Traces in Deformation Quantization and a Riemann-Roch-Hirzebruch Formula for Differential Operators

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Abstract

The text consists of two parts. In the first two chapters, we construct a trace on the deformed algebra of functions on a supersymplectic manifold. It is a generalization of the trace that has been given by Feigin, Felder and Shoikhet [16] for symplectic manifolds. The main ingredients are a generalization of Fedosov’s connection to supersymplectic manifolds found by Bordemann [9], which is explained in chapter 1, and a Hochschild cocycle for the Weyl-Clifford algebra, which is constructed in chapter 2. Using then a general construction for an isomorphism between the Hochschild homologies of a differential graded algebra and the corresponding twisted algebra, we find a chain map from the Hochschild homology of the algebra of deformed functions to the de Rham cohomology of the symplectic manifold. In Hochschild degree 0, we get a volume form on the symplectic manifold depending on a deformed function. Its integral over the manifold defines a unique normalized trace.

In the chapters 3 and 4, we prove a generalization of the Riemann-Roch-Hirzebruch theorem for holomorphic differential operators on a holomorphic vector bundle over a compact, connected, complex manifold. First a trace on the algebra of these operators is constructed. The procedure is similar to the one for the trace in chapter 2, but instead of a Fedosov connection we use a flat connection induced by a Maurer-Cartan form in formal geometry. As we consider holomorphic differential operators, they extend in a natural way to chain maps on the Dolbeault complex. Thus there is a Lefschetz number associated to such a differential operator, which is the alternating sum of the traces of the differential operator restricted to the cohomologies. There is also a natural trace for holomorphic differential operators that can be found by a “climbing the staircase” argument in the Hochschild-Čech double complex. The main result of chapter 3 is that this Hochschild-Čech trace is equal to the trace constructed by the method of formal geometry. In chapter 4 it is shown that also the Lefschetz number is equal to the Hochschild-Čech trace, which finally proves our generalization of the Riemann-Roch-Hirzebruch theorem.

Introduction

The text consists of two parts, the first two chapters and the last two chapters. We briefly describe the main results and mention the methods that have been used.

First part

In [16] Felder, Feigin and Shoikhet gave an explicit formula for a Hochschild cocycle of degree $2n$ for the $2n$-dimensional Weyl algebra and applied it to construct a trace on the algebra of deformed functions on a symplectic manifold. Furthermore they also computed the trace of the function 1, which turns out to be the integral over the manifold of a certain characteristic class. The result is

$$\text{Tr}(1) = (-\varepsilon)^n \int_M \hat{A}(TM) \exp(-\Omega/\varepsilon).$$

where $\hat{A}$ is the $\hat{A}$-genus and where $\Omega$ denotes the characteristic class of the $\star$-product.

The goal of the first part of this thesis is to extend this construction to the case of supersymplectic manifolds.

Batchelor’s theorem states that every smooth supermanifold can be written as a bundle $E \to M$ where $M$ is a manifold. Rothstein [8] found a similar theorem for supersymplectic manifolds: Every supersymplectic manifolds corresponds, in a non-canonical way, to the data $(M, \omega, E, g, \nabla)$ where $(M, \omega)$ is a symplectic manifold, $E \to M$ a bundle with nondegenerate metric $g$ and connection $\nabla$ on $E$, which is compatible with $g$.

Bordemann showed [9] that Fedosov’s construction for the deformation quantization of symplectic manifolds can be generalized to supersymplectic manifolds. In the symplectic case, one considers the formal Weyl bundles where the fibers are Weyl algebras $A_{2n}$ over the ring of formal power series $\mathbb{C}[[\varepsilon]]$. In the graded case, the Weyl algebra is replaced by Weyl-Clifford algebra $A_{2n|m} := A_{2n} \otimes \mathcal{C}l_m$. In particular, the Weyl bundle $W$ is again a

$^1$We use a normalization different from the one in [16].
bundle over the manifold $M$. An Abelian connection on $W$, which is a derivation for the fiberwise product, is constructed. The flat sections, denoted by $\Gamma_D(M, W)$, then form an algebra that is in one-to-one correspondence with the deformed algebra of functions $A = C^\infty(M) \otimes \mathcal{C}_m[[\varepsilon]]$. The isomorphism map $Q$ from $A$ to $\Gamma_D(M, W)$ is called quantization map. The Abelian connection on $W$ can be written in the form $D = \nabla + \text{ad}(A)$ where $\nabla$ is the canonical connection induced by a symplectic connection chosen on $M$ and the connection on $E$ and $A$ is a 1-form on $M$ with values in the Weyl bundle.

The symplectic Lie algebra is embedded in the Weyl algebra by the elements that are quadratic in the generators. Similarly, the product of the symplectic and the orthogonal Lie algebra is embedded in the Weyl-Clifford algebra, also by the quadratic elements. We write $A_{2n}^{\text{pol}}$ for the polynomial Weyl algebra over the ring $\mathbb{C}(\varepsilon)$. In [16] an explicit formula for an sp$(2n, \mathbb{C})$-basic Hochschild cocycle $\tau \in \text{HH}_2(A_{2n}^{\text{pol}})$ is given. We generalize this formula to an sp$(2n, \mathbb{C}) \times \text{so}(m, \mathbb{C})$-basic Hochschild cocycle $\tau \in \text{HH}_2(A_{2n|m}^{\text{pol}})$ where $A_{2n|m}^{\text{pol}} = A_{2n}^{\text{pol}} \otimes \mathcal{C}_m$.

Let the bundle $E$ be oriented. Our main result is that the maps

$$
\chi_k : \Gamma_D(M, W)^{(k+1)} \to \Omega^{2n-k}(M, \mathbb{C}[[\varepsilon]]),
\chi_k(f_0, \ldots, f_k) = \tau((f_0, \ldots, f_k) \times (A)_{2n-k}),
$$

where $\times$ denotes the shuffle map, define a chain map from the Hochschild homology of $A$, which is identified with the flat sections, to the de Rham cohomology of $M$ with values in $\mathbb{C}[[\varepsilon]]$. Furthermore, we show that on the homologies these maps are invariant under the flow of the Heisenberg equation. In particular, the integral over $\chi_0$ gives a formula for the unique normalized trace on $A$.

Finally, we consider the case where the Grassmann dimension of the supersymplectic manifold is even. There is a function $\Theta := \theta_1 \cdots \theta_{2m}$ on the manifold which is in local coordinates the product of all Grassmann variables and which corresponds to the orientation on $E$. We compute the trace of this section and get an integral over a characteristic class:

$$
\text{Tr}(\Theta) = (-\varepsilon)^n \int_M \hat{A}(R_1) \hat{B}(R_2) \text{tr} \left( \exp(R_3 - \Omega/\varepsilon) \right).
$$

where $\hat{A}(R_1) := \det \left( \frac{R_1}{2 \sinh(R_1/2)} \right)^{1/2}$ is the $\hat{A}$-genus and $\hat{B}(R_2) := \det \left( \cosh(R_2/2) \right)^{1/2}$. $R_1, R_2, R_3$ are the curvatures of the connections on $TM$, $E$ and on the coefficient bundle of the Weyl bundle respectively. $\Omega$ denotes the characteristic class of the star product.
Second part

We consider a connected, compact, complex manifold $X$ of complex dimension $n$ with a holomorphic vector bundle $E$ of rank $r$ and the algebra of holomorphic differential operators acting on section of this bundle. We use an idea very similar to the one described above in part I to construct a trace on the algebra of holomorphic differential operators.

The construction starts with formal differential geometry (see [13]). We consider the $\infty$-jet $J_\infty E$ of formal holomorphic trivializations $\mathbb{C}^n \times \mathbb{C}^r \rightarrow E$. We denote by $W_{n,r}$ the Lie algebra which is the semidirect product of the Lie algebra of formal holomorphic vector fields on $\mathbb{C}^n$ and the holomorphic matrices. The vector fields act on the matrices by derivation. Note that $W_{n,r}$ is embedded in $M_r(\mathcal{D}_n)$, which is the algebra of $r \times r$-matrices of holomorphic differential operators. $J_\infty E$ is a homogeneous $W_{n,r}$ space, which means that there is an injective Lie algebra homomorphism from $W_{n,r}$ to the Lie algebra of vector fields on $J_\infty E$. This map is in each point $\phi \in J_\infty E$ an isomorphism from $W_{n,r}$ to the tangent space $T_{\phi}^{(1,0)}J_\infty E$ and its inverse defines a Maurer-Cartan form, which is a 1-form on $J_\infty E$ with values in $W_{n,r}$ satisfying the Maurer-Cartan equation

$$d\Omega_{MC} + \frac{1}{2}[\Omega_{MC},\Omega_{MC}] = 0.$$ 

There is a natural action of the group $G := GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$ on $J_\infty E$ and the fibers of the quotient $J_\infty E/G$ are contractible, thus there exists a global $G$-equivariant section $\phi \in \Gamma(J_1 E, J_\infty E)^G$, where $J_1 E$ is the extended frame bundle. The pullback $\omega = \phi^*\Omega_{MC} \in \Omega^1(J_1 E, W_{n,r})^G$ also obeys a Maurer-Cartan equation and defines a connection on the bundle

$$J_1 E \times_G M_r(\mathcal{D}_n).$$

The flat sections are in 1 to 1 correspondence with the holomorphic differential operators on $E$. Therefore we have map $D \mapsto \hat{D}$ from the differential operators to the flat sections. If we consider $\phi$ as a formal map $TE \rightarrow E$, we can consider the map $D \mapsto \hat{D}$ as the (formal) pullback by $\phi$.

