it possible to establish existence, uniqueness and stability of solutions.

2.2.2 Direct Methods

Direct methods are based on the Helmholtz representation formula (1.7). For impenetrable obstacles, where no interior field exists, the representation formula can be cast into the following simplified form

\[U^s = \Psi_{DI}^s(\gamma_D^+U^s) - \Psi_{SI}^s(\gamma_N^+U^s).\]

An application of the exterior Dirichlet and Neumann traces yields the well known exterior Calderón projector \(P_\kappa\)

\[\gamma_D^+U^s = -V_\kappa(\gamma_N^+U^s) + ((\frac{1}{2}I + K'_\kappa)(\gamma_D^+U^s),\]

\[\gamma_N^+U^s = (\frac{1}{2}I + K_\kappa)(\gamma_N^+U^s) - W_\kappa(\gamma_D^+U^s).\]

(2.14, 2.15)
From these two identities boundary integral equations for the exterior Dirichlet and Neumann problems for the Helmholtz equation can be derived. However, a straightforward use of these equations will be affected by spurious resonances.

In 1971 Burton and Miller [24] had the idea to consider complex linear combinations of (2.14) and (2.15), which results in the following boundary integral equation for the sound-soft scattering problem

\[
(V_\kappa + i\eta (\frac{1}{2} \text{Id} + K'_\kappa))\vartheta = (K_\kappa - \frac{1}{2} \text{Id} - i\eta W_\kappa)(g_D),
\]

with a fixed coupling parameter \(\eta\) satisfying \(0 < \eta \in \mathbb{R}\). Moreover, the unknown \(\vartheta := \gamma^+_N U^\oplus \in H^{-1/2}(\Gamma)\) evaluates to the exterior Neumann trace of the scattered field. In order to obtain a variational formulation which is amenable to a Galerkin discretization, by means of conforming boundary elements, we must aim at matching trial and test spaces. This would force us to consider the boundary integral equation in \(H^{1/2}(\Gamma)\), which is not possible. Thus, we have to abandon the idea of working with the natural trace space for problem (2.1) and lift the whole equation to \(L^2(\Gamma)\). Unfortunately, the double-layer operator \(K_\kappa\) fails to be compact on non-smooth domains in three dimensions, thus a Gårding inequality for merely Lipschitz continuous boundaries remains elusive. This bars us from applying the powerful Fredholm alternative and prevents us from establishing existence and stability of solutions. On the other hand, it is well known that variational formulations based on the Burton-Miller equation produce unique solutions, cf. [24].

The results obtained for the stabilised combined field integral equation in section 2.2.1 motivate the application of regularising operators in the case of direct methods. Hence, following the idea of Burton and Miller we consider a complex combination of (2.15) and (2.14)

\[
(K_\kappa - \frac{1}{2} \text{Id} - i\eta M \circ W_\kappa)(\gamma D U) - (i\eta M \circ (\frac{1}{2} \text{Id} + K'_\kappa) + V_\kappa)(\gamma^+_N U) = 0,
\]

where a regularising operator has been applied to lift (2.14) to \(H^{1/2}(\Gamma)\). The boundary integral equation holds in \(H^{1/2}(\Gamma)\) and can be tested by functions \(\varphi \in H^{-1/2}(\Gamma)\):

\[
(\varphi, V_\kappa(\vartheta))_\Gamma - i\eta \langle \varphi, (M \circ (\frac{1}{2} \text{Id} + K'_\kappa))(\vartheta) \rangle_\Gamma = \langle \varphi, (K_\kappa - \frac{1}{2} \text{Id} - i\eta M \circ W_\kappa)(g_D) \rangle_\Gamma. \tag{2.16}
\]

Furthermore, the application of a regularising operator, which fits into the framework of assumption 2.1 does not destroy uniqueness of solutions for problem (2.16).

**Lemma 2.7.** Provided that the regularising operator \(M\) satisfies assumption 2.1, the boundary integral operator

\[
V_\kappa + i\eta M \circ (\frac{1}{2} \text{Id} + K'_\kappa) : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma),
\]

is injective, for all \(0 < \eta \in \mathbb{R}\).

**Proof.** For a proof see [21, Lem. 4.1]. \(\square\)

Due to item 1. of assumption 2.1 we conclude that the boundary integral operator \(M \circ (\frac{1}{2} \text{Id} + K'_\kappa) : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)\) is compact, which makes it possible to establish a Gårding inequality for the sesquilinear form underlying the variational formulation (2.16). Hence existence and stability of solutions follows immediately from their uniqueness.

Unfortunately, due to the operator products \(M \circ W_\kappa\) and \(M \circ K'_\kappa\), the variational formulation (2.16) is not suited for a discretization by means of conforming boundary element spaces. Thus we have to resort to a mixed variational formulation by choosing the regularising operator (2.8) and introducing the auxiliary variable \(p := M((\frac{1}{2} \text{Id} + K'_\kappa)(\vartheta) + W_\kappa(g_D)) \in H^1(\Omega)\):
Find \( \vartheta \in H^{-1/2}(\Gamma) \) and \( p \in H^1(\Gamma) \) such that for all \( \varphi \in H^{-1/2}(\Gamma) \) and \( q \in H^1(\Gamma) \) there holds

\[
\begin{align*}
(\varphi, \mathcal{V}_\kappa(\vartheta))_\Gamma - i\eta (\varphi, p)_\Gamma &= (\varphi, (\mathcal{K}_\kappa - \frac{1}{2}\mathbb{I}) (g_D))_\Gamma, \\
-((\frac{1}{2}\mathbb{I} + \mathcal{K}_\kappa')(\vartheta), q)_\Gamma + b(p, q) &= (\mathcal{W}_\kappa(g_D), q)_\Gamma.
\end{align*}
\] (2.17)

Uniqueness of solutions is inherited from the original variational problem. Moreover, due to lemma 1.5, the sesquilinear form corresponding to the single-layer operator \( \mathcal{V}_\kappa \) satisfies a Gårding inequality on \( H^{-1/2}(\Gamma) \) and \( b \) is clearly elliptic since it gives raise to an inner product on \( H^1(\Gamma) \). Finally, the embedding \( H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \) turns all off-diagonal terms into compact sesquilinear forms. Thus we are able to establish a Gårding inequality for the sesquilinear form underlying (2.17).

**Remark 2.8.** The second equation of the mixed variational formulation can be recast into the following form

\[
p = M((\frac{1}{2}\mathbb{I} + \mathcal{K}_\kappa')(\vartheta) + \mathcal{W}_\kappa(g_D)).
\]

Moreover, since \((g_D, \vartheta)\) are Cauchy data of a distribution \( U \), solving problem (2.1). Hence, they belong to the kernel of the interior Calderón projector, from which we derive the following identity

\[
(\frac{1}{2}\mathbb{I} + \mathcal{K}_\kappa')(\vartheta) + \mathcal{W}_\kappa(g_D) = 0.
\]

We conclude \( p = 0 \) independently of the wave number \( \kappa \), the boundary data \( g_D \), or the shape of the obstacle. In short, \( p \) is a “dummy variable”.

### 2.3 Galerkin Discretization

After having removed all operator products from the problems (2.6), (2.12) and (2.16), all the variational formulations (2.9), (2.13), and (2.17) from can be discretized by restricting them to conforming finite dimensional subspaces \( \Theta_h \) of \( H^{-1/2}(\Gamma) \) and \( Q_h \) of \( H^1(\Gamma) \) and \( H^{1/2}(\Gamma) \), respectively. A powerful theorem about Galerkin discretizations of coercive variational formulations will yield an asymptotic quasi-optimality estimate of the discretization error, cf. [77, 89].

Assuming a minimal resolution of the discrete spaces \( \Theta_h \) and \( Q_h \) we conclude:

- **Existence and uniqueness of discrete solutions** \( (\vartheta_h, p_h) \in \Theta_h \times Q_h \) to the regularised combined field integral equation (2.9) and the a-priori error estimate

\[
\|\vartheta - \vartheta_h\|_{H^{-1/2}(\Gamma)} + \|p - p_h\|_{H^1(\Gamma)} \leq C \left( \inf_{\varphi_h \in \Theta_h} \|\vartheta - \varphi_h\|_{H^{-1/2}(\Gamma)} + \inf_{q_h \in Q_h} \|p - q_h\|_{H^1(\Gamma)} \right).
\]

- **Existence and uniqueness of discrete solutions** \( (\vartheta_h, u_h) \in \Theta_h \times Q_h \) to the regularised combined field integral equation (2.13) and the a-priori error estimate

\[
\|\vartheta - \vartheta_h\|_{H^{-1/2}(\Gamma)} + \|u - u_h\|_{H^{1/2}(\Gamma)} \leq C \left( \inf_{\varphi_h \in \Theta_h} \|\vartheta - \varphi_h\|_{H^{-1/2}(\Gamma)} + \inf_{v_h \in Q_h} \|u - v_h\|_{H^{1/2}(\Gamma)} \right).
\]

- **Existence and uniqueness of discrete solutions** \( (\vartheta_h, p_h) \in \Theta_h \times Q_h \) to the regularised Burton-Miller equation (2.17) and the a-priori error estimate

\[
\|\vartheta - \vartheta_h\|_{H^{-1/2}(\Gamma)} + \|p - p_h\|_{H^1(\Gamma)} \leq C \inf_{\varphi_h \in \Theta_h} \|\vartheta - \varphi_h\|_{H^{-1/2}(\Gamma)}. \tag{2.18}
\]
First, we equip the boundary $\Gamma$ by a family $\{\mathcal{M}\}_h$ of surface triangulations consisting of (curved) triangles or quadrilaterals. The meshes $\mathcal{M}_h$ have to resolve the shape of the Lipschitz polyhedron $\Omega$ in the sense that none of their elements may reach across an edge of $\Omega$. This leads to the definition of the boundary element spaces $\Theta_h$ and $Q_h$

$$
\Theta_h := \{ \varphi \in L^2(\Gamma); \varphi|_K \in \mathcal{P}_{k-1}(K) \ \forall K \in \mathcal{M} \},
$$

$$
Q_h := \{ q \in C^0(\Gamma); q|_K \in \mathcal{P}_k(K) \ \forall K \in \mathcal{M} \},
$$

where $\mathcal{P}_k(K)$ stands for the space of polynomials of degree $\leq k$ on the cell $K$. This refers to the total degree in the case of triangles and the degree in each variable in the case of quadrilaterals. Now we can employ the usual best approximation estimates for the $h$-version of boundary elements, cf. [10, Sect. 4.4] and [76, Sect. 4.3].

Lemma 2.9. For $0 < t < s < k + 1$ and all $\vartheta \in H^s(\Gamma)$ there holds

$$
\inf_{\varphi_h \in \Theta_h} \| \vartheta - \varphi_h \|_{H^{-t}(\Gamma)} \leq C h^{s-t} \| \vartheta \|_{H^s(\Gamma)},
$$

where the constant $C > 0$ is independent of $h$ and $\vartheta$.

Lemma 2.10. For $0 < t < s < k + 1$ and all $u \in H^s(\Gamma)$ there holds

$$
\inf_{v_h \in Q_h} \| u - v_h \|_{H^{-t}(\Gamma)} \leq C h^{s-t} \| u \|_{H^s(\Gamma)},
$$

where the constant $C > 0$ is independent of $h$ and $u$.

Remark 2.11. Due to remark 2.8 we already know that the dummy variable $p$ in (2.17) is equal to zero and the choice of the finite element space $Q_h$ has no impact on the overall convergence rate. Nevertheless, we cannot just simply drop $p$ from the variational formulation, since (2.18) is only an asymptotic estimate, which depends on sufficiently good approximation properties of $Q_h$.

2.4 Numerical Experiments

Limited computational resources allow the numerical exploration of asymptotic convergence rates and extensive parameter studies only in two dimensions. Fortunately, the theoretical developments in this section hold in a two as well as in a three dimensional setting. For the numerical experiments we considered:

- the unit circle $\Omega_\circ := \{ x \in \mathbb{R}^2 : |x| < 1 \}$ as a specimen of a domain with smooth boundary. The two resonant frequencies $\kappa_1 = 5.5201$ and $\kappa_2 = 11.7915$ were used, which correspond to the second and fourth zero of the Bessel function $J_0(x)$.

- the unit square $\Omega_{\square} := \{ x \in \mathbb{R}^2 : -1/2 < x_1, x_2 < 1/2 \}$ as representative of polygonal domains. The associated two resonant frequencies are $\kappa_3 = \sqrt{5\pi}$ and $\kappa_4 = \sqrt{13\pi}$.

The boundary $\Gamma$ of each domain is approximated by a family of shape-regular boundary element meshes $\{\mathcal{M}\}_h$, $h > 0$, consisting of piecewise flat panels. On each mesh we used piecewise constants for $\Theta_h$, and linear surface elements for $Q_h$, that is, the case $k = 1$ of (2.19). The dense matrices of the discrete boundary integral operators were computed using Duffy’s trick and highly accurate adaptive composite Gauss-Legendre quadrature as proposed in [76, Ex. 5.1.9] and [78]. All computations were done in MATLAB and a direct solver was used whenever we aimed to study discretization errors.
Experiment 2.12. We study the discretization error for point evaluations of boundary element solutions to the sound-soft scattering problem (2.1). For simplicity reasons we limit ourselves to indirect methods and consider the regularised combined field integral equation (2.9) as our method of choice. As excitation we use the fundamental solution from (1.4)

$$g_D(x) := -G_\kappa(x - x_{\text{int}}), \quad x \in \Gamma,$$

where the interior evaluation point $x_{\text{int}}$ is set to $x_{\text{int}} := (0.25, 0)^T$. Thus, $G_\kappa(x - x_{\text{int}})$ provides an exact solution to problem (2.1). The discretization error for point evaluations of the numerical solution

$$U_h(x) := \Psi_{\text{SL}}(\vartheta_h)(x) + i\eta\Psi_{\text{DL}}(p_h)(x),$$

is measured on a series of shape-regular meshes of the unit square $\Omega$ for the frequencies $\kappa_3$ and $\kappa_4$, and on the unit circle $\Omega_0$ for the frequencies $\kappa_1$ and $\kappa_2$. The single-layer and double-layer potentials are numerically evaluated at the exterior point $x := (2, 0)^T$ using a high-order Gauss-Legendre quadrature. For the unit square $\Omega$ the observed convergence rates are recorded in figure 2.3 and for the unit circle $\Omega_0$ they can be found in figure 2.2.

Experiment 2.13. Using the same excitation as before, we measure the discretization error for the regularised Burton-Miller equation (2.17) on a series of shape-regular meshes of the unit circle $\Omega$ for the two frequencies $\kappa_1$ and $\kappa_2$ and the coupling parameter $\eta = 1$. The discretization error measured w.r.t the $L^2(\Gamma)$-norm is computed by means of a two-point Gauss-Legendre quadrature. The $H^{-1/2}(\Gamma)$-norm is evaluated by means of the discrete single-layer potential operator on the current mesh after the exact solution $\vartheta$ has been projected onto $\Theta_h$. On the unit circle both Dirichlet and Neumann trace of the scattered wave are smooth, which translates into optimal convergence rates for $\vartheta$, which are recorded in figure 2.4.
Figure 2.3: Potential evaluation for the regularised CFIE and the wave numbers $\kappa_3$ (−) and $\kappa_4$ (•−) on the unit square $\Omega$.

If we repeat the same experiment on a series of shape-regular meshes of the unit square $\Omega$ for the two frequencies $\kappa_3$ and $\kappa_4$, we arrive at the discretization errors recorded in figure 2.5. Now the observed convergence rates are very low because of the corner discontinuities of the Neumann trace of the exact solution.

Experiment 2.14. We examine the dependence of the spectral condition number of the system matrices on the wave number for

1. the single-layer ansatz (2.2),
2. the regularised combined field integral equation (2.9) and
3. the regularised Burton-Miller equation (2.17).

In each case the condition numbers have been evaluated in the neighbourhood of the first resonant frequency of $\Omega$ and $\Omega_0$ using two shape-regular meshes consisting of approximately 500 panels each. On both domains the extremal eigenvalues were computed by means of direct and inverse power iterations. The results are recorded in figure 2.6 and 2.7. In case of the regularised variational formulations the power iterations have been applied to the Schur complement system after elimination of the auxiliary variables $p$. Obviously, regularisation manages to suppress the pronounced peak in the condition number of the single-layer ansatz (2.2).
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Figure 2.4: $L^2(\Gamma)$ and energy errors for the regularised Burton-Miller equation and the wave numbers $\kappa_1 (-)$ and $\kappa_2 (-)$ on the unit circle $\Omega_\circ$.

Figure 2.5: $L^2(\Gamma)$ and energy errors for the regularised Burton-Miller equation and the wave numbers $\kappa_3 (-)$ and $\kappa_4 (-)$ on the unit square $\Omega_\square$. 
Figure 2.6: Estimated spectral condition number on the unit square $\Omega_\square$.

Figure 2.7: Estimated spectral condition number on the unit circle $\Omega_\circ$. 

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**CHAPTER 2. THE HELMHOLTZ SCATTERING PROBLEM**

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Chapter 3

The Helmholtz Transmission Problem

3.1 Problem Formulation

In this section we introduce the mathematical model underlying the acoustic transmission problem, which describes the scattering of incident waves $U^i$ from a penetrable obstacle or scatterer $\Omega$. It can be viewed as an interior source problem for the total field inside $\Omega$ combined with a scattering problem for the scattered field $U^s$ in the air region $\Omega^+$. Both problems are coupled by the so called transmission conditions, which ensure global continuity of the total field and fluxes across the interface boundary $\Gamma$.

A typical setting for the interior source problem are scatterers consisting of various types of inhomogeneous media, as well as compactly supported sources, whereas the exterior scattering problem assumes homogeneous material and total absence of any source terms.

Thus the mathematical model for the acoustic scattering problem boils down to the following transmission problem for the Helmholtz equation

$$\begin{align*}
  -\Delta U - \kappa^2 n(x) U &= f(x) \quad \text{in } \Omega, \\
  -\Delta U^s - \kappa^2 U^s &= 0 \quad \text{in } \Omega^+, \\
  \gamma_D^U U^s - \gamma_D^U U &= g_D \quad \text{on } \Gamma, \\
  \gamma_N^U U^s - \gamma_N^U U &= g_N \quad \text{on } \Gamma, \\
  \frac{\partial U^s}{\partial r} - i\kappa U^s &= o\left(r^{(1-d)/2}\right) \quad \text{uniformly for } r := |x| \to \infty,
\end{align*}$$

Figure 3.1: Helmholtz Transmission Problem.
with the refractive index \( n \in L^\infty(\Omega) \), the source term \( f \in L^2(\Omega) \) and the wave number \( \kappa > 0 \), cf. [76, Sect. 2.9]. In the case of excitation by an incident field \( U^i \) the generic jump data \( g_D \in H^{1/2}(\Gamma) \) and \( g_N \in H^{-1/2}(\Gamma) \) evaluate to the Dirichlet and Neumann data of \( U^i \) on the boundary \( \Gamma \):

\[
g_D := -\gamma_D^i U^i, \quad g_N := -\gamma_N^i U^i.
\]

It is well known that the transmission problem (3.1) has a unique solution \( U \in H_{loc}(\Delta, \mathbb{R}^3) \), cf. [76, Sect. 2.10].

**Remark 3.1.** Please note that inside \( \Omega \) the field \( U \) in (3.1) refers to the total field, whereas in \( \Omega^+ \) we write \( U^s \) for the scattered field. There the total field can be recovered through \( U = U^s + U^i \).

## 3.2 Overview on FEM-BEM Coupling Methods

The main purpose of this section is to give a brief overview on some of the methods used for the coupling of finite elements and boundary elements to the Helmholtz transmission problem.

### 3.2.1 Non-Symmetric Coupling

The non-symmetric coupling method of finite elements and boundary elements was first proposed by Silvester and Hsieh [79] and later on by Zienkiewicz et al. [90]. The first rigorous analysis of the variational formulation underlying the non-symmetric coupling approach was carried out by Johnson and Nédélec [56]. Although, the authors have limited themselves to the case of the Laplace equation, after some minor changes all results carry over to the case of the Helmholtz transmission problem. For numerical experiments and references into the engineering literature we refer to [90].

In what follows, we denote by \( (\cdot, \cdot)_\Omega \) the following \( L^2(\Omega) \)-inner product

\[
(U, V)_\Omega := \int_\Omega U \overline{V} \, dx, \quad U, V \in L^2(\Omega).
\]

We depart from an integration by parts formula, which confirms that any solution \( U \) of problem (3.1) fulfills

\[
q_\kappa(U, V) - (\gamma_D^i U, \gamma_D^i V)_\Gamma = (f, V)_\Omega \quad \forall V \in H^1(\Omega), \quad (3.2)
\]

where we have used the abbreviation

\[
q_\kappa(U, V) := \int_\Omega \text{grad} U \cdot \text{grad} \overline{V} - \kappa^2 n(x) U \overline{V} \, dx, \quad U, V \in H^1(\Omega).
\]

**Lemma 3.2.** The sesquilinear form \( q_\kappa \) is coercive in the sense that there exist constants \( C_1, C_2 > 0 \) such that

\[
\text{Re} \{ q_\kappa(U, U) \} \geq C_1 \| U \|^2_{H^1(\Omega)} - C_2 \| U \|^2_{L^2(\Omega)},
\]

holds true for all \( U \in H^1(\Omega) \).

**Proof.** The lemma is a straightforward consequence of the following estimate

\[
\text{Re} \{ q_\kappa(U, U) \} \geq \| U \|^2_{H^1(\Omega)} - (1 + \kappa^2 \| n \|^2_{\infty}) \| U \|^2_{L^2(\Omega)},
\]

which holds true for all \( U \in H^1(\Omega) \).
Due to the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ we immediately conclude that the sesquilinear form $q_\kappa$ also satisfies a Gårding inequality on $H^1(\Omega)$.

The variational problem corresponding to the non-symmetric coupling approach arises by employing the transmission conditions of (3.1) and using the non-symmetric Dirichlet-to-Neumann map (1.14) to express $\gamma_N U$ in (3.2). In order to avoid the operator products occurring in (1.14) we also introduce $\gamma_N U^s$ as the new variable, which amounts to

$$\vartheta := (V_\kappa^{-1} \circ (K_\kappa - \frac{1}{2} \text{id})) (\gamma_D U + g_D) \in H^{-1/2}(\Gamma).$$

Thus, we end up with:

Find $U \in H^1(\Omega)$ and $\vartheta \in H^{-1/2}(\Gamma)$ such that for all $V \in H^1(\Omega)$ and $\varphi \in H^{-1/2}(\Gamma)$ there holds

$$q_\kappa(U, V) - (\vartheta, \gamma_D V)_\Gamma = f(V),$$

$$\langle \varphi, (\text{id} - 2 K_\kappa)(\gamma_D U) \rangle_\Gamma + 2 \langle \varphi, V_\kappa(\vartheta) \rangle_\Gamma = g(\varphi),$$

where the right hand sides are given by

$$f(V) := \langle f, V \rangle_\Omega - \langle g_N, \gamma_D V \rangle_\Gamma,$$

$$g(\varphi) := \langle \varphi, (2 K_\kappa - \text{id})(g_D) \rangle_\Gamma.$$

On smooth surfaces or curves in three or two dimensions, respectively, it can be shown that the operator $K_\kappa : H^{1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$ is compact, cf. [56, Sect. 2]. Unfortunately, this result does not carry over to the case of non-smooth surfaces or curves. Nevertheless, combining this result together with lemma 3.2 and the results from section 1.3 yields a Gårding inequality, in the sense of item 1. of definition 1.10, for the sesquilinear form underlying the variational formulation (3.3) on the product space $H^1(\Omega) \times H^{-1/2}(\Gamma)$ on smooth surfaces.

### 3.2.2 Symmetric Coupling

The symmetric coupling approach for finite elements and boundary elements was first used by Costabel [29] and is based on a symmetric representation of the exterior Dirichlet-to-Neumann map. Although, the author has established all results for the general case of strongly elliptic systems, we limit ourselves to the case of the Helmholtz transmission problem, which fits perfectly into the framework of [29].

The variational problem associated with the symmetric coupling approach emerges by employing the transmission conditions of (3.1) and using the Dirichlet-to-Neumann map (1.15) to express $\gamma_N U$ in (3.2). Again we avoid all operator products occurring in (1.15) by introducing $\gamma_N U^s$ as the new variable

$$\vartheta := (V_\kappa^{-1} \circ (K_\kappa - \frac{1}{2} \text{id})) (\gamma_D U + g_D) \in H^{-1/2}(\Gamma).$$

Thus results in the following variational formulation:

Find $U \in H^1(\Omega)$, $\vartheta \in H^{-1/2}(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\varphi \in H^{-1/2}(\Gamma)$ there holds

$$q_\kappa(U, V) + (W_\kappa(\gamma_D U), \gamma_D V)_\Gamma - ((\frac{1}{2} \text{id} - K_\kappa')(\vartheta), \gamma_D V)_\Gamma = f(V),$$

$$\langle \varphi, (\frac{1}{2} \text{id} - K_\kappa)(\gamma_D U) \rangle_\Gamma + (\varphi, V_\kappa(\vartheta))_\Gamma = g(\varphi),$$

where

$$f(V) := \langle f, V \rangle_\Omega - \langle g_N, \gamma_D V \rangle_\Gamma - \langle W_\kappa(g_D), \gamma_D V \rangle_\Gamma,$$

$$g(\varphi) := \langle \varphi, (K_\kappa - \frac{1}{2} \text{id})(g_D) \rangle_\Gamma.$$
Using the results from section 1.3 it is not difficult to verify that the sesquilinear form associated with the variational problem (3.4) satisfies a Gårding inequality according to item 1. of definition 1.10 on the product space $H^1(\Omega) \times H^{-1/2}(\Gamma)$. The main benefit of the symmetric compared to the non-symmetric coupling approach is that in the first case, the sesquilinear form of the underlying variational formulation satisfies a Gårding inequality, not only on smooth but on Lipschitz domains as well. However, this is no safeguard against spurious modes:

Assume the resonant case, then we can find $W \in H^1(\Omega) \setminus \{0\}$ such that

$$\Delta W + \kappa^2 W = 0 \quad \text{in } \Omega \quad \text{and} \quad \gamma_D^- W = 0 \quad \text{on } \Gamma.$$ 

Due to the fact that $\gamma_D^- W = 0$ and by using theorem 1.8 we have that

$${\begin{bmatrix} 0 \\ -\gamma_N^- W \end{bmatrix}} = P_{\gamma}^{-1} {\begin{bmatrix} 0 \\ -\gamma_N^- W \end{bmatrix}} = {\begin{bmatrix} V_{\kappa}(\gamma_N^- W) \\ (\frac{1}{2}\text{id} + K'_{\kappa})(\gamma_N^- W) \end{bmatrix}},$$

which means that $(0, \gamma_N^- W)$ provides a solution to the variational formulation (3.4) in the case of $f = g = 0$.

Even in the resonant case, the right hand sides of problem (3.4) will be consistent and thus solutions $(U, \vartheta)$ to the variational problem still exist. Furthermore, the first component of the solution will be unique. Unfortunately, this is of little comfort as far as numerical methods are concerned: First, inevitable perturbations introduced by discretization will destroy the consistency of the right hand side. Second, whenever $\kappa$ is close to a resonant frequency, the resulting linear system will be extremely ill-conditioned; see the profound analysis of the impact of spurious resonances in the case of electromagnetic scattering given in [26].

**Remark 3.3.** Under the assumptions made in remark 1.9 we may use a symmetric Dirichlet-to-Neumann map derived from (1.18). This will lead to a coupled variational problem of the form (3.4) with much simpler right hand sides $f(V) = (f, V)_\Omega - (g_N, \gamma_D^- V)_\Gamma$ and $g(\varphi) = - (\varphi, g_D)_\Gamma$.

### 3.3 Stabilised Coupling

The contents of this section have been published in SIAM J. Numer. Anal. [53].

#### 3.3.1 Stabilised Variational Formulation

As pointed out in section 3.2.2, the existence of spurious resonances is directly linked to the fact that for certain $\kappa$ there are non-trivial interior Helmholtz solutions $U$ that satisfy $\gamma_D^- U = 0$. On the other hand, we know that there exist Robin-type (mixed) boundary conditions that ensure unique solvability of the corresponding boundary value problem for $-\Delta U - \kappa^2 U = 0$ in $\Omega$. Note that we can rely on two Robin-type boundary operators to state the transmission conditions of (3.1) as long as we can recover the conventional Dirichlet and Neumann trace from them. In fact, this idea can serve as the starting point for the derivation of all combined field integral equations. Here, it motivates the introduction of the following generic trace transformation operator

$$\mathcal{T} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma). \quad (3.5)$$

We demand that the interior homogeneous “Dirichlet problem” for $-\Delta U - \kappa^2 U = 0$ together with the modified traces has a unique solution for every $\kappa$. In light of theorem 1.8 this amounts to the following assumption:
Assumption 3.4. The trace transformation operator $T$ satisfies

$$\text{Range } (T \circ P_\pm) \cap \left\{ 0 \times H^{-1/2}(\Gamma) \right\} = \{0\}.$$  

Then one can use $T$, build associated Calderón projectors for the modified traces, derive symmetrically coupled variational problems and check their properties. Here, we would like to skip this tedious process of creative discovery and present the final finding on what is required for $T$:

Assumption 3.5. The blocks of the transformation operator $T$ from (3.5) are assumed to possess the following properties

1. $T : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ is bijective,
2. $A : H^{1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$ is bounded and bijective,
3. $B : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$ is compact,
4. $C : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$ is compact,
5. $D : H^{-1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$ is bounded and bijective.

The first requirement enables us to retrieve the conventional Dirichlet and Neumann trace from their transformed counterparts. This is essential, because it is these traces that will invariably occur in (3.2) so that we have to resort to them in one way or another when pursuing the coupling of (3.2) with boundary integral equations. Switching back and forth between conventional and transformed traces employs the following splitting of the trace transformation operator

$$T = \mathcal{R} + \mathcal{S}, \quad \mathcal{R} := \begin{bmatrix} A & \cdot \\ \cdot & D \end{bmatrix}, \quad \mathcal{S} := \begin{bmatrix} \cdot & \cdot \\ \cdot & C \end{bmatrix}. \quad (3.6)$$

Based on it we define the following generalised Calderón projectors

$$P_\pm := \mathcal{R}^{-1} \circ (T \circ P_\pm - \mathcal{S}). \quad (3.7)$$

Note that they are meant to act on conventional traces. Let us make the transformed exterior Calderón projector more explicit: an elementary computation yields

$$P_+ = \begin{bmatrix} \frac{1}{2} \text{Id} + \mathcal{K}_\kappa, & -\mathcal{V}_\kappa \\ -\mathcal{W}_\kappa, & \frac{1}{2} \text{Id} - \mathcal{K}'_\kappa \end{bmatrix}. \quad (3.8)$$

where the entries of the operator matrix are given by

$$\mathcal{V}_\kappa := \mathcal{V}_\kappa + A^{-1} \circ \mathcal{B} \circ \left( \frac{1}{2} \text{Id} + \mathcal{K}'_\kappa \right), \quad (3.9)$$
$$\mathcal{K}_\kappa := \mathcal{K}_\kappa - A^{-1} \circ \mathcal{B} \circ \mathcal{W}_\kappa,$$
$$\mathcal{K}'_\kappa := \mathcal{K}'_\kappa + D^{-1} \circ \mathcal{C} \circ \mathcal{V}_\kappa,$$
$$\mathcal{W}_\kappa := \mathcal{W}_\kappa + D^{-1} \circ \mathcal{C} \circ \left( \frac{1}{2} \text{Id} - \mathcal{K}_\kappa \right). \quad (3.10)$$

An analogue of theorem 1.8 still holds for the transformed Calderón projectors.

**Lemma 3.6.** If and only if $U$ is an exterior/interior radiating Helmholtz solution we have

$$P_\pm \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix} = \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix}.$$
Proof. As $T$ is one-to-one, we immediately conclude from theorem 1.8 that $U$ is an exterior/interior radiating Helmholtz solution if and only if

$$P_\pm \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix} = \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix}$$

$$\Downarrow$$

$$(T \circ P_\pm) \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix} = T \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix} = (R + S) \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix} = R^{-1} \circ (T \circ P_\pm - S) \begin{bmatrix} \gamma_D^\pm U \\ \gamma_N^\pm U \end{bmatrix}.$$  (3.11)

Now, the same formal manipulations as in section 1.4 yield the following operator expression for the Dirichlet-to-Neumann map

$$\text{DtN}_k^+ := -\mathcal{W}_\kappa + (\frac{1}{2} I_d - \mathbb{K}_\kappa) \circ \mathcal{V}_\kappa^{-1} \circ (\mathbb{K}_\kappa - \frac{1}{2} I_d),$$

which maps exterior Dirichlet traces of radiating Helmholtz solutions $U$ to exterior Neumann traces.

Remark 3.7. Again, if the incident wave $U^1$ can be extended to an interior Helmholtz solution, then we can apply the trace transformation operator to (1.18) and we end up with

$$T \begin{bmatrix} \gamma_D^+ U \\ \gamma_N^+ U \end{bmatrix} = (T \circ P_+) \begin{bmatrix} \gamma_D^+ U \\ \gamma_N^+ U \end{bmatrix} - T \begin{bmatrix} g_D \\ g_N \end{bmatrix}.$$  (3.13)

Using the operator splitting (3.6) and definition (3.7) of the generalized Calderón projector we can eliminate the trace transformation operator $T$ from the left hand side of (3.13) and we obtain

$$\begin{bmatrix} \gamma_D^+ U \\ \gamma_N^+ U \end{bmatrix} = P_+ \begin{bmatrix} \gamma_D^+ U \\ \gamma_N^+ U \end{bmatrix} - (R^{-1} \circ T) \begin{bmatrix} g_D \\ g_N \end{bmatrix}.$$  (3.14)

As in remark 1.9 this relationship can be used to construct new Dirichlet-to-Neumann maps for the total field.

