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On the Algebraic Foundation of Bounded Cohomology

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für urs

If I were to rewrite it now I should not write (to use Prof. Littlewood's simile) like 'a missionary talking to cannibals', but with decent terseness and restraint; and, writing more shortly, I should be able to include a great deal more.

Godfrey Harold Hardy

Abstract

Bounded cohomology for topological spaces was introduced by Gromov in the late seventies, mainly to describe the simplicial volume invariant. It is an exotic cohomology theory for spaces in that it fails excision and thus cannot be represented by spectra. Gromov's basic vanishing result of bounded cohomology for simply connected spaces implies that bounded cohomology for spaces is an invariant of the fundamental group. To prove this, one is led to introduce a cohomology theory for groups and the present work is concerned with the latter, which has been studied by Gromov, Brooks, Ivanov and Noskov to name but the most important initial contributors. Generalizing these ideas from discrete groups to topological groups, Burger and Monod have developed continuous bounded cohomology in the late nineties.

It is a widespread opinion among experts that (continuous) bounded cohomology cannot be interpreted as a derived functor and that triangulated methods break down. We prove that this is wrong.

We use the formalism of exact categories and their derived categories in order to construct a classical derived functor on the category of Banach G-modules with values in Waelbroeck's abelian category. This gives us an axiomatic characterization of this theory for free and it is a simple matter to reconstruct the classical seminormed cohomology spaces out of Waelbroeck's category.

We prove that the derived categories of right bounded and of left bounded complexes of Banach G-modules are equivalent to the derived category of two abelian categories (one for each boundedness condition), a consequence of the theory of abstract truncation and hearts of t-structures. Moreover, we prove that the derived categories of Banach G-modules can be constructed as the homotopy categories of model structures on the categories of chain complexes of Banach G-modules thus proving that the theory fits into yet another standard framework of homological and homotopical algebra.

Zusammenfassung

Unter Benutzung des Formalismus der exakten Kategorien und der derivierten Kategorien zeigen wir, dass die weitverbreitete Meinung, beschränkte Cohomologie lasse sich nicht via derivierte Funktoren interpretieren, ganz einfach falsch ist. Mittels der Theorie der abstrakten Trunkierung rekonstruieren wir die Waelbroecksche Kategorie von Banachquotienten und einige Verallgemeinerungen. Diese Kategorien treten als Wertebereich unserer Definition der Banachcohomologie für Gruppen in Erscheinung, woraus wir die Theorie von Gromov-Brooks-Ivanov sowie die Burger-Monodsche Theorie rekonstruieren können. Weiter zeigen wir, dass sich diese Theorien auch mittels Modellkategorien behandeln lassen.

Contents

Introduction and Main Results

This work is based on the remarkable thesis of N. Monod [**Mon01**] which laid out the foundations for the theory of continuous bounded cohomology with a view towards applications in rigidity theory. It is the detailed presentation of joint results with his advisor M. Burger. The power and the merits of the machinery are well-known and the reader should consult the recent works of Burger, Monod and their collaborators in order to get an idea of the state of the art. Since Monod's thesis is quite heavy-going and the present work is probably even worse in this respect, we recommend the novice to ℓ^1-homology and bounded cohomology to consult Löh's beautiful thesis [**Löh07**], the declared—and happily also achieved—goal of which is to provide a lightweight approach to the theory.

We shall focus on the algebraic aspects of the theory in order to provide a good reason for the close resemblance of the fundamental *homological* results in bounded cohomology to their classical algebraic counterparts. Burger and Monod point out that the definitions of the central notions are *inspired* by relative homological algebra—cf. also N. V. Ivanov [**Iva85**]. Our point is that they are nothing but *instances* of the categorical notions up to a slight detail which stems from the interest in semi-norms.

Keeping track of semi-norms in homology is certainly of utmost importance to most applications up to date. Many technicalities are due to them and it seems that they are nothing but a nuisance for theoretical considerations. A clear-cut approach is probably best achieved by dropping the semi-norms and by recovering them only after the cohomology theory has been developed in full.

The main upshot of the present work is as follows:

(1) Underlying continuous bounded cohomology there is an exact category in the sense of Quillen with enough injectives.
(2) Continuous bounded cohomology arises from the total derived functor of the maximal invariant submodule functor.
(3) The heart of the canonical left t-structure on the derived category of Banach spaces is Waelbroeck's abelian category of quotient Banach spaces. Continuous bounded cohomology factors over this abelian category and can thus be viewed as a classical derived functor—modulo the fact that its domain of definition is not an abelian category. An ad hoc construction gives back the semi-normed space-valued functor of Burger and Monod.

Continuous bounded cohomology is a functional analytic variant of group cohomology. Objects of study are Hausdorff topological groups and their strongly continuous and isometric representations on Banach spaces. Since the category of Banach spaces is not abelian, the classical language does not apply. However, the category of Banach G-modules can be equipped with various exact structures in the sense of Quillen, for which the theory of the derived category allows us to speak of derived functors.

At the heart of the matter is Burger-Monod's "functorial characterization" which tells us that continuous bounded cohomology may be *computed* as if it were the classical derived functor of the G-invariants. By the work of Burger-Monod we know that cohomology in degree zero coincides with the G-invariants, cohomology vanishes on injectives in higher degrees and that there is a natural long exact sequence for every short exact sequence of G-modules. Thus, bounded cohomology should be the derived functor of the G-invariants and the main purpose of this work is to make this precise and to show that bounded cohomology for groups is not as exotic as is often claimed.

1. Reconstructing the Burger-Monod Theory

Our construction of bounded cohomology involves three essential choices and one *ad hoc* construction. These choices are justified by the original work of Burger and Monod, see [**Mon01**] and [**BM02**].

Fix as ground field k either the real or complex numbers. Let **Ban** be the category of Banach spaces over k and bounded k-linear maps.

Let G be a Hausdorff topological group and define the category $G-\mathbf{Ban}$ of *Banach G-modules* to be the strongly continuous and isometric representations of G on Banach spaces over k together with the bounded linear G-equivariant maps. This is an additive category which is finitely bicomplete and whose class of kernel-cokernel pairs coincides with the short sequences of Banach G-modules whose underlying sequence of k-vector spaces is short exact.

Notice that there is a functor $(-)^G : G-\mathbf{Ban} \to \mathbf{Ban}$ associating to a Banach G-module E the closed subspace E^G of G-invariant vectors. It is right adjoint to the trivial module functor $\mathbf{Ban} \to G-\mathbf{Ban}$ which arises from considering a Banach space as a Banach G-module with the trivial G-action.

The three choices are:

(i) On $G-\mathbf{Ban}$ choose the exact structure given by the class $\mathscr{E}_{\mathrm{rel}}^G$ of short sequences whose underlying sequence of Banach spaces is split exact. Burger and Monod have essentially shown that $(G-\mathbf{Ban}, \mathscr{E}_{\mathrm{rel}}^G)$ has enough injectives. Therefore the derived category $\mathbf{D}^+ (G-\mathbf{Ban}, \mathscr{E}_{\mathrm{rel}}^G)$ of right bounded complexes is equivalent to the homotopy category $\mathbf{K}^+ (\mathscr{I} (\mathscr{E}_{\mathrm{rel}}^G))$ of right bounded complexes of $\mathscr{E}_{\mathrm{rel}}^G$-injective Banach G-modules. In particular $\mathbf{D}^+ (G-\mathbf{Ban}, \mathscr{E}_{\mathrm{rel}}^G)$ has small Hom-sets and right derived functors exist.

(ii) On **Ban** choose the exact structure given by the class $\mathscr{E}_{\max}$ consisting of *all* kernel-cokernel pairs. It is well-known that $(\mathbf{Ban}, \mathscr{E}_{\max})$ has enough of both projective and injective objects.

(iii) Choose the *left* truncation functor $\tau_\ell^{\leq 0}$ on $\mathbf{Ch}(\mathbf{Ban})$ given on objects by

$$\tau_\ell^{\leq 0} E = (\cdots \to E^{-2} \to E^{-1} \to \operatorname{Ker} d_E^0 \to 0 \to \cdots).$$

It induces the *left* t-structure $(\mathbf{D}_\ell^{\leq 0}, \mathbf{D}_\ell^{\geq 0})$ on $\mathbf{D}^+ (\mathbf{Ban}, \mathscr{E}_{\max})$ whose heart is equivalent to Waelbroeck's abelian category **qBan**. Let

$$H_\ell^0 := \tau_\ell^{\geq 0} \tau_\ell^{\leq 0} \cong \tau_\ell^{\leq 0} \tau_\ell^{\geq 0}$$

be the associated homological functor $H_\ell^0 : \mathbf{D}^+ (\mathbf{Ban}, \mathscr{E}_{\max}) \to \mathbf{qBan}$ and write as usual $H_\ell^n(A) = H_\ell^0(A[n])$ for $n \in \mathbb{Z}$.

Remark. For the moment it suffices to know about **qBan** that it is abelian and that its objects are represented by pairs $(E^{-1} \hookrightarrow E^0)$ of Banach spaces and an injective bounded linear map between them. Moreover, there is an embedding (a fully faithful functor) $\iota : \mathbf{Ban} \to \mathbf{qBan}$ given on objects by $\iota(E) = (0 \hookrightarrow E)$ and that ι admits a left adjoint because **Ban** has cokernels, see Lemma 2.2.6 of Chapter III.

Let us give a simple proof of the existence of enough injectives in $(G\text{–}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}})$. Underlying the argument is the concept of a monad (triple) arising from an adjoint pair of functors, see Weibel [**Wei94**, Section 8.6].

Recall that a function $f : G \to E$ from a Hausdorff topological group G into a Banach space E is called *left uniformly continuous* if for every net $g_\lambda \to 1$ in G one has $\sup_{x\in G} \|f(g_\lambda x) - f(x)\|_E \xrightarrow{\lambda\to\infty} 0$.

Lemma. *The forgetful functor* $\downarrow : G\text{–}\mathbf{Ban} \to \mathbf{Ban}$ *has a right adjoint given on objects* $E \in \mathbf{Ban}$ *by the Banach space*

$$C_b^{\mathrm{lu}}(G,E) = \{f : G \to E \,:\, f \textit{ is bounded and left uniformly continuous}\}$$

equipped with the supremum norm and the G*-action* $(gf)(x) = f(g^{-1}x)$.

Proof. Let L be the forgetful functor and let $R = C_b^{\mathrm{lu}}(G,-)$. Let $M \in G\text{–}\mathbf{Ban}$ and let $E \in \mathbf{Ban}$ be any two objects. The unit and counit of the adjunction are the natural transformations

$$\eta : \mathrm{id}_{G-\mathbf{Ban}} \Rightarrow RL \qquad \text{and} \qquad \varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{Ban}}$$

given by

$$\eta_M(m) = (x \mapsto x^{-1}m) \in RL(M) = C_b^{\mathrm{lu}}(G, \downarrow M) \qquad \text{and} \qquad \varepsilon_E(f) = f(1) \in E$$

for $m \in M$ and for $f \in LR(E) = \downarrow C_b^{\mathrm{lu}}(G,E)$. Since $\|m\|_M = \sup_{x\in G} \|\eta_M(m)(x)\|_M$, we see that $\eta_M(m)$ is a bounded function. Let us check that $\eta_M(m)$ is left uniformly continuous, so let $g_\lambda \to 1$ be a net and compute for $m \in M$

$$\sup_{x\in G} \|\eta_M(m)(g_\lambda x) - \eta_M(m)(x)\|_M = \sup_{x\in G} \|x^{-1}g_\lambda^{-1}m - x^{-1}m\|_M = \|g_\lambda^{-1}m - m\|_M$$

the last term converges to zero because the action of G on M is strongly continuous. Moreover, $\eta_M : M \to RL(M)$ is G-equivariant because

$$\eta_M(gm)(x) = x^{-1}gm = (g^{-1}x)^{-1}m = \eta_M(m)(g^{-1}x) = [g\eta_M(m)](x).$$

The fact that η and ε are natural transformations is obvious as well as the one that the compositions

$$L(M) \xrightarrow{L(\eta_M)} LRL(M) \xrightarrow{\varepsilon_{L(M)}} L(M) \quad \text{and} \quad R(E) \xrightarrow{\eta_{R(E)}} RLR(E) \xrightarrow{R(\varepsilon_E)} R(E)$$

are equal to $\mathrm{id}_{L(M)}$ and to $\mathrm{id}_{R(E)}$. Thus we have $L \dashv R$ by [**ML98**, Ch. IV.1]. □

Corollary. *There are enough injectives in* $(G\text{–}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}})$.

Proof. Choose the split exact structure $\mathscr{E}_{\min}$ on $\mathbf{Ban}$ so that every Banach space is (projective and) injective. Observe that the forgetful functor

$$\downarrow : (G\text{–}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}}) \longrightarrow (\mathbf{Ban}, \mathscr{E}_{\min})$$

is exact and thus its right adjoint $C_b^{\mathrm{lu}}(G,-)$ preserves injectives. In particular, $C_b^{\mathrm{lu}}(G,E)$ is $\mathscr{E}^G_{\mathrm{rel}}$-injective for all $E \in \mathbf{Ban}$. For every $M \in G\text{–}\mathbf{Ban}$ the unit of the adjunction provides us with an admissible monic

$$\eta_M : M \rightarrowtail C_b^{\mathrm{lu}}(G, \downarrow M)$$

into an injective because $\varepsilon_{\downarrow M}$ is a left inverse of $\downarrow\eta_M$ in $\mathbf{Ban}$. □

Having settled this point, we can proceed to our main result.

The three choices stated at the beginning of the section lead to the following diagram of categories:

$$\begin{array}{ccccc}
G\text{–}\mathbf{Ban} & \xrightarrow{\quad(-)^G\quad} & \mathbf{Ban} & & \\
\big\downarrow{\scriptstyle c_0} & & \big\downarrow{\scriptstyle c_0} & & \\
\mathbf{K}^+(G\text{–}\mathbf{Ban}) & \xrightarrow{\quad(-)^G\quad} & \mathbf{K}^+(\mathbf{Ban}) & & \\
\big\downarrow{\scriptstyle q_G} & \overset{\alpha}{\Longleftarrow} & \big\downarrow{\scriptstyle q} & & \\
\mathbf{D}^+(G\text{–}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}}) & \xrightarrow{\mathbf{R}^+(-)^G} & \mathbf{D}^+(\mathbf{Ban}, \mathscr{E}_{\max}) & \xrightarrow{H^n_\ell(-)} & \mathbf{qBan}
\end{array}$$

where c_0 arises from considering an object of an additive category as a complex concentrated in degree zero, q_G and q are the localization functors and finally $(\mathbf{R}^+(-)^G, \alpha)$ is the right derived functor of $(-)^G$ together with its universal natural transformation

$$\alpha : q \circ (-)^G \Longrightarrow \mathbf{R}^+(-)^G \circ q_G.$$

For $E \in G\text{–}\mathbf{Ban}$ and $n \in \mathbb{Z}$ let

$$\mathscr{H}^n(G, E) := H^n_\ell\big(\mathbf{R}^+(-)^G(q_G \circ c_0(E))\big).$$

By its definition, this family of functors has the following properties:

Theorem. *Let G be a Hausdorff topological group. Up to unique isomorphism of δ-functors there is a unique family of functors*

$$\mathscr{H}^n(G, -) : G\text{–}\mathbf{Ban} \longrightarrow \mathbf{qBan}, \qquad n \geq 0,$$

having the following properties:

(i) *(Normalization) $\mathscr{H}^0(G, E) = (0 \hookrightarrow E^G)$ for all $E \in G\text{–}\mathbf{Ban}$.*
(ii) *(Vanishing) $\mathscr{H}^n(G, I) = 0$ for all injective objects I in $(G\text{–}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}})$ and all $n > 0$.*
(iii) *(Long exact sequence) For every short exact sequence $E' \rightarrowtail E \twoheadrightarrow E''$ in $\mathscr{E}^G_{\mathrm{rel}}$ there is a family of connecting morphisms $\mathscr{H}^n(G, E'') \xrightarrow{\delta^n} \mathscr{H}^{n+1}(G, E')$ depending naturally on the sequence and fitting into a long exact sequence*

$$\cdots \xrightarrow{\delta^{n-1}} \mathscr{H}^n(G, E') \to \mathscr{H}^n(G, E) \to \mathscr{H}^n(G, E'') \xrightarrow{\delta^n} \mathscr{H}^{n+1}(G, E') \to \cdots$$

in **qBan**.

Proof. The classical uniqueness proof for universal δ-functors goes through *verbatim*, see for instance [**Lan67**, Ch. I.1]. The only necessary modification is that exactness and injectivity in $G\text{–}\mathbf{Ban}$ are defined using the exact structure $\mathscr{E}^G_{\mathrm{rel}}$.

So let us prove that $\mathscr{H}^n(G, -)$ has the desired properties.

(i) To compute the right derived functor $\mathbf{R}^+(-)^G$ on a given complex E, we replace E by a quasi-isomorphic complex with injective components I (such a complex exists because there are enough injectives, see [**Kel90**, 4.1, Lemma, b)]) and apply $(-)^G$ to I. If the complex E is concentrated in degree zero, this reduces to taking an injective resolution

$$E \rightarrowtail I^0 \xrightarrow{d^0} I^1 \to \cdots$$

and applying $(-)^G$ to $I^\bullet$. Now notice $E = \operatorname{Ker}(d^0)$ and that the functor

$$G\text{–}\mathbf{Ban} \xrightarrow{(-)^G} \mathbf{Ban} \xrightarrow{\iota} \mathbf{qBan}$$

has a left adjoint, in particular it preserves kernels. Moreover, $\iota = H^0_\ell \circ q \circ c_0$, so

$$(0 \hookrightarrow E^G) = \mathrm{Ker}\left(\iota \circ (-)^G(d^0)\right).$$

is the value of $\mathscr{H}^0(G, E)$.

(ii) This is clear: An injective object I is quasi-isomorphic to itself considered as a complex concentrated in degree zero, so $\mathscr{H}^n(G, I) = 0$ for all $n \neq 0$.

(iii) Let $\sigma = (E' \rightarrowtail E \twoheadrightarrow E'')$ be a short exact sequence in $\mathscr{E}^G_{\mathrm{rel}}$. Using the functoriality of the mapping cone construction we obtain a diagram in $\mathbf{Ch}^+(G\text{–}\mathbf{Ban})$

$$\begin{array}{ccccccc}
E' & \overset{m}{\rightarrowtail} & E & \xrightarrow{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & \mathrm{cone}\,(m) & \xrightarrow{[\,1\ 0\,]} & E'[1] \\
\| & & \| & & \downarrow{\scriptstyle [\,0\ e\,]} & \overset{\delta}{\nearrow} & \\
E' & \overset{m}{\rightarrowtail} & E & \overset{e}{\twoheadrightarrow} & E'' & &
\end{array}$$

depending naturally on σ. Because

$$\sigma = \mathrm{cone}\,([\,0\ e\,]),$$

$[\,0\ e\,]$ is a quasi-isomorphism and hence becomes invertible in the derived category. Thus at the level of $\mathbf{D}^+(G\text{–}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}})$ the morphism

$$\delta := [\,1\ 0\,] \circ [\,0\ e\,]^{-1}$$

is defined and its natural dependence on σ is clear. By applying the homological functor $H^0_\ell \circ \mathbf{R}^+(-)^G$ to the distinguished triangle

$$E' \to E \to E'' \xrightarrow{\delta} E'[1],$$

we obtain the desired long exact sequence.

□

REMARK. Because $\mathscr{H}^n(G, -) = 0$ for $n < 0$ there is not much interest in characterizing these functors.

Now we turn to the *ad hoc* construction. Let **Csn** be the category of complete semi-normed spaces and continuous linear maps. Completeness is to be understood in the sense that every Cauchy sequence has an accumulation point.

LEMMA. *There is a "realization functor"* $\mathbf{qBan} \xrightarrow{\mathrm{real}} \mathbf{Csn}$ *given on objects by*

$$(E^{-1} \hookrightarrow E^0) \mapsto \mathrm{Coker}_{\mathbf{Csn}}\,(E^{-1} \hookrightarrow E^0).$$

It is exact in the sense that short exact sequences in **qBan** *are transformed to short sequences in* **Csn** *whose underlying sequence of k-vector spaces is exact.*

PROOF. Taking cokernels in **Csn** amounts to taking the cokernel in **Ab** and putting the quotient semi-norm on the resulting space, so real is well-defined (cf. the proof of Lemma 2.2.6 of Chapter III). The forgetful functor $(\mathbf{Ban}, \mathscr{E}_{\max}) \to \mathbf{Ab}$ is exact and preserves monics, so the composition of the realization functor real with the forgetful functor $\mathbf{Csn} \to \mathbf{Ab}$ is its unique exact prolongation $\mathbf{qBan} \to \mathbf{Ab}$ according to Theorem 2.2.3 of Chapter III. □

Define for $n \in \mathbb{Z}$

$$\mathscr{H}^n_{cb}(G, -) := \mathrm{real} \circ \mathscr{H}^n(G, -).$$

The following Corollary is a trivial consequence of the previous theorem and the previous lemma.

Corollary. *Let G be a Hausdorff topological group. There is a family of functors*

$$\mathscr{H}^{n}_{cb}(G,-) : G{-}\mathbf{Ban} \longrightarrow \mathbf{Csn}, \qquad n \geq 0,$$

having the following properties:

- *(i) (Normalization) $\mathscr{H}^{0}_{cb}(G,E) = E^{G}$ for all $E \in G{-}\mathbf{Ban}$.*
- *(ii) (Vanishing) $\mathscr{H}^{n}_{cb}(G,I) = 0$ for all injective objects I in $(G{-}\mathbf{Ban}, \mathscr{E}^{G}_{\mathrm{rel}})$ and all $n > 0$.*
- *(iii) (Long exact sequence) For every short exact sequence $E' \rightarrowtail E \twoheadrightarrow E''$ in $\mathscr{E}^{G}_{\mathrm{rel}}$ there is a family of connecting morphisms $\mathscr{H}^{n}_{cb}(G,E'') \xrightarrow{\delta^n} \mathscr{H}^{n+1}_{cb}(G,E')$ depending naturally on the sequence and fitting into a long sequence*

$$\cdots \xrightarrow{\delta^{n-1}} \mathscr{H}^{n}_{cb}(G,E') \to \mathscr{H}^{n}_{cb}(G,E) \to \mathscr{H}^{n}_{cb}(G,E'') \xrightarrow{\delta^{n}} \mathscr{H}^{n+1}_{cb}(G,E') \to \cdots$$

in $\mathbf{Csn}$ whose underlying sequence of k-vector spaces is exact. □

Corollary. *Our functors $\mathscr{H}^{n}_{cb}(G,-)$ and Burger-Monod's $H^{n}_{cb}(G,-)$ are isomorphic on the category $G{-}\mathbf{Ban}$.*

Proof. This is obvious: according to the "functorial characterization" of bounded cohomology (see [**Mon01**, Theorem 7.2.1]) one may compute $H^{n}_{cb}(G,M)$ by taking a "strong resolution $I^{\bullet}$ of M by relatively injectives" of E, applying $(-)^{G}$ and taking "homology" in degree n. By definition, such a resolution is an $\mathscr{E}^{G}_{\mathrm{rel}}$-resolution by $\mathscr{E}^{G}_{\mathrm{rel}}$-injectives. It remains to observe that $\mathscr{H}^{n}_{cb}(G,M) = \ker d^{n+1} / \operatorname{im} d^{n}$ in Monod's notation, thus we are done. □

Remark. The Burger-Monod functors are defined on the class of *all* Banach G-modules E, no continuity requirement is imposed. However, this essentially rests on the following simple observation: if

$$E^{c} = \{ e \in E \, : \, (g \mapsto ge) \in C(G,E) \}$$

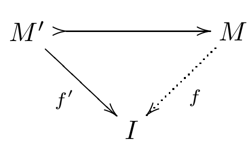
is the *maximal continuous submodule* of E consisting of the G-continuous elements (cf. [**Mon01**, Lemma 1.2.4]) then the inclusion $E^{c} \hookrightarrow E$ induces an isomorphism

$$H^{n}_{cb}(G,E^{c}) \cong H^{n}_{cb}(G,E)$$

because $F^{G} = (F^{c})^{G}$ for all Banach G-module, see [**Mon01**, Proposition 6.1.5].

We can therefore recover the most general incarnation of Burger-Monod's functors by considering the functors $\mathscr{H}^{n}_{cb}(G,(-)^{c})$ on the category of Banach G-modules without continuity requirement. A proper treatment of these functors would entail an examination of the Grothendieck spectral sequence for the composition $(-)^{G} \circ (-)^{c}$. This appears to be only of marginal interest to the present work, so we content ourselves with the remark that the functor $(-)^{c}$ is right adjoint to the inclusion functor of the category of strongly continuous Banach G-modules in the category of all Banach G-modules.

Remark. It should be noted that Burger-Monod's definition of "relative injectivity" does not coincide with the one provided by the exact structure $\mathscr{E}^{G}_{\mathrm{rel}}$ because they require that the splitting map of an admissible monic has norm at most one and that the extension problem



can be solved in such a way that $\|f\| = \|f'\|$, thus yielding a stronger property than $\mathscr{E}^{G}_{\mathrm{rel}}$-injectivity by the open mapping theorem.

2. More Functoriality

Notice that a continuous group homomorphism $\varphi : G \to G'$ between Hausdorff topological groups induces an exact functor

$$\varphi^* : (G'-\mathbf{Ban}, \mathscr{E}^{G'}_{\mathrm{rel}}) \to (G-\mathbf{Ban}, \mathscr{E}^{G}_{\mathrm{rel}})$$

as follows: given a G'-module M', let $\varphi^* M'$ be the G-module with the G-action defined by $gm' := \varphi(g)m'$.

Let $\mathfrak{GBan}$ be the category whose objects are pairs (G, M) where G is a Hausdorff topological group and $M \in G-\mathbf{Ban}$ and whose morphisms are given by $(\varphi, f) : (G, M) \to (G', M')$ where $\varphi : G \to G'$ is a continuous group homomorphism and $f : \varphi^* M' \to M$ is a morphism in $G-\mathbf{Ban}$.

Theorem. *The functor* $\mathscr{H}^n(G, -) : G-\mathbf{Ban} \to \mathbf{qBan}$ *induces a contravariant functor* $\mathscr{H}^n : \mathfrak{GBan} \to \mathbf{qBan}$.

Proof. For a morphism of Hausdorff topological groups, $\varphi : G \to G'$, the family of functors

$$\mathscr{H}^n(G, \varphi^*(-)) : (G'-\mathbf{Ban}, \mathscr{E}^{G'}_{\mathrm{rel}}) \to \mathbf{qBan}, \qquad n \geq 0,$$

defines a δ-functor on $G'-\mathbf{Ban}$. Because $\mathscr{H}^n(G', -) : (G'-\mathbf{Ban}, \mathscr{E}^{G'}_{\mathrm{rel}}) \to \mathbf{qBan}$ is effaçable for $n \geq 1$ and because there is a natural transformation

$$\mathscr{H}^0(G', -) \Rightarrow \mathscr{H}^0(G, \varphi^*(-))$$

induced by the inclusion $(M')^{G'} \subset (\varphi^* M')^G$, we conclude that there is a unique morphism of δ-functors

$$\alpha^n_\varphi : \mathscr{H}^n(G', -) \Rightarrow \mathscr{H}^n(G, \varphi^*(-)), \qquad n \geq 0,$$

which in addition depends functorially on φ in the sense that $\alpha^n_{\varphi\circ\psi} = \alpha^n_\psi \circ \alpha^n_\varphi$.

Now given a morphism $(\varphi, f) : (G, M) \to (G', M')$ of $\mathfrak{GBan}$, this yields a morphism

$$\mathscr{H}^n(G', M') \xrightarrow{(\varphi,f)^*} \mathscr{H}^n(G, M)$$

and it is clear that this defines a functor. □

3. Discrete Groups

Let G be a group and let $G-\mathbf{Ban}$ be the category of isometric representations of G on Banach spaces and continuous linear G-equivariant maps. Let $\ell^1(G)$ be the Banach group algebra.

Notation. Let E be a Banach space. The *induced Banach G-module* is

$$\uparrow E = \ell^1(G) \widehat{\otimes} E$$

with the left G-action on the factor $\ell^1(G)$. The *coinduced Banach G-module* is

$$\Uparrow E := \mathrm{Hom}_{\mathbf{Ban}}(\ell^1(G), E)$$

with the action coming from the right action of G on $\ell^1(G)$.

Notation. Let M be a Banach G-module. The *module of coinvariants* of M is the Banach space

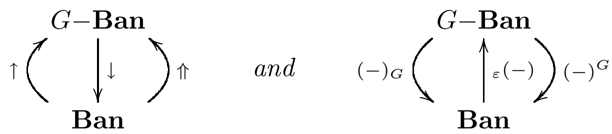
$$M_G = M/\overline{\{m - gm \,:\, m \in M,\, g \in G\}}$$

and the *module of invariants* is the Banach space

$$M^G = \{m \in M \,:\, gm = m\}.$$

The following two theorems are proved by direct inspection:

THEOREM (Fundamental Adjunctions). *Let* $\downarrow\colon G\text{–}\mathbf{Ban} \to \mathbf{Ban}$ *be the forgetful functor and let* ${}_{\varepsilon}(-) : \mathbf{Ban} \to G\text{–}\mathbf{Ban}$ *be the trivial module functor. There are two adjoint triples of functors*



that is to say $\uparrow$ *is left adjoint to* $\downarrow$ *and* $\downarrow$ *is left adjoint to* $\Uparrow$. □

THEOREM. *The class* $\mathscr{E}^{G}_{\max}$ *of all kernel-cokernel pairs is an exact structure on* $G\text{–}\mathbf{Ban}$. □

PROPOSITION. *Equip* $G\text{–}\mathbf{Ban}$ *and* $\mathbf{Ban}$ *with the maximal exact structures.*

(i) The forgetful functor $\downarrow$, *induction* $\uparrow$ *and coinduction* $\Uparrow$ *are exact.*
(ii) The trivial module functor ${}_{\varepsilon}(-)$ *is exact.*

PROOF. That the forgetful functor and the trivial module functor are exact is obvious from the fundamental adjunctions: as right adjoints they preserve kernels and as left adjoints they preserve cokernels, hence they preserve kernel-cokernel pairs. To see that induction and coinduction are exact, it suffices to notice that $\ell^1(G)$ is projective and hence flat as a Banach space. □

REMARK (Helemskiĭ). The coinvariants $(-)_G$ and the invariants $(-)^G$ are of course not exact in general. We will construct the G-tensor product $\widehat{\otimes}_G$ in Chapter V for which we have $(-)_G = k \mathbin{\widehat{\otimes}_G} -$, where the ground field k is considered as a trivial G-module, as always. Similarly, we have $(-)^G = \operatorname{Hom}_G(k, -)$. It follows that $(-)_G$ is exact if and only if k is flat and this happens if and only if the group G is amenable. Similarly, $(-)^G$ is exact if and only if k is projective and this happens if and only if G is finite.

PROOF OF THE REMARK. To see the relation to amenable and finite groups, notice that $\ell^1(G)$ is a projective G-module by the fundamental adjunction—the ground field k is a projective Banach space and $\ell^1(G) = \ell^1(G) \mathbin{\widehat{\otimes}} k = \uparrow k$. Similarly, since k is an injective Banach space, $\ell^\infty(G) = \operatorname{Hom}_{\mathbf{Ban}}(\ell^1(G), k) = \Uparrow k$ is an injective Banach G-module.

We claim that k is a projective Banach G-module if and only if G is finite and that k is an injective G-module if and only if G is amenable. To see the first part, consider the augmentation (summation) $\varepsilon : \ell^1(G) \to k$. It is an admissible epic and it is a split epic if and only if k is projective. However, there is a right inverse for ε if and only if there is a non-zero invariant function in $\ell^1(G)$, and this can happen if and only if G is finite. Similarly, the inclusion $\varepsilon^* : k \to \ell^\infty(G)$ is an admissible monic, it is a split monic if and only if k is injective and this is equivalent to the amenability of G—if there is a G-invariant left inverse $m : \ell^\infty(G) \to k$ of ε^*, simply take the "total variation" of m and renormalize if necessary to find an invariant mean on $\ell^\infty(G)$; conversely, an invariant mean on $\ell^\infty(G)$ is by definition a G-invariant left inverse of ε^*. Finally, k is flat if and only if $k \cong k^*$ is injective by Proposition 1.4.2 of Chapter V. □

COROLLARY. *There are enough projectives and enough injectives in the exact category* $(G\text{–}\mathbf{Ban}, \mathscr{E}^{G}_{\max})$.

PROOF. First recall that there are enough projectives and injectives in **Ban** since every Banach space is a quotient of an ℓ^1-space and a subspace of an ℓ^∞-space.

We only prove that there are enough projectives, the proof that there are enough injectives being dual. Let M be a Banach G-module. Choose an ℓ^1-space (= a projective Banach space) L and an admissible epic $L \twoheadrightarrow \downarrow M$. Induction being a left adjoint, we get an admissible epic $\uparrow L \twoheadrightarrow \uparrow\!\downarrow M$ and because induction is left adjoint to an exact functor we know that $\uparrow L$ is projective in $(G\!-\!\mathbf{Ban}, \mathscr{E}^G_{\max})$. The counit $\uparrow\!\downarrow M \to M$ of the adjunction $\uparrow(-) \dashv \downarrow(-)$ is an admissible epic since it splits over **Ban**, so its composition with $\uparrow L \twoheadrightarrow \uparrow\!\downarrow M$ is an admissible epic $\uparrow L \twoheadrightarrow M$. □

Definition. The *relative exact structure* $\mathscr{E}^G_{\mathrm{rel}}$ on $G\!-\!\mathbf{Ban}$ is the pullback of the split exact structure on **Ban** under the forgetful functor, see Appendix B. More explicitly,

$$\mathscr{E}^G_{\mathrm{rel}} = \{\sigma = (M' \to M \to M'') : \downarrow\sigma \text{ is split exact}\}.$$

Corollary. *There are enough projectives and injectives in the exact category* $(G\!-\!\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}})$.

Proof. By definition, the forgetful functor

$$\downarrow : (G\!-\!\mathbf{Ban}, \mathscr{E}^G_{\mathrm{rel}}) \longrightarrow (\mathbf{Ban}, \mathscr{E}_{\min})$$

is exact, where $\mathscr{E}_{\min}$ denotes the split exact structure on **Ban**. Notice that all objects are projective and injective in $(\mathbf{Ban}, \mathscr{E}_{\min})$. Because induction is left adjoint to the forgetful functor, it preserves projectives and hence $\uparrow E$ is relatively projective for all Banach spaces E. For each Banach G-module the counit of the adjunction $\uparrow\!\downarrow M \to M$ is an $\mathscr{E}^G_{\mathrm{rel}}$-admissible epic from a projective. The claim about enough injectives follows from duality. □

Corollary (Transfer). *A Banach G-module M is $\mathscr{E}^G_{\max}$-projective if and only if it is $\mathscr{E}^G_{\mathrm{rel}}$-projective and $\downarrow M$ is an $\mathscr{E}_{\max}$-projective Banach space.*

Proof. Let M be $\mathscr{E}^G_{\max}$-projective, then M is also $\mathscr{E}^G_{\mathrm{rel}}$-projective. Since coinduction is exact with respect to the maximal structures, its left adjoint $\downarrow$ preserves projectives, hence $\downarrow M$ is $\mathscr{E}_{\max}$-projective.

Conversely, suppose that M is $\mathscr{E}^G_{\mathrm{rel}}$-projective and that $\downarrow M$ is $\mathscr{E}_{\max}$-projective. Then $\uparrow\!\downarrow M$ is $\mathscr{E}^G_{\max}$-projective. Because the counit $\uparrow\!\downarrow M \to M$ is an $\mathscr{E}^G_{\mathrm{rel}}$-admissible epic onto a $\mathscr{E}^G_{\mathrm{rel}}$-projective, it splits, and hence M is $\mathscr{E}^G_{\max}$-projective. □

4. Amenable Groups

Various notions of amenability pervade the theory of bounded cohomology. We have seen in the previous section that amenable groups can be defined by the injectivity or flatness of the trivial Banach G-module k—*in this section we will only work with the maximal exact structures on* **Ban** *and* $G\!-\!\mathbf{Ban}$.

For $n \in \mathbb{Z}$ we define the functors

$$\mathscr{H}^n_{\max}(G,-) := H^n_\ell(\mathbf{R}^+(-)^G) : G\!-\!\mathbf{Ban} \longrightarrow \mathbf{qBan}\,.$$

We write $\mathscr{H}^n_{\max}$ instead of $\mathscr{H}^n$ in order to indicate that we work with the maximal exact structure on $G\!-\!\mathbf{Ban}$.

Remark. There is of course an axiomatic description of $\mathscr{H}^n_{\max}$, as it is a universal δ-functor.

There is a well-known slogan to the end that amenable groups are completely trivial in bounded cohomology, see e.g. [**Mon01**, p.18]. This is a good slogan since it summarizes the situation well even if it is slightly inaccurate. The ℓ^1-homology of amenable groups is trivial, since $(-)_G$ is exact and hence its own (left or right) derived functor. For bounded cohomology, the situation is more delicate and the rest of this section is devoted to a discussion of this.