We see an analogy to the construction in the first part of this thesis: $M_r(\mathcal{D}_n)$ corresponds to the Weyl algebra, $J_1 E \times_G M_r(\mathcal{D}_n)$ corresponds to the Weyl bundle, $\omega$ corresponds to the connection 1-form $A$ and the map $D \mapsto \hat{D}$ corresponds to the quantization map. Also the formula for the Hochschild cocycle of degree 2n for $M_r(\mathcal{D}_n)$ is the same as for the Weyl algebra. As in part one, we obtain a map

$$\chi_0(D) := \tau(\hat{D}, \omega, \ldots, \omega) \in \Omega^{2n}(X)$$

and get a trace by integrating $\chi_0(D)$ over $X$. 
INTRODUCTION

There is a second natural way to define a trace on the algebra of holomorphic differential operators. As they commute with the Dolbeault differential and therefore act on its cohomology, we may define the Lefschetz number

$$L(D) = \sum_{i=0}^{n} (-1)^{i} \text{tr}(H^{i}(D))$$

which is well defined as the cohomology spaces are finite dimensional. The theorem that we prove in the second part of this thesis states that the two traces for holomorphic differential operators are equal up to a factor:

$$L(D) = \frac{1}{(2\pi i)^{n}} \int_{X} \chi_{0}(D).$$

In the case $D = \text{id}$ this theorem reduces to the Riemann-Roch-Hirzebruch theorem.

The rough idea of the proof is to use a ”climb-the-staircase” argument for the differential operator $D$ in the Hochschild-Čech double complex (see section 3.4). The computation stops when the Hochschild- and the Čech-degree reach $2n$. It is then possible to compare the two traces and show that they are equal.
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Chapter 1

Deformation Quantization

"Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop."

Alice’s Adventures in Wonderland by Lewis Carroll

In this chapter we will essentially recall well known facts about deformation quantization and therefore omit the proofs. The main references are [5] and [9].

1.1 Introduction

In physics, quantization describes the transition from a classical mechanical system to a corresponding quantum mechanical system. From the algebraic point of view, this means to pass from a commutative (associative) algebra of observables to a noncommutative one. In physics, the noncommutativity of the quantized algebra is controlled by the Planck constant $\hbar$. In the limit $\hbar \to 0$ one recovers the classical algebra. Deformation quantization is a method to construct such noncommutative, associative algebras, but the parameter that controls the noncommutativity – here called $\varepsilon$ – is only a formal parameter, i.e. we only consider formal power series in $\varepsilon$. The concrete mathematical problem of deformation quantization can be formulated as follows: The initial data is a Poisson manifold $(M, \{,\})$ on which one considers the commutative algebra of smooth functions with product given by the pointwise multiplication. The question is now if there is an associative product $\star$ on the set $C^\infty(M)[[\varepsilon]]$ of the form

$$f \star g = fg + \varepsilon B_1(f,g) + \varepsilon^2 B_2(f,g) + \ldots$$

where the $B_i$ are bidifferential operators and $B_1(f,g) - B_1(g,f) = \{f,g\}$. 

1
Deformation quantization started 1978 with the seminal paper by Bayen, Flato, Fronsdal, Lichnerovicz, and Sternheimer [1]. In the 1980s the existence of \( \star \)-products has been proved for the special case of symplectic manifolds. The final answer was given 1997 in the famous preprint of Kontsevich [3] where he proved the formality conjecture as a special case of which the existence of a \( \star \)-product for every Poisson structure follows.

1.2 Weyl algebra

As a prototype for deformation quantization of arbitrary Poisson manifolds, we consider \( M = T\mathbb{R}^n \cong \mathbb{R}^{2n} \) with the Poisson structure coming from the standard symplectic structure \( \alpha = \omega^{-1} \):

\[
\alpha = \sum_{i,j=1}^{n} \alpha_{ij} \partial_i \otimes \partial_j \quad \text{where} \quad (\alpha_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The Poisson bracket for two functions \( f, g \in C^\infty(M) \) is then given by

\[
m \circ \alpha(f \otimes g).
\]

where \( m : f \otimes g \mapsto fg \) is the pointwise product. An explicit formula for a \( \star \)-product can then be written as

\[
f \star g = m \circ \exp(\varepsilon \alpha/2)(f \otimes g).
\]

It is easy to verify that \( \star \) is an associative product and that

\[
[f, g] = \{f, g\} + O(\varepsilon^2),
\]

where we defined \( [f, g] := (f \star g - g \star f)/\varepsilon \). In other words, we have a deformation of the commutative algebra in the direction of the Poisson bracket \( \{f, g\} := m \circ \alpha(f \otimes g) \). The above \( \star \)-product is usually called Weyl-Moyal product (or Weyl-Groenewold product) and we will call the corresponding algebra \( (C^\infty(\mathbb{R}^{2n})[[\varepsilon]], \star) \) the Weyl algebra and denote it by \( \mathcal{A}_{2n} \).

There is a natural grading on the Weyl algebra. If we write \( y \) for the coordinates on \( \mathbb{R}^{2n} \), this grading is given by \( \deg(y) = 1 \) and \( \deg(\varepsilon) = 2 \). The Moyal product preserves this degree as \( \deg(\varepsilon \alpha) = 0 \).

The Moyal product is invariant under linear symplectic transformations. The corresponding action of the symplectic Lie algebra \( \text{sp}(2n, \mathbb{C}) \) can be written as \( \text{ad}(a) \) where \( a = \frac{1}{2}a_{ij}y^i y^j \), \( (a_{ij}) \) is symmetric matrix with values in \( \mathbb{C} \) and \( \text{ad}(a)f := [a, f] \). Explicitly for \( f = f_\ell y^\ell \), we get \( \text{ad}(a)f = \sum_{k, \ell} a_{k\ell} \alpha^{k\ell} f_\ell y^\ell \) where the matrix \( \sum_k a_{k\ell} \alpha^{k\ell} \) lies in \( \text{sp}(2n, \mathbb{C}) \).
Later, we will also consider the algebra of $A_{2n}$-valued $r \times r$ matrices and denote it by
$$A_{2n}^r := A_{2n} \otimes \mathcal{C} M_r(\mathbb{C}).$$

### 1.3 Weyl bundle and Fedosov connection

In the year 1994, Fedosov found an elegant geometric way to construct $\star$-products on symplectic manifolds (see [4, 5]) which we are now going to describe briefly. The geometric idea is to consider an infinite dimensional bundle – the formal Weyl bundle – over the symplectic manifold, whose fibers are Weyl algebras. Then an Abelian connection on this bundle is constructed and the functions $C^\infty(M)[[\varepsilon]]$ are identified with flat sections in this bundle. For these sections the wanted $\star$-product is simply given by the fiberwise Moyal product. The two non-trivial steps in this construction are, first, to find the Abelian connection, and second, to find the algebra isomorphism between the functions and the flat sections.

Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold with complexified tangent bundle and $\mathcal{E} \to M$ a complex vector bundle of rank $r$. We then define the formal Weyl bundle with coefficients in $\text{End}(\mathcal{E})$ as the associated bundle
$$W(M) = F_G(M) \times_G A_{2n}^r,$$
where $G = \text{GL}_r(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C})$ and $F_G(M)$ is the bundle of frames of $\mathcal{E}$ and symplectic frames of $TM$. For a point $(x, e, f) \in F_G$ where $x \in M$, $e : \mathbb{R}^r \to \mathcal{E}_x$ and $f : \mathbb{R}^{2n} \to T_xM$, the action of $g = (g_1, g_2) \in \text{GL}_r(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C})$ is given by $(x, e, f)g = (x, eg_1, fg_2)$. For $a \in A_{2n}$, it is given by $(ga)(y) = g_1a(g_2^{-1}y)g_1^{-1}$. We now consider forms with values in the Weyl bundle. In local coordinates, an element $a \in \Omega^k(M, W)$ can be written as
$$a(x, y, \varepsilon) = \sum_{j \geq 0, |\alpha| \geq 0, |\beta| = k} \varepsilon^j a_{j\alpha\beta}(x) y^\alpha dx^\beta,$$
where $x$ are coordinates on $M$, $y^i \in T^*_x M$ coordinates on the tangent bundle and $a_{j\alpha\beta}(x) \in \text{End}(\mathcal{E}_x)$. We define the operator $\delta^{-1} : \Omega^k(M, W) \to \Omega^{k-1}(M, W)$ that acts on an element $a$ of the above form as $\sum_{k=1}^{2n} y^k i_{\partial_k/\partial x_k} /(|\alpha| + |\beta|)$.

We write $\circ$ (instead of $\star$) for the fiberwise Moyal product. On forms with values in the Weyl bundle, this induces a map:
$$\circ : \Omega^k(M, W) \times \Omega^\ell(M, W) \to \Omega^{k+\ell}(M, W).$$

Assume that $\nabla^{TM}$ is a torsion-free, symplectic connection on $TM$ and $\nabla^\mathcal{E}$ a connection on $\mathcal{E}$. Such connections always exist and they define in a natural way a covariant derivative
on $\Omega^\bullet(M,W)$ that can locally be written in the form
\[ \nabla a = da + [\Gamma, a] \]
where $\Gamma = \Gamma^T_M + \varepsilon \Gamma_E$ is a 1-form with values in the Weyl bundle. The commutator is extended as a superbracket to the algebra $\Omega^\bullet(M,W)$, that is $[a,b] = (a \circ b - (-1)^{k\ell} b \circ a)/\varepsilon$ for $a \in \Omega^k(M,W)$ and $b \in \Omega^\ell(M,W)$. If $\Gamma^i_{jk}$ are the Christoffel symbols for the symplectic connection, then $\Gamma^T_M = \frac{1}{2} \omega_{ij} \Gamma^i_{k\ell} y^i y^k dx^\ell$. Note that $\Gamma$ does not transform like a 1-form and therefore is only defined locally. Let $A \in \Omega^1(M,W)$, then
\[ Da = \nabla a + [A, a] = da + [\Gamma + A, a] \]
is again a covariant derivative on $\Omega^\bullet(M,W)$ and a superderivation of the algebra $\Omega^\bullet(M,W)$. We write $R^T_{jkl}$ for the coefficients of the curvature tensor of the connection $\nabla^T_M$ and $R^E_{k\ell}$ for the curvature of the connection $\nabla^E$. We then define $R = \frac{1}{4} \omega_{im} R^n_{jkl} y^i y^n dx^k \wedge dx^l + \frac{\varepsilon}{2} R^E_{k\ell} dx^k \wedge dx^\ell$ and call the 2-form
\[ \Omega = R + \nabla A + \frac{1}{2} [A, A] \]
the curvature of the connection $D$. It is easy to verify that with the above definitions $D^2 a = [\Omega, a]$. We are now interested in the case where the connection $D$ is Abelian, that is the curvature is a central form: $\Omega \in \Omega^2(M)[[\varepsilon]]$. The flat sections in $\Gamma(M,W)$ then form a subalgebra which we denote by $\Gamma_D(M,W)$. Fedosov proved that an Abelian connection can be constructed iteratively with respect to the degree in the Weyl bundle. There is a similar construction for the isomorphism $Q$ – also called quantization map – from the function on $M$ to the flat sections on the Weyl bundle. The results are summarized in the following theorem:

**Theorem 1.1. (Fedosov)** Let $\Omega = \Omega_0 + \varepsilon \Omega_1 + \varepsilon^2 \Omega^2 + \cdots \in \Omega^2(M)[[\varepsilon]]$ be a closed 2-form with $\Omega_0 = -\omega$ and $\mu \in \Gamma(M,W)$ a section of degree $\geq 3$ and $\mu|_{y=0} = 0$. Then there exists a unique $A \in \Omega^1(M,W)$ with $\delta^{-1} A = \frac{1}{2} \omega_{ij} y^i y^j + \mu$ such that the connection $D = \nabla + ad(A)$ is Abelian, i.e. $D^2 a = 0$ for all $a \in \Omega^k(M,W)$, and such that its curvature is equal to the given $\Omega$. Furthermore there is an isomorphism $Q$ called quantization map from $\Gamma(M, \text{End}(E))[\varepsilon]$ to the algebra $\Gamma_D(M,W)$ of flat sections in the Weyl bundle such that the inverse of $Q$ is the map that evaluates a section $a(x,y,\varepsilon)$ in $y = 0$.

The map $Q$ defines an isomorphism of algebras if we endow the space $C^\infty(M)[[\varepsilon]]$ with the product
\[ f \ast g = Q^{-1}(Q(f) \circ Q(g)) , \]
which is a $\ast$-product for the symplectic manifold $M$. 

1.4 Weyl-Clifford algebra

Instead of $\mathbb{R}^{2n}$, we consider the superspace $\mathbb{R}^{2n|m}$ with coordinates $(y_1, \ldots, y_{2n}, \theta_1, \ldots, \theta_m)$ where $y_i$ are commutative variables and $\theta_i$ are Grassmann variables, which generate the algebra $\Lambda^\bullet(\mathbb{R}^m)$. We define the standard supersymplectic structure by

$$\alpha^{ij} \frac{\partial}{\partial y^i} \otimes \frac{\partial}{\partial y^j} + g^{ij} \frac{\partial}{\partial \theta^i} \otimes \frac{\partial}{\partial \theta^j}$$

where $(\alpha^{ij})$ is as defined as for the Weyl algebra and $(g^{ij}) = 1$. It is then straightforward to define an associative product on $C^\infty(\mathbb{R}^{2n|m}[[\varepsilon]]) = C^\infty(\mathbb{R}^{2n}) \otimes \Lambda^\bullet(\mathbb{R}^m)[[\varepsilon]]$:

$$f \star g = m \circ \exp(\varepsilon(\alpha + g)/2)f \otimes g.$$ 

When we deal with Grassmann variables, we always use the Koszul sign convention, which means that we multiply an expression by $(-1)^{k\ell}$ whenever an expression of Grassmann degree $k$ moves past an expression of degree $\ell$. As $\deg \theta^i = 1$ and $\deg \partial/\partial \theta^i = -1$, this implies for example that

$$\theta^i \star \theta^j = \theta^i \theta^j - \varepsilon g^{ij},$$

where the minus sign comes from passing $\partial/\partial \theta^i$ by $\theta_i$. We write $A_{2n|m}$ for the super-Weyl algebra. It is easy to see that $A_{2n|m} = A_{2n} \otimes C\ell_m$ where $C\ell_m$ is the Clifford algebra with respect to the quadratic form defined by $g^{ij}$.

As in the case of the Weyl algebra there is a natural grading where $\deg(y) = \deg(\theta) = 1$ and $\deg(\varepsilon) = 2$.

The super-Moyal product is invariant under transformations of the group $\text{Sp}_{2n}(\mathbb{C}) \times O_m(\mathbb{C})$. As in the case of the Weyl algebra, there is an infinitesimal action of this group.

The symplectic action is exactly the same and the orthogonal action can be written as $\text{ad}(a)$ where $a = \frac{1}{2} a_{ij} \theta^i \theta_j$ and $(a_{ij})$ is an antisymmetric matrix. Explicitly for $f = f_\ell \theta^\ell$, we get $\text{ad}(a)f = \sum_{i,j,\ell} g^{i\ell} a_{ij} \theta^j f_\ell = \sum_{j,\ell} a_{\ell j} \theta_j f_\ell$ where $(a_{ij}) \in \text{so}(m, \mathbb{C})$.

A generalization of the Weyl-Clifford algebra are $A_{2n|m}^r$-valued $r \times r$ matrices:

$$A_{2n|m}^r := A_{2n|m} \otimes \mathbb{C} M_r(\mathbb{C}).$$

1.5 Supersymplectic supermanifolds

For a general introduction to smooth supermanifolds see for example [6, 7]. Batchelor’s theorem tells us that a smooth supermanifold can be represented by a manifold $M$ and a bundle $E \to M$. The functions on the supermanifold are then given by the sections of the
CHAPTER 1. DEFORMATION QUANTIZATION

bundle $\Lambda \bullet E$. There is an analog of this theorem for supersymplectic supermanifolds by Rothstein (see [8]) which states that a supermanifold with a closed, even, nondegenerate 2-form can be described by the differential geometric data $(M, \omega, E, g, \nabla)$ where $(M, \omega)$ is a symplectic manifold, $E \to M$ a bundle with nondegenerate metric $g$ and connection $\nabla$ on $E$ that is compatible with $g$.