We end this section with an easily verifiable criterion telling us when assumption 3.4 is satisfied:

Lemma 3.8. If the following equivalence holds

$$\text{Im} \left\{ (\vartheta, (A^{-1} \circ B)(\vartheta))_\Gamma \right\} = 0 \iff \vartheta = 0,$$

then

$$\text{Range} \left( T \circ P_- \right) \cap \left( \{0\} \times H^{-1/2}(\Gamma) \right) = \{0\}.$$

Proof. If $\xi \in H^{-1/2}(\Gamma)$ satisfies

$$\begin{bmatrix} 0 \\ \xi \end{bmatrix} \in \text{Range} \left( T \circ P_- \right),$$
then there exists \( \vartheta \in H^{1/2}(\Gamma) \) and \( \varphi \in H^{-1/2}(\Gamma) \) such that

\[
\begin{bmatrix}
0 \\
\xi
\end{bmatrix} = (T \circ P_\perp) \begin{bmatrix}
\vartheta \\
\varphi
\end{bmatrix}.
\]

Taking the transformed interior traces of the function

\[ U(x) := \Psi_{\text{SL}}^\kappa(\varphi)(x) - \Psi_{\text{DL}}^\kappa(\vartheta)(x), \quad x \in \Omega, \]

gives us the following set of equations

\[
T \begin{bmatrix}
\gamma_D^ U \\
\gamma_N^ U
\end{bmatrix} = (T \circ P_\perp) \begin{bmatrix}
\vartheta \\
\varphi
\end{bmatrix} = \begin{bmatrix}
0 \\
\xi
\end{bmatrix}. \tag{3.15}
\]

Thus \( U \) is a solution to the boundary value problem

\[
\Delta U + \kappa^2 U = 0 \quad \text{in} \ \Omega,
\]

\[
A(\gamma_D^ U) + B(\gamma_N^ U) = 0 \quad \text{on} \ \Gamma. \tag{3.16}
\]

Recalling equation (3.2) and using that \( A \) is bijective, we obtain

\[
0 = \int_\Omega |\nabla U|^2 - \kappa^2 |U|^2 \, dx - (\gamma_N^ U, \gamma_D^ U)_\Gamma,
\]

\[
= \int_\Omega |\nabla U|^2 - \kappa^2 |U|^2 \, dx + (\gamma_N^ U, (A^{-1} \circ B)(\gamma_N^ U))_\Gamma.
\]

Since the integral assumes only real values, taking the imaginary part, we get

\[
\text{Im} \{ (\gamma_N^ U, (A^{-1} \circ B)(\gamma_N^ U))_\Gamma \} = 0.
\]

Hence, the assumption of the lemma implies \( \gamma_N^ U = 0 \), and via (3.16) we conclude \( \gamma_D^ U = 0 \). Eventually, (3.15) shows that \( \xi = 0 \). \( \square \)

Parallel to the approach in section 3.2.2, we use equation (3.2) in combination with the transformed Dirichlet-to-Neumann map (3.12) and introduce the new variable

\[
\vartheta := (\nabla^\kappa^{-1} \circ (\nabla^\kappa - \frac{1}{2} \mathbb{I}))((\gamma_D^ U + g_D) \in H^{-1/2}(\Gamma)).
\]

If \( U \) solves the Helmholtz transmission problem (3.1), then \( \gamma_D^ U + g_D = \gamma_D^ U^g \), and we learn from lemma 3.6 and (3.8) that actually \( \vartheta = \gamma_N^ U^g \). As in the case of classical coupling, \( \vartheta \) will supply the exterior Neumann trace of the scattered field. Thus we arrive at the following regularised variational formulation:

Find \( U \in H^1(\Omega) \) and \( \vartheta \in H^{-1/2}(\Gamma) \) such that for all \( V \in H^1(\Omega) \) and \( \varphi \in H^{-1/2}(\Gamma) \) there holds

\[
q_{\kappa}(U, V) + (\mathcal{W}_{\kappa}(\gamma_D^ U), \gamma_D^ V)_\Gamma - \left( \left( \frac{1}{2} \mathbb{I} - \nabla^\kappa \right)(\vartheta), \gamma_D^ V \right)_\Gamma = f(V),
\]

\[
(\varphi, (\frac{1}{2} \mathbb{I} - \nabla^\kappa)(\gamma_D^ U))_\Gamma + (\varphi, \nabla^\kappa(\vartheta))_\Gamma = g(\varphi), \tag{3.17}
\]

where the right hand sides are given by

\[
f(V) := (f, V)_\Omega - (g_N, \gamma_D^ V)_\Gamma - (\mathcal{W}_{\kappa}(g_D), \gamma_D^ V)_\Gamma,
\]

\[
g(V) := (\varphi, (\nabla^\kappa - \frac{1}{2} \mathbb{I})(g_D))_\Gamma.
\]

We first investigate the \( H^1(\Omega) \times H^{-1/2}(\Gamma) \)-coercivity of the sesquilinear form underlying the variational formulation (3.17). From assumption 3.5 it is immediate that the operators \( A^{-1} \circ B, D^{-1} \circ C \) are compact. This plays a key role in the proofs of the following results.
Lemma 3.9. There exists a constant $C > 0$ and compact operators

$$\mathcal{T}_V : H^{-1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma), \quad \mathcal{T}_W : H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$$

such that

$$\text{Re} \{ (\varphi, (\nabla \kappa + \mathcal{T}_V)(\varphi))_\Gamma \} \geq C \| \varphi \|^2_{H^{1/2}(\Gamma)},$$

$$\text{Re} \{ (\varphi, (\mathcal{W}_\kappa + \mathcal{T}_W)(\varphi))_\Gamma \} \geq C \| \varphi \|^2_{H^{1/2}(\Gamma)},$$

for all $\varphi \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$.

Proof. Using (3.9) and (3.10) a straightforward application of lemma 1.5 yields

$$\text{Re} \{ (\varphi, (\mathcal{V}_\kappa)(\varphi))_\Gamma - (\varphi, (A^{-1} \circ B \circ (\frac{1}{2}I + \mathcal{K}_\kappa'))(\varphi))_\Gamma + (\varphi, (\mathcal{T}_V)(\varphi))_\Gamma \}$$

$$= \text{Re} \{ (\varphi, (\mathcal{V}_\kappa + \mathcal{T}_V)(\varphi))_\Gamma \} \geq C \| \varphi \|^2_{H^{-1/2}(\Gamma)},$$

$$\text{Re} \{ ((\mathcal{W}_\kappa)(\varphi), \varphi)_\Gamma + ((C^{-1} \circ C \circ (\mathcal{K}_\kappa - \frac{1}{2}I))(\varphi), \varphi)_\Gamma + (\mathcal{T}_W)(\varphi), \varphi)_\Gamma \}$$

$$= \text{Re} \{ ((\mathcal{W}_\kappa + \mathcal{T}_W)(\varphi), \varphi)_\Gamma \} \geq C \| \varphi \|^2_{H^{1/2}(\Gamma)},$$

for all $\varphi \in H^{-1/2}(\Gamma)$ and $\varphi \in H^{1/2}(\Gamma)$.

Lemma 3.10. There exist compact operators $\mathcal{T}_\kappa : H^{1/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$, such that

$$((\mathcal{K}_\kappa + \mathcal{T}_\kappa)(\varphi), \varphi)_\Gamma = (\mathcal{K}_\kappa^*(\varphi), \varphi)_\Gamma$$

for all $\varphi \in H^{-1/2}(\Gamma)$, $\varphi \in H^{1/2}(\Gamma)$, where $\mathcal{K}_\kappa^*$ denotes the $L^2(\Omega)$-adjoint of $\mathcal{K}_\kappa$.

Proof. We begin with an application of lemma 1.6 and obtain

$$((\mathcal{K}_\kappa + \mathcal{T}_\kappa)(\varphi), \varphi)_\Gamma = (\mathcal{K}_\kappa^*(\varphi), \varphi)_\Gamma$$

for all $\varphi \in H^{-1/2}(\Gamma)$, $\varphi \in H^{1/2}(\Gamma)$. Using this result we finally arrive at the following equation

$$((\varphi, (\mathcal{K}_\kappa)(\varphi))_\Gamma - (\mathcal{K}_\kappa^*(\varphi), \varphi)_\Gamma = ((\mathcal{K}_\kappa - D^{-1} \circ C \circ \mathcal{V}_\kappa)(\varphi), \varphi)_\Gamma - ((A^{-1} \circ B \circ \mathcal{W}_\kappa)^*(\varphi), \varphi)_\Gamma,$$

which holds for arbitrary $\varphi \in H^{-1/2}(\Gamma)$, $\varphi \in H^{1/2}(\Gamma)$.

Summing up, from these lemmata and lemma 3.2 we conclude that the sesquilinear form of the regularised variational problem (3.17) satisfies a Gårding inequality, in the sense of item 1. of definition 1.10, on the product space $H^1(\Omega) \times H^{-1/2}(\Gamma)$. It remains to establish uniqueness of solutions, which amounts to confirming that the variational formulation (3.17) is really immune to spurious resonances.

Theorem 3.11. Solutions to the regularised variational problem (3.17) are unique.

Proof. In order to establish uniqueness of solutions for the variational formulation (3.17) we consider the case $f = g = 0$: seek $U \in H^1(\Omega)$, $\varphi \in H^{-1/2}(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\varphi \in H^{-1/2}(\Gamma)$ there holds

$$q_\kappa(U, V) + \langle \mathcal{W}_\kappa(\gamma_D U), \gamma_D V \rangle_\Gamma - ((\frac{1}{2}I - \mathcal{K}_\kappa)(\varphi), \gamma_D V)_\Gamma = 0, \quad (3.18)$$

$$\langle \varphi, (\frac{1}{2}I - \mathcal{K}_\kappa)(\gamma_D U) \rangle_\Gamma + \langle \varphi, \mathcal{V}_\kappa(\varphi) \rangle_\Gamma = 0. \quad (3.19)$$
Using integration by parts, we obtain
\[ \Delta U + \kappa^2 n(x) U = 0 \quad \text{in } \Omega. \]

As a consequence, \( q_k(U, V) = (\gamma_D^+ U, \gamma_D^+ V)_I \). Plugging this identity into (3.18) and using the definition of \( \mathcal{P}_+ \) from (3.7), the identity
\[
\begin{bmatrix}
\gamma_D^+ U \\
\gamma_N^+ U
\end{bmatrix}
= \mathcal{P}_+ \begin{bmatrix}
\gamma_D^+ U \\
\vartheta
\end{bmatrix}
\]  
(3.20)

is immediate. By the definition of \( \mathcal{P}_+ \) and (1.13)
\[
\mathcal{P}_+ = \mathcal{R}^{-1} \circ (\mathcal{T} \circ \mathcal{P}_+ - \mathcal{S}) \\
= \mathcal{R}^{-1} \circ (\mathcal{T} \circ (\text{id} - \mathcal{P}_-) - \mathcal{S}) \\
= \mathcal{R}^{-1} \circ (\mathcal{R} + \mathcal{S} - \mathcal{T} \circ \mathcal{P}_- - \mathcal{S}) \\
= \text{id} - \mathcal{R}^{-1} \circ \mathcal{T} \circ \mathcal{P}_-,
\]
and we infer
\[
\mathcal{T} \circ \mathcal{P}_- = \mathcal{R} \circ (\text{id} - \mathcal{P}_+).
\]

Together with (3.20) this identity confirms
\[
(\mathcal{T} \circ \mathcal{P}_-) \begin{bmatrix}
\gamma_D^+ U \\
\vartheta
\end{bmatrix}
= \mathcal{R} \begin{bmatrix}
0 \\
\vartheta - \gamma_N^+ U
\end{bmatrix}
\in \{0\} \times H^{-1/2}(I),
\]
and, by assumption 3.4,
\[
\mathcal{R} \begin{bmatrix}
0 \\
\vartheta - \gamma_N^+ U
\end{bmatrix}
= 0 \Rightarrow \mathcal{D}(\vartheta - \gamma_N^+ U) = 0.
\]

Next, from assumption 3.5, item 5., we conclude that
\[
\gamma_N^+ U = \vartheta.
\]

From this and (3.20) we directly obtain as in (3.11) in the proof of lemma 3.6
\[
\begin{bmatrix}
\gamma_D^+ U \\
\gamma_N^+ U
\end{bmatrix}
= \mathcal{P}_+ \begin{bmatrix}
\gamma_D^+ U \\
\vartheta
\end{bmatrix}.
\]

Hence, by virtue of theorem 1.8, setting
\[
W(x) := \begin{cases}
U(x), & x \in \Omega \\
\Psi_{DL}^k(\gamma_D^+ U)(x) - \Psi_{SL}^k(\vartheta)(x), & x \in \Omega^+
\end{cases}
\]
provides us with a solution to the Helmholtz transmission problem with zero right hand side, and uniqueness of solutions to the Helmholtz transmission problem ensures \( U = 0 \), and, by (3.21), \( \vartheta = 0 \). We have demonstrated that (3.18) and (3.19) only possess the trivial solution and this finishes the proof. \( \square \)

Eventually, the existence of solutions to the variational problem (3.17) follows from theorem 3.11 and a Fredholm argument, see theorem 1.11. Finally, the arguments in the proof of theorem 3.11 have also confirmed that we really get information about the solution to the Helmholtz transmission problem from the variational problem (3.17):
Corollary 3.12. If \((W, \vartheta) \in H^1(\Omega) \times H^{-1/2}(\Gamma)\) solves the variational problem (3.17), then \(W = U\) and \(\vartheta = \gamma_N U^\diamond\) with \((U, U^\diamond)\) solving (3.1).

In this section we present a rather simple specimen of a trace transformation operator \(T\), which satisfies all the assumptions 3.4 and 3.5. Its main ingredient are the regularising operators

\[
M : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma),
\]

already introduced in section 2.2.1. For arbitrary \(\eta \in \mathbb{R} \setminus \{0\}\) we chose the following trace transformation operators

\[
T_1 := \begin{bmatrix} \text{Id} & i\eta M \\ i\eta & \text{Id} \end{bmatrix}, \quad T_2 := \begin{bmatrix} \text{Id} & i\eta M \\ -i\eta & \text{Id} \end{bmatrix}. \tag{3.22}
\]

To begin with, we have to verify the assumptions 3.5 and 3.4. We note that assumption 3.4 can instantly be concluded from assumption 2.1, item 2., and lemma 3.8. Items 2. through 5. of assumption 3.5 are evident appealing to assumption 2.1, item 1. It is also obvious that \(T_2\) is bijective with

\[
T_2^{-1} = \begin{bmatrix} \text{Id} & -i\eta M \\ -i\eta & \text{Id} \end{bmatrix}.
\]

It remains to establish that \(T_1\) is bijective, too. Key will be the following lemma.

Lemma 3.13. For \(\zeta \in \mathbb{R}_+\) or \(\zeta \in i\mathbb{R}\) the following operators are bijective

\[
\text{Id} + \zeta M : H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma), \quad \text{Id} + \zeta M : H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma).
\]

Proof. It is sufficient to verify that the operators have trivial kernel. In the first case we find that \((\text{Id} + \zeta M)(\vartheta) = 0\) implies

\[
(\vartheta, \varphi) + \zeta(M(\vartheta), \varphi) = 0,
\]

which holds true for all \(\varphi \in H^{1/2}(\Gamma)\). We choose \(\varphi := M(\vartheta)\) and we obtain

\[
(\vartheta, M(\vartheta)) + \zeta\|M(\vartheta)\|^2 = 0.
\]

For either \(\zeta > 0\) or \(\zeta \in i\mathbb{R}\) assumption 2.1, item 2., implies

\[
(\vartheta, M(\vartheta)) = 0 \quad \Leftrightarrow \quad \vartheta = 0.
\]

Thanks to assumption 2.1, item 1., we have a Fredholm alternative argument [64, Thm. 2.27] at our disposal and conclude that the operator \(\text{Id} + \zeta M : H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)\) is surjective from the fact that it is injective.

In the \(H^{1/2}(\Gamma)\)-setting \((\text{Id} + \zeta M)(\varphi) = 0\) is equivalent to

\[
(\vartheta, \varphi) + \zeta(\vartheta, M(\varphi)) = 0 \quad \forall \vartheta \in H^{-1/2}(\Gamma).
\]

The same reasoning as above also settles this case. \(\Box\)

The lemma tells us that the formal inverse

\[
T_1^{-1} = (\text{Id} + \eta^2 M)^{-1} \circ \begin{bmatrix} \text{Id} & -i\eta M \\ -i\eta & \text{Id} \end{bmatrix}
\]

is well defined, which implies assumption 3.5, item 1., for \(T_1\).
A particularly convenient regularising operator can be obtained from the variational definition 2.7 in section 2.2.1: there, $M : H^{-1}(\Omega) \rightarrow H^1(\Gamma)$ is implicitly defined by
\[
(\text{grad}_\Gamma M(p), \text{grad}_\Gamma q) + (M(p), q) = (p, q) \quad \forall q \in H^1(\Gamma),
\]
and all $p \in H^1(\Gamma)$.

Using the two trace transformation operators we obtain two variational formulations which are free from spurious resonances. However, from the point of view of boundary element discretization, they are not yet useful, because they still contain products of (non-local) operators that elude a straightforward Galerkin discretization. To get rid of the operator products, we rely on the usual trick and introduce extra unknowns. We discuss the resulting variational problems for the trace transformation operators $T_1$ and $T_2$ from (3.22) and $M$ given by the variational definition 2.7:

**Case $T = T_1$:** Find $U \in H^1(\Omega)$, $\varphi \in H^{1/2}(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\psi \in H^{1/2}(\Gamma)$ there holds
\[
q(K(U, V) + (\{W_\kappa + i\eta (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D U), \gamma_D V\})_\Gamma
- ((\frac{1}{2} \text{Id} - K_\kappa' - i\eta V_\kappa)(\varphi), \gamma_D V)_\Gamma = f_1(V), \quad (3.23)
\]
\[
(\varphi, (\frac{1}{2} \text{Id} - K_\kappa + i\eta M \circ W_\kappa)(\gamma_D U))_\Gamma + (\varphi, (V_\kappa + i\eta M \circ (\frac{1}{2} \text{Id} + K_\kappa'))(\varphi))_\Gamma = g_1(\varphi),
\]
where the right hand sides are given by
\[
f_1(V) := (f, V)_\Omega - (g_N, \gamma_D V)_\Gamma - ((W_\kappa + i\eta(\frac{1}{2} \text{Id} - K_\kappa))(\gamma_D V), \gamma_D V)_\Gamma,
\]
\[
g_1(\varphi) := (\varphi, (K_\kappa - \frac{1}{2} \text{Id} - i\eta M \circ W_\kappa)(\gamma_D V))_\Gamma.
\]

**Case $T = T_2$:** Find $U \in H^1(\Omega)$, $\varphi \in H^{1/2}(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\psi \in H^{1/2}(\Gamma)$ there holds
\[
q(K(U, V) + (\{W_\kappa(\gamma_D U), \gamma_D V\})_\Gamma - ((\frac{1}{2} \text{Id} - K_\kappa')(\varphi), \gamma_D V)_\Gamma = f_2(V), \quad (3.24)
\]
\[
(\varphi, (\frac{1}{2} \text{Id} - K_\kappa + i\eta M \circ W_\kappa)(\gamma_D U))_\Gamma + (\varphi, (V_\kappa + i\eta M \circ (\frac{1}{2} \text{Id} + K_\kappa'))(\varphi))_\Gamma = g_2(\varphi),
\]
where the right hand sides are given by
\[
f_2(V) := (f, V)_\Omega - (g_N, \gamma_D V)_\Gamma - ((W_\kappa(g_D), \gamma_D V)_\Gamma,
\]
\[
g_2(\varphi) := (\varphi, (K_\kappa - \frac{1}{2} \text{Id} - i\eta M \circ W_\kappa)(\gamma_D V))_\Gamma.
\]
Both regularised variational formulations contain the same operator products, namely
\[
\forall \kappa \in V_\kappa = V_\kappa + i\eta M \circ (\frac{1}{2} \text{Id} + K_\kappa'),
\]
\[
\frac{1}{2} \text{Id} - K_\kappa = \frac{1}{2} \text{Id} - K_\kappa + i\eta M \circ W_\kappa.
\]
This suggests that we introduce the new variable
\[
p := (M \circ (\frac{1}{2} + K_\kappa'))(\varphi) + (M \circ W_\kappa)(\gamma_D U + g_D) \in H^1(\Gamma), \quad (3.25)
\]
which converts problem (3.17) into the following two variational problems. The first arises from using $T_1$:

Find $U \in H^1(\Omega)$, $\varphi \in H^{1/2}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega)$, $\varphi \in H^{1/2}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds
\[
q(K(U, V) + i\eta((\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D U), \gamma_D V)_\Gamma + (W_\kappa(\gamma_D U), \gamma_D V)_\Gamma
- ((\frac{1}{2} \text{Id} - K_\kappa')(\varphi), \gamma_D V)_\Gamma + i\eta(V_\kappa(\varphi), \gamma_D V)_\Gamma = f_1(V), \quad (3.26)
\]
\[
(\varphi, (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D U))_\Gamma + (\varphi, (V_\kappa(\varphi))_\Gamma - i\eta(\varphi, p)_\Gamma = g_1(\varphi),
\]
\[
- (W_\kappa(\gamma_D U), q)_\Gamma - ((K_\kappa + \frac{1}{2} \text{Id})(\varphi), q)_\Gamma + b(p, q) = h_1(q),
\]
\[
- (W_\kappa(\gamma_D U), q)_\Gamma - ((K_\kappa + \frac{1}{2} \text{Id})(\varphi), q)_\Gamma + b(p, q) = h_1(q),
\]
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with right hand sides

\[ f_1(V) := (f, V)_\Omega - (g_N, \gamma_D V)_\Gamma - i\eta((\frac{1}{2}\mathbb{I} - K_\kappa)(g_D), \gamma_D V)_\Gamma - (W_\kappa(g_D), \gamma_D V)_\Gamma, \]

\[ g_1(\varphi) := (\varphi, (K_\kappa - \frac{1}{2}\mathbb{I})(g_D))_\Gamma, \]

\[ h_1(q) := (W_\kappa(g_D), q)_\Gamma. \]

The second arises from using \( T_2 \):

Find \( U, \vartheta \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma) \) and \( p \in H^1(\Gamma) \) such that for all \( V \in H^1(\Omega), \varphi \in H^{-1/2}(\Gamma) \) and \( q \in H^1(\Gamma) \) there holds

\[ q_\kappa(U, V) + (W_\kappa(\gamma_D U), \gamma_D V)_\Gamma - ((\frac{1}{2}\mathbb{I} - K_\kappa)(\vartheta), \gamma_D V)_\Gamma = f_2(V), \]

\[ (\varphi, (\frac{1}{2}\mathbb{I} - K_\kappa)(\gamma_D U))_\Gamma + (\varphi, V_\kappa(\vartheta))_\Gamma - i\eta(\varphi, p)_\Gamma = g_2(\varphi), \]

\[ - (W_\kappa(\gamma_D U), q)_\Gamma - ((K_\kappa + \frac{1}{2}\mathbb{I})(\vartheta), q)_\Gamma + b(p, q) = h_2(q), \]

with right hand sides

\[ f_2(V) := (f, V)_\Omega - (g_N, \gamma_D V)_\Gamma - (W_\kappa(g_D), \gamma_D V)_\Gamma, \]

\[ g_2(\varphi) := (\varphi, (K_\kappa - \frac{1}{2}\mathbb{I})(g_D))_\Gamma, \]

\[ h_2(q) := (W_\kappa(g_D), q)_\Gamma. \]

In order to settle the issue of existence and uniqueness of solutions of the variational problems (3.26) and (3.27) we first observe that by the very definition of \( M \) in (2.7) and (3.25) the first two components of any solution \((U, \vartheta, p)\) of the problems (3.26) and (3.27) will also solve the problems (3.23) and (3.24), respectively. Since these special cases are of the variational problem (3.17) and both \( T_1 \) and \( T_2 \) are valid trace transformation operators, theorem 3.11 yields uniqueness.

It follows directly from the embedding \( H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \) and \( H^{-1/2}(\Gamma) \hookrightarrow H^{-1}(\Gamma) \) that all new off diagonal terms are compact sesquilinear forms. Since \( b \) is \( H^1(\Gamma) \)-elliptic, we obtain that the sesquilinear forms for both variational formulations satisfy a Gårding inequality, according to item 1. of definition 1.10 on the product space \( H^1(\Omega) \times H^{-1/2}(\Gamma) \times H^1(\Gamma) \).

Again, a Fredholm argument ensures the existence of solutions from the uniqueness result. The statement of corollary 3.12 directly carries over to the \((U, \vartheta)\)-components of the variational problems (3.23) and (3.24). Thus we have obtained two well-posed variational formulations which yield weak solutions to the Helmholtz transmission problem (3.1) and which are also amenable to standard Ritz-Galerkin discretizations.

We finish this section by an important observation: (3.25) can be recast into

\[ p = (M \circ (\frac{1}{2} + K_\kappa))(\vartheta) + (M \circ W_\kappa)(\gamma_D U + g_D). \]

At second glance, we realise that \( p = 0 \), if \((U, \vartheta)\) solve the problems (3.23) and (3.24), respectively. This directly follows from corollary 3.12, theorem 1.8 and the definition of the exterior Calderón projector \( P_+ \). In short, \( p \) is a “dummy variable”.

**Remark 3.14.** Under the assumptions made in remark 3.3 we can derive a Dirichlet-to-Neumann map from (3.14), to obtain coupled variational problems of the form (3.26) with much simpler right hand sides given by

\[ f_1(V) = (f, V)_\Omega - i\eta(g_D, \gamma_D V)_\Gamma - (g_N, \gamma_D V)_\Gamma, \]

\[ g_1(\varphi) = (\varphi, g_D)_\Gamma, \]

\[ h_1(q) = -(g_N, q)_\Gamma. \]
The same approach can be used for coupled variational problems of the form (3.27), where the right hand sides take on the following simplified forms

\[
\begin{align*}
    f_2(V) &= (f, V)_\Omega - (g_N, \gamma_D V)_\Gamma, \\
    g_2(\varphi) &= (\varphi, g_D)_\Gamma, \\
    h_2(q) &= -(g_N, q)_\Gamma.
\end{align*}
\]

(3.28)

In both cases the solution \( U \) in \( \Omega \) will remain the same.

### 3.3.2 Galerkin Discretization

With operator products removed, the Galerkin discretization of the variational problems (3.26) and (3.27) is easily achieved by restricting them to finite element subspaces \( V_h \) of \( H^1(\Omega) \) and boundary element subspaces \( \Theta_h \) and \( Q_h \) of \( H^{-1/2}(\Gamma) \) and \( H^1(\Gamma) \), respectively. A powerful theorem about the Galerkin approximation of coercive variational problems, see [77, 89], will then yield the asymptotic quasi-optimality of the Galerkin solutions: Assuming a minimal resolution of \( V_h, \Theta_h, \) and \( Q_h \), existence and uniqueness of discrete solutions \( (U_h, \vartheta_h, p_h) \in V_h \times \Theta_h \times Q_h \) to the variational problems (3.26) and (3.27) is guaranteed and we have the a-priori error estimate

\[
\|U - U_h\|_{H^1(\Omega)} + \|\vartheta - \vartheta_h\|_{H^{-1/2}(\Gamma)} + \|p - p_h\|_{H^1(\Gamma)} \leq C \left( \inf_{V_h \in V_h} \|U - V_h\|_{H^1(\Omega)} + \inf_{\vartheta_h \in \Theta_h} \|\vartheta - \vartheta_h\|_{H^{-1/2}(\Gamma)} \right),
\]

where the constant \( C > 0 \) does not depend on the discrete trial spaces.

The standard choices for \( V_h, \Theta_h, \) and \( Q_h \) are based on a tetrahedral or quadrilateral meshes \( \mathcal{M} \) of \( \Omega \), which induce a mesh \( \mathcal{M}_\Gamma \) of \( \Gamma \) by plain restriction to \( \Gamma \). Then we may pick

\[
\begin{align*}
    V_h &:= \{ V \in C^0(\Omega) : V|_K \in \mathcal{P}_k(K) \ \forall K \in \mathcal{M} \}, \\
    \Theta_h &:= \{ \varphi \in L^2(\Gamma) : \varphi|_K \in \mathcal{P}_{k-1}(K) \ \forall K \in \mathcal{M}_\Gamma \}, \\
    Q_h &:= \{ q \in C^0(\Gamma) : q|_K \in \mathcal{P}_k(K) \ \forall K \in \mathcal{M}_\Gamma \}.
\end{align*}
\]

(3.29)

Here, \( \mathcal{P}_k(K) \) stands for the space of polynomials of degree \( \leq k \) on the cell \( K \). This refers to the total degree in the case of tetrahedra and the degree in each variable in the case of hexahedra.

Then, the usual best approximation estimates [76] for the \( h \)-version of finite elements and boundary elements give us

\[
\begin{align*}
    \inf_{V_h \in V_h} \|U - V_h\|_{H^1(\Omega)} &\leq Ch_{\min\{s-1,k\}} \|U\|_{H^s(\Omega)}, \\
    \inf_{\vartheta_h \in \Theta_h} \|\vartheta - \vartheta_h\|_{H^{-1/2}(\Gamma)} &\leq Ch_{\min\{s+1/2,k\}} \|\vartheta\|_{H^s(\Gamma)},
\end{align*}
\]

with constants depending on the shape regularity of \( \mathcal{M} \) and \( h > 0 \) denoting the mesh width of \( \mathcal{M} \).

### 3.3.3 Numerical Examples

Limited computational resources allow the numerical exploration of asymptotic convergence rates only in two dimensions. Fortunately, the theoretical developments in this chapter hold in two as well as in three dimensions. For the numerical experiments we considered

- the unit circle \( \Omega_0 := \{ x \in \mathbb{R}^2 : |x| < 1 \} \) as a specimen of a domain with smooth boundary. The two interior resonant frequencies \( \kappa_1 = 5.5201 \) and \( \kappa_2 = 11.7915 \) were used, which correspond to the second and fourth zero of the Bessel function \( J_0(x) \).
the unit square $\Omega_{\square} := \{x \in \mathbb{R}^2 : -1/2 < x_1, x_2 < 1/2\}$ as representative of polygonal domains. The associated two lowest resonant frequencies are $\kappa_3 = \sqrt{2\pi}$ and $\kappa_4 = \sqrt{13\pi}$.

On each domain finite element meshes $\mathcal{M}_l$, $l \in \mathbb{N}$, consisting of quadrilaterals with straight edges were used. In the case of $\Omega_0$ the triangulation $\mathcal{M}_l$ is created by inscribing $\Omega_0$ a regular $2^{l+3}$-gon and a centred unit square. The portions of the line segments from the center to the corners of the polygon are split into $2^l$ equal parts, whose endpoints are connected to form a quadrilateral mesh outside the unit square. This is extended by an orthogonal tensor product mesh inside the unit square. The family of meshes arising from this construction will be quasi-uniform and shape-regular with mesh width $h$ of $\mathcal{M}_l$ being proportional to $2^{-(l+1)}$.

![Figure 3.2: Quadrilateral mesh of the unit circle.](image)

On $\Omega_{\square}$ the mesh $\mathcal{M}_l$ is a plain uniform orthogonal tensor product grid with mesh width $h = 2^{-(l+1)}$.

We used mapped bilinear Lagrangian finite elements to build $\mathcal{V}_h$, piecewise constants on $\mathcal{M}_l$ for $\Theta_h$, and linear surface elements for $\mathcal{Q}_h$, that is, the case $k = 1$ of (3.29). The finite element stiffness matrix was assembled using a four-point Gaussian quadrature rule on the reference element. The dense matrices of the discrete boundary integral operators were computed using Duffy’s trick and highly accurate adaptive composite Gauss-Legendre quadrature as proposed in [76, Ex. 5.1.9] and [78]. All computations were done in MATLAB and a direct solver was used whenever we aimed to study discretization errors. Extensive parameter studies were performed in parallel, using the MATLAB message passing interface standard MatlabMPI [57].

In all the experiments we used $n(x) = 1$ in $\Omega$ and excitation by incident plane waves. These will also provide the exact solutions. Please note that in this setting $\vartheta = \gamma_n^U U^1$, because there is no scattered field. As far as the stable regularised coupled schemes are concerned we consistently use the second regularised variational formulation (3.27) together with the simple right hand sides (3.28).

When analytic solutions are known, we measure the discretization error in the interior total field in either the $H^1(\Omega)$ or the $L^2(\Omega)$-norm and the error in $\vartheta$ in either the $H^{-1/2}(\Gamma)$ or the $L^2(\Gamma)$-norm. Integer Sobolev norms are calculated by means of four-point Gaussian quadrature. The $H^{-1/2}(\Gamma)$-norm is evaluated by means of the discrete single-layer potential operator on the current mesh after the exact solution for $\vartheta$ has been projected onto $\Theta_h$. 
Experiment 3.15. A plane incident wave \( U^I(x) = \exp(i\kappa d \cdot x) \), \(|d| = 1\), is used, where the incident angle between the propagation direction \( d \) and the \( x \)-axis is \( 5\pi/4 \). We measure the discretization errors in different norms on the domain \( \Omega_\Box \) for the two frequencies \( \kappa_3 \) and \( \kappa_4 \) on the series of shape-regular meshes and for a regularisation parameter \( \eta = 1 \), see figures 3.3, 3.4 for results. Table 3.1 lists the observed convergence rates, which are very low because of the corner discontinuity of \( \vartheta \).

![Figure 3.3: Energy errors for \( \kappa_3 (-) \) and \( \kappa_4 (-.) \) on the unit square \( \Omega_\Box \).](image)

Experiment 3.16. Using the same excitation as before, we measure the discretization errors on the series of shape-regular meshes of the unit circle \( \Omega_0 \) for the two frequencies \( \kappa_1 \) and \( \kappa_2 \), see figures 3.5 and 3.6. Now both \( \vartheta \) and \( U \) are smooth, which translates into optimal convergence rates, see table 3.1.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>( |U - U_h|_{H^1(\Omega)} )</th>
<th>( |\vartheta - \vartheta_h|_{H^{-1/2}(\Gamma)} )</th>
<th>( |U - U_h|_{L^2(\Omega)} )</th>
<th>( |\vartheta - \vartheta_h|_{L^2(\Gamma)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.15</td>
<td>( O(h) )</td>
<td>( O(h) )</td>
<td>( O(h^2) )</td>
<td>( O(h^2) )</td>
</tr>
<tr>
<td>3.16</td>
<td>( O(h) )</td>
<td>( O(h^2) )</td>
<td>( O(h^2) )</td>
<td>( O(h) )</td>
</tr>
</tbody>
</table>

Table 3.1: Observed convergence rates for stabilised coupling.