Remark (Finite Groups). We already discussed that G is finite if and only if k a projective Banach G-module. Hence G is finite if and only if $(-)^G = \mathrm{Hom}_G(k,-)$ is an exact functor $G{-}\mathbf{Ban} \to \mathbf{Ban}$. In turn, this is equivalent to saying that $(-)^G$ is its own derived functor. Because by Theorem 2.2.3 of Chapter III the embedding $\mathbf{Ban} \to \mathbf{qBan}$ preserves and reflects exactness, G is finite if and only if

$$\mathscr{H}^n_{\max}(G,-) = 0$$

for all $n \geq 1$. By dimension shifting one shows that one may replace the last condition by $\mathscr{H}^1_{\max}(G,-) = 0$.

The following examples concerning the augmentation ideal were explained to the author by Nicolas Monod:

Example (The Augmentation Ideal, I). Consider the *augmentation sequence*

$$\ell^1_0(G) \rightarrowtail \ell^1(G) \overset{\varepsilon}{\twoheadrightarrow} k$$

where $\varepsilon : \ell^1(G) \to k$ is the *augmentation* (summation) and $\ell^1_0(G) = \mathrm{Ker}(\varepsilon)$ is the *augmentation ideal* consisting of ℓ^1-sequences of sum zero. Trivially, $(\ell^1(G))^G = k$ if and only if G is finite while $(\ell^1(G))^G = 0$ if G is infinite. By our remark on finite groups we know that $\mathscr{H}^1_{\max}(G,\ell^1_0(G)) = 0$. Conversely, if $\mathscr{H}^1_{\max}(G,\ell^1_0(G)) = 0$, the long exact sequence implies that $(\ell^1(G))^G \cong k$, so G must be finite.

Remark (Johnson's Theorem [**Joh72**]). Consider the composition of functors

$$(G{-}\mathbf{Ban})^{\mathrm{op}} \xrightarrow{(-)^*} G{-}\mathbf{Ban} \xrightarrow{(-)^G} \mathbf{Ban}\,.$$

The dual space functor is exact and it transforms projective Banach G-modules to injective G-modules by Proposition 1.4.3 of Chapter V. Hence we may apply the composition theorem for derived functors, see [**Wei94**, Theorem 10.8.2], to get

$$(\mathbf{R}^+(-)^G) \circ (-)^* \cong (\mathbf{R}^+(-)^G) \circ (\mathbf{R}^+(-)^*) \cong \mathbf{R}^+((-)^G \circ (-)^*).$$

Moreover,

$$(-)^G \circ (-)^* \cong \mathrm{Hom}_G(k,(-)^*) \cong \mathrm{Hom}_G(-,k).$$

Therefore we have for all $n \in \mathbb{Z}$ and all $E \in G{-}\mathbf{Ban}$

$$\mathscr{H}^n_{\max}(G,E^*) \cong H^n_\ell\big(\mathbf{R}^+\left((-)^G \circ (-)^*\right)(E)\big) \cong H^n_\ell(\mathbf{R}^+\,\mathrm{Hom}_G(E,k)).$$

Now by Helemskiĭ's remark G is amenable if and only if k is injective. So G is amenable if and only if $\mathrm{Hom}_G(-,k) : G{-}\mathbf{Ban} \to \mathbf{Ban}$ is exact, and this is equivalent to the vanishing

$$\mathscr{H}^n_{\max}(G,E^*) = 0$$

for all $E \in G{-}\mathbf{Ban}$ and all $n \geq 1$ because by Theorem 2.2.3 of Chapter III the embedding $\mathbf{Ban} \to \mathbf{qBan}$ preserves and reflects exactness. Again by dimension shifting one gets that G is amenable if and only if $\mathscr{H}^1_{\max}(G,E^*) = 0$ for all E.

Example (The Augmentation Ideal, II). Recall that $\ell^1(G)$ is the dual space of $c_0(G)$. Johnson's theorem implies that $\mathscr{H}^1_{\max}(G,\ell^1(G)) \cong 0$ for all amenable groups. Now if G is an infinite amenable group, the long exact sequence associated to the augmentation sequence gives that $\mathscr{H}^1_{\max}(G,\ell^1_0(G)) \cong (0 \hookrightarrow k)$ in $\mathbf{qBan}$. In particular, $\ell^1_0(G)$ is not a dual Banach G-module if G is infinite.

Remark (cf. [**Mon01**, Section 6.2]). From the (inhomogeneous) bar resolution one obtains the following explicit interpretation of

$$H^1_b(G,E) \cong \mathrm{real}\,\mathscr{H}^1_{\max}(G,E).$$

A 1-cocycle is a bounded function $f : G \to E$ such that $f(gh) = gf(h) + f(g)$ for all $g,h \in G$. A 1-cocycle is a 1-coboundary if it is of the form $f(g) = gv - v$

for some $v \in E$. Every 1-cocycle gives rise to a *bounded* affine isometric action $\alpha : G \to \mathrm{Isom}\,(E)$ via

$$\alpha(g)v = gv + f(g),$$

i.e., α has bounded orbits. Call two bounded affine isometric actions equivalent if they differ only by a translation. One easily establishes a bijection between cohomology classes in $H^1_b(G, E)$ and equivalence classes of bounded affine isometric actions. A bounded affine action corresponds to the zero class if and only if it fixes a vector.

Remark (Fixed Point Property of Amenable Groups). A group G is amenable if and only if every affine G-action on a dual Banach space which leaves a weak*-compact convex set K invariant has a fixed point in K. This is Day's fixed point theorem [**Day61**]. In contrast, the previous remark implies that Johnson's theorem is equivalent to the following statement: a group is amenable if and only if every bounded affine isometric action on a dual Banach space has a fixed point.

Example (The Augmentation Ideal, III). As an affine G-space the augmentation ideal is isometrically isomorphic to the G-invariant affine subspace L of $\ell^1(G)$ consisting of sequences of sum one. Obviously, the induced G-action on L has bounded orbits. There is a G-invariant vector in L if and only if G is finite.

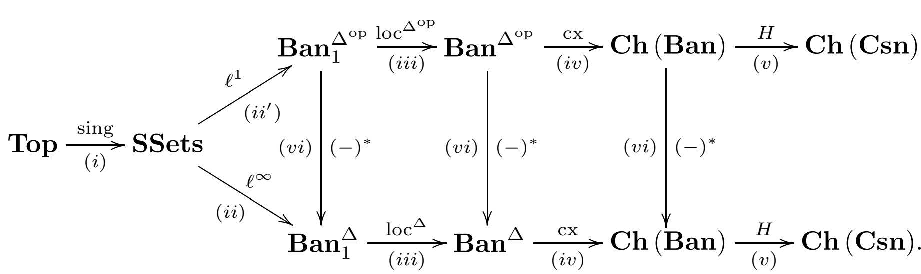
5. ℓ^1-Homology and Bounded Cohomology for Spaces

Let **Top** be the category of topological spaces and continuous maps. Let **SSets** be the category of simplicial sets; that is, the category $\mathbf{Sets}^{\Delta^{\mathrm{op}}}$ of contravariant functors from the simplicial category Δ to **Sets**.

The class of all Banach spaces forms on the one hand the bicomplete, pointed, non-additive category $\mathbf{Ban}_1$ in which the morphisms are the linear maps of norm at most one. On the other hand, there is the additive but only finitely bicomplete version **Ban** of Banach spaces and bounded linear maps. Let k be either the field of real or complex numbers.

Let X be a topological space. Consider the simplicial set $\mathrm{sing}\,(X)$ of singular simplexes in X. Apply the contravariant functor ℓ^∞ to $\mathrm{sing}\,(X)$ so as to obtain a cosimplicial object $\ell^\infty\mathrm{sing}\,(X)$ in $\mathbf{Ban}_1$. Under the canonical functor $\mathbf{Ban}_1 \to$ **Ban** this becomes a cosimplicial object in the additive category **Ban**. Look at the associated complex. Take homology. The resulting complex of complete seminormed spaces is called *bounded cohomology* of X and denoted by $H^\bullet_b(X)$.

Let us once again have a look at the definition, now in diagrammatic terms:



A contemplation of the individual functors and categories involved reveals the following:

(i) The *singular functor* $\mathbf{Top} \xrightarrow{\mathrm{sing}} \mathbf{SSets}$ is right adjoint to the geometric realization functor $\mathbf{SSets} \xrightarrow{|\cdot|} \mathbf{Top}$. This is completely standard and needs no further commenting upon.

(ii) The ground field k is both a generator and a cogenerator of $\mathbf{Ban}_1$. It is therefore worthwhile to look at its represented and corepresented functors. The functor $B_{\leq 1} := \mathbf{Ban}_1(k, -)$ associates to a Banach space the set underlying its closed unit ball and acts on morphisms by restriction. The corepresented functor $B^*_{\leq 1} := \mathbf{Ban}_1(-, k)$ sends a Banach space to the unit ball of its dual space X^*, and a morphism f of $\mathbf{Ban}_1^{\mathrm{op}}$ is sent to the restriction of its Banach adjoint f^*.

It turns out that both functors $B_{\leq 1}$ and $B^*_{\leq 1}$ have a left adjoint. The functor $\ell^1 : \mathbf{Sets} \to \mathbf{Ban}_1$ is left adjoint to $B_{\leq 1}$ while $\ell^\infty : \mathbf{Sets} \to \mathbf{Ban}_1^{\mathrm{op}}$ is left adjoint to $B^*_{\leq 1}$.

The functors ℓ^1 and ℓ^∞ therefore preserve colimits and in particular coproducts. What happens with products is more delicate and requires the investigation of monoidal structures, that is to say "tensor products".

(iii) This is the extension of the obvious functor

$$\mathbf{Ban}_1 \xrightarrow{\mathrm{loc}} \mathbf{Ban}$$

to the corresponding functor categories by post-composition.

As already mentioned, $\mathbf{Ban}_1$ has the drawback of being non-additive. Nevertheless it is not very far from being additive. It has a zero object. It makes sense to take convex combinations of morphisms: two parallel morphisms f, g have "half a sum", that is $\frac{1}{2}(f+g)$ is defined. Therefore, it is reasonable to expect that some localization of $\mathbf{Ban}_1$ which inverts in particular $\frac{1}{2}$ should be additive.

Let Σ be the class of all bijective linear maps of norm at most one. This is a locally small multiplicative system with cancellation and satisfies both Ore conditions. This means that one may describe the morphisms in the quotient category $\mathbf{Ban}_1[\Sigma^{-1}]$ by an explicit calculus of fractions. On the other hand, the obvious functor $\mathbf{Ban}_1 \to \mathbf{Ban}$ sends precisely the class Σ to isomorphisms, hence one may identify $\mathbf{Ban}$ with the localization $\mathbf{Ban}_1[\Sigma^{-1}]$.

Having thus characterized the obvious functor $\mathbf{Ban}_1 \to \mathbf{Ban}$ as solution to a universal problem, we denote it by loc. Notice that it cannot have an adjoint on either side because of the absence of non-trivial infinite (co-)products in **Ban**.

(iv) So far, we have described the components of two functors which associate to a topological space a simplicial and a cosimplicial object in the additive category of Banach spaces. Next, we can pass to the associated complexes which are obtained by taking as differentials the alternating sums of the face maps. The resulting complexes are usually denoted by $C^{\ell^1}_*(X)$ and $C^*_b(X)$.

(v) Having finally obtained two complexes of Banach spaces, we may take the homology of these complexes. The resulting complexes of complete seminormed spaces are the ℓ^1-homology and the bounded cohomology of X.

(vi) Applying the dual space functor at each stage, we can always pass from the ℓ^1-side to the ℓ^∞-side of the diagram. This breaks down only at the last stage when taking homology. See [**Löh07**, Remark (3.4), p.35] for a discussion of the issues at hand.

The basic result of Gromov in bounded cohomology is:

Theorem ([**Gro82**], [**Iva85**]). *Let X be a connected simply connected topological space which is homotopy equivalent to a countable CW-complex. The bounded cohomology of X vanishes in all degrees $n > 0$.* □

Remark. The proof of this result is quite difficult and not very well understood as is indicated by the strange hypothesis on X (Gromov does not make this assumption explicit, his proof is however rather sketchy to say the least). The reason for this is the fact that the complete proof given by Ivanov uses the Dold-Thom construction and an old version of the Brown representability theorem which both necessitate the countability assumption.

The reduction of bounded cohomology for spaces to group cohomology is done as follows:

THEOREM (Brooks [**Bro81**], Ivanov [**Iva85**]). *Let X be a connected and countable CW-complex and let $G = \pi_1(X)$ be its fundamental group.*

(i) The complex $C_b^*(\widetilde{X})$ of bounded cochains on the universal covering space $\widetilde{X}$ of X consists of $\mathscr{E}_{\mathrm{rel}}^G$-injective Banach G-modules (the action is inherited from the action of G on $\widetilde{X}$).

(ii) The inclusion of constant functions $k \to C_b^0(\widetilde{X})$ yields a quasi-isomorphism $k \to C_b^*(\widetilde{X})$, i.e., an isomorphism in the derived category $\mathbf{D}^+(G{-}\mathbf{Ban}, \mathscr{E}_{\max}^G)$.

(iii) The complexes $C_b^*(X)$ and $C_b(\widetilde{X})^G$ are isometrically isomorphic.

In particular, $H_b^*(X)$ is isomorphic to $H_b^*(G,k)$. □

COROLLARY. *Let X be a connected and countable CW-complex and let $G = \pi_1(X)$ be its fundamental group. Equip $G{-}\mathbf{Ban}$ and $\mathbf{Ban}$ with the maximal exact structures.*

(i) There is an isomorphism $C_*^{\ell^1}(X) \cong \mathbf{L}^-(-)_G(k)$ in $\mathbf{D}^-(\mathbf{Ban}, \mathscr{E}_{\max})$, in particular the ℓ^1-chain complex $C_*^{\ell^1}(X)$ only depends on the fundamental group.

(ii) There is an isomorphism $C_b^*(X) \cong \mathbf{R}^+(-)^G(k)$ in $\mathbf{D}^+(\mathbf{Ban}, \mathscr{E}_{\max})$, in particular, the bounded cochain complex $C_b^*(X)$ only depends on the fundamental group.

PROOF. The Brooks-Ivanov theorem implies that the chain map $\varepsilon^* : k \to C_b^*(\widetilde{X})$ is a quasi-isomorphism in $\mathbf{D}^+(G{-}\mathbf{Ban}, \mathscr{E}_{\max}^G)$. Moreover by the dual of the last corollary in section 3, the complex $C_b^*(\widetilde{X})$ consists of $\mathscr{E}_{\max}^G$-injectives, because it consists of $\mathscr{E}_{\mathrm{rel}}^G$-injectives and the underlying Banach spaces are $\mathscr{E}_{\max}$-injective. Therefore $C_b^*(X) = C_b^*(\widetilde{X})^G \cong \mathbf{R}^+(-)^G(k)$, and this establishes (ii).

The duality functor preserves and reflects exactness. The dual of the augmentation $\varepsilon : C_0^{\ell^1}(\widetilde{X}) \to k$ is the inclusion of constants, so the chain map $C_*^{\ell^1}(\widetilde{X}) \to k$ is a quasi-isomorphism in $\mathbf{D}^-(G{-}\mathbf{Ban}, \mathscr{E}_{\max}^G)$. Now by a result of Park [**Par04**, p.611], the complex $C_*^{\ell^1}(\widetilde{X})$ consists of $\mathscr{E}_{\mathrm{rel}}^G$-projective Banach G-modules and since it consists of ℓ^1-spaces (which are projective as Banach spaces), our last corollary of section 3 lets us conclude that it is even a complex of $\mathscr{E}_{\max}^G$-projective G-modules. Now $C_*^{\ell^1}(X) = C_*^{\ell^1}(\widetilde{X})_G \cong \mathbf{L}^-(-)_G(k)$ proves part (i). □

REMARK. Part (i) implies the Matsumoto-Morita conjecture to the end that ℓ^1-homology of sufficiently nice topological spaces only depends on the fundamental group [**MM85**, Remark 2.6]. The first complete proof of the Matsumoto-Morita conjecture was given in [**Löh07**, Corollary 4.14, p.52], she even proved an isometric version of it. Proofs of the Matsumoto-Morita conjecture were claimed earlier in [**Par04**] as well as [**Bou04**]. The former is certainly incomplete, see [**Löh07**, Caveats (4.13), (4.15)], while the latter is so cluttered with mistakes that the author was unable to decide whether it can be made into a complete proof or not[1].

[1]The title does not exactly add to the author's confidence in the validity of the arguments in the article.

6. The Canonical Semi-Norm

Much effort has been invested in axiomatizing the semi-norm arising from the bar-resolution. The following two points remain unclear to the present author:

- Why are semi-norm considerations so powerful?
- Where do the semi-norms actually come from?

One could now go about and reproduce Sections 7.3–7.5 of [**Mon01**]. However, this does not seem to be worth the effort since we cannot improve on Monod's results, anyway. We content ourselves with pointing out that all complexes known to compute the semi-norm have a semi-simplicial nature: For a suitable category $\mathscr{C}$ of Banach G-modules, there is a *resolvent functor* F and a natural transformation $\delta : \mathrm{id}_{\mathscr{C}} \Rightarrow F$ satisfying:

(i) For each $M \in \mathscr{C}$ the morphism $\delta_M : M \to F(M)$ is of norm at most one.
(ii) The "restriction" $(\delta_M)^c : M^c \to F(M)^c$ to the maximal continuous submodules is a split monic in $\mathbf{Ban}_1$, that is to say, it has a left inverse of norm at most one.

Using this, one builds up a semi-simplicial resolution for each $M \in \mathscr{C}$

$$M \xrightarrow{\delta_M} F(M) \underset{\delta_{F(M)}}{\overset{F(\delta_M)}{\rightrightarrows}} F^2(M) \begin{array}{c} \xrightarrow{F^2(\delta_M)} \\ \xrightarrow{F(\delta_{F(M)})} \\ \xrightarrow[\delta_{F^2(M)}]{} \end{array} F^3(M) \quad \begin{array}{c} \cdots \\ \cdots \\ \cdots \\ \cdots \end{array}$$

and then takes the alternating sum of the co-face maps in order to get a complex. There are essentially two examples:

(i) The category $\mathscr{C}$ is the category of *all* Banach G-modules (not necessarily continuous) over the locally compact group G and $F(-) = C_b(X, -)$, where X is a locally compact continuous proper G-space such that X/G is paracompact. See [**Mon01**, Theorem 7.4.5] for details.
(ii) Let G be a locally compact and second countable group. The category $\mathscr{C}$ is Burger-Monod's category of *coefficient G-modules*, that is to say the duals of separable and continuous G-modules, and $F(-) = L^\infty_{w^*}(S, -)$ for some amenable standard Borel G-space[2] S with quasi-invariant measure. Details can be found in [**Mon01**, Theorem 7.5.3].

There are some variants obtained by taking the *inhomogeneous resolutions* and alternating versions of the complexes. We refer the interested reader to [**Mon01**] for a careful study of the conditions.

For countable discrete groups G, and this is the original motivation, one has of course the complex $C_b^*(\widetilde{X})$, where X is any connected and countable CW-complex having G as a fundamental group. This is proved in [**Iva85**]. See also [**Löh07**] for detailed information on semi-norms both in homology and cohomology and various beautiful applications to geometry.

[2] Monod assumes further that the G-space S is regular, this is however automatic, see Appendix D.

Part 1

Triangulated Categories

CHAPTER I

Triangulated Categories

In this chapter we introduce the axioms of triangulated categories and draw the most basic consequences. These are of course all well-known, however one is forced to dig deep in the literature in order to extract the elementary proofs presented here, so the author found it worthwhile to collect them together.

Contents

1. Definition and Elementary Properties

1.1. Pre-Triangulated Categories. Let $\mathscr{T}$ be an additive category. A *suspension* on $\mathscr{T}$ is an additive auto-equivalence $\Sigma : \mathscr{T} \to \mathscr{T}$. Every suspension on $\mathscr{T}$ gives rise to a notion of *triangles* on $(\mathscr{T}, \Sigma)$, namely the diagrams in $\mathscr{T}$ which are of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A.$$

We will write $[f, g, h]$ for the triangles on $(\mathscr{T}, \Sigma)$. A *morphism of triangles* is the same thing as a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

so, in particular, the triangles on $(\mathscr{T}, \Sigma)$ form an additive category.

Definition 1.1.1. A *pre-triangulated category* $(\mathscr{T}, \Sigma, \Delta)$ is a triple consisting of an additive category $\mathscr{T}$, a suspension $\Sigma : \mathscr{T} \to \mathscr{T}$ and a class Δ of *distinguished* triangles on $(\mathscr{T}, \Sigma)$. This data is subject to the following conditions:

[TR 1] For each object $A \in \mathscr{T}$, the triangle $A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \Sigma A$ is distinguished. For every morphism $A \xrightarrow{f} B$ there exists a distinguished triangle containing f, i.e., there is a diagram

$$A \xrightarrow{f} B \to C \to \Sigma A$$

in Δ. Moreover, the class Δ is closed under isomorphisms of triangles.

[TR 2] The triangle $[f, g, h]$ is distinguished if and only if the triangle $[g, h, -\Sigma f]$ is distinguished.

[TR 3] Consider the following diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots{\scriptstyle \exists c} & & \downarrow{\scriptstyle \Sigma a} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

in $\mathscr{T}$. If the left hand square is commutative and the rows are distinguished triangles, then there exists a [not necessarily unique] morphism c making the diagram commutative.

Let us draw some consequences of the axioms. It follows from **[TR 1]** alone that one may change the sign of any two morphisms in a distinguished triangle. For example, the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{f} & B & \xrightarrow{-g} & C & \xrightarrow{-h} & \Sigma A \end{array}$$

proves that $[f, g, h]$ is a distinguished triangle if and only if $[f, -g, -h]$ is a distinguished triangle.

Remark 1.1.2. It is *not* true in general that changing the sign of one morphism in a distinguished triangle yields a distinguished triangle [**Ive86**, Example 4.21, p. 32].

Making heavy use of **[TR 2]** it is easy to prove that the axioms of pre-triangulated categories are self-dual. This helps saving quite some work by admitting reasoning by duality:

Lemma 1.1.3. *The triple* $(\mathscr{T}, \Sigma, \Delta)$ *is a pre-triangulated category if and only if* $(\mathscr{T}^{\mathrm{op}}, \Sigma^{-1}, \Delta^{\mathrm{op}})$ *is a pre-triangulated category.* □

Let $[f, g, h]$ be a distinguished triangle. By **[TR 1]** and **[TR 3]** there is a morphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \end{array}$$

and thus $gf = 0$. It follows from **[TR 2]** that $hg = 0$ as well as $\Sigma(f)h = 0$. In brief:

Lemma 1.1.4. *The composition of any two consecutive morphisms in a distinguished triangle is zero.* □

The fundamental fact on pre-triangulated categories is the following result which gives among many other things an explanation for the notorious analogy "distinguished triangles should be considered as a replacement for exact sequences".

Proposition 1.1.5 ([**Bal00**, Section 0]). *Given a distinguished triangle* $[f, g, h]$, *the morphism* f *is a* weak kernel *of* g *in the sense that there exists a factorization* $x = fx'$

$$\begin{array}{ccccccc} & & X & & & & \\ & {\scriptstyle \exists x'}\swarrow & \downarrow{\scriptstyle x} & \searrow{\scriptstyle 0} & & & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \end{array}$$

for every $X \xrightarrow{x} B$ *such that* $gx = 0$.

Similarly, h is a weak cokernel *of g in that there is a factorization $y = y'h$*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ & & & \searrow^{0} & \downarrow{\scriptstyle y} & \swarrow_{\exists y'} & \\ & & & & Y & & \end{array}$$

for every $C \xrightarrow{y} Y$ such that $yg = 0$.

Proof. By **[TR 1]** we can embed the identity of X into a distinguished triangle. So we have a diagram

$$\begin{array}{ccccccc} X & \xrightarrow{1} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle \exists ?} & & \downarrow{\scriptstyle x} & & \downarrow & & \downarrow{\scriptstyle \exists ?} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \end{array}$$

in which the rows are distinguished triangles.

Rotating twice to the left using **[TR 2]**, the question becomes:

$$\begin{array}{ccccccc} \Sigma^{-1}X & \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{1} & X \\ \downarrow{\scriptstyle \Sigma^{-1}x} & & \downarrow & & \downarrow{\scriptstyle \exists x'} & & \downarrow{\scriptstyle x} \\ \Sigma^{-1}B & \xrightarrow{-\Sigma^{-1}g} & \Sigma^{-1}C & \xrightarrow{-\Sigma^{-1}h} & A & \xrightarrow{f} & B \end{array}$$

so that the existence of x' follows from **[TR 3]**.

The weak cokernel property follows by duality. □

Remark 1.1.6. Notice that the factorizations x' and y' are *not* unique, so that f is not a kernel of g and h is not a cokernel of g in general.

Corollary 1.1.7. *In a pre-triangulated category, a morphism is epic if and only if it has a right inverse and it is monic if and only if it has a left inverse.*

Proof. By duality it suffices to prove the first statement. The "if" part is true in every category, so it remains to prove that every epic splits: Let f be an epic and embed it in a distinguished triangle $[f, g, h]$. Because $gf = 0$ and f is epic we must have $g = 0$, so apply Proposition 1.1.5 with $X = B$ and $x = 1_B$ to obtain a right inverse x' of f. □

Remark 1.1.8. In triangulated categories with countable (co-)products, the above corollary admits a substantial refinement, see [**Ver96**, Ch 2, 1.2.9].

Lemma 1.1.9 ([**Bal00**, Section 0]). *Let $[f, g, h]$ be a distinguished triangle. If*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 0} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \end{array}$$

is a morphism of triangles then $c^2 = 0$.

Proof. We have $cg = 0$, so by the weak cokernel property there is $\Sigma A \xrightarrow{c'} C$ such that $c = c'h$, so $c^2 = c'hc = 0$. □

For pre-triangulated categories, there is the following version of the five lemma:

Corollary 1.1.10. *Consider a morphism of distinguished triangles*

$$\begin{CD} A @>f>> B @>g>> C @>h>> \Sigma A \\ @VaVV @VbVV @VcVV @VV\Sigma aV \\ A' @>f'>> B' @>g'>> C' @>h'>> \Sigma A'. \end{CD}$$

If two out of a, b and c are isomorphisms then so is the third.

Proof. Using [**TR 2**] one reduces to the situation in which the assumption holds for a and b, so suppose that a and b are isomorphisms. In the following diagram the rows are distinguished triangles

$$\begin{CD} A @>f>> B @>g>> C @>h>> \Sigma A \\ @VaVV @VbVV @VcVV @VV\Sigma aV \\ A' @>f'>> B' @>g'>> C' @>h'>> \Sigma A' \\ @Va^{-1}VV @Vb^{-1}VV @V\exists\tilde{c}VV @VV\Sigma a^{-1}V \\ A @>f>> B @>g>> C @>h>> \Sigma A \end{CD}$$

and $\tilde{c}$ exists by [**TR 3**] since the lower left square commutes by the computation

$$fa^{-1} = b^{-1}bfa^{-1} = b^{-1}f'aa^{-1} = b^{-1}f'.$$

Subtracting $(1_A, 1_B, 1_C)$ from

$$\begin{CD} A @>f>> B @>g>> C @>h>> \Sigma A \\ @V1VV @V1VV @V\tilde{c}cVV @VV\Sigma(1)V \\ A @>f>> B @>g>> C @>h>> \Sigma A \end{CD}$$

and putting $\varepsilon = \tilde{c}c - 1_C$, we obtain a morphism of distinguished triangles $(0, 0, \varepsilon)$, so $\varepsilon^2 = 0$ holds by the previous lemma. In particular

$$(1-\varepsilon)\tilde{c}c = (1-\varepsilon)(1+\varepsilon) = 1$$

and thus

$$\bar{c} = (1-\varepsilon)\tilde{c}$$

is a left inverse of c.

Similarly, $\varepsilon' = c\tilde{c} - 1_{C'}$ satisfies $(\varepsilon')^2 = 0$ and thus

$$\bar{\bar{c}} = \tilde{c}(1-\varepsilon')$$

is a right inverse of c. Therefore $\bar{c} = \bar{c}(c\bar{\bar{c}}) = (\bar{c}c)\bar{\bar{c}} = \bar{\bar{c}}$ is the inverse of c. □

Corollary 1.1.11. *Let $A \xrightarrow{f} B$ be a morphism in a pre-triangulated category $(\mathscr{T}, \Sigma, \Delta)$. The distinguished triangle*

$$A \xrightarrow{f} B \to C \to \Sigma A$$

whose existence is granted by [**TR 1**] *is unique up to* non-unique *isomorphism.* □

Corollary 1.1.12. *A morphism $A \xrightarrow{f} B$ in a pre-triangulated category is an isomorphism if and only if*

$$A \xrightarrow{f} B \to 0 \to \Sigma A$$

is a distinguished triangle.

Proof. If f is an isomorphism then

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & 0 & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle f} & & \| & & \| & & \downarrow{\scriptstyle \Sigma f} \\ B & = & B & \longrightarrow & 0 & \longrightarrow & \Sigma B \end{array}$$

is an isomorphism of triangles and hence the top row is a distinguished triangle because the bottom row is a distinguished triangle by [**TR 1**].

Conversely, if in the diagram above both rows are distinguished triangles then f must be an isomorphism by the five lemma. □

Definition 1.1.13. Let $\mathscr{T}$ be a pre-triangulated category and let $\mathscr{A}$ be an abelian category. A functor $F : \mathscr{T} \to \mathscr{A}$ is *homological* if for each distinguished triangle $A \to B \to C \to \Sigma A$ the sequence

$$F(A) \to F(B) \to F(C)$$

is exact at $F(B)$.

It is customary to write $F = F^0$ and $F^n(A) = F(\Sigma^n A)$ for $n \in \mathbb{Z}$. It then follows from [**TR 2**] that a homological functor yields a long exact sequence

$$\cdots \to F^{n-1}(C) \to F^n(A) \to F^n(B) \to F^n(C) \to F^{n+1}(A) \to \cdots$$

in $\mathscr{A}$ for each distinguished triangle $A \to B \to C \to \Sigma A$.

By far the most important homological functor is $\mathrm{Hom} = \mathrm{Hom}_{\mathscr{T}}$, one writes $\mathrm{Hom}^n(A, B) := \mathrm{Hom}(A, \Sigma^n B)$.

Corollary 1.1.14. *For each object X in a pre-triangulated category* $\mathrm{Hom}(X, -)$ *is a homological functor* $\mathscr{T} \to \mathbf{Ab}$.

Proof. This follows from the weak kernel property and the fact that the composition of two consecutive morphisms in a distinguished triangle is zero. □

Now we can prove an important self-strengthening of the weak kernel property.

Corollary 1.1.15 ([**BBD82**, 1.1.9]). *Consider the following diagram in which the rows are distinguished triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \vdots{\scriptstyle \exists a} & & \downarrow{\scriptstyle b} & & \vdots{\scriptstyle \exists c} & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \end{array}$$

The following four conditions are equivalent:

(1) $g'bf = 0$.
(2) There exists a morphism $a : A \to A'$ making the left hand square commutative.
(3) There exists a morphism $c : C \to C'$ making the right hand square commutative.
(4) The morphism b fits into a morphism of triangles (a, b, c).

If in addition to these conditions $\mathrm{Hom}^{-1}(A, C') = 0$ *then a and c are unique.*

Proof. Condition (1) is a consequence of the three other conditions since the composition of consecutive maps in distinguished triangles is zero. Clearly (4) implies (2) and (3). Condition (2) follows from (1) by applying the weak kernel property of the bottom row to bf and (3) follows from (1) by the weak cokernel property of the top row applied to $g'b$. Condition (4) follows from (2) and [**TR 3**].

Since $\mathrm{Hom}(A, -)$ is a homological functor, we have a long exact sequence

$$\mathrm{Hom}^{-1}(A, C') \to \mathrm{Hom}^0(A, A') \to \mathrm{Hom}^0(A, B') \to \mathrm{Hom}^0(A, C').$$

If $\mathrm{Hom}^{-1}(A, C') = 0$ and the conditions are satisfied, then a is unique and similarly, the long exact sequence

$$\mathrm{Hom}^{-1}(A, C') \to \mathrm{Hom}^0(C, C') \to \mathrm{Hom}^0(B, C') \to \mathrm{Hom}^0(A, C')$$

implies the uniqueness of c. □

REMARK 1.1.16. Notice that the equivalence of the first three conditions also follows from contemplating the long exact sequences used in the second part of the proof.

PROPOSITION 1.1.17. *The direct sum of two distinguished triangles is a distinguished triangle.*

PROOF. Given two distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \qquad \text{and} \qquad A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

we claim that

$$A \oplus A' \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} B \oplus B' \xrightarrow{\left[\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right]} C \oplus C' \xrightarrow{\left[\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right]} \Sigma A \oplus \Sigma A'$$

is a distinguished triangle as well.

By **[TR 1]** we can embed $\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]$ in a distinguished triangle

$$A \oplus A' \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} B \oplus B' \longrightarrow Z \longrightarrow \Sigma A \oplus \Sigma A'.$$

Applying **[TR 3]** to the situation

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow{\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & & \vdots{\scriptstyle \exists c} & & \downarrow \\
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \longrightarrow & Z & \longrightarrow & \Sigma A \oplus \Sigma A'.
\end{array}$$

and to

$$\begin{array}{ccccccc}
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A' \\
\downarrow{\scriptstyle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & & \vdots{\scriptstyle \exists c'} & & \downarrow \\
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \longrightarrow & Z & \longrightarrow & \Sigma A \oplus \Sigma A'.
\end{array}$$

we obtain a morphism of triangles

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \xrightarrow{\left[\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right]} & C \oplus C' & \xrightarrow{\left[\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right]} & \Sigma A \oplus \Sigma A' \\
\| & & \| & & \downarrow{\scriptstyle [\, c \;\, c' \,]} & & \| \\
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \longrightarrow & Z & \longrightarrow & \Sigma A \oplus \Sigma A'.
\end{array}$$

We claim that $[\, c \;\, c' \,]$ is an isomorphism and the proposition is proved as soon as this is established because the class of distinguished triangles is closed under isomorphisms of triangles.

Applying the homological functor $\mathrm{Hom}(Z, -) : \mathscr{T} \to \mathbf{Ab}$ to the diagram

$$\begin{array}{ccccccccc}
A \oplus A' & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C' & \longrightarrow & \Sigma A \oplus \Sigma A' & \longrightarrow & \Sigma B \oplus \Sigma B' \\
\| & & \| & & \downarrow & & \| & & \| \\
A \oplus A' & \longrightarrow & B \oplus B' & \longrightarrow & Z & \longrightarrow & \Sigma A \oplus \Sigma A' & \longrightarrow & \Sigma B \oplus \Sigma B'
\end{array}$$

yields a diagram in **Ab** in which we know the rows to be exact and every vertical arrow except the middle one to be an isomorphism. The five lemma in **Ab** applies to prove that

$$\operatorname{Hom}(Z, C \oplus C') \to \operatorname{Hom}(Z, Z)$$

is an isomorphism. Therefore $\left[\begin{smallmatrix} c & c' \end{smallmatrix}\right]$ has a right inverse. Dually, applying the cohomological functor $\operatorname{Hom}(-, C \oplus C')$ proves the existence of a left inverse of $\left[\begin{smallmatrix} c & c' \end{smallmatrix}\right]$. A morphism having both a left and a right inverse is an isomorphism. □

Proposition 1.1.18 ([**BS01**, 1.6]). *Two triangles are distinguished if and only if their sum is distinguished.*

Proof. The only if part is already settled. So suppose we are given triangles

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \qquad \text{and} \qquad A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

such that

$$A \oplus A' \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} B \oplus B' \xrightarrow{\left[\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right]} C \oplus C' \xrightarrow{\left[\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right]} \Sigma A \oplus \Sigma A'$$

is a distinguished triangle. There are two distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{k} D \xrightarrow{l} \Sigma A \qquad \text{and} \qquad A' \xrightarrow{f'} B' \xrightarrow{k'} D' \xrightarrow{l'} \Sigma A'$$

by [**TR 1**]. By [**TR 3**] there is a morphism of triangles

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \xrightarrow{\left[\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right]} & C \oplus C' & \xrightarrow{\left[\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right]} & \Sigma A \oplus \Sigma A' \\
\downarrow{\scriptstyle [1\ 0]} & & \downarrow{\scriptstyle [1\ 0]} & & \vdots{\scriptstyle [c\ d]} & & \downarrow{\scriptstyle [1\ 0]} \\
A & \xrightarrow{f} & B & \xrightarrow{k} & D & \xrightarrow{l} & \Sigma A
\end{array}$$

and a morphism of triangles

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \xrightarrow{\left[\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right]} & C \oplus C' & \xrightarrow{\left[\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right]} & \Sigma A \oplus \Sigma A' \\
\downarrow{\scriptstyle [0\ 1]} & & \downarrow{\scriptstyle [0\ 1]} & & \vdots{\scriptstyle [d'\ c']} & & \downarrow{\scriptstyle [0\ 1]} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{k'} & D' & \xrightarrow{l'} & \Sigma A'.
\end{array}$$

Given this, we construct a morphism of distinguished triangles

$$\begin{array}{ccccccc}
A \oplus A' & \xrightarrow{\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]} & B \oplus B' & \xrightarrow{\left[\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right]} & C \oplus C' & \xrightarrow{\left[\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right]} & \Sigma A \oplus \Sigma A' \\
\| & & \| & & \downarrow{\left[\begin{smallmatrix} c & 0 \\ 0 & c' \end{smallmatrix}\right]} & & \| \\
A \oplus A' & \xrightarrow[\left[\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right]]{} & B \oplus B' & \xrightarrow[\left[\begin{smallmatrix} k & 0 \\ 0 & k' \end{smallmatrix}\right]]{} & D \oplus D' & \xrightarrow[\left[\begin{smallmatrix} l & 0 \\ 0 & l' \end{smallmatrix}\right]]{} & \Sigma A \oplus \Sigma A'
\end{array}$$

which together with the five lemma proves $\left[\begin{smallmatrix} c & 0 \\ 0 & c' \end{smallmatrix}\right]$ to be an isomorphism. Therefore c and c' are isomorphisms. It follows that the triangles

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \qquad \text{and} \qquad A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

are distinguished because they are isomorphic to the distinguished triangles

$$A \xrightarrow{f} B \xrightarrow{k} D \xrightarrow{l} \Sigma A \qquad \text{and} \qquad A' \xrightarrow{f'} B' \xrightarrow{k'} D' \xrightarrow{l'} \Sigma A'$$

via $(1_A, 1_B, c)$ and $(1_{A'}, 1_{B'}, c')$. □

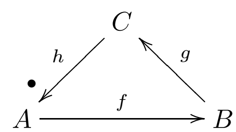
1.2. The Octahedral Axiom. The salient ingredient of Verdier's axiomatics for triangulated categories is the octahedral axiom. It seems fair to say that every serious application of triangulated categories involves octahedra: for instance, only pale shades of Verdier's localization theory (see Chapter II) as well as the theory of abstract truncation (see Chapter III) remain without it. In this section we only give two of the many equivalent versions, both will be used in later chapters.