For $X \in \Gamma(M, TM)$, we denote the lift of $X$ to $E$ by $\nabla_X \in \Gamma(E, TE)$. Let $R \in \Gamma(M, \operatorname{End} E \otimes \Lambda^2 T^*M)$ be the curvature of $\nabla$ and let $\tilde{R} \in \Gamma(M, \Lambda^2 E \otimes \Lambda^2 T^*M)$ denote the contraction of the curvature by $g$. In local coordinates $R = R^i_{jk\ell} e_i^k dx^k \wedge dx^\ell$ where $e_1, \ldots, e_m$ is an orthogonal basis of $E_x$ and $e^1, \ldots, e^m$ the dual basis. Then we find

$$\tilde{R} = g_{is} R^s_{jk\ell} \theta^i \theta^j dx^k \wedge dx^\ell,$$

where the $\theta^i$, $i = 1, \ldots, m$ are the Grassmann coordinates on the fibers of $E$. We define the supersymplectic form $\varpi$ corresponding to the given data by

$$\varpi(\nabla_X, \nabla_Y) = \omega(X, Y) + \frac{1}{2} \tilde{R}(X, Y),$$

$$\varpi(\phi, \psi) = g(\phi, \psi),$$

$$\varpi(\nabla_X, \phi) = 0.$$

where $X, Y$ are vector fields on $M$ and $\phi, \psi$ are vertical vector fields on $E$.

1.6 Weyl-Clifford bundle and Fedosov connection

We now summarize a generalization of Fedosov’s construction to supersymplectic manifolds found by Bordemann (see [9, 10]). First, we have to define the Weyl-Clifford bundle. Not as one would expect, it is a bundle over the base manifold $M$ rather than over the supermanifold. Assume that – as in the last section – the data $(M, \omega, E, g, \nabla)$ for a supersymplectic manifold is given. Further let $\mathcal{E} \to M$ be a complex bundle of rank $r$. By giving in each point $x \in M$ a basis of $\mathcal{E}_x$, a symplectic basis of $T_x M$ and an orthogonal basis of $E_x$, we get the base bundle $F_G(M)$ where $G = \operatorname{GL}_r(\mathbb{C}) \times \operatorname{Sp}_{2n}(\mathbb{C}) \times \operatorname{O}_m(\mathbb{C})$. We then define the Weyl-Clifford bundle with coefficients in $\operatorname{End}(\mathcal{E})$ as

$$W(M, E) = F_G(M) \times_G \mathcal{A}_{2n|m}^r.$$

We will just write $W$ instead of $W(M, E)$ if this doesn’t lead to any confusion. As in the case of the usual Weyl bundle, we consider forms on $M$ with values in $W$ and again denote the fiberwise super-Moyal product by $\circ$:

$$\circ : \Omega^k(M, W) \times \Omega^\ell(M, W) \to \Omega^{k+\ell}(M, W).$$
We choose a symplectic, torsion-free connection $\nabla_M$ on $TM$ and a connection $\nabla_E$ on $E$. Then the connections $\nabla_M$, $\nabla_E$ and $\nabla_E$ combine in a canonical way to a connection $\nabla$ on $W$. For given $A \in \Omega^1(M,W)$, there is a connection $D = \nabla + \text{ad}(A)$ on $W$, which is a covariant derivative on $\Omega^*(M,W)$ and a derivation on the algebra $\Omega^*(M,W)$. We write $R_{jk}^k$ for the curvature tensor of the connection $\nabla_T M$ and $R_{E}^k$ for the curvature of the connection $\nabla_E$. Let $\hat{R}$ be defined as in the previous section. We then define $R = \frac{1}{4} \omega_{ij} R_{jk}^i y^j dx^k \wedge dx^k + \frac{2}{3} R_{E}^k dx^k \wedge dx^k + \frac{1}{4} \hat{R}$ and call the 2-form
\[
\Omega = R + \nabla A + \frac{1}{2}[A, A]
\]
the curvature of the connection $\nabla$.

There is a theorem analogous to Fedosov’s theorem (see [9, 10]):

**Theorem 1.2. (Bordemann)** Let $\Omega = \Omega_0 + \varepsilon \Omega_1 + \varepsilon^2 \Omega_2 + \cdots \in \Omega^2(M)[[\varepsilon]]$ be a closed 2-form with $\Omega_0 = -\omega$ and $\mu \in \Gamma(M,W)$ a section of degree $\geq 3$ and $\mu |_{y=0} = 0$. Then there exists a unique $A \in \Omega^1(M,W)$ with $\delta^{-1} A = \frac{1}{2} \omega_{ij} y^i y^j + \mu$ such that the connection $D = \nabla + \text{ad}(A)$ is Abelian, i.e. $D^2 a = 0$ for all $a \in \Omega^k(M,W)$, and such that its curvature is equal to the given $\Omega$. Furthermore there is an isomorphism $Q$ called quantization map from $\Gamma(M, \Lambda^* E^* \otimes \text{End}(E))[[\varepsilon]]$ to the algebra $\Gamma_D(M,W)$ of flat sections in the Weyl bundle such that the inverse of $Q$ is the map that evaluates a section $a(x,y,\theta,\varepsilon)$ in $y=0$.

As in the non-super case the $\star$-product for $f, g \in \Gamma(M, \Lambda^* E^* \otimes \text{End}(E))[[\varepsilon]]$ is now given by
\[
f \star g = Q^{-1}(Q(f) \circ Q(g)).
\]

### 1.7 Heisenberg equation

The proofs to what we are going to explain in this section are given in great detail in the book of Fedosov [5] for the case of a symplectic manifold, but the same proofs apply in a straightforward way to the supersymplectic case.

Assume that we have a smooth family of symplectic forms $\omega_t$, of metrics $g_t$ on $E \to M$ and of Abelian connections $D_t = \nabla_t + \text{ad}(A_t)$ with curvature $\Omega_t$. A Hamiltonian is a smooth family of sections $H_t \in \Gamma(M,W)$ such that $\lambda_t := D_t H_t - \dot{A}_t \in \Omega^1(M)[[\varepsilon]]$ is a 1-form with values in the center of the Weyl algebra and such that there exists a family of vector fields $X_t$ on $M$ so that $\deg(i_{X_t} A_t + H_t) \geq 2$. For $a(t) \in \Omega^*(M,W)$, we then consider the Heisenberg equation
\[
\frac{da}{dt} + (i_{X_t} D_t + D_t i_{X_t}) a + [H_t, a] = 0.
\]
If \( a(t) \in \Gamma_{D_t}(M,W) \) is a flat section, the equation reduces to

\[
\frac{da}{dt} + [H_t, a] = 0.
\]

For given \( a(0) \), this equation has a unique solution for \( t \geq 0 \). The flatness of \( a(t) \) is preserved because

\[
\frac{d}{dt}(D_t a) = -[\lambda_t, D_t a].
\]

The flow \( \Phi_t \) of the Heisenberg equation generates automorphisms of the algebra \( \Omega^\bullet(M,W) \) of the form

\[
\Phi_t(a) = (f_t)_*(U^{-1}_t \circ a \circ U_t).
\]

\( f_t \) is the flow of the vector field \( X_t \) with adequate liftings \( \sigma_t, s_t, v_t \) to the bundles \( TM, E \) and \( E \). We define the natural push forward by \( f_t \) as

\[
((f_t)_*a)(x, y, \theta) = v_t a(f_t^{-1}(x), \sigma_t^{-1}y, s_t^{-1}\theta)v_t^{-1}.
\]

\( U_t \) is given by

\[
U(t) = P \exp \left( \frac{1}{\varepsilon} \int_0^t f_s^* H_3(s) dt \right)
\]

where \( H_3 \) are the terms of \( H \) with degree \( \geq 3 \) and \( P \) is the time ordering operator.