Experiment 3.17. We examine the dependence of the discretization error, measured in the \( H^1(\Omega) \) and \( H^{-1/2}(\Gamma) \)-norms, respectively, on the wave number for a mesh of the domain \( \Omega_\Box \) with 14161 elements. The results for conventional symmetric FEM-BEM coupling (3.4) are recorded in figure 3.8. In figure 3.7 the discretization errors are plotted for the second version of regularised FEM-BEM coupling (3.27). We note that the discretization errors for both methods are of exactly the same size. Moreover, they grow as \( \kappa \) increases. This is hardly surprising, because this is already observed for low order finite element discretizations of the Helmholtz equation [6].
Figure 3.4: $L^2(\Gamma)$-errors for $\kappa_3$ (–) and $\kappa_4$ (–) on the unit square $\Omega_\square$.

Figure 3.5: Energy errors for $\kappa_1$ (–) and $\kappa_2$ (–) on the unit circle $\Omega_\circ$. 
The pronounced spikes in the discretization error graph for the $\theta$-component in figure 3.8 are due to the resonant frequencies which affect the conventional symmetric FEM-BEM coupling. The dotted lines indicate the location of interior spurious modes. On the other hand, the spurious modes are completely suppressed when using the second version of regularised FEM-BEM coupling (3.27) as we can see in figure 3.7.

**Experiment 3.18.** In this experiment we study the convergence of the GMRES [75, Sect. 6.5] iterative solver applied to the linear systems arising from a Galerkin discretization of symmetric FEM-BEM coupling (3.4) and the second version of the regularised coupling approach (3.27). Both matrices were computed for the wave number $\kappa_3$ with the coupling parameter $\eta$ set to 1 on a mesh of the unit square $\Omega_o$ consisting of 9801 elements. In the case of regularised coupling the iteration was applied to the entire system matrix and the Schur complement system arising after elimination of the unknowns corresponding to the dummy variable $p$. For the GMRES method the relative tolerance was set to $10^{-6}$ and a maximum number 2000 outer and 100 inner iterations was used.

The relative residuals after each inner iteration are recorded in figure 3.9. Obviously, the GMRES method applied to the entire system matrix of the regularised coupling approach fails to converge to the desired tolerance within the specified number of iterations. On the other hand, the Schur complement system retains the convergence behaviour of the symmetric coupling approach.

**Experiment 3.19.** We recorded the dependence of the spectral condition number of the entire system matrix on the wave number for

1. the symmetric FEM-BEM coupling (3.4) and
2. the second version of regularised FEM-BEM coupling (3.27)
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Figure 3.7: Energy errors for stabilised coupling on the unit square $\Omega$.

Figure 3.8: Energy errors for symmetric coupling on the unit square $\Omega$. 
in the neighbourhood of the resonant frequency $\kappa_3$ for a mesh of the domain $\Omega$ with 22201 elements, see figure 3.10 for results. In each case the extremal eigenvalues were computed by means of direct and inverse power iterations. Obviously, regularisation manages to suppress the pronounced peak in the condition number in the case of the symmetrically coupled problem.

### 3.4 On the Choice of the Coupling Parameter

In this section we will analyse the impact of the coupling parameter $\eta$ on the eigenvalues of the coupled variational problem. Our investigation will not be carried out for the coupled finite element boundary element Galerkin schemes introduced in section 3.3.2, but will be limited to spherical geometries and a spectral discretization scheme based on eigenfunctions of the Laplacian and the boundary integral operators from section 1.3. Thus, for the rest of this section we assume $\Omega$ to be the unit ball $B := \{x \in \mathbb{R}^2; |x| < 1 \}$ and $\Gamma$ the unit sphere $S := \{x \in \mathbb{R}^2; |x| = 1 \}$. For simplicity reasons we will restrict ourselves to a spectral analysis of the symmetric coupling approach (3.4) and the second version of regularised FEM-BEM coupling (3.27).

In [58] a spectral discretization scheme was used to estimate the condition number of the system matrix underlying the combined field integral equation. The same technique was further refined to derive a-priori error estimates, which are explicit with respect to the mesh width $h$ and the wave number $\kappa$, cf. [38, 23, 7].

#### 3.4.1 Eigenvalues and Eigenfunctions

The main goal of this section is to derive explicit expressions for the eigenvalues and eigenfunctions for the Laplace operator and the boundary integral operators from section 1.3 assuming the special case of a spherical domain. We depart from an abstract perspective and consider first
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the eigenvalue problem related to the shifted Laplace operator with zero Neumann boundary conditions:

Find $\lambda \in \mathbb{R}$ and $0 \neq U \in H^1(\Omega)$ such that for all $V \in H^1(\Omega)$, there holds

$$\langle \nabla U, \nabla V \rangle_\Omega + \langle U, V \rangle_\Omega = \lambda \langle U, V \rangle_\Omega.$$

**Theorem 3.20.** Let $H$ be a separable Hilbert space, and suppose $S : H \rightarrow H$ is a compact and symmetric operator. Then there exists a countable orthonormal basis of $H$ consisting of eigenvectors of $S$.

**Proof.** For a proof see [36, Thm. 5, Sect. 4, Chap. VII].

This theorem motivates the variational definition of the following auxiliary operator: For any $U \in L^2(\Omega)$, find $S(U) \in H^1(\Omega)$ such that

$$\langle \nabla S(U), \nabla V \rangle_\Omega + \langle S(U), V \rangle_\Omega = \langle U, V \rangle_\Omega \quad \forall V \in H^1(\Omega). \quad (3.30)$$

The sesquilinear form underlying the left hand side of (3.30) is clearly elliptic, since it give rise to an inner product on $H^1(\Omega)$. By virtue of the Lax-Milgram lemma, $S$ is a well-defined bounded operator, satisfying

$$\|S(U)\|_{H^1(\Omega)} \leq C \|U\|_{L^2(\Omega)} \quad \forall U \in L^2(\Omega).$$

Moreover, due to the compact inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$, $S : L^2(\Omega) \hookrightarrow L^2(\Omega)$ is compact. For arbitrary $f, g \in L^2(\Omega)$, we define $U := S(f)$, $V := S(g)$ and conclude

$$S(f) = U \iff \langle S(f), g \rangle_\Omega = \langle \nabla U, \nabla V \rangle_\Omega + \langle U, V \rangle_\Omega,$$

$$S(g) = V \iff \langle f, S(g) \rangle_\Omega = \langle \nabla U, \nabla V \rangle_\Omega + \langle U, V \rangle_\Omega,$$
which implies the identity
\[
(f, S^*(g))_\Omega = (S(f), g)_\Omega = (f, S(g))_\Omega \quad \forall f, g \in L^2(\Omega),
\]
where \( * \) denotes the \( L^2(\Omega) \)-adjoint of \( S \). Hence, \( S \) is symmetric and theorem 3.20 ensures existence of an \( L^2(\Omega) \)-orthonormal basis of \( H^1(\Omega) \), consisting of eigenfunctions of the Laplace operator with zero Neumann boundary data.

Explicit expressions for the eigenvalues and their corresponding eigenfunctions cannot be derived for Lipschitz continuous domains of arbitrary shape in general, but need to be based on special geometrical features and symmetries. A special case, where such explicit expressions can be derived and are relatively easy to compute, is the case of the unit Ball \( \mathbb{B} \), cf. [32, Chap. 5]. In this specific setting, the eigenvalue problem for the Laplace operator with Neumann boundary conditions assumes the following form
\[
\Delta U + \lambda U = 0 \quad \text{in } \mathbb{B}, \quad \frac{\partial U}{\partial r} = 0 \quad \text{on } S := \partial \mathbb{B}. \tag{3.31}
\]

If we solve the eigenvalue problem (3.31) based on the trial expression \( U(r, \theta) := f(r) h(\theta) \), with a \( 2\pi \)-periodic function \( h \), we immediately obtain
\[
\frac{f''(r) + rf'(r) + \lambda f(r)}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = \text{Const.} = C,
\]
where the constant \( C \) assumes the value \( n^2 \), with \( n \in \mathbb{Z} \). Hence, we conclude that the angular part satisfies \( h(\theta) = \exp(in \theta) \) and the radial part \( f \) provides a solution to the ordinary differential equation
\[
r^2 y''(r) + ry'(r) + (r^2 \lambda - n^2)y(r) = 0. \tag{3.32}
\]
The eigenvalue problem (3.31) can now be cast into a much simpler form: We are looking for eigenvalues \( \lambda \), for which there exists a solution \( y \) to the ordinary differential equation (3.32), bounded in \([0,1)\), that satisfies the boundary condition \( f'(1) = 0 \). In light of the transformations \( kr = \rho \) and \( \lambda = k^2 \), we conclude that \( z(\rho) = y(r) \) solves Bessel’s differential equation
\[
z''(\rho) + z'(\rho) + \left(1 - \frac{n^2}{\rho^2}\right)z(\rho) = 0 \quad \text{in } (0,1). \tag{3.33}
\]
Solutions to (3.33), which are continuous in \([0,1)\) are provided by the Bessel functions \( J_n(x) \), see [88, Chap. 3] for a definition. Moreover, any solution to (3.32) can be written as \( y_n(r) = J_n(kr) \), where the constant \( k \) has to be inferred from the boundary condition \( y_n'(1) = 0 \), that is \( k J_n'(k) = 0 \). Hence, the eigenvalues \( \lambda = k^2 \) of (3.32) are squares of zeros of the first derivatives of the Bessel functions. In fact, each of the functions \( J_n'(x) \) has infinitely many zeros \( k_{nm} \), \( m \in \mathbb{N} \), which are numbered in ascending order (see Table 3.2 for a few zeros of \( J_n'(x) \)). Based this notation the eigenfunctions from (3.31) can be written in the following form
\[
J_n(k_{nm}r)e^{in\theta}, \quad n \in \mathbb{Z}, m \in \mathbb{N}. \tag{3.34}
\]
For further details on zeros of Bessel functions and related topics we refer to the monograph [1] and the article [37].

In order to construct an explicit \( L^2(\mathbb{B}) \)-orthonormal Hilbert basis, which consists of eigenfunctions to problem (3.31), it remains to normalize all functions from (3.34) with respect to the \( L^2(\mathbb{B}) \)-norm. Hence following the idea of [32, Sect. 5], we consider the differential equation for \( y(r) := J_n(kr) \) and by means of partial integration we derive the identity
\[
0 = \int_0^1 (ry')' ry \, dr + \int_0^1 \left( k^2 r - \frac{n^2}{r} \right) ryy' \, dr = \left( (ry')^2 + (k^2 r^2 - n^2) y^2 \right) \bigg|_0^1 - 2k^2 \int_0^1 ry^2 \, dr,
\]
3.4. ON THE CHOICE OF THE COUPLING PARAMETER

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Table 3.2: Zeros of Bessel function $J_{n}^{'}(x)$ taken from [1, Tab. 9.5].

which holds true for all $n \in \mathbb{Z}$. If $k \neq 0$ is a zero of the first derivative of the Bessel function of order $n$, we conclude

$$\int_{0}^{1} rJ_{n}(kr)^{2} dr = \frac{1}{2} \left( 1 - \frac{n^{2}}{k^{2}} \right) J_{n}(k)^{2},$$

and thus the set of $L^{2}(\mathbb{B})$-normalized eigenfunctions is given by

$$U_{nm}(r, \vartheta) := \frac{1}{\sqrt{\pi}} \left( 1 - \frac{n^{2}}{k_{nm}^{2}} \right)^{-1/2} \frac{J_{n}(k_{nm}r)}{J_{n}(k_{nm})} e^{in\vartheta}, \quad U_{01}(r, \vartheta) := \frac{1}{\sqrt{2\pi}},$$

with $(0, 1) \neq (n, m) \in \mathbb{Z} \times \mathbb{N}$. Moreover, a $H^{1}(\mathbb{B})$-orthonormal Hilbert basis of $L^{2}(\mathbb{B})$ can be obtained from the following definition

$$V_{nm}(r, \vartheta) := (1 + k_{nm}^{2})^{-1/2} U_{nm}(r, \vartheta). \quad (3.35)$$

A simple algorithm for the computation of zeros of Bessel functions and their derivatives based on reversion of asymptotic expansions and Newton-Raphson iterations has been introduced by Temme in [84].

Next we derive eigenvalues and eigenfunctions for all boundary integral operators from section 1.3. In contrast to the construction above, we can rely on an orthonormal system in $L^{2}(\mathbb{S})$ and do not need to establish sophisticated spectral theory. Again we will limit ourselves to two spatial dimensions and a spherical geometry. Generalizations to three dimensions are straightforward and can be found in [23, Sect. 5.1] and [7, Sect. 3.2].

The Fourier coefficients of a function $\varphi \in L^{2}(\mathbb{S})$ are defined by $\varphi_{n} := (y_{n}, \varphi)_{\mathbb{S}}$, for $n \in \mathbb{Z}$, where $y_{n}$ are the trigonometric monomials, defined by

$$y_{n}(\vartheta) := \frac{1}{\sqrt{2\pi}} e^{in\vartheta}, \quad n \in \mathbb{Z}, \quad (3.36)$$

which form a complete orthonormal system in $L^{2}(\mathbb{S})$. Equivalent spaces to the standard Sobolev spaces $H^{s}(\mathbb{S})$ can be defined by imposing certain growth conditions on the Fourier coefficients of their elements, cf. [59, Sect. 8] and [3, Sect. 3].

**Definition 3.21.** For $0 \leq s < \infty$ the space $\mathcal{H}^{s}(\mathbb{S})$ is defined as the subspace of all functions $\varphi \in L^{2}(\mathbb{S})$ with the property

$$\sum_{n \in \mathbb{Z}} (1 + n^{2})^{s} |\varphi_{n}|^{2} < \infty, \quad (3.37)$$
for the Fourier coefficients \( \varphi_n \) of \( \varphi \). With the inner product defined by

\[
(\varphi, \psi)_{H^s(S)} := \sum_{n \in \mathbb{Z}} (1 + n^2)^s \varphi_n \overline{\psi_n},
\]

and the induced norm, \( H^s(S) \) is a Hilbert space. For negative \( s < 0 \), \( H^s(S) \) is the dual space of \( H^{-s}(S) \). For a functional \( F \in H^s(S) \), its norm is given by (3.37), where the Fourier coefficients \( F_n \) are given by \( F_n := F(y_n) \).

The following lemma bridges the gap between \( H^s(S) \) and \( H^s(S) \) and promotes trigonometric monomials (3.36) to a pivotal role for the discretization of boundary integral equations on spheres.

**Lemma 3.22.** For \( s \in \mathbb{R} \), the space \( H^s(S) \) is a Hilbert space and is isomorphic to \( H^s(S) \), that is the norms induced by the inner products are equivalent and the sets \( H^s(S) \) and \( H^s(S) \) coincide. Moreover, The trigonometric polynomials form a complete orthogonal system in \( H^s(S) \) and are eigenfunctions of the operators \( \mathcal{V}_\kappa \), \( \mathcal{K}_\kappa \), \( \mathcal{K}'_\kappa \), and \( \mathcal{W}_\kappa \):

\[
\mathcal{V}_\kappa(y_n) = \lambda_n^{(V)} y_n, \quad \lambda_n^{(V)} := \frac{i \pi \kappa}{2} J_n(\kappa) H_n^{(1)}(\kappa),
\]

\[
\mathcal{K}_\kappa(y_n) = \lambda_n^{(K)} y_n, \quad \lambda_n^{(K)} := \frac{i \pi \kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i \pi \kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2},
\]

\[
\mathcal{K}'_\kappa(y_n) = \lambda_n^{(K')} y_n, \quad \lambda_n^{(K')} := \frac{i \pi \kappa}{2} J_n(\kappa) H_n^{(1)'}(\kappa) + \frac{1}{2} = \frac{i \pi \kappa}{2} J_n'(\kappa) H_n^{(1)}(\kappa) - \frac{1}{2},
\]

\[
\mathcal{W}_\kappa(y_n) = \lambda_n^{(W)} y_n, \quad \lambda_n^{(W)} := \frac{i \pi \kappa^2}{2} J_n'(\kappa) H_n^{(1)'}(\kappa),
\]

for all \( n \in \mathbb{Z} \).
Proof. For a proof see [3, Thm. 2].

In light of definition 3.21, complete orthonormal systems for $\mathcal{H}^1(\mathbb{S})$ and $\mathcal{H}^{-1/2}(\mathbb{S})$, can be easily obtained by re-normalization of the trigonometric monomials defined in (3.36). Thus we are lead to the following definition

$$p_n := \left(1 + n^2\right)^{-1/2} y_n, \quad \vartheta_n := \left(1 + n^2\right)^{-1/4} y_n,$$

and conclude that \{\(p_n : n \in \mathbb{Z}\)\} and \{\(\vartheta_n : n \in \mathbb{Z}\)\} form complete orthonormal systems in $\mathcal{H}^1(\mathbb{S})$ and $\mathcal{H}^{-1/2}(\mathbb{S})$, respectively. Moreover, for all $n, n' \in \mathbb{Z}$, $n \neq n'$, the following orthogonality relation holds

$$\langle \gamma_D^{-1} V_{nm}, \vartheta_{n'} \rangle_{\mathbb{S}} = \langle \gamma_D^{-1} V_{nm}, p_{n'} \rangle_{\mathbb{S}} = 0,$$

for all $m \in \mathbb{N}$, where \(\langle \cdot, \cdot \rangle_{\mathbb{S}}\) denotes the $L^2(\mathbb{S})$-inner product.

### 3.4.2 Spectral Analysis

For a spectral analysis of the symmetric and regularised coupling approach it is convenient to consider the variational formulations on the Sobolev spaces $\mathcal{H}^{-1/2}(\mathbb{S})$ and $\mathcal{H}^1(\mathbb{S})$, since each space can be equipped with an orthonormal system consisting of eigenfunctions to the boundary integral operators from section 1.3. In the first case, such a system is provided by \{\(\vartheta_n : n \in \mathbb{Z}\)\}, and for the latter by \{\(p_n : n \in \mathbb{Z}\)\}, respectively. Moreover, for arbitrary $\varphi \in \mathcal{H}^{-1/2}(\mathbb{S})$ and $q \in \mathcal{H}^1(\mathbb{S})$ the following Fourier series expansions hold

$$\varphi = \sum_{n \in \mathbb{Z}} \varphi_n \vartheta_n, \quad \varphi_n := \langle \varphi, \vartheta_n \rangle_{\mathcal{H}^{-1/2}(\mathbb{S})},$$
$$q = \sum_{n \in \mathbb{Z}} q_n p_n, \quad q_n := \langle q, p_n \rangle_{\mathcal{H}^1(\mathbb{S})}.$$

In contrast to variational formulations based on boundary integral equations only, a spectral analysis of the variational formulations (3.4), (3.26), or (3.27) will also need a complete orthonormal system for the analysis of the total field in $\Omega$. Such a system is provided by the orthonormal basis \(\{V_{nm} : n \in \mathbb{Z}, m \in \mathbb{N}\}\), consisting of eigenfunctions to the Laplace operator satisfying zero Neumann boundary data (see (3.35) for a definition). Moreover, the corresponding eigenvalue evaluates to the square of the $m$-th zero $k_{nm}$ of the first derivative of the Bessel function $J_n'(x)$ of order $n$. By definition of the trace scaling constants

$$\gamma_{nm} := \left(1 - \frac{n^2}{k_{nm}^2}\right)^{-1/2} (1 + k_{nm}^2)^{-1/2}, \quad \Gamma_n := \left(\sum_{m \in \mathbb{N}} \gamma_{nm}^2\right)^{1/2},$$

the following Fourier series expansion holds for arbitrary $U \in H^1(\mathbb{B})$ and its Dirichlet trace

$$U = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} a_{nm} V_{nm}, \quad \gamma_D^{-1} U = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} a_{nm} \gamma_{nm} y_n, \quad a_{nm} := \langle U, V_{nm} \rangle_{H^1(\mathbb{B})}.$$

Unfortunately, the trace scaling constants $\gamma_{nm}$ and $\Gamma_n$ feature the eigenvalues $k_{nm}^2$ of the Laplacian, which cannot be computed analytically but need to be approximated using Newton-Raphson iterations [84]. The asymptotic behaviour of the trace scaling constants $\gamma_{\text{sym}}$ and $\gamma_{\text{reg}}$ is plotted in figure 3.12 and 3.13.

On the one hand, we consider the symmetric coupling approach (3.4) and denote by $a_{\text{sym}}$ the corresponding sesquilinear form. Furthermore, we define the vector space $\mathcal{V} := H^1(\mathbb{B}) \times$
Figure 3.12: Values of the trace scaling constants $\gamma_{nm}$.

Figure 3.13: Values of the trace scaling constants $\Gamma_{nm}$. 
\( \mathcal{H}^{-1/2}(\mathcal{S}) \), which turns into a Hilbert space when endowed with the natural graph norm. Based on this definition (3.4) can be cast into the following form:

Find \( (U, \vartheta) \in \mathcal{V} \) such that for all \( (V, \phi) \in \mathcal{V} \), there holds

\[
a_{\text{sym}}((U, \vartheta), (V, \phi)) = f_{\text{sym}}(V, \phi) \quad (3.41)
\]

On the other hand, the stabilised variational formulation (3.27) can be written as a variational problem for a continuous sesquilinear form \( a_{\text{reg}} \) on the Hilbert space \( \mathcal{W} := \mathcal{V} \times \mathcal{H}^1(\mathcal{S}) \), that is endowed with the natural graph norm:

Find \( (U, \vartheta, p) \in \mathcal{W} \) in such that for all \( (V, \phi, q) \in \mathcal{W} \), there holds

\[
a_{\text{reg}}((U, \vartheta, p), (V, \phi, q)) = f_{\text{reg}}(V, \phi, q) \quad (3.42)
\]

In section 3.2.2 and 3.3 it was shown that the sesquilinear forms \( a_{\text{sym}} \) and \( a_{\text{reg}} \) are continuous and satisfy Gårding inequalities in the sense of item 1. of definition 1.10, on \( \mathcal{V} \) and \( \mathcal{W} \) with continuity constants \( C_{\text{sym}} > 0 \) and \( C_{\text{reg}} > 0 \), respectively. Both sesquilinear forms are injective, if the wave number \( \kappa \) does not agree with an internal resonant frequency. In addition, regularisation provides unique solutions for all frequencies \( \kappa > 0 \), which confirms injectivity of \( a_{\text{reg}} \) even in the resonant case.

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Table 3.3: Zeros of Bessel function \( J_n(x) \) taken from [1, Tab. 9.5].

To derive discrete versions of (3.41) and (3.42) we need to identify suitable finite-dimensional subspaces of \( H^1(\mathbb{B}) \), \( \mathcal{H}^{-1/2}(\mathcal{S}) \), and \( \mathcal{H}^1(\mathcal{S}) \). An easy way to obtain such spaces, is based on truncation of Fourier series expansions: For fixed truncation levels \( N, M \in \mathbb{N} \), we define the index sets

\[
\mathcal{I}_N := \{ n \in \mathbb{Z} : |n| \leq N \} \subset \mathbb{Z}, \quad \mathcal{J}_M := \{ m \in \mathbb{N} : m \leq M \} \subset \mathbb{N},
\]

which give rise to the following finite-dimensional subspaces

\[
S^1_{NM}(\mathbb{B}) := \left\{ \sum_{n \in \mathcal{I}_N} \sum_{m \in \mathcal{J}_M} a_{nm} V_{nm} : a_{nm} \in \mathbb{C}, n \in \mathcal{I}_N, m \in \mathcal{J}_M \right\},
\]

\[
S^{-1/2}_N(\mathcal{S}) := \left\{ \sum_{n \in \mathcal{I}_N} \varphi_n \vartheta_n : \varphi_n \in \mathbb{C}, n \in \mathcal{I}_N \right\}, \quad S^1_N(\mathcal{S}) := \left\{ \sum_{n \in \mathcal{I}_N} q_n p_n : q_n \in \mathbb{C}, n \in \mathcal{I}_N \right\},
\]

that can be used to construct spectral discretization schemes for both symmetric and regularised coupling. Thus, we arrive at the discrete test and trial spaces for (3.41) and (3.42)

\[
\mathcal{V}_{NM} := S^1_{NM}(\mathbb{B}) \times S^{-1/2}_N(\mathcal{S}), \quad \mathcal{W}_{NM} := \mathcal{V}_{NM} \times S^1_N(\mathcal{S}),
\]
which provide us with conforming finite-dimensional subspaces of $V$ and $W$, that inherit the approximation property (1.20) from $S_{MN}^{S}(B)$, $S_{N}^{-1/2}(S)$, and $S_{N}(S)$. Hence, we can rely on theorem 1.12 to establish asymptotic quasi-optimality estimates for the sesquilinear forms $a_{\text{sym}}$ and $a_{\text{reg}}$. There exist $0 < N_{0} \in \mathbb{N}$ and $0 < M_{0} \in \mathbb{N}$, and constants $\gamma_{\text{sym}} > 0$ and $\gamma_{\text{reg}} > 0$, such that for all $N > N_{0}$ and $M > M_{0}$ the following estimates hold true

$$
\sup_{(U_{h}, \vartheta_{h}) \in \mathcal{V}_{NM}} \left| a_{\text{sym}}((U_{h}, \vartheta_{h}), (V_{h}, \varphi_{h})) \right| \geq \gamma_{\text{sym}} \| (U_{h}, \vartheta_{h}) \|_{V}, \quad \forall (U_{h}, \vartheta_{h}) \in \mathcal{V}_{NM},
$$

and

$$
\sup_{(U_{h}, \vartheta_{h}, p_{h}) \in \mathcal{W}_{NM}} \left| a_{\text{reg}}((U_{h}, \vartheta_{h}, p_{h}), (V_{h}, \varphi_{h}, q_{h})) \right| \geq \gamma_{\text{reg}} \| (U_{h}, \vartheta_{h}, p_{h}) \|_{W}, \quad \forall (U_{h}, \vartheta_{h}, p_{h}) \in \mathcal{W}_{NM},
$$

where the first estimate for $a_{\text{sym}}$ can only be established, if $\kappa$ does not agree with an interior resonant frequency. The inf-sup constants $\gamma_{\text{sym}}$ and $\gamma_{\text{reg}}$ can now be numerically evaluated based on a singular value decomposition (SVD) of the Galerkin matrix, cf. [40, Sect. 2.3]. Recalling the definition of $\mathcal{V}_{NM}$, we obtain the following series expansions for $(U_{h}, \vartheta_{h}) \in \mathcal{V}_{NM}$

$$
U_{h} = \sum_{n \in I_{N}} \sum_{m \in J_{M}} a_{nm} V_{nm}, \quad \vartheta_{h} = \sum_{n \in I_{N}} \varphi_{n} \vartheta_{n}, \quad (3.43)
$$

with coefficients $a_{nm} \in \mathbb{C}$ and $\varphi_{n} \in \mathbb{C}$, for $n \in I_{N}$ and $m \in J_{M}$. Obviously, the vectors $\{V_{nm} : n \in I_{N}, m \in J_{M}\}$, and $\{\vartheta_{n} : n \in I_{N}\}$, form an orthonormal basis of $S_{NM}^{1}(B)$ and $S_{N}^{-1/2}(S)$, respectively. By grouping together all coefficients at level $n \in I_{N}$ according to $X_{n} := (a_{n1}, \ldots, a_{nM}, \varphi_{n})^{T}$, we obtain the coefficient vector $X := (X_{n})_{n \in I_{N}}$ corresponding to $(U_{h}, \vartheta_{h}) \in \mathcal{V}_{nm}$ and the Galerkin matrix $A_{(\text{sym})}$. Furthermore, due to the orthogonality relation (3.38) we conclude that $A_{(\text{sym})}$ assumes a block-diagonal structure

$$
A_{(\text{sym})} = \text{diag} \left( A_{n}^{(\text{sym})} \right)_{n \in I_{N}},
$$

where the matrix blocks $A_{n}^{(\text{sym})}$ can be obtained from a separate assembly procedure at each level $n \in I_{N}$. Based on a singular value decomposition of the Galerkin matrix $A_{(\text{sym})}$, we arrive at the following estimate

$$
\sup_{(V_{h}, \varphi_{h}) \in \mathcal{V}_{NM}} \left| a_{\text{sym}}((U_{h}, \vartheta_{h}), (V_{h}, \varphi_{h})) \right| \leq \sup_{Y \neq 0} \frac{|Y^{H} A_{(\text{sym})} X|}{\| Y \|_{2}} \geq \min_{n \in I_{N}} \left\{ \left| \sigma_{\text{min}} \left( A_{n}^{(\text{sym})} \right) \right| \right\} \|X\|_{2} = \min_{n \in I_{N}} \left\{ \left| \sigma_{\text{min}} \left( A_{n}^{(\text{sym})} \right) \right| \right\} \| (U_{h}, \varphi_{h}) \|_{V},
$$

where $X^{H}$ denotes the hermitian adjoint of $X$ and $\sigma_{\text{min}}(M)$ the singular value of smallest magnitude of an arbitrary matrix $M$. Hence, it follows that the inf-sup constant $\gamma_{\text{sym}}$ can be estimated from above by the singular values of each block $A_{n}^{(\text{sym})}$ of the Galerkin matrix $A_{(\text{sym})}$. Next, for an arbitrary element $(U_{h}, \vartheta_{h}, p_{h}) \in \mathcal{W}_{NM}$ we have the following series expansions for the component $p_{h}$

$$
p_{h} = \sum_{n \in I_{N}} q_{n} p_{n},
$$

with coefficients $q_{n} \in \mathbb{C}$, $n \in I_{N}$. Moreover, the vectors $\{p_{n} : n \in I_{N}\}$ clearly form an orthonormal basis of $S_{N}(S)$. Recalling the series expansion (3.43) for the components $U_{h}$ and $\vartheta_{h}$, we can
3.4. ON THE CHOICE OF THE COUPLING PARAMETER

Group the coefficients on each level \( n \in \mathcal{I}_N \) according to \( X_n := (a_{n1}, \ldots, a_{nM}, \varphi_n, q_n)^T \) and obtain the coefficient vector \( X := (X_n)_{n \in \mathcal{I}_N} \), corresponding to \( (U_h, \varphi_h, p_h) \in \mathcal{W}_{NM} \), together with the Galerkin matrix \( A^{(\text{reg})} \). Again, the orthogonality relation \((3.38)\) implies a block-diagonal structure of \( A^{(\text{reg})} \)

\[
A^{(\text{reg})} = \text{diag} \left( A_n^{(\text{reg})} \right)_{n \in \mathcal{I}_N},
\]

where the matrix blocks \( A_n^{(\text{reg})} \) can be obtained from an assembly procedure at each level \( n \in \mathcal{I}_N \). Hence, we can rely on a block-wise singular value decomposition of the Galerkin matrix \( A^{(\text{reg})} \) to estimate the discrete inf-sup constant \( \gamma^{\text{reg}} \)

\[
\sup_{(V_h, \varphi_h, p_h) \in \mathcal{W}_{NM}} \frac{a^{(\text{reg})}(U_h, \varphi_h, p_h), (V_h, \varphi_h, p_h))}{\|(V_h, \varphi_h, p_h)\|_{\mathcal{W}}} = \sup_{\|Y\|_2 \neq 0} \frac{|Y^H A^{(\text{reg})} X|}{\|Y\|_2} \geq \min_{n \in \mathcal{I}_N} \left\{ \sigma_{\min} \left( A_n^{(\text{reg})} \right) \right\} \|X\|_2
\]

\[
= \min_{n \in \mathcal{I}_N} \left\{ \sigma_{\min} \left( A_n^{(\text{reg})} \right) \right\} \|(U_h, \varphi_h, p_h)\|_{\mathcal{W}}.
\]

On each level \( n \in \mathcal{I}_N \), the matrix blocks \( A_n^{(\text{sym})} \) and \( A_n^{(\text{reg})} \) have been evaluated symbolically and their singular values have been computed numerically using MATLAB. For the wave number \( \kappa = 40 \) and various sets of truncation levels, the behaviour of the smallest singular value for symmetric and regularised coupling has been recorded in figure 3.14 and 3.15, respectively. For the stabilised version of FEM-BEM coupling the truncation parameter \( \eta \) was set to 1, as in the experiments in section 3.3.3. Both figures indicate that once a minimal resolution of the discrete test and trial spaces \( \mathcal{V}_{NM} \) and \( \mathcal{W}_{NM} \) has been achieved, the discrete inf-sup constants \( \gamma^{\text{sym}} \) and \( \gamma^{\text{reg}} \) can be computed from the smallest singular values of the Galerkin matrices \( A^{(\text{reg})} \) and \( A^{(\text{sym})} \), respectively. It is not surprising that the minimal resolution strongly depends on the wave number \( \kappa \).