Let $(\mathscr{T}, \Sigma, \Delta)$ be a pre-triangulated category. Recall the consequence of the five lemma stated in Corollary 1.1.11: for a morphism $A \xrightarrow{f} B$ there is a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and the isomorphism class of C is uniquely determined by f. It is therefore natural to ask how this isomorphism class behaves under composition of morphisms and the octahedral axiom gives an answer to precisely this question.

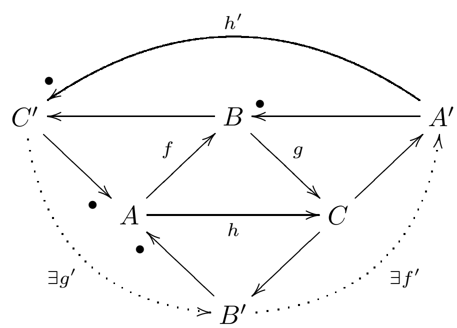
Notation 1.2.1. It is often convenient to depict distinguished triangles diagrammatically as actual triangles



where the bullet indicates that h is a morphism of degree 1 in that it is not an arrow from C to A but rather from C to ΣA.

Definition 1.2.2. A pre-triangulated category $(\mathscr{T}, \Sigma, \Delta)$ is called *triangulated* if it satisfies in addition the *octahedral axiom:*

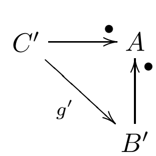
[TR 4] Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be morphisms and embed f, g and $h = gf$ into distinguished triangles so as to obtain the following diagram in which h' is the composite $A \to \Sigma B \to \Sigma C'$:



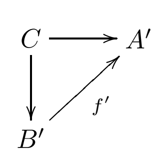
There exist morphisms f' and g' such that

(i) The triangle $C' \xrightarrow{g'} B' \xrightarrow{f'} A' \xrightarrow{h'} \Sigma C'$ is distinguished.

(ii) The triangles



and



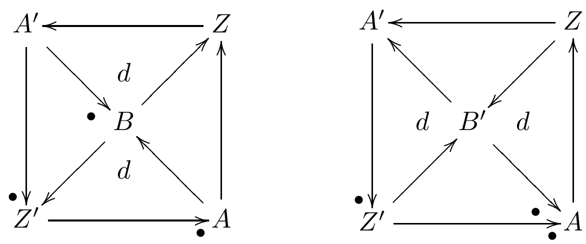
are commutative.

(iii) The two morphisms $B \to B'$ (via C and C') coincide as well as the two morphisms $B' \to \Sigma B$ (via ΣA and A').

Remark 1.2.3. Even though there appears to be no known example of a pre-triangulated category which fails to satisfy the octahedral axiom it is conceivable that such examples exist.

Remark 1.2.4. Point (i) of [**TR 4**] is certainly the most important one because it allows to "construct" distinguished triangles in a non-trivial way.

According to [**BBD82**, 1.1.6] an *octahedron* is given by two diagrams



with the same surrounding squares and in which the triangles marked with a "d" are distinguished and the others are commutative. It is furthermore required that the two morphisms from B to B' (via Z and Z') coincide as well as the two morphisms from B' to ΣB (via ΣA and A').

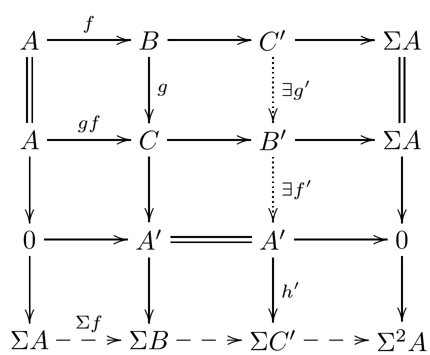
The left hand diagram is called the *upper cap* and the right hand diagram is called the *lower cap* of the octahedron.

Proposition 1.2.5 ([**BBD82**, 1.1.6]). *In a pre-triangulated category the following statements are equivalent:*

(i) The octahedral axiom [**TR 4**].
(ii) Every diagram of type upper cap *can be completed to an octahedron.*
(iii) Every diagram of type lower cap *can be completed to an octahedron.* □

Remark 1.2.6. We refer the reader to [**BBD82**, 1.1.6] for a study of the symmetries of the octahedron.

Remark 1.2.7. In the literature one often encounters the diagram of the octahedral axiom in guise of the commutative diagram



in which the first three rows and all columns are a distinguished triangles, the last row is obtained from the first by applying Σ. The octahedral axiom can thus be seen as a refinement of [**TR 3**]: the existence of f' and g' and of such a commutative diagram follow from [**TR 3**], but [**TR 4**] says moreover that f' and g' can be chosen in such a way that the third column is a distinguished triangle.

The drawback of this presentation of the octahedron is that one does not see the second part of condition (iii) of [**TR 4**] directly. The point is that in the above heuristic f' should be thought of as coming from applying **[TR 3]** to

$$\begin{array}{ccccccc} A & \xrightarrow{gf} & C & \longrightarrow & B' & \longrightarrow & \Sigma A \\ \downarrow{\scriptstyle f} & & \| & & \vdots{\scriptstyle \exists f'} & & \downarrow{\scriptstyle \Sigma f} \\ B & \xrightarrow{g} & C & \longrightarrow & A' & \longrightarrow & \Sigma B \end{array}$$

rather than the possibly more tempting passage from the second to the third line.

Remark 1.2.8. It is actually not difficult to prove that **[TR 3]** follows from **[TR 1]** and **[TR 4]**, we leave this as an exercise for the interested reader, who—in case of emergency—may consult [**BBD82**, 1.1.11] or [**May01**, Lemma 2.2].

Remark 1.2.9. By rotating the diagram in Remark 1.2.7 one gets

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{-1}A' & = & \Sigma^{-1}A' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle -\Sigma^{-1}h'} & & \downarrow \\ A & \xrightarrow{f} & B & \longrightarrow & C' & \longrightarrow & \Sigma A \\ \| & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle \exists g'} & & \| \\ A & \xrightarrow{gf} & C & \longrightarrow & B' & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & \vdots{\scriptstyle \exists f'} & & \downarrow \\ 0 & \longrightarrow & A' & = & A' & \longrightarrow & 0, \end{array}$$

a diagram which should prompt the reader to think of the middle square as bicartesian. We refer the reader to [**PS88**, (1.11), p.17] and [**Nee01**, Theorem 1.4.6] for equivalent versions of the octahedral axiom involving *homotopy bicartesian squares*. Compare also the strengthened octahedral axiom given in [**BBD82**, 1.1.13].

Remark 1.2.10 (Verdier's Exercise, [**BBD82**, 1.1.11]). Every commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

in a triangulated category can be embedded into a diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \Sigma A' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & \Sigma A'' \\ \downarrow & & \downarrow & & \downarrow & (-) & \downarrow \\ \Sigma A & \dashrightarrow & \Sigma B & \dashrightarrow & \Sigma C & \dashrightarrow & \Sigma^2 A \end{array}$$

in which the first three rows and the first three columns are distinguished triangles, all unlabeled squares are commutative and the square labeled with $(-)$ is anti-commutative. The row and the column with dashed arrows are obtained from distinguished triangles by applying Σ.

1.3. Triangulated Subcategories. Let $(\mathscr{T}, \Sigma, \Delta)$ be a (pre-)triangulated category. A full subcategory $\mathscr{S} \subset \mathscr{T}$ is called a (pre-)triangulated subcategory if it is stable under $\Sigma^{\pm 1}$ and if for every distinguished triangle in $\mathscr{T}$ with two objects in $\mathscr{S}$ there exists an isomorphic triangle with all three objects in $\mathscr{S}$.

Proposition 1.3.1. *A (pre-)triangulated subcategory of a (pre-)triangulated category is itself (pre-)triangulated.* □

Remark 1.3.2. It is more convenient to work with *strictly full* subcategories $\mathscr{S}$, for which the condition in the definition reads: $\mathscr{S}$ is stable under Σ and if two objects of a distinguished triangle in $\mathscr{T}$ are in $\mathscr{S}$ then so is the third.

2. Triangle Functors

2.1. Definition.

Definition 2.1.1. Let $(\mathbf{K}, \Sigma, \Delta)$ and $(\mathbf{K}', \Sigma', \Delta')$ be triangulated categories. A *triangle functor* is a pair (F, α) consisting of an additive functor $F : \mathbf{K} \to \mathbf{K}'$ and a natural transformation $\alpha : F\Sigma \Rightarrow \Sigma' F$ such that for all distinguished triangles $(u, v, w) \in \Delta$

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{\alpha_X Fw} \Sigma' FX$$

is a distinguished triangle in $\mathbf{K}'$.

Remark 2.1.2. Let (F, α) be a triangle functor. Consider the distinguished triangle

$$X \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{-1} \Sigma X$$

and apply (F, α) to obtain the distinguished triangle

$$FX \longrightarrow 0 \longrightarrow F\Sigma X \xrightarrow{-\alpha_X} \Sigma' FX.$$

Therefore the five lemma for triangulated categories implies that α is an isomorphism of functors.

Remark 2.1.3. Triangulated categories form a large category whose morphisms are the triangle functors.

2.2. Adjoint Pairs of Triangle Functors.

Definition 2.2.1 ([**KV87**, 1.6]). Let $(R, \rho) : \mathbf{K} \to \mathbf{K}'$ and $(L, \lambda) : \mathbf{K}' \to \mathbf{K}$ be triangle functors such that L is left adjoint to R. Let $\varphi : LR \Rightarrow \mathrm{id}_{\mathbf{K}}$ and $\psi : \mathrm{id}_{\mathbf{K}'} \Rightarrow RL$ be the adjunction morphisms. The following four conditions are equivalent:

(i) $\lambda = (\varphi \Sigma L)(L\varrho^{-1} L)(L\Sigma' \psi)$;
(ii) $\varrho^{-1} = (R\Sigma\varphi)(R\lambda R)(\psi \Sigma' R)$;
(iii) $\varphi\Sigma = (\Sigma\varphi)(\lambda R)(L\varrho)$;
(iv) $\Sigma'\psi = (\varrho L)(R\lambda)(\psi\Sigma')$.

If these conditions are satisfied, the triangle functors (L, λ) and (R, ϱ) are called an *adjoint pair of triangle functors*.

Proposition 2.2.2 ([**KV87**, 1.6]). *Let $(R, \varrho) : \mathbf{K} \to \mathbf{K}'$ be a triangle functor and suppose that it has a left adjoint $L \dashv R$. Let*

$$\psi : \mathrm{id}_{\mathbf{K}'} \Rightarrow RL \qquad \text{and} \qquad \varphi : LR \Rightarrow \mathrm{id}_{\mathbf{K}}$$

be the adjunction morphisms. For $X \in \mathbf{K}'$ put

$$\lambda_X = \varphi_{\Sigma L X} \circ L\varrho^{-1}_{LX} \circ L\Sigma' \psi_X.$$

Then $\lambda : L\Sigma' \Rightarrow \Sigma L$ is a natural transformation such that (L, λ) is a triangle functor and (L, λ) and (R, ϱ) are an adjoint pair of triangle functors. □

CHAPTER II

The Derived Category of an Exact Category

In this section we briefly review the construction of the derived category of an exact category following [**Nee90**]. In order to do this, we need to discuss the central topic of Verdier localization first. The only reasonably complete exposition of the derived category of an exact category the author knows of is [**Kel96**]. There are some remarks to be found in [**BBD82**, 1.1.4,1.3.22], see also [**Lau83**, Chapitre 1], however, both these sources assume that the exact category at hand has kernels, which is somewhat too stringent for our purposes.

Contents

1. Verdier Localization

1.1. Thick Subcategories and Rickard's Criterion.

LEMMA 1.1.1. *Let* (F, α) *be a triangle functor* $\mathbf{K} \to \mathbf{K}'$. *Consider*

$$\mathscr{T} = \{X \in \mathbf{K} : F(X) \cong 0\}.$$

Then $\mathscr{T}$ *is a strictly full triangulated subcategory such that*

$$X \oplus Y \in \mathscr{T} \quad \Rightarrow \quad X, Y \in \mathscr{T}$$

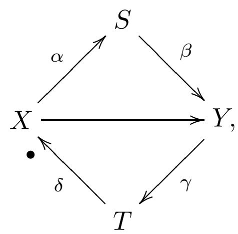
for all $X, Y \in \mathbf{K}$.

PROOF. By its definition $\mathscr{T}$ is closed under isomorphisms in $\mathbf{K}$. To prove $\mathscr{T}$ triangulated, apply the five lemma for triangulated categories. The last fact follows from the additivity of F: the only direct summands of zero objects are zero objects. □

DEFINITION 1.1.2 (Verdier). A strictly full triangulated subcategory $\mathscr{T}$ of the triangulated category $\mathbf{K}$ is called *thick* if $X \oplus Y \in \mathscr{T}$ implies $X, Y \in \mathscr{T}$ for all $X, Y \in \mathbf{K}$.

REMARK 1.1.3. By Lemma 1.1.1, the kernel of a triangle functor is thick. The basic observation of Verdier is that the converse is true (modulo set-theoretic difficulties): every thick subcategory $\mathscr{T}$ of a triangulated category $\mathbf{K}$ is the kernel of a triangle functor $\mathbf{K} \to \mathbf{K} / \mathscr{T}$, and this triangle functor is a solution of a universal problem.

Definition 1.1.4 (Verdier). A triangulated subcategory $\mathscr{T}$ of $\mathbf{K}$ is called *épaisse* if for all diagrams



in which the upper triangle is commutative and the lower triangle is distinguished, $S, T \in \mathscr{T}$ implies $X, Y \in \mathscr{T}$. "If a morphism of $\mathbf{K}$ factors through an object of $\mathscr{T}$ and has its cone in $\mathscr{T}$ then it is a morphism of $\mathscr{T}$."

Remark 1.1.5. An épaisse subcategory is necessarily strictly full. To see this, suppose that $X \to Y$ is an isomorphism and $Y \in \mathscr{T}$, take $S = Y$, notice that the cone of an isomorphism is zero and apply the definition of an épaisse subcategory.

Proposition 1.1.6 ("Rickard's Criterion"). *A triangulated subcategory $\mathscr{T}$ of a triangulated category $\mathbf{K}$ is thick if and only if it is épaisse.*

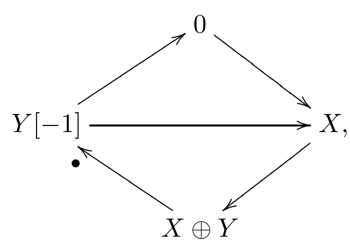
Proof. Suppose first that $\mathscr{T}$ is épaisse. We have already observed that $\mathscr{T}$ must be strictly full, so we have to show that $X \oplus Y \in \mathscr{T}$ implies $X, Y \in \mathscr{T}$. The direct sum of the distinguished triangles

$$X \to X \to 0 \to X[1] \qquad \text{and} \qquad 0 \to Y \to Y \to 0$$

yields the distinguished triangle

$$Y[-1] \xrightarrow{0} X \to X \oplus Y \to Y$$

by rotation. Therefore there is the diagram



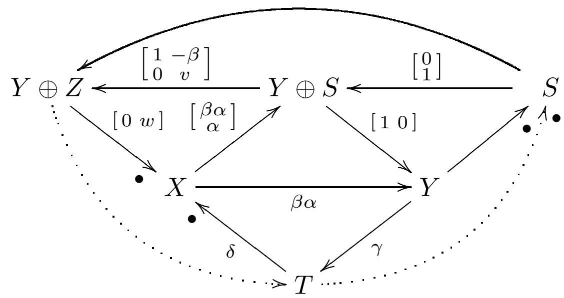
which implies $Y[-1], X \in \mathscr{T}$ because $0, X \oplus Y \in \mathscr{T}$ and $\mathscr{T}$ is épaisse.

Conversely, suppose that $\mathscr{T}$ is thick and that we are given a diagram as in Definition 1.1.4 with $S, T \in \mathscr{T}$. We have to prove that then $X, Y \in \mathscr{T}$ as well. Choose a distinguished triangle $X \xrightarrow{\alpha} S \xrightarrow{v} Z \xrightarrow{w} X[1]$. Add the distinguished triangle $0 \to Y \to Y \to 0$ to it in order to obtain the first row in the following diagram which displays an isomorphism of triangles:

$$\begin{array}{ccccccc}
X & \xrightarrow{\left[\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}\right]} & Y \oplus S & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right]} & Y \oplus Z & \xrightarrow{[0 \ w]} & X[1] \\
\| & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right]} & & \| & & \| \\
X & \xrightarrow{\left[\begin{smallmatrix} \beta\alpha \\ \alpha \end{smallmatrix}\right]} & Y \oplus S & \xrightarrow{\left[\begin{smallmatrix} 1 & -\beta \\ 0 & v \end{smallmatrix}\right]} & Y \oplus Z & \xrightarrow{[0 \ w]} & X[1].
\end{array}$$

Since the first row is a distinguished triangle, so is the second row.

Now construct the following octahedron



which yields a distinguished triangle $S \to Y \oplus Z \to T \to S[1]$, so $Y \oplus Z \in \mathscr{T}$ because $\mathscr{T}$ is a strictly full triangulated subcategory and $S, T \in \mathscr{T}$. Since $\mathscr{T}$ is thick we have $Y \in \mathscr{T}$. Finally, the distinguished triangle $X \to Y \to T \to X[1]$ implies $X \in \mathscr{T}$. $\square$

Remark 1.1.7. Unfortunately, Verdier's thesis [**Ver96**]—of which only the preface was submitted to the university—was made accessible to the broad mathematical community only in the mid-nineties. The definition of a thick subcategory is contained in *loc. cit.* (Chapitre 2, 2.1.6). The definition of an épaisse subcategory is contained in Verdier's early draft "Catégories Dérivées (Etat 0)" (see [**Dea77**, p.276]) and is more difficult to work with. Proposition 1.1.6 was rediscovered by Rickard in [**Ric89**, Proposition 1.4]. The simple proof presented here may be found in [**Nee90**, Criterion 1.3] in somewhat more condensed form.

1.2. Verdier Localization. We now turn to the description of the converse of Lemma 1.1.1 already alluded to in Remark 1.1.3. Suppose we are given a triangulated category $\mathbf{K}$ and a thick subcategory $\mathscr{T}$. We want to construct a triangulated category $\mathbf{K}/\mathscr{T}$ and a triangle functor $(F, \alpha) : \mathbf{K} \to \mathbf{K}/\mathscr{T}$ such that

$$\mathscr{T} = \{X \in \mathbf{K} \,:\, F(X) \cong 0\}.$$

Assume for the moment that $\mathbf{K}/\mathscr{T}$ and (F, α) exist and suppose u is a morphism in $\mathbf{K}$ such that there exists a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} T \xrightarrow{w} X[1]$$

with $T \in \mathscr{T}$ —any distinguished triangle over u will do because $\mathscr{T}$ is strictly full by definition. Applying the triangle functor (F, α) we obtain a distinguished triangle

$$F(X) \xrightarrow{F(u)} F(Y) \longrightarrow 0 \longrightarrow F(X)[1]$$

and conclude:

Lemma 1.2.1. *Provided that $\mathbf{K}/\mathscr{T}$ and (F, α) exist, a morphism u of $\mathbf{K}$ is mapped to an isomorphism of $\mathbf{K}/\mathscr{T}$ under F if and only if its cone is in $\mathscr{T}$.* $\square$

Thus, let $\mathscr{W}$ be the class of morphisms whose cone is in $\mathscr{T}$.

Proposition 1.2.2 ([**Ver96**, Ch II, 2.1.8]). *The class $\mathscr{W}$ satisfies:*

(i) $1_X \in \mathscr{W}$ *for all* $X \in \mathbf{K}$.
(ii) *The class $\mathscr{W}$ is closed under composition.*

(iii) (Left Ore Condition) *Given a diagram* $X' \xleftarrow{s} X \xrightarrow{f} Y$ *with* $s \in \mathscr{W}$ *there exists a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \vdots\,{\scriptstyle t} \\ X' & \overset{f'}{\cdots\!\!\to} & Y' \end{array}$$

with $t \in \mathscr{W}$.

(iii$^{\mathrm{op}}$) (Right Ore Condition) *Given a diagram* $X' \xrightarrow{f'} Y' \xleftarrow{t} Y$ *with* $s \in \mathscr{W}$ *there exists a commutative diagram*

$$\begin{array}{ccc} X & \overset{f}{\cdots\!\!\to} & Y \\ {\scriptstyle s}\,\vdots & & \downarrow{\scriptstyle t} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

with $s \in \mathscr{W}$.

(iv) (Left Cancellation) *If f and g are morphisms and there exists $s \in \mathscr{W}$ such that $fs = gs$ then there exists $t \in \mathscr{W}$ such that $tf = tg$.*

(iv$^{\mathrm{op}}$) (Right Cancellation) *If f and g are morphisms and there exists $t \in \mathscr{W}$ such that $tf = tg$ then there exists $s \in \mathscr{W}$ such that $fs = gs$.*

(v) (Saturation) *Let f, g, h be composeable morphisms. If gf and hg are in $\mathscr{W}$ then $g \in \mathscr{W}$.*

(vi) *If $s \in \mathscr{W}$ then $s[\pm 1] \in \mathscr{W}$.*

(vii) *Consider the following diagram whose rows are distinguished triangles:*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} & & \vdots\,{\scriptstyle \exists u} & & \downarrow{\scriptstyle s[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

If $s, t \in \mathscr{W}$ then there exists $u \in \mathscr{W}$ such that (s, t, u) is a morphism of triangles.

We only sketch the proof. A much more complete proof (and an improved statement) can be found in *loc. cit.*

Proof. Properties (i) and (vi) are obvious. Property (ii) is a direct consequence of the octahedral axiom. Let us prove (iv): Suppose there are a morphism f of $\mathbf{K}$ and a morphism $s \in \mathscr{W}$ such that $fs = 0$. We have to establish the existence of $t \in \mathscr{W}$ such that $tf = 0$. Choose a distinguished triangle with base s and observe that $T \in \mathscr{T}$ by definition of $\mathscr{W}$. Apply the weak cokernel property as follows:

$$\begin{array}{ccccccc} X & \xrightarrow{s} & Y & \xrightarrow{v} & T & \xrightarrow{w} & X[1] \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle \exists f'} & & & \\ & & Z & & & & \end{array}$$

Embed f' into a distinguished triangle

$$T \xrightarrow{f'} Z \xrightarrow{t} C \to T[1]$$

and notice that $t \in \mathscr{W}$ because $T \in \mathscr{T}$ and $tf = (tf')v = 0$ because the composition of two consecutive morphisms in a distinguished triangle is zero. Property (iv$^{\mathrm{op}}$) follows by duality.

Property (vii) is easy as soon as one notices that every morphism of morphisms decomposes as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \| & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{tf} & Y' \\ {\scriptstyle s}\downarrow & & \| \\ X' & \xrightarrow{f'} & Y' \end{array}$$

Applying the octahedral axiom to each of the two squares one obtains two morphisms in $\mathscr{W}$, and composing these (using (ii)), one finds the required morphism u.

Now let us prove that (i), (vi) and (vii) imply (iii$^{\mathrm{op}}$). Given the diagram

$$\begin{array}{ccc} & & X' \\ & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow{f} & X \end{array}$$

embed $-f$ into a triangle $Z \xrightarrow{-f} X \xrightarrow{u} Y \xrightarrow{v} Z[1]$ and rotate it to the right. Embedding us into a triangle we end up with the diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{us} & Y & \xrightarrow{v'} & Z' & \xrightarrow{f'[1]} & X'[1] \\ {\scriptstyle s}\downarrow & & \| & & \vdots{\scriptstyle \exists s' \in \mathscr{W}} & & \downarrow{\scriptstyle s[1]} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z[1] & \xrightarrow{f[1]} & X[1] \end{array}$$

in which the existence of $s' \in \mathscr{W}$ follows from (vii). Using (vi) we can apply Σ^{-1} to the rightmost square so as to obtain the required square. By duality, (iii) holds as well.

It remains to prove (v) which follows again from a repeated use of the octahedral axiom, see [**Ver96**, Ch. II, 2.1.13]. □

DEFINITION 1.2.3. Let $\mathscr{C}$ be any category, let $\mathscr{W}$ be a class of morphisms and consider the conditions of Proposition 1.2.2.

(1) If $\mathscr{W}$ satisfies conditions (i), (ii), (iii) and (iv) then $\mathscr{W}$ is said to *admit a calculus of left fractions*.
(2) If $\mathscr{W}$ satisfies conditions (i), (ii), (iii$^{\mathrm{op}}$) and (iv$^{\mathrm{op}}$) then $\mathscr{W}$ is said to *admit a calculus of right fractions*.
(3) If $\mathscr{W}$ admits both a calculus of right and left fractions then it is a *multiplicative system*.
(4) If $\mathscr{W}$ admits a calculus of left or right fractions and satisfies condition (v) then it is said to be *saturated*.
(5) If $\mathscr{C}$ is a triangulated category and if $\mathscr{W}$ satisfies conditions (i) to (vii) of Proposition 1.2.2 then it is a multiplicative system which is *compatible with the triangulation*.

Proposition 1.2.2 has the following converse:

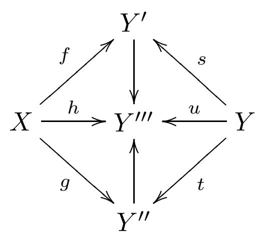
PROPOSITION 1.2.4 (Verdier). *Let* $\mathbf{K}$ *be a triangulated category. Let* $\mathscr{W}$ *be a multiplicative system which is compatible with the triangulation. Then the full subcategory* $\mathscr{T}$ *of* $\mathbf{K}$ *whose objects are the ones isomorphic to the cone of a morphism in* $\mathscr{W}$ *is a thick triangulated subcategory.* □

Remark 1.2.5. Thus there is a canonical one-to-one correspondence between thick triangulated subcategories and multiplicative systems compatible with the triangulation.

Let $\mathscr{C}$ be a category and let $\mathscr{W}$ be a class of morphisms in $\mathscr{C}$ admitting a calculus of left fractions. Then it is possible to "invert the morphisms of $\mathscr{W}$ formally" as follows. Let $\mathscr{C}[\mathscr{W}^{-1}]$ be the category whose objects are the objects of $\mathscr{C}$ and whose morphisms are the equivalence classes of diagrams (s, f)

$$X \xrightarrow{f} Y' \xleftarrow{s} Y$$

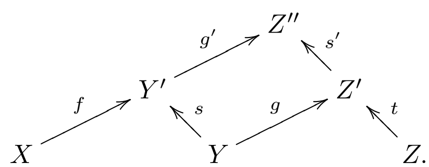
with f an arbitrary morphism and $s \in \mathscr{W}$. Such a diagram should be thought of as "$s^{-1} \circ f$" and is called a *left fraction*. Two left fractions (s, f) and (t, g) are *equivalent* if and only if there exists a commutative diagram



in $\mathscr{C}$ with $u \in \mathscr{W}$. The composition of two left fractions $(s, f), (t, g)$ is defined as

$$(s, f) \circ (t, g) = (s't, g'f)$$

where (s', g') stems from the left Ore condition as indicated by the diagram



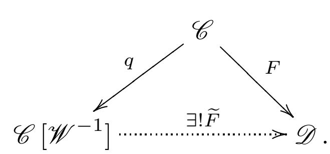
Modulo some set-theoretic reservations (there is no reason for the *classes* of equivalence classes to be *sets*), the category $\mathscr{C}[\mathscr{W}^{-1}]$ is well-defined as a (possibly large) category.

There is a functor $q : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ given by the identity on objects and given by $q(f) = (1, f)$ on morphisms. It is easily checked that $(s, 1)$ is the inverse of $q(s) = (1, s)$. Moreover, q is the solution of a universal problem:

Theorem 1.2.6 (Gabriel-Zisman [**GZ67**]). *Let $\mathscr{C}$ be a category and let $\mathscr{W}$ be a class of morphisms in $\mathscr{C}$ admitting a calculus of left fractions. The functor*

$$\mathscr{C} \xrightarrow{q} \mathscr{C}[\mathscr{W}^{-1}]$$

sends the morphisms in $\mathscr{W}$ to isomorphisms in $\mathscr{C}[\mathscr{W}^{-1}]$ and is universal with respect to this property: For every functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ such that $F(w)$ is invertible for all $w \in \mathscr{W}$ there exists a unique functor $\mathscr{C}[\mathscr{W}^{-1}] \xrightarrow{\widetilde{F}} \mathscr{D}$ such that $F = \widetilde{F}q$, or, in diagrammatic terms,



Moreover, the category $\mathscr{C}[\mathscr{W}^{-1}]$ and the quotient functor are additive if $\mathscr{C}$ is additive. ☐

Definition 1.2.7. A pair $(\mathscr{C}[\mathscr{W}^{-1}], q)$ satisfying the universal property stated in the Gabriel-Zisman theorem is uniquely determined up to equivalence of categories. By abuse of language, any such pair is called 'the' category obtained from $\mathscr{C}$ by *inverting the morphisms of* $\mathscr{W}$ or the category obtained from $\mathscr{C}$ by localization with respect to $\mathscr{W}$.

Coming back to the triangulated setting, it is not hard (but somewhat tedious) to check that a multiplicative system compatible with the triangulation gives rise to a triangulation on its localization. More precisely:

Theorem 1.2.8 (Verdier Localization). *Let* $\mathbf{K}$ *be a triangulated category and let* $\mathscr{T}$ *be a thick triangulated subcategory. Let* $\mathscr{W}$ *be the multiplicative system associated to* $\mathscr{T}$*, let* $\mathbf{K}/\mathscr{T} := \mathbf{K}[\mathscr{W}^{-1}]$ *and let* q *be the quotient functor* $\mathbf{K} \xrightarrow{q} \mathbf{K}/\mathscr{T}$*.*

(i) The suspension functor induces an auto-equivalence of $\mathbf{K}/\mathscr{T}$*.*
(ii) Define the distinguished triangles in $\mathbf{K}/\mathscr{T}$ *to be the triangles in* $\mathbf{K}/\mathscr{T}$ *which are isomorphic to images of distinguished triangles under* q*.*
(iii) With the structure from (i) and (ii), $\mathbf{K}/\mathscr{T}$ *is a triangulated category.*
(iv) Together with the identity transformation $1 : q\Sigma_{\mathbf{K}} \Rightarrow \Sigma_{\mathbf{K}/\mathscr{T}}q$ *the quotient functor is a triangle functor* $\mathbf{K} \xrightarrow{(q,1)} \mathbf{K}/\mathscr{T}$*. It is the* universal triangle functor annihilating $\mathscr{T}$ *in that every triangle functor* $\mathbf{K} \xrightarrow{(F,\alpha)} \mathbf{K}'$ *with* $\mathscr{T} \subset \operatorname{Ker}(F)$ *factors uniquely over* $(q, 1)$

$$\begin{array}{ccc} & \mathbf{K} & \\ {\scriptstyle (q,1)} \swarrow & & \searrow {\scriptstyle (F,\alpha)} \\ \mathbf{K}/\mathscr{T} & \xrightarrow{\exists!(\widetilde{F},\widetilde{\alpha})} & \mathbf{K}' \end{array}$$

via a triangle functor $(\widetilde{F}, \widetilde{\alpha})$*.*
(v) A morphism f *of* $\mathbf{K}$ *is mapped to an isomorphism in* $\mathbf{K}/\mathscr{T}$ *by* q *if and only if* $f \in \mathscr{W}$*.* □

Probably the best and most detailed proof of Verdier's localization theorem in the literature (up to the fact that Neeman is always somewhat sloppy when it comes to triangle functors) can be found in [**Nee01**, Section 2.1].

Remark 1.2.9. The hypothesis that $\mathscr{T}$ be thick is unnecessarily restrictive: As Neeman proves in *loc. cit.* the triangulated category $\mathbf{K}/\mathscr{T}$ can be constructed for every strictly full triangulated subcategory $\mathscr{T}$ of $\mathbf{K}$ and enjoys properties (i) to (iv) in Verdier's localization theorem. However, it turns out that $\mathbf{K}/\mathscr{T}$ coincides with $\mathbf{K}/\bar{\mathscr{T}}$ where

$$\bar{\mathscr{T}} = \{X \in \mathscr{K} \,:\, X \oplus Y \in \mathscr{T} \text{ for some } Y \in \mathbf{K}\}$$

is the *thick closure* of $\mathscr{T}$ in $\mathbf{K}$. This accounts for the failure of (v) in this more general setting, which must be modified by replacing $\mathscr{T}$ by $\bar{\mathscr{T}}$.

2. The Derived Category of an Exact Category

2.1. The Homotopy Category of an Additive Category. Let $\mathscr{A}$ be an additive category. Let $\mathbf{Ch}(\mathscr{A})$ be the category of chain complexes over $\mathscr{A}$ with morphisms the chain maps. To wit, a complex is a diagram

$$\cdots \xrightarrow{d_A^{-2}} A^{-1} \xrightarrow{d_A^{-1}} A^0 \xrightarrow{d_A^0} A^1 \xrightarrow{d_A^1} \cdots$$

subject to the condition $d_A^{n+1} d_A^n = 0$ for all $n \in \mathbb{Z}$, chain maps and chain homotopies are defined in the usual way.

Let $\mathbf{K}(\mathscr{A})$ be the category whose objects are complexes over $\mathscr{A}$ and whose morphisms are the homotopy classes of chain maps. Notice that $\mathbf{K}(\mathscr{A})$ is additive. This can be seen by noticing that $\mathbf{Ch}(\mathscr{A})$ is additive and that the chain maps homotopic to zero are an *ideal* in $\mathbf{Ch}(\mathscr{A})$ (i.e., if f and g are composeable and if one out of f, g is homotopic to zero, then the composition gf is homotopic to zero; moreover, if f and g are homotopic to zero, then $f \oplus g$ is homotopic to zero). Thus, $\mathbf{K}(\mathscr{A})$ can be obtained from $\mathbf{Ch}(\mathscr{A})$ by factoring out the ideal of chain maps which are homotopic to zero.

The suspension functor on $\mathbf{Ch}(\mathscr{A})$ is defined by $\Sigma : A \mapsto A[1]$ where $(A[1])^n = A^{n+1}$, $d^n_{A[1]} = -d^{n+1}_A$. It is an automorphism of $\mathbf{Ch}(\mathscr{A})$ leaving the ideal of chain maps homotopic to zero invariant, therefore it induces an automorphism of $\mathbf{K}(\mathscr{A})$, which we still denote by Σ.

The triangulated structure on $\mathbf{K}(\mathscr{A})$ is defined using the mapping cone construction. Given a chain map $f : A \to B$ (not a homotopy class!), we define the mapping cone of f to be the complex given by

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n \qquad \text{and} \qquad d^n_f = \begin{bmatrix} -d^{n+1}_A & 0 \\ f & d_B \end{bmatrix} : \operatorname{cone}(f)^n \to \operatorname{cone}(f)^{n+1}.$$

The mapping cone of f yields a *strict triangle* in $\mathbf{Ch}(\mathscr{A})$

$$A \xrightarrow{f} B \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[1\ 0]} \Sigma A.$$

It is easy to see that the composition of two consecutive maps in a strict triangle is homotopic to zero.