The Heisenberg equation can be used to show a generalization of Darboux' theorem: The algebra \( \Gamma_{D_t}(M,W) \) is locally isomorphic to the (trivial) algebra \( \Gamma_{D_0}(U,W) \) where \( U \subset \mathbb{R}^{2n} \) is an open subset. On the trivial algebra the symplectic form and the metric are constant, \( E \to M \) is trivial and the flat connection is given by \( D_0 = d + \text{ad}(\omega_{ij} y^i dx^j) \). The flat sections are then of the form \( a(x, y, \theta, \varepsilon) = f(x + y, \theta, \varepsilon) \) where the right hand side has to be understood as the Taylor series with respect to \( y \).
Chapter 2

Hochschild homology and traces of deformed algebras

"It is a mistake to think you can solve any major problems just with potatoes."

Life, the Universe and Everything (The Hitchhiker’s Guide to the Galaxy)

2.1 Definitions and Preliminaries about Hochschild homology

Let $A$ be an associative but not necessarily commutative algebra with unit 1 over a commutative ring $k$. The Hochschild homology $\text{HH}_*(A, M)$ with coefficients in the $A$-$A$-bimodule $M$ is the homology of the chain complex

$$
\cdots \xrightarrow{b} C_n(A, M) \xrightarrow{b} C_{n-1}(A, M) \xrightarrow{b} \cdots \xrightarrow{b} C_1(A, M) \xrightarrow{b} M \xrightarrow{b} 0
$$

where $C_n(A, M) = M \otimes A^\otimes n$ and $b$ is given by

$$
b(m, a_1, \ldots, a_n) := (ma_1, a_2, \ldots, a_n) + \sum_{j=1}^{n-1} (-1)^j (m, a_1, \ldots, a_j a_{j+1}, \ldots, a_n) + (-1)^n (a_n m, a_1, \ldots, a_{n-1}) .
$$

As above, we will usually replace "$\otimes$" by commas to save space. If we replace in the above definition $M \otimes A^n$ by $M \otimes (A/k1)^n$, we get the so called normalized Hochschild complex that has the same homology. In many cases, it is more convenient to use the normalized complex. If $M = A$, we abbreviate $\text{HH}_*(A, A)$ by $\text{HH}_*(A)$. If our algebra $A$ is $\mathbb{Z}$- or $\mathbb{Z}_2$-graded, we use the same definition of the differential $b$, but the last factor gets an additional
sign $(-1)^{|a_n|} \sum_{i=1}^{n-1} |a_i|$ which is obviously consistent with the Koszul sign convention. Also with this additional sign, the complex still has the structure of a simplicial module and therefore the above statement about the normalized complex remains true in the graded case (see [22], 1.6.4-1.6.6, p. 46). Unless said otherwise, we from now on always use the normalized complex.

The Hochschild cohomology $HH^*(A, M)$ is the cohomology of the complex with chains $C^*(A, M) = \text{Hom}(A^n, M)$. The differential $\beta : C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$\beta f(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{j=1}^{n} (-1)^j f(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} f(a_1, \ldots, a_n)a_{n+1}$$

$C^*(A, A^*)$ is the complex dual to $C_*(A) := C_*(A, A)$. 

**Morita invariance**

We write $M_r(k)$ for the $r \times r$ matrices with entries in $k$. Instead of $M$ and $A$, we now consider matrices with entries in $M$ and $A$:

$$\mathcal{M} = M_r(k) \otimes_k M, \quad \mathcal{A} = M_r(k) \otimes_k A$$

We define the trace map $\text{tr} : \mathcal{M} \otimes \mathcal{A}^n \to M \otimes A^n$ by

$$\text{tr}(M_0 \otimes a_0, \ldots, M_n \otimes a_n) = \text{tr}(M_0 \cdots M_n)(a_0, \ldots, a_n),$$

where the trace on the right hand side is the usual trace for matrices. Morita invariance of matrices means that this map induces an isomorphism from $H_*(\mathcal{A}, \mathcal{M})$ to $H_*(A, M)$. The inverse map is induced by the embedding of $k$ into $M_r(k)$ as the matrix entry in the upper left corner.

Morita invariance still holds in the case of graded algebras.

**Tensor products**

Let $A$ and $B$ be $\mathbb{Z}_2$-graded algebras with unit over a commutative ring $k$. We define the following complexes:

1. $C_*(A) \otimes C_*(B)$ where $(C_*(A) \otimes C_*(B))_n = \oplus_{p+q=n} C_p(A) \otimes C_q(B)$ and $b(c_1 \otimes c_2) = bc_1 \otimes c_2 + (-1)^p c_1 \otimes bc_2$ for $c_1 \in C_p(A)$ and $c_2 \in C_q(B)$. 

2. $(C_\bullet(A) \times C_\bullet(B))_n = C_n(A) \otimes C_n(B)$, and $b_i(c_1 \otimes c_2) = b_i(c_1) \otimes b_i(c_2)$.

3. $C_\bullet(A \otimes B)$ where the product is defined using the Koszul sign rule: $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|}(a_1 a_2 \otimes b_1 b_2)$.

**Theorem 2.1.** Let $A, B$ be as above. The computation of the (graded) Hochschild homology of $A \otimes B$ can be reduced to the computation of the Hochschild homologies of $A$ and $B$:

$$H_n(C_\bullet(A \otimes B)) \cong \bigoplus_{p+q=n} H_p(C_\bullet(A)) \otimes H_q(C_\bullet(B)).$$

The isomorphism is given by the shuffle map:

$$sh : \bigoplus_{p+q=n} H_p(C_\bullet(A)) \otimes H_q(C_\bullet(B)) \to H_n(C_\bullet(A \otimes B))$$

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_p) \otimes (b_0 \otimes b_1 \otimes \cdots \otimes b_q) \mapsto \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) \sigma((a_0 \otimes b_0) \otimes (a_1 \otimes 1) \otimes \cdots \otimes (a_p \otimes 1) \otimes (1 \otimes b_1) \otimes \cdots \otimes (1 \otimes b_q))$$

where the sum goes over all $(p, q)$-shuffles. $\sigma$ acts on the tensor product by permutation of the elements $\{(a_1 \otimes 1), \ldots, (a_p \otimes 1), (1 \otimes b_1), \ldots, (1 \otimes b_q)\}$ and the natural sign that comes from the Koszul rule.

**Proof.** We split the equivalence of the homologies in the following steps

$$H_n(C_\bullet(A \otimes B)) \cong H_n(C_\bullet(A) \times C_\bullet(B)) \cong H_n(C_\bullet(A) \otimes C_\bullet(B))$$

$$\cong \bigoplus_{p+q=n} H_p(C_\bullet(A)) \otimes H_q(C_\bullet(B))$$

It is easy to see that the first equivalence is merely a reordering of the terms in the tensor products. One gets the isomorphism by taking into account the Koszul sign convention. The second equivalence is due to the Eilenberg-Zilber theorem (see e.g. [21], chapter 8, p.239ff) which holds for simplicial modules. The map from right to left is the shuffle map. The last equivalence comes from the Künneth short exact sequence and the fact that we deal with projective modules (see e.g. [22], 1.0.16, p.7). The isomorphism is the canonical map. 

The **Hochschild-Kostant-Rosenberg theorem**

We will only need the following special case of the HKR theorem (see e.g. [22], 3.2.2, p. 93f):
CHAPTER 2. HOCHSCHILD HOMOLOGY AND TRACES

Theorem 2.2. $HH_k^{\text{poly}}(\mathbb{C}^n)$, the $k$-th Hochschild homology of the commutative algebra of polynomials on $\mathbb{C}^n$, is spanned by all elements of the form

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) f(x_1, \ldots, x_n) \otimes x_{i_{\sigma(1)}} \otimes \cdots \otimes x_{i_{\sigma(k)}},$$

where $f$ is a polynomial on $\mathbb{C}^n$, $i_1 < \cdots < i_k$.

There is an analogous theorem for the Grassmann algebra:

Theorem 2.3. Let $G_n$ be the Grassmann algebra over $\mathbb{C}$ generated by the elements $\{\theta_1, \ldots, \theta_n\}$. Then a basis for $HH_k(G_n)$ is given by the elements of the form

$$\sum_{\sigma \in S_k} f(\theta_1, \ldots, \theta_n) \otimes \theta_{i_{\sigma(1)}} \otimes \cdots \otimes \theta_{i_{\sigma(k)}}$$

where $f \in G_n$ and $i_1 \leq \cdots \leq i_k$.

Proof. We use the normalized complex $\overline{C}_*(G_n)$. In the case $n = 1$, $\overline{C}_k(G_1)$ is spanned by the elements $1 \otimes \theta \otimes \cdots \otimes \theta$ and $\theta \otimes \theta \otimes \cdots \otimes \theta$. Obviously both elements lie in the kernel of $b$ and therefore form a basis for the homology.

The case $n > 1$ follows from theorem 2.1 by taking tensor products: As the sign coming from the permutation is exactly canceled by the Koszul sign, the shuffle map symmetrizes the terms which proves the claim.