The continuity constants \( C^{\text{sym}} \) and \( C^{\text{reg}} \) of both sesquilinear forms can be estimated in a block-wise fashion, by the maximum of the continuity constants of all operator blocks involved. For each block, sharp estimates can be established using Fourier series expansions for all variables from the corresponding trial and test spaces. As an example, we consider the sesquilinear form

\[
(a, (\frac{1}{2} I - K_\kappa)(\gamma_D(U)))_\Sigma, \quad a \in \mathcal{H}^{-1/2}(\Sigma), \quad U \in H^1(\mathcal{B}).
\]

which is part of both \( a^{\text{sym}} \) and \( a^{\text{reg}} \). Analogous estimates can be derived for all other blocks in a similar way. Thus, substituting the Fourier series expansions \((3.40)\) and \((3.39)\) into \((3.44)\), the following estimate is straightforward

\[
\|a, (\frac{1}{2} I - K_\kappa)(\gamma_D(U))\|_\Sigma \leq \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \gamma_{nm} (1 + n^2)^{1/4} |\lambda_n^{(K)}| - \frac{1}{2} \|\varphi_n a_{nm}\|
\]

\[
\leq \left( \sup_{n \in \mathbb{Z}} (1 + n^2)^{1/4} \Gamma_n |\lambda_n^{(K)}| - \frac{1}{2} \right) \|\varphi\|_{\mathcal{H}^{-1/2}(\Sigma)} \|U\|_{H^1(\mathcal{B})},
\]

where \( a_{nm} \) and \( \varphi_n \) denote the Fourier coefficients of \( U \in H^1(\mathcal{B}) \) and \( \varphi \in \mathcal{H}^{-1/2}(\Sigma) \). Moreover, the continuity constant

\[
C = \sup_{n \in \mathbb{Z}} (1 + n^2)^{1/4} \Gamma_n |\lambda_n^{(K)}| - \frac{1}{2},
\]

can be evaluated very efficiently, since convergence of the series \( \Gamma_n \) is very fast and the supremum turns out to be a maximum, which is attained for small values of \( |n| \), for \( n \in \mathbb{Z} \). Hence, reliable numerical estimates of the continuity constants of both variational formulations can be
Figure 3.14: Smallest singular value of $A^{(sym)}$ for various truncation levels and $\kappa = 40$.

Figure 3.15: Smallest singular value of $A^{(reg)}$ for various truncation levels and $\kappa = 40$. 
established. In case of the regularised coupling approach one operator block deserves special attention, namely the one given by
\[ \iota \eta (\varphi, p) \in \mathcal{H}^{-1/2}(S), \quad p \in \mathcal{H}^{1}(S). \]

Again, we can rely on Fourier series expansions of the variables \( \varphi \) and \( p \) to derive the following estimate
\[
|\iota \eta (\varphi, p)| \leq |\eta| \sum_{n \in \mathbb{Z}} (1 + n^2)^{-1/4} \left| \varphi_n q_n \right| \leq |\eta| \left\| \varphi \right\|_{\mathcal{H}^{-1/2}(S)} \left\| p \right\|_{\mathcal{H}^{1}(S)},
\]
where \( \varphi_n \) and \( q_n \) denote the Fourier coefficients of \( \varphi \) and \( q \), respectively. Hence, we conclude that for a fixed \( \kappa \) the continuity constant of the stabilised formulation should grow linearly with \( |\eta| \), provided that the coupling parameter \( \eta \) is sufficiently large.

![Figure 3.16: Behaviour of the inf-sup constants in the neighbourhood of the resonant frequency \( \kappa = 5.52007 \).](image)

**Experiment 3.23.** We study the behaviour of the inf-sup constants \( \gamma_{\text{sym}} \) and \( \gamma_{\text{reg}} \) in the neighbourhood of the resonant frequency \( \kappa = 5.52007 \), which corresponds to the second zero of the Bessel function \( J_0(x) \). We have used the truncation levels \( N = 20 \) and \( M = 50 \), which should be large enough to guarantee a sufficient resolution of the discrete inf-sup constants \( \gamma_{\text{sym}} \) and \( \gamma_{\text{reg}} \). In addition, we have set the coupling parameter \( \eta \) equal to 1, which corresponds to the choice made in section 3.3.3. A similar numerical experiment based on a coupled finite element boundary element Galerkin scheme has already been carried out in section 3.3.3, where numerical estimates of the spectral condition number of the Galerkin matrix in the neighbourhood of a resonant frequency were derived (see experiment 3.19 for details). In case of the spectral discretization scheme, the results are recorded in figure 3.16. Both inf-sup constants display some mild growth as the wave number increases. The constant \( \gamma_{\text{sym}} \) of the symmetric coupling approach shows a pronounced spike at \( \kappa \approx 5.52 \), which confirms the presence of resonant
frequencies even in the case of a spectral discretization. On the other hand, the regularised coupling approach is insensitive to spurious resonances.

**Experiment 3.24.** Discrete solutions \((U_h, \vartheta_h, p_h) \in \mathcal{W}_{NM}\) to the regularised variational problem (3.27) obtained from a spectral discretization of \(\mathcal{W}\) satisfy the asymptotic quasi-optimality estimate

\[
\|U - U_h\|_{H^1(\Omega)} + \|\vartheta - \vartheta_h\|_{H^{-1/2}(\Omega)} + \|p - p_h\|_{H^1(\Omega)} \leq C_{\text{opt}} \left( \inf_{V_h \in \mathcal{S}_{NM}(\Omega)} \|U - V_h\|_{H^1(\Omega)} + \inf_{\varphi_h \in \mathcal{S}_{N}^{1/2}(\Omega)} \|\vartheta - \varphi_h\|_{H^{-1/2}(\Omega)} \right),
\]

with a constant \(C_{\text{opt}}\) given by \(C_{\text{opt}} := 1 + C_{\text{reg}}/\gamma_{\text{reg}}\), see section 3.3.2 and the proof of lemma lemma 1.13 for details. This experiment considers the behaviour of the quasi-optimality constant \(C_{\text{opt}}\) for a fixed wave number and variable values of the coupling parameter \(\eta\). For the spectral discretization scheme we used the truncation levels \(N = 20\) and \(M = 50\). The results of this experiment are recorded in figure 3.17 for \(\kappa = 5.52007\), in figure 3.18 for \(\kappa = 5.33144\), and in figure 3.19 for \(\kappa = 4.71239\).

![Figure 3.17: Coupling parameter dependency of the quasi-optimality constant for \(\kappa = 5.52007\).](image)

Figure 3.17 indicates that, whenever \(\kappa\) agrees with one of the resonant frequencies the quasi-optimality constant displays some algebraic growth, whenever \(\eta\) tends to zero or infinity. This may seem surprising at first sight, but if \(\eta\) tends to zero then the first two equations of the regularised coupled variational formulation (3.27) become more and more decoupled from the third. Hence, in the limit, the first two components of the discrete solution \((U_h, \vartheta_h, p_h) \in \mathcal{W}_{NM}\) completely decouple from \(p_h\) and thus \(U_h\) and \(\vartheta_h\) will solve the discrete symmetric coupled problem (3.4). Moreover, the blow-up of the quasi-optimality constant for small values of \(\eta\) is clearly attributed to the behaviour of the inf-sup constant \(\gamma_{\text{reg}}\). Nevertheless, small values of \(C_{\text{opt}}\) are guaranteed if we choose the coupling parameter \(\eta\) between \(10^{-1}\) and \(10^1\).
3.4. ON THE CHOICE OF THE COUPLING PARAMETER

Figure 3.18: Coupling parameter dependency of the quasi-optimality constant for $\kappa = 5.33144$.

On the other hand, as indicated by figure 3.18, if $\kappa^2$ corresponds to a Dirichlet eigenvalue of the Laplace operator on $\mathbb{B}$, the linear growth of the quasi-optimality constant for small values of $\eta$ is completely suppressed. However, in light of the previous paragraph, this is no surprise at all, since $\kappa$ does not correspond to a resonant frequency, the symmetric coupled problem is well-posed and the inf-sup constant $\gamma_{\text{reg}}$ does not blow-up as $\eta$ tends to zero. In contrast to the resonant case, the inf-sup constant tends to zero as $\eta$ approaches infinity. This results in a stronger growth rate of the quasi-optimality constant $C_{\text{opt}}$ for large values of $\eta$. Again, choosing the coupling parameter somewhere in between $10^{-1}$ and $10^1$ will yield optimal values for the quasi-optimality constant.

The behaviour of the quasi-optimality constant, if the wave number is neither a resonant frequency nor related to an interior Dirichlet eigenvalue of the Laplace operator, is recorded in figure 3.19. The most striking difference to figure 3.17 and 3.18 is that the inf-sup constant stays bounded away from zero, whether the coupling parameter $\eta$ approaches infinity or zero. Moreover, the value of $\gamma_{\text{reg}}$ is more or less constant for all $\eta$ between $10^{-6}$ and $10^1$. Hence, the growth of $C_{\text{opt}}$ as $\eta$ tends to infinity is attributed to the growth of the continuity constant. Thus an upper bound of $10^1$ for the coupling parameter $\eta$ will be sufficient to guarantee small values of the quasi-optimality constant.

**Experiment 3.25.** Figures 3.20 and 3.21 record the quantitative behaviour of the quasi-optimality constant $C_{\text{opt}}$ for various values of the wave number $\kappa$ and the coupling parameter $\eta$, in the low and mid frequency regime. This experiment was carried out with truncation levels set to $N = 20$ and $M = 50$. By and large this experiment confirms the statements already made in experiment 3.24. Moreover, we can clearly identify $\eta \approx 1$ to be a good choice for the coupling parameter, which yields small values for $C_{\text{opt}}$ for all wave numbers under consideration.
Figure 3.19: Coupling parameter dependency of the quasi-optimality constant for $\kappa = 4.71239$.

Figure 3.20: Coupling parameter and wave number dependency of the quasi-optimality constant.
3.5 Conclusion

We have derived a regularised version of the exterior Calderón projector for the scattered field $U^s$, which is based on the trace transformation operator $T$. Furthermore, two concrete specimens of $T$ have been constructed based on regularising operators $M$, that are already widely used for the stabilisation of boundary element methods for pure scattering problems (see chapter 2 for details). Based on the regularised Calderón projector we have derived a novel type of Dirichlet-to-Neumann maps, that can be used to couple boundary integral equations with variational equations on $\Omega$.

The two variational formulations derived from the trace transformation operators feature operator products of local and non-local operators on the interface boundary $\Gamma$ and thus elude a standard Galerkin discretization by means of finite elements and boundary elements. However, a special choice of the regularisation operator and the introduction of an auxiliary, non-physical variable $p$ made it possible to get rid of all remaining operator products. Thus we have introduced mixed regularised Galerkin schemes for the approximate solution of Helmholtz transmission problems. A special feature of these variational formulations is that the auxiliary variable $p$ evaluates to zero, independent of the wave number, the material parameters, the incident field, and the shape of the obstacle.

For the mixed Galerkin schemes we have proved Gårding inequalities for the underlying sesquilinear forms and established uniqueness of solutions for all positive wave numbers. This made it possible to apply standard theory for coercive variational formulations and derive an asymptotic inf-sup estimates for the sesquilinear forms and establish existence, uniqueness and stability of discrete solutions. In addition, we have derived an asymptotic quasi-optimality estimate for the discretization error, that does not depend on the auxiliary variable $p$, which confirms that the overall convergence rate of the symmetric approach is retained.

Finally, we have justified our theoretical discoveries by a variety of numerical examples, which confirm the theoretical convergence rates of the discretization error for the $U$ and $\theta$-components.

Figure 3.21: Coupling parameter and wave number dependency of the quasi-optimality constant.
of the solution in various norms on smooth and non-smooth domains.

Our investigation concerning wave number dependency of the discretization errors clearly revealed the ill-posed character of the symmetric approach in the neighbourhood of the so-called resonant frequencies. In case of the symmetrically coupled problem the \( \varphi \)-component of the discretization error displayed clearly pronounced spikes, which have been completely suppressed, when either one of the mixed regularised variational formulations was used. This did not only hold true for the discretization errors but carried over to the spectral condition numbers of the underlying linear systems as well.

### 3.6 Open Questions

The following questions and points regarding FEM-BEM coupling for time-harmonic acoustic scattering remained open. These might be subjects for future research:

- So far we have not addressed the issue of fast and efficient implementations for the symmetric and the stabilised variational formulations. Efficient techniques for the assembly of the system matrices and the evaluation of the matrix-vector products for boundary integral equations are well developed. One could use for instance \( \mathcal{H} \)-matrices [43, 44], panel clustering [45], fast multipole methods (FMM) [41], or wavelet compression techniques [8, 86, 33, 46] to reduce the memory cost and increase efficiency.

However, employing fast and efficient boundary element schemes means switching to a perturbed variational problem, for which we no longer have discrete inf-sup conditions and quasi-optimality estimates at hand to establish well-posedness and optimal convergence rates. Nevertheless, the situation may not be completely lost but the numerical analysis of the variational formulation must now be based on Strang’s lemma.

- As recorded in figure 3.10 the spurious modes have no impact on the condition number of the stabilised variational formulations. However, due to the implicit definition of the regularising operator, the overall spectral condition number of the system matrix is by a factor 100 larger than in the symmetric case. This will lead to a substantial increase in the number of iterations an iterative solver needs to solve the system.

Instead of solving the entire linear system for the regularised coupling approach, we can switch to the Schur complement system, which reduces the spectral condition number to the level of symmetric FEM-BEM coupling. Unfortunately, the Schur complement features the inverse of a sparse, positive definite matrix \( B \), which arises from a Galerkin discretization of the sesquilinear form \( b \) in (3.27). For an intermediate number of unknowns, \( B^{-1} \) can be evaluated directly based on a QR or Cholesky decomposition, [40, Chap. 5]. For a larger number of unknowns, the Schur complement system might still be useful, when a Krylov solver is used to evaluate the inverse of \( B \).

Although the Schur complement system greatly reduces the condition number, the resulting set of equations still needs a large number of GMRES iterations to be solved (see figure 3.9 for details). Hence, the use of a suitable preconditioner for the entire system matrix might substantially decrease the total number of iterations and reduce the overall runtime.

- Figure 3.19 indicates that the choice of the coupling parameter \( \eta \) has a strong effect on the constant in the quasi-optimality estimate of the Galerkin solution. However, on the sphere \( S \), explicit a-priori error estimates with respect to the mesh width \( h \) and the wave number \( \kappa \), as in [23, 7] could not be established by means of a Fourier analysis only.
Part II

The Maxwell Case
Chapter 4

Introduction

So far, all our considerations have been limited to direct acoustic scattering from penetrable or impenetrable obstacles, which, although interesting, are of limited relevance in modern physical practice. Of more importance nowadays are electromagnetic scattering problems due to their wider range of practical applications, which range from magnetic resonance imaging (MRI) in computerized tomography, which describes a non-invasive method used to render images of living tissue, to optical remote sensing systems such as LIDAR (Light Detection and Ranging) or RADAR (Radio Detection and Ranging), which measure properties of scattered electromagnetic waves to determine range, altitude, direction, or speed of both stationary and moving obstacles. Perhaps one of the most important applications are global positioning systems (GPS), where dozens of satellites in earth orbit transmit signals, which allow GPS receivers to determine their location, speed and direction.

Since we are mainly concerned with scattering problems, we focus again on the effect an inhomogeneous object has on an incident electromagnetic field or particle. In principal, the same classification as in the Helmholtz case is applicable to Maxwell scattering problems, roughly dividing them into direct and indirect ones. To be more specific, if the total field \( E \) is split into an incident and a scattered field \( E^i \) and \( E^s \), respectively, then the focus of the direct scattering problem is to compute \( E^s \) from the knowledge of a prescribed incident field \( E^i \) and the physical laws that determine the wave motion. Direct electromagnetic scattering problems can be further subdivided into scattering from bounded, inhomogeneous media, which we refer to as Maxwell transmission problems and scattering from impenetrable objects, for which we adopt the name Maxwell scattering problems.

4.1 Electromagnetic Waves

We start with a brief overview of classical electromagnetic scattering theory as presented in [65, Chap. 1] and [47, Chap. 1]. The fundamental set of equations describing the dynamics of all macroscopic electromagnetic phenomena have been derived by Maxwell [62] in 1873. The proposed set of equations can be specified in either integral as well as in differential form. We restrict ourselves to representations in differential form, since they gives rise to variational formulations, which can be dealt with by finite element and boundary element Galerkin schemes.

We consider the Maxwell transmission problem, that is, the scattering of time-dependent electromagnetic waves from a bounded, penetrable obstacle \( \Omega \subset \mathbb{R}^3 \), where the electromagnetic fields are usually modeled in terms of the following six quantities: For any \((x, t) \in \mathbb{R}^3 \times \mathbb{R}_+\) let \( E = E(x, t) \) and \( H = H(x, t) \) denote the electric and magnetic intensities, \( D = D(x, t) \) and \( B = B(x, t) \) the electric and magnetic flux densities, \( J = J(x, t) \) the electric current density, and \( \rho = \rho(x, t) \) the electric charge density. These variables are related by Maxwell's equations, which apply in the region of \( \mathbb{R}^3 \) occupied by the electromagnetic field.
4.1. ELECTROMAGNETIC WAVES

Faraday’s law of induction  \[
\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (4.1)
\]

Gauss law  \[
\text{div } \mathbf{D} = \rho, \quad (4.2)
\]

Ampère’s circuital law  \[
\text{curl } \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \quad (4.3)
\]

\[
\text{div } \mathbf{B} = 0. \quad (4.4)
\]

First of all, \textit{Faraday’s law of induction} (4.1) states that changes in \( \mathbf{B} \) induce an electromotive force in any loop surrounding it. Second, \textit{Gauss law} (4.2) indicates that the field lines of \( \mathbf{E} \) begin and end on electric charge. Finally, (4.4) states that \( \mathbf{B} \) is solenoidal. A fifth fundamental equation, which is usually called \textit{equation of continuity} and links together electric charge and current densities, is given by

\[
\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}, \quad (4.5)
\]

and enforces charge conservation in a physical system. Unfortunately, not all of these five equations are independent and formally taking the divergence of (4.1) and (4.3) together with (4.5) yields

\[
-\frac{\partial}{\partial t} \text{div } \mathbf{B} = \frac{\partial}{\partial t} \left( \text{div } \mathbf{D} - \rho \right) = 0.
\]

Nonetheless, one can either choose (4.1), (4.2), and (4.3) or (4.1), (4.3), and (4.5) as sets of independent equations. Assuming the special case, were all quantities involved are harmonically oscillating fields or functions with a single angular frequency \( \omega > 0 \), we can switch to complex phasor notation [47, Sect. 1.7] and arrive at the expressions

\[
\mathbf{E}(\mathbf{x}, t) = \text{Re} \left\{ \mathbf{E}(\mathbf{x}) \exp(-i\omega t) \right\}, \quad \mathbf{D}(\mathbf{x}, t) = \text{Re} \left\{ \mathbf{D}(\mathbf{x}) \exp(-i\omega t) \right\},
\]

\[
\mathbf{H}(\mathbf{x}, t) = \text{Re} \left\{ \mathbf{H}(\mathbf{x}) \exp(-i\omega t) \right\}, \quad \mathbf{B}(\mathbf{x}, t) = \text{Re} \left\{ \mathbf{B}(\mathbf{x}) \exp(-i\omega t) \right\},
\]

\[
\mathbf{J}(\mathbf{x}, t) = \text{Re} \left\{ \mathbf{J}(\mathbf{x}) \exp(-i\omega t) \right\}, \quad \mathbf{\rho}(\mathbf{x}, t) = \text{Re} \left\{ \mathbf{\rho}(\mathbf{x}) \exp(-i\omega t) \right\}.
\]

Note that all quantities introduced in the previous equation are now complex-valued vector fields or scalar functions only depending on space but not on time. Hence, complex phasor notation allows us to cast Maxwell’s equations into the following time-harmonic system

\[
\text{curl } \mathbf{\hat{E}} - i\omega \mathbf{\hat{B}} = 0, \quad \text{div } \mathbf{\hat{D}} = \mathbf{\hat{\rho}}, \quad (4.6)
\]

\[
\text{curl } \mathbf{\hat{H}} + i\omega \mathbf{\hat{D}} = \mathbf{\hat{J}}, \quad \text{div } \mathbf{\hat{B}} = 0.
\]

Moreover, translating (4.5) into time-harmonic form provides us with the following identity

\[
\text{div } \mathbf{\hat{J}} = i\omega \mathbf{\hat{\rho}}.
\]

In addition to the set of Maxwell’s equations constitutive laws, which specify the characteristics of the medium, are indispensable. This amounts to expressing the electric and magnetic flux densities \( \mathbf{D} \) and \( \mathbf{B} \) and the electric current density \( \mathbf{J} \) as functions of the electric and magnetic intensity \( \mathbf{E} \) and \( \mathbf{H} \), respectively. Most macroscopic phenomena can be described by the following three cases:

1. \textit{Homogeneous, isotropic materials}. The term homogeneous is used to label media, which consist of one single type of material. In that case and if the material is linear, the field quantities are related by the following equations

\[
\mathbf{\hat{D}} = \varepsilon_0 \mathbf{\hat{E}}, \quad \mathbf{\hat{B}} = \mu_0 \mathbf{\hat{H}},
\]

where the constants \( \varepsilon_0 \) and \( \mu_0 \) are called electric permittivity and magnetic permeability.
2. \textit{Inhomogeneous, isotropic materials.} The most prominent practical case is that of an inhomogeneous medium where various materials with different properties occupy the domain of the electromagnetic field. Moreover, if the material is linear and its properties do not depend on the direction of the field, we have
\[
\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H},
\] (4.7)
where $\varepsilon$ and $\mu$ are positive, bounded, scalar functions.

In addition to the constitutive laws already discussed above, one further relation needs to be considered. According to Ohm’s law, which holds for moderate field strengths, the electromagnetic field induces currents
\[
\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_a. \quad (4.8)
\]
Here, $\sigma$ denotes the conductivity, which is a non-negative function, and the vector field $\mathbf{J}_a$ refers to the applied current density. Since we focus on scattering of electromagnetic waves from penetrable, inhomogeneous obstacles, further simplifications of (4.6) are possible. Thus, following the ideas of [28, Sect. 9.1], normalizing the field quantities according to $\mathbf{E} = \sqrt{\varepsilon_0} \mathbf{E}$, $\mathbf{H} = \sqrt{\mu_0} \mathbf{H}$, switching to relative material parameters
\[
\varepsilon_r(x) := \frac{\varepsilon(x)}{\varepsilon_0} + i \frac{\sigma(x)}{\omega}, \quad \mu_r(x) := \frac{\mu(x)}{\mu_0},
\]
and employing the constitutive laws (4.7) and (4.8) finally turns (4.6) into the following first-order Maxwell system
\[
\text{curl } \mathbf{E} - i\kappa \mu_r \mathbf{H} = 0, \quad \text{curl } \mathbf{H} + \varepsilon_r \mathbf{E} = \frac{1}{i\kappa} \mathbf{F},
\]
where the source term $\mathbf{F}$ and the wave number $\kappa$ are given by $\mathbf{F} := i\kappa \sqrt{\mu_0} \mathbf{J}_a$, and $\kappa := \omega \sqrt{\varepsilon_0 \mu_0}$, respectively. Note that the relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$ might display some spatial variation inside the scatterer $\Omega$, but assume the constant value $\varepsilon_r = \mu_r = 1$ in the exterior air region $\Omega^+$. In addition, the source term $\mathbf{F}$ is assumed to be compactly supported inside $\Omega$. Moreover, the divergence conditions
\[
\text{div } (\mu_r \mathbf{E}) = -\frac{1}{\kappa^2} \mathbf{F}, \quad \text{div } \varepsilon_r \mathbf{H} = 0,
\]
can be immediately derived from (4.6) and the constitutive laws (4.7) and (4.8). By eliminating the $\mathbf{H}$ field from the first-order Maxwell system, we arrive at the electric field equation
\[
\text{curl } \mu_r^{-1} \text{curl } \mathbf{E} - \kappa^2 \varepsilon_r \mathbf{E} = \mathbf{F}. \quad (4.9)
\]
As in the Helmholtz case it turns out that an additional condition is needed to obtain a well-posed problem. To do so, we adopt the following splitting of the total field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_s$ into a prescribed incident field $\mathbf{E}_1$, which satisfies the homogeneous electric field equation
\[
\text{curl } \text{curl } \mathbf{E}_1 - \kappa^2 \mathbf{E}_1 = 0 \quad \text{in } \mathbb{R}^3,
\]
in the background medium, and a resulting scattered wave $\mathbf{E}_s$. Prominent examples of incident fields are so-called plane waves given by
\[
\mathbf{E}_1(x) = \mathbf{p} \exp(ik \mathbf{d} \cdot x) \quad x \in \mathbb{R}^3,
\]
where $\mathbf{d} \in \mathbb{R}^3$, $|\mathbf{d}| = 1$, denotes the direction of propagation of the wave, and the vector $\mathbf{p} \neq 0$ is called the polarization and must be orthogonal to $\mathbf{d}$. Finally, by imposing the Silver-Müller radiation condition [66]
\[
|\text{curl } \mathbf{E}_s \times x - i\kappa |x| \mathbf{E}_s| = o(r^{-1}) \quad \text{uniformly for } r := |x| \to \infty, \quad (4.10)
\]
we enforce the scattered field $\mathbf{E}_s$ to be purely out-going.
4.2 Sobolev Spaces, Traces, and Differential Operators

The main purpose of this section is to introduce suitable Sobolev spaces, trace operators, and trace spaces for time-harmonic electromagnetic scattering and transmission problems. The main references here are the articles [15, 16, 17].

The natural Hilbert space for an analysis of domain based time-harmonic Maxwell problems is the space

$$H_{\text{loc}}(\text{curl}, D) := \{ V \in L^2_{\text{loc}}(D); \text{curl} V \in L^2_{\text{loc}}(D) \},$$

here and below $D$ denotes a generic domain, which can be either $\Omega$ or $\Omega^+$, and the subscript $\text{loc}$ will be dropped if $D$ itself is bounded. For a thorough examination of these spaces we refer to [39, Chap. 1]. The Sobolev spaces of scalar functions and their dual spaces, $H^s(\Gamma)$ and $H^{-s}(\Gamma)$, can be invariantly defined for $0 \leq s \leq 1$, see [42, Thm. 1.3.3]. Furthermore, we denote by

$$\gamma : H^s_{\text{loc}}(D) \mapsto H^{s-1/2}(\Gamma), \quad \frac{1}{2} < s < \frac{3}{2},$$

the natural trace operator from lemma 1.1 and attach superscripts $-$ and $+$ to distinguish whether they act from $\Omega$ or $\Omega^+$. Moreover, $(\cdot, \cdot)_r$ denotes the inner product for the space of square integrable, scalar functions on the interface boundary $\Gamma$

$$(u, v)_r := \int_{\Gamma} u v dS, \quad u, v \in L^2(\Gamma),$$

which can be extended to a sesquilinear duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, when $L^2(\Gamma)$ is taken as pivot space. Furthermore, we define

$$L^2_t(\Gamma) := \{ u \in L^2(\Gamma); u \cdot n = 0 \},$$

to be the space of tangential, square integrable vector fields with inner product

$$(u, v)_t := \int_{\Gamma} u \cdot \nabla v dS, \quad u, v \in L^2_t(\Gamma).$$

If $\Omega$ is a curvilinear Lipschitz polyhedron in the parlance of [31], then the boundary $\Gamma$ can be split into $N_\Gamma$ smooth and open faces $\Gamma_j, j = 1, \ldots, N_\Gamma$, which motivates the definition of

$$H^s(\Gamma) := \left\{ u \in H^1(\Gamma); u |_{\Gamma_j} \in H^s(\Gamma_j), \quad j = 1, \ldots, N_\Gamma \right\} \quad \text{for } s > 1,$$

$$H^s_t(\Gamma) := \left\{ u \in L^2_t(\Gamma); u |_{\Gamma_j} \in H^s(\Gamma_j), \quad j = 1, \ldots, N_\Gamma \right\} \quad \text{for } s \geq 0,$$

where all spaces will be equipped with their natural graph norms.

Next, we aim at identifying suitable tangential trace spaces for $H^1_{\text{loc}}(D)$. In order to do so, we adopt the ideas and notion of [15, Sect. 2]: For two adjacent faces $\Gamma_i$ and $\Gamma_j$, we denote by $e_{ij}$ their common edge, and by $\tau_{ij}$ a unit vector parallel to $e_{ij}$. Moreover, $n_j$ denotes the restriction of the outer unit normal $n$ onto the face $\Gamma_j$ and $\tau_i := \tau_{ij} \times n_i$. For all functions $u \in L^2(\Gamma)$ and $u \in L^2(\Gamma)$ we adopt the notion $u_i$ and $u_j$ for their restriction onto the face $\Gamma_j$. Now let $\Gamma_i$ and $\Gamma_j$ be two adjacent faces and $u_i \in H^{1/2}(\Gamma_i), \quad v_j \in H^{1/2}(\Gamma_j)$, then we define the equivalence relation

$$u_i \overset{1/2}{=} v_j \quad \text{at } e_{ij} \quad \Leftrightarrow \quad \int_{\Gamma_i} \int_{\Gamma_j} \frac{|u_i(x) - v_j(y)|^2}{|x - y|^3} dS(x) dS(y) < \infty.$$

Furthermore, for fixed $j$ let $I_j$ stand for the set of indices $i$ such that the faces $\Gamma_i$ and $\Gamma_j$ share a common edge $e_{ij}$. Thus we have established all the prerequisites, which are needed for a proper definition of the following trace spaces

$$H^{1/2}_{\parallel}(\Gamma) := \{ u \in H^{1/2}_t(\Gamma); \quad u_j \cdot \tau_{ij} \overset{1/2}{=} u_i \cdot \tau_{ij} \quad \text{at } e_{ij} \quad \text{for } j = 1, \ldots, N_\Gamma, \quad i \in I_j \},$$

$$H^{1/2}_{\perp}(\Gamma) := \{ u \in H^{1/2}_t(\Gamma); \quad u_i \cdot \tau_i \overset{1/2}{=} u_j \cdot \tau_j \quad \text{at } e_{ij} \quad \text{for } j = 1, \ldots, N_\Gamma, \quad i \in I_j \}.$$
Theorem 4.1. The spaces $H_{\|}^{1/2}(\Gamma)$ and $H_{\bot}^{1/2}(\Gamma)$ are Hilbert spaces when endowed with the
natural graph norms
\[ \|u\|_{H_{\|}^{1/2}(\Gamma)}^2 := \sum_{j=1}^{N_{\Gamma}} \|u_j\|_{H_{\|}^{1/2}(\Gamma)}^2 + \sum_{j=1}^{N_{\Gamma}} \sum_{i \in I_j} \mathcal{N}_{ij}^\| (u), \quad \|v\|_{H_{\|}^{1/2}(\Gamma)}^2 := \sum_{j=1}^{N_{\Gamma}} \|v_j\|_{H_{\|}^{1/2}(\Gamma)}^2 + \sum_{j=1}^{N_{\Gamma}} \sum_{i \in I_j} \mathcal{N}_{ij}^\| (v), \]
where
\[ \mathcal{N}_{ij}^\| (u) := \left\{ \int_{\Gamma_j} \int_{\Gamma_j} \frac{|u_i(x) \cdot \tau_{ij}(x) - u_j(y) \cdot \tau_{ij}(y)|^2}{|x - y|^3} dS(y) dS(x) \right\}, \]
\[ \mathcal{N}_{ij}^\| (v) := \left\{ \int_{\Gamma_j} \int_{\Gamma_j} \frac{|v_i(x) \cdot \tau_{ij}(x) - v_j(y) \cdot \tau_{ij}(y)|^2}{|x - y|^3} dS(y) dS(x) \right\}. \]

Proof. A proof can be found in [15, Prop. 2.6]. □

Results analogous to the classical Rellich embedding theorem can also be established for
tangential trace spaces.

Lemma 4.2. The embeddings $H_{\|}^{1/2}(\Gamma) \hookrightarrow L^2_\ell(\Gamma)$ and $H_{\bot}^{1/2}(\Gamma) \hookrightarrow L^2_\ell(\Gamma)$ are compact.

In addition, we denote by $H_{\|}^{-1/2}(\Gamma)$ and $H_{\bot}^{-1/2}(\Gamma)$ the dual spaces corresponding to $H_{\|}^{1/2}(\Gamma)$ and $H_{\bot}^{1/2}(\Gamma)$, respectively. Note that $(\cdot, \cdot)_t$ can be extended to a sesquilinear duality pairing on
\[ (\cdot, \cdot)_t : H_{\|}^{-1/2}(\Gamma) \times H_{\bot}^{1/2}(\Gamma) \rightarrow \mathbb{C}, \quad (\cdot, \cdot)_t : H_{\bot}^{-1/2}(\Gamma) \times H_{\|}^{1/2}(\Gamma) \rightarrow \mathbb{C}, \]
when $L^2_\ell(\Gamma)$ is used as pivot space.

For any $U \in C^\infty(D)$ the tangential components trace $\gamma_t$ and the twisted tangential trace $\gamma_x$ can be defined a.e. on $\Gamma$ by
\[ (\gamma_t U)(x) := n(x) \times (U(x) \times n(x)), \quad (\gamma_x U)(x) := U(x) \times n(x). \]

For Lipschitz domains, the generalization of these traces onto $H_{\text{loc}}^1(D)$, has been achieved in [15, Prop. 2.7].

Lemma 4.3. The tangential components trace $\gamma_t^\pm : H_{\text{loc}}^1(D) \hookrightarrow H_{\|}^{1/2}(\Gamma)$ and the twisted tangential trace $\gamma_x^\pm : H_{\text{loc}}^1(D) \hookrightarrow H_{\bot}^{1/2}(\Gamma)$ are continuous, surjective and possess continuous right inverses.