Remark 2.1.1. As the notation suggests, the mapping cone does depend functorially on f *on the level of* $\mathbf{Ch}(\mathscr{A})$. It is also easily checked that homotopic chain maps have cones which are isomorphic, however, the isomorphism is *not* canonical in that it depends on the *choice* of a chain homotopy in general.

The reader unfamiliar with the mapping cone construction should consult [**Wei94**, Chapters 1.5 and 10.1] before reading further in order to get himself acquainted with its basic properties. See also Appendix A.

Definition 2.1.2. A *distinguished triangle* in $\mathbf{K}(\mathscr{A})$ is a triangle which is isomorphic to the image of a strict triangle under the canonical quotient functor $\mathbf{Ch}(\mathscr{A}) \to \mathbf{K}(\mathscr{A})$. We denote the class of distinguished triangles by Δ.

A straightforward but somewhat tedious verification shows:

Theorem 2.1.3 (Verdier). *The category* $(\mathbf{K}(\mathscr{A}), \Sigma, \Delta)$ *is triangulated.*

Proof. Proofs can be found e.g. in [**BGK⁺87**], [**GM03**], [**Ver96**], and, modulo sign conventions, in [**Wei94**]. □

Depending on what one wants to do, it is convenient to introduce various boundedness conditions. The three most common boundedness conditions are:

Definition 2.1.4. A complex is *bounded below,* or $A \in \mathbf{Ch}^+(\mathscr{A})$, if $A^n = 0$ for $n \ll 0$. A complex is *bounded above,* or $A \in \mathbf{Ch}^-(\mathscr{A})$ if $A^n = 0$ for $n \gg 0$. A complex is *bounded,* or $A \in \mathbf{Ch}^b(\mathscr{A})$ if it is both bounded above and bounded below.

It is plain that the chain maps homotopic to zero form an ideal in $\mathbf{Ch}^*(\mathscr{A})$ for $* \in \{+,-,b\}$ and that the suspension induces an automorphism of each of these full subcategories of $\mathbf{Ch}(\mathscr{A})$. The corresponding quotient categories are denoted by $\mathbf{K}^*(\mathscr{A})$ for $* \in \{+,-,b\}$ and we have:

Proposition 2.1.5. *For* $* \in \{\ \ ,+,-,b\}$*, a functor* $\mathbf{Ch}^*(\mathscr{A}) \to \mathscr{B}$ *to an additive category* $\mathscr{B}$ *factors over a functor* $\mathbf{K}^*(\mathscr{A}) \to \mathscr{B}$ *if and only if it sends null-homotopic chain maps to zero morphisms.* □

Corollary 2.1.6. *There is a commutative diagram of fully faithful functors*

$$\begin{array}{ccccc} & & \mathbf{K}^-(\mathscr{A}) & & \\ & \nearrow & & \searrow & \\ \mathbf{K}^b(\mathscr{A}) & & & & \mathbf{K}(\mathscr{A}) \\ & \searrow & & \nearrow & \\ & & \mathbf{K}^+(\mathscr{A}) & & \end{array}$$

which can be considered as inclusions of triangulated subcategories. □

2.2. The Derived Category of an Exact Category. Let $(\mathscr{A},\mathscr{E})$ be an exact category—the definition is given in Section 2.3 of Chapter IV. A complex over $\mathscr{A}$ is called *acyclic* (or $\mathscr{E}$-acyclic) if for all $n \in \mathbb{Z}$ there are short exact sequences $Z^n A \overset{i^n}{\rightarrowtail} A^n \overset{p^n}{\twoheadrightarrow} Z^{n+1}A$ such that $d^n = i^{n-1}p^n$. Let $\mathbf{Ac}(\mathscr{A},\mathscr{E})$ be the full subcategory of $\mathbf{K}(\mathscr{A})$ consisting of acyclic complexes. Imposing boundedness conditions, one obtains full subcategories $\mathbf{Ac}^*(\mathscr{A})$ of $\mathbf{K}^*(\mathscr{A})$ for $* \in \{+,-,b\}$.

Recall that an additive category has *split idempotents* if every idempotent has a kernel. Thomason [**TT90**, Appendix A] defines *weakly split idempotents* in an *exact* category by requiring that every retraction is an admissible epic and proves that this is equivalent (modulo the "obscure axiom"—which is automatic—see [**Kel90**, Appendix A]) to the requirement that every co-retraction is an admissible monic.

Theorem 2.2.1 ([**Nee90**]). *Let* $(\mathscr{A},\mathscr{E})$ *be an exact category.*

- *(i) The category* $\mathbf{Ac}(\mathscr{A},\mathscr{E})$ *is a full triangulated subcategory of* $\mathbf{K}(\mathscr{A})$*. In particular,* $\mathbf{Ac}^*(\mathscr{A},\mathscr{E})$ *is triangulated for* $* \in \{+,-,b\}$*.*
- *(ii) The following conditions are equivalent:*
  - *(a) Idempotents split in* $\mathscr{A}$*.*
  - *(b) The category* $\mathbf{Ac}(\mathscr{A},\mathscr{E})$ *is thick in* $\mathbf{K}(\mathscr{A})$*.*
  - *(c) All null-homotopic complexes are in* $\mathbf{Ac}(\mathscr{A},\mathscr{E})$*.*
  - *(d) The category* $\mathbf{Ac}(\mathscr{A},\mathscr{E})$ *is strictly full.*
- *(iii) The following are equivalent:*
  - *(a) The category* $\mathscr{A}$ *has weakly split idempotents.*
  - *(b) The category* $\mathbf{Ac}^b(\mathscr{A},\mathscr{E})$ *is thick in* $\mathbf{K}^b(\mathscr{A})$*.*
  - *(c) The categories* $\mathbf{Ac}^+(\mathscr{A},\mathscr{E})$ *and* $\mathbf{Ac}^-(\mathscr{A},\mathscr{E})$ *are thick in* $\mathbf{K}^+(\mathscr{A})$ *and* $\mathbf{K}^-(\mathscr{A})$*.*

Proof. Statement (i) is Neeman's Lemma 1.1.

Statement (ii): Neeman's Lemma 1.2 states that (a) implies (b) and his Remark 1.8 shows that (b) implies (a). The equivalence of (a), (c) and (d) may be found in [**Kel96**, Example 11.2].

Statement (iii): Neeman's Remark 1.9 establishes that (b) implies (a). That (a) implies (b) is due to Thomason and can be found in [**TT90**, 1.11.6]. Neeman proves in Remark 1.10 that (a) implies (c). That (c) implies (b) follows from the facts that the intersection of thick subcategories is again thick and $\mathbf{Ac}^b = \mathbf{Ac}^+ \cap \mathbf{Ac}^-$. □

Definition 2.2.2 ([**Nee90**]). Let $(\mathscr{A},\mathscr{E})$ be an exact category in which idempotents split. The derived category of $(\mathscr{A},\mathscr{E})$ is defined to be the Verdier quotient

$$\mathbf{D}(\mathscr{A},\mathscr{E}) := \mathbf{K}(\mathscr{A}) / \mathbf{Ac}(\mathscr{A},\mathscr{E}).$$

We denote the quotient functor by $\mathbf{K}(\mathscr{A}) \xrightarrow{q} \mathbf{D}(\mathscr{A},\mathscr{E})$.

A *quasi-isomorphism* is a chain map whose mapping cone is acyclic.

Remark 2.2.3. A chain map becomes an isomorphism in the derived category if and only if it is a quasi-isomorphism by Lemma 1.2.1 and Verdier's localization theorem.

3. Derived Functors

One of the main motivations for introducing the derived category is to treat derived functors right—as Verdier puts it in [**Ver96**]: "Dans ce travail on se propose d'introduire un formalisme de l'hyperhomologie". The formalism of triangles permits a succinct definition of derived functors replacing the notion of universal δ-functors. Of course, the main properties of δ-functors are built into the formalism.

3.1. Motivation. Suppose first that $\mathscr{A}$ and $\mathscr{B}$ are exact categories with split idempotents and that we are given an additive functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$. Because additive functors preserve homotopies, we obtain the following diagram:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{B} \\ \downarrow & & \downarrow \\ \mathbf{K}^{*}(\mathscr{A}) & \xrightarrow{\mathbf{K}^{*}F} & \mathbf{K}^{*}(\mathscr{B}) \end{array}$$

for all decorations $* \in \{\ \ , +, -, b\}$.

Now the question arises whether $\mathbf{K}^{*}F$ induces a functor on the level of the derived categories.

The answer is not clear in general. However, it is obviously "yes" if F is in addition exact, because then it preserves the subcategory of acyclic complexes: $\mathbf{K}^{*}F(\mathbf{Ac}^{*}(\mathscr{A})) \subset \mathbf{Ac}^{*}(\mathscr{B})$ and thus the universal property of the derived category yields a diagram

$$\begin{array}{ccc} \mathbf{K}^{*}(\mathscr{A}) & \xrightarrow{\mathbf{K}^{*}F} & \mathbf{K}^{*}(\mathscr{B}) \\ \downarrow & & \downarrow \\ \mathbf{D}^{*}(\mathscr{A}) & \xrightarrow{\mathbf{D}^{*}F} & \mathbf{D}^{*}(\mathscr{B}) \end{array}$$

and $\mathbf{D}^{*}F$ is computed on a complex by applying F in each degree.

If, however, the functor F fails to be exact then the question of extending F to the derived category is more subtle.

3.2. Definition. We follow [**Mal07**]. Let $\mathscr{C}$ be a category and let $\mathscr{W}$ we be a class of morphisms in $\mathscr{C}$. To avoid cumbersome language, call the pair $(\mathscr{C}, \mathscr{W})$ a *localizer* if the localization $\mathscr{C}[\mathscr{W}^{-1}]$ of $\mathscr{C}$ with respect to $\mathscr{W}$ exists in the sense that the class of morphisms between any two objects is a set.

Definition 3.2.1. Let $(\mathscr{C}, \mathscr{W})$ be a localizer and let $P : \mathscr{C} \to \mathscr{C}[\mathscr{W}^{-1}]$ be the canonical quotient functor. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. A *right derived functor* of F is a pair $(\mathbf{R}F, \alpha)$ consisting of a functor $\mathbf{R}F : \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ and a natural transformation $\alpha : F \Rightarrow \mathbf{R}F \circ P$

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ {\scriptstyle P}\downarrow & \Downarrow\alpha & \nearrow{\scriptstyle \mathbf{R}F} \\ \mathscr{C}[\mathscr{W}^{-1}] & & \end{array}$$

with the following universal property: for every pair (G, γ) consisting of a functor $G : \mathscr{C}[\mathscr{W}^{-1}] \to \mathscr{D}$ and a natural transformation $\gamma : F \Rightarrow G \circ P$ there is a unique natural transformation $\delta : \mathbf{R}F \Rightarrow G$ such that $\gamma = (\delta \star P)\alpha$.

A right derived functor $(\mathbf{R}F, \alpha)$ is called *absolute* if for every functor $H : \mathscr{D} \to \mathscr{E}$ the pair $(H \circ \mathbf{R}F, H \star \alpha)$ is a right derived functor of $H \circ F$.

The definition of (absolute) left derived functors is obtained by dualization.

Remark 3.2.2. The universal property of a right derived functor guarantees that it is unique up to unique isomorphism.

Remark 3.2.3 (Maltsiniotis). The above definition of a derived functor exhibits it as a special instance of a *Kan extension* and thus fits into Mac Lane's paradigm "every concept is a Kan extension".

Remark 3.2.4 (Maltsiniotis). A general condition ensuring the existence of an absolute right derived functor is the following: Let $\mathscr{C}$ be a *model category* in the sense of Quillen [**Qui67**] (assumed neither *closed* nor *(co-)complete*) and denote its class of weak equivalences by $\mathscr{W}$. Then $(\mathscr{C}, \mathscr{W})$ is a localizer by [**Qui67**, Ch. I, 1.13, Theorem 1']. Suppose that $F : \mathscr{C} \to \mathscr{D}$ is a functor that sends weak equivalences between fibrant objects of $\mathscr{C}$ to isomorphisms in $\mathscr{D}$. Then F admits an absolute right derived functor $(\mathbf{R}F, \alpha)$. Indeed, by [**Qui67**, Ch. I, 4.2, Proposition 1] one may construct $\mathbf{R}F$ by *choosing* for each object $X \in \mathscr{C}$ a fibrant resolution $i_X : X \to X'$ (that is i_X is a weak equivalence and X' is fibrant) and putting $\mathbf{R}F(X) = F(X')$ and $\alpha_X = F(i_X) : F(X) \to F(X') = \mathbf{R}F(X)$. To see that the pair $(\mathbf{R}F, \alpha)$ is indeed an *absolute* right derived functor, simply notice that for every functor $H : \mathscr{D} \to \mathscr{E}$ the composed functor $H \circ F$ transforms weak equivalences between fibrant objects to isomorphisms, and the construction of the right derived functor of $H \circ F$ yields that it is nothing but $(H \circ \mathbf{R}F, H \star \alpha)$.

Remark 3.2.5. In Appendix C we show that the previous remark can be applied to $\mathbf{Ch}^{+}(\mathscr{A})$ and $\mathbf{Ch}^{-}(\mathscr{A})$ over an exact category $\mathscr{A}$ satisfying several further conditions.

3.3. Enough Injective Objects.

The goal of this section is to explain:

Theorem 3.3.1. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough injectives and split idempotents. Let $\mathscr{I}$ be the full subcategory of injective objects in $(\mathscr{A}, \mathscr{E})$.*

(i) Let $\mathscr{W}$ be the class of quasi-isomorphisms—that is to say the multiplicative system associated with the thick triangulated subcategory $\mathbf{Ac}^{+}(\mathscr{A}, \mathscr{E})$) of $\mathbf{K}^{+}(\mathscr{A})$. The pair $(\mathbf{K}^{+}(\mathscr{A}), \mathscr{W})$ is a localizer. More precisely, the composite

$$\mathbf{K}^{+}(\mathscr{I}) \to \mathbf{K}^{+}(\mathscr{A}) \to \mathbf{D}^{+}(\mathscr{A}, \mathscr{E})$$

is an equivalence of triangulated categories.

(ii) For every additive functor $F : \mathscr{A} \to \mathscr{A}'$ into an additive category $\mathscr{A}'$ the right derived functor of $\mathbf{K}^{+}F : \mathbf{K}^{+}(\mathscr{A}) \to \mathbf{K}^{+}(\mathscr{A}')$ exists and is a triangle functor $\mathbf{D}^{+}(\mathscr{A}, \mathscr{E}) \to \mathbf{K}^{+}(\mathscr{A}')$. Moreover, it is the absolute right derived functor of $\mathbf{K}^{+}F$.

Construction 3.3.2. For every object in $A \in \mathbf{K}^{+}(\mathscr{A})$ *choose* a quasi-isomorphism $i_A : A \to I_A$ to a complex in $\mathbf{K}^{+}(\mathscr{I}) \subset \mathbf{K}^{+}(\mathscr{A})$ and *define* the right derived functor to be

$$\mathbf{RK}^{+}F(A) := \mathbf{K}^{+}F(I_A)$$

together with

$$\alpha_A := \mathbf{K}^{+}F(i_A) : \mathbf{K}^{+}F(A) \to \mathbf{K}^{+}F(I_A) = \mathbf{RK}^{+}F(A).$$

LEMMA 3.3.3. *Let I be a bounded below complex of injectives and let $I \xrightarrow{f} A$ be a quasi-isomorphism. Then f has a left inverse in $\mathbf{K}(\mathscr{A})$.*

PROOF. Because f is a quasi-isomorphism, cone (f) is acyclic. By the comparison theorem, see [**Wei94**, 2.2.6, 2.2.7], the morphism

$$\operatorname{cone}(f) \xrightarrow{[1\ 0]} I[1]$$

is null homotopic by a map $[\,k\ s\,] : \operatorname{cone}(f) \to I$ of graded objects (*not* a chain map!). The second coordinate of the equation

$$[\,1\ 0\,] = [\,k\ s\,] \begin{bmatrix} -d_I & 0 \\ f & d_A \end{bmatrix} - d_I\,[\,k\ s\,] = [\,-kd_I + sf - d_I k \quad sd_A - d_I s\,]$$

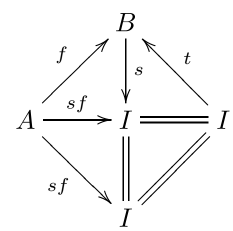
shows that $s : A \to I$ is a chain map and the first coordinate of the equation proves that sf is homotopic to the identity of I via k. □

COROLLARY 3.3.4. *If I is a bounded below complex of injectives then*

$$\operatorname{Hom}_{\mathbf{D}(\mathscr{A},\mathscr{E})}(A, I) \cong \operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A, I)$$

for each complex A.

PROOF. Let the right fraction $A \xrightarrow{f} B \xleftarrow{t} I$ represent a morphism $A \to I$ in $\mathbf{D}$, so suppose t is a quasi-isomorphism. By the previous lemma, there is a left inverse s of t and the diagram



proves that $t^{-1}f$ is equivalent to sf, in other words: every morphism $A \to I$ in $\mathbf{D}$ is represented by a morphism $A \to I$ in $\mathbf{K}$.

On the other hand, if two parallel morphisms $f, g : A \to I$ of $\mathbf{K}$ are identified in $\mathbf{D}$ then there exists a quasi-isomorphism $I \xrightarrow{t} B$ such that $tf = tg$. Again by the previous lemma t has a left inverse s, so $f = stf = stg = g$ in $\mathbf{K}$. □

COROLLARY 3.3.5. *The composite functor $\mathbf{K}^+(\mathscr{I}) \to \mathbf{K}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{A},\mathscr{E})$ is a fully faithful triangle functor.* □

PROOF OF THEOREM 3.3.1. To see that the functor $i : \mathbf{K}^+(\mathscr{I}) \to \mathbf{D}^+(\mathscr{A},\mathscr{E})$ is an equivalence of categories, it suffices to prove it to be essentially surjective given the previous corollary—this is done in [**Kel90**, 4.1, Lemma, b)] using the assumption on enough injectives.

Constructing a quasi-inverse k for i amounts to choosing for each object in $\mathbf{D}^+(\mathscr{A},\mathscr{E})$ an isomorphic object in the image of i. There is a unique natural transformation turning k into a triangle functor by appealing to Proposition 2.2.2. This establishes point (i).

Point (ii) is standard, see [**Wei94**, Section 10.5], again one uses the assumption on enough injectives. □

3.4. Equivalences of Derived Categories. The above construction can be seen as a very special instance of the following general construction: Let us be given an exact category $(\mathscr{A},\mathscr{E})$ and a full additive subcategory $\mathscr{B} \subset \mathscr{A}$ which is closed under extensions in the sense that for each sequence in $\mathscr{E}$ of the form

$$B' \rightarrowtail A \twoheadrightarrow B''$$

with B' and B'' in $\mathscr{B}$ the object A belongs to $\mathscr{B}$ as well. The exact structure $\mathscr{E}$ then restricts to an exact structure $\mathscr{E}_{\mathscr{B}}$ on $\mathscr{B}$ and it is natural to ask about the properties of the functor

$$\mathbf{D}^+(\mathscr{B}, \mathscr{E}_{\mathscr{B}}) \longrightarrow \mathbf{D}^+(\mathscr{A}, \mathscr{E})$$

induced by the (exact) inclusion $\mathscr{B} \subset \mathscr{A}$.

Two useful conditions can be found in [**Kel96**, Section 12] and read:

(C1) For each $A' \in \mathscr{A}$ there exists a short exact sequence $A' \rightarrowtail B \twoheadrightarrow A''$ with $B \in \mathscr{B}$.

(C2) For each short exact sequence $B' \rightarrowtail A \twoheadrightarrow A''$ with $B \in \mathscr{B}$ there exists a commutative diagram

$$\begin{array}{ccccc} B' & \rightarrowtail & A & \twoheadrightarrow & A'' \\ \| & & \downarrow & & \downarrow \\ B' & \rightarrowtail & B & \twoheadrightarrow & B'' \end{array}$$

whose second row is an exact sequence of $\mathscr{B}$.

THEOREM 3.4.1 ([**Kel96**, Theorem 12.1]). *Using the above notation, consider the functor*

$$\mathbf{D}^+(\mathscr{B}, \mathscr{E}_{\mathscr{B}}) \to \mathbf{D}^+(\mathscr{A}, \mathscr{E}).$$

(i) In presence of condition (C1) *the functor is essentially surjective.*

(ii) In presence of condition (C2) *the functor is fully faithful.* □

EXAMPLE 3.4.2. Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough injectives and let $\mathscr{B} = \mathscr{I}$ be the full additive subcategory of injectives. Then $\mathscr{E}$ induces the split exact structure on $\mathscr{I}$, hence $\mathbf{K}^+(\mathscr{I}) = \mathbf{D}^+(\mathscr{I})$ and the theorem gives us part (i) of Theorem 3.3.1.

EXAMPLE 3.4.3. Let $\mathscr{A} = G{-}\mathbf{Ban}$ and equip it with the maximal exact structure. Let $\mathscr{F}(G)$ be the full additive subcategory of flat modules. The maximal exact structure induces the *pure* exact structure on $\mathscr{F}$, and (C1$^{\mathrm{op}}$) and (C2$^{\mathrm{op}}$) are satisfied, see Chapter V, Section 1.4. Therefore we obtain an equivalence

$$\mathbf{D}^-(\mathscr{F}(G), \mathscr{E}^G_{\mathrm{pure}}) \to \mathbf{D}^-(G{-}\mathbf{Ban}, \mathscr{E}^G_{\mathrm{max}})$$

of derived categories.

3.5. The Generalized Existence Theorem. The definitive treatment of derived functors in the triangulated setting is due to Deligne and can be found in [**Kel96**, Sections 13–15]. It does not make much sense to repeat the construction here, just because it seems hard to improve upon Keller's exposition.

CHAPTER III

Abstract Truncation: t-Structures and Hearts

In the derived category $\mathbf{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$, good truncation of complexes yields two families of endofunctors $\tau^{\leq n}$ and $\tau^{\geq n}$ indexed by integers—truncation above and below degree n. The complexes which are truncated above and below degree 0 form a pair of full subcategories $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ of $\mathbf{D}(\mathscr{A})$ whose main properties were formalized by Bernstein-Beĭlinson-Deligne [**BBD82**] with the notion of a *t-structure*. The *heart* of a t-structure is the intersection $\mathscr{C} = \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$ and turns out to be abelian. The truncation functors give rise to the homological functor $H^0 = \tau^{\geq 0}\tau^{\leq 0} : \mathbf{D}(\mathscr{A}) \to \mathscr{C}$. It can be shown that in the derived category of an exact category which is sufficiently close to being abelian admits a natural t-structure and that the heart is the solution of a universal problem. The material in this chapter is mainly taken from *loc. cit.* and [**Lau83**].

Contents

1. Introduction

1.1. Recognizing Abelian Subcategories. In order to motivate the approach, we look at the "classical case" of the derived category $\mathbf{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$. We consider $\mathscr{A}$ as a full subcategory of $\mathbf{D}(\mathscr{A})$ by viewing an object of $\mathscr{A}$ as a complex concentrated in degree zero. We start with the following easy observation.

Proposition 1.1.1. *The subcategory $\mathscr{A}$ of $\mathbf{D}(\mathscr{A})$ satisfies:*

(i) It is a full additive subcategory.

(ii) $\operatorname{Hom}^i(X,Y) := \operatorname{Hom}_{\mathbf{D}(\mathscr{A})}(X, Y[i]) = 0$ *for all* $i < 0$ *and all* $X, Y \in \mathscr{A}$.

(iii) For every short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathscr{A}$ there exists a unique connecting morphism $Z \to X[1]$ in $\mathbf{D}(\mathscr{A})$ such that

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle in $\mathbf{D}(\mathscr{A})$.

Proof. Point (i) is obvious, point (ii) is easy to see and point (iii) follows from [**BBD82**, 1.1.10], which in turn follows from an application of the homological functor $\operatorname{Hom}(X, -)$. □

Given a morphism $X \xrightarrow{f} Y$ in $\mathscr{A}$, its mapping cone is the complex

$$S = \cdots \to 0 \to X \xrightarrow{f} Y \to 0 \to \cdots$$

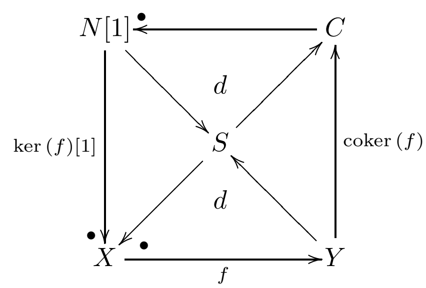
in which X sits in degree -1 and Y in degree 0. By definition of the triangulation on $\mathbf{D}(\mathscr{A})$, it gives rise to a distinguished triangle

$$X \xrightarrow{f} Y \to S \to X[1].$$

Let $N = \mathrm{Ker}(f)$ and $C = \mathrm{Coker}(f)$. In $\mathbf{Ch}(\mathscr{A})$, we may consider $N[1]$ as a subcomplex of S and $S/N[1]$ is quasi-isomorphic to C. Therefore there is a distinguished triangle

$$N[1] \to S \to C \to N[2]$$

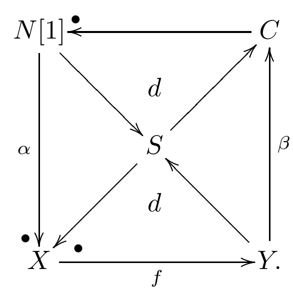
in $\mathbf{D}(\mathscr{A})$. Plugging these two distinguished triangles together yields the diagram



in which the two triangles marked with a "d" are the distinguished triangles above and the other two are commutative. Notice that this diagram may be considered as the lower cap of an octahedron.

Hypothesis 1.1.2. Let $\mathbf{D}$ be a triangulated category and let $\mathscr{C} \subset \mathbf{D}$ be a full additive subcategory such that $\mathrm{Hom}^i(X,Y) = 0$ for all $i < 0$ and all $X, Y \in \mathscr{C}$.

Proposition 1.1.3 ([**BBD82**, 1.2.2]). *Under Hypothesis 1.1.2, let $X \xrightarrow{f} Y$ be a morphism and complete it to a distinguished triangle (X, Y, S). Suppose that there exist objects $N, C \in \mathscr{C}$ and a distinguished triangle $(N[1], S, C)$. Consider the diagram*



Then $\alpha[-1]$ is a kernel of f in $\mathscr{C}$ and β is a cokernel of f in $\mathscr{C}$.

Proof. Let $Z \in \mathscr{C}$. Apply the homological functor $\mathrm{Hom}(Z, -)$ to the upper and lower triangle in the diagram in order to obtain two exact sequences of abelian groups

$$0 \longrightarrow \mathrm{Hom}^0(Z,N) \longrightarrow \mathrm{Hom}^{-1}(Z,S) \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Hom}^{-1}(Z,S) \longrightarrow \mathrm{Hom}^0(Z,X) \longrightarrow \mathrm{Hom}^0(Z,Y)$$

which combine to prove the assertion on α. A dual argument yields the assertion on β. □

Definition 1.1.4 ([**BBD82**, 1.2.3]). Assume Hypothesis 1.1.2.

(i) A morphism f of $\mathscr{C}$ is *admissible* if it is the base of a diagram as in Proposition 1.1.3.

(ii) A sequence $X \to Y \to Z$ in $\mathscr{C}$ is *admissible* if it is obtained from a distinguished triangle in $\mathbf{D}$ by suppressing the morphism $Z \to X[1]$.

REMARK 1.1.5 ([**BBD82**, 1.2.3]). If f is admissible and either monic or epic in $\mathscr{C}$ then the diagram in Proposition 1.1.3 reduces to a single distinguished triangle because then either $N = 0$ or $C = 0$ which implies either $S \cong C$ or $N \cong S$.

Conversely, if $X \to Y \to Z$ is admissible, then it consists of admissible morphisms and it is necessarily a kernel-cokernel pair.

PROPOSITION 1.1.6 ([**BBD82**, 1.2.4]). *Under Hypothesis 1.1.2 the following assertions are equivalent:*

(i) Every morphism of $\mathscr{C}$ is admissible.
(ii) The category $\mathscr{C}$ is abelian and its short exact sequences are admissible.

PROOF. This is a straightforward application of the octahedral axiom. See *loc. cit.* for details. ☐

DEFINITION 1.1.7 ([**BBD82**, 1.2.6]). A full additive subcategory $\mathscr{C}$ of a triangulated category $\mathbf{D}$ is called *admissible* if it satisfies Hypothesis 1.1.2 and the equivalent conditions of Proposition 1.1.6.

1.2. Concrete Truncation. Throughout this section we will work under the following assumption.

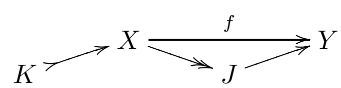
HYPOTHESIS 1.2.1. The category $\mathscr{A}$ is additive.

(i) Every morphism of $\mathscr{A}$ has a kernel.
(ii) Every kernel in $\mathscr{A}$ has a cokernel and is the kernel of its cokernel.
(iii) The class of all kernel-cokernel pairs is an exact structure.

We will always consider $\mathscr{A}$ as an exact category with the structure as provided by (iii). Notice that by (i) idempotents split in $\mathscr{A}$.

REMARK 1.2.2. In Hypothesis 1.2.1 (i) and (ii) are not self-dual. If (i^{op}) and (ii^{op}) are satisfied, then $\mathscr{A}^{\mathrm{op}}$ satisfies (i) and (ii), so the entire section is subject to dualization.

REMARK 1.2.3. Under Hypothesis 1.2.1, every morphism has a factorization



where K is a kernel of f and J is a coimage of f. The morphism $J \to Y$ is always monic in presence of enough projectives but it is not necessarily an admissible monic (i.e., a kernel). In fact, if $\mathscr{A}$ is not abelian, then it cannot always be an admissible monic because of Freyd's recognition theorem for abelian categories, see [**Fre66**, Proposition 3.1].

REMARK 1.2.4. Hypothesis 1.2.1 is equivalent to the one made in [**BBD82**, 1.3.22]. We prefer our formulation because it shows that contrary to the usual situation for exact categories, there is *no* choice of the exact structure in this context, one has to consider the *maximal* one.

DEFINITION 1.2.5 (Canonical Truncation Functors). Define two functors

$$\tau^{\leq 0}, \tau^{\geq 0} : \mathbf{Ch}(\mathscr{A}) \longrightarrow \mathbf{Ch}(\mathscr{A})$$

by

$$\tau^{\leq 0} E = (\cdots \to E^{-2} \to E^{-1} \to \operatorname{Ker} d_E^0 \to 0 \to \cdots)$$

and

$$\tau^{\geq 0} E = (\cdots \to 0 \to \operatorname{Coim} d_E^{-1} \to E^0 \to E^1 \to \cdots).$$

Moreover for $n \in \mathbb{Z}$ put

$$\tau^{\leq n} = \Sigma^{-n} \circ \tau^{\leq 0} \circ \Sigma^{n} \qquad \text{and} \qquad \tau^{\geq n} = \Sigma^{-n} \circ \tau^{\geq 0} \circ \Sigma^{n}.$$

Clearly, $\tau^{\leq n}$ and $\tau^{\geq n}$ perform the same operation as $\tau^{\leq 0}$ and $\tau^{\geq 0}$, only in degree n.

Remark 1.2.6. The superscript of the truncation functors indicates which part of the homology of E is left untouched when applying a truncation functor.

The following yields another explanation for the notation used in the truncation functors.

Proposition 1.2.7.

(i) *The functor $\tau^{\leq n}$ is right adjoint to the inclusion $\mathbf{Ch}^{\leq n}(\mathscr{A}) \subset \mathbf{Ch}(\mathscr{A})$.*
(ii) *The functor $\tau^{\geq n}$ is left adjoint to the inclusion $\mathbf{Ch}^{\geq n}(\mathscr{A}) \subset \mathbf{Ch}(\mathscr{A})$.* □

Remark 1.2.8. For $n \leq m$ the complexes $\tau^{\geq n}\tau^{\leq m}E$ and $\tau^{\leq m}\tau^{\geq n}E$ both coincide with the complex

$$\cdots \to 0 \to \operatorname{Coim}(d_E^{n-1}) \to E^n \to \cdots \to E^{m-1} \to \operatorname{Ker} d_E^m \to 0 \to \cdots$$

and there is a commutative diagram of endofunctors of $\mathbf{Ch}(\mathscr{A})$

$$\begin{array}{ccccc}
\tau^{\leq m} & \longrightarrow & \mathrm{id} & \longrightarrow & \tau^{\geq n} \\
\downarrow & & & & \uparrow \\
\tau^{\geq n}\tau^{\leq m} & & \xrightarrow[\cong]{\qquad} & & \tau^{\leq m}\tau^{\geq n}.
\end{array}$$

Proposition 1.2.9 ([**Lau83**, Lemme 1.3.1]).

(i) *The functors $\tau^{\leq n}$ and $\tau^{\geq n}$ induce endofunctors of $\mathbf{K}(\mathscr{A})$.*
(ii) *There are the inclusions*

$$\begin{aligned}
\tau^{\leq n}\,\mathbf{K}(\mathscr{A}) &\subset \mathbf{K}^{-}(\mathscr{A}), & \qquad \tau^{\geq n}\,\mathbf{K}(\mathscr{A}) &\subset \mathbf{K}^{+}(\mathscr{A}), \\
\tau^{\leq n}\,\mathbf{K}^{+}(\mathscr{A}) &\subset \mathbf{K}^{b}(\mathscr{A}), & \qquad \tau^{\geq n}\,\mathbf{K}^{-}(\mathscr{A}) &\subset \mathbf{K}^{b}(\mathscr{A}).
\end{aligned}$$

Definition 1.2.10. A complex E is *acyclic in degrees $> n$ $(< n)$* if for all $i > n$ $(i < n)$ the morphism

$$\operatorname{Coim} d_E^{i-1} \to \operatorname{Ker} d_E^{i}$$

is an isomorphism.

Proposition 1.2.11 ([**Lau83**, Lemme 1.3.2]).

(i) *If E is acyclic in degrees $> n$ then the morphism*

$$\tau^{\leq n}E \to E$$

is a quasi-isomorphism.
(ii) *If E is acyclic in degrees $< n$ then the morphism*

$$E \to \tau^{\geq n}E$$

is a quasi-isomorphism. □

Corollary 1.2.12. *The truncation functors preserve $\mathbf{Ac}(\mathscr{A})$ and hence induce endofunctors of $\mathbf{D}(\mathscr{A})$.* □

Proposition 1.2.13 ([**Lau83**, Proposition 1.3.3]). *Let $0 \to E \xrightarrow{u} F \xrightarrow{v} G \to 0$ be a sequence in $\mathbf{Ch}(\mathscr{A})$ which is exact in each degree. The morphism*

$$\operatorname{cone}(u) \to G$$

is a quasi-isomorphism. □

Corollary 1.2.14 ([**Lau83**, 1.4]). *Let $0 \to E \xrightarrow{u} F \xrightarrow{v} G \to 0$ be a sequence in $\mathbf{Ch}(\mathscr{A})$ which is exact in each degree. There is a distinguished triangle*

$$E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{w} E[1]$$

in $\mathbf{D}(\mathscr{A})$.

Proof. Consider the diagram

$$\begin{array}{ccccccc} E & \xrightarrow{u} & F & \longrightarrow & \operatorname{cone}(u) & \longrightarrow & E[1] \\ \| & & \| & & \downarrow{\scriptstyle \mathrm{qi}} & & \| \\ E & \xrightarrow{u} & F & \xrightarrow{v} & G & \xrightarrow{w} & E[1]. \end{array}$$

The morphism w of $\mathbf{D}(\mathscr{A})$ is given by the fraction

$$G \xleftarrow{\mathrm{qi}} \operatorname{cone}(u) \longrightarrow E[1]$$

and the fact that the class of distinguished triangles is closed under isomorphisms implies the result. □

Proposition 1.2.11 and Corollary 1.2.14 combine to give:

Corollary 1.2.15 ([**Lau83**, Proposition 1.4.1]). *For $E \in \mathbf{Ch}(\mathscr{A})$ the sequence of complexes*

$$\tau^{\leq 0}E \to E \to \tau^{\geq 1}E$$

defines a distinguished triangle

$$\tau^{\leq 0}E \to E \to \tau^{\geq 1}E \to (\tau^{\leq 0}E)[1]$$

in $\mathbf{D}(\mathscr{A})$. □

Notation 1.2.16. We put

$$\mathbf{D}^{\leq 0} = \{\text{complexes which are acyclic in degrees} > 0\}$$

and

$$\mathbf{D}^{\geq 0} = \{\text{complexes which are acyclic in degrees} < 0\}.$$

Moreover let

$$\mathbf{D}^{\leq n} := \Sigma^{-n}\,\mathbf{D}^{\leq 0} \qquad \text{and} \qquad \mathbf{D}^{\geq n} := \Sigma^{-n}\,\mathbf{D}^{\geq 0}$$

for $n \in \mathbb{Z}$.