Hochschild homology with values in modules

Let $M_1, M_2, M_3$ be $A$-bimodules and $\bullet : M_1 \otimes_A M_2 \to M_3$ a homomorphism of $A$-bimodules. Then there is a chain map $\cup : C^p(A, M_1) \otimes C^q(A, M_2) \to C^{p+q}(A, M_3)$ defined by

$$\varphi \cup \psi(a_1, \ldots, a_{p+q}) = \varphi(a_1, \ldots, a_p) \bullet \psi(a_{p+1}, \ldots, a_{p+q}).$$

In chapter 4, we will deal with a special bicomplex where the module $M = \otimes M^j$ is itself a finite complex of $A$-bimodules. We then denote by $C^*(A, M)$ the total complex of the double complex

$$C^{p,q}(A, M) = \text{Hom}(A^{\otimes p}, M^q)$$

with differential $\delta = d_H + (-1)^p d_M : C^{p,q} \to C^{p+1,q} \oplus C^{p,q+1}$. The complex of $A$-bimodules dual to $M$ is $(M^* = \oplus (M^*)^j, d_{M^*})$ with $(M^*)^j = (M^{-j})^*$, $d_{M^*} \varphi = (-1)^j \varphi \circ d_M$ for $\varphi \in (M^j)^*$ and action of $A$ defined by $a \cdot \varphi(x) = \varphi(xa), \varphi \cdot a(x) = \varphi(ax), a \in A, x \in M$. With these definitions, $C^*(A, A^*)$ is the complex dual to the Hochschild chain complex $C_*(A)$. 
2.2 Twist by a Maurer-Cartan element

With any homomorphism \( \bullet : M_1 \otimes_A M_2 \to M_3 \) of complexes of \( A \)-bimodules is associated a chain map, the \emph{cup product} \( \cup : C^{p,q}(A, M_1) \otimes C^{p',q'}(A, M_2) \to C^{p+p',q+q'}(A, M_3) \). In addition to the above definition of the product, we have to add a sign:

\[
\varphi \cup \psi(a_1, \ldots, a_{p+p'}) = (-1)^{q'p} \varphi(a_1, \ldots, a_p) \bullet \psi(a_{p+1}, \ldots, a_{p+p'}).
\]

We will use this construction in two special cases: (a) \( M_1 = M_2 = M_3 = M \) is a differential graded algebra whose product factors through \( M \otimes_A M \) defining thus a map \( \bullet : M \otimes_A M \to M \). (b) \( M_1 = M \) is a complex of \( A \)-bimodules, \( M_2 = M^*, \ M_3 = A^* \) with zero differential and \( \bullet : M^* \otimes_A M \to A^* \) is the map \((\varphi, x) \mapsto (y \mapsto \varphi(xy))\).

2.2 Twist by a Maurer-Cartan element

In this section, we would like to explain a general construction on the chain complex of an arbitrary differential graded algebra that will be used later in the sections 2.5 and 3.6.

Let \( A = \oplus_{j \in \mathbb{Z}} A^j \) be a differential graded algebra with unit and with differential \( d : A^j \to A^{j+1} \). We denote by \( |a| = j \) the degree of a homogeneous element \( a \in A^j \). The Hochschild chain complex of \( A \) is \( C^\bullet(A) = \oplus_{p \in \mathbb{Z}} C^p(A) \) with

\[
C^p(A) = \Pi_{j_r - q = p} \bar{A}^{j_1} \otimes \cdots \otimes \bar{A}^{j_q}.
\]

The Hochschild differential \( b \) is defined as in (2.1) except that the last term has an additional sign \((-1)^{|a_p|(|a_0| + \cdots + |a_{p-1}|)}\) and the differential \( d \) is extended as a derivation of degree 1 for the tensor product:

\[
d(a_0, \ldots, a_p) = \sum_{j=0}^p (-1)^{|a_0| + \cdots + |a_{j-1}|} \langle a_0, \ldots, da_j, \ldots, a_p \rangle.
\]

A \emph{Maurer-Cartan element} of \( A \) is an element \( \omega \in A^1 \) of degree 1 obeying the Maurer-Cartan equation

\[
d\omega + \omega^2 = 0.
\]

The Maurer-Cartan equation implies that \( d_\omega a = da + \omega a - (-1)^{|a|} \omega \omega = : da + [\omega, a] \) is a differential. Moreover \( d_\omega \) is a derivation of degree 1 of the algebra \( A \) and therefore the algebra \( A \) with differential \( d_\omega \) is a differential graded algebra. We call this differential graded algebra the twist of \( A \) by \( \omega \) and denote it \( A_\omega \).

\(^1\)We introduce the upper index notation \( C^q \) to have a differential of degree one as \( d \) is. Thus in the ungraded case we have \( C^q(A) = C_{-q}(A) \), concentrated in negative degrees.
The symmetric group $S_p$ acts on $A \otimes \bar{A}^p$ by permutations of the last $p$ factors with Koszul signs: the transposition of neighboring factors $a$ and $b$ is accompanied by the sign $(-1)^{|a||b|}$. Recall that the shuffle product $C_p(A) \otimes C_q(A) \to C_{p+q}(A)$ is defined by

$$(a_0, \ldots, a_p) \times (b_0, \ldots, b_q) = (-1)^{|b_0|} \sum_{j} |a_j| \text{sgn}(\pi) \text{sh}_{p,q}(a_0 b_0, a_1, \ldots, a_p, b_1, \ldots, b_q)$$

where $\text{sh}_{p,q} x = \sum_{\pi \in S_{p+q}} \text{sgn}(\pi) \pi \cdot x$, with sum over $(p, q)$-shuffles in $S_{p+q}$. Note that the sign $\text{sgn}(\pi)$ is additional to the above explained Koszul sign. The shuffle product is associative, and if $A$ is Abelian (which we do not assume), it is a homomorphism of complexes, see [21, 22].

**Proposition 2.4.** Let $A_\omega$ be the twist of $A$ by a Maurer–Cartan element $\omega \in A^1$. Let $(\omega)_k = (1, \omega, \ldots, \omega)$ with $k$ factors of $\omega$. Then the map

$$(a_0, \ldots, a_p) \mapsto \sum_{k \geq 0} (-1)^k (a_0, \ldots, a_p) \times (\omega)_k$$

is an isomorphism of complexes $C(A_\omega) \to C(A)$.

We split the proof into a few steps.

**Lemma 2.5.** $b(\omega)_0 = 0$ and, for $k \geq 1$, $b(\omega)_k = d(\omega)_{k-1}$.

**Proof.** The first statement is obvious. Let $k \geq 1$. Then

$$b(\omega)_k = b(1, \omega, \ldots, \omega) = (\omega, \ldots, \omega) + \sum_{j=1}^{k-1} (-1)^j (1, \omega, \ldots, \omega^2, \ldots, \omega) + (-1)^k (-1)^{k-1} (\omega, \ldots, \omega)$$

$$= - \sum_{j=1}^{k-1} (-1)^j (1, \omega, \ldots, d\omega, \ldots, \omega)$$

$$= d(\omega)_{k-1}.$$ 

□

**Lemma 2.6.** Let $a \in C^p(A)$. Then

$$b(a \times (\omega)_k) = b a \times (\omega)_k + (-1)^p a \times b(\omega)_k$$

$$- (-1)^p \sum_{j=0}^{p} (-1)^{|a_0| + \cdots + |a_j-1|} ((a_0, \ldots, [\omega, a_j], \ldots, a_p) \times (\omega)_{k-1},$$

where $[a, a'] = aa' - (-1)^{|a||a'|} a'a$ is the graded commutator.
2.2. TWIST BY A MAURER-CARTAN ELEMENT

Proof. For simplicity, we give the proof in the case where all $a_j$ are of degree 0, which is the case appearing in our application. The additional signs appearing in the general case can be treated easily.

If we write out the sum over shuffles, we see that there are four types of terms appearing on the left-hand side: those containing the products $a_j a_{j+1}$, $\omega^2$, $\omega a_j$ and $a_j \omega$. The terms of the first and of the second type combine to give the first two terms on the right-hand side. The proof that the signs match is the same as in the proof of the homomorphism property for commutative algebras, see [22], proposition 4.2.2, so we consider only the last two types. Consider a shuffle $\pi$ appearing on the left-hand side such that $l$ out of the $k$ factors $\omega$ have been shuffled to the left of $a_j$. Then the term containing the product $\omega a_j$ comes with a sign $\text{sgn}(\pi)(-1)^{j-1+l}$. The same term occurs for a shuffle $\pi'$ in $(a_0, \ldots, [\omega, a_j], \ldots, a_p) \times (\omega)^{k-1}$ with a sign equal to $\text{sgn}(\pi')(1)^{l-1}$, where $(-1)^{l-1}$ is the Koszul sign coming from the fact that $l - 1$ factors $\omega$ are permuted by $\pi'$ to the left of the odd element $[\omega, a_j]$. The signs of the shuffles are related by $\text{sgn}(\pi) = \text{sgn}(\pi')(1)^{p-j+1}$. The ratio of signs is thus $(-1)^{p-1}$, as claimed. The same reasoning can be applied to $a_j \omega$.