On smooth domains the definitions of the surface differential operators are based on paramet-
ric representations of the surface, see [65, Sect. 3.4] for details. However, on merely Lipshitz continuous domains these definitions are useless and have to be adapted. We start with an alterna-
tive, local definition of the tangential components trace, cf. [15, Sect. 3.1]. For any $U \in H^1(\Omega)$, we define
\[ (\gamma_{t,j} U)(x) := U(x) - (U(x) \cdot n_j(x)) n_j(x), \quad \text{a.e. } x \in \Gamma_j, \]
for $j = 1, \ldots, N_{\Gamma}$, which can be used as an equivalent definition of $\gamma_t$
\[ \gamma_t : H^1(\Omega) \hookrightarrow H_{\|}^{1/2}(\Gamma), \quad (\gamma_t U)(x) = (\gamma_{t,j} U)(x) \quad \text{a.e. } x \in \Gamma_j \quad \forall j = 1, \ldots, N_{\Gamma}. \]

Now the surface gradient operator can be defined face-by-face fashion as follows
\[ \text{grad}_{\Gamma_j} U = \gamma_{t,j}(\text{grad} U) \quad \forall U \in H^2(\Omega). \]
Clearly, there holds \( \text{grad}_j : H^2(\Omega) \mapsto H^1/2(\Gamma_j) \) and we define \( \text{grad}_\Gamma : H^2(\Omega) \mapsto H^1/2(\Gamma) \) by
\[
\text{grad}_\Gamma U(x) := \text{grad}_j U(x) \quad \text{a.e. } x \in \Gamma_j \quad \forall j = 1, \ldots, N_\Gamma. 
\]
Moreover, the following identity holds true \( \text{grad}_\Gamma U = \gamma_\Gamma(\text{grad} U) \). In the same way we can define the surface vector curl operator \( \text{curl}_\Gamma \) by setting \( \text{curl}_\Gamma U = \gamma_\times(\text{grad} U) \). Furthermore, it can be shown that the operators
\[
\text{grad}_\Gamma : H^{3/2}(\Gamma) \mapsto H^1/2(\Gamma), \quad \text{curl}_\Gamma : H^{3/2}(\Gamma) \mapsto H^1/2(\Gamma),
\]
are linear and continuous mappings, see [15, Props. 3.1, 3.2]. Now we can introduce the surface divergence \( \text{div}_\Gamma : H^{-1/2}(\Gamma) \mapsto H^{-3/2}(\Gamma) \) and the scalar surface curl operator \( \text{curl}_\Gamma : H^{-1/2}(\Gamma) \mapsto H^{-3/2}(\Gamma) \) as the adjoint operators of \( -\text{grad}_\Gamma \) and \( \text{curl}_\Gamma \).

\[
\text{div}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), \quad \text{curl}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma),
\]
are linear and continuous. Analogously, the adjoint operators are also linear and continuous for the following choice of spaces:
\[
\text{div}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), \quad \text{curl}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma).
\]

**Lemma 4.4.** The operators \( \text{grad}_\Gamma \) and \( \text{curl}_\Gamma \) can be extended to \( H^{1/2}(\Gamma) \). Moreover,

\[
\text{grad}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), \quad \text{curl}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)
\]
are linear and continuous. Analogously, the adjoint operators are also linear and continuous for the following choice of spaces:

\[
\text{div}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma), \quad \text{curl}_\Gamma : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma).
\]

**Proof.** For a proof see [18, Prop. 3.6].

Based on lemma 4.4 we are able to define trace spaces
\[
H^{-1/2}(\text{curl}_\Gamma, \Gamma) := \{ v \in H^{-1/2}_\perp(\Gamma); \text{curl}_\Gamma v \in H^{-1/2}(\Gamma) \},
\]
\[
H^{-1/2}(\text{div}_\Gamma, \Gamma) := \{ \zeta \in H^{-1/2}_\parallel(\Gamma); \text{div}_\Gamma \zeta \in H^{-1/2}(\Gamma) \},
\]
which are endowed with their corresponding graph norms.

**Theorem 4.5.** The tangential components trace \( \gamma_{\parallel}^\pm : H_{\text{loc}}^1(\text{curl}, D) \mapsto H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) and the twisted tangential trace \( \gamma_{\times}^\pm : H_{\text{loc}}^1(\text{curl}, D) \mapsto H^{-1/2}(\text{div}_\Gamma, \Gamma) \) are continuous, surjective with continuous right inverses.

**Proof.** A proof can be found in [15, Thms. 2.7], [16, Thm. 4.5], or [12, Sect. 4].

From this theorem we conclude that \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) is exactly the right space for tangential components of field quantities in \( H(\text{curl}, \Omega) \). Thus we adopt the alternative notation \( \gamma_D \) for \( \gamma_{\parallel} \). The usual \( L^2(\Gamma) \)-inner product can be extended to a sesquilinear duality pairing
\[
(\cdot, \cdot)_t : H^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{-1/2}(\text{curl}_\Gamma, \Gamma) \mapsto \mathbb{C}
\]
by means of a Green’s formula
\[
\mp \int_D U \cdot \text{curl} \nabla \cdot \text{curl} U - \text{curl} U \cdot \nabla \text{d}x = (\gamma_{\times}^\pm U, \gamma_{\times}^\pm V)_t \quad \forall U, V \in H_{\text{loc}}^1(\text{curl}, D),
\]
where an overbar denotes complex conjugation, cf. [16, Sect. 4]. In the previous formula \( - \) applies in case of \( \Omega \) and + for \( \Omega^+ \). For continuous tangential vector fields \( u \) we define the surface twist operator by

\[
R(u)(x) := n(x) \times u(x),
\]

which can be extended to an isometric isomorphism \( R : H^{-1/2}(\text{curl}, \Gamma) \rightarrow H^{-1/2}(\text{div}, \Gamma) \). Moreover, the surface twist operator \( R \) is invertible, satisfying \( R^{-1} = R^* = -R \), and for arbitrary \( \lambda \in H^{-1/2}(\text{div}, \Gamma) \) and \( v \in H^{-1/2}(\text{curl}, \Gamma) \) there holds

\[
curl \Gamma v = -\text{div} \Gamma R(v), \quad \text{div} \Gamma \lambda = \text{curl} \Gamma R(\lambda),
\]

see [17, Sect. 3] and [17, Eq. 26] for details. Besides the tangential traces defined above, we also need the normal components trace \( \gamma_n \) defined by

\[
(\gamma_n U)(x) := n(x) \cdot U(x),
\]

for almost all \( x \in \Gamma \) and \( U \in C^\infty(\bar{D}) \), which can be extended to a continuous and surjective mapping \( \gamma_n : H^1_{\text{loc}}(\text{div}, D) \rightarrow H^{-1/2}(\Gamma) \), cf. [39, Thm. 2.5]. Finally, an equivalent to the Neumann trace is indispensable and has to be provided in a weak sense: For

\[
U \in H^1_{\text{loc}}(\text{curl}^2, D) := \{ V \in H^1_{\text{loc}}(\text{curl}, D); \ \text{curl} \ \text{curl} V \in L^2_{\text{loc}}(D) \},
\]

we define \( \gamma_N U \in H^{-1/2}(\text{div}, \Gamma) \) by

\[
\pm \int_D \text{curl} U \cdot \text{curl} \mathbf{V} - \text{curl} \text{curl} U \cdot \mathbf{V} \, dx = (\gamma_N^+ U, \gamma_N^+ \mathbf{V})_t \quad \forall \mathbf{V} \in H^1_{\text{loc}}(\text{curl}, D),
\]

where a \( - \) applies in case of \( \Omega \) and + for \( \Omega^+ \). Moreover, for smooth vector fields \( U \) the Neumann trace \( \gamma_N U \) evaluates to \( \gamma_N (\text{curl} U) = \text{curl} U \times n \).

**Lemma 4.6.** The traces \( \gamma_N^\pm : H^1_{\text{loc}}(\text{curl}^2, D) \rightarrow H^{-1/2}(\text{div}, \Gamma) \) furnish continuous mappings.

*Proof.* For a proof see [51, Lem 3.3]. \( \square \)

Finally, for an arbitrary trace operator \( \gamma \) we introduce jump and average operators \( [\cdot]_\Gamma \) and \( \{\cdot\}_\Gamma \), respectively, according to the following definition

\[
[\gamma]_\Gamma := \gamma^+ - \gamma^-, \quad \{\gamma\}_\Gamma := \frac{1}{2}(\gamma^+ + \gamma^-).
\]

**Remark 4.7.** An equivalent definition of the surface gradient \( \text{grad} \Gamma : H^1(\Gamma) \rightarrow L^2(\Gamma) \) and the vector surface \( \text{curl} \text{curl} \Gamma : H^1(\Gamma) \rightarrow L^2(\Gamma) \) was given by Nècas [67] and is based on local Lipschitz continuous charts. Their adjoint operators \( \text{div} \Gamma : L^2(\Gamma) \rightarrow H^{-1}(\Gamma) \) and \( \text{curl} \Gamma : L^2(\Gamma) \rightarrow H^{-1}(\Gamma) \) are defined by

\[
(\text{div} \Gamma u, p)_\Gamma = -(u, \text{grad} \Gamma p)_t \quad \forall p \in H^1(\Gamma), \ u \in L^2(\Gamma),
\]

\[
(\text{curl} \Gamma u, p)_\Gamma = (u, \text{curl} \Gamma \varphi)_t \quad \forall p \in H^1(\Gamma), \ u \in L^2(\Gamma).
\]

These definitions can be extended to \( H^{3/2}(\Gamma) \) and such that they coincide with (4.11) and (4.12).
4.3 Potentials and Boundary Integral Operators

In this section we introduce the surface potentials and boundary integral operators related to the electric field equation. The main references here are the articles [17, 52, 19]. Any distribution $U \in H_\text{loc}(\text{curl}^2, \Omega \cup \Omega^+)$ which satisfies the electric field equation

$$\text{curl} \text{curl} U - \kappa^2 U = 0 \quad \text{in } \Omega \cup \Omega^+, \quad (4.14)$$

together with the Silver-Müller radiation condition (4.10) can be written using the Stratton-Chu representation formula (cf. [83], [17, Sect. 3], [25, Chap. 3, Sect. 1.3.2], and [69, Sect. 5.5])

$$U = \Psi_{\text{DL}}^\kappa(\gamma_D U|_\Gamma) - \Psi_{\text{SL}}^\kappa(\gamma_N U|_\Gamma) - \text{grad} \Psi_{\text{SL}}^\kappa(\gamma_n U|_\Gamma) \quad \text{in } \Omega \cup \Omega^+, \quad (4.15)$$

with the potentials defined by:

- scalar single-layer potential $\Psi_{\text{SL}}^\kappa(\varphi)(x) := \int_{\Gamma} G_\kappa(x - y) \varphi(y) dS(y), \quad x \notin \Gamma,$
- vectorial single-layer potential $\Psi_{\text{SL}}^\kappa(\mu)(x) := \int_{\Gamma} G_\kappa(x - y) \mu(y) dS(y), \quad x \notin \Gamma,$
- Maxwell double-layer potential $\Psi_{\text{DL}}^\kappa(v)(x) := \text{curl} \Psi_{\text{SL}}^\kappa(R(v))(x), \quad x \notin \Gamma,$

based on the fundamental solution for the Helmholtz equation defined in (1.4)

$$G_\kappa(z) := \frac{1}{4\pi} \frac{\exp(\kappa |z|)}{|z|}, \quad z \neq 0.$$  

Following the idea of [18, Eq. 26], the representation formula (4.15) can be further simplified to

$$U = \Psi_{\text{DL}}^\kappa(\gamma_D U|_\Gamma) - \Psi_{\text{SL}}^\kappa(\gamma_N U|_\Gamma) \quad \text{in } \Omega \cup \Omega^+, \quad (4.16)$$

based on the Maxwell single-layer potential

$$\Psi_{\text{SL}}^\kappa(\mu)(x) := \Psi_{\text{SL}}^\kappa(\mu)(x) + \frac{1}{\kappa^2} \text{grad} \Psi_{\text{SL}}^\kappa(\text{div}_\Gamma \mu)(x), \quad x \notin \Gamma,$$

and the following observation $\text{div}_\Gamma (\gamma_D^\kappa U) = \gamma_D^\kappa(\text{curl} \text{curl} U) = \kappa^2 \gamma_D^\kappa U$, which holds true for all vector fields $U \in H_\text{loc}(\text{curl}^2, \Omega \cup \Omega^+)$ satisfying the electric field equation (4.14).

**Lemma 4.8.** The scalar and vectorial single-layer potentials $\Psi_{\text{SL}}^\kappa$ and $\Psi_{\text{DL}}^\kappa$ give rise to continuous mappings $\Psi_{\text{SL}}^\kappa : H^{-1/2}(\Gamma) \mapsto H^1_\text{loc}(\mathbb{R}^3)$, $\Psi_{\text{DL}}^\kappa : H_{\parallel}^{-1/2}(\Gamma) \mapsto H^1_\text{loc}(\mathbb{R}^3)$.

**Proof.** For a proof see [30] or [51, Thm. 5.1]. \hfill \Box

**Lemma 4.9.** For $v \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ we have $\text{div} \Psi_{\text{SL}}^\kappa(v) = \Psi_{\text{SL}}^\kappa(\text{div}_\Gamma v)$ in $L^2_\text{loc}(\mathbb{R}^3)$.

**Proof.** A proof can be found in [63, Lem. 2.3]. \hfill \Box

From [52, Eq. 4.8, 4.9] we conclude that the following two identities hold

$$\begin{align*}
(\text{curl} \text{curl} - \kappa^2 \text{Id}) \Psi_{\text{SL}}^\kappa(\mu)(x) &= \text{grad} \Psi_{\text{SL}}^\kappa(\text{div}_\Gamma \mu)(x) \quad \forall \mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma), \\
(\text{curl} \text{curl} - \kappa^2 \text{Id}) \Psi_{\text{DL}}^\kappa(v)(x) &= 0 \quad \forall v \in H^{-1/2}(\text{curl}_\Gamma, \Gamma),
\end{align*}$$

for all $x \notin \Gamma$ and globally in $L^2_\text{loc}(\Omega \cup \Omega^+)$. Hence we conclude that both $\Psi_{\text{SL}}^\kappa$ and $\Psi_{\text{DL}}^\kappa$ provide radiating solutions to the electric field equation (4.14) in $\Omega \cup \Omega^+$, which satisfy the following continuity properties, cf. [52, Thm. 4.3].
Lemma 4.10. The Maxwell single-layer potential \( \Psi_{SL}^\kappa : H^{-1/2}(\text{div}\Gamma,\Gamma) \rightarrow H_{0c}(\text{curl}^2,\Omega \cup \Omega^+) \) and the Maxwell double-layer potential \( \Psi_{DL}^\kappa : H^{-1/2}(\text{curl}\Gamma,\Gamma) \rightarrow H_{0c}(\text{curl}^2,\Omega \cup \Omega^+) \) are continuous mappings.

Similar jump relations as in (1.9) also hold for the potentials related to the electric field equation, cf. [28, Thm. 6.11], [69, Thm. 5.5.1] or [51, Sect. 5].

Lemma 4.11. The interior and exterior Dirichlet and Neumann traces of the potentials \( \Psi_{SL}^\kappa \) and \( \Psi_{DL}^\kappa \) are well-defined and satisfy

\[
\begin{align*}
[\gamma_D \Psi_{SL}^\kappa (\mu)]_\Gamma &= 0, & [\gamma_N \Psi_{SL}^\kappa (\mu)]_\Gamma &= -\mu, & \forall \mu \in H^{-1/2}(\text{div}\Gamma,\Gamma), \\
[\gamma_D \Psi_{DL}^\kappa (\nu)]_\Gamma &= \nu, & [\gamma_N \Psi_{DL}^\kappa (\nu)]_\Gamma &= 0, & \forall \nu \in H^{-1/2}(\text{curl}\Gamma,\Gamma).
\end{align*}
\]

A straightforward application of averaged Dirichlet and Neumann traces to the potentials \( \Psi_{SL}^\kappa \) and \( \Psi_{DL}^\kappa \) yields the boundary integral operators for the electric field equation, cf. [52, Lem. 5.1, Thm. 5.2].

Lemma 4.12. The integral operators

\[
\begin{align*}
S_\kappa & : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma), & S_\kappa := \{\gamma\}_\Gamma \circ \Psi_{SL}^\kappa, \\
S_\kappa & : H_\parallel^{-1/2}(\Gamma) \rightarrow H_\parallel^{1/2}(\Gamma), & S_\kappa := \{\gamma_D\}_\Gamma \circ \Psi_{SL}^\kappa, \\
S_\kappa & : H_\perp^{-1/2}(\Gamma) \rightarrow H_\perp^{1/2}(\Gamma), & S_\kappa := \{\gamma_N\}_\Gamma \circ \Psi_{DL}^\kappa,
\end{align*}
\]

are continuous.

Theorem 4.13. The following integral operators are continuous:

\[
\begin{align*}
V_\kappa & : H^{-1/2}(\text{div}\Gamma,\Gamma) \rightarrow H^{-1/2}(\text{curl}\Gamma,\Gamma), & V_\kappa := \{\gamma_D\}_\Gamma \circ \Psi_{SL}^\kappa, \\
K_\kappa & : H^{-1/2}(\text{div}\Gamma,\Gamma) \rightarrow H^{-1/2}(\text{div}\Gamma,\Gamma), & K_\kappa := \{\gamma_N\}_\Gamma \circ \Psi_{SL}^\kappa, \\
K_\kappa & : H^{-1/2}(\text{curl}\Gamma,\Gamma) \rightarrow H^{-1/2}(\text{curl}\Gamma,\Gamma), & K_\kappa := \{\gamma_D\}_\Gamma \circ \Psi_{DL}^\kappa, \\
W_\kappa & : H^{-1/2}(\text{curl}\Gamma,\Gamma) \rightarrow H^{-1/2}(\text{div}\Gamma,\Gamma), & W_\kappa := -\{\gamma_N\}_\Gamma \circ \Psi_{DL}^\kappa.
\end{align*}
\]

Note that the definitions of the potentials \( \Psi_{SL}^\kappa \) and \( \Psi_{DL}^\kappa \) together with Lemma 4.12 allow an evaluation of \( S_\kappa, S_\kappa, \) and \( S_\kappa^\times \) also in the case \( \kappa = 0. \)

For the implementation of boundary element methods, explicit integral representations of the operators \( V_\kappa, K_\kappa, K_\kappa', \) and \( W_\kappa \) are essential, cf. [19, Sect. 5], [26, Eq. 2.22]. In case of \( S_\kappa \) and \( S_\kappa \) the following formulas can be derived

\[
\begin{align*}
S_\kappa(\varphi)(x) &= \int_{\Gamma} G_\kappa(x - y) \varphi(y) \, dS(y) & \varphi \in L^\infty(\Gamma), \\
S_\kappa(\mu)(x) &= \int_{\Gamma} G_\kappa(x - y) \mu(y) \, dS(y) & \mu \in L^\infty(\Gamma),
\end{align*}
\]

which hold on the boundary \( \Gamma \) in a point-wise sense. Moreover, the definitions of \( V_\kappa \) and \( W_\kappa \) together with the formulas \( \gamma_\kappa \circ \text{grad} = \text{grad}_\Gamma \circ \gamma \) and \( W_\kappa = -\kappa^2 R^* \circ V_\kappa \circ R \) imply the identities

\[
V_\kappa = S_\kappa + \frac{1}{\kappa^2} \text{grad}_\Gamma \circ S_\kappa \circ \text{div}_\Gamma, \quad W_\kappa = \text{curl}_\Gamma \circ S_\kappa \circ \text{curl}_\Gamma - \kappa^2 S_\kappa^\times. \quad (4.17)
\]

Based on the definitions of the boundary integral operators in theorem 4.13 and the jump relations from lemma 4.11, we immediately derive the following trace relations

\[
\begin{align*}
\gamma_D^\pm \Psi_{SL}^\kappa &= V_\kappa, & \gamma_N^\pm \Psi_{SL}^\kappa &= K_\kappa \pm \frac{1}{2} \text{Id}, \\
\gamma_D^\pm \Psi_{DL}^\kappa &= K_\kappa \pm \frac{1}{2} \text{Id}, & \gamma_N^\pm \Psi_{DL}^\kappa &= -W_\kappa.
\end{align*}
\]
Since we aim to apply the Fredholm alternative, the identification of boundary integral operators that amount to compact perturbations deserves special attention, cf. \cite[Lem. 5.4]{52}, \cite[Thm. 3.12]{22} and \cite[Lem. 3.2]{55}.

**Lemma 4.14.** There exists a compact linear operator $T_\kappa : H^{-1/2}(\text{div}_{\Gamma}, \Gamma) \mapsto H^{-1/2}(\text{div}_{\Gamma}, \Gamma)$ such that

$$(K'_\kappa(\mu), v)_\kappa = (\mu, K_\kappa(v))_\kappa - (T_\kappa(\mu), v)_\kappa \quad \forall \mu \in H^{-1/2}(\text{div}_{\Gamma}, \Gamma), \; v \in H^{-1/2}(\text{curl}_{\Gamma}, \Gamma).$$

**Lemma 4.15.** The following integral operators are compact:

\begin{align*}
\delta S_\kappa &:= S_\kappa - S_0 : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma), \\
\delta S_\kappa^\times &:= S_\kappa^\times - S_0^\times : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma), \\
\delta S_\kappa^\times &:= S_\kappa^\times - S_0^\times : H^{1/2}_\perp(\Gamma) \mapsto H^{1/2}_\perp(\Gamma).
\end{align*}

Finally, ellipticity of the sesquilinear forms related to $S_0$, $S_0^\times$, and $S_0^\times$ provides a crucial tool for the numerical analysis of variational formulations based on boundary integral equations, cf. \cite[Cor. 8.13]{64}, \cite[Chap. XI, Sect. 2, Thm. 3]{34} or \cite[Prop. 4.1]{17}.

**Lemma 4.16.** The operators $S_0$, $S_0^\times$, and $S_0^\times$ are continuous, self-adjoint and fulfill

\begin{align*}
\Re \{ (\varphi, S_0(\varphi))_\Gamma \} &\geq C \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma), \\
\Re \{ (\mu, S_0(\mu))_\Gamma \} &\geq C \|\mu\|^2_{H^{-1/2}(\Gamma)} \quad \forall \mu \in H^{-1/2}(\Gamma), \; \text{div}_{\Gamma} \mu = 0, \\
\Re \{ (v, S_0^\times(v))_\Gamma \} &\geq C \|v\|^2_{H^{1/2}_\perp(\Gamma)} \quad \forall v \in H^{1/2}_\perp(\Gamma), \; \text{curl}_{\Gamma} v = 0,
\end{align*}

with constants $C > 0$ depending only on $\Gamma$.

### 4.4 Calderón Projectors

In the case of the Helmholtz transmission problem the Calderón projectors provided indispensable tools for the coupling of finite element and boundary element methods, which indicates their importance for time-harmonic Maxwell transmission problems. Both Calderón projectors can obtained by a straightforward application of the interior/exterior Dirichlet and Neumann traces to the representation formula (4.16) using the trace relations (4.18)

$$P_\pm : H^{-1/2}(\text{div}_{\Gamma}, \Gamma) \times H^{-1/2}(\text{curl}_{\Gamma}, \Gamma) \mapsto H^{-1/2}(\text{div}_{\Gamma}, \Gamma) \times H^{-1/2}(\text{curl}_{\Gamma}, \Gamma),$$

$$P_\pm := \begin{bmatrix}
\frac{1}{2} I_d + K_\kappa \\
\mp \frac{1}{2} I_d + K_\kappa^\prime
\end{bmatrix},$$

\begin{equation}
(4.19)
\end{equation}

cf. \cite[Sect. 3.3]{22}, \cite[Eq. 29]{35}, and \cite[Sect. 5.5]{69}. By definition, the identity $P_- + P_+ = I_d$ holds and the kernel of $P_-$ coincides with the range of $P_+$ and vice versa. The following lemma promotes the Calderón projectors to a pivotal rule in the derivation of boundary integral equations, cf. \cite[Thm. 3.7]{85}.

**Lemma 4.17.** If and only if $(v, \mu) \in H^{-1/2}(\text{curl}_{\Gamma}, \Gamma) \times H^{-1/2}(\text{div}_{\Gamma}, \Gamma)$ belongs to the range of $P_\pm$, then there exists a Maxwell solution $E$ such that $\gamma_{\perp}^E E = v$ and $\gamma_{\parallel}^E E = \mu$. 

As in the case of the Helmholtz equation, this lemma provides a way for constructing explicit expressions for the exterior Dirichlet-to-Neumann map

\[ \text{DtN}_\kappa^{+} : \mathcal{H}^{-1/2}(\text{curl}, \Gamma) \mapsto \mathcal{H}^{-1/2}(\text{div}, \Gamma), \]

for time-harmonic Maxwell problems in \( \Omega^+ \). From (4.19) and lemma 4.17 we formally derive the following three expressions for the Dirichlet-to-Neumann map

\[ \begin{align*}
\text{DtN}_\kappa^{+} &:= \mathcal{V}_\kappa^{-1} \circ (K_\kappa - \frac{1}{2} \text{Id}), \\
\text{DtN}_\kappa^{+} &:= -\left( \frac{1}{2} \text{Id} + K_\kappa^{-1} \right) \circ \mathcal{W}_\kappa, \\
\text{DtN}_\kappa^{+} &:= -\mathcal{W}_\kappa + \left( \frac{1}{2} \text{Id} - K_\kappa^{-1} \right) \circ \mathcal{V}_\kappa^{-1} \circ (K_\kappa - \frac{1}{2} \text{Id}).
\end{align*} \]

Only the third formula reflects the essential symmetry of the boundary value problem in the case \( \kappa = 0 \). It will again be the starting point for symmetric coupling.

### 4.5 Functional Analytic Framework

This section provides a brief overview over the theoretical framework, which used to establish existence and uniqueness of solutions, and asymptotic quasi-optimality of the discretization error for certain non-coercive variational formulations. The interior source problem as well as variational formulations based on boundary integral equations for Maxwell's equation fit into this framework. The main references here are the articles [13, 26, 22, 14].

Let \( W \) and \( H \) be two complex, separable Hilbert spaces with dense injection \( W \subset H \). Moreover, we are provided with a \( W \)-stable decomposition \( W = X \oplus N \), such that for all \( w \in W \) we have uniquely determined \( u \in X \), \( v \in N \) with \( w = u + v \) and

\[ C^{-1} \|w\|_W \leq \|u\|_W + \|v\|_W \leq C \|w\|_W, \]

where the constant \( C > 0 \) does not depend on \( u \), \( v \), or \( w \). Based on this splitting we define the isomorphism \( X : W \mapsto W \) by \( X(w) := u - v \). Although, we could allow for more generalized expressions of \( X \), it is sufficient to consider the case of a simple sign flip operator. On the other hand, since we have practical applications in mind, more general expressions would only artificially complicate the numerical analysis. Furthermore, we introduce the sesquilinear pairing \( \langle \cdot, \cdot \rangle_{W' \times W} : W' \times W \mapsto \mathbb{C} \) defined by

\[ \langle w', w \rangle_{W' \times W} := \langle w', \overline{w} \rangle_{W' \times W} \quad \forall w' \in W', \ w \in W, \]

where \( \langle \cdot, \cdot \rangle_{W' \times W} \) denotes the duality pairing between \( W' \) and \( W \). The following assumption states the essential requirements for conforming discretization schemes in case of non-coercive variational formulations.

**Assumption 4.18.** Assume that there exists a sequence of closed subspaces \( W_h \subset W \) with decompositions \( W_h = X_h \oplus N_h \), satisfying the following assumption:

1. The family \( W_h \) is approximating in \( W \), i.e.

\[ \lim_{h \to 0} \inf_{w_h \in W_h} \|w - w_h\|_W = 0. \]

2. \( W_h \) satisfies a gap property, i.e. the two subsets \( X_h, N_h \) of \( W_h \) fulfill

\[ \delta_h := \max \{ \delta(X, X_h), \delta(N, N_h) \} \to 0, \quad \text{as} \ h \to 0, \]

where

\[ \delta(X, X_h) := \sup_{u_h \in X_h} \inf_{u \in X} \|u_h - u\|_W, \quad \text{and} \ \delta(N, N_h) := \sup_{v_h \in N_h} \inf_{v \in N} \|v_h - v\|_W. \]
Now suppose we are given a bounded, non-coercive sesquilinear form \( a : W \times W \to \mathbb{C} \) and a linear functional \( f : W \to \mathbb{C} \). We consider the following variational problem on the Hilbert space \( W \):

For any \( f \in W' \), find \( u \in W \) such that for all \( v \in W \) there holds

\[
a(u, v) = (f, v)_{W' \times W}.
\]

Since we aim at establishing existence and uniqueness of discrete solutions and asymptotic quasi-optimality of the discretization error for conforming Galerkin schemes, further assumptions on the sesquilinear form underlying the variational formulation are indispensable.

**Definition 4.19.** The sesquilinear form \( a : W \times W \to \mathbb{C} \) is said to

1. satisfy a **generalized Gårding inequality**, if there exists a compact operator \( T : W \to W' \) and a constant \( C > 0 \), such that for all \( w \in W \) there holds

\[
\Re \left\{ a(w, X(w)) + (T(w), X(w))_{W' \times W} \right\} \geq C\|w\|_W^2.
\]

2. be **injective**, if \( a(u, v) = 0 \) for all \( v \in W \), implies \( u = 0 \).

Now a straightforward application of the Fredholm alternative yields existence and uniqueness of solutions on the continuous level, cf. [77, 89]. By choosing a sequence of closed, finite-dimensional subspaces \( W_h \subset W \) which satisfy item 1. and 2. from assumption 4.18, we arrive at the following Galerkin problem:

Find \( u_h \in W_h \) such that for all \( v_h \in W_h \) there holds

\[
a(u_h, v_h) = (f, v_h)_{W' \times W}. \tag{4.20}
\]

Finally, we the following theorem ensures existence and uniqueness of discrete solutions, and quasi-optimality of the discretization error on a sequence of conforming, finite dimensional subspaces, cf. [22, Thm. 4.1] or [13, Thm. 3.7].

**Theorem 4.20.** Let the sesquilinear form \( a \) be injective and satisfy a generalized Gårding inequality in the sense of item 1. of definition 4.19. Moreover, we assume that the family of closed, finite-dimensional subspaces \( W_h \) is approximating in \( W \) and satisfies a gap property in the sense of assumption 4.18. Then there exists \( h_0 > 0 \) such that for all \( 0 < h < h_0 \) the following discrete inf-sup estimate holds

\[
\sup_{w_h \in W_h} \frac{\Re \left\{ a(v_h, w_h) \right\}}{\|w_h\|_W} \geq \gamma \|v_h\|_W \quad \forall v_h \in W_h. \tag{4.21}
\]

Condition (4.21) immediately implies uniqueness of solutions \( u_h \in W_h \) for the Galerkin problem (4.20) and the following quasi-optimality estimate of the discretization error

\[
\|u - u_h\|_W \leq C \inf_{v_h \in W_h} \|u - v_h\|_W,
\]

with a constant \( C > 0 \) provided, that a minimal resolution is guaranteed.

**Remark 4.21.** The approximation and the gap property of assumption 4.18 of the family of subspaces \( W_h \subset W \) are equivalent to the existence of two bounded, linear operators, namely an **interpolation operator** \( \Pi_h : W \to W_h \), and a **bridge mapping** \( B_h : W_h \to W \), which satisfy

\[
\forall w \in W : \|w - \Pi_h(w)\|_W \to 0, \quad \sup_{w_h \in W_h} \frac{\|w_h - B_h(w_h)\|_W}{\|w_h\|_W} \to 0,
\]
as \( h \to 0 \), see [22, Sect. 4.1] and [52, Sect. 11]. Provided we have two such operators at hand, then the following two estimates are straightforward

\[
\lim_{h \to 0} \inf_{w_h \in W_h} \|w - w_h\|_W \leq C \lim_{h \to 0} \|w - \Pi_h(w)\|_W,
\]

\[
\delta(X, X_h) = \sup_{u_h \in X_h} \inf_{u \in X} \left\| u_h - u \right\|_W \leq C \sup_{w_h \in W_h} \frac{\|w_h - B_h(w_h)\|_W}{\|w_h\|_W},
\]

\[
\delta(N, N_h) = \sup_{v_h \in N_h} \inf_{v \in N} \left\| v_h - v \right\|_W \leq C \sup_{w_h \in W_h} \frac{\|w_h - B_h(w_h)\|_W}{\|w_h\|_W}.
\]

Thus, item 1. and 2. of assumption 4.18 hold. In a finite element/boundary element framework the existence of an interpolation estimate is straightforward.

### 4.6 Decompositions

This section provides stable splittings for \( H(\text{curl}, \Omega) \), and the Dirichlet and Neumann trace spaces \( H^{-1/2}(\text{curl}, \Gamma) \) and \( H^{-1/2}(\text{div}, \Gamma) \), which are needed to establish generalized Gårding inequalities for the sesquilinear forms underlying the coupled variational formulations. The main references here are the articles [35, 4, 17, 26] and the monograph [69].

First, let us consider the sesquilinear form associated with the electric field equation (4.9)

\[
q_\kappa(E, V) := (\mu_r^{-1} \text{curl} E, \text{curl} V)_{\Omega} - \kappa^2 (\varepsilon_r E, V)_{\Omega}, \quad E, V \in H(\text{curl}, \Omega),
\]

where \((\cdot, \cdot)_{\Omega}\) denotes the \(L^2(\Omega)\)-inner product, defined by

\[
(U, V)_{\Omega} := \int_{\Omega} U \cdot V \, dx, \quad U, V \in L^2(\Omega).
\]

Although, \(q_\kappa\) shares some similarity with the sesquilinear form associated with the Helmholtz equation, we cannot hope to establish a Gårding inequality on \( H(\text{curl}, \Omega) \), due to the lack of compactness of the embedding \( H(\text{curl}, \Omega) \) into \( L^2(\Omega) \). In light of the theory presented in section 4.5, this difficulty can be overcome by a suitable splitting of the fields into two components \( H(\text{curl}, \Omega) = X(\Omega) \oplus N(\Omega) \). For the electric field equation this idea has been introduced by Nédélec and was first applied to boundary integral operators in [35]. Since then, it has been gradually extended to a powerful tool in numerical analysis, cf. [4, 17, 26] and the monograph [69]. The following features of a splitting prove essential:

1. the subspace \( N(\Omega) \) in the splitting agrees with the kernel of \( \text{curl} \),
2. stability of the splitting w.r.t. the norm on \( H(\text{curl}, \Omega) \),
3. extra regularity of vector fields in the complement space \( X(\Omega) \).