Definition 1.2.17. Let $\mathscr{A}$ satisfy Hypothesis 1.2.1. The pair $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is called the *canonical t-structure* on $\mathbf{D}(\mathscr{A})$.

Theorem 1.2.18 ([**BBD82**, 1.3.22]). *The pair $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ in $\mathbf{D}(\mathscr{A})$ satisfies*

(i) $\operatorname{Hom}(X, Y) = 0$ *for all $X \in \mathbf{D}^{\leq 0}$ and all $Y \in \mathbf{D}^{\geq 1}$;*
(ii) $\mathbf{D}^{\leq 0} \subset \mathbf{D}^{\leq 1}$ *and* $\mathbf{D}^{\geq 0} \supset \mathbf{D}^{\geq 1}$*;*
(iii) For all $X \in \mathbf{D}(\mathscr{A})$ there exist $A \in \mathbf{D}^{\leq 0}$, $B \in \mathbf{D}^{\geq 1}$ and a distinguished triangle

$$A \to X \to B \xrightarrow{\delta} A[1].$$

*In particular, it is a t-*structure *in the sense of* [**BBD82**, 1.3.1]. *Moreover, this t-structure is* non-degenerate *in the sense that the subcategories*

$$\bigcap_{n\in\mathbb{Z}} \mathbf{D}^{\leq n} \qquad \text{and} \qquad \bigcap_{n\in\mathbb{Z}} \mathbf{D}^{\geq n}$$

are equivalent to the zero category. □

2. Abstract Truncation

2.1. Definition and Main Properties. The results of the previous section can be axiomatized. The crucial definition is the following:

DEFINITION 2.1.1 ([**BBD82**, 1.3.1]).

(i) A *t-structure* on a triangulated category $\mathbf{D}$ is a pair of strictly full subcategories $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ satisfying conditions (i), (ii), (iii) of Theorem 1.2.18, where as in Notation 1.2.16 we put

$$\mathbf{D}^{\leq n} := \Sigma^{-n}\,\mathbf{D}^{\leq 0} \qquad \text{and} \qquad \mathbf{D}^{\geq n} := \Sigma^{-n}\,\mathbf{D}^{\geq 0}\,.$$

(ii) The t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is *non-degenerate* if it satisfies moreover

$$\bigcap_{n\in\mathbb{Z}} \mathbf{D}^{\leq n} \cong 0 \cong \bigcap_{n\in\mathbb{Z}} \mathbf{D}^{\geq n}\,.$$

(iii) The *heart* of the t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is the subcategory

$$\mathscr{C} = \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}\,.$$

EXAMPLES 2.1.2.

(i) If $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is a t-structure then so are its *translates* $(\mathbf{D}^{\leq n}, \mathbf{D}^{\geq n})$ for all $n \in \mathbb{Z}$.

(ii) On every triangulated category $\mathbf{D}$ there are two *trivial t-structures*

$$(0, \mathbf{D}) \qquad \text{and} \qquad (\mathbf{D}, 0)$$

If $\mathbf{D} \neq 0$ then these structures are degenerate.

(iii) If $\mathscr{A}$ satisfies Hypothesis 1.2.1, the canonical t-structure $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ on $\mathbf{D}(\mathscr{A})$ (see Notation 1.2.16 and Definition 1.2.17) is a non-degenerate t-structure by Theorem 1.2.18.

THEOREM 2.1.3 (Bernstein-Beĭlinson-Deligne). *Let $\mathbf{D}$ be a triangulated category and let $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ be a t-structure on $\mathbf{D}$.*

(i) The inclusion $\mathbf{D}^{\leq n} \subset \mathbf{D}$ has a right adjoint $\tau^{\leq n}$ and the inclusion $\mathbf{D}^{\geq n} \subset \mathbf{D}$ has a left adjoint $\tau^{\geq n}$.

(ii) For each $X \in \mathbf{D}$ there exists a unique morphism $\delta \in \operatorname{Hom}^1(\tau^{\geq 1}X, \tau^{\leq 0}X)$ such that

$$\tau^{\leq 0}X \to X \to \tau^{\geq 1}X \xrightarrow{\delta} (\tau^{\leq 0}X)[1]$$

is a distinguished triangle. Up to unique isomorphism it is the only distinguished triangle (A, X, B) with $A \in \mathbf{D}^{\leq 0}$ and $B \in \mathbf{D}^{\geq 1}$.

(iii) For $n \leq m$ there exists a unique natural transformation $\tau^{\geq n}\tau^{\leq m} \to \tau^{\leq m}\tau^{\geq n}$ such that the diagram

$$\begin{array}{ccccc} \tau^{\leq m} & \longrightarrow & \mathrm{id} & \longrightarrow & \tau^{\geq n} \\ \downarrow & & & & \uparrow \\ \tau^{\geq n}\tau^{\leq m} & & \longrightarrow & & \tau^{\leq m}\tau^{\geq n} \end{array}$$

commutes. It is an isomorphism of functors.

(iv) The heart $\mathscr{C}$ is an admissible abelian subcategory of $\mathbf{D}$ and it is closed under extensions. This means that for all distinguished triangles (X, Y, Z) with $X, Z \in \mathscr{C}$, necessarily $Y \in \mathscr{C}$.

(v) The functor $H^0_\tau := \tau^{\geq 0}\tau^{\leq 0} : \mathbf{D} \to \mathscr{C}$ is a homological functor. Put $H^n_\tau(X) := H^0_\tau(X[n])$. If furthermore $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ is non-degenerate then

(a) $X = 0$ if and only if $H^n_\tau(X) = 0$ for all $n \in \mathbb{Z}$;

(b) $X \in \mathbf{D}^{\leq 0}$ if and only if $H^n_\tau(X) = 0$ for all $n > 0$;

(c) $X \in \mathbf{D}^{\geq 0}$ if and only if $H^n_\tau(X) = 0$ for all $n < 0$.

PROOF. These statements are proved in this order in [**BBD82**, 1.3.3-1.3.7]. □

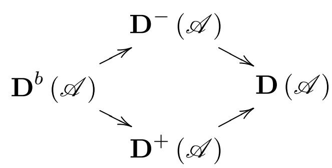
PROPOSITION 2.1.4 ([**BBD82**, 1.3.19]). *Let* $\mathbf{D}$ *be a triangulated category equipped with a* t*-structure* $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$ *and let* $\mathscr{T} \subset \mathbf{D}$ *be a full triangulated subcategory. Put*

$$\mathscr{T}^{\leq 0} := \mathbf{D}^{\leq 0} \cap \mathscr{T} \qquad \text{and} \qquad \mathscr{T}^{\geq 0} := \mathbf{D}^{\geq 0} \cap \mathscr{T} .$$

The pair $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ *is a* t*-structure on* $\mathscr{T}$ *if and only if* $\tau^{\leq 0}_{\mathbf{D}} \mathscr{T} \subset \mathscr{T}$*.* □

DEFINITION 2.1.5. Under the assumptions of Proposition 2.1.4, $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is called the *induced* t*-structure* on $\mathscr{T}$.

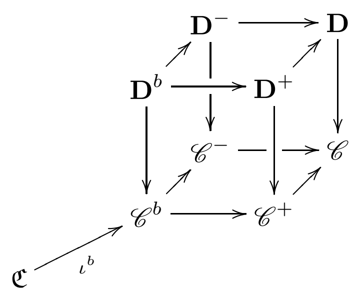
COROLLARY 2.1.6. *Let* $\mathscr{A}$ *be an additive category satisfying Hypothesis 1.2.1 and let* $\mathbf{D}(\mathscr{A})$ *be equipped with the canonical* t*-structure. The inclusions*



induce non-degenerate t*-structures on* $\mathbf{D}^*(\mathscr{A})$ *for* $* \in \{+, -, b\}$ *which will also be called* canonical t-structures. □

2.2. The Hearts of the Canonical t-Structures. This section is taken from [**BBD82**, 1.3.22] and the more detailed exposition [**Lau83**, Sections 1.4, 1.5].

Let $\mathscr{A}$ be an additive category satisfying Hypothesis 1.2.1 and consider the canonical t-structures on $\mathbf{D}^*(\mathscr{A})$ for $* \in \{\ \ , +, -, b\}$ and denote by $\mathscr{C}^*$ their hearts. There is a commutative cube of functors (ignore $\mathfrak{C}$ for the moment)



in which the vertical functors are the homological functors H^0_τ given by $\tau^{\geq 0}\tau^{\leq 0}$ and all the others are fully faithful inclusions. The goal of this section is to construct a rather explicit subcategory $\mathfrak{C}$ of $\mathscr{C}^b$ and it turns out that it is equivalent to $\mathscr{C}^*$ for $* \in \{\ \ , +, -, b\}$, more precisely, the functors at bottom of the above diagram are equivalences.

The category $\mathfrak{C}$ is nothing but the image of H^0_τ, notice that it does not matter on which derived category of $\mathscr{A}$ we work. We now give a more explicit description.

CONSTRUCTION 2.2.1. The category $\mathfrak{C}$ is given as follows:

Objects of $\mathfrak{C}$. For every $A \in \mathbf{D}(\mathscr{A})$, the complex $H^0_\tau(A)$ is a complex

$$\cdots \to 0 \to A^{-1} \xrightarrow{d_A^{-1}} A^0 \to 0 \to \cdots$$

concentrated in degrees -1 and 0 which we will write as

$$(A^{-1} \xrightarrow{d_A} A^0)$$

and since $H^0(A)$ is acyclic in all degrees $\neq 0$ the differential d_A *must be monic* (not necessarily a kernel).

Morphisms of $\mathfrak{C}$*, I.* A morphism in $\mathbf{Ch}\,(\mathscr{A})$ between two objects of $\mathfrak{C}$, from $(A^{-1} \xrightarrow{d_A} A^0)$ to $(B^{-1} \xrightarrow{d_B} B^0)$, say, is uniquely determined by its component f^0 in degree 0 because d_B is monic. Moreover, $A^0 \xrightarrow{f^0} B^0$ is the component of a chain map $A \to B$ if and only if $f^0 d_A$ factors (necessarily in an unique way) over d_B which may be written as $f^0(A^{-1}) \subset B^{-1}$.

Morphisms of $\mathfrak{C}$*, II.* Next, a chain map between two objects of $\mathfrak{C}$ is chain homotopic to zero if and only if $f^0(A^0) \subset A^{-1}$ or, more precisely,

$$0 \simeq \begin{array}{ccc} (A^{-1} & \xrightarrow{d_A} & A^0) \\ \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} \\ (B^{-1} & \xrightarrow{d_B} & B^0) \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} & & A^0 \\ & \overset{\exists}{\swarrow} & \downarrow{\scriptstyle f^0} \\ B^{-1} & \xrightarrow{d_B} & B^0. \end{array}$$

Morphisms of $\mathfrak{C}$*, III.* The mapping cone of a chain map between two objects of $\mathfrak{C}$ is

$$\cdots \to 0 \to A^{-1} \xrightarrow{\left[\begin{smallmatrix} -d_A \\ f^{-1} \end{smallmatrix}\right]} A^0 \oplus B^{-1} \xrightarrow{[\, f^0 \;\; d_B \,]} B^0 \to 0 \to \cdots$$

which is concentrated in degrees $-2, -1, 0$. Recall that a chain map is a quasi-isomorphism if and only if the mapping cone is exact and in our current framework this amounts to saying that the diagram

$$\begin{array}{ccc} A^{-1} & \xrightarrow{d_A} & A^0 \\ \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} \\ B^{-1} & \xrightarrow{d_B} & B^0 \end{array}$$

is bicartesian.

Morphisms of $\mathfrak{C}$*, IV.* The morphisms between two objects of $\mathfrak{C}$ can now be constructed in three steps: take the group of chain maps, divide out the subgroup of chain maps which are homotopic to zero and then localize with respect to quasi-isomorphisms. Let A and B be two objects of $\mathfrak{C}$. By Step *I* we have

$$\mathrm{Hom}_{\mathbf{Ch}\,(\mathscr{A})}\,(A, B) = \{f^0 : A^0 \to B^0 \ : \ f^0(A^{-1}) \subset B^{-1}\},$$

by Step *II*

$$\mathrm{Hom}_{\mathbf{K}\,(\mathscr{A})}\,(A, B) = \frac{\{f^0 : A^0 \to B^0 \ : \ f^0(A^{-1}) \subset B^{-1}\}}{\{f^0 : A^0 \to B^0 \ : \ f^0(A^0) \subset B^{-1}\}}$$

and by Step *III*

$$\mathrm{Hom}_{\mathbf{D}\,(\mathscr{A})}\,(A, B) = \varinjlim \mathrm{Hom}_{\mathbf{K}\,(\mathscr{A})}\,(A', B)$$

where the colimit is taken over all quasi-isomorphisms $A' \to A$ with $A' \in \mathfrak{C}$. This leads us to the explicit identification

$$\mathrm{Hom}_{\mathfrak{C}}\,(A, B) = \varinjlim \mathrm{Hom}_{\mathbf{K}\,(\mathscr{A})}\,(A', B)$$

with the usual caveat that if $\mathscr{A}$ is not small then the colimit yields *a priori* an abelian group which is large.

By definition, $\mathscr{C}^*$ is a subcategory of $\mathbf{D}^*\,(\mathscr{A})$ and it is abelian by Theorem 2.2.3. Moreover $\mathfrak{C}$ is a full subcategory of $\mathscr{C}^*$. By Proposition 1.2.11 every object of $\mathscr{C}^*$ is isomorphic to an object of $\mathfrak{C}$, hence $\mathfrak{C}$ and $\mathscr{C}^*$ are equivalent and in particular $\mathfrak{C}$ is abelian.

REMARK 2.2.2. By Remark 1.1.5 a sequence $A \to B \to C$ in $\mathfrak{C}$ is short exact if and only if it can be extended to a distinguished triangle in $\mathbf{D}\,(\mathscr{A})$ by a necessarily unique morphism $C \to A[1]$ in $\mathbf{D}\,(\mathscr{A})$.

THEOREM 2.2.3. *The abelian category $\mathfrak{C}$ has the following properties:*

(i) The composite functor $i : \mathscr{A} \xrightarrow{c_0} \mathbf{D}^* \xrightarrow{H^0_\tau} \mathfrak{C}$ is fully faithful, exact and reflects exactness and preserves monics. It realizes $\mathscr{A}$ as an extension closed subcategory of $\mathfrak{C}$ and commutes with the formation of kernels and coimages.

(ii) Let $\mathscr{B}$ be an abelian category and let $\mathscr{A} \xrightarrow{F} \mathscr{B}$ be an exact functor which preserves monics. Then F extends uniquely to an exact functor $\mathfrak{C} \xrightarrow{\widetilde{F}} \mathscr{B}$.

REMARK 2.2.4. The functor i is thus the universal exact and monic-preserving functor from $\mathscr{A}$ to an abelian category.

PROOF OF *(i)*. The composition

$$\mathscr{A} \xrightarrow{c_0} \mathbf{D}(\mathscr{A}) \xrightarrow{H^0_\tau} \mathfrak{C}$$

is of course nothing but

$$A \mapsto (0 \to A)$$

and it is fully faithful, thus we view $\mathscr{A}$ as a full subcategory of $\mathfrak{C}$. An object $(A^{-1} \xrightarrow{d_A} A^0)$ of $\mathfrak{C}$ is isomorphic to an object of $\mathscr{A}$ if and only if d_A is an admissible monic. In particular, if $\mathscr{A}$ is abelian, we have that $\mathscr{A}$ and $\mathfrak{C}$ are equivalent since every monic in $\mathscr{A}$ is admissible.

Let us prove that the inclusion $\mathscr{A} \subset \mathfrak{C}$ preserves and reflects exactness. The inclusion is exact by the very definition of triangles in $\mathbf{D}(\mathscr{A})$. If $A' \xrightarrow{u} A \xrightarrow{v} A''$ becomes exact in $\mathfrak{C}$, the canonical morphism $\mathrm{cone}(u) \xrightarrow{\varphi} A''$ must be a quasi-isomorphism, hence $\mathrm{cone}(\varphi) \in \mathbf{Ac}(\mathscr{A})$, but $\mathrm{cone}(\varphi)$ is the sequence we started with.

Let us prove that $\mathscr{A}$ is closed under extensions in $\mathfrak{C}$. Given a short exact sequence in $\mathfrak{C}$ of the form

$$\begin{array}{ccc} (0 & \xrightarrow{d_A} & A) \\ \downarrow & & \downarrow f \\ (B^{-1} & \xrightarrow{d_B} & B^0) \\ \downarrow & & \downarrow g \\ (0 & \xrightarrow{d_C} & C) \end{array}$$

we have to prove that $(B^{-1} \xrightarrow{d_B} B^0)$ is isomorphic to an object of $\mathscr{A}$. Represent the morphism $B^\bullet \to C^\bullet$ by a fraction

$$\begin{array}{ccccc} & & D^\bullet & & \\ & \swarrow^{\mathrm{qi}} & & \searrow^{h^\bullet} & \\ B^\bullet & & & & C^\bullet \end{array}$$

put $K^0 = \mathrm{Ker}(h^0)$ and $K^{-1} = D^{-1}$. Since $D^{-1} \to C$ is zero, the universal property of the kernel yields a unique morphism $K^{-1} \to K^0$ such that the following diagram is commutative:

$$\begin{array}{ccc}
(K^{-1} & \overset{d_K}{\rightarrowtail} & K^0) \\
\Vert & & \downarrow{\scriptstyle k^0} \\
(D^{-1} & \xrightarrow{d_D} & D^0) \\
\downarrow & & \downarrow{\scriptstyle h^0} \\
(0 & \xrightarrow{d_C} & C)
\end{array}$$

Since $h^\bullet$ and $g^\bullet$ are isomorphic morphisms of $\mathfrak{C}$, they have isomorphic kernels, so $K^\bullet$ is isomorphic to $A^\bullet \in \mathscr{A}$, thus d_K must be an admissible monic (*cf.* the first paragraph of the proof). Moreover, k^0 is an admissible monic by definition, so the commutativity of the diagram implies that $d_D = k^0 d_K$ is an admissible monic as well and hence $D^\bullet$ is isomorphic to an object in $\mathscr{A}$. But $D^\bullet$ is also isomorphic to $B^\bullet$, so we are done.

That i preserves monics follows from Remark 2.2.8 below.

To see that i commutes with the formation of kernels and coimages, notice that by exactness of i it suffices to check that it preserves kernels. This is easily achieved by checking the universal property of the kernel. □

REMARK 2.2.5. Let f be a morphism in $\mathscr{A}$ such that $i(f)$ is epic. Let k be a kernel of f. Because i commutes with the formation of kernels, the sequence $(i(k), i(f))$ is exact. It follows from the fact that i reflects exactness that f is an admissible epic.

We defer the proof of Theorem 2.2.3 *(ii)* to the end of this section.

LEMMA 2.2.6. *If $\mathscr{A}$ has cokernels the functor $i : \mathscr{A} \to \mathfrak{C}$, $A \mapsto (0 \to A)$ has a left adjoint. In other words: $\mathscr{A}$ is a full reflective subcategory of the abelian category $\mathfrak{C}$.*

PROOF. Suppose that $\mathscr{A}$ has cokernels. Then the functor

$$\mathfrak{C} \xrightarrow{q} \mathscr{A}, \quad q(A^{-1} \xrightarrow{d_A} A^0) = \operatorname{Coker}(d_A)$$

is well-defined: it is a functor on the level of chain maps between objects of $\mathfrak{C}$, it maps null-homotopic maps to zero and, finally, the horizontal morphisms in a bicartesian square have canonically isomorphic cokernels. The counit is $\varepsilon : qi \xrightarrow{\cong} \mathrm{id}_{\mathscr{A}}$ and the diagram

$$\begin{array}{ccc}
(A^{-1} & \xrightarrow{d_A} & A^0) \\
\downarrow & & \downarrow \\
(0 & \longrightarrow & \operatorname{Coker} d_A)
\end{array}$$

displays the unit $\eta : \mathrm{id}_{\mathfrak{C}} \to iq$. It remains to check that the two compositions of natural transformations

$$q \xrightarrow{q\eta} qiq \xrightarrow{\varepsilon q} q \qquad \text{and} \qquad i \xrightarrow{\eta i} iqi \xrightarrow{i\varepsilon} i$$

are the identity, which is obvious. □

REMARK 2.2.7. The previous proof is somewhat too categorical, but it seems to be the shortest one to write down. It is illuminating to make the isomorphism

$$\operatorname{Hom}_{\mathscr{A}}(q(A^\bullet), B) \cong \operatorname{Hom}_{\mathfrak{C}}(A^\bullet, i(B))$$

explicit by contemplating the definitions.

REMARK 2.2.8. Let $(A^{-1} \xrightarrow{d_A} A^0) \in \mathfrak{C}$. The diagram

$$\begin{array}{ccc}
(0 & \longrightarrow & A^{-1}) \\
\downarrow & & \downarrow{\scriptstyle d_A} \\
(0 & \longrightarrow & A^0) \\
\downarrow & & \downarrow{\scriptstyle 1_{A^0}} \\
(A^{-1} & \xrightarrow{d_A} & A^0)
\end{array}$$

is a short exact sequence in $\mathfrak{C}$ by Remark 1.1.5.

COROLLARY 2.2.9. *Suppose $\mathscr{A}$ has cokernels and let q be the left adjoint of the inclusion $\mathscr{A} \xrightarrow{i} \mathfrak{C}$ provided by Lemma 2.2.6. The objects in $i(\mathscr{A})$ are left q-acyclic and there are enough q-acyclic objects. In particular, the total left derived functor $\mathbf{L}^{-}q$ exists.* □

THEOREM 2.2.10 ([**BBD82**, 1.3.23 Exercice c)]). *The inclusion $\mathscr{A} \xrightarrow{i} \mathfrak{C}$ induces an equivalence of categories*

$$\mathbf{D}^{-}(\mathscr{A}) \xrightarrow[\cong]{\mathbf{D}^{-}(i)} \mathbf{D}^{-}(\mathfrak{C}).$$

PROOF. We view $\mathscr{A} \subset \mathfrak{C}$ as a full subcategory via i and recall that $\mathscr{A}$ is closed under extensions by Theorem 2.2.3 (i). We wish to apply [**Kel96**, Theorem 12.1] and notice that by the first sentence in the proof $\mathscr{A}$ is a *fully exact* subcategory of $\mathfrak{C}$ in Keller's terminology (*loc. cit.* Section 4). We need to verify the duals of Keller's conditions which read:

- (C1$^{\mathrm{op}}$) For each $C'' \in \mathfrak{C}$, there exists a short exact sequence $C' \rightarrowtail A \twoheadrightarrow C''$ with $A \in \mathscr{A}$.
- (C2$^{\mathrm{op}}$) For each short exact sequence $C' \rightarrowtail C \twoheadrightarrow A''$ with $A'' \in \mathscr{A}$ there exists a diagram

$$\begin{array}{ccccc}
A' & \rightarrowtail & A & \twoheadrightarrow & A'' \\
\downarrow & & \downarrow & & \| \\
C' & \rightarrowtail & C & \twoheadrightarrow & A''
\end{array}$$

  whose first row is exact in $\mathscr{A}$.

Condition (C1$^{\mathrm{op}}$) is taken care of by Remark 2.2.8 and it implies that $\mathbf{D}^{-}(i)$ is essentially surjective.

It remains to verify (C2$^{\mathrm{op}}$) which implies that $\mathbf{D}^{-}(i)$ is fully faithful. To this end, let

$$\begin{array}{ccc}
(C'^{-1} & \xrightarrow{d'} & C'^{0}) \\
\downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} \\
(C^{-1} & \xrightarrow{d} & C^0) \\
\downarrow & & \downarrow{\scriptstyle g} \\
(0 & \longrightarrow & A^0)
\end{array}$$

be a short exact sequence in $\mathfrak{C}$. Let us represent the morphism $C^\bullet \to A^\bullet$ as a fraction

$$C^\bullet \xleftarrow{\ \text{qi}\ } B^\bullet \xrightarrow{\ h^\bullet\ } A^\bullet$$

and put $K^0 = \operatorname{Ker}(h^0)$ and $K^{-1} = B^{-1}$. Because $B^{-1} \to A^0$ is zero, it factors uniquely over K^0 and hence we get a short sequence of chain maps

$$\begin{array}{ccc}
(K^{-1} & \xrightarrow{d_K} & K^0) \\
\| & & \downarrow{\scriptstyle k^0} \\
(B^{-1} & \xrightarrow{d_B} & B^0) \\
\downarrow & & \downarrow{\scriptstyle h^0} \\
(0 & \longrightarrow & A^0)
\end{array}$$

which we claim to be short exact in $\mathfrak{C}$. By construction, $h^\bullet$ is epic, so it suffices to show that $k^\bullet = \ker(h^\bullet)$ which is easily seen by checking the universal property. Moreover, $K^\bullet$ is isomorphic to $C'^\bullet$ in $\mathfrak{C}$.

This implies that

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & K^{-1} & \xrightarrow{\left[\begin{smallmatrix} -d_K \\ 1 \end{smallmatrix}\right]} & K^0 \oplus B^{-1} & \xrightarrow{[\,k^0\ d_B\,]} & B^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h^0} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

is a quasi-isomorphism by Remark 1.1.5. In other words, the sequence

$$\begin{array}{ccccccc}
K^{-1} & \xrightarrow{\left[\begin{smallmatrix} -d_K \\ 1 \end{smallmatrix}\right]} & K^0 \oplus B^{-1} & \xrightarrow{\qquad [\,k^0\ d_B\,] \qquad} & B^0 & \xrightarrow{h^0} & A^0 \\
 & & & \searrow \quad Z \quad \nearrow{\scriptstyle \exists z} & & &
\end{array}$$

is exact and hence $[\,k^0\ d_B\,]$ factors over some object $Z \in \mathscr{A}$ via an admissible epic and an admissible monic. Therefore we have a commutative diagram in $\mathfrak{C}$

$$\begin{array}{ccccc}
i(Z) & \xrightarrow{i(z)} & i(B^0) & \xrightarrow{i(h^0)} & i(A^0) \\
\exists!\,\downarrow & & \downarrow & & \| \\
C'^\bullet & \xrightarrow{f^\bullet} & C^\bullet & \xrightarrow{g^\bullet} & A^\bullet
\end{array}$$

and the dotted arrow exists since the lower row is short exact. The upper row is short exact because i is exact. □

Proof of Theorem 2.2.3*(ii)*. Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor into an abelian category and suppose that F preserves monics. First notice that an exact functor on $\mathscr{A}$ induces a functor $\widetilde{F} : \mathfrak{C} \to \mathscr{B}$ by

$$\widetilde{F}(A^{-1} \xrightarrow{d_A} A^0) = \operatorname{Coker} F(d_A)$$

on objects—this is because it induces zero on morphisms homotopic to zero and because of its exactness it preserves bicartesian diagrams. It is clear that $\widetilde{F} \circ i = F$, so $\widetilde{F}$ is an extension of F and we will prove that $\widetilde{F} = L_0\widetilde{F}$ and that $L_n\widetilde{F} = 0$ for $n \neq 0$, so $\widetilde{F}$ is indeed exact.

Having proved that $i(\mathscr{A}) \subset \mathfrak{C}$ is a fully exact subcategory of $\mathfrak{C}$ satisfying conditions ($C1^{op}$) and ($C2^{op}$) of the proof of Theorem 2.2.10, we have proved in particular that there are enough left $\widetilde{F}$-acyclic objects, namely the objects of $i(\mathscr{A})$, see [**Kel96**, Section 15]. This applies to any extension of F, and the following argument will prove exactness of $\widetilde{F}$ as well as its uniqueness. Since $\mathbf{L}^- \widetilde{F}$ exists on $\mathbf{D}^-(\mathfrak{C})$, it can be computed on objects $C = (A^{-1} \to A^0) \in \mathfrak{C}$ by choosing a quasi-isomorphism $A' \to C$ where A' is a complex of $\widetilde{F}$-acyclic objects and $\mathbf{L}^- \widetilde{F}(C) = \widetilde{F}(A')$. To do this, we use the resolution provided by Remark 2.2.8 and conclude that $\mathbf{L}^- \widetilde{F}(C)$ is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow F(A^{-1}) \xrightarrow{F(d_A)} F(A^0) \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degrees -1 and 0. Since F preserves monics, the homology of this complex is concentrated in degree zero, so

$$L_0\widetilde{F}(C) = \operatorname{Coker} F(d_A) \qquad \text{and} \qquad L_n\widetilde{F}(C) = 0 \quad \text{for } n \neq 0,$$

as claimed earlier, therefore we have established exactness and uniqueness of $\widetilde{F}$. □

This argument gives us some more information:

COROLLARY 2.2.11. *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor to an abelian category sending exact sequences in $\mathscr{A}$ to right exact sequences in $\mathscr{B}$. Then F extends to a functor $\widetilde{F} : \mathfrak{C} \to \mathscr{B}$.*

Moreover, if F is exact then $\widetilde{F}$ is right exact and of homological dimension ≤ 1, and, in that case, $\widetilde{F}$ is of homological dimension 0 if and only if F preserves monics.

PROOF. The condition on F assures that it preserves push-out diagrams. Therefore the definition

$$\widetilde{F}(A^{-1} \xrightarrow{d_A} A^0) = \operatorname{Coker} F(d_A)$$

makes sense and is indeed an extension of F. The rest follows from the argument in the previous proof. □

Part 2

Homological Algebra for Bounded Cohomology

CHAPTER IV

Categories of Banach Spaces

There are two standard ways of turning the class of all Banach spaces over the ground field k into a category:

(1) Take as morphisms the linear maps of norm at most one: $\mathbf{Ban}_1$.
(2) Take as morphisms all bounded linear maps: **Ban**.

The category $\mathbf{Ban}_1$ has the virtue of being bicomplete and the drawback of being non-additive. The category **Ban** is additive, however it is only finitely bicomplete. We will give a more or less systematic presentation of the basic categorical properties of $\mathbf{Ban}_1$ and **Ban**.

We will only consider the Archimedean case in which the ground field k is either the field of real numbers or the one of complex numbers.

Contents

1. The Bicomplete Category of Banach Spaces

Our reference to this section is Cigler-Losert-Michor [**CLM79**, Chapter I.1]. The objects of the category $\mathbf{Ban}_1$ are the Banach spaces over the ground field k and the morphisms are the linear maps of norm at most one. We insist that the zero space is a Banach space. It is a zero object in the sense that it is both initial and terminal in $\mathbf{Ban}_1$.

1.1. Monics and Epics. In an arbitrary category $\mathscr{C}$, a morphism m is *monic* if $mf = mg$ implies $f = g$ for all parallel morphisms f and g and a morphism e is *epic* if $fe = ge$ implies $f = g$. A morphism is an *isomorphism* if it has a two-sided inverse.

PROPOSITION 1.1.1. *Consider a morphism* $E \xrightarrow{f} F$ *in* $\mathbf{Ban}_1$.

(i) f *is monic if and only if it is injective.*
(ii) f *is epic if and only if it has dense range.*

(iii) f is an isomorphism if and only if it is isometric and surjective.

Proof. See [**CLM79**, I.1.3, I.1.4, I.1.5]. □

Remark 1.1.2. The inclusion $\ell^1 \subset c_0$ is both monic and epic but it is not an isomorphism.

1.2. Equalizers and Coequalizers. Let I be the category $\bullet \rightrightarrows \bullet$ with two objects and two parallel non-identity morphisms. An *equalizer* in the category $\mathscr{C}$ is a limit of a functor $I \to \mathscr{C}$ and a *coequalizer* in $\mathscr{C}$ is a colimit of a functor $I \to \mathscr{C}$.

Proposition 1.2.1. *Let $E \overset{f}{\underset{g}{\rightrightarrows}} F$ be two parallel morphisms in $\mathbf{Ban}_1$.*

(i) The equalizer of f and g is $\{x \in E : f(x) = g(x)\}$.
(ii) The coequalizer of f and g exists and may be identified with

$$C = F/N,$$

where $N = \overline{\{f(x) - g(x) : x \in E\}}$.

Proof. See [**CLM79**, I.1.13, I.1.18]. □

Taking $g = 0$ yields:

Corollary 1.2.2. *Let $E \xrightarrow{f} F$ be a morphism in $\mathbf{Ban}_1$.*

(i) The kernel of f exists and may be identified with the space

$$\{x \in E : f(x) = 0\}.$$

(ii) The cokernel of f exists and may be identified with the space

$$F/\overline{\{f(x) : x \in E\}}.$$

1.3. Products and Coproducts.

Proposition 1.3.1. *Let $\{(E_i, \|\cdot\|_i)\}_{i \in I}$ be an arbitrary set of Banach spaces.*

(i) The linear space

$$E = \{x = (x_i)_{i \in I} : x_i \in E_i,\ \|x\|_\infty := \sup_{i \in I} \|x_i\|_i < \infty\}$$

is a Banach space with the norm $\|\cdot\|_\infty$ and together with the projections $E \to E_i, (x_i)_{i \in I} \mapsto x_i$ it represents the product

$$E = \prod_{i \in I} E_i$$

in $\mathbf{Ban}_1$.

(ii) The linear space

$$F = \{x = (x_i)_{i \in I} : x_i \in E_i, \|x\|_1 := \sum_{i \in I} \|x_i\|_i < \infty\}$$

is a Banach space with the norm $\|\cdot\|_1$ and together with the obvious inclusions $E_i \to E$ it represents the coproduct

$$F = \coprod_{i \in I} E_i$$

in $\mathbf{Ban}_1$.

Proof. See [**CLM79**, I.1.9, I.1.10]. □

REMARK 1.3.2. If the set $\{E_i\}_{i\in I}$ contains at least two non-zero spaces then the canonical map

$$\coprod_{i\in I} E_i \to \prod_{i\in I} E_i$$

is a monomorphism but it is *not* an isomorphism in $\mathbf{Ban}_1$.

REMARK 1.3.3. Let S be a set. Then $\ell^1(S) = \coprod_S k$ and $\ell^\infty(S) = \prod_S k$.

1.4. Limits and Colimits.

PROPOSITION 1.4.1. *In* $\mathbf{Ban}_1$ *all small limits and all small colimits exist.*

PROOF. See [**CLM79**, I.1.12, I.1.17] for a detailed construction which may be obtained by making the following abstract nonsense explicit:

The zero space is both an initial and a terminal object in $\mathbf{Ban}_1$. All small products and coproducts exist by Proposition 1.3.1. Equalizer and coequalizer of all pairs of parallel arrows exist by Proposition 1.2.1. Hence the hypotheses of Corollary 2 (and its dual) on page 113 in Mac Lane [**ML98**] are satisfied. □

COROLLARY 1.4.2. *The category* $\mathbf{Ban}_1$ *is not equivalent to a small category.*

PROOF. Since for a non-zero Banach space E, the set $\mathbf{Ban}_1(E, E)$ contains the distinct elements 1_E and 0_E, the category $\mathbf{Ban}_1$ cannot be equivalent to a pre-order. Hence Freyd's observation [**ML98**, Proposition 3, p. 114] yields the claim. □

1.5. The Ground Field. Recall that we work over the ground field k which is either the field of real numbers or the field of complex numbers.

A *generator* in a category is an object G having the following property: given two distinct morphisms $f, g : E \to F$ there exists a morphism $h : G \to E$ such that $fh \neq gh$. Dually, a cogenerator is an object H such that for two distinct morphisms $f, g : E \to F$ there is a morphism $h : F \to H$ such that the composite $hf \neq hg$.

PROPOSITION 1.5.1. *The ground field k is both a generator and a cogenerator of* $\mathbf{Ban}_1$.

PROOF. Let $E \xrightarrow{f} F$ be a non-zero morphism. There exists $x \in E$ with $\|x\| = 1$ such that $f(x) \neq 0$. Sending $1 \in k$ to $x \in E$ yields a morphism $k \to E$ such that the composite $k \to E \to F$ is non-zero and hence k is a generator. On the other hand, the Hahn-Banach theorem implies that there exists a functional $F \to k$ of norm one such that the composite $E \to F \to k$ is nonzero and hence k is a cogenerator. □

NOTATION 1.5.2. The (co-)represented functor of a (co-)generator is distinguished in any pointed category. In $\mathbf{Ban}_1$, the represented functor

$$B_{\leq 1}(-) := \mathbf{Ban}_1(k, -) : \mathbf{Ban}_1 \longrightarrow \mathbf{Sets}$$

associates to a Banach space the set underlying its closed unit ball and the corepresented functor

$$B^*_{\leq 1}(-) := \mathbf{Ban}_1(-, k) : \mathbf{Ban}_1^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

associates to a Banach space the set underlying the closed unit ball of its dual.

PROPOSITION 1.5.3.

(i) *The functor* $\ell^1 : \mathbf{Sets} \to \mathbf{Ban}_1$ *is left adjoint to* $B_{\leq 1}$.
(ii) *The functor* $\ell^\infty : \mathbf{Sets} \to \mathbf{Ban}_1^{\mathrm{op}}$ *is left adjoint to* $B^*_{\leq 1}$.