Lemma 2.7. Let $a \in C^p(A)$ and set $\delta a = b a + (-1)^p d a$. Then

$$\delta \sum_{k \geq 0} (-1)^k a \times (\omega)_k = \sum_{k \geq 0} (-1)^k \delta a \times (\omega)_k.$$ 

Proof. This follows from the previous lemma by inserting the definitions and summing over $k$. □

Lemma 2.8. The map $C(A) \to C(A_\omega)$ of proposition 2.4 is an isomorphism.

Proof. The map is the shuffle multiplication by $\psi = \sum (-1)^k (\omega)_k$. We claim that the inverse map is the shuffle multiplication by $\bar{\psi} = \sum (\omega)_k$. Using the fact that the shuffle product is associative, it suffices to show that $\psi \times \bar{\psi} = 1$. This follows from

$$(\omega)_k \times (\omega)_l = \sum_{\pi \in S_{k,l}} (\omega)_{k+l} = \binom{k+l}{k} (\omega)_{k+l}.$$ 

□

Proposition 2.4 follows from the last two lemmata.
2.3 Hochschild homology of the Weyl-Clifford algebra

We consider the polynomial Weyl algebra $A_{2n}^{pol} = \mathbb{C}[y](\varepsilon)$, $y \in \mathbb{R}^{2n}$ over the ring $k = \mathbb{C}(\varepsilon)$ of Laurent polynomials in $\varepsilon$ and with product defined in the same way as the product for the usual Weyl algebra. We also consider the Clifford algebra $\mathcal{C}_m$ as algebra over the ring $k = \mathbb{C}(\varepsilon)$ and define $A_{2n|m}^{pol} = A_{2n}^{pol} \otimes_k \mathcal{C}_m$.

We will show here that the Hochschild homology of the polynomial Weyl-Clifford algebra $A_{2n|m}^{pol}$ vanishes except in degree $2n$ where it is generated by the cycle

$$c_{2n|m} = \sum_{\sigma \in S_{2n}} \theta_1 \cdots \theta_m \otimes y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(2n)}.$$

We only have to compute the Hochschild homologies of $A_{2n}^{pol}$ and $\mathcal{C}_m$. The above statement then follows immediately from theorem 2.1. The following fact is well known

**Theorem 2.9.** The Hochschild homology of $A_{2n}^{pol}$ is zero except in degree $2n$ where $HH^{2n}(A_{2n}^{pol}) \cong \mathbb{C}(\varepsilon)$ is generated by the (class of the) cycle

$$c_{2n} = \sum_{\sigma \in S_{2n}} 1 \otimes y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(2n)}.$$

**Proof.** It is sufficient to show the case $n = 1$ because the general case then follows easily using theorem 2.1.

The Weyl algebra $A_{2n}^{pol}$ has a filtration $F$ with respect to the degree in $\varepsilon$, more explicitly $A_2^{pol} = F^0 \supset F^1 \supset \cdots \supset \{0\}$. $E^0 = F_0/F_0+1$ is then $\varepsilon^p$ times the algebra of polynomials over $\mathbb{C}$ with the standard commutative product. Therefore $E^p_{1}$ is the homology of the polynomials which is given by the HKR theorem. We compute the homology of $E^p_{1}$ where the differential map is $b : E^p_{1} \to F^p \mod F^{p+2}$. Short calculations show that

$$b(f(y_1, y_2)) = 0$$
$$b(f(y_1, y_2) \otimes y_1 + g(y_1, y_2) \otimes y_2) = -\partial f/\partial y_2 + \partial g/\partial y_1 = -\text{rot}(f, g)$$
$$b(f(y_1, y_2) \otimes (y_1 \otimes y_2 - y_1 \otimes y_2)) = -(\partial f/\partial y_1 \otimes y_1 + \partial f/\partial y_2 \otimes y_2) = -\text{grad}(f) \otimes (y_1, y_2)^T$$

It follows from the de Rham cohomology of $\mathbb{R}^2$ that the homology vanishes, except in degree 2, where it is generated by $1 \otimes (y_1 \otimes y_2 - y_2 \otimes y_1)$. \hfill \Box

**Theorem 2.10.** The Hochschild homology of $A_{0|m}$ is zero, except in degree 0, where $HH_0(A_{0|m}) \cong \mathbb{C}(\varepsilon)$ is generated by (the class of) $c_{0|m} = \theta_1 \cdots \theta_m$. 
2.4. HOCHSCHILD COCYCLE OF THE WEYL-CLIFFORD ALGEBRA

Proof. It is again sufficient to consider the case \( m = 1 \). Analogously to the previous section, we consider the filtration in \( \varepsilon \) and use "HKR" (see theorem 2.3) for the \( E^p_1 \)-terms in the spectral sequence. \( E_1 \) is spanned by the elements

\[
A^j_1 = \theta \otimes \cdots \otimes \theta, \quad A^j_2 = \theta \otimes \cdots \otimes \theta
\]

As \( \theta \star \theta = -\varepsilon \), we get for \( j \geq 0 \)

\[
dA^j_1 = 0, \quad dA^{j+1}_2 = -2\varepsilon A^j_1.
\]

It is therefore clear that the homology of \( E_1 \) only consists of the class of \( A^0_2 = \theta =: c_{0\mid 1} \).

2.4 Hochschild cocycle of the Weyl-Clifford algebra

We give in this chapter an explicit formula for a Hochschild cocycle of the Weyl-Clifford algebra and show that it satisfies some additional invariance properties.

We first introduce some notation. Let \( \psi = 2B_1 \) where \( B_1 \) is the first 1-periodically continued Bernoulli polynomial. Explicitly \( \psi \) is the 1-periodic function that is defined as \( \psi(u) = 1 + 2u \) for \(-1 \leq u < 0\). We write \( \alpha_{jk} \) for the bivector field \( \alpha \) acting on the \( j \)-th and \( k \)-th factor of a tensor product and likewise for \( g \). We then define for \( u_i \in [0, 1] \) the following maps that act on a tensor product of \( 2n+1 \) sections in the Weyl bundle where the numbering starts with 0.

\[
\omega_{2n} = \exp \left( \sum_{0 \leq j < k \leq 2n} \varepsilon \psi(u_j - u_k)(\alpha_{jk} + g_{jk}) \right)
\]

\[
\pi_{2n} = \frac{1}{n!} \left( \sum_{1 \leq j < k \leq 2n} \alpha_{jk} du_j \wedge du_k \right)^n
\]

\[
\mu_{2n|m}(a_0 \otimes \cdots \otimes a_{2n}) = \int (a_0 \cdots a_{2n})(y, \theta) \bigg|_{y=0} d\theta_1 \cdots d\theta_m.
\]

The integral in \( \mu_{2n|m} \) is the Berezin integral with sign so that

\[
\int \theta_1 \cdots \theta_m d\theta_1 \cdots d\theta_m = 1.
\]

Theorem 2.11. There is a cocycle \( \tau_{2n|m} \in HH^{2n}(A^{pol}_{2n|m}) \) given by the formula

\[
\tau_{2n|m} = \mu_{2n|m} \circ \int_{\Delta_{2n}} \omega_{2n} \circ \pi_{2n}
\]

where \( u_0 = 0, \Delta_{2n} = \{(u_1, \ldots, u_{2n}) \in [0, 1]^{2n} \mid j < k \Rightarrow u_j < u_k\} \). The cocycle \( \tau_{2n|m} \) satisfies the additional properties:
(i) $\tau_{2n|m}(c_{2n|m}) = 1$. In particular the cocycle is nontrivial.

(ii) It is $sp_{2n}(\mathbb{C})$- and $so_{m}(\mathbb{C})$-invariant:

$$\sum_{i=0}^{2n} \tau_{2n|m}(a_0 \otimes \cdots \otimes [a, a_i] \otimes \cdots \otimes a_{2n}) = 0$$

where $a = \frac{1}{2}a_{jk}y^j y^k$ or $a = \frac{1}{2}a_{jk}\theta^j \theta^k$ and $[\ldots]$ is the super-commutator in the Weyl-Clifford algebra.