Thus, any \( E \in H(\text{curl}, \Omega) \) can be decomposed into two components \( E = E^0 + E^\perp \), where \( E^0 \in N(\Omega) \) and \( E^\perp \in X(\Omega) \). This naturally leads us to the definition of a bounded, linear isomorphism \( X : H(\text{curl}, \Omega) \mapsto H(\text{curl}, \Omega) \), given by \( X(E) := E^\perp - E^0 \), which can be employed to “flip signs” on the product space \( X(\Omega) \times N(\Omega) \). Moreover, the sesquilinear form \( q_\kappa(\cdot, X(\cdot)) \) assumes the following block structure

\[
(\mu_r^{-1} \text{curl} E^\perp, \text{curl} V^\perp)_\Omega - \kappa^2 (\varepsilon_r E^\perp, V^\perp)_\Omega - \kappa^2 (\varepsilon_r E^0, V^\perp)_\Omega,
\]

\[
+ \kappa^2 (\varepsilon_r E^\perp, V^0)_\Omega + \kappa^2 (\varepsilon_r E^0, V^0)_\Omega.
\]
Extra regularity of the arguments in \( \mathbf{X}(\Omega) \) provides us with the following compact sesquilinear forms

\[
\begin{align*}
&(\mathbf{E}^\perp, \mathbf{V}^\perp)_\Omega, \\
&(\varepsilon \mathbf{E}^\perp, \mathbf{V}^\perp)_\Omega, \\
&(\varepsilon \mathbf{E}^\parallel, \mathbf{V}^\parallel)_\Omega, \\
&(\mathbf{E}^\parallel, \mathbf{V}^\parallel)_\Omega,
\end{align*}
\]

\( \forall \mathbf{E}, \mathbf{V} \in H(\text{curl}, \Omega) \).

Thus, there exists a compact sesquilinear form \( c : H(\text{curl}, \Omega) \times H(\text{curl}, \Omega) \to \mathbb{C} \) and a constant \( C > 0 \), such that the following estimate holds

\[
\text{Re} \left\{ q_\kappa(\mathbf{E}, \mathbf{X}(\mathbf{E})) + c(\mathbf{E}, \mathbf{X}(\mathbf{E})) \right\} \geq C \left( \| \mathbf{E} \|^2_{H(\text{curl}, \Omega)} + \| \mathbf{E}^\parallel \|^2_{H(\text{curl}, \Omega)} \right),
\]

for all \( \mathbf{E} \in H(\text{curl}, \Omega) \). Hence, we immediately derive a generalized Gårding inequality for the sesquilinear form \( q_\kappa \) on the product space \( \mathbf{X}(\Omega) \times \mathbf{N}(\Omega) \). This motivates the use of a Helmholtz-type regular splitting, whose construction is based on the existence of vector potentials in \( H^1(\Omega) \), cf. [5, Lem. 3.5] and [50, Lem. 2.5].

**Lemma 4.22.** There exists a continuous mapping

\[
\mathbf{L} : H_{\text{loc}}(\text{div} 0, \mathbb{R}^3) := \{ \mathbf{V} \in L^2_{\text{loc}}(\mathbb{R}^3); \text{div} \mathbf{V} = 0 \} \to H^1_{\text{loc}}(\mathbb{R}^3),
\]

such that \( \text{div} \circ \mathbf{L} \)(\( \mathbf{U} \)) = 0 and \( \text{curl} \circ \mathbf{L} \)(\( \mathbf{U} \)) = \( \mathbf{U} \) for all \( \mathbf{U} \in H_{\text{loc}}(\text{div} 0, \mathbb{R}^3) \).

Following the idea of [52, Sect. 7], we introduce the operator

\[
\mathbf{P} : H(\text{curl}, \Omega) \to H^1(\Omega), \quad \mathbf{P}(\mathbf{U}) := (\mathbf{L} \circ \text{curl})(\mathbf{U}),
\]

and conclude that \( \mathbf{P} \) is a continuous projection that preserves the \( \text{curl} \) and satisfies \( \text{Ker}(\mathbf{P}) = \text{Ker}(\text{curl}) \cap H(\text{curl}, \Omega) \), cf. [52, Lem. 7.2]. Based on this device, we define the following closed subspaces

\[
\mathbf{X}(\text{curl}, \Omega) := \mathbf{P}(H(\text{curl}, \Omega)), \quad \mathbf{N}(\text{curl}, \Omega) := \text{Ker}(\text{curl}) \cap H(\text{curl}, \Omega),
\]

which provide us with a stable and direct Helmholtz-type splitting

\[
H(\text{curl}, \Omega) = \mathbf{X}(\text{curl}, \Omega) \oplus \mathbf{N}(\text{curl}, \Omega).
\]

For both components we retain the \( H(\text{curl}, \Omega) \)-norm and extra regularity of field components contained in \( \mathbf{X}(\text{curl}, \Omega) \) yields the following compact embedding

\[
\mathbf{X}(\text{curl}, \Omega) \hookrightarrow L^2(\Omega),
\]

cf. [52, Cor. 7.3]. This construction provides us with a stable decomposition of \( H(\text{curl}, \Omega) \), which can be used to establish generalized a Gårding inequality for the sesquilinear form underlying the electric field equation. In addition, the splittings of \( H(\text{curl}, \Omega) \) and \( H^{-1/2}(\text{div} \Gamma, \Gamma) \) allow us to establish the following compactness results, cf. [52, Lem. 8.1, 8.2].

**Lemma 4.23.** The following mappings

\[
(\gamma_D)^* \circ (\mathbf{K}_\kappa - \frac{1}{2} \mathbf{I}) : \mathbf{X}(\text{div} \Gamma, \Gamma) \leftrightarrow \mathbf{X}(\text{curl}, \Omega)',
\]

\[
(\gamma_\Gamma)^* \circ (\mathbf{K}_\kappa - \frac{1}{2} \mathbf{I}) : \mathbf{N}(\text{div} \Gamma, \Gamma) \leftrightarrow \mathbf{N}(\text{curl}, \Omega)',
\]

\[
(\frac{1}{2} \mathbf{I} - \mathbf{K}_\kappa) \circ \gamma_D : \mathbf{X}(\text{curl}, \Omega) \leftrightarrow \mathbf{X}(\text{curl}, \Gamma)',
\]

\[
(\frac{1}{2} \mathbf{I} - \mathbf{K}_\kappa) \circ \gamma_D : \mathbf{N}(\text{curl}, \Omega) \leftrightarrow \mathbf{N}(\text{curl}, \Gamma)',
\]

are compact.
It is hardly surprising that the splitting idea has to be adopted for the treatment of boundary integral operators as well. This time we consider the sesquilinear form associated with the single-layer operator $V_\kappa$, which evaluates to

$$\langle \lambda, V_\kappa(\mu) \rangle_t = \langle \lambda, S_\kappa(\mu) \rangle_t - \frac{1}{\kappa^2} \langle \text{div}_\Gamma \lambda, S_\kappa(\text{div}_\Gamma \mu) \rangle_{\Gamma}, \quad \lambda, \mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma).$$

Slightly abusing notations, we define the boundary integral operator, cf. [19, Eq. 31]

$$V_0 := S_0 + \frac{1}{\kappa^2} \text{grad}_\Gamma \circ S_0 \circ \text{div}_\Gamma.$$

Now from lemma 4.15 and (4.17) we immediately conclude compactness of the mapping $V_\kappa - V_0 : H^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma)$.

Hence, switching between $V_\kappa$ and $V_0$ amounts to a compact perturbation and the numerical analysis only needs to take into account the properties of $V_0$. Unfortunately, this is not sufficient to establish a Gårding inequality for

$$\langle \lambda, V_\kappa(\mu) \rangle_t = \langle \lambda, S_\kappa(\mu) \rangle_t - \frac{1}{\kappa^2} \langle \text{div}_\Gamma \lambda, S_\kappa(\text{div}_\Gamma \mu) \rangle_{\Gamma},$$

on $H^{-1/2}(\text{div}_\Gamma, \Gamma)$, since the second term has an infinite dimensional kernel. This time we opt for a Hodge-type splitting of the Neumann trace space into the following two components $H^{-1/2}(\text{div}_\Gamma, \Gamma) = X(\Gamma) \oplus N(\Gamma)$, to overcome the lack of coercivity of the single-layer operator $V_\kappa$ on the natural trace space. In contrast to the splitting employed in [22, Thm. 3.4], we will trade orthogonality for increased regularity in the complement subspace $X(\Gamma)$:

1. the subspace $N(\Gamma)$ in the splitting agrees with the kernel of $\text{div}_\Gamma$,
2. stability of the splitting w.r.t. the norm on $H^{-1/2}(\text{div}_\Gamma, \Gamma)$,
3. extra regularity of vector fields in the complement space $X(\Gamma)$.

Again, any $\lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ can be decomposed into $\lambda = \lambda^\perp + \lambda^0$, where $\lambda^\perp \in X(\Gamma)$ and $\lambda^0 \in N(\Gamma)$. This time, the sign-flip isomorphism $X^\Gamma : H^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\text{div}_\Gamma, \Gamma)$ is defined by $X^\Gamma(\lambda) := \lambda^0 - \lambda^\perp$. Thus, the sesquilinear form $\langle X^\Gamma(\cdot), V_0(\cdot) \rangle_t$ assumes the following block structure on the decomposed space $X(\Gamma) \times N(\Gamma)$

$$\begin{align*}
\frac{1}{\kappa^2} \langle \text{div}_\Gamma \lambda^\perp, S_0(\text{div}_\Gamma \mu^\perp) \rangle_{\Gamma} - \langle \lambda^\perp, S_0(\mu^\perp) \rangle_t - \langle \lambda^0, S_0(\mu^0) \rangle_t, \\
\langle \lambda^0, S_0(\mu^\perp) \rangle_t + \langle \lambda^0, S_0(\mu^0) \rangle_t.
\end{align*}$$

Furthermore, the embedding $X(\Gamma) \subset H^{1/2}(\Gamma) \hookrightarrow L_2^\gamma(\Gamma)$ allows us to identify the compact sesquilinear forms

$$\langle \lambda^\perp, S_0(\mu^\perp) \rangle_t, \quad \langle \lambda^\perp, S_0(\mu^0) \rangle_t, \quad \langle \lambda^0, S_0(\mu^\perp) \rangle_t, \quad \langle \lambda^0, S_0(\mu^0) \rangle_t, \quad \forall \lambda, \mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma).$$

Together with lemma 4.16, this implies existence of a compact operator $T_\mathbf{V} : H^{-1/2}(\text{div}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ and a constant $C > 0$, such that for all $\lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ the following estimate holds

$$\text{Re} \left\{ \langle X^\Gamma(\lambda), (V_0 + T_\mathbf{V})(\lambda) \rangle_t \right\} \geq C \left( \| \lambda^\perp \|^2_{H^{-1/2}(\text{div}_\Gamma, \Gamma)} + \| \lambda^0 \|^2_{H^{-1/2}(\text{div}_\Gamma, \Gamma)} \right).$$

Hence, we obtain a generalized Gårding inequality for the sesquilinear form corresponding to the single-layer operator $V_\kappa$ on the product space $X(\Gamma) \times N(\Gamma)$. This motivates the construction...
of the following Hodge-type decomposition of the Neumann trace space \( H^{-1/2}(\text{div}_\Gamma, \Gamma) \), cf. [52, Sect. 7]: Pick an arbitrary \( \lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \), set \( \omega := \text{div}_\Gamma \lambda \in H^{-1/2}(\Gamma) \) and solve the the Neumann problem

\[
\Psi \in H^1(\Omega)/\mathbb{R}: \quad \Delta \Psi = 0 \text{ in } \Omega, \quad \gamma_n^{-} (\text{grad } \Psi) = \omega \text{ on } \Gamma.
\]

Obviously, \( W := \text{grad } \Psi \in H(\text{div} 0, \Omega) \) belongs to the domain of \( L \), which allows us to introduce the operator \( J : H^{-1/2}(\Gamma) \mapsto H^1(\Omega) \) by \( J(\omega) := L(W) \). Moreover, continuity of \( J \) is straightforward. This motivates the definition of the following operator

\[
\mathcal{P}^\Gamma : H^{-1/2}(\text{div}_\Gamma, \Gamma) \mapsto \mathbb{H}^{1/2}(\Gamma), \quad \mathcal{P}^\Gamma := \gamma_x \circ J \circ \text{div}_\Gamma,
\]

which provides us with a continuous projector \( \mathcal{P}^\Gamma \) that preserves the \( \text{div} \) and satisfies \( \text{Ker}(\mathcal{P}^\Gamma) = \text{Ker}(\text{div}_\Gamma) \cap H^{-1/2}(\text{div}_\Gamma, \Gamma) \), cf. [52, Lem. 7.4]. Finally, the definition of the following subspaces

\[
X(\text{div}_\Gamma, \Gamma) := \mathcal{P}^\Gamma(H^{-1/2}(\text{div}_\Gamma, \Gamma)), \quad N(\text{div}_\Gamma, \Gamma) := \text{Ker}(\text{div}_\Gamma) \cap H^{-1/2}(\text{div}_\Gamma, \Gamma),
\]

yields a stable and direct Hodge-type decomposition of the Neumann trace space \( H^{-1/2}(\text{div}_\Gamma, \Gamma) \)

\[
H^{-1/2}(\text{div}_\Gamma, \Gamma) = X(\text{div}_\Gamma, \Gamma) \oplus N(\text{div}_\Gamma, \Gamma).
\]

As above, the extra regularity of traces from \( X(\text{div}_\Gamma, \Gamma) \) rewards us with the following compact embedding

\[
X(\text{div}_\Gamma, \Gamma) \hookrightarrow L^2_\xi(\Gamma),
\]

cf. [52, Cor. 7.5]. The previous construction provides us with a stable splitting of \( H^{-1/2}(\text{div}_\Gamma, \Gamma) \), which can be used to establish a generalized Gårding inequality for the sesquilinear form associated with the single-layer operator \( V_\kappa \) on the decomposed product space \( X(\text{div}_\Gamma, \Gamma) \times N(\text{div}_\Gamma, \Gamma) \).

Based on the splitting of the Neumann trace space \( H^{-1/2}(\text{div}_\Gamma, \Gamma) \) and the surface twist operator \( R \), a stable decomposition of \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) can be established based on the definition of the following closed subspaces

\[
X(\text{curl}_\Gamma, \Gamma) := R^{-1}(X(\text{div}_\Gamma, \Gamma)) \quad \text{and} \quad N(\text{curl}_\Gamma, \Gamma) := R^{-1}(N(\text{div}_\Gamma, \Gamma)).
\]

Since \( R^{-1} \) is an isometric isomorphism we immediately conclude

\[
H^{-1/2}(\text{curl}_\Gamma, \Gamma) = X(\text{curl}_\Gamma, \Gamma) \oplus N(\text{curl}_\Gamma, \Gamma),
\]

and the identities (4.13) imply that \( N(\text{curl}_\Gamma, \Gamma) \) coincides with the kernel of the \( \text{curl}_\Gamma \). Furthermore, the extra regularity of the complement subspace \( X(\text{curl}_\Gamma, \Gamma) \subset H^{1/2}_\parallel(\Gamma) \) provides us with the following compact embedding

\[
X(\text{curl}_\Gamma, \Gamma) \subset H^{1/2}_\parallel(\Gamma) \hookrightarrow L^2_\xi(\Gamma).
\]

Finally, stability of the decomposition follows from the definition of the norms in theorem 4.1 and the identities (4.13). The splitting of the Dirichlet trace space \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) enables us to establish a generalized Gårding inequality for the sesquilinear form associated with the hypersingular operator \( W_\kappa \). To do so, we decompose \( u \in H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) into \( u = u^0 + u^\perp \), with components \( u^0 \in N(\text{curl}_\Gamma, \Gamma) \) and \( u^\perp \in X(\text{curl}_\Gamma, \Gamma) \), and introduce the sign-flip isomorphism \( X^\Gamma : H^{-1/2}(\text{curl}_\Gamma, \Gamma) \mapsto H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) by \( X^\Gamma(u) := u^\perp - u^0 \). Deliberately misusing notations, we introduce the boundary integral operator

\[
W_0 := \text{curl}_\Gamma \circ S_0 \circ \text{curl}_\Gamma - \kappa^2 S_0^\times,
\]
and conclude that, due to lemma 4.15, switching between $\mathbf{W}_\kappa$ and $\mathbf{W}_0$ amounts to a compact perturbation. Hence, adopting the splitting idea for both arguments and employing the isomorphism $\mathbf{X}^F$, the sesquilinear form $(\mathbf{W}_0(\cdot), \mathbf{X}^F(\cdot))$ assumes the following block structure

$$
\begin{align*}
\kappa^2 (\mathbf{W}_0(\mathbf{v}), \mathbf{X}^F(\mathbf{u}))_{\Gamma} + \kappa^2 (\mathbf{W}_0(\mathbf{v}), \mathbf{X}^F(\mathbf{u}))_{\Gamma} = - \kappa^2 (\mathbf{W}_0(\mathbf{v}), \mathbf{X}^F(\mathbf{u}))_{\Gamma} + \kappa^2 (\mathbf{W}_0(\mathbf{v}), \mathbf{X}^F(\mathbf{u}))_{\Gamma}.
\end{align*}
$$

Moreover, extra regularity of the arguments contained in $\mathbf{X}(\mathbf{curl}_\Gamma, \Gamma)$ provides us with the compact sesquilinear forms

$$
\begin{align*}
(u, S_0^\lambda(v))^t, \quad (u, S_0^\lambda(v))^t, \quad (u, S_0^\lambda(v))^t, \quad \forall u, v \in H^{-1/2}(\mathbf{curl}_\Gamma, \Gamma).
\end{align*}
$$

Now lemma 4.16 yields existence of a compact operator $T_\mathbf{W} : H^{-1/2}(\mathbf{curl}_\Gamma, \Gamma) \rightarrow H^{-1/2}(\mathbf{div}_\Gamma, \Gamma)$ and a constant $C > 0$ such that

$$
\mathrm{Re} \left\{ ((\mathbf{W}_\kappa + T_\mathbf{W}) (u), \mathbf{X}^F(u))_{\Gamma} \right\} \geq C \left( \|u\|_{H^{-1/2}(\mathbf{curl}_\Gamma, \Gamma)}^2 + \|v\|_{H^{-1/2}(\mathbf{curl}_\Gamma, \Gamma)}^2 \right),
$$

for all $u \in H^{-1/2}(\mathbf{curl}_\Gamma, \Gamma)$, which amounts to a generalized Gårding inequality for the sesquilinear form associated with the boundary integral operator $\mathbf{W}_\kappa$ on the product space $\mathbf{X}(\mathbf{curl}_\Gamma, \Gamma) \times \mathbf{N}(\mathbf{curl}_\Gamma, \Gamma)$. We finish this section with the following lemma, which provides a key feature for the numerical analysis of coupled variational formulations.

\textbf{Lemma 4.24.} For any $E \in H(\mathbf{curl}, \Omega)$, the Helmholtz-type splitting $E = E^0 + E^\perp$, with components $E^\perp \in \mathbf{X}(\mathbf{curl}, \Omega)$, $E^0 \in \mathbf{N}(\mathbf{curl}, \Omega)$, provides a valid Hodge-type decomposition of the Dirichlet trace $\gamma_D E$, that is $\gamma_D E^0 \in \mathbf{N}(\mathbf{curl}_\Gamma, \Gamma)$ and $\gamma_D E^\perp \in \mathbf{X}(\mathbf{curl}_\Gamma, \Gamma)$.

\textit{Proof.} We start from the Helmholtz decomposition of an arbitrary field $E \in H(\mathbf{curl}, \Omega)$ into $E = E^\perp + E^0$, with components $E^\perp \in \mathbf{X}(\mathbf{curl}, \Omega)$ and $E^0 \in \mathbf{N}(\mathbf{curl}, \Omega)$. By recalling the formula $\mathbf{curl}_\Gamma \circ \gamma_D = \gamma_n \circ \mathbf{curl}$, we conclude that $\gamma_D E^0$ belongs to the kernel of the $\mathbf{curl}_\Gamma$. Moreover, since $\mathbf{X}(\mathbf{curl}_\Gamma, \Omega) \subset H^1(\Omega)$ lemma 4.3 implies that for any $E^\perp \in \mathbf{X}(\mathbf{curl}, \Omega)$ we have $\gamma_D E^\perp \in H^{1/2}_\sigma(\Gamma)$. Hence, we derive $\gamma_D E^\perp \in \mathbf{X}(\mathbf{curl}_\Gamma, \Gamma)$, which finishes the proof. \hfill $\square$
4.6. DECOMPOSITIONS
Chapter 5

The Maxwell Transmission Problem

5.1 Problem Formulation

In this section we describe the model underlying the time-harmonic Maxwell transmission problem, which describes the scattering of incident electromagnetic waves $E^i$ from a penetrable obstacle or scatterer $\Omega$. It can be viewed as an interior source problem for the total field $E$ inside $\Omega$ coupled together with an exterior scattering problem for the scattered field $E^s$ on $\Omega^+$ via the so-called transmission conditions.

A typical setting for the interior source problem are inhomogeneous, isotropic material parameters $\varepsilon$ and $\mu$, that may display some spatial variation inside $\Omega$, as well as compactly supported sources. On the other hand, the exterior scattering problem can be characterized by homogeneous, isotropic materials $\varepsilon_0 > 0$ and $\mu_0 > 0$, which are constant throughout the whole exterior air region, and the total absence of any sources.

\[ \Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega} \]

Figure 5.1: Maxwell Transmission Problem.

Hence, the transmission problem for the time-harmonic Maxwell equations can be described
by the following mathematical model

\[
\begin{align*}
\text{curl} \mu_r(x)^{-1} \text{curl} E - \kappa^2 \varepsilon_r(x) E &= F(x) \quad \text{in } \Omega, \\
\text{curl} \text{curl} E^s - \kappa^2 E^s &= 0 \quad \text{in } \Omega^+, \\
\gamma_D^+ E^s - \gamma_D^- E &= g_D \quad \text{on } \Gamma, \\
\gamma_N^+ E^s - \mu_r^{-1} \gamma_N^- E &= g_N \quad \text{on } \Gamma,
\end{align*}
\]

(5.1)

where \( \kappa \) denotes the wave number, cf. [69, Sect. 5.6.3]. Here and below, we assume the relative permittivity \( \varepsilon_r \) and the relative permeability \( \mu_r \) to be \( L^\infty(\Omega) \) complex-valued functions such that

\[
\begin{align*}
\Re\{\varepsilon_r(x)\} &\geq \alpha, \quad \text{and } \Im\{\varepsilon_r(x)\} \geq 0, \\
\Re\{\mu_r(x)\} &\geq \alpha, \quad \text{and } \Im\{\mu_r(x)\} \geq 0,
\end{align*}
\]

almost everywhere in \( \Omega \) (\( \alpha \) is a real positive constant). In the case of excitation by plane electric waves, the generic jump data \( g_D \) and \( g_N \) evaluate to the following traces

\[
g_D := -\gamma_D E^i, \quad g_N := -\gamma_N E^i.
\]

Using Rellich’s lemma and unique continuation techniques, the following result can be established, cf. [48, Thm. 3.1].

**Theorem 5.1.** Provided that the relative material parameters \( \varepsilon_r \) and \( \mu_r \) are \( L^\infty(\Omega) \) complex-valued functions satisfying (5.2), the Maxwell transmission problem (5.1) has a unique solution.

**Remark 5.2.** Note that inside the scatterer \( \Omega \) the field \( E \) in (5.1) denotes the total electric field, whereas in \( \Omega^+ \) we write \( E^s \) for the scattered electric field. There the total field can be recovered by \( E = E^i + E^s \).

### 5.2 Overview on FEM-BEM Coupling Methods

The main purpose of this section is to provide a brief overview on some of the methods used for the coupling of finite elements and boundary elements for the Maxwell transmission problem.

**5.2.1 Symmetric Coupling**

The derivation of coupled formulations usually departs from a domain-based variational equation, which can be obtained by applying Green’s formula to the electric field equation (4.9) in \( \Omega \): Find \( E \in H(\text{curl}, \Omega) \) such that

\[
\begin{align*}
(\mu_r^{-1} \text{curl} E, \text{curl} V)_\Omega - \kappa^2 (\varepsilon_r E, V)_\Omega - (\mu_r^{-1} \gamma_N^- E, \gamma_D^- V)_t &= (F, V)_\Omega, \quad (5.3)
\end{align*}
\]

for all \( V \in H(\text{curl}, \Omega) \). This equation can now be coupled to the scattered field \( E^s \) by means of the transmission conditions

\[
\begin{align*}
\mu_r^{-1} \gamma_N^- E &= \gamma_D^+ E^s - g_N, \\
\gamma_D^+ E^s &= \gamma_D^- E + g_D.
\end{align*}
\]

(5.4)

Furthermore, some information about the physical behaviour of the scattered field on the exterior domain \( \Omega^+ \) must be taken into account. In light of lemma 4.17, this information is provided by the exterior Calderón projector \( P_+ \) (see section 4.4 for details and a definition). In variational form the corresponding identities are given by

\[
\begin{align*}
(\mu, \gamma_D^+ E^s)_t &= (\mu, \left(\frac{1}{2} \text{Id} + K_\kappa\right)(\gamma_D^+ E^s))_t - (\mu, V_\kappa(\gamma_D^+ E^s))_t, \\
(\gamma_D^+ E^s, v)_t &= -(W_\kappa(\gamma_D^+ E^s), v)_t + (\left(\frac{1}{2} \text{Id} - K_\kappa\right)(\gamma_D^+ E^s), v)_t,
\end{align*}
\]

(5.5)
for all $\mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ and $v \in H^{-1/2}(\text{curl}_\Gamma, \Gamma)$. Now we can use the transmission conditions (5.4) and the second equation of the Calderón projector to replace the boundary term in (5.3) and combine the resulting equation with the first equation of (5.5), cf. [35, Sect. 4] or [52, Sect. 6]. Finally, by introducing the new variable $\lambda := \gamma^+_N E^x \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ we arrive at the following variational formulation:

Find $E \in H(\text{curl}, \Omega)$ and $\lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ such that for all $V \in H(\text{curl}, \Omega)$ and $\mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ there holds

$$q_K(E, V) + (W_K(\gamma_D E), \gamma_D V)_t + ((K'K - \frac{1}{2} \text{Id})(\lambda), \gamma_D V)_t = f(V),$$

$$((\mu, (\frac{1}{2} \text{Id} - K\gamma)(\gamma_D E)), \gamma_D V)_t = g_1(\mu),$$

with right hand sides given by

$$f(V) := (F, V)_\Omega - (g_N, \gamma_D V)_t - (W_K(g_D), \gamma_D V)_t,$$

$$g_1(\mu) := (\mu, (\frac{1}{2} \text{Id} - K\gamma)(g_D))_t,$$

and $q_K$ representing the interior sesquilinear form,

$$q_K(E, V) := (\mu^{-1}_r \text{curl} E, \text{curl} V)_\Omega - \kappa^2 (\varepsilon_r E, V)_\Omega.$$

Moreover, the following lemma guarantees uniqueness of solution $E \in H(\text{curl}, \Omega)$ and $\lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ to the variational formulation (5.6), cf. [52, Lem. 6.1].

**Lemma 5.3.** Provided that $\text{curl} \text{curl} U - \kappa^2 U = 0$ in $\Omega$ and $\gamma_D^- U = 0$ on $\Gamma$ implies $U = 0$, then any solution to (5.6) provides a solution to (5.1) by retaining $E$ in $\Omega$ and using the representation formula (4.16) for the Cauchy data $(\gamma_D^- E + g_D, \lambda)$ in $\Omega^+$. 

On the other hand, if $\kappa^2$ coincides with an eigenvalue of

$$\text{curl} \text{curl} U - \lambda U = 0 \text{ in } \Omega, \quad \gamma_D^- U = 0 \text{ on } \Gamma,$$

then solutions to (5.6) are unique up to contributions $(0, \eta)$, where $\eta$ corresponds to the Neumann trace $\gamma_N^- U$ of an eigenfunction $U$ of (5.7). Even in the resonant case, the right hand sides $f_1$ and $g_1$ of (5.6) are consistent and hence solutions $(E, \lambda)$ still exist. Moreover, the interior electric field $E$ and its Dirichlet trace $\gamma_D^- E$ are unique. Unfortunately, this is of little comfort as far as numerical methods are concerned: First, inevitable perturbations introduced by the discretization scheme will destroy the consistency of the right hand side. Second, whenever $\kappa$ is close to one of the resonant frequencies, the system matrix, which arises from a conforming discretization of (5.6), will be extremely ill-conditioned, cf. [26] for Maxwell scattering problems.

Since the sesquilinear form underlying the variational problem (5.6) features the domain-based part $q_K$, as well as the Maxwell single-layer operator $V_K$, our considerations from section 4.6 clearly indicate that coercivity on $H(\text{curl}, \Omega)$ and $H^{-1/2}(\text{div}_\Gamma, \Gamma)$ cannot hold. Thus, based on the splittings provided in section 4.6, we can decompose the trial and test functions in (5.6) according to:

$$E = E^\perp + E^0, \quad E^\perp \in X(\text{curl}, \Omega), \quad E^0 \in N(\text{curl}, \Omega),$$

$$V = V^\perp + V^0, \quad V^\perp \in X(\text{curl}, \Omega), \quad V^0 \in N(\text{curl}, \Omega),$$

$$\lambda = \lambda^\perp + \lambda^0, \quad \lambda^\perp \in X(\text{div}_\Gamma, \Gamma), \quad \lambda^0 \in N(\text{div}_\Gamma, \Gamma),$$

$$\mu = \mu^\perp + \mu^0, \quad \mu^\perp \in X(\text{div}_\Gamma, \Gamma), \quad \mu^0 \in N(\text{div}_\Gamma, \Gamma).$$
5.2. OVERVIEW ON FEM-BEM COUPLING METHODS

Furthermore, we sort the resulting variables according to their electric and magnetic nature, grouping them into $(\lambda^\perp, E^0)$ and $(\lambda^0, E^\perp)$, cf. [52, Sect. 8], and obtain a variational formulation with a distinct block structure on the Hilbert space

$$\hat{V} := X(\text{div}_T, \Gamma) \times \mathcal{N} (\text{curl}, \Omega) \times \mathcal{N} (\text{div}_T, \Gamma) \times X(\text{curl}, \Omega),$$

that is endowed with its natural graph norm:

Find $(\lambda^\perp, E^0, \lambda^0, E^\perp) \in \hat{V}$ such that for all $(\mu^\perp, V^0, \mu^0, V^\perp) \in \hat{V}$ there holds

$$\hat{a}_{\text{sym}} ((\lambda^\perp, E^0, \lambda^0, E^\perp), (\mu^\perp, V^0, \mu^0, V^\perp)) = \hat{f}_{\text{sym}} (\mu^\perp, V^0, \mu^0, V^\perp).$$

The sesquilinear form $a_{\text{sym}} : \hat{V} \times \hat{V} \rightarrow \mathbb{C}$ and the linear form $f_{\text{sym}} : \hat{V} \rightarrow \mathbb{C}$ are defined by

$$a_{\text{sym}} ((\lambda^\perp, E^0, \lambda^0, E^\perp), (\mu^\perp, V^0, \mu^0, V^\perp)) := a_{\text{sym}} ((E^\perp + E^0, \lambda^\perp + \lambda^0), (V^\perp - V^0, -\mu^\perp + \mu^0)),$$

$$f_{\text{sym}} (\mu^\perp, V^0, \mu^0, V^\perp) := f_{\text{sym}} (V^\perp - V^0, -\mu^\perp + \mu^0),$$

where $a_{\text{sym}}$ and $f_{\text{sym}}$ are the sesquilinear and linear forms underlying (5.6). Clearly the split variational formulation (5.8) produces the same solutions as (5.6) for $\lambda = \lambda^\perp + \lambda^0$ and $E = E^\perp + E^0$. Note that switching between $a_{\text{sym}}$ and $\hat{a}_{\text{sym}}$ involves the application of two sign-flip isomorphisms on $H(\text{curl}, \Omega)$ and $H^{-1/2}(\text{div}_T, \Gamma)$, as defined in section 4.6.