PROOF. Assertion (i) boils down to establishing a natural isomorphism

$$\mathbf{Ban}_1(\ell^1(S), E) \cong \mathbf{Sets}(S, B_{\leq 1}(E))$$

for sets S and Banach spaces E. See [**CLM79**, I.1.11 Remark 2] for full details.

Assertion (ii) can be seen from the following chain of natural isomorphisms

$$\begin{aligned}\mathbf{Ban}_1^{\mathrm{op}}\left(\ell^\infty(S), E\right) &\cong \mathbf{Ban}_1\left(E, \ell^\infty(S)\right)\\ &\cong \mathbf{Ban}_1\left(\ell^1(S), E^*\right)\\ &\cong \mathbf{Sets}\left(S, B_{\leq 1}(E^*)\right)\\ &\cong \mathbf{Sets}\left(S, B^*_{\leq 1}(E)\right)\end{aligned}$$

which completes the proof. □

Corollary 1.5.4. *Every Banach space is a quotient of an ℓ^1-space and a subspace of an ℓ^∞-space.*

Proof. Let E be a Banach space. The counit of the adjunction $\ell^1 \dashv B_{\leq 1}$ yields the first claim: Let

$$\eta_E \in \mathbf{Ban}_1\left(\ell^1(B_{\leq 1}(E)), E\right)$$

be the morphism corresponding to $1_{B_{\leq 1}(E)}$ under the natural isomorphism

$$\mathbf{Ban}_1\left(\ell^1(B_{\leq 1}(E)), E\right) \cong \mathbf{Sets}\left(B_{\leq 1}(E), B_{\leq 1}(E)\right);$$

it is easily checked that η_E is surjective.

The second assertion is proved similarly: The counit of the adjunction $\ell^\infty \dashv B^*_{\leq 1}$ yields

$$\varphi_E \in \mathbf{Ban}_1^{\mathrm{op}}\left(\ell^\infty(B^*_{\leq 1}(E)), E\right) = \mathbf{Ban}_1\left(E, \ell^\infty(B^*_{\leq 1}(E))\right)$$

and we leave it to the reader to check that φ_E may be considered as "evaluation". Thus the Hahn-Banach theorem implies that φ_E is isometric. □

Remark 1.5.5. A slightly different proof is given in [**CLM79**, I.1.11].

Remark 1.5.6. The corollary shows that the (co-)generator k can be used to construct the whole category $\mathbf{Ban}_1$. Every Banach space can be realized as a quotient of a coproduct of copies of k and a subspace of a product. Moreover, this can be done in a functorial way.

1.6. A Multiplicative System. We exhibit a multiplicative system Σ in the sense of Gabriel-Zisman [**GZ67**] (see also [**Wei94**, Section 10.2]) on $\mathbf{Ban}_1$ whose category of fractions is the additive category $\mathbf{Ban}$. The basic idea is that by inverting the homotheties in $\mathbf{Ban}_1$ one should obtain all bounded linear maps.

Proposition 1.6.1. *Let* $\operatorname{loc} : \mathbf{Ban}_1 \to \mathbf{Ban}$ *be the obvious functor and let*

$$\begin{aligned}\Sigma &= \{f \in \mathbf{Ban}_1^{\to} \;:\; \operatorname{loc}(f) \textit{ is an isomorphism}\}\\ &= \{\textit{bijective morphisms}\}.\end{aligned}$$

The class Σ satisfies:

(1) *It contains all identity morphisms and is closed under composition.*
(2) *Given $s \in \Sigma$ and $f \in \mathbf{Ban}_1^{\to}$ with the same domain, there exists a diagram*

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & F \\ {\scriptstyle s\in\Sigma}\downarrow & & \vdots\downarrow{\scriptstyle s'\in\Sigma} \\ E' & \overset{f'}{\cdots\cdots\to} & F'. \end{array}$$

(2^{op}) *Given $s \in \Sigma$ and $f \in \mathbf{Ban}_1^{\to}$ with the same codomain, there exists a diagram*

$$\begin{array}{ccc} E' & \overset{f'}{\dashrightarrow} & F' \\ {\scriptstyle s'\in\Sigma}\, \downarrow & & \downarrow\, {\scriptstyle s\in\Sigma} \\ E & \xrightarrow{f} & F. \end{array}$$

(3) *Given two parallel morphisms f and g, there exists $s \in \Sigma$ such that $fs = gs$ if and only if there exists $t \in \Sigma$ such that $tf = tg$.*

In other words, Σ is a multiplicative system *in the sense of Gabriel-Zisman. The properties* (2) *and* (2^{op}) *are called the* Ore conditions *and property* (3) *is called the* cancellation condition *for a multiplicative system.*

Moreover, the multiplicative system Σ is saturated *in the sense that*

(4) *Let f, g and h be composeable morphisms. If gf and hg are in Σ then g is in Σ as well.*

Finally, for each $E \in \mathbf{Ban}_1$ let $\Sigma_E = \Sigma \cap \mathbf{Ban}_1(E, E)$ be the set *of bijective endomorphisms of E. Then*

(5) *For all $s : F \to E$ in Σ there exists a morphism $s' : E \to F$ such that $ss' \in \Sigma_E$.*

(5^{op}) *For all $s : E \to F$ there exists a morphism $s' : F \to E$ such that $s's \in \Sigma_E$.*

In particular, the multiplicative system Σ is locally small on both sides.

Proof. Points (1), (3) and (4) are trivial. For (5) and (5^{op}) it suffices to take $s' = \frac{s^{-1}}{\|s^{-1}\|}$ where the inverse is taken in **Ban**.

It remains to check the Ore conditions. To check (2), construct the push-out diagram under f and s, so as to obtain f' and s'. Then notice that it is also a push-out diagram of the underlying vector spaces, hence s' must be a bijection. The proof of (2^{op}) is dual. □

Corollary 1.6.2. *The category of fractions $\mathbf{Ban}_1[\Sigma^{-1}]$ has small* Hom*-sets and its morphisms may be represented by both a calculus of left fractions and a calculus of right fractions. Moreover, there is a commutative diagram of categories*

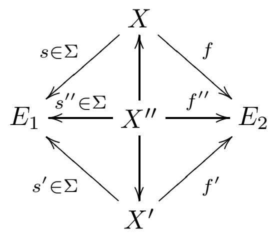
$$\begin{array}{ccccc} & & \mathbf{Ban}_1 & & \\ & {\scriptstyle \mathrm{loc}}\swarrow & & \searrow{\scriptstyle i} & \\ \mathbf{Ban}_1[\Sigma^{-1}] & & \overset{\exists! F}{\underset{\cong}{\dashrightarrow}} & & \mathbf{Ban} \end{array}$$

and the horizontal functor is an equivalence of categories.

Proof. The first part is a consequence of the Gabriel-Zisman Theorem on the localization of categories. The horizontal functor is given by the universal property of the category of fractions $\mathbf{Ban}_1[\Sigma^{-1}]$ since $i : \mathbf{Ban}_1 \to \mathbf{Ban}$ sends bijective morphisms in $\mathbf{Ban}_1$ to isomorphisms in **Ban** (open mapping theorem).

Let us check that the horizontal functor F is fully faithful (bijective on Hom-sets) and essentially surjective (every object in **Ban** is isomorphic to an object in $F(\mathbf{Ban}_1[\Sigma^{-1}])$) so that by Freyd's criterion F is an equivalence of categories.

To this end, recall that the objects of $\mathbf{Ban}_1[\Sigma^{-1}]$ are the objects of $\mathbf{Ban}_1$ and that the morphisms in $\mathbf{Ban}_1[\Sigma^{-1}]$ are represented by fractions $E_1 \xleftarrow{s\in\Sigma} X \xrightarrow{f} E_2$ which we will abbreviate by f/s. Two fractions f/s and f'/s' are equivalent if and only if there exists a commutative diagram



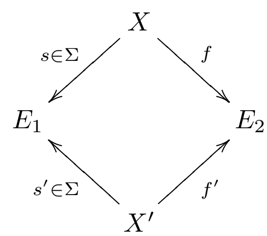
in $\mathbf{Ban}_1$. The functor F is the identity on objects and is given by

$$F(f/s) = i(f)i(s)^{-1}$$

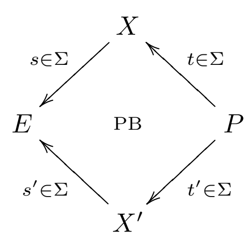
on morphisms.

To see that F is full, let f be a morphism in $\mathbf{Ban}$ and observe that if $\|f\| \le 1$ then $f = F(f/1)$ and if $\|f\| \ge 1$ then $f = F((f/\|f\|)/(1/\|f\|))$.

To see that F is faithful, assume that $F(f/s) = F(f'/s')$, so we have a diagram in $\mathbf{Ban}_1$



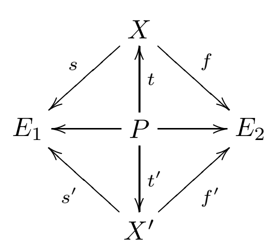
and moreover $i(f)i(s)^{-1} = i(f')i(s')^{-1}$. We want to prove that the fractions f/s and f'/s' are equivalent. Form the pull-back over s and t to obtain the diagram



so $st = s't' \in \Sigma$. We claim that $ft = f't'$: this follows from

$$i(ft)i(st)^{-1} = i(f)i(s)^{-1} = i(f')i(s')^{-1} = i(f't')i(s't')^{-1},$$

so that $i(ft) = i(f't')$, and the fact that i is faithful. Thus we have a commutative diagram



proving that f/s is equivalent to f'/s', so F is faithful.

It is trivial that F is essentially surjective because $\mathbf{Ban}_1$ and $\mathbf{Ban}$ have the same class of objects and F is the identity on objects. □

1.7. The Category of Waelbroeck Duals. Waelbroeck has given a nice description of the opposite category of $\mathbf{Ban}_1$. The basic idea is to realize it via the dual space functor.

DEFINITION 1.7.1. A *Waelbroeck dual space* is a triple (E, K, τ) consisting of a k-vector space E, a convex, circled and absorbing subset[1] $K \subset E$ and a compact Hausdorff topology τ on K such that:

(i) For every $x \in K$ the map $y \mapsto \frac{1}{2}(x+y)$ is continuous on K.
(ii) The origin of K has a neighbourhood base for τ consisting of circled convex sets.

A *morphism* of Waelbroeck duals $(E, K, \tau) \xrightarrow{f} (E', K', \tau')$ is a linear map such that $f(K) \subset K'$ and f is τ-τ'-continuous.

We denote the category of Waelbroeck duals by **WD**.

EXAMPLE 1.7.2. Let E be a Banach space. The dual space E^* together with its unit ball and the weak*-topology is a Waelbroeck dual. Correspondingly, if $f : F \to E$ is a morphism in $\mathbf{Ban}_1$ then its Banach space dual $f^* : E^* \to F^*$ is a morphism of Waelbroeck duals because $\|f^*\| = \|f\| \leq 1$ and f^* is weak*-continuous.

THEOREM 1.7.3 ([**Wae67**]). *The category* **WD** *is isomorphic to the opposite category of* $\mathbf{Ban}_1$*.* □

The proof is quite non-trivial and uses a number of results from the basic arsenal of a well-educated functional analyst, see [**CLM79**, Section I.2]. The work can be considerably reduced by assuming right away that the map $(x, y) \mapsto \frac{1}{2}(x+y)$ is continuous as a function $K \times K \to K$, a fact which follows from Waelbroeck's definition.

COROLLARY 1.7.4. *Let* Σ *be the class of bijective linear maps in* **WD***. It is a saturated locally small multiplicative system satisfying cancellation and both Ore conditions. The category of fractions* $\mathbf{WD}[\Sigma^{-1}]$ *is equivalent to* $\mathbf{Ban}^{\mathrm{op}}$*.* □

2. The Additive Category of Banach Spaces

We now turn to the investigation of the category **Ban** over the ground field k.

2.1. Basic Properties.

REMARK 2.1.1. The category **Ban** is k-linear and has kernels and cokernels. In particular idempotents split and all *finite* (co-)limits exist.

REMARK 2.1.2. The monics are precisely the injective linear maps, the epics are precisely the linear maps with dense range. A monic is a kernel if and only if it has closed range. An epic is a cokernel if and only if it has closed range which amounts to the same as being surjective.

EXAMPLE 2.1.3. The inclusion $\ell^1 \subset c_0$ is monic and epic but not an isomorphism. In particular **Ban** is not abelian.

REMARK 2.1.4. Infinite coproducts do not exist: Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of non-zero Banach spaces. Suppose that there exists a coproduct represented by a Banach space F and (non-zero) morphisms $f_n : E_n \to F$. Applying the universal property of the coproduct to the system $nf_n : E_n \to F$ yields a morphism $g : F \to F$ such that $nf_n = gf_n$ for all $n \in \mathbb{N}$. Hence the absurdity $n \leq \|g\|$ for all $n \in \mathbb{N}$. A similar argument yields the non-existence of non-trivial infinite products.

[1]Let $S \subset E$ be a subset of a k-vector space E. The set S is called *circled* if $\lambda s \in S$ for all $s \in S$ and all $\lambda \in k$ with $|\lambda| = 1$. The set S is *absorbing* if for all $e \in E$ there exist $s \in S$ and $\lambda > 0$ such that $e = \lambda s$.

Remark 2.1.5. The categorical definition of projectivity by lifting morphisms over all epics is useless: sending $1 \in k$ to a sequence in c_0 which is not in ℓ^1 yields a morphism $k \to c_0$ which cannot be lifted to ℓ^1, so k is not categorically projective.

Similarly, categorical injectivity is useless: summation $\ell^1 \to k$ yields a morphism which cannot be extended to c_0, so k is not categorically injective.

Since k is a direct summand of every nonzero Banach space, the zero objects in **Ban** are the only categorically projectives and the only categorically injectives.

2.2. A Monoidal Structure. We will only use the monoidal structure given by the *projective tensor product*. There are of course many others and the corresponding internal hom-functors are treated in the literature under the name of *operator ideals*.

Let E and F be two Banach spaces. Equip the algebraic tensor product $E \otimes_k F$ (that is to say the tensor product of the underlying k-vector spaces) with the projective tensor norm, which is defined for $\omega \in E \otimes_k F$ as

$$\|\omega\|_\pi := \inf \left\{ \sum \|e_i\|_E \, \|f_i\|_F \, : \, \omega = \sum e_i \otimes f_i \right\}$$

where the infimum is taken over all representations of ω as a finite sum of elementary tensors. It is obvious that $\| \cdot \|_\pi$ defines a semi-norm. A little more effort establishes:

Lemma 2.2.1. *The expression $\| \cdot \|_\pi$ defines a norm on $E \otimes_k F$ satisfying the identity $\|e \otimes f\|_\pi = \|e\|_E \, \|f\|_F$ for all elementary tensors.* □

Definition 2.2.2. The *projective tensor product* $E \mathbin{\widehat{\otimes}} F$ of E and F is defined to be the *completion* of $E \otimes_k F$ with respect to $\| \cdot \|_\pi$.

The main properties of the projective tensor product are summarized in the following theorem:

Theorem 2.2.3 (Grothendieck). *The projective tensor product $\widehat{\otimes}$ is a symmetric and associative monoidal structure on* **Ban** *whose unit is the ground field k and whose internal* hom *is* $\mathbf{Ban}\,(E, F)$ *equipped with the operator norm.* □

Remark 2.2.4. Because of the adjointness $\mathbf{Ban}\,(E \mathbin{\widehat{\otimes}} F, G) \cong \mathbf{Ban}\,(E, \mathbf{Ban}\,(F, G))$ the projective tensor product is additive and preserves colimits, in particular cokernels.

The following definition has many equivalent formulations and it will only be used to prove that even though the projective tensor product has a right adjoint it fails to be right exact.

Definition 2.2.5 (Grothendieck). A Banach space F is said to have the *approximation property* if $- \mathbin{\widehat{\otimes}} F$ preserves monics.

Remark 2.2.6. P. Enflo [**Enf73**] has given an example of a separable and reflexive Banach space which fails to have the approximation property. Subsequent to his work, A. Szankowski [**Sza79**] has shown that $\mathbf{Ban}\,(\ell^2, \ell^2)$ does not have the approximation property. However, all classical Banach spaces (e.g. ℓ^p, $1 \leq p \leq \infty$) are known to have the approximation property, see [**LT77**].

Example 2.2.7 ([**Wae05**, 3.2.7]). Let F be a Banach space which is infinite-dimensional, reflexive and has the approximation property. For instance, $F = \ell^p$ with $1 < p < \infty$ has the required properties.

Claim. The functor $- \mathbin{\widehat{\otimes}} F$ fails to be exact in the middle.

Proof. Put $E_1 = F^*$ and $E_0 = \ell^\infty(B)$, where $B = B_{\leq 1}(F)$ is the closed unit ball in $F = F^{**}$.

There is an obvious inclusion $i : E_1 \rightarrowtail E_0$ given by the fact that every $x \in E_1$ can be considered as a bounded function on B by evaluation. The Hahn-Banach theorem implies that i is isometric.

Next observe that *i cannot have a left inverse* because E_1 is reflexive and infinite-dimensional. If i had a left inverse p, this left inverse would be weakly compact because E_1 is reflexive. Since ip is a weakly compact idempotent of a $C(K)$-space (take K to be the Stone-Čech compactification of B with the discrete topology), it is even compact by the Dunford-Pettis theorem (see e.g. [**DS58**, VI.7.4, Corollary 5, p. 494]). By the open mapping theorem, the open unit ball of E_0 is mapped to an open set under ip which is at the same time relatively compact because ip is compact, hence $i(E_1)$ must be finite dimensional, contradicting our assumptions.

Because F has the approximation property, the map $E_1 \mathbin{\widehat{\otimes}} F \xrightarrow{i \mathbin{\widehat{\otimes}} F} E_0 \mathbin{\widehat{\otimes}} F$ is a monic.

Now supose $- \mathbin{\widehat{\otimes}} F$ is exact in the middle. In particular, $i \mathbin{\widehat{\otimes}} F$ must have closed range. Because $E_0 \mathbin{\widehat{\otimes}} F = F^* \mathbin{\widehat{\otimes}} F$, evaluation yields a continuous linear functional $\varphi \in (F^* \mathbin{\widehat{\otimes}} F)^* \cong \mathbf{Ban}\,(F^*, F^*)$ which corresponds to 1_{F^*}. Since $i \mathbin{\widehat{\otimes}} F$ has closed range, it is a homeomorphism onto its image, and hence the Hahn-Banach theorem allows us to extend φ to a functional $\psi \in (E_0 \mathbin{\widehat{\otimes}} F)^* \cong \mathbf{Ban}\,(E_0, F^*)$ and it is clear that $\psi|_{E_0 = F^*} = 1_{F^*}$, hence ψ is a left inverse to i, a contradiction. □

2.3. Three Exact Structures. The failure of **Ban** to be abelian forces us to change the setting. Basic homological algebra in abelian categories actually goes through without much pain in the much more general setting of exact categories.

Let us recall the definition (see [**Kel90**, Appendix A] or [**Qui73**, § 2]):

DEFINITION 2.3.1. An *exact structure* $\mathscr{E}$ on an additive category $\mathscr{A}$ is a class $\mathscr{E}$ of short sequences in $\mathscr{A}$ which is closed under isomorphisms in $\mathscr{A}^{\to\to}$ and consists of kernel-cokernel pairs and is subject to axioms [Ex 0] to [Ex 2^{op}] below. Elements of $\mathscr{E}$ are displayed diagrammatically as

$$A' \overset{m}{\rightarrowtail} A \overset{e}{\twoheadrightarrow} A''.$$

Morphisms m of $\mathscr{A}$ appearing on the left in $\mathscr{E}$ are called *admissible monics* and morphisms e of $\mathscr{A}$ appearing on the right are called *admissible epics*. The axioms are:

[Ex 0] Identities are admissible monics.
[Ex 0^{op}] Identities are admissible epics.
[Ex 1] Admissible monics are closed under composition.
[Ex 1^{op}] Admissible epics are closed under composition.
[Ex 2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.
[Ex 2^{op}] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Axioms [Ex 2^{op}] and [Ex 2^{op}] are illustrated by the diagrams:

$$\begin{array}{ccc} A' & \overset{m'}{\rightarrowtail} & A \\ {\scriptstyle f'}\downarrow & \text{PO} & \downarrow{\scriptstyle f} \\ B' & \overset{m}{\rightarrowtail} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \overset{e}{\twoheadrightarrow} & A'' \\ {\scriptstyle f}\downarrow & \text{PB} & \downarrow{\scriptstyle f''} \\ B & \overset{e''}{\twoheadrightarrow} & B''. \end{array}$$

According to Keller, *loc. cit.*, it suffices to require [Ex 0^{op}], [Ex 1^{op}], [Ex 2] and a weakening of [Ex 2^{op}].

The pair $(\mathscr{A}, \mathscr{E})$ is called an *exact category* if $\mathscr{E}$ is an exact structure on $\mathscr{A}$.

Example 2.3.2. The class $\mathscr{E}_{\min}$ of *split exact sequences* is an exact structure on every additive category $\mathscr{A}$. This applies of course to **Ban** as well and gives us the first exact structure.

The following result is at the heart of our approach:

Theorem 2.3.3. *The class $\mathscr{E}_{\max}$ of all kernel-cokernel pairs is an exact structure on* **Ban**.

Proof. First let us notice that the class $\mathscr{E}$ coincides with the class of short sequences of Banach spaces such that the underlying sequence of k-vector spaces is short exact by the explicit description of kernels and cokernels in Remark 2.1.2. Therefore $\mathscr{E}$ is closed under isomorphisms, identities are admissible monics and epics. Moreover, the composition of two admissible monics or two admissible epics is again an admissible monic or an admissible epic.

By Remark 2.1.1 all push-outs and pull-backs exist, hence we need only check that admissible monics and epics are closed under base-change along an arbitrary morphism. Let us first deal with the slightly more subtle case of an admissible monic, so consider the diagram

$$\begin{array}{ccc} A' & \overset{m'}{\rightarrowtail} & A \\ {\scriptstyle f'}\downarrow & \text{PO} & \vdots\, {\scriptstyle f} \\ B' & \overset{m}{\cdots\rightarrowtail} & B \end{array}$$

in which $B = \operatorname{Coker}\big(A' \xrightarrow{\left[\begin{smallmatrix} -m' \\ f' \end{smallmatrix}\right]} A \oplus B'\big)$ and $[\, f \;\, m\,] := \operatorname{coker}\left[\begin{smallmatrix} -m' \\ f' \end{smallmatrix}\right]$. We assert that B is also the push-out in the category of k-vector spaces. This follows from the fact that $\left[\begin{smallmatrix} -m' \\ f' \end{smallmatrix}\right]$ has closed range[2]. In particular, $[\, f \;\, m\,]$ is also the cokernel of $\left[\begin{smallmatrix} -m' \\ f' \end{smallmatrix}\right]$ in the category of k-vector spaces. Therefore the left square below is a push-out diagram both in **Ban** and the category of k-vector spaces:

$$\begin{array}{ccccc} A' & \overset{m'}{\rightarrowtail} & A & \overset{e'}{\twoheadrightarrow} & C \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \| \\ B' & \xrightarrow{m} & B & \xrightarrow{e} & C. \end{array}$$

The latter fact implies that (m, e) is an exact sequence of k-vector spaces and hence m must be an admissible monic in **Ban** by the first observation of our proof.

The case of the pull-back of an admissible epic along an arbitrary morphism is dual. □

Remark 2.3.4. Theorem 2.3.3 is stated e.g. in [**BBD82**, 1.1.4, Exemple 2][3] or [**Kel96**, Example 4.6].

The *duality principle,* see e.g. [**Rud91**, Ch. 4], has the following consequence, to which we will refer loosely as the duality principle throughout this chapter and the next one.

[2]Let $a'_n \in A'$ be such that $(-m'(a'_n), f'(a'_n))$ converges in $A \oplus B'$. Then $-m'(a_n)$ converges and because $-m'$ is a homeomorphism onto its range by the open mapping theorem, a'_n must converge to some a'. Thus $(-m'(a'_n), f'(a'_n)) \to (-m'(a'), f'(a'))$.

[3]In *loc. cit.* it is suggested to consider also another exact structure on **Ban**, which at first glance seems to be different. However, Michael's selection theorem ([**Mic56**, Proposition 7.2], see also [**BL00**, Chapter 1, Proposition 1.19, (ii)]) together with the open mapping theorem implies that it coincides with our $\mathscr{E}_{\max}$.

PROPOSITION 2.3.5 (Duality Principle). *The dual space functor preserves and reflects $\mathscr{E}_{\max}$-exact sequences in* **Ban**. □

COROLLARY 2.3.6. *The class $\mathscr{E}_{\text{pure}} = \{(m,e) \in \mathscr{E}_{\max} : (e^*, m^*) \in \mathscr{E}_{\min}\}$ is an exact structure which we call the* pure exact structure. □

REMARK 2.3.7. We have the strict inclusions $\mathscr{E}_{\min} \subsetneqq \mathscr{E}_{\text{pure}} \subsetneqq \mathscr{E}_{\max}$ of exact structures.

To see this, recall Phillips' lemma [**Phi40**, 7.5, p.539] (see [**Whi66**] for a simple proof) which states that the canonical inclusion $c_0 \subset \ell^\infty = (c_0)^{**}$ is not split. It is however a pure monic—the canonical inclusion into the bidual is always pure—so that $\mathscr{E}_{\min} \subsetneqq \mathscr{E}_{\text{pure}}$. To see that the second inclusion is strict, apply Corollary 1.5.4 to find an admissible epic $\ell^1(S) \twoheadrightarrow c_0$ and if it were pure, we would conclude the absurdity that $\ell^1 = (c_0)^*$ is a direct summand of $\ell^\infty(S)$.

REMARK 2.3.8. Phillips' lemma implies that c_0 cannot be $\mathscr{E}_{\max}$-injective. However, there is the curiosity that c_0 is $\mathscr{E}_{\max}$-injective in the category of all separable Banach spaces, see [**LT77**, 2.f.5].

2.4. Projective, Injective and Flat Banach Spaces. In this section we only consider the exact category $(\mathbf{Ban}, \mathscr{E}_{\max})$. But first we recall the general definition:

DEFINITION 2.4.1. Let $(\mathscr{A}, \mathscr{E})$ be an exact category.

- (i) An object $P \in \mathscr{A}$ is called *projective* if $\operatorname{Hom}_{\mathscr{A}}(P, -) : (\mathscr{A}, \mathscr{E}) \to \mathbf{Ab}$ is exact. There are *enough projectives* in $(\mathscr{A}, \mathscr{E})$ if every $A \in \mathscr{A}$ there exists a projective P and an admissible epic $P \twoheadrightarrow A$.
- (ii) An object $I \in \mathscr{A}$ is called *injective* if $\operatorname{Hom}_{\mathscr{A}}(-, I) : (\mathscr{A}, \mathscr{E}) \to \mathbf{Ab}$ is exact. There are *enough injectives* in $(\mathscr{A}, \mathscr{E})$ if for every $A \in \mathscr{A}$ there exists an injective I and an admissible monic $A \rightarrowtail I$.

THEOREM 2.4.2. *There are enough projectives and injectives in* $(\mathbf{Ban}, \mathscr{E}_{\max})$.

PROOF. This follows immediately from Corollary 1.5.4 and the classical facts that ℓ^1-spaces are $\mathscr{E}_{\max}$-projective (this is trivial) and that ℓ^∞-spaces are injective (this is almost trivial, use the Hahn-Banach Theorem). □

The projectives in $(\mathbf{Ban}, \mathscr{E}_{\max})$ are completely classified:

THEOREM 2.4.3 (Köthe [**Köt66**], Pełczyński [**Peł60**]). *A Banach space is $\mathscr{E}_{\max}$-projective if and only if it is isomorphic to a space $\ell^1(S)$ for some set S.* □

THEOREM 2.4.4. *The following classes of Banach spaces are $\mathscr{E}_{\max}$-injective:*

- *(i) ℓ^∞-spaces;*
- *(ii) L^∞-spaces;*
- *(iii) $C(K)$-spaces, K compact Hausdorff and extremely disconnected (the closure of every open set is open).* □

PROOF. That $\ell^\infty(S)$ is injective is a simple consequence of the Hahn-Banach theorem. That L^∞ of a probability space is injective is proved in [**BL00**, p.32]. Finally, the case of $C(K)$-spaces is dealt with in [**Day73**, Ch. VI, § 3, (2)]. □

REMARK 2.4.5. Every ℓ^∞-space is an L^∞-space (simply take the counting measure). By the spectral theorem, every L^∞-space is a $C(K)$-space, where K is compact Hausdorff and extremely disconnected.

REMARK 2.4.6. The $C(K)$-spaces with K compact Hausdorff and extremely disconnected are in fact characterized up to isometry by an *isometric* injectivity property by a result of Kelley, improving upon earlier results by Akilov, Goodner and Nachbin, see [**Day73**, Ch. V, § 4, Theorem 3; Ch. VI, § 3, (2)].

DEFINITION 2.4.7. A Banach space F *flat* if the functor $F \mathbin{\widehat{\otimes}} - : \mathbf{Ban} \to \mathbf{Ban}$ is exact (with respect to $\mathscr{E}_{\max}$).

The following result can be found in [**Wae05**, 3.3.11, 3.1.12]. It is the Banach space analog of the classical Lambek theorem in the theory of modules over a ring, see e.g. [**Lam99**, Theorem 4.9, p.125].

THEOREM 2.4.8. *For a Banach space F the following are equivalent:*

(i) The Banach space F is flat.
(ii) The dual space F^* is injective.
(iii) The bidual space F^{**} is flat.
(iv) The space F is an $\mathscr{L}_1$-space in the sense of Lindenstrauss-Pełczyński [**LP68**].

PROOF. To see the equivalence $(i) \Leftrightarrow (ii)$, let

$$E' \rightarrowtail E \twoheadrightarrow E''$$

be a short exact sequence in $\mathscr{E}_{\max}$. By the duality principle 2.3.5, the sequence

$$F \mathbin{\widehat{\otimes}} E' \to F \mathbin{\widehat{\otimes}} E \to F \mathbin{\widehat{\otimes}} E''$$

is exact if and only if

$$(F \mathbin{\widehat{\otimes}} E'')^* \to (F \mathbin{\widehat{\otimes}} E)^* \to (F \mathbin{\widehat{\otimes}} E')^*$$

is exact, but the latter sequence coincides with

$$\mathrm{Hom}_{\mathbf{Ban}}(E'', F^*) \to \mathrm{Hom}_{\mathbf{Ban}}(E, F^*) \to \mathrm{Hom}_{\mathbf{Ban}}(E', F^*)$$

which is exact if and only if F^* is injective. Therefore F is flat if and only if F^* is injective.

$(iii) \Rightarrow (ii)$ We have just proved that F^{**} is flat if and only if F^{***} is injective. Because F^* is always a complemented subspace in F^{***}, we see that flatness of F^{**} implies injectivity of F^*.

For the only non-obvious implication, $(ii) \Rightarrow (iii)$, one first notices that F^* is complemented in some L^∞-space and then applies the theory of Riesz lattices (in particular Kakutani's representation theorem for abstract L-spaces) in order to show that F^{**} is a complemented subspace of some L^1-space. By $(i) \Leftrightarrow (ii)$ and Theorem 2.4.4 any L^1-space is flat, and it is clear that a complemented subspace of a flat Banach space is again flat. The interested reader may consult [**Wae05**, 3.3.11] for further details.

The remaining equivalence $(iii) \Leftrightarrow (iv)$ is essentially a cheat: The $\mathscr{L}_1$-spaces of Lindenstrauss-Pełczyński are characterized by the fact that their bidual is complemented in some L^1-space, see [**Wae05**, 3.1.12]. □

Combining our information on projective, injective and flat Banach spaces we have the following characterizations:

COROLLARY 2.4.9.

(i) A Banach space is projective if and only if it is isomorphic to $\ell^1(S)$ for some set S.
(ii) A Banach is injective if and only if it is complemented in some L^∞-space.
(iii) A Banach space is flat if and only if its bidual is complemented in some L^1-space. □

2.5. The Pure Exact Structure and Flatness. The following results are analogs of the classical theory of pure exact sequences, a nice account of which may be found in [**Lam99**, §4J, pp.153ff.].

In Corollary 2.3.6 we introduced the *pure exact structure* of short sequences whose dual sequence splits:

$$\mathscr{E}_{\mathrm{pure}} = \{(m, e) \in \mathscr{E}_{\max} : (e^*, m^*) \in \mathscr{E}_{\min}\}.$$

Convention 2.5.1. Let us use the adjectives *maximal*, *pure* or *split* to design notions corresponding to the exact structures $\mathscr{E}_{\max}$, $\mathscr{E}_{\mathrm{pure}}$ or $\mathscr{E}_{\min}$, respectively—similarly for the corresponding adverbs.

Proposition 2.5.2. *A Banach space F is maximally flat if and only if every maximal epic $X \twoheadrightarrow F$ is pure.*

Proof. Suppose F is flat and that $e : X \twoheadrightarrow F$ is a maximal epic. By the duality principle 2.3.5, we have that $e^* : F^* \rightarrowtail X^*$ is a maximal monic. By Theorem 2.4.8, the space F^* is injective, hence e^* is a split monic. This implies that e is pure.

Conversely, suppose that every maximal epic $X \twoheadrightarrow F$ is pure. By Corollary 1.5.4 there is a set S and a maximal epic $e : \ell^1(S) \twoheadrightarrow F$. By assumption on F, the epic e is pure, so F^* is a direct summand of the maximally injective Banach space $\ell^\infty(S)$, so F^* is maximally injective and hence F is flat by Theorem 2.4.8. □

Corollary 2.5.3. *Consider a maximally exact sequence $\sigma = (A \rightarrowtail B \twoheadrightarrow C)$ of Banach spaces.*

- *(i) Assume B is maximally flat. Then σ is pure if and only if C is maximally flat.*
- *(ii) Assume C is maximally flat. Then A is maximally flat if and only if B is maximally flat.*

Proof. In both cases it suffices to consider the dual sequence

$$\sigma^* = (C^* \rightarrowtail B^* \twoheadrightarrow A^*).$$

To see (i), we apply Theorem 2.4.8 to see that B^* is maximally injective, hence σ^* is split exact if and only if C^* is maximally injective and this happens if and only if C is maximally flat, again by Theorem 2.4.8.

For (ii), the flatness of C implies that σ is pure by Proposition 2.5.2 and hence σ^* is a split extension of A^* by the injective C^* by Theorem 2.4.8. Therefore A^* is injective if and only if B^* is injective and we conclude by applying Theorem 2.4.8 once again. □

The following result states that every Banach space is flat with respect to the pure structure and that the pure exact structure is the largest exact structure on **Ban** such that every Banach space is flat.

Proposition 2.5.4. *For every Banach space X, the functor*

$$X \mathbin{\widehat{\otimes}} - : (\mathbf{Ban}, \mathscr{E}_{\mathrm{pure}}) \longrightarrow (\mathbf{Ban}, \mathscr{E}_{\max})$$

is exact. Conversely, if $\sigma = (A' \rightarrowtail A \twoheadrightarrow A'')$ is $\mathscr{E}_{\max}$-exact and for all Banach spaces X the sequence

$$X \mathbin{\widehat{\otimes}} A' \to X \mathbin{\widehat{\otimes}} A \to X \mathbin{\widehat{\otimes}} A''$$

is in $\mathscr{E}_{\max}$ then σ is in $\mathscr{E}_{\mathrm{pure}}$.

Proof. Let $\sigma = (A' \rightarrowtail A \twoheadrightarrow A'')$ be pure. Applying the dual space functor to

$$X \mathbin{\widehat{\otimes}} A' \to X \mathbin{\widehat{\otimes}} A \to X \mathbin{\widehat{\otimes}} A''$$

we get the sequence

$$(X \mathbin{\widehat{\otimes}} A'')^* \to (X \mathbin{\widehat{\otimes}} A)^* \to (X \mathbin{\widehat{\otimes}} A')^*,$$

which is isomorphic to the sequence

$$\operatorname{Hom}_{\mathbf{Ban}}(X, (A'')^*) \to \operatorname{Hom}_{\mathbf{Ban}}(X, A^*) \to \operatorname{Hom}_{\mathbf{Ban}}(X, (A')^*)$$

using the adjointness of $X \mathbin{\widehat{\otimes}} -$ and $\operatorname{Hom}_{\mathbf{Ban}}(X, -)$. The last sequence is exact because σ is pure so that

$$(A'')^* \to A^* \to (A')^*$$

is split exact. Because the dual space functor preserves and reflects exactness, we conclude that

$$X \mathbin{\widehat{\otimes}} A' \to X \mathbin{\widehat{\otimes}} A \to X \mathbin{\widehat{\otimes}} A''$$

is exact.

To prove the second part, assume that $\sigma = (A' \rightarrowtail A \twoheadrightarrow A'')$ is in $\mathscr{E}_{\max}$ and that

$$X \mathbin{\widehat{\otimes}} A' \rightarrowtail X \mathbin{\widehat{\otimes}} A \twoheadrightarrow X \mathbin{\widehat{\otimes}} A''$$

is in $\mathscr{E}_{\max}$ as well for all Banach spaces X. Taking $X = (A')^*$ and applying the dual space functor and adjointness of $(A')^* \mathbin{\widehat{\otimes}} -$ and $\operatorname{Hom}_{\mathbf{Ban}}((A')^*, -)$ we get the short exact sequence

$$\operatorname{Hom}_{\mathbf{Ban}}((A')^*, (A'')^*) \rightarrowtail \operatorname{Hom}_{\mathbf{Ban}}((A')^*, A^*) \twoheadrightarrow \operatorname{Hom}_{\mathbf{Ban}}((A')^*, (A')^*)$$

which in turn implies that the maximal epic

$$A^* \twoheadrightarrow (A')^*$$

has a right inverse and this suffices to conclude that σ is pure. □

Corollary 2.5.5. *Every dual Banach space is purely injective.*

Proof. Let X^* be a dual Banach space and let $\sigma = (A' \rightarrowtail A \twoheadrightarrow A'')$ be a pure exact sequence. We know that

$$X \mathbin{\widehat{\otimes}} A' \rightarrowtail X \mathbin{\widehat{\otimes}} A \twoheadrightarrow X \mathbin{\widehat{\otimes}} A''$$

is exact. Since

$$\operatorname{Hom}_{\mathbf{Ban}}(X \mathbin{\widehat{\otimes}} -, k) \cong \operatorname{Hom}_{\mathbf{Ban}}(-, \operatorname{Hom}_{\mathbf{Ban}}(X, k)) = \operatorname{Hom}_{\mathbf{Ban}}(-, X^*)$$

we conclude that

$$\operatorname{Hom}_{\mathbf{Ban}}(A'', X^*) \rightarrowtail \operatorname{Hom}_{\mathbf{Ban}}(A, X^*) \twoheadrightarrow \operatorname{Hom}_{\mathbf{Ban}}(A', X^*)$$

is exact, so X^* is purely injective. □

Corollary 2.5.6. *A Banach space X is purely injective if and only if the canonical inclusion $\iota : X \to X^{**}$ is a split monic.*

Proof. Since ι is a pure monic and X^{**} is purely injective, ι is split if and only if X is purely injective. □

Remark 2.5.7. An example of a purely injective space which is not a dual space is $L^1(\Omega, \mu)$ for some non-atomic measure space.

2.6. The Category of Waelbroeck Quotients. The reader should consult Chapter III before proceeding with this subsection.

The category **Ban** satisfies Hypothesis 1.2.1 of Chapter III.

DEFINITION 2.6.1. The category **Wael** is the abelian category $\mathfrak{C}\,(\mathbf{Ban})$ which is equivalent to the heart of the canonical t-structure on $\mathbf{D}\,(\mathbf{Ban})$.

The following result is suggested in [**BBD82**, 1.3.24].

THEOREM 2.6.2. *The category* **Wael** *is equivalent to Waelbroeck's category* **qBan** *of Banach quotients.*

PROOF. Waelbroeck's recognition theorem [**Wae05**, 2.1.25] for **qBan** applies: It suffices to notice that our construction of **Wael** has the desired properties and that Waelbroeck's pseudo-isomorphisms are precisely our quasi-isomorphisms. □

CHAPTER V

Derived Categories of Banach G-Modules

In this chapter we tie up some loose ends. We prove various equivalences of derived categories. Perhaps the most surprising fact is that for the maximal exact structure, the derived category of bounded above complexes of Banach G-modules is equivalent to the derived category of bounded above complexes of an explicit abelian category. Similarly, there is another abelian category whose derived category of bounded below complexes is equivalent to the derived category of bounded below complexes of Banach G-modules. In a more standard vein, we prove that the derived category of bounded above complexes of Banach G-modules is equivalent to the homotopy category of bounded above complexes of projective Banach G-modules and to the derived category of bounded above complexes of flat Banach G-modules with the pure exact structure.

Contents

1. Discrete Groups

1.1. Basics. Let G be a discrete group and let $G{-}\mathbf{Ban}_1$ be the category of functors $G \to \mathbf{Ban}_1$ and natural transformations, where the group G is considered as a category with one object. A more classical description of $G{-}\mathbf{Ban}_1$ is: objects are the isometric representations of G on Banach spaces and morphisms are the G-equivariant linear contractions of norm at most one. Let Σ be the class of bijective linear maps in $G{-}\mathbf{Ban}_1$.

Proposition 1.1.1. *The class Σ is a saturated and locally small multiplicative system satisfying both Ore conditions and cancellation. The localization $G{-}\mathbf{Ban}_1[\Sigma^{-1}]$ is the additive category $G{-}\mathbf{Ban}$ of isometric representations of G on Banach spaces and G-equivariant bounded linear maps.*

Proof. The proof of Proposition 1.6.1 and Corollary 1.6.2 of Chapter IV carry over to this more general setting without any pain. □

Remark 1.1.2. Consider the category $G{-}\mathbf{Ban}$. A morphism is monic if and only if it is injective. A morphism is epic if and only if it has dense range. A morphism is a kernel if and only if it is injective with closed range. A morphism is a cokernel if and only if it is surjective.

Theorem 1.1.3. *The category $G{-}\mathbf{Ban}$ is additive, has kernels, cokernels and the class $\mathscr{E}^G_{\max}$ of all kernel-cokernel pairs is an exact structure.*

Proof. By the explicit description of kernels and cokernels in $G-\mathbf{Ban}$, the elements in $\mathscr{E}^G_{\max}$ coincide with the short sequences of Banach G-modules whose underlying sequence of kG-modules is exact. We leave the details to the reader. $\square$

Let $\ell^1(G)$ be the group Banach algebra. Let $\ell^1(G)-\mathbf{Ban}$ be the category of left Banach $\ell^1(G)$-modules.

Proposition 1.1.4. *The categories $G-\mathbf{Ban}$ and $\ell^1(G)-\mathbf{Ban}$ are equivalent.* $\square$

Definition 1.1.5. Let M be a Banach G-module.

(i) The submodule of G-invariant vectors is given by
$$M^G = \{m \in M \,:\, gm = m \text{ for all } g \in G\}.$$
It may be identified with $\mathrm{Hom}_G(k, M)$ when k is considered as a trivial G-module.

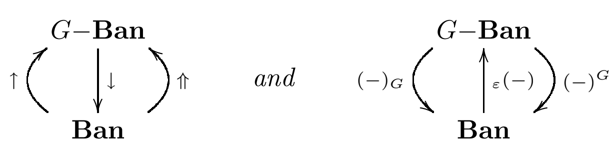
(ii) Similarly, we define the quotient module of G-coinvariant vectors by
$$M_G = M/\overline{\{gm - m \,:\, m \in M,\, g \in G\}}.$$

Notation 1.1.6. Let E be a Banach space. The *induced Banach G-module* is
$$\uparrow E = \ell^1(G)\,\widehat{\otimes}\,E$$
with the left G-action on the factor $\ell^1(G)$. The *coinduced Banach G-module* is
$$\Uparrow E := \mathrm{Hom}_{\mathbf{Ban}}(\ell^1(G), E)$$
with the action coming from the right action of G on $\ell^1(G)$.

Notation 1.1.7. If M is a G-module then its Banach dual $M^* = \mathrm{Hom}_{\mathbf{Ban}}(M, k)$ is also a (left) G-module if we equip it with the action $(gm^*)(m) = m^*(g^{-1}m)$. One has the identity
$$(M^*)^G \cong (M_G)^*$$
by elementary duality theory, see [**Rud91**, 4.6ff].

More generally, $\mathrm{Hom}_{\mathbf{Ban}}(M, N)$ becomes a Banach G-module by setting
$$(gf)(m) = g(f(g^{-1}m)).$$
Using this notation, $\mathrm{Hom}_G(M, N) = (\mathrm{Hom}_{\mathbf{Ban}}(M, N))^G$.

Theorem 1.1.8 (Fundamental Adjunctions). *Let $\downarrow\colon G-\mathbf{Ban} \to \mathbf{Ban}$ be the forgetful functor and let ${}_\varepsilon(-) : \mathbf{Ban} \to G-\mathbf{Ban}$ be the trivial module functor. There are two adjoint triples of functors*



that is to say $\uparrow$ is left adjoint to $\downarrow$ and $\downarrow$ is left adjoint to $\Uparrow$.

Proof. This is completely straightforward. To exemplify the argument, we check that induction $\uparrow$ is left adjoint to the forgetful functor $\downarrow$. Let M be a Banach G-module and E be a Banach space. Let $1 \in \ell^1(G)$ be the identity and let
$$\alpha_M : \uparrow\downarrow M \to M,\ f \otimes m \mapsto fm \qquad \text{and} \qquad \beta_E : E \to \downarrow\uparrow E,\ e \mapsto 1 \otimes e.$$
Now observe that
$$(\downarrow \alpha_M) \circ \beta_{\downarrow M} = \mathrm{id}_{\downarrow M} \qquad \text{and} \qquad \alpha_{\uparrow E} \circ (\uparrow \beta_E) = \mathrm{id}_{\uparrow E},$$
so by general nonsense we indeed have an adjunction, see [**ML98**, Chapter IV.1]. $\square$

COROLLARY 1.1.9. *Equip* $G-\mathbf{Ban}$ *and* $\mathbf{Ban}$ *with the maximal exact structures.*

(i) The forgetful functor $\downarrow$*, induction* $\uparrow$ *and coinduction* $\Uparrow$ *are exact.*
(ii) The trivial module functor ${}_{\varepsilon}(-)$ *is exact.*

PROOF. Since left adjoints preserve cokernels and right adjoints preserve kernels, we have that the forgetful functor and the trivial module functor preserve kernel-cokernel pairs—they have both a right adjoint and a left adjoint. That coinduction (induction) is exact follows from the fact that $\ell^1(G)$ is projective (flat) as a Banach space, see Corollary 2.4.9 of Chapter IV. □

COROLLARY 1.1.10. *There are enough projectives and injectives in* $(G-\mathbf{Ban}, \mathscr{E}^G_{\max})$.

PROOF. Let us prove that there are enough injectives. Let M be a Banach G-module and let B be the closed unit ball of its dual. Evaluation yields an isometric embedding $e : \downarrow M \to \ell^\infty(B)$ and notice that $\ell^\infty(B)$ is an injective Banach space. Since coinduction is a right adjoint, it preserves kernels and because its left adjoint is the (exact) forgetful functor, coinduction preserves injectives, so

$$\Uparrow(e) : \Uparrow\downarrow M \to \Uparrow \ell^\infty(B)$$

is an admissible monic into an injective Banach G-module. The unit of the adjunction is an admissible monic as well, since it is a split monic in $\mathbf{Ban}$, hence the composition

$$M \to \Uparrow\downarrow M \to \Uparrow \ell^\infty(B)$$

is an admissible monic into an injective Banach G-module. □

Let $\mathscr{P}^G_{\max}$ be the full subcategory of $\mathscr{E}^G_{\max}$-projectives in $G-\mathbf{Ban}$ and let $\mathscr{I}^G_{\max}$ be the full subcategory of $\mathscr{E}^G_{\max}$-injectives in $G-\mathbf{Ban}$.

COROLLARY 1.1.11.

(i) The inclusion $\mathscr{P}^G_{\max} \subset G-\mathbf{Ban}$ *induces an equivalence of triangulated categories*

$$\mathbf{K}^-(\mathscr{P}^G_{\max}) \xrightarrow{\cong} \mathbf{D}^-(G-\mathbf{Ban}, \mathscr{E}^G_{\max}).$$

In particular, the latter has small Hom*-sets and left derived functors exist.*

(ii) The inclusion $\mathscr{I}^G_{\max} \subset G-\mathbf{Ban}$ *induces an equivalence of triangulated categories*

$$\mathbf{K}^+(\mathscr{I}^G_{\max}) \xrightarrow{\cong} \mathbf{D}^+(G-\mathbf{Ban}, \mathscr{E}^G_{\max}).$$

In particular, the latter has small Hom*-sets and right derived functors exist.*

1.2. The Canonical t-Structures. The reader unacquainted with abstract truncation should consult Chapter III before reading this section. We will equip $G-\mathbf{Ban}$ with the exact structure $\mathscr{E}^G_{\max}$ throughout.

DEFINITION 1.2.1 (Truncation Functors). Let E be in $\mathbf{Ch}(G-\mathbf{Ban})$. The *left truncation functors* are given by

$$\tau_\ell^{\leq 0} E = (\cdots \to E^{-2} \to E^{-1} \to \operatorname{Ker} d_E^0 \to 0 \to \cdots)$$

and

$$\tau_\ell^{\geq 0} E = (\cdots \to 0 \to \operatorname{Coim} d_E^{-1} \to E^0 \to E^1 \to \cdots),$$

while the *right truncation functors* are given by

$$\tau_r^{\leq 0} = (\cdots \to E^{-1} \to E^{-0} \to \operatorname{Im} d_E^0 \to 0 \to \cdots)$$

and

$$\tau_r^{\geq 0} = (\cdots \to 0 \to \operatorname{Coker} d_E^{-1} \to E^0 \to E^1 \to \cdots).$$

DEFINITION 1.2.2. The *left t-structure* on $\mathbf{D}(G{-}\mathbf{Ban})$ is

$$(\mathbf{D}_\ell^{\leq 0}, \mathbf{D}_\ell^{\geq 0})$$

and the *right t-structure* on $\mathbf{D}(G{-}\mathbf{Ban})$ is

$$(\mathbf{D}_r^{\leq 0}, \mathbf{D}_r^{\geq 0}).$$

The corresponding *hearts* are denoted by

$$\mathfrak{C}_\ell = \mathbf{D}_\ell^{\leq 0} \cap \mathbf{D}_\ell^{\geq 0} \qquad \text{and} \qquad \mathfrak{C}_r = \mathbf{D}_r^{\leq 0} \cap \mathbf{D}_r^{\geq 0},$$

or $\mathfrak{C}_\ell(G)$ and $\mathfrak{C}_r(G)$ if the group needs to be specified. The *inclusion functors* are

$$i_\ell : G{-}\mathbf{Ban} \to \mathfrak{C}_\ell, M \mapsto (0 \to M) \qquad \text{and} \qquad i_r : G{-}\mathbf{Ban} \to \mathfrak{C}_r : M \mapsto (M \to 0).$$

NOTATION 1.2.3. As usual, we put for $n \in \mathbb{Z}$

$$\tau_\ell^{\leq n} = \Sigma^{-n} \circ \tau_\ell^{\leq 0} \circ \Sigma^n \qquad \text{and} \qquad \tau_\ell^{\geq n} = \Sigma^{-n} \circ \tau_\ell^{\geq 0} \circ \Sigma^n$$

as well as

$$\tau_r^{\leq n} = \Sigma^{-n} \circ \tau_r^{\leq 0} \circ \Sigma^n \qquad \text{and} \qquad \tau_r^{\geq n} = \Sigma^{-n} \circ \tau_r^{\geq 0} \circ \Sigma^n.$$

The corresponding homological functors are

$$H_\ell^0 = \tau_\ell^{\geq 0}\tau_\ell^{\leq 0} \cong \tau_\ell^{\leq 0}\tau_\ell^{\geq 0} \qquad \text{and} \qquad H_r^0 = \tau_r^{\geq 0}\tau_r^{\leq 0} \cong \tau_r^{\leq 0}\tau_r^{\geq 0}$$

and we put

$$H_\ell^n = H_\ell^0 \circ \Sigma^n \qquad \text{and} \qquad H_r^n = H_r^0 \circ \Sigma^n.$$

THEOREM 1.2.4. *The hearts $\mathfrak{C}_\ell(G)$ and $\mathfrak{C}_r(G)$ are abelian categories and the functors*

$$H_\ell^0 : \mathbf{D}(G{-}\mathbf{Ban}) \to \mathfrak{C}_\ell(G) \qquad \textit{and} \qquad H_r^0 : \mathbf{D}(G{-}\mathbf{Ban}) \to \mathfrak{C}_r(G)$$

are conservative homological functors. Moreover, the inclusion functors induce equivalences of triangulated categories

$$\mathbf{D}^-(G{-}\mathbf{Ban}) \to \mathbf{D}^-(\mathfrak{C}_\ell(G)) \qquad \textit{and} \qquad \mathbf{D}^+(G{-}\mathbf{Ban}) \to \mathbf{D}^+(\mathfrak{C}_r(G)).$$

PROOF. This follows immediately from Theorem 2.1.3 and Theorem 2.2.10 of Chapter III. ☐

1.3. The G-Tensor Product and Flatness.

DEFINITION 1.3.1. Let M and N be Banach G-modules then the projective tensor $M \mathbin{\widehat{\otimes}} N$ becomes a Banach G-module by acting diagonally

$$g(m \otimes n) = gm \otimes gn.$$

We define the projective G-tensor product by setting

$$M \mathbin{\widehat{\otimes}}_G N := (M \mathbin{\widehat{\otimes}} N)_G.$$

DEFINITION 1.3.2. Let $\mathscr{E}$ be an exact structure on $G{-}\mathbf{Ban}$. A Banach G-module X is called *flat* if the functor

$$X \mathbin{\widehat{\otimes}}_G - : (G{-}\mathbf{Ban}, \mathscr{E}) \to (\mathbf{Ban}, \mathscr{E}_{\max})$$

is exact.

There are *enough $\mathscr{E}$-flat objects* if for every $M \in G{-}\mathbf{Ban}$ there exists an $\mathscr{E}$-flat object and an $\mathscr{E}$-admissible epic $M \twoheadrightarrow F$.

REMARK 1.3.3. Let M and N be Banach G-modules. By duality theory [**Rud91**, 4.6ff], there is a natural isomorphism

$$(M \mathbin{\widehat{\otimes}_G} N)^* \cong ((M \mathbin{\widehat{\otimes}} N)^*)^G$$

and there are natural isomorphisms

$$(M \mathbin{\widehat{\otimes}} N)^* \cong \mathrm{Hom}_{\mathbf{Ban}}(M, N^*),$$

and

$$\mathrm{Hom}_{\mathbf{Ban}}(M, N^*)^G \cong \mathrm{Hom}_G(M, N^*),$$

so there are natural isomorphisms

$$(M \mathbin{\widehat{\otimes}_G} N)^* \cong \mathrm{Hom}_G(M, N^*) \cong \mathrm{Hom}_G(N, M^*).$$

1.4. The Pure Exact Structure and Flatness.

DEFINITION 1.4.1. Let $\mathscr{E}^G_{\mathrm{min}}$ be the split exact structure on $G{-}\mathbf{Ban}$.

The *pure exact structure* on $G{-}\mathbf{Ban}$ is defined to be

$$\mathscr{E}^G_{\mathrm{pure}} = \{(m, e) \in \mathscr{E}^G_{\mathrm{max}} \,:\, (e^*, m^*) \in \mathscr{E}^G_{\mathrm{min}}\}$$

that is to say the exact structure of short sequences of G-modules whose dual sequence splits, see Appendix B.

PROPOSITION 1.4.2. *Let $\mathscr{E}$ be an exact structure on $G{-}\mathbf{Ban}$. Consider the duality functor*

$$(-)^* : (G{-}\mathbf{Ban}, \mathscr{E}) \longrightarrow (G{-}\mathbf{Ban}, \mathscr{E}).$$

(i) Assume $(-)^*$ is exact. If F is $\mathscr{E}$-flat then F^* is $\mathscr{E}$-injective.
(ii) Assume $(-)^*$ reflects exactness. If F^* is $\mathscr{E}$-injective then F is $\mathscr{E}$-flat.

PROOF. Let $M' \rightarrowtail M \twoheadrightarrow M''$ be $\mathscr{E}$-exact. Consider the sequence of Banach spaces

$$(F \mathbin{\widehat{\otimes}_G} M'')^* \to (X \mathbin{\widehat{\otimes}_G} M)^* \to (F \mathbin{\widehat{\otimes}_G} M')^*$$

and observe that by Remark 1.3.3 it is isomorphic to

$$\mathrm{Hom}_G(M'', F^*) \to \mathrm{Hom}_G(M, F^*) \to \mathrm{Hom}_G(M', F^*).$$

To prove (i), assume F is $\mathscr{E}$-flat, then the first sequence is exact because $(-)^*$ is exact, hence the second sequence is exact, so F^* is $\mathscr{E}$-injective.

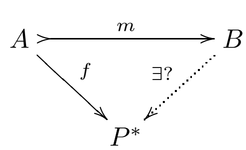
To prove (ii), assume that F^* is $\mathscr{E}$-injective, so the second sequence is exact, so

$$F \mathbin{\widehat{\otimes}_G} M' \to F \mathbin{\widehat{\otimes}_G} M \to F \mathbin{\widehat{\otimes}_G} M''$$

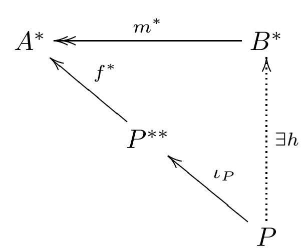
is exact because $(-)^*$ reflects exactness and therefore F is flat. □

PROPOSITION 1.4.3. *Let $\mathscr{E}$ be an exact structure on $G{-}\mathbf{Ban}$ such that the dual of an admissible monic is an admissible epic. If P is projective then P^* is injective.*

PROOF. The point is that the canonical inclusion $\iota = \iota_{P^*} : P^* \to P^{***}$ has ι_P^* as a left inverse. Every extension problem



yields



by dualization and h exists because P is $\mathscr{E}$-projective. We claim that $h^*\iota_B$ is a solution of the extension problem on the left. Indeed, since $\iota : \mathrm{id}_{G-\mathbf{Ban}} \Rightarrow (-)^{**}$ is a natural transformation and $\iota_P^*\iota_{P^*} = \mathrm{id}_{P^*}$, we have

$$f = \iota_P^*(\iota_{P^*}f) = \iota_P^*(f^{**}\iota_A) = (f^*\iota_P)^*\iota_A = (m^*h)^*\iota_A = h^*m^{**}\iota_A = (h^*\iota_B)m$$

as we claimed, hence P^* is injective. □

Proposition 1.4.4. *Suppose that $(-)^*$ is $\mathscr{E}$-exact. If a Banach G-module F is $\mathscr{E}$-flat then every admissible epic $M \twoheadrightarrow F$ is pure. The converse is true if $(-)^*$ preserves and reflects exactness and if there are enough $\mathscr{E}$-flat objects.*

Proof. Suppose that F is $\mathscr{E}$-flat and let $e : M \twoheadrightarrow F$ be an admissible epic. We know that F^* is $\mathscr{E}$-injective and that $e^* : F^* \rightarrowtail M^*$ is $\mathscr{E}$-admissible, so it splits and hence e is a pure epic.

Conversely, assume that there are enough $\mathscr{E}$-flat objects and that M has the property that every $\mathscr{E}$-admissible epic onto it is pure. Choose an $\mathscr{E}$-flat object F and an $\mathscr{E}$-admissible epic $e : F \twoheadrightarrow M$. By assumption on M we have that $\mathscr{E}$ is pure. Therefore $e^* : M^* \rightarrowtail F^*$ is a split monic into an $\mathscr{E}$-injective, hence M^* is $\mathscr{E}$-injective, so M is $\mathscr{E}$-flat by Proposition 1.4.2 (ii). □

Corollary 1.4.5. *Suppose that $(-)^*$ preserves and reflects exactness with respect to $\mathscr{E}$. Consider a short exact sequence $\sigma = (A \rightarrowtail B \twoheadrightarrow C)$.*

(i) Assume that B is $\mathscr{E}$-flat. Then σ is pure if and only if C is $\mathscr{E}$-flat.
(ii) Assume that C is $\mathscr{E}$-flat. Then A is $\mathscr{E}$-flat if and only if B is $\mathscr{E}$-flat.

Proof. In both cases it suffices to consider the dual sequence

$$\sigma^* = (C^* \rightarrowtail B^* \twoheadrightarrow A^*).$$

For (i), notice that B^* is $\mathscr{E}$-injective, hence σ^* is split exact if and only if C^* is $\mathscr{E}$-injective and this happens if and only if C is $\mathscr{E}$-flat.

For (ii), notice that the flatness of C implies that σ is pure and hence that σ^* is split exact. Therefore σ^* is a split extension of A^* by the $\mathscr{E}$-injective C^*, so A^* is $\mathscr{E}$-injective if and only if B^* is injective and hence A is $\mathscr{E}$-flat if and only if B is $\mathscr{E}$-flat. □

Corollary 1.4.6. *Let $\mathscr{E}$ be an exact structure on $G-$**Ban** and suppose that $(-)^*$ preserves and reflects exactness with respect to $\mathscr{E}$. Let $\mathscr{F}$ be the full subcategory of $\mathscr{E}$-flat objects.*

(i) The restriction of $\mathscr{E}$ to $\mathscr{F}$ consists of purely exact sequences.
*(ii) The category $\mathscr{F}$ is closed under extensions in $G-$**Ban**: If*

$$F' \rightarrowtail M \twoheadrightarrow F''$$

is $\mathscr{E}$-exact and F', F'' are $\mathscr{E}$-flat, then M is $\mathscr{E}$-flat. In particular, the pure exact sequences form an exact structure on $\mathscr{F}$.

Proof. Point (i) and the first part of (ii) are only restatements of the previous corollary, while the second part of (ii) follows from the first part by a direct verification of the axioms[1]. □

In presence of enough $\mathscr{E}$-flat objects we can realize the derived category of right bounded complexes $\mathbf{D}^-(G-\mathbf{Ban}, \mathscr{E})$ using the derived category of $\mathscr{E}$-flat objects. Functors which restrict to purely exact functors on flat objects can be derived using flat resolutions. The main example is of course $M \,\widehat{\otimes}_G -$ for any Banach G-module M as we will see shortly.

[1]If $(\mathscr{A}, \mathscr{E})$ is an exact category and $\mathscr{B} \subset \mathscr{A}$ is a full additive subcategory which is closed under extensions then the restriction of $\mathscr{E}$ to $\mathscr{B}$ is an exact structure on $\mathscr{B}$.

COROLLARY 1.4.7. *Let $\mathscr{E}$ be an exact structure on G*$-$**Ban** *and let $\mathscr{F}$ be the full subcategory of $\mathscr{E}$-flat objects equipped with the pure exact structure. Suppose moreover that there are enough $\mathscr{E}$-flat objects in $(G-\mathbf{Ban},\mathscr{E})$ and that the duality functor $(-)^*$ preserves and reflects exactness.*

(i) The inclusion $\mathscr{F}\subset G-\mathbf{Ban}$ induces an equivalence

$$\mathbf{D}^-(\mathscr{F})\xrightarrow{\cong}\mathbf{D}^-(G-\mathbf{Ban},\mathscr{E})$$

of the derived categories of bounded above complexes.

(ii) Suppose that $T: G-\mathbf{Ban}\to(\mathscr{A},\mathscr{S})$ is a functor to an exact category and assume that the restriction of T to $\mathscr{F}$ transforms purely exact sequences to exact sequences in $\mathscr{A}$. Then for every $M\in\mathbf{Ch}^-(G-\mathbf{Ban})$ every $\mathscr{E}$-quasi-isomorphism $F\to M$ with $F\in\mathbf{Ch}^-(\mathscr{F})$ induces an isomorphism

$$T(F)\xrightarrow{\sim}\mathbf{L}^-T(M),$$

so the left derived functor of T can be computed using flat resolutions.

PROOF. Part (i) is a direct consequence of [**Kel96**, Theorem 12.1] and part (ii) is a consequence of Lemma 15.4 in *loc. cit.* □

REMARK 1.4.8. By Proposition 2.3.5 of Chapter IV, the duality functor $(-)^*$ preserves and reflects $\mathscr{E}^G_{\max}$-exactness, in particular, all results in this section apply to $(G-\mathbf{Ban},\mathscr{E}^G_{\max})$.

The following result implies that the pure exact structure on $G-\mathbf{Ban}$ is the largest exact structure $\mathscr{E}$ for which all Banach G-modules are flat.

COROLLARY 1.4.9. *For every Banach G-module M the functor*

$$M\mathbin{\widehat{\otimes}_G}-:(G-\mathbf{Ban},\mathscr{E}^G_{\mathrm{pure}})\longrightarrow(\mathbf{Ban},\mathscr{E}_{\mathrm{pure}})$$

is exact. Conversely, if $\sigma=(N'\rightarrowtail N\twoheadrightarrow N'')$ is $\mathscr{E}^G_{\max}$-exact and for all Banach G-modules M the sequence of Banach spaces

$$M\mathbin{\widehat{\otimes}_G}N'\rightarrowtail M\mathbin{\widehat{\otimes}_G}N\twoheadrightarrow M\mathbin{\widehat{\otimes}_G}N''$$

is $\mathscr{E}_{\max}$-exact then σ is pure.

PROOF. Let $\sigma=(N'\rightarrowtail N\twoheadrightarrow N'')$ be pure and let M be any Banach G-module. By Remark 1.3.3, applying the dual space functor $(-)^*$ to the sequence

$$M\mathbin{\widehat{\otimes}_G}N'\to M\mathbin{\widehat{\otimes}_G}N\to M\mathbin{\widehat{\otimes}_G}N''$$

yields

$$\mathrm{Hom}_G(M,(N'')^*)\to\mathrm{Hom}_G(M,N^*)\to\mathrm{Hom}_G(M,(N')^*).$$

Now the last sequence is split exact in **Ban** because σ is assumed to be pure and $\mathrm{Hom}_G(M,-)$ is additive, so $M\mathbin{\widehat{\otimes}_G}-$ is exact with respect to the pure structures.

To prove the converse, assume that $N'\rightarrowtail N\twoheadrightarrow N''$ is a maximally exact sequence of Banach G-modules and that for all Banach G-modules M the sequence

$$M\mathbin{\widehat{\otimes}_G}N'\to M\mathbin{\widehat{\otimes}_G}N\to M\mathbin{\widehat{\otimes}_G}N''$$

is a maximally exact sequence of Banach spaces. Take $M=(N')^*$ to conclude that

$$\mathrm{Hom}_G((N')^*,(N'')^*)\rightarrowtail\mathrm{Hom}_G((N')^*,N^*)\twoheadrightarrow\mathrm{Hom}_G((N')^*,(N')^*)$$

is exact by Remark 1.3.3. It follows that $N^*\to(N')^*$ has a left inverse, so σ^* is split exact. □

2. Some Remarks on Topological Groups

There are several difficulties in extending the results of the previous section to non-discrete groups. First, if G is not locally compact we lack an adequate group algebra in the Banach setting. If G is locally compact, one may work with the convolution algebra $L^1(G)$, but there is the difficulty that this algebra has no unit if G is non-discrete, so that the category of $L^1(G)$-modules splits up into various subcategories. The picture is not quite clear to the present author.

In the introduction, we decided to work with the category of strongly continuous isometric representations, simply because this is technically the easiest to deal with. For G locally compact, this category is equivalent to the category of *essential Banach $L^1(G)$-modules* by a theorem of Johnson [**Joh72**]—recall that for a Banach algebra A (with two-sided bounded approximate unity) the essential A-modules are given by the class of Banach A-modules M satisfying $A \widehat{\otimes}_A M \cong M$. Incidentally, the functor $L^1(G) \widehat{\otimes}_{L^1(G)} -$ is Monod's *maximal continuous submodule functor* $\mathcal{C}_G$.

Dually, one should look at the category of *complete A-modules,* (in the terminology of [**CLM79**, Chapter 2]) that is to say the category of A-modules M satisfying $\mathrm{Hom}_A(A, M) \cong M$—this is most likely the category that should replace Burger-Monod's coefficient G-modules in a more conceptual framework.

2.1. A Result on Enough Injectives. As in the introduction, let G be a Hausdorff topological group and let $G{-}\mathbf{Ban}$ be the category of strongly continuous isometric representations of G on Banach spaces with bounded G-equivariant maps as morphisms. Let $\mathscr{E}^G_{\max}$ be the exact structure consisting of all kernel-cokernel pairs.

THEOREM 2.1.1. *There are enough injectives in* $(G{-}\mathbf{Ban}, \mathscr{E}^G_{\max})$.

PROOF. We have proved in the introduction that the forgetful functor

$$\downarrow\colon (G{-}\mathbf{Ban}, \mathscr{E}^G_{\max}) \to (\mathbf{Ban}, \mathscr{E}_{\max})$$

has $C^{\mathrm{lu}}_b(G, -)$ as a right adjoint. Since the forgetful functor is exact, $C^{\mathrm{lu}}_b(G, -)$ preserves injectives and kernels, and since there are enough injectives in $(\mathbf{Ban}, \mathscr{E}_{\max})$, we conclude that the same is the case in $(G{-}\mathbf{Ban}, \mathscr{E}^G_{\max})$. ☐

COROLLARY 2.1.2. *The derived category* $\mathbf{D}^+(G{-}\mathbf{Ban}, \mathscr{E}^G_{\max})$ *has small* Hom*-sets and right derived functors are defined on all bounded below complexes.* ☐

REMARK 2.1.3. The author does not know whether there are enough projectives in $(G{-}\mathbf{Ban}, \mathscr{E}^G_{\max})$, because he was unable to decide whether the forgetful functor has a left adjoint.

REMARK 2.1.4. By using the right derived functor

$$\mathbf{R}^+(-)^G : \mathbf{D}^+(G{-}\mathbf{Ban}, \mathscr{E}^G_{\max}) \to \mathbf{D}^+(\mathbf{Ban}, \mathscr{E}_{\max})$$

and the left t-structure on the latter, one can now prove the existence and uniqueness of a classical derived functor $R^i(-)^G : (G{-}\mathbf{Ban}, \mathscr{E}^G_{\max}) \to \mathbf{qBan}$ as in the introduction. We leave the details to the interested reader.

Part 3

Appendices

APPENDIX A

Mapping Cones, Homotopy Push-Outs, Mapping Cylinders

The mapping cone construction is at the very basis of our work. Particular cases are homotopy push-outs and mapping cylinders. We collect the basic facts on these constructions, valid over any additive category.

Contents

1. The Mapping Cone

Throughout this section we work with complexes over an additive category $\mathscr{A}$.

1.1. Definition. Given a chain map $A \xrightarrow{f} B$, its *mapping cone* is defined by

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n \qquad \text{and} \qquad d^n = \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}.$$

It gives rise to a *strict triangle*

$$A \xrightarrow{f} B \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \operatorname{cone}(f) \xrightarrow{[1\ 0]} A[1]$$

which is used to define the triangulated structure on $\mathbf{K}(\mathscr{A})$. Given a morphism of chain maps

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{f'} & B' \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{cone}(f) & \xrightarrow{[1\ 0]} & A[1] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\left[\begin{smallmatrix} \alpha[1] & 0 \\ 0 & \beta \end{smallmatrix}\right]} & & \downarrow{\scriptstyle\alpha[1]} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{cone}(f') & \xrightarrow{[1\ 0]} & A'[1] \end{array}$$

in $\mathbf{Ch}(\mathscr{A})$. In particular, the strict triangle construction is functorial.

If f and g are homotopic maps then $\operatorname{cone}(f)$ is isomorphic to $\operatorname{cone}(g)$ in $\mathbf{Ch}(\mathscr{A})$, however the isomorphism depends on the choice of a homotopy in general. Notice also that the composite of two consecutive morphisms in a strict triangle is homotopic to zero.

Example 1.1.1. The mapping cone of the identity $A \xrightarrow{1} A$ is null-homotopic, a contracting homotopy is given by $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Assuming that idempotents split in $\mathscr{A}$, we conclude that the mapping cone of the identity is split exact.

Example 1.1.2. The mapping cone of $A \xrightarrow{0} B$ is the complex $A[1] \oplus B$.

Finally, the mapping cone yields the following push-out diagram in $\mathbf{Ch}(\mathscr{A})$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \big\downarrow \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] & \text{PO} & \big\downarrow \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \\ \operatorname{cone}(1_A) & \xrightarrow[\left[\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right]]{} & \operatorname{cone}(f) \end{array}$$

as is easily checked.

1.2. The Homotopy Push-Out. In general, a commutative square in a triangulated category

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow f' & & \big\downarrow g \\ B' & \xrightarrow{g'} & C \end{array}$$

is called a *homotopy push-out diagram* if there exists a morphism $C \xrightarrow{\delta} A[1]$ such that

$$A \xrightarrow{\left[\begin{smallmatrix} f \\ -f' \end{smallmatrix}\right]} B \oplus B' \xrightarrow{[\,g \;\, g'\,]} C \xrightarrow{\delta} A[1]$$

is a distinguished triangle.

Remark 1.2.1. By **[TR 1]** a homotopy push-out of two morphisms $f : A \to B$ and $f' : A \to B'$ exists—simply embed $\left[\begin{smallmatrix} f \\ -f' \end{smallmatrix}\right]$ into a distinguished triangle.

Remark 1.2.2. It follows from the five lemma for triangulated categories that the homotopy push-out C of the morphisms f and f' is unique up to non-unique isomorphism.