Since we aim at applying the Fredholm alternative and deduce existence of solutions from their uniqueness, a generalized Gårding inequality for the sesquilinear form $a_{\text{sym}}$ on the Hilbert space

$$V := H(\text{curl}, \Omega) \times H^{-1/2}(\text{div}_T, \Gamma),$$

that is endowed with its natural graph norm, will prove indispensable. In order to do so we establish coercivity of $a_{\text{sym}}$ on $V$, which implies a generalized Gårding inequality for the sesquilinear form $a_{\text{sym}}$ on $V$. We start by rearranging the variational formulation (5.8) in the following block-wise fashion:

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} b_{12} & b_{11} \\ b_{22} & b_{21} \end{bmatrix} = \begin{bmatrix} g_1 (\mu^\perp) \\ f_1 (V^0) \\ g_1 (\mu^0) \\ f_1 (V^\perp) \end{bmatrix},$$

for all $\begin{bmatrix} \mu^\perp & V^0 & \mu^0 & V^\perp \end{bmatrix} \in V$. The operator blocks $b_{11}, b_{12}, b_{21},$ and $b_{22}$ are defined by the following expressions

$$b_{11} := \begin{cases} - (\mu^\perp, V^0 (\lambda^\perp))_t - (\mu^\perp, (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D E^0))_t \\ - ((K_\kappa - \frac{1}{2} \text{Id})(\lambda^\perp), \gamma_D V^0)_t + \kappa^2 (\gamma_D E^0, S_\kappa (\gamma_D V^0))_t + \kappa^2 (\varepsilon_r E^0, V^0)_\Omega, \end{cases}$$

$$b_{12} := \begin{cases} - (\mu^\perp, S_\kappa (\lambda^\perp))_t - (\mu^\perp, (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D E^\perp))_t \\ - ((K_\kappa - \frac{1}{2} \text{Id})(\lambda^\perp), \gamma_D V^\perp)_t + \kappa^2 (\gamma_D E^\perp, S_\kappa (\gamma_D V^\perp))_t + \kappa^2 (\varepsilon_r E^\perp, V^\perp)_\Omega, \end{cases}$$

$$b_{21} := \begin{cases} (\mu^0, S_\kappa (\lambda^\perp))_t + (\mu^0, (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D E^0))_t \\ ((K_\kappa - \frac{1}{2} \text{Id})(\lambda^\perp), \gamma_D V^0)_t - \kappa^2 (\gamma_D E^0, S_\kappa (\gamma_D V^0))_t - \kappa^2 (\varepsilon_r E^0, V^\perp)_\Omega, \end{cases}$$

$$b_{22} := \begin{cases} (\mu^0, S_\kappa (\lambda^\perp))_t + (\mu^0, (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D E^\perp))_t \\ ((K_\kappa - \frac{1}{2} \text{Id})(\lambda^\perp), \gamma_D V^\perp)_t + (W_\kappa (\gamma_D E^\perp), \gamma_D V^\perp)_t + q_\kappa (E^\perp, V^\perp). \end{cases}$$
Here, we have used that the order 1 terms contained in $\mathbf{V}_k$ and $\mathbf{W}_k$ vanish, when they are applied to fields contained in $\mathbf{N}$ (divr, $\Gamma$) or $\gamma_D^{-1}(\mathbf{N}$ (curl, $\Omega$)), respectively (cf. lemma 4.24 for details). Next, we consider the sesquilinear forms $b_{12}$ and $b_{21}$ and conclude that the compact embeddings $X (\text{curl} \Omega) \hookrightarrow L^2(\Omega)$ and $X (\text{div} \Gamma) \hookrightarrow L^2(\Gamma)$ combined with the continuity properties from lemma 4.12 imply compactness of the mappings

$$S_k : X (\text{div} \Gamma, \Gamma) \mapsto H^{-1/2}_\parallel (\Gamma), \quad S_k^{-1} \circ \gamma_D : X (\text{curl} \Omega) \mapsto H^{1/2}_\perp (\Gamma).$$

Moreover, by recalling lemma 4.23, we obtain that the remaining sesquilinear forms contained in the blocks $b_{12}$ and $b_{21}$ are compact. In addition, we can employ (4.17) and lemma 4.15 and switch from asym to

$$b((\lambda^+, E^0, \lambda^0, E^\perp), (\mu^+, V^0, \mu^0, V^\perp)) :=$$

$$(\mu_\perp^{-1}\text{curl} E^\perp, \text{curl} V^\perp)_{\Omega} + \kappa^2(\varepsilon, E^0, V^0)_{\Omega} + \kappa^2(\gamma_D^{-1} E^0, S_0^{-1}(\gamma_D^{-1} V^0))_{\Gamma} + (\text{curl} \mu^{-1}, S_0(\text{curl} \lambda^{-1}))_{\Gamma} + (\mu^+, (\kappa - \frac{1}{2} \mathbb{I})^{-1} S_0(\gamma_D^{-1} E^0))_{\Gamma} + ((\frac{1}{2} \mathbb{I} - Q) \lambda^0, \gamma_D^{-1} E^0)_{\Gamma} + ((\frac{1}{2} \mathbb{I} - Q) \lambda^{-1}, S_0(\gamma_D^{-1} V^0))_{\Gamma} + ((\frac{1}{2} \mathbb{I} - Q) \lambda^{-1}, S_0(\gamma_D^{-1} V^0))_{\Gamma} + ((\frac{1}{2} \mathbb{I} - Q) \lambda^0, \gamma_D^{-1} V^0)_{\Gamma},$$

by means of a compact perturbation. Finally, we can apply the cancellation argument from lemma 4.14 and sort out all terms, which do not give rise to an elliptic sesquilinear form. Now, the following result is straightforward, cf. [52, Thm. 8.3].

**Theorem 5.4.** The sesquilinear form $\hat{a}_{\text{sym}} : \hat{V} \times \hat{V} \mapsto \mathbb{C}$ satisfies a Gårding inequality; that is, it can be written as a sum $\hat{a}_{\text{sym}} = \hat{a}_E + \hat{a}_C$ of a $\mathbf{V}$-elliptic sesquilinear form $\hat{a}_E : \mathbf{V} \times \mathbf{V} \mapsto \mathbb{C}$ and a compact sesquilinear form $\hat{a}_C : \mathbf{V} \times \mathbf{V} \mapsto \mathbb{C}$. This confirms a generalized Gårding inequality in the sense of item 1. of definition 4.19 for the sesquilinear form $a_{\text{sym}}$ on $\mathbf{V}$. Hence, lemma 5.3 together with a Fredholm argument (cf. [77, 89]) ensures existence of solutions to (5.6), provided that $\kappa$ does not correspond to one of the interior resonant frequencies.

### 5.2.2 Stabilised Symmetric Coupling – An Engineering Approach

Recently a new symmetric FEM-BEM formulation for solving unbound three dimensional electromagnetic problems has been proposed by Lee [87]. Their approach arises from two “cement” variables, $\lambda^- := \mu_\perp^{-1}\gamma^- N E$ and $\lambda^+ := -\gamma^+ N E^0$, which correspond to the interior and exterior magnetic traces on $\Gamma$, cf. [60]. In combination with $e := \gamma_D^{-1} E$, the transmission conditions of (5.1) can be rewritten in the following way

$$\gamma_D^{-1} E - e = \gamma_D E^0, \quad \lambda^- + \lambda^+ = \gamma_N E^0.$$ 

Note that we can rely on complex Robin-type boundary conditions to obtain an alternative but equivalent set of transmission conditions on the interface boundary $\Gamma$

$$-i \gamma_D E + \lambda^- = -i e - \lambda^+ + f, \quad -i e + \lambda^+ = -i \gamma_D E - \lambda^- - g,$$

with the boundary data given by $f := -i \gamma_D E^0 + \gamma_N E^0$ and $g := -i \gamma_D E^0 - \gamma_N E^0$. Due to the mismatch in spaces in (5.9) we can no longer work with the natural Dirichlet and Neumann
traces spaces $H^{-1/2}({\text{curl}}, \Gamma)$ and $H^{-1/2}({\text{div}}, \Gamma)$, but we have to lift all traces onto $L^2_\delta(\Gamma)$. Furthermore, to get meaningful tangential traces of the total field $E$, we have to replace $H({\text{curl}}, \Omega)$ by $H_t({\text{curl}}, \Omega) := \{U \in H({\text{curl}}, \Omega); \gamma_\delta^E U \in L^2_\delta(\Gamma)\}$, which is a Hilbert space when endowed with the graph norm

$$\|U\|_{H_t({\text{curl}}, \Omega)} := \|U\|_{H({\text{curl}}, \Omega)} + \|\gamma_\delta^E U\|_{L^2_\delta(\Gamma)},$$

cf. [65, Sect. 4]. Now we can couple the exterior field on the unbound domain $\Omega^+$ to the interior field on $\Omega$ based on the transmission condition (5.9) and the exterior Calderón projector (5.5). This leaves us with the following variational problem, cf. [87, Eq. 4.20]:

Find $E \in H_t({\text{curl}}, \Omega)$, $\lambda^+ \in L^2_\delta(\Gamma)$, $e \in L^2_\delta(\Gamma)$, and $\lambda^- \in L^2_\delta(\Gamma)$ such that for all $V \in H_t({\text{curl}}, \Omega)$, $\mu^- \in L^2_\delta(\Gamma)$, $v \in L^2_\delta(\Gamma)$, and $\mu^+ \in L^2_\delta(\Gamma)$ there holds

$$q_\kappa(E, V) - \frac{1}{2}(\gamma_\delta^E, \gamma_\delta^V)_{\Gamma} - \frac{1}{2}(\lambda^-, \gamma_\delta^V)_{\Gamma} + \frac{1}{2}(\mu^-, v)_{\Gamma} + \frac{1}{2}(\mu^-, e)_{\Gamma} - \frac{1}{2}(\mu^-, \lambda^+)_{\Gamma} = -\frac{1}{2}(\mu^-, \lambda^+)_{\Gamma},$$

$$\frac{1}{2}((\kappa V - i\kappa_\kappa)(\mu^+)_{\Gamma} + \frac{1}{2}((\kappa V + i\kappa_\kappa)(\mu^+)_{\Gamma} = -\frac{1}{2}(g, e)_{\Gamma}.$$
Considering the imaginary part only, we arrive at $\gamma_N^E E = 0$ and $\gamma_D^E E = 0$ immediately follows from the boundary condition. Since both traces are equal to zero we conclude that $E$ must vanish on $\Omega$, which establishes uniqueness of solutions to the boundary value problem (5.10). Note that we can rely on a Robin-type boundary operator to state the transmission conditions of (5.1), as long as we are able to recover the conventional traces.

In order to obtain a stable coupled variational formulation for the Maxwell transmission problem (5.1), we will make use of the idea of complex linear combinations of traces underlying the boundary value problem (5.10). Unfortunately, the trace spaces $H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ and $H^{-1/2}(\text{div}_\Gamma, \Gamma)$ do not match, which means that complex linear combinations of Dirichlet and Neumann traces are not well-defined. The problem concerning the non-matching Dirichlet and Neumann trace spaces and the lack of coercivity can be overcome by introducing a special trace transformation operator

$$ T : H^{-1/2}(\text{curl}_\Gamma, \Gamma) \times H^{-1/2}(\text{div}_\Gamma, \Gamma) \mapsto H^{-1/2}(\text{curl}_\Gamma, \Gamma) \times H^{-1/2}(\text{div}_\Gamma, \Gamma) $$

defined by

$$ T \begin{bmatrix} u \\ \mu \end{bmatrix} := \begin{bmatrix} u + i\eta M(\mu) \\ \mu \end{bmatrix}, \quad \eta > 0, $$

(5.11)

for all $u \in H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ and $\mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$. The main ingredient here is a regularising operator

$$ M : H^{-1/2}(\text{div}_\Gamma, \Gamma) \mapsto H^{-1/2}(\text{curl}_\Gamma, \Gamma), $$

which satisfies the following assumption.

**Assumption 5.5.** We suppose that

1. $M : H^{-1/2}(\text{div}_\Gamma, \Gamma) \mapsto H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ is compact, and
2. $\text{Re} \{ \langle \mu, M(\mu) \rangle \} > 0$ for all $\mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \setminus \{0\}$.

After a straightforward application of the trace transformation operator (5.11) to the exterior Calderón projector (4.19) the first equation changes into

$$ (\mu, \gamma_D^E E^s + i\eta M(\gamma_N^E E^s) )_t = (\mu, (\frac{1}{2} \text{Id} + K_\kappa - i\eta M \circ W_\kappa)(\gamma_D^E E^s) )_t $$

$$ + \langle \mu, (i\eta M \circ (\frac{1}{2} \text{Id} - K_\kappa') - V_\kappa)(\gamma_N^E E^s) \rangle_t. $$

A simple algebraic transformation yields the following variational identities for the Dirichlet and Neumann traces $\gamma_D^E E^s$ and $\gamma_N^E E^s$

$$ (\mu, \gamma_D^E E^s)_t = (\mu, (\frac{1}{2} \text{Id} + K_\kappa - i\eta M \circ W_\kappa)(\gamma_D^E E^s) )_t $$

$$ - (\mu, (V_\kappa + i\eta M \circ (\frac{1}{2} \text{Id} + K_\kappa'))(\gamma_N^E E^s) )_t, $$

(5.12)

$$ (\gamma_N^E E^s, v)_t = -(W_\kappa(\gamma_D^E E^s), v)_t + ((\frac{1}{2} \text{Id} - K_\kappa)(\gamma_N^E E^s), v)_t, $$

for all $\mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ and $v \in H^{-1/2}(\text{curl}_\Gamma, \Gamma)$, cf. [53, Sect. 6] for the Helmholtz case. The identity (5.12) provides an alternative realisation of the Dirichlet-to-Neumann map.

For the construction of regularising operators we strongly rely on the techniques already established for stabilisation of time-harmonic Maxwell scattering problems, cf. [20, Sect. 4]. A crucial tool in the construction of a suitable regularising operator will be the following trace space

$$ H(\text{curl}_\Gamma, \Gamma) := \{ \mu \in L^2(\Gamma); \text{curl}_\Gamma \mu \in L^2(\Gamma) \} \subset L^2(\Gamma). $$

**Lemma 5.6.** The space $H(\text{curl}_\Gamma, \Gamma) \subset H^{-1/2}(\text{curl}_\Gamma, \Gamma)$ is a dense subspace.
Proof. We start from the dense inclusions \( C^\infty(\Omega) \subset H(\text{curl}, \Omega) \). By definition and due to theorem 4.5 we conclude that \( V_\gamma := \gamma_D(C^\infty(\Omega)) \subset H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) is dense. Since the following inclusions hold \( V_\gamma \subset H(\text{curl}_\Gamma, \Gamma) \subset H^{-1/2}(\text{curl}_\Gamma, \Gamma) \), the statement is proved. \( \square \)

Lemma 5.7. The embedding \( H(\text{curl}_\Gamma, \Gamma) \hookrightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) is compact.

Proof. Let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence in \( H(\text{curl}_\Gamma, \Gamma) \) such that

\[
\|u_n\|_{H(\text{curl}_\Gamma, \Gamma)} \leq 1 \quad \text{for all } n \in \mathbb{N}.
\]

The compact embedding \( L^2(\Gamma) \hookrightarrow H^{-1/2}(\Gamma) \) directly implies, that there exists \( u \in H^{-1/2}(\Gamma) \) and a subsequence \( u_{n_k} \) of \( u_n \) such that \( u_{n_k} \to u \) strongly in \( H^{-1/2}(\Gamma) \). Due to the continuity of \( \text{curl}_\Gamma : H^{-1/2}(\Gamma) \hookrightarrow H^{-3/2}(\Gamma) \) we get \( \text{curl}_\Gamma u_{n_k} \to \text{curl}_\Gamma u \) strongly in \( H^{-3/2}(\Gamma) \). On the other hand, we know that

\[
\|\text{curl}_\Gamma u_n\|_{L^2(\Gamma)} \leq 1 \quad \text{for all } n \in \mathbb{N},
\]

which implies up to extraction of a subsequence \( \text{curl}_\Gamma u_{n_k} \), is strongly converging to an element in \( H^{-1/2}(\Gamma) \). By uniqueness of the limit we conclude that \( \text{curl}_\Gamma u \in H^{-1/2}(\Gamma) \), and, up to selecting a proper subsequence \( u_{n_k} \to u \in H^{-1/2}(\Gamma) \), strongly. \( \square \)

A simple eligible operator \( M \) can be introduced through a variational definition:

For any \( \zeta \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \) find \( M(\zeta) \in H(\text{curl}_\Gamma, \Gamma) \) such that

\[
(M(\zeta), \eta)_\Gamma + (\text{curl}_\Gamma M(\zeta), \text{curl}_\Gamma \eta)_\Gamma = (\zeta, \eta)_\Gamma, \quad \forall \eta \in H(\text{curl}_\Gamma, \Gamma).
\]

In order to simplify notations we introduce the associated sesquilinear form

\[
b(p, q) := (p, q)_\Gamma + (\text{curl}_\Gamma p, \text{curl}_\Gamma q)_\Gamma.
\]

Moreover, compactness of \( M : H^{-1/2}(\text{div}_\Gamma, \Gamma) \hookrightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) can be immediately derived from lemma 5.7.

Lemma 5.8. The regularising operator \( M : H^{-1/2}(\text{div}_\Gamma, \Gamma) \hookrightarrow H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) defined by (5.13) is injective and thus item 2. of assumption 5.5 holds for all \( \mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \).

Proof. Assume that \( M(\zeta) = 0 \) from which we conclude that \( (\zeta, \eta)_\Gamma = 0 \) for all \( \eta \in H(\text{curl}_\Gamma, \Gamma) \). Now choose \( \eta \in H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) such that

\[
(\zeta, \eta)_\Gamma = \|\zeta\|^2_{H^{-1/2}(\text{div}_\Gamma, \Gamma)}
\]

and since \( H(\text{curl}_\Gamma, \Gamma) \subset H^{-1/2}(\text{curl}_\Gamma, \Gamma) \) is dense, there exists a sequence \( \{\eta_n\}_{n \in \mathbb{N}} \) in \( H(\text{curl}_\Gamma, \Gamma) \), such that \( \eta_n \to \eta \) strongly in \( H^{-1/2}(\text{curl}_\Gamma, \Gamma) \), as \( n \to \infty \). From the definition of the regularising operator we infer \( 0 = (\zeta, \eta_n)_\Gamma \), for all \( n \in \mathbb{N} \). Thus taking the limit yields \( \zeta = 0 \), which finishes the proof. \( \square \)

Hence we conclude that both items of assumption 5.5 are satisfied and \( M \) given by (5.13) qualifies as a regularising operator. Introducing the new variable \( \lambda := \gamma_N^* E^s \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \) and using the same procedure as in section 5.2.1 to couple the boundary integral equations (5.12) together with the variational problem on \( \Omega \), we finally end up with the following formulation:

Find \( E \in H(\text{curl}, \Omega) \) and \( \lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \) such that for all \( V \in H(\text{curl}, \Omega) \) and \( \mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma) \) there holds

\[
q_k(E, V) + (W_\kappa(\gamma_D^*)E, \gamma_D^*V)_\Gamma + ((K'_\kappa - \frac{1}{2} \text{Id})(\lambda), \gamma_D^*V)_\Gamma = f_2(V),
\]

\[
(\mu, (\frac{1}{2} \text{Id} - K_\kappa + i\eta M \circ W_\kappa)(\gamma_D^*E))_\Gamma + (\mu, (V + i\eta M(\frac{1}{2} \text{Id} + K'_\kappa))(\lambda))_\Gamma = g_2(\mu),
\]

(5.14)
with the right hand sides given by
\[ f_2(V) := (F, V)_\Omega - (g_N, \gamma_D V)_t - (W_\kappa(g_D), \gamma_D V)_t, \]
\[ g_2(\mu) := (\mu, (K_\kappa - \frac{1}{2}I) - i\eta M \circ W_\kappa(g_D))_t. \]

Summing up, due to the compactness of the regularising operator \( M \) we conclude that all additional off-diagonal terms in the sesquilinear form underlying the regularised variational formulation, compared to the symmetric formulation, are compact. In combination with theorem 5.4 this results again in a generalized Gårding inequality on \( V \). It remains to establish uniqueness of solutions, which amounts to confirming that (5.14) is really immune to spurious resonances.

**Lemma 5.9.** Any solution of (5.14) provides a solution to (5.1) by retaining \( E \) in \( \Omega \) and using the representation formula (4.16) for the Cauchy data \((\gamma_D E + g_D, \lambda)\) in \( \Omega^+ \).

**Proof.** Our approach is based on [85, Sect. 4.3] and [22, Sect. 5]. Testing with \( V \) that is compactly supported in \( \Omega \) confirms that \( E \) satisfies (5.1) in \( \Omega \). We conclude (5.3) for any admissible \( V \). This renders (5.14) equivalent to
\[
(\xi, \gamma_D V)_t + (W_\kappa(u), \gamma_D V)_t + ((K_\kappa - \frac{1}{2}I)(\lambda), \gamma_D V)_t = 0, \\
(\mu, (\frac{1}{2}I - K_\kappa + i\eta M \circ W_\kappa(u))_t - (\mu, (V_\kappa + i\eta M \circ (\frac{1}{2}I + K_\kappa))(\lambda))_t = 0,
\]
with \( \xi := \mu^{-1}\gamma_N E + g_N \) and \( u := \gamma_D E + g_D \). Translated into operator notation this yields
\[
(\mathcal{T} \circ \begin{bmatrix} \frac{1}{2}I - K_\kappa & V_\kappa \\ V_\kappa & \frac{1}{2}I + K_\kappa \end{bmatrix}) \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda - \xi \end{bmatrix},
\]
where the second operator in the product we recognize as an interior Calderón projector (4.19).

By applying the trace transformation operator (5.11) to the Dirichlet and Neumann traces of the following function
\[
U(x) := \Psi_{SL}^\kappa(\lambda)(x) - \Psi_{DL}^\kappa(u)(x), \quad x \in \Omega,
\]
we obtain the traces
\[
\gamma_D U + i\eta M(\gamma_N U) = 0, \quad \gamma_N U = \lambda - \xi.
\]
Furthermore, since \( U \) is a solution to the boundary value problem
\[
\text{curl curl } U - \kappa^2 U = 0 \text{ in } \Omega, \quad \gamma_D U + i\eta M(\gamma_N U) = 0 \text{ on } \Gamma,
\]
integration by parts together with the “test function” \( V := \overline{U} \) yields
\[
\eta (\gamma_N U, M(\gamma_N U)) = (\gamma_N U, \gamma_D U)_t = \int_{\Omega} |\text{curl } U|^2 - \kappa^2 |U|^2 \, dx \in \mathbb{R}.
\]
Considering the imaginary part of the previous equation we finally arrive at
\[
0 = \eta \text{Re} \{(\gamma_N U, M(\gamma_N U))_t\}
\]
and item 2. of assumption 5.5 immediately implies \( \lambda = \xi \). From (5.15) we conclude that \((u, \lambda)\) belongs to the kernel of the interior Calderón projector, which implies that they represent Cauchy data of an exterior Maxwell solution. Hence, due to the following definition
\[
W(x) := \Psi_{DL}^\kappa(u)(x) - \Psi_{SL}^\kappa(\lambda)(x), \quad x \in \Omega^+.
\]
we obtain a pair of solutions \((E, W)\) to the Maxwell transmission problem (5.1). Finally, uniqueness of solutions to the transmission problem carries over to the variational formulation (5.14).
Eventually, existence of solutions to the variational problem (5.14) follows from their uniqueness and a Fredholm argument, cf. [64, Thm. 2.33]. Although (5.14) provides us with a stable variational formulation, it cannot be discretized by means of a straightforward Galerkin scheme, due to the various operator products involved. This suggests to introduce the auxiliary variable

\[ p := M((\frac{1}{2} \text{Id} + K'_\kappa)(\lambda) + W_\kappa(\gamma_D^c E + g_D)) \in H(\text{curl}_\Gamma, \Gamma), \]

which converts (5.14) into the following mixed variational formulation:

Find \( E \in H(\text{curl}, \Omega), \lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma), \) and \( p \in H(\text{curl}_\Gamma, \Gamma) \) such that for all \( V \in H(\text{curl}, \Omega), \mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma), \) and \( q \in H(\text{curl}_\Gamma, \Gamma) \) there holds

\[ q_k(E, V) + (W_\kappa(\gamma_D^c E), \gamma_D^c V)_t + ((K'_\kappa - \frac{1}{2} \text{Id})(\lambda), \gamma_D^c V)_t = f_3(V), \]
\[ (\mu, (\frac{1}{2} \text{Id} - K_\kappa)(\gamma_D^c E))_t + (\mu, V_\kappa(\lambda))_t - \eta(\mu, p)_t = g_3(\mu), \]
\[ -(W_\kappa(\gamma_D^c E), q)_t - ((\frac{1}{2} \text{Id} + K'_\kappa)(\lambda), q)_t + b(p, q) = h_3(q), \]

with right hand sides given by

\[ f_3(V) := (F, V)_\Omega - (g_N, \gamma_D^c V)_t - (W_\kappa(g_D), \gamma_D^c V)_t, \]
\[ g_3(\mu) := (\mu, (K_\kappa - \frac{1}{2} \text{Id})(g_D))_t, \]
\[ h_3(q) := (W_\kappa(g_D), q)_t. \]

An essential feature of the stabilised variational formulation is that the auxiliary unknown \( p \) obtained from the solutions \( (E, \lambda, p) \) to the mixed variational formulation (5.17) can be recast into the following expression

\[ p = M((\frac{1}{2} \text{Id} + K'_\kappa)(\lambda) + W_\kappa(\gamma_D^c E + g_D)). \]

At second glance, we realize that \( p = 0 \), if \( (E, \theta) \) solves (5.17). This follows directly from lemma 5.9 and (5.5). Summing up, \( p \) is a “dummy variable”.

It remains to establish a generalized Gårding inequality for the sesquilinear form \( a_{\text{reg}} \) on the \( \mathcal{W} \). To do so, we use again the splittings from section 4.6 and group the components according to their nature into electric \((\lambda^\perp, E^0)\), magnetic \((\lambda^0, E^\perp)\), and auxiliary ones \( p \). Thus we arrive at a variational formulation on the Hilbert space

\[ \mathcal{W} := \mathcal{V} \times H(\text{curl}_\Gamma, \Gamma), \]

that is endowed with its natural graph norm:

Find \( (\lambda^\perp, E^0, \lambda^0, E^\perp, p) \in \mathcal{W} \) such that for all \( (\mu^\perp, V^0, \mu^0, V^\perp, q) \in \mathcal{W} \) there holds

\[ a_{\text{reg}}((\lambda^\perp, E^0, \lambda^0, E^\perp, p), (\mu^\perp, V^0, \mu^0, V^\perp, q)) = \hat{f}_{\text{reg}}((\mu^\perp, V^0, \mu^0, V^\perp, q)). \]

Again, the sesquilinear form and the linear form of the split variational equation are related to those underlying (5.17), namely \( a_{\text{reg}} \) and \( f_{\text{reg}} \), through the following equations

\[ \hat{a}_{\text{reg}}((\lambda^\perp, E^0, \lambda^0, E^\perp, p), (\mu^\perp, V^0, \mu^0, V^\perp, q)) := a_{\text{reg}}((B^0 + E^0, \lambda^\perp + \lambda^0, p), (V^\perp - V^0, -\mu^\perp + \mu^0, q)), \]
\[ \hat{f}_{\text{reg}}((\mu^\perp, V^0, \mu^0, V^\perp, q)) := f_{\text{reg}}((V^\perp - V^0, -\mu^\perp + \mu^0, q)). \]

As in the symmetric case, switching between \( \hat{a}_{\text{reg}} \) and \( a_{\text{reg}} \) involves the application of two sign-flip isomorphisms, as defined in section 4.6.

In order to settle the issue of existence and uniqueness of solutions to (5.17) we first observe that by the very definition of \( M \) in (5.13) and (5.16) the first two components of \( (U, \lambda, p) \) of (5.17) will also solve (5.14) and thus lemma 5.9 ensures uniqueness. The next lemma tells us that we do not need to worry about the new terms introduced into the variational equations.
Lemma 5.10. The following sesquilinear forms are compact
\[ (\cdot, \cdot)_t : H^{-1/2}(\text{div}, \Gamma) \times H(\text{curl}, \Gamma) \to \mathbb{C}, \]
\[ (\mathcal{W}_\kappa(\cdot), \cdot)_t : H^{-1/2}(\text{curl}, \Gamma) \times H(\text{curl}, \Gamma) \to \mathbb{C}, \]
\[ \left( \left( \frac{1}{2} \text{Id} + K'_\kappa \right)(\cdot), \cdot \right)_t : H^{-1/2}(\text{div}, \Gamma) \times H(\text{curl}, \Gamma) \to \mathbb{C}. \]

Proof. It is sufficient to note that the sesquilinear forms
\[ (\cdot, \cdot)_t, \left( \left( \frac{1}{2} \text{Id} + K'_\kappa \right)(\cdot), \cdot \right)_t : H^{-1/2}(\text{div}, \Gamma) \times H^{-1/2}(\text{curl}, \Gamma) \to \mathbb{C} \]
and
\[ (\mathcal{W}_\kappa(\cdot), \cdot)_t : H^{-1/2}(\text{curl}, \Gamma) \times H^{-1/2}(\text{curl}, \Gamma) \to \mathbb{C} \]
are continuous and that the injection from \( H(\text{curl}, \Gamma) \) into \( H^{-1/2}(\text{curl}, \Gamma) \) is compact due to lemma 5.7.

As an immediate consequence we note that all additional off-diagonal terms of (5.17) are compact. Furthermore, the sesquilinear form
\[ b : H(\text{curl}, \Gamma) \times H(\text{curl}, \Gamma) \to \mathbb{C} \]
is clearly elliptic, since it gives rise to an inner product on \( H(\text{curl}, \Gamma) \). In combination with theorem 5.4 we conclude that the sesquilinear form \( \mathcal{a}_{\text{reg}} : \mathcal{W} \times \mathcal{W} \to \mathbb{C} \) from (5.18) satisfies a Gårding inequality on \( \mathcal{W} \), which is equivalent to a generalized Gårding inequality for the sesquilinear form \( \mathcal{a}_{\text{reg}} \) on the Hilbert space
\[ \mathcal{W} := H(\text{curl}, \Omega) \times H^{-1/2}(\text{div}, \Gamma) \times H(\text{curl}, \Gamma), \]
that is endowed with its natural graph norm. In both cases, a Fredholm argument ensures existence of solutions from the uniqueness result. Thus we have obtained a well-posed variational formulation which yields weak solutions to the transmission problem. Moreover, with operator products removed, (5.17) is amenable to standard Galerkin discretizations by means of conforming finite element and boundary element spaces.

Remark 5.11. The stabilised variational formulation (5.17) could be derived in the same way as in the case of the Helmholtz transmission problem in section 3.3, using a bijective trace transformation operator \( T \) of the following form
\[ T := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : H^{-1/2}(\text{curl}, \Gamma) \times H^{-1/2}(\text{div}, \Gamma) \to H^{-1/2}(\text{curl}, \Gamma) \times H^{-1/2}(\text{div}, \Gamma), \]
whose operator blocks satisfy the assumptions:

1. The operators
\[ A : H^{-1/2}(\text{curl}, \Gamma) \to H^{-1/2}(\text{curl}, \Gamma), \]
\[ D : H^{-1/2}(\text{div}, \Gamma) \to H^{-1/2}(\text{div}, \Gamma), \]
are bounded and bijective.

2. The operators
\[ B : H^{-1/2}(\text{div}, \Gamma) \to H^{-1/2}(\text{curl}, \Gamma), \]
\[ C : H^{-1/2}(\text{curl}, \Gamma) \to H^{-1/2}(\text{div}, \Gamma), \]
are compact.
5.3. STABILISED COUPLING

The same trick as in section 3.3 allows us to construct generalized versions of the exterior Calderón projector $P_+$, that takes into account the scattered field on $\Omega^+$ in a coupled variational formulation. Finally, introducing the new unknown $\lambda := \gamma_N^+ E^s$ we arrive at a formally regularised problem:

Find $E \in H(\text{curl}, \Omega)$ and $\lambda \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ such that for all $V \in H(\text{curl}, \Omega)$ and $\mu \in H^{-1/2}(\text{div}_\Gamma, \Gamma)$ there holds

\[
q_k(E, V) + \left((W_k + D^{-1} \circ C \circ (1/2 \text{Id} - K_k))(\gamma_D^+ E), \gamma_D^+ V\right)_t \\
+ \left((K_k' - 1/2 \text{Id} + D^{-1} \circ C \circ V_k(\lambda), \gamma_D^+ V\right)_t = f_4(V), \\
(\mu, (1/2 \text{Id} - K_k + A^{-1} \circ B \circ W_k)(\gamma_D^+ E))_t \\
+ (\mu, (V_k + A^{-1} \circ B \circ (1/2 \text{Id} + K_k))(\lambda))_t = g_4(\mu),
\]

with right hand sides given by

\[
f_4(V) := (F, V)_{\Omega^+} - (g_N, \gamma_N^+ V)_t - ((W_k + D^{-1} \circ C \circ (1/2 \text{Id} - K_k))(g_D), \gamma_D^+ V)_t,
\]

\[
g_4(\mu) := (\mu, (K_k - 1/2 \text{Id} - A^{-1} \circ B \circ W_k)(g_D))_t.
\]

By construction and due to theorem 5.4, the sesquilinear form underlying the variational formulation (5.19) satisfies a generalized Gårding on $V$. Unfortunately, uniqueness of solutions does not hold in general, but needs to be enforced by a careful choice of the operators $A$, $B$, $C$, and $D$. The following range criterion

\[
\text{Range } (T \circ P_+) \cap \{0\} \times H^{-1/2}(\text{div}_\Gamma, \Gamma) = \{0\}. 
\]

turns out to be a sufficient condition, which guarantees uniqueness of solutions to (5.19). A closer look reveals that (5.20) can be derived from

\[
\text{Im } \{ (\mu, (A^{-1} \circ B)(\mu))_t \} = 0 \iff \mu = 0,
\]

which, in general, is much easier to verify.

5.3.2 Galerkin Discretization

This section introduces discrete spaces for a conforming discretization of the symmetric and regularised coupling approach from sections 5.2.1 and 5.3. The main references here are the articles [73, 68].

Our construction of finite element and boundary element spaces for the approximation of the total field $E$ in $H(\text{curl}, \Omega)$ and the Neumann trace $\lambda$ in $H^{-1/2}(\text{div}_\Gamma, \Gamma)$ strongly relies on the discretization schemes introduced in [52, Sect. 9]. To start with, we equip the possibly curvilinear polyhedron $\Omega$ with a family of shape-regular, tetrahedral meshes $\{\Omega_h\}_h$. Here and below, $h \in \mathbb{R}$ denotes the mesh width, which evaluates to the length of the longest edge contained in $\Omega_h$. In addition, we write $T_h$ for the collection of all tetrahedra contained in $\Omega_h$. By plain restriction of $\Omega_h$ to its boundary we obtain a sequence $\{\Gamma_h\}_h$ of surface triangulations which inherit shape-regularity from $\{\Omega_h\}_h$. Moreover, we assume that all surface meshes are aligned with edges of $\Gamma$.

Due to their exceptional algebraic properties, discrete 1-forms or edge elements should be used to approximate electric fields in $\Omega$: For a fixed polynomial degree $\nu \in \mathbb{N}_0$ and any tetrahedron $T \in T_h$, the local spaces are defined by

\[
S_{\nu+1}(T) := \{ U \in (P_{\nu+1}(T))^3; U(x) \cdot x = 0 \ \forall x \in T \},
\]

\[
E_{\nu+1}(T) := (P_{\nu}(T))^3 \oplus S_{\nu+1}(T),
\]
where $\mathcal{P}_k(T)$ denotes the space of multivariate polynomials of total degree $k \in \mathbb{N}_0$ on $T$ (cf. [68]), which gives rise to the global finite element space

$$\mathcal{E}_{\nu+1}(\Omega_h) := \{ U \in H(\text{curl}, \Omega); \, U|_T \in \mathcal{E}_{\nu+1}(T) \, \forall T \in \mathcal{T}_h \}.$$  

Tangential continuity of $\mathcal{E}_{\nu+1}(\Omega_h)$ across inter-element faces renders degrees of freedom based on moments of tangential components on edges, faces, and elements well-defined (see [49, 27] for details). This allows us to introduce nodal interpolation operators $\Pi_{\nu+1}$, mapping onto $\mathcal{E}_{\nu+1}(\Omega_h)$, which have been initially defined for continuous vector fields only, but have been extended to less regular settings, cf. [27, Lem. 3.2, 3.3].

**Lemma 5.12.** If $s > \frac{1}{2}$, then for all $U \in H^s(\Omega)$ such that $\text{curl} U \in H^s(\Omega)$

$$\|U - \Pi_{\nu+1}(U)\|_{L^2(\Omega)} \leq C h^{\min\{\nu+1,s\}} \left( \|U\|_{H^s(\Omega)} + \|\text{curl} U\|_{H^s(\Omega)} \right),$$

$$\|\text{curl}(U - \Pi_{\nu+1}(U))\|_{L^2(\Omega)} \leq C h^{\min\{\nu+1,s\}} \|\text{curl} U\|_{H^s(\Omega)},$$

with constants $C > 0$ depending only on $\Omega, \nu, s$ and the shape-regularity of the meshes.

Note that the previous lemma does not guarantee continuity of the nodal interpolation operator $\Pi_{\nu+1}$ on the entire space $H(\text{curl}, \Omega)$. In addition to edge elements, we also introduce $H(\text{div}, \Omega)$-conforming finite element spaces $\mathcal{F}_{\nu+1}(\Omega_h)$ of discrete 2-forms or face elements, cf. [11, Chap. 3] and [68]. Here and below $\mathcal{P}_k(T)$ denotes the set of multivariate homogeneous polynomials of total degree $k \in \mathbb{N}_0$ on $T$. This allows us to introduce local finite element space

$$\mathcal{F}_{\nu+1}(T) := (\mathcal{P}_\nu(T))^3 \oplus \mathcal{P}_{\nu+1}(T), \quad x \in T,$$

which serves as a starting point for an element-by-element definition of the global of finite element space

$$\mathcal{F}_{\nu+1}(\Omega_h) := \{ U \in H(\text{div}, \Omega); \, U|_T \in \mathcal{F}_{\nu+1}(T) \, \forall T \in \mathcal{T}_h \}.$$  

Suitable degrees of freedom for these spaces are provided by moments of face fluxes and weighted integrals over elements, which allow us to introduce the nodal interpolation operators $\Upsilon_{\nu+1}$ onto $\mathcal{F}_{\nu+1}(\Omega_h)$. Now a straightforward application of the Stokes theorem, cf. [49], yields the following **commuting diagram property**

$$\text{curl} \circ \Pi_{\nu+1} = \Upsilon_{\nu+1} \circ \text{curl}, \quad (5.21)$$

which holds for sufficiently smooth vector fields in $\Omega$. From relation (5.21) we immediately conclude that $\Pi_{\nu+1}$ leaves the kernel of the $\text{curl}$ invariant. Furthermore, a direct application of the Gauss theorem provides us with another valuable property

$$\text{div} \circ \Upsilon_{\nu+1} = Q_\nu \circ \text{div}, \quad (5.22)$$

which holds for sufficiently smooth vector fields in $\Omega$, where $Q_\nu$ denotes the $L^2(\Omega)$-projection onto the space $Q_\nu(\Omega_h)$ of piecewise polynomial functions of total degree $\nu$ in $\Omega_h$. Hence, we obtain that $\Upsilon_{\nu+1}$ preserves the kernel of the $\text{div}$. Now a straightforward combination of both (5.21) and (5.22) yields the following commuting diagram, cf. [50, Sect. 3]
The construction of conforming boundary element spaces for $H^{-1/2}(\text{div}, \Gamma)$ can also be based on differential forms. Since $H^{-1/2}(\text{div}, \Gamma)$ consists of tangential traces of the magnetic field, which can also be described by 1-forms, we conclude that $H^{-1/2}(\text{div}, \Gamma)$ should be approximated by tangential traces of edge elements $\gamma_\times(\mathcal{E}_{\nu+1}(\Omega_h))$, which boils down to two-dimensional face elements $\mathcal{F}_{\nu+1}(\Gamma_h)$ on the surface mesh $\Gamma_h$, cf. [73]. Furthermore, the degrees of freedom are inherited from $\mathcal{E}_{\nu+1}(\Omega_h)$ and by the very construction of $\mathcal{F}_{\nu+1}(\Gamma_h)$ the induced nodal interpolation operator $\gamma_{\nu+1}^\Gamma$ satisfies

$$\gamma_{\nu+1}^\Gamma \circ \gamma_\times = \gamma_\times \circ \Pi_{\nu+1},$$

which implies further commuting diagram properties, namely

$$\gamma_{\nu+1}^\Gamma \circ \gamma_\times = \gamma_\times \circ \Pi_{\nu+1},$$

that are valid for sufficiently smooth tangential vector fields on the boundary $\Gamma_h$

$$C^\infty(\Gamma) \xrightarrow{\text{curl}} C^\infty(\Gamma) \xrightarrow{\text{div}} C^\infty(\Gamma)$$

$$\Pi_{\nu+1}^\Gamma \xrightarrow{\text{curl}} \mathcal{E}_{\nu+1}(\Gamma_h) \xrightarrow{\gamma_{\nu+1}^\Gamma} \mathcal{F}_{\nu+1}(\Gamma_h) \xrightarrow{\Pi_{\nu+1}^\Gamma} Q_{\nu}^\Gamma,$$

cf. [11, Prop. 3.7]. Here and below, $Q_{\nu}^\Gamma$ denotes the $L^2(\Gamma)$-orthogonal projection onto the space $Q_{\nu}(\Gamma_h)$ of piecewise polynomial functions of degree $\nu$ on $\Gamma_h$. Moreover, we immediately conclude that $\gamma_{\nu+1}^\Gamma$ preserves the kernel of the $\text{div}$ and satisfies the following interpolation error estimate, cf. [73] and [55, Sect. 5].

**Lemma 5.13.** If $\mu \in H^s(\Gamma)$, $\text{div} \mu \in H^s(\Gamma)$ for some $s > 0$, then

$$\| \mu - \gamma_{\nu}^\Gamma(\mu) \|_{L^2(\Gamma)} \leq C h^{\min\{s,\nu+1\}} \left( \| \mu \|_{H^s(\Gamma)} + \| \text{div} \mu \|_{H^s(\Gamma)} \right),$$

$$\| \text{div}(\mu - \gamma_{\nu}^\Gamma(\mu)) \|_{L^2(\Gamma)} \leq C h^{\min\{s,\nu+1\}} \| \text{div} \mu \|_{H^s(\Gamma)}.$$

Finally, it remains to choose a suitable conforming discrete trial space for $H(\text{curl}, \Gamma)$. Picking an arbitrary function $q_h$ from such a space, it must feature $q_h \in L^2(\Gamma)$, as well as $\text{curl} \ q_h \in L^2(\Gamma)$. Thus a suitable choice would be to take $\gamma_{\times}(\mathcal{E}_{\nu+1}(\Omega_h))$, which creates exactly the space of $H(\text{curl}, \Gamma)$-conforming surface edge elements $\mathcal{E}_{\nu+1}(\Gamma_h)$, cf. [73, Rem. 5.29]. The degrees of freedom are again inherited from $\mathcal{E}_{\nu+1}(\Omega_h)$. Note that this boundary element space could also be derived by plain rotation of all vector quantities in lemma 5.13 in combination with (4.13) we obtain the following interpolation estimates on $H(\text{curl}, \Gamma)$.

$$\gamma_{\nu+1}^\Gamma \circ \Pi_{\nu+1}^\Gamma = Q_{\nu}^\Gamma \circ \gamma_{\times} \circ \Pi_{\nu+1}^\Gamma,$$

which translates into the following commuting diagram

$$C^\infty(\Gamma) \xrightarrow{\text{curl}} C^\infty(\Gamma)$$

$$\Pi_{\nu+1}^\Gamma \xrightarrow{\text{curl}} \mathcal{E}_{\nu+1}(\Gamma_h) \xrightarrow{\gamma_{\nu+1}^\Gamma} \mathcal{F}_{\nu+1}(\Gamma_h) \xrightarrow{\Pi_{\nu+1}^\Gamma} Q_{\nu}(\Gamma),$$

Again, we conclude invariance of $\text{Ker} (\text{curl}) \cap \text{Dom} (\Pi_{\nu+1}^\Gamma)$ under $\Pi_{\nu+1}^\Gamma$. Moreover, by plain rotation of all vector quantities in lemma 5.13 in combination with (4.13) we obtain the following interpolation estimates on $H(\text{curl}, \Gamma)$. 

\begin{align*}
\gamma_{\nu+1}^\Gamma \circ \gamma_\times &= \gamma_\times \circ \Pi_{\nu+1}, \\
\gamma_{\nu+1}^\Gamma \circ \gamma_\times &= \gamma_\times \circ \Pi_{\nu+1}, \\
\gamma_{\nu+1}^\Gamma \circ \gamma_\times &= \gamma_\times \circ \Pi_{\nu+1}, \\
\gamma_{\nu+1}^\Gamma \circ \gamma_\times &= \gamma_\times \circ \Pi_{\nu+1},
\end{align*}
Lemma 5.14. If \( q \in H^s(\Gamma) \), \( \text{curl}_\Gamma q \in H^s(\Gamma) \) for some \( s > 0 \), then
\[
\|q - \Pi^{\Gamma}_{s+1}(q)\|_{L^2(\Gamma)} \leq C h^\min\{s,s+1\} \left( \|q\|_{H^s(\Gamma)} + \|\text{curl}_\Gamma q\|_{H^s(\Gamma)} \right),
\]
\[
\|\text{curl}_\Gamma(q - \Pi^{\Gamma}_{s+1}(q))\|_{L^2(\Gamma)} \leq C h^\min\{s,s+1\} \|\text{curl}_\Gamma q\|_{H^s(\Gamma)}.
\]

Based on the conforming finite element spaces, the Galerkin discretization of the variational problem (5.17) is straightforward:

Find \( E_h \in \mathcal{E}_{s+1}(\Omega_h) \), \( \lambda_h \in \mathcal{F}_{s+1}(\Gamma_h) \), and \( p_h \in \mathcal{E}_{s+1}(\Omega_h) \) such that for all \( V_h \in \mathcal{E}_{s+1}(\Omega_h) \), \( \mu_h \in \mathcal{F}_{s+1}(\Gamma_h) \), and \( q_h \in \mathcal{E}_{s+1}(\Gamma_h) \) there holds
\[
q_{\kappa}(E_h, V_h) + (W_{\kappa}(\gamma D E_h), \gamma D V_h)_t + (\langle K_{\kappa} - \frac{1}{2}I \rangle (\lambda_h), \gamma D V_h)_t = f_3(V_h),
\]
\[
(\langle \mu_h, (\frac{1}{2}I - K_{\kappa}) (\gamma D E_h) \rangle)_t + (\mu_h, \mathcal{V}_h(\lambda_h))_t - \eta(\mu_h, p_h)_t = g_3(\mu_h),
\]
\[
- (W_{\kappa}(\gamma D E_h), q_h)_t - \left( \langle \frac{1}{2}I + K_{\kappa} \rangle (\lambda_h), q_h \right)_t + b(p_h, q_h) = h_3(q_h).
\]

5.3.3 Discrete Inf-Sup Estimates

The splitting idea, which was used to prove coercivity on the continuous level, has to be adopted for a numerical analysis on the discrete level as well. Following the ideas of [52, Sect. 10], we apply the nodal interpolation operators to the Helmholtz and Hodge-type splittings from section 4.6.

We start with the construction of a discrete counterpart to \( X(\text{curl}, \Omega) \). To do so, we formally define the discrete analogue of the projector \( P \) defined in section 4.6
\[
P_{s+1} : \mathcal{E}_{s+1}(\Omega_h) \mapsto \mathcal{E}_{s+1}(\Omega_h), \quad P_{s+1}(U_h) := (\Pi_{s+1} \circ P)(U_h),
\]
for \( U_h \in \mathcal{E}_{s+1}(\Omega_h) \). Unfortunately, even on \( H^1(\Omega) \) the nodal interpolation operator \( \Pi_{s+1} \) fails to be bounded, since the regularity of the \( \text{curl} \) is not controlled. However, since \( P_{s+1} \) will only be applied to finite element functions from \( \mathcal{E}_{s+1}(\Omega_h) \), the following lemma saves the idea, cf. [50, Lem. 4.6] and [5, Sect. 4].

Lemma 5.15. If \( U \in H^s(\Omega) \) for some \( \frac{1}{2} \leq s \leq 1 \) and \( \text{curl} U \in \mathcal{F}_{s+1}(\Omega_h) \), then \( U \in \text{Dom} \left( \Pi_{s+1} \right) \) and
\[
\|U - \Pi_{s+1}(U)\|_{L^2(\Omega)} \leq C h^s \|U\|_{H^s(\Omega)},
\]
with \( C > 0 \) depending only on \( \Omega, \nu, s \), and the shape regularity of \( \Omega_h \).

The features of \( P \) derived in section 4.6 and the commuting diagram property (5.21) imply that \( \text{curl} P(U_h) \in \mathcal{F}_{s+1}(\Omega_h) \) for \( U_h \in \mathcal{E}_{s+1}(\Omega_h) \). Hence, we conclude that \( P_{s+1} \) is a \( h \)-uniformly continuous projection, which preserves the \( \text{curl} \) and satisfies \( \text{Ker} (P_{s+1}) = \text{Ker} (\text{curl}) \cap \mathcal{E}_{s+1}(\Omega_h) \). Moreover, the following definition
\[
X_h(\text{curl}, \Omega_h) := P_{s+1}(\mathcal{E}_{s+1}(\Omega_h)), \quad N_h(\text{curl}, \Omega_h) := \text{Ker} (\text{curl}) \cap \mathcal{E}_{s+1}(\Omega_h),
\]
provides us with a \( h \)-uniformly \( H(\text{curl}, \Omega) \)-stable splitting
\[
\mathcal{E}_{s+1}(\Omega_h) = X_h(\text{curl}, \Omega_h) \oplus N_h(\text{curl}, \Omega_h).
\]

The following lemma provides us with the necessary means to pursue the same strategy for the face element spaces \( \mathcal{F}_{s+1}(\Gamma_h) \), cf. [55, Lem. 6.2].

Lemma 5.16. If \( \mu \in H^s(\Gamma) \) and \( \text{div}_\Gamma \mu \in \mathcal{Q}_s(\Gamma_h) \), for some \( 0 < s \leq 1 \), then \( \mu \in \text{Dom} \left( \Upsilon^\Gamma_{s+1} \right) \) and
\[
\|\mu - \Upsilon^\Gamma_{s+1}(\mu)\|_{L^2(\Gamma)} \leq C h^s \|\mu\|_{H^s(\Gamma)},
\]
with \( C > 0 \) only depending on \( \Gamma, \nu, s \), and the shape-regularity of the meshes.
Thus, the following operator

\[ P_{\nu+1}^\Gamma : \mathcal{F}_{\nu+1} (\Gamma_h) \mapsto \mathcal{F}_{\nu+1} (\Gamma_h) , \quad P_{\nu+1}^\Gamma (\mu_h) := (\Psi_{\nu+1}^\Gamma \circ P^\Gamma) (\mu_h) , \]

for \( \mu_h \in \mathcal{F}_{\nu+1} (\Gamma_h) \), provides us with a \( h \)-uniformly continuous projector, which preserves \( \text{div}_\Gamma \) and fulfills \( \text{Ker} (P_{\nu+1}^\Gamma) = \text{Ker} (\text{div}_\Gamma) \cap \mathcal{F}_{\nu+1} (\Gamma_h) \). Here \( P^\Gamma \) denotes the projection onto \( X (\text{div}_\Gamma, \Gamma) \) defined in section 4.6. Moreover, \( P_{\nu+1}^\Gamma \) yields the desired \( h \)-uniformly stable splitting of the discrete Neumann trace space

\[ \mathcal{F}_{\nu+1} (\Gamma_h) = X_h (\text{div}_\Gamma, \Gamma_h) \oplus N_h (\text{div}_\Gamma, \Gamma_h) , \]

with

\[ X_h (\text{div}_\Gamma, \Gamma_h) := P_{\nu+1}^\Gamma (\mathcal{F}_{\nu+1} (\Gamma_h)) , \quad N_h (\text{div}_\Gamma, \Gamma_h) := \text{Ker} (\text{div}_\Gamma) \cap \mathcal{F}_{\nu+1} (\Gamma_h) . \]

Notice that vector fields in \( \mathcal{E}_{\nu+1} (\Omega_h) \) only have continuous tangential components across inter-element faces, whereas any vector field in \( H^1 (\Omega) \) is necessarily continuous. On the other hand, there are elements in \( X_h (\text{div}_\Gamma, \Gamma_h) \) that are not twisted tangential traces of continuous vector fields. Summing up, we have

\[ X_h (\text{curl}_\Gamma, \Omega_h) \not\subset X (\text{curl}_\Gamma, \Omega) , \quad X_h (\text{div}_\Gamma, \Gamma_h) \not\subset X (\text{div}_\Gamma, \Gamma) , \]
\[ N_h (\text{curl}_\Gamma, \Omega_h) \subset N (\text{curl}_\Gamma, \Omega) , \quad N_h (\text{div}_\Gamma, \Gamma_h) \subset N (\text{div}_\Gamma, \Gamma) . \]

Thus by choosing

\[ \mathcal{W}_h := X_h (\text{div}_\Gamma, \Gamma_h) \times N_h (\text{curl}_\Gamma, \Omega_h) \times N_h (\text{div}_\Gamma, \Gamma_h) \times X_h (\text{curl}_\Gamma, \Omega_h) \times \mathcal{E}_{\nu+1} (\Gamma_h) , \]

as a discrete approximation space for \( \mathcal{W} \), we have made a non-conforming choice, since \( \mathcal{W}_h \not\subset \mathcal{W} \). Note that this is a special type of non-conformity, since it does not arise from the choice of discrete spaces, but from the way they are split. However, the Gårding inequality for \( \mathcal{a}_{\text{reg}} \) was only established with respect to the split space \( \mathcal{W}_h \). This prevents us from applying the well-known results about convergence of conforming Galerkin discretizations of coercive variational problems.

In light of the theory presented in section 4.5, the situation is not completely lost. For a numerical analysis, we can switch back to the sesquilinear form \( \mathcal{a}_{\text{reg}} \), which satisfies a generalized Gårding inequality on the Hilbert space \( \mathcal{W} \), by an application of two sign-flip isomorphisms (see sections 4.6 for details). According to assumption 4.18 and theorem 4.20, discrete inf-sup estimates can be established by constructing suitable decompositions of \( \mathcal{W} \). We start by splitting the continuous trial space \( \mathcal{W} = \mathcal{X} \oplus \mathcal{N} \) into the following components

\[ \mathcal{X} := X (\text{curl}_\Gamma, \Omega) \times X (\text{div}_\Gamma, \Gamma) \times H (\text{curl}_\Gamma, \Gamma) , \]
\[ \mathcal{N} := N (\text{curl}_\Gamma, \Omega) \times N (\text{div}_\Gamma, \Gamma) \times \{0\} . \]

Moreover, we introduce the discrete trial space

\[ \mathcal{W}_h := \mathcal{E}_{\nu+1} (\Omega_h) \times \mathcal{F}_{\nu+1} (\Gamma_h) \times \mathcal{E}_{\nu+1} (\Gamma_h) , \]

and conclude that, due to lemmata 5.12, 5.13, and 5.14, \( \mathcal{W}_h \) is approximating in \( \mathcal{W} \) and thus item 1. of assumption 4.18 holds. In addition, we also split the discrete trial space \( \mathcal{W}_h = \mathcal{X}_h \oplus \mathcal{N}_h \) according to

\[ \mathcal{X}_h := X_h (\text{curl}_\Gamma, \Omega_h) \times X_h (\text{div}_\Gamma, \Gamma_h) \times \mathcal{E}_{\nu+1} (\Gamma_h) , \]
\[ \mathcal{N}_h := N_h (\text{curl}_\Gamma, \Omega_h) \times N_h (\text{div}_\Gamma, \Gamma_h) \times \{0\} . \]
into a non-conforming component \( X_h \subsetneq X \) and a conforming component \( N_h \subset N \), for which \( \delta(N, N_h) = 0 \) holds. According to remark 4.21, we can rely on a suitable bridge mapping to establish the gap property of assumption 4.18. A suitable operator can be constructed in a component-wise fashion on the discrete trial spaces \( X_h (\text{curl}, \Omega_h) \) and \( X_h (\text{div}, \Gamma_h) \).

We start with the following definition of a bridge mapping for \( X_h (\text{curl}, \Omega_h) \)

\[
B_{\nu+1} : X_h (\text{curl}, \Omega_h) \to X (\text{curl}, \Omega), \quad B_{\nu+1}(U_h) := P(U_h),
\]

for \( U_h \in X_h (\text{curl}, \Omega_h) \), cf. [52, Sect. 11]. Moreover, from the properties of \( P \) and \( P_{\nu+1} \) we immediately derive the identities

\[
\begin{align*}
(\Pi_{\nu+1} \circ B_{\nu+1})(U_h) &= U_h, \quad \text{curl} B_{\nu+1}(U_h) = \text{curl} U_h \in F_{\nu+1}(\Omega_h),
\end{align*}
\]

for all \( U_h \in X_h (\text{curl}, \Omega_h) \). Hence, we can employ lemma 5.15 and establish the following estimate

\[
\left\| U_h - B_{\nu+1}(U_h) \right\|_{L^2(\Omega)} \leq C h \left\| B_{\nu+1}(U_h) \right\|_{H^1(\Omega)} \leq C h \left\| \text{curl} U_h \right\|_{L^2(\Omega)},
\]

where the constant \( C > 0 \) only depends on \( \Omega, \nu \), and the shape regularity of \( \Omega_h \). A similar construction can also be used to introduce a bridge mapping for \( X_h (\text{div}, \Gamma_h) \)

\[
B_{\nu+1}^\Gamma : X_h (\text{div}, \Gamma_h) \to X (\text{div}, \Gamma), \quad B_{\nu+1}^\Gamma(\mu_h) := P\Gamma(\mu_h),
\]

for \( \mu_h \in X_h (\text{div}, \Gamma_h) \), cf. [52, Sect. 11]. As in the previous case, the properties of both \( P\Gamma \) and \( P_{\nu+1}^\Gamma \) provide us with the following identities

\[
\begin{align*}
(T_{\nu+1}^\Gamma \circ B_{\nu+1}^\Gamma)(\mu_h) &= \mu_h, \quad \text{div} \Gamma B_{\nu+1}^\Gamma(\mu_h) = \text{div} \Gamma \mu_h \in Q_v(\Gamma_h),
\end{align*}
\]

which hold for all \( \mu_h \in X_h (\text{div}, \Gamma_h) \). Thus, a straightforward application of lemma 5.16 allows us to establish the following estimate

\[
\left\| \mu_h - B_{\nu+1}^\Gamma(\mu_h) \right\|_{L^2(\Gamma)} = \left\| (T_{\nu+1}^\Gamma - \text{Id}) \circ B_{\nu+1}^\Gamma(\mu_h) \right\|_{L^2(\Gamma)} \leq C h^{1/2} \left\| B_{\nu+1}^\Gamma(\mu_h) \right\|_{H^{1/2}(\Gamma)} \leq C h^{1/2} \left\| \text{div} \Gamma \mu_h \right\|_{H^{-1/2}(\Gamma)},
\]

with \( C > 0 \) depending only on \( \Gamma, \nu \) and the shape regularity of the meshes. These results motivate the following definition of a bridge mapping for \( \mathcal{W} \)

\[
B_h : \mathcal{W}_h \to \mathcal{W}, \quad B_h(U_h, \mu_h, p_h) := (B_{\nu+1}(U_h^\perp) + U_h^\perp, B_{\nu+1}^\Gamma(\mu_h^\perp) + \mu_h^\perp, p_h),
\]

for \( U_h \in \mathcal{E}_{\nu+1}(\Omega_h), \mu_h \in \mathcal{F}_{\nu+1}(\Gamma_h), \) and \( p_h \in \mathcal{E}_{\nu+1}(\Gamma_h) \). Combining the estimates (5.24) and (5.25) we end up with

\[
\delta(\mathcal{X}, \mathcal{X}_h) = \sup_{u_h \in \mathcal{X}_h} \inf_{u \in \mathcal{X}} \left\| u_h - u \right\|_{\mathcal{W}} \leq \sup_{w_h \in \mathcal{W}_h} \left\| w_h - B_h(w_h) \right\|_{\mathcal{W}} \leq C h^{1/2} \to 0,
\]

as \( h \to 0 \) and since \( \mathcal{N}_h \subset \mathcal{N} \), there holds \( \delta(\mathcal{N}, \mathcal{N}_h) = 0 \), which confirms the desired gap property for the discrete space \( \mathcal{W}_h \). Hence, we can apply theorem 4.20 and establish a discrete inf-sup estimate for the sesquilinear form \( a_{\text{reg}} \) on \( \mathcal{W}_h \): There exist minimal mesh width \( h_0 > 0 \) such that for all \( 0 < h < h_0 \) and the whole family \( \mathcal{W}_h \) the following estimate holds

\[
\sup_{0 \neq (v_h, \eta_h, r_h) \in \mathcal{W}_h} \frac{\text{Re} \left\{ a_{\text{reg}}((U_h, \mu_h, q_h), (V_h, \eta_h, r_h)) \right\}}{\left\| (V_h, \eta_h, r_h) \right\|_{\mathcal{W}}} \geq \gamma \left\| (U_h, \mu_h, q_h) \right\|_{\mathcal{W}},
\]

for all \((U_h, \mu_h, q_h) \in \mathcal{W}_h \) where the constant \( \gamma > 0 \) does not depend on \( h, U_h, \mu_h, \) or \( q_h \). Moreover, based on the discrete inf-sup estimate for the sesquilinear form \( a_{\text{reg}} \), the following asymptotic quasi-optimality estimate of the discretization error is straightforward.
Theorem 5.17. There exists a minimal mesh width $h_0$, depending only on $\Omega$, $\kappa$, $\nu$ and the shape regularity of the meshes $\Omega_h$, such that for every $h < h_0$ the discrete problem (5.23) has a unique solution $(E_h, \theta_h, p_h) \in E_{\nu+1}(\Omega_h) \times F_{\nu+1}(\Gamma_h) \times E_{\nu+1}(\Gamma_h)$, which is quasi-optimal in the following sense

$$
\|E - E_h\|_{H(\text{curl}, \Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\text{div}, \Gamma)} + \|p - p_h\|_{H(\text{curl}, \Gamma)} \\
\leq C \left( \inf_{V_h \in E_{\nu+1}(\Omega_h)} \|E - V_h\|_{H(\text{curl}, \Omega)} + \inf_{\mu_h \in F_{\nu+1}(\Gamma_h)} \|\lambda - \mu_h\|_{H^{-1/2}(\text{div}, \Gamma)} \right),
$$

with a constant $C > 0$ independent of $(E, \theta, p)$ and $h$.

Note that the auxiliary variable $p$ does not show up on the right hand side of the quasi-optimality estimate, which is guaranteed by $p = 0 \in E_{\nu+1}(\Gamma_h)$. However, we can not simply drop the auxiliary variable from (5.17), since the quasi-optimality estimate in theorem 5.17 is an asymptotic estimate, which only holds under sufficient approximation properties of the space $W_h$.

Remark 5.18. In the case of symmetric coupling, a conforming discretization for (5.6) is given by $E_{\nu+1}(\Omega_h) \times F_{\nu+1}(\Gamma_h)$. Based on the results above, existence and uniqueness of solutions to (5.6) on the continuous as well as on the discrete level can be established, provided that $\kappa$ does not correspond to one of the interior resonant frequencies. Moreover, the following quasi-optimality estimate of the discretization error holds

$$
\|E - E_h\|_{H(\text{curl}, \Omega)} + \|\lambda - \lambda_h\|_{H^{-1/2}(\text{div}, \Gamma)} \\
\leq C \left( \inf_{V_h \in E_{\nu+1}(\Omega_h)} \|E - V_h\|_{H(\text{curl}, \Omega)} + \inf_{\mu_h \in F_{\nu+1}(\Gamma_h)} \|\lambda - \mu_h\|_{H^{-1/2}(\text{div}, \Gamma)} \right),
$$

where the constant $C > 0$ is independent of the mesh width $h$ and the solutions $E$ and $\lambda$, [52, Thm. 11.1].

5.4 Conclusion

We have derived a regularised version of the exterior Calderón projector for the scattered field $E^S$. The construction is based on a trace transformation operator $T$, which features regularising operators $M$ that provide compact mappings from $H^{-1/2}(\text{div}, \Gamma)$ onto $H^{-1/2}(\text{curl}, \Gamma)$. The regularised Calderón projector provides us with a novel type of Dirichlet-to-Neumann maps, that can be used to couple boundary integral equations on the interface boundary $\Gamma$ together with variational equations in $\Omega$.

The stabilised variational formulation derived from the regularised Calderón projector features operator products of local and non-local operators on the interface boundary $\Gamma$ and thus eludes a straightforward Galerkin discretization by means of finite elements and boundary elements. However, a special choice of the regularising operator together with the introduction of an auxiliary, non-physical variable $p$ made it possible to get rid of all occurring operator products. Thus we have arrived at a mixed regularised Galerkin scheme for the approximate solution of time-harmonic Maxwell transmission problems. A special feature of these formulations is that the auxiliary variable $p$ evaluates to zero, independent of the wave number, the material parameters, the incident field, and the shape of the scatterer.

For the sesquilinear form underlying the mixed regularised variational formulation we have established a generalized Gårding inequality based on a Helmholtz-type decomposition of the total field in $\Omega$ and a Hodge-type decomposition of the Neumann trace on $\Gamma$. Furthermore, the
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special intrinsic structure of the trace transformation operator, made it possible to establish uniqueness of solutions for all positive frequencies. Hence, the proposed variational formulation manages to suppress all spurious modes related to interior resonant frequencies.

We have proposed a conforming discretization of the mixed regularised variational formulation, featuring edge elements on tetrahedra and triangles for the discretization of the total field in $\Omega_h$ and the auxiliary variable $p$ on $\Gamma_h$. In addition, we have introduced $\text{div}_\Gamma$-conforming Raviart-Thomas elements on triangles for the discretization of the Neumann trace on the interface boundary $\Gamma_h$. Furthermore, we have made use of bridge mappings to establish a gap property for $H(\text{curl}, \Omega) \times H^{-1/2}(\text{div}_\Gamma, \Gamma) \times H(\text{curl}_\Gamma, \Gamma)$. This made it possible to apply standard theory for certain non-coercive variational formulations and derive an asymptotic inf-sup estimate for the sesquilinear form underlying the coupled variational equations. Based on this estimate we have established existence, uniqueness and stability of discrete solutions together with an asymptotic quasi-optimality estimate for the discretization error, whose right hand side does not depend on the auxiliary variable $p$. Hence, we have established a coupled variational formulation which produces unique discrete solutions for all positive wave numbers, which converge with the optimal rate towards the exact solutions of the time-harmonic Maxwell transmission problem.

5.5 Open Questions

The following question concerning FEM-BEM coupling for time-harmonic Maxwell transmission problems remained open:

- So far we have not addressed implementation issues for time-harmonic Maxwell transmission problems. Straightforward implementations of the symmetric or regularised coupled variational formulations feature several fully-populated complex matrix blocks, due to the non-local character of the boundary integral operators associated with the electric field equation. These matrices are typically of size $N \times N$, where $N$ is proportional to the number of triangles contained in the surface mesh $\Gamma_h$. Moreover, since Maxwell’s equations are inherently three dimensional, $N$ typically ranges from a few ten thousand up to several millions panels, which clearly indicates the need for fast implementations of boundary element methods with low memory consumption.

Efficient techniques for the assembly of the system matrices and the evaluation of the matrix-vector products for boundary integral equations are well developed and we could use $\mathcal{H}$-matrices [43, 44], panel clustering [45] or fast multipole methods [41] to reduce the memory cost and speed up the assembly procedure and the evaluation of the matrix-vector product.

However, employing fast and efficient boundary element schemes means switching to a perturbed variational formulation, for which we no longer have discrete inf-sup conditions and quasi-optimality estimates at hand. Nevertheless, the resulting variational formulation may not be completely lost but the numerical analysis must now be based on Strang’s lemma.

- The regularised coupled variational formulation (5.14) features a continuous sesquilinear form $b : H(\text{curl}_\Gamma, \Gamma) \times H(\text{curl}_\Gamma, \Gamma) \rightarrow \mathbb{C}$, defined by

$$b(p, q) := (\text{curl}_\Gamma p, \text{curl}_\Gamma q)_\Gamma + (p, q)_\Gamma,$$

for $p, q \in H(\text{curl}_\Gamma, \Gamma)$, which gives rise to an inner product on $H(\text{curl}_\Gamma, \Gamma)$. On the discrete level, $b$ corresponds to a symmetric positive definite matrix $B$. Hence, the spectral condition number of system matrix corresponding to the stabilised variational formulation (5.14)
will substantially increase, which results in painfully long runtimes, when iterative solvers are used to compute approximate solutions for (5.14).

On one hand, we can switch to the Schur complement system by eliminating all degrees of freedom corresponding to the auxiliary solution component $p_h$. As in the Helmholtz case, the Schur complement system features the inverse of a sparse, symmetric positive definite matrix $B$, which corresponds to a Galerkin discretization of $b$ by means of $\text{curl}_p$-conforming edge elements on $\Gamma_h$. For an intermediate number of unknowns, $B^{-1}$ can be evaluated directly based on a QR or Cholesky decomposition, whereas for large numbers of unknowns a Krylov solver has to be employed.

On the other hand, a suitable preconditioner for the entire system matrix could greatly decrease the number of iterations an iterative solver needs to solve the linear system. Hence, the use of a preconditioner might be feasible, since the Schur complement only reduces the condition number down to the level of the symmetric coupling approach.
Bibliography


# Curriculum Vitae

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