Given two chain maps $f : A \to B$ and $f' : A \to B'$, we define the complex

$$\operatorname{hpo}(f, f') = \operatorname{cone}\left[\begin{smallmatrix} f \\ -f' \end{smallmatrix}\right]$$

or, explicitly,

$$\operatorname{hpo}(f, f')^n = A^{n+1} \oplus B^n \oplus B'^n \qquad \text{and} \qquad d^n = \begin{bmatrix} -d_A^{n+1} & 0 & 0 \\ f^{n+1} & d_B^n & 0 \\ -f'^{n+1} & 0 & d_{B'}^n \end{bmatrix}.$$

It follows that we have a strict triangle

$$A \xrightarrow{\left[\begin{smallmatrix} f \\ -f' \end{smallmatrix}\right]} B \oplus B' \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} \operatorname{hpo}(f, f') \xrightarrow{[\,1 \;\, 0 \;\, 0\,]} A[1].$$

and a homotopy push-out diagram in $\mathbf{K}(\mathscr{A})$:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ f' \big\downarrow & & \big\downarrow \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \\ B' & \xrightarrow{\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]} & \operatorname{hpo}(f, f') \end{array}$$

Note that this is not in general a push-out diagram in $\mathbf{Ch}(\mathscr{A})$.

REMARK 1.2.3. It can be checked either by direct inspection or by appealing to the weak cokernel property of distinguished triangles that this diagram is a weak push-out in the sense that it has the mapping property of a push-out, but without uniqueness.

REMARK 1.2.4. For the sake of symmetry it is customary to "place A in the middle", that is, one puts

$$(B \cup_A B')^n = B^n \oplus A^{n+1} \oplus B'^n \qquad \text{and} \qquad d^n = \begin{bmatrix} d_B^n & f^{n+1} & 0 \\ 0 & d_A^{n+1} & 0 \\ 0 & -f'^{n+1} & d_{B'}^n \end{bmatrix},$$

see for instance [**TT90**, 1.1.2], but this blurs the relation to the mapping cone construction.

1.3. The Mapping Cylinder. The most famous instance of a homotopy push-out is the mapping cylinder construction. For a chain map $f : A \to B$ one puts

$$\mathrm{cyl}\,(f) = \mathrm{hpo}\,(1_A, f),$$

or, explicitly,

$$\mathrm{cyl}\,(f)^n = A^{n+1} \oplus B^n \oplus A^n \qquad \text{and} \qquad d^n = \begin{bmatrix} -d_A^{n+1} & 0 & 0 \\ 1_{A^{n+1}} & d_B^n & 0 \\ -f^{n+1} & 0 & d_A^n \end{bmatrix}.$$

The important technical feature of the mapping cylinder is that it is homotopy equivalent to B. We refer the reader to [**Wei94**, Chapter 1.5] for further information on the cone and the cylinder.

1.4. The Eckmann-Hilton Duals. The constructions of the previous section are subject to Eckmann-Hilton dualization.

In topology, the Eckmann-Hilton dual of the mapping cone is the homotopy fiber and the dual of the mapping cylinder is the mapping path space. In the category of chain complexes, the suspension is an automorphism, and Σ^{-1} can be thought of as the loop-space functor.

Using this, one can construct the homotopy fiber simply as

$$\Sigma^{-1}\mathrm{cone}\,(f)$$

and then go on to define a homotopy pull-back square, a homotopy pull-back construction and a mapping path space construction. We leave it to the reader to figure out the explicit formulae and to dualize the previous sections.

APPENDIX B

Pull-Back of Exact Structures

In order to recognize exact structures, the following remark is quite convenient:

THEOREM. *Let* $(\mathscr{A}, \mathscr{E})$ *and* $(\mathscr{A}', \mathscr{E}')$ *be exact categories and let* $F : \mathscr{A} \to \mathscr{A}'$ *be an exact functor. Let* $\mathscr{E}'' \subset \mathscr{E}'$ *be another exact structure on* $\mathscr{A}'$*. Put*

$$F^* \mathscr{E}'' = \mathscr{E} \cap F^{-1}(\mathscr{E}'').$$

The category $(\mathscr{A}, F^* \mathscr{E}'')$ *is exact.*

DEFINITION. The exact structure $F^* \mathscr{E}''$ is called the exact structure on $\mathscr{A}$ obtained by *pull-back via* F or simply the *pull-back exact structure*.

PROOF. Let us check the axioms. By definition, $F^* \mathscr{E}''$ is closed under isomorphisms, consists of kernel-cokernel pairs, and identity morphisms are both admissible monics and epics in $F^* \mathscr{E}''$. Since F is exact, it is clear that the classes of admissible monics and the classes of admissible epics are closed under composition. Finally, F preserves push-outs under admissible monics, so the class of admissible monics in $F^* \mathscr{E}''$ is closed under push-outs. Dually, the class of admissible epics is closed under pull-backs. □

EXAMPLE. The duality functor $F = (-)^* : (G\text{--}\mathbf{Ban}, \mathscr{E}^G_{\text{max}}) \to (G\text{--}\mathbf{Ban}, \mathscr{E}^G_{\text{max}})$ is exact. The pure exact structure on $G\text{--}\mathbf{Ban}$ is the pull-back

$$\mathscr{E}^G_{\text{pure}} = F^* \mathscr{E}^G_{\text{min}}$$

of the split exact structure on $G\text{--}\mathbf{Ban}$.

EXAMPLE. The forgetful functor $F = \downarrow : (G\text{--}\mathbf{Ban}, \mathscr{E}^G_{\text{max}}) \to (\mathbf{Ban}, \mathscr{E}_{\text{max}})$ is exact. The *relative exact structure* on $G\text{--}\mathbf{Ban}$ is the pull-back

$$\mathscr{E}^G_{\text{rel}} = F^* \mathscr{E}_{\text{min}}$$

of the split exact structure on $\mathbf{Ban}$.

APPENDIX C

Model Categories

Besides triangulated categories, Quillen's theory of model categories [**Qui67**] provides another standard framework of homological (or rather homotopical) algebra. As already remarked by Quillen [**Qui67**, Ch. I, 1.2, Examples, B.], there is a model structure on the category of right bounded complexes over an abelian category $\mathscr{A}$ with enough projectives. This structure can be generalized to a class of additive categories which are sufficiently close to being abelian with enough projectives. We do not assume that $\mathscr{A}$ is more than *finitely bicomplete*.

Contents

1. Statement of the Result

1.1. Assumptions. Let $\mathscr{A}$ be an additive category such that:

(i) Every morphism has a kernel and a cokernel.
(ii) The class of all kernel-cokernel pairs is an exact structure $\mathscr{E}$ on $\mathscr{A}$.
(iii) There are enough projectives in $(\mathscr{A}, \mathscr{E})$.

We will fix the exact structure $\mathscr{E}$ on $\mathscr{A}$ throughout this appendix. Notice that it follows from (i) and the additivity of $\mathscr{A}$ that all finite limits and colimits in $\mathscr{A}$ exist.

1.2. The Definition of the Model Structure. Let $\mathbf{C} = \mathbf{Ch}_+(\mathscr{A})$ be the category of *right bounded complexes*—notice that we use *homological indexing* contrary to the rest of this work.

Definition 1.2.1. A chain map $A \xrightarrow{f} B$ is called a

(i) *weak equivalence* if it is a quasi-isomorphism, that is to say $\operatorname{cone}(f)$ is exact.
(ii) *fibration* if its components are admissible epics (= cokernels).

(iii) *cofibration* if its components are admissible monics (= kernels) with projective objects of $\mathscr{A}$ as cokernels.

THEOREM 1.2.2. *With fibrations, cofibrations and weak equivalences as in the definition, the category* $\mathbf{C}$ *is a closed model category.*

REMARK 1.2.3. A nice and gentle introduction to model categories can be found in [**DS95**]. Unfortunately, their "model structure" on $\mathbf{Ch}_{\geq 0}(\mathscr{A})$ turns out to be wrong in the sense that axiom **MC4**(ii) breaks down (the mistake in the verification occurs in the last line of section 7.11 and seems to be unrepairable). One must replace the condition $k > 0$ in their "Theorem" 7.2.(iii) by $k \geq 0$, but this is not just a typo (cf. their proof of **MC4**(i)). Also, in their verification of the fifth axiom they use Quillen's highbrow small object argument. We are obviously not in position to use this line of attack simply because we do not assume the existence of infinite limits or colimits in $\mathscr{A}$, so we have to resort to a straightforward explicit construction.

1.3. The Axioms. Recall that a closed model category is a category $\mathbf{C}$ with three distinguished classes of morphisms

(i) *weak equivalences* ($\xrightarrow{\sim}$)
(ii) *fibrations* ($\twoheadrightarrow$)
(iii) *cofibrations* ($\rightarrowtail$)

each of which is closed under composition and contains all identity morphisms. An *acyclic fibration* is a fibration which is at the same time a weak equivalence, and an *acyclic cofibration* is a cofibration which is at the same time a weak equivalence. This datum is subject to the following axioms:

MC1 Finite limits and colimits exist in $\mathbf{C}$.

MC2 Let f and g be composeable morphisms. If two out of f, g and gf are weak equivalences then so is the third.

MC3 Let f be a retract of g. If g is a weak equivalence, or a fibration, or a cofibration then so is f.

MC4 Consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & \overset{\exists h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y. \end{array}$$

A lifting h exists if either (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

MC5 Every morphism f can be factored as $f = pi$ in two ways: first (i) i is a cofibration and p is an acyclic fibration and second (ii) i is an acyclic cofibration and p is a fibration.

2. Proof of Theorem 1.2.2

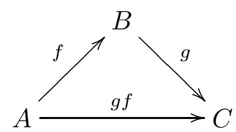
2.1. Preliminaries. Identity morphisms are weak equivalences since the mapping cone of an identity morphism is homotopic to zero, hence acyclic (idempotents split in $\mathscr{A}$). It is clear that identity morphisms are fibrations and cofibrations.

Let us check that the three classes of morphisms are closed under composition. For fibrations this is trivial. For cofibrations this follows from the fact that the direct sum of projectives is again projective. For weak equivalences this is the octahedral axiom for $\mathbf{Ac}(\mathscr{A}, \mathscr{E})$, more precisely:

LEMMA 2.1.1 ([**Nee90**, 1.1]). *The category* $\mathbf{Ac}_+(\mathscr{A}, \mathscr{E})$ *is a strictly full triangulated subcategory of* $\mathbf{K}_+(\mathscr{A})$*. In other words: The mapping cone of a chain map between acyclic complexes is acyclic.* □

2.2. MC1. As in the setting of abelian categories, limits and colimits in $\mathbf{C}$ can be taken degreewise, hence the fact that $\mathscr{A}$ is finitely bicomplete implies that $\mathbf{C}$ is finitely bicomplete.

2.3. MC2. This is essentially the octahedral axiom for $\mathbf{K}_+(\mathscr{A})$ together with the fact that $\mathbf{Ac}_+(\mathscr{A}, \mathscr{E})$ is a strictly full triangulated subcategory. Consider the commutative diagram



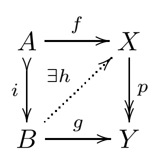
and form the mapping cones over f, g and gf. The octahedral axiom for $\mathbf{K}_+(\mathscr{A})$ tells us that these cones fit into a distinguished triangle in $\mathbf{K}_+(\mathscr{A})$ and the assumption that two out of f, g and gf are weak equivalences (= quasi-isomorphisms) tells us that two of the cones are acyclic, hence so is the third because $\mathbf{Ac}_+(\mathscr{A}, \mathscr{E})$ is a strictly full triangulated subcategory.

2.4. MC3. In an exact category a direct summand of a short exact sequence is short exact. This implies that a direct summand of an acyclic complex is acyclic if idempotents split.

We need to show that the fibrations, cofibrations and weak equivalences are closed under taking retractions. For fibrations and weak equivalences this is clear from the above. For cofibrations one needs to make the additional observation that a retract of a complex with projective components is again a complex with projective components.

2.5. MC4. So far, everything was relatively painless. The verification of the remaining two axioms requires some more work.

We have to check that a lifting h in the diagram below exists



in two situations:

(i) Suppose that i is a cofibration and p is an acyclic fibration.

Let P be the cokernel of i, which is a complex with projective components because i is assumed to be a cofibration. Let K be the kernel of p, this is an acyclic complex because p is an acyclic fibration. We may assume that $A_k = B_k = P_k = 0$ for $k < 0$. Notice that for all $k \in \mathbb{Z}$

$$A_k \xrightarrow{i_k} B_k \longrightarrow P_k$$

is a split exact complex because P_k is projective, so we can and will identify $B_k \cong A_k \oplus P_k$.

We construct the lift h by induction and take $h_k = 0$ for $k < 0$. So let $k \geq 0$ and suppose by induction that for $j < k$ we have constructed morphisms $h_j : B_j \to X_j$ satisfying for all $j < k$

(a) $dh_j = h_{j-1}d$,
(b) $p_j h_j = g_j$,
(c) $h_j i_j = f_j$.

Using the identification $B_k \cong A_k \oplus P_k$ we put $\tilde{h}_k = [\, f_k \ \bar{h}_k \,]$ where $\bar{h}_k : P_k \to X_k$ is some lift of $g_k : P_k \to Y_k$. The problem we run into is that $\tilde{h}_k$ has no reason to satisfy (a).

So define $u = d\tilde{h}_k - h_{k-1}d : B_k \to X_{k-1}$ and observe that
(a') $du = 0$ because h_{k-1} satisfies (a),
(b') $p_{k-1}u = 0$ by construction and (b),
(c') $ui_k = 0$ by construction as well.

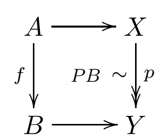
Notice that u measures the failure of $\tilde{h}_k$ to satisfy (a). Using (b') and (c') we see that u factors over a map $u' : P_k \to K_{k-1}$. Because K is an acyclic complex, there is a short exact sequence $Z_{k-1}K \rightarrowtail K_{k-1} \twoheadrightarrow Z_{k-2}K$ and the composite $P_k \xrightarrow{u'} K_{k-1} \twoheadrightarrow Z_{k-2}K$ is zero by (a'). Therefore u' factors over $u'' : P_k \to Z_{k-1}K$ and using the projectivity of P_k again we obtain a morphism $u''' : P_k \to K_k \to X_k$. Putting $h_k = \tilde{h}_k - u'''$ we have constructed a morphism such that (a), (b) and (c) holds for all $j \le k$, completing the induction.

(ii) Suppose that i is an acyclic cofibration and p is a fibration.

The cokernel of i is a right bounded complex P with projective components and it is acyclic because i is an acyclic cofibration. It follows that P is a split exact complex with projective components and therefore it is a projective object in $\mathbf{C}$ with the degreewise exact structure [**Wei94**, Exercise 2.2.1, p.34]. In particular, the complex B decomposes as a direct sum $A \oplus P$ and we can lift the restriction of g to P over p. This immediately yields the claim.

2.6. MC5. We need to produce two factorizations of the morphism $A \xrightarrow{f} B$. We will make use of the following:

LEMMA 2.6.1. *Consider a pull-back diagram in* $\mathbf{C}$

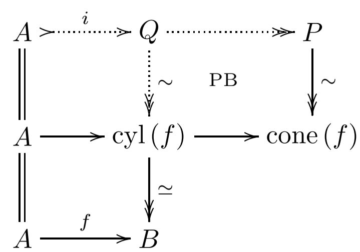


and assume that p is an acyclic fibration. Then f is an acyclic fibration as well.

PROOF. Since we have a pull-back diagram, we know that f is a fibration and that $\operatorname{Ker}(f) \cong \operatorname{Ker}(p)$.

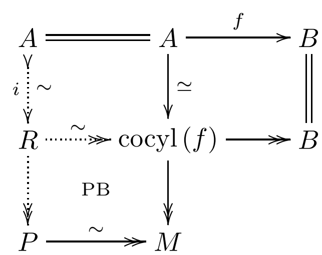
The construction of the mapping cone yields a morphism $\operatorname{cone}(p)[-1] \to \operatorname{Ker}(p)$ which is seen to be a quasi-isomorphism using the Quillen embedding theorem and the long exact homology sequence. Thus $\operatorname{Ker}(p) \cong \operatorname{Ker}(f)$ is acyclic, because $\mathbf{Ac}_+(p)$ is a triangulated subcategory of $\mathbf{K}_+(\mathscr{A})$. Repeating the argument for the quasi-isomorphism $\operatorname{cone}(f)[-1] \to \operatorname{Ker}(f)$ proves that $\operatorname{cone}(f)$ is acyclic, hence f is an acyclic fibration. ☐

(i) Let us prove that f may be factored over a cofibration and an acyclic fibration. The construction is summarized in the following diagram:



Recall that the canonical projection of the mapping cylinder onto B is a homotopy equivalence and that the middle row of the diagram is degreewise split exact. Since there are enough projectives in $\mathscr{A}$, we may choose a complex P with projective components and a quasi-isomorphism $P \twoheadrightarrow \operatorname{cone}(f)$ which is an admissible epic in every degree, see [**Kel90**, 4.1, Lemma, b)]. We may now form the pull-back in order to obtain Q and an acyclic fibration $Q \to \operatorname{cyl}(f)$ by Lemma 2.6.1. Because fibrations and weak equivalences are closed under composition, the morphism $Q \to B$ is an acyclic fibration. Since the top right square is a pull-back, we see that the top row is degreewise exact, hence i is a cofibration.

(ii) Let us prove that f may be factored over an acyclic cofibration and a fibration. The construction is summarized in the following diagram (here, cocyl denotes the mapping path space construction, see Appendix A):



The argument is similar to case (i). Recall that $A \to \operatorname{cocyl}(f)$, the inclusion of A in the cocylinder of f, is a degreewise monic and a homotopy equivalence. It is the kernel of its cokernel $\operatorname{cocyl}(f) \twoheadrightarrow M$. Again because there are enough projectives we may choose a complex with projective components and a degreewise admissible epic quasi-isomorphism $P \twoheadrightarrow M$. Therefore the pullback yields a complex R and a quasi-isomorphism $R \to \operatorname{cocyl}(f)$ and a degreewise split epic $R \twoheadrightarrow P$. By **MC2** we deduce that i is a quasi-isomorphism and it is a cofibration because it is the kernel of $R \twoheadrightarrow P$.

This completes the proof of Theorem 1.2.2. □

3. Remarks

3.1. A Generalization. The assumption that $\mathscr{A}$ has kernels and cokernels entered only in the verification of **MC1** and was otherwise only used in the much weaker form that idempotents split in $\mathscr{A}$. Thus, the verification of the other axioms is valid in much greater generality:

Corollary 3.1.1. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough projectives and suppose that idempotents split. Define the classes of fibrations, cofibrations and weak equivalences in* $\mathbf{C} = \mathbf{Ch}_+(\mathscr{A})$ *as in Definition 1.2.1. Then* $\mathbf{C}$ *satisfies the axioms of a closed model category up to* **MC1**. □

3.2. Dualization. The dual of a closed model category is again a model category up to interchanging the classes of fibrations and cofibrations, hence there is a model structure on $\mathbf{Ch}^+(\mathscr{A})$ if $\mathscr{A}$ has enough injectives.

3.3. Fibrant and Cofibrant Objects. An object in a closed model category is called *fibrant* if the unique morphism to 'the' terminal object is a fibration. Similarly, an object is called *cofibrant* if the unique morphism from 'the' initial object is a cofibration. Thus, all objects in $\mathbf{Ch}_+(\mathscr{A})$ are fibrant and the cofibrant objects are precisely the complexes with projective components.

3.4. The Homotopy Category. By localizing a model category $\mathbf{C}$ with respect to the class of weak equivalences one obtains the *homotopy category* $\mathrm{Ho}(\mathbf{C})$. The following result is essentially due to Quillen.

THEOREM 3.4.1. *Let* $\mathbf{C} = \mathbf{Ch}_+(\mathscr{A})$ *for* $\mathscr{A}$ *as in Section 1. The derived category* $\mathbf{D}_+(\mathscr{A})$ *and the category* $\mathrm{Ho}(\mathbf{C})$ *are equivalent.*

PROOF. This follows from [**Qui67**, Ch. I, 1.13, Theorem 1'], because all notions of homotopy coincide for objects which are both fibrant and cofibrant. Hence (using Quillen's notation) $\pi\,\mathbf{C}_{cf}$ coincides with the category $\mathbf{K}_+(\mathscr{P})$, but the former is equivalent to $\mathrm{Ho}(\mathbf{C})$ and the latter is equivalent to $\mathbf{D}_+(\mathscr{A})$. □

REMARK 3.4.2. It is conceivable that the theorem remains true under the much weaker assumptions that the exact category $(\mathscr{A}, \mathscr{E})$ has split idempotents and enough projectives. The author has not checked this in full detail.

APPENDIX D

Standard Borel G-Spaces are Regular

Standard Borel G-spaces with a quasi-invariant probability measure are regular in the sense of Monod [**Mon01**, Definition 2.1.1], provided G is a Polish group. Marc Burger suggested that [**Tak03**, Proposition 1.1] should imply this result for locally compact second countable groups. This is indeed true. However, Takesaki glosses over a somewhat delicate measurability issue which is at the heart of the matter as witnessed by the present proof. A clever trick communicated by Nicolas Monod considerably shortened our embarrassingly clumsy original argument.

Contents

1. Representations Associated to a Group Action

1.1. Basics. Let us introduce the setting in somewhat more detail. We assume G to be a Polish group, i.e., a topological group whose topology is second countable and completely metrizable (see e.g. [**Kec95**, Chapter I] or [**Arv76**, Section 3.1]). Consider a Borel G-action $G \times X \to X, (g,x) \mapsto gx$ on a standard Borel space (X, Σ) which is equipped with a quasi-invariant probability measure μ, i.e., $g_*\mu \sim \mu$ for all $g \in G$. The *Radon-Nikodým cocycle* is defined by

$$\rho(g,x) = \frac{d(g_*\mu)}{d\mu}(x)$$

or, in other words, by the *"transformation formula"*

$$\int_X \rho(g,x) f(x)\, d\mu(x) = \int_X f(gx)\, d\mu(x) \qquad \forall\, f \in L^1(X,\mu)$$

and thus it satisfies the *cocycle condition*

$$\rho(gh,x) = \rho(g,hx)\rho(h,x).$$

for all $g, h \in G$ and μ-almost all $x \in X$. Strictly speaking, ρ yields for each g a μ-equivalence class of Borel measurable functions $(X,\mu) \to \mathbb{R}_{\geq 0}$. We will prove in a moment that there exists a representative of ρ which is a Borel measurable function $G \times X \to \mathbb{R}_{\geq 0}$.

By the transformation formula and the cocycle identity, the Radon-Nikodým cocycle yields representations $\pi^p : G \to \mathscr{L}(L^p(X,\mu))$ for all $1 \leq p \leq \infty$ given by

$$(\pi^p(g)f)(x) := \rho(g^{-1},x)^{\frac{1}{p}} f(g^{-1}x),$$

where, of course, $(\pi^\infty(g)f)(x) = f(g^{-1}x)$ is understood.

Exploiting the duality between L^p and L^q for $\frac{1}{p} + \frac{1}{q} = 1$, we obtain the *contragredient representation* $(\pi^p)^\# : G \to \mathscr{L}(L^q(X,\mu))$ given by $(\pi^p)^\#(g) := (\pi^p(g^{-1}))^*$ which is easily seen to coincide with π^q, provided $1 \leq p < \infty$.

The following lemma is folklore and probably due to Mackey. The proof given here is extracted from the proof of [**FMW04**, Proposition 2.22]. The reader unacquainted with measure theory in standard Borel spaces is urged to either skip it or else to organize a copy of [**Kec95**] before delving into the proof.

LEMMA 1.1.1. *Given a measure-class preserving Borel action of a Polish group G on a standard probability space (X, Σ, μ), there is a Borel measurable version of the Radon-Nikodým cocycle, i.e., a Borel map $\rho : G \times X \to X$ such that*

$$\int_X \rho(g,x) f(x)\, d\mu(x) = \int_X f(gx)\, d\mu(x)$$

for all $g \in G$ and all $f \in L^1(X, \mu)$.

PROOF. It is well-known that the space $\mathscr{F}$ of μ-equivalence classes of Borel measurable functions $X \to \mathbb{R}_{\geq 0}$ becomes Polish when equipped with the topology of convergence in measure, see e.g. [**Moo76**, Proposition 7]. Let $\{A_n\}_{n\in\mathbb{N}}$ be a separating and generating set of Σ (cf. [**Kec95**, Proposition 12.1 (iii)]). The set

$$\mathscr{R} := \bigcap_{n\in\mathbb{N}} \left\{ (g,f) \in G \times \mathscr{F} \;:\; \mu(gA_n) = \int_{A_n} f\, d\mu \right\}$$

is the graph of a function $r : G \to \mathscr{F}$ because the fiber of $\mathscr{R}$ over each $g \in G$ consists of the Radon-Nikodým derivative of the automorphism $x \mapsto gx$ of (X, μ).

By the simple [**FMW04**, Lemma 2.7], we are done as soon as we can show that r is Borel measurable. By the Borel graph theorem [**Kec95**, Theorem 14.12], we need only check that $\mathscr{R}$ is a Borel set. First, the map $\psi : (g,x) \mapsto (g, gx)$ is a Borel automorphism of $G \times X$, so it maps Borel sets to Borel sets. It follows from [**Kec95**, Theorem 17.25] that $g \mapsto \mu(gA_n) = \mu(\psi(\{g\} \times A_n) \cap \mathrm{pr}_G^{-1}(g))$ is Borel measurable. Second, by [**FMW04**, Lemma 2.17] the function $f \mapsto \int_{A_n} f\, d\mu$ is Borel on $\mathscr{F}$. Thus $\mathscr{R}$ is indeed a Borel set. □

1.2. The Continuity Theorem.

Our main result is:

THEOREM 1.2.1. *Let (X, Σ, μ) be a standard probability space equipped with a measure class-preserving Borel G-action of a Polish group. For $1 < p < \infty$ the Banach G-module $(L^p(X, \mu), \pi^p)$ is strongly continuous.*

PROOF. Using [**Kec95**, Theorem 17.25] and Lemma 1.1.1 it is easy to check that

$$g \mapsto \langle \pi^p(g)\xi, \eta \rangle_{L^p, L^q} = \int_X \rho(g^{-1}, x)^{\frac{1}{p}} \xi(g^{-1}x) \overline{\eta(g^{-1}x)}\, d\mu(x)$$

is Borel measurable for all $\xi \in L^p$ and all $\eta \in L^q$. It follows that for all $\xi \in L^p(X, \mu)$ the orbit map $g \mapsto \pi^p(g)\xi$ is weak*-measurable. Now by [**Mon01**, Lemma 3.3.3], the norm and the weak* topologies induce the same Borel structure on the reflexive and *separable* Banach space $L^p(X, \mu)$. In particular, the orbital maps $g \mapsto \pi^p(g)\xi$ are also norm-measurable for all $\xi \in L^p(X, \mu)$ and by [**Mon01**, Lemma 1.1.3] it follows that $L^p(X, \mu)$ is a strongly continuous Banach G-module. □

REMARK 1.2.2. Instead of invoking the Baire category theorem (which is used in the proof of [**Mon01**, Lemma 1.1.3]) and the theory of standard Borel spaces, one could resort to Fubini's theorem. This argument was explained to the author by Claire Anantharaman-Delaroche and can be found in [**AAB$^+$08**, Appendice A].

COROLLARY 1.2.3. *Let (X, Σ, μ) be a standard probability space equipped with a measure-class preserving Borel action of a Polish group G.*

(i) For $1 \leq p < \infty$ the Banach G-module $(L^p(X, \mu), \pi^p)$ is strongly continuous.
(ii) For $1 < p \leq \infty$ the Banach G-module $(L^p(X, \mu), \pi^p)$ is a coefficient G-module.

PROOF (MONOD). By the definition of a coefficient G-module, see [**Mon01**, Definition 1.2.1] point (i) is equivalent to point (ii), so the only thing we need to prove is that $(L^1(X,\mu),\pi^1)$ is strongly continuous. But this is obvious due to a neat trick: For $1 \leq p, q < \infty$ the G-modules $L^p(X,\mu)$ and $L^q(X,\mu)$ are G-equivariantly homeomorphic via the *Mazur map* given by $\varphi(f) = |f|^{p/q}\,\mathrm{sign}\,(f)$ for $f \in L^p$. □

REMARK 1.2.4. The point of Monod's definition of a regular Borel G-space is *exactly* to ensure that the G-action on $L^1(X,\mu)$ is strongly continuous and hence $L^\infty(X,\mu)$ is a coefficient G-module. Since in [**Mon01**] the regularity hypothesis almost always comes in conjunction with the assumption that the group be locally compact second countable, it can usually be ignored. Of course, the definition of regularity is still vital in order to deal with groups which fail to be discrete or Polish.

1.3. The Homological Characterization of Amenability. At the very heart of many applications of bounded cohomology to rigidity theory is Burger-Monod's homological characterization of amenability in the sense of Zimmer, which now becomes perfect thanks to our continuity result:

THEOREM 1.3.1 (Burger-Monod, see [**Mon01**, Theorem 5.7.1]). *Let G be a locally compact and second countable group and let (X,μ) be a standard Borel G-space with quasi-invariant probability measure. The following assertions are equivalent:*

(i) The G-space (X,μ) is amenable in the sense of Zimmer.
(ii) The coefficient G-module $L^\infty(X,\mu)$ is relatively injective. □

APPENDIX E

The Existence of Bruhat Functions

In this chapter we give an elementary proof of the existence of Bruhat functions for proper actions. The result is of course well-known but the proofs in the literature depend on results in harmonic analysis. Bruhat functions are extensively used in Monod's treatment of bounded cohomology because proper group actions give rise to relatively injective Banach G-modules via $C_b(X)$ and its Banach-space valued siblings.

STANDING ASSUMPTION. *For simplicity of the exposition, we assume in this appendix that all locally compact spaces under consideration are second countable unless explicitly specified otherwise.*

However, we remark that most that is said can naturally be expressed in the language of groupoids and can be generalized in many directions beyond the restrictive viewpoint adopted here.

1. Proper Group Actions

1.1. First Remarks. Recall that a continuous map $f : X \to Y$ between two locally compact spaces is *proper* if the pre-image of every compact set in Y is compact in X. It easily follows from the definition that a proper map is closed. Let now G be a locally compact group acting continuously on a locally compact space X. The action is said to be *proper* if the map

$$G \times X \to X \times X, \quad (g, x) \mapsto (x, gx)$$

is proper.

Because the image of $G \times X$ in $X \times X$ is the orbit equivalence relation, it follows that the latter is closed in $X \times X$, which is in turn equivalent to saying that $G \setminus X$ is a Hausdorff space. From this it follows easily that every orbit is closed. So the main advantage of assuming properness is the existence of a well-behaved quotient space, or what amounts to the same in some sense, of well-behaved orbits. For instance, if X is a manifold and the action of some Lie group G is proper and free, then the quotient $G \setminus X$ is canonically a manifold. Another fundamental property of proper actions is that the stabilizer of each point is compact.

EXAMPLES 1.1.1. We give a brief list of examples:

(i) It is clear from the definitions that G acts properly on itself. More generally, if $H < G$ is a closed subgroup, then G acts properly on the homogeneous space G/H if and only if H is compact.

(ii) If G acts properly on X and if $H < G$ is a closed subgroup then H acts properly on X. If H is moreover normal then G/H acts properly on $H \setminus X$.

(iii) The Arzelà-Ascoli Theorem implies that the isometry group G of a proper metric space X [i.e., closed bounded sets are compact] is locally compact and acts properly on X, when G equipped with the topology of pointwise convergence which coincides with the topology of uniform convergence on compact subsets.

1.2. Existence of Bruhat Functions. One of the most useful general phenomena that are characteristic of proper group actions is the following:

THEOREM 1.2.1. *Consider an action of a locally compact group G on a locally compact space X and fix a left Haar measure on G. The action of G on X is proper if and only if there exists a* Bruhat function*, that is, a continuous non-negative function $\beta : X \to \mathbb{R}$ such that the following holds:*

(i) The saturation of every compact set $K \subset X$ intersects the support of β in a compact set, i.e., $\operatorname{supp}\beta \cap GK$ *is compact.*
(ii) For every $x \in X$, the relation $\int_G \beta(g^{-1}x)\, dg = 1$ holds.

REMARK 1.2.2. First observe that the statement in (i) implies that the integral in (ii) is finite. Condition (i) can obviously be modified to demanding instead of compactness of K that K maps to a compact set in $G \setminus X$ under the canonical projection. Condition (ii) in turn ensures that the support of β contains at least one point of every G-orbit.

REMARK 1.2.3. The interested reader might want to convince himself that the argument only needs a minor adjustment in the non-separable situation: one has to assume that $G \setminus X$ is *paracompact*, which is automatic given our standing assumption.

REMARK 1.2.4. In the groupoid setting one has to require furthermore – and this is somewhat stringent – that there exists a *continuous Haar system*, which implies in particular that the canonical projection onto the orbit space is *open*. Of course, for action groupoids and also Lie groupoids [even without Hausdorff assumption], this is not really an issue.

PROOF OF THEOREM 1.2.1. First we prove necessity of (i) and (ii). The slogan is the following: Take a locally finite partition of unity in $G \setminus X$ with compact supports and lift it to a family of functions with compact supports on X, sum these lifted functions up and normalize appropriately.

It is an easy exercise in point set topology that an open and continuous map $f : X \to Y$ from one locally compact space onto another has the following property: Let $W \subset Y$ be open and relatively compact and let K be a compact subset of W, then there exist $V \subset X$ open and relatively compact and $C \subset V$ compact such that $f(V) = W$ and $f(C) = K$.

We will apply this remark to the canonical projection $\pi : X \to G \setminus X$. Note that the quotient $G \setminus X$ is *paracompact*: It is Hausdorff by properness of the action and hence it is also locally compact and second countable by openness of π. Now choose a "good cover" of $G \setminus X$, more precisely, let $\{W_i\}_{i\in I}$ a locally finite cover of $G \setminus X$ by open and relatively compact sets containing a locally finite subcover $\{K_i\}_{i\in I}$ of $G \setminus X$ by compact sets, that is, $K_i \subset W_i$. Now pick compact sets $C_i \subset X$ and open and relatively compact sets $V_i \supset C_i$ such that $\pi(C_i) = K_i$ and $\pi(V_i) = W_i$. Observe that $\{V_i\}_{i\in I}$ is locally finite, so $A := \bigcup_{i\in I} C_i$ is closed. Put $B := \bigcup_{i\in I} V_i$, which is an open subset containing A.

The Bruhat function will be a continuous function with support in B and strictly positive on A. At the moment we put tentatively

$$\widetilde{\beta}(x) := \sum_{i\in I} f_i(x),$$

where f_i is a non-negative continuous function satisfying $f_i(c) > 0$ for all $c \in C_i$ and furthermore $\operatorname{supp} f_i \subset V_i$. By local finiteness again, the sum defines a non-negative continuous function $X \to \mathbb{R}$. Obviously, $\widetilde{\beta}$ is strictly positive on A and its support is contained in B.

Before proving that $\widetilde{\beta}$ yields a Bruhat function by applying an appropriate normalization, we extract some further information out of the above construction and the properness assumption. For two sets $U, V \subset X$ we write

$$G_U^V := \{(g, x) \in G \times X \,:\, (x, gx) \in U \times V\}.$$

Notice that properness of the action is equivalent to the assertion that G_U^V is compact for every pair U, V of *compact* subsets of X. Notice also that the set G_U^V is non-empty if and only if $\pi(U) \cap \pi(V) \neq \emptyset$. We claim that for every compact $K \subset X$ the set $G_K^{V_i}$ is non-empty only for finitely many $i \in I$. Indeed, by local finiteness of $\{W_i\}_{i\in I}$ we have that $\pi(K) \cap W_i = \pi(K) \cap \pi(V_i)$ is non-empty only for finitely many $i \in I$. By properness of the action it follows in particular that $G_K^B = \bigcup_{i\in I} G_{K_i}^B$ is relatively compact for every compact set $K \subset X$. Moreover, G_K^A is non-empty and compact for every compact $K \subset X$.

This establishes parts (i) and (ii) up to normalization. More precisely, $\widetilde{\beta}$ is strictly positive at some point of GK and $GK \cap \operatorname{supp} \widetilde{\beta}$ is compact. It follows in particular that for every $x \in X$

$$0 < \int \widetilde{\beta}(g^{-1}x)\, dg < \infty.$$

By left-invariance of the Haar measure, this integral depends only on the G-orbit of x, hence it descends to a continuous function $b : G \setminus X \to \mathbb{R}$. Now

$$\beta(x) = \widetilde{\beta}(x)/b(Gx)$$

satisfies (i) and (ii).

The sufficiency of (i) and (ii) is quite obvious if one forms the transformation groupoid $G \ltimes X$ [we only need its multiplication $m((g, x), (g', x')) = (gg', x')$ defined whenever $x = g'x'$]. The sets $G_K^{\operatorname{supp}\beta}$ and $G_{\operatorname{supp}\beta}^K$ are both compact for every compact set $K \subset X$. Now make use of the groupoid multiplication and note that

$$G_{K_1}^{K_2} = G_{\operatorname{supp}\beta}^{K_1} \cdot G_{K_2}^{\operatorname{supp}\beta}$$

is compact. ☐

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