(iii) If $a$ is again as in (ii), then

$$\sum_{i=1}^{2n} (-1)^i \tau_{2n|m}(a_0 \otimes \cdots \otimes a_{i-1} \otimes a \otimes a_i \otimes \cdots \otimes a_{2n-1}) = 0.$$ 

The above theorem is a straightforward generalization of a theorem in [16] where it was shown for $m = 0$. As in the mentioned article, we will use the above cocycle with its additional properties to construct maps from the Hochschild homology of the algebra $(C^\infty(M, \Lambda^* E^*), \star)$ to the de Rham cohomology $H_{dR}(M[[\varepsilon]])$ (see the following section). Also for the proof of the theorem we follow [16].

Proof. First, we show the cocycle property. We define an object

$$\eta \in \Omega^{2n}(\Delta_{2n+1}, \text{Hom}(A_{2n|m})^{2n+1}, \mathbb{C}[[\varepsilon]])$$

$$\eta = \mu_{2n+1|m} \circ \omega_{2n+1} \circ \pi_{2n+1}$$

where the map $\mu_{2n+1|m} : (A_{2n|m})^{\otimes (n+1)} \rightarrow \mathbb{C}[[\varepsilon]]$ is defined analogously to $\mu_{2n|m}$ and the maps $\omega_{2n+1}$ and $\pi_{2n+1}$ are defined as for the index $2n$ but with sum until $2n + 1$ instead of $2n$. We first show that $\eta$ is closed, i.e. $d\eta = 0$. Differentiating $\eta$ leads to a multiplication of the above exponential by

$$\sum_{0 \leq j < k \leq 2n+1} (du_j - du_k)(\alpha_{jk} + g_{jk}) = \sum_{0 \leq j, k \leq 2n+1} du_j(\alpha_{jk} + g_{jk}).$$

We used that $\alpha_{jk} = -\alpha_{kj}$ and $g_{jk} = -g_{kj}$. The $\alpha$-terms then vanish because the expression

$$du_t \alpha_{st} \wedge \left( \sum_{1 \leq j < k \leq 2n+1} \alpha_{jk} du_j \wedge du_k \right)^n$$

is antisymmetric under permutation of the components in $y = (y_1, \ldots, y_{2n})$, but as there are $2n + 1$ arguments, it must therefore be zero. For the $g$-term, we consider the composition with the Berezin integral:

$$\int \sum_{0 \leq k \leq 2n+1} du_j g_{jk}.$$
2.4. HOCHSCHILD COCYCLE OF THE WEYL-CLIFFORD ALGEBRA

Consider for some fixed \( j \) the application of this operator to an element \( a := a_0 \otimes \cdots \otimes a_{2n+1} \), where \( a_i = f_i(\theta)g_i(y) \) and we may assume that \( f_i(\theta) \) are monomials. If the result doesn’t vanish then one Grassmann variable – say \( \theta_1 \) – must appear exactly three times in \( a \). Once in \( a_j \) and also once in \( a_{k_1} \) and \( a_{k_2} \). Therefore we get exactly two terms corresponding to \( k = k_1 \) and \( k = k_2 \). It’s not difficult to check that these two terms have opposite sign and therefore cancel. We now use Stokes’ theorem:

\[
0 = \int_{\Delta_{2n+1}} d\eta = \sum_{k=0}^{2n+1} (-1)^k \int_{\Delta_{2n}} i^*_k \eta \tag{2.1}
\]

where

\[
i_0(u_1, \ldots, u_{2n}) = (0, u_1, \ldots, u_{2n}),
\]

\[
i_k(u_1, \ldots, u_{2n}) = (u_1, \ldots, u_k, u_k, \ldots, u_{2n}),
\]

\[
i_{2n+1}(u_1, \ldots, u_{2n}) = (u_1, \ldots, u_{2n}, 1).
\]

If we apply equation (2.1) to \( (a_0, \ldots, a_{2n+1}) \), we get

\[
\sum_{i=0}^{2n} (-1)^i \tau_{2n|m}(a_0, \ldots, a_i \ast a_{i+1}, \ldots, a_{2n+1}) + \left( -1 \right)^{2n+1+\lfloor a_{2n+1} \rfloor} \sum_{i=0}^{2n} \lfloor a_i \rfloor \tau_{2n|m}(a_{2n+1} \ast a_0, a_1, \ldots, a_{2n}) = 0
\]

which is exactly the cocycle condition for \( \tau_{2n|m} \).

The identity \( \tau_{2n|m}(c_{2n|m}) = 1 \) is almost obvious. The next step is to verify the \( \mathfrak{sp}_{2n}(\mathbb{C}) \times \mathfrak{so}_m(\mathbb{C}) - \)invariance. \( A_{2n|m} \) and \( \mathbb{C}[\varepsilon] \) are \( \mathfrak{sp}_{2n}(\mathbb{C}) \times \mathfrak{so}_m(\mathbb{C}) \)-modules. The first one with the action \( \text{ad}(a) \) for \( a \in \mathfrak{sp}_{2n}(\mathbb{C}) \times \mathfrak{so}_m(\mathbb{C}) \) and the second with the trivial action. This also makes \( A_{2n|m}^{\otimes 2n} \) to an \( \mathfrak{sp}_{2n}(\mathbb{C}) \times \mathfrak{so}_m(\mathbb{C}) \)-module. First we verify that \( \alpha_{jk} \) and \( g_{jk} \) are module morphisms. We consider \( \alpha_{jk} \) and \( a \in \mathfrak{sp}_{2n}(\mathbb{C}) \). It suffices to do check the following case \( (a \in \mathfrak{sp}_{2n}(\mathbb{C})) \):

\[
\alpha_{12} \text{ad}(a)(a_1 \otimes a_2) = \text{ad}(a)\alpha_{12}(a_1 \otimes a_2)
\]

\[
= \alpha^{ij} \frac{\partial^2 a}{\partial y^i \partial y^j} \alpha^{kl} \frac{\partial}{\partial y^l} \otimes \frac{\partial}{\partial y^j} + \alpha^{ij} \frac{\partial}{\partial y^i} \otimes \frac{\partial^2 a}{\partial y^j \partial y^k} \alpha^{kl} \frac{\partial}{\partial y^j} = 0.
\]

The computation for \( a \in \mathfrak{so}_m(\mathbb{C}) \) and \( g_{ij} \) is almost the same. As \( a \in \mathfrak{so}_m(\mathbb{C}) \) doesn’t depend on \( y \), its action commutes with \( \alpha_{jk} \) for trivial reasons, and likewise for \( a \in \mathfrak{sp}_{2n}(\mathbb{C}) \) and \( g_{ij} \).

It remains to check that \( \mu_{2n|m} \) is a module morphism. As \( [a, a_j]_{y=0} = 0 \) for \( a \in \mathfrak{sp}_{2n}(\mathbb{C}) \), it is an \( \mathfrak{sp}_{2n}(\mathbb{C}) \)-morphism. Now consider \( a = \theta_j \theta_k \). For \( \mu_{2n|m} \circ \text{ad}(a)(a_0 \otimes \cdots \otimes a_{2n}) \) not to vanish, we need the product \( (a_0 \otimes \cdots \otimes a_{2n}) \) to possess no factor \( \theta_j \) and exactly two factors \( \theta_k \) in different \( a_i \)'s (or the same with \( j \leftrightarrow k \)) to survive the Berezin integral. Then
ad(a) produces exactly two terms corresponding to the two factors \( \theta_k \). As the order of the factors \( \theta_1 \ldots \theta_m \) of the two terms differ by a transposition, the terms cancel. Therefore we have shown that \( \tau_{2n|m} \) is a composition of \( \text{sp}_{2n}(\mathbb{C}) \times \text{so}_m(\mathbb{C}) \)-module morphisms, the last having trivial action. This proves (ii).

As in [16] (lemma 2.2), we show the behavior of \( \tau_{2n|m} \) under permutations of the arguments in \((\mathcal{A}_{2n|m})_{2n+1}\). Let the permutations \( S_{2n} \) act on \((\mathcal{A}_{2n|m})_{2n+1}\) by permuting the last \( 2n \) arguments taking into account the Koszul sign rule. We write \( \rho(\sigma) \) for this action:

\[
\tau \circ \rho(\sigma)(a_0, a_1, \ldots, a_{2n}) = \pm \tau(a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(2n)}) .
\]

We now omit the indices "2n and m" in the notation and write

\[
\tau = \mu \circ \int_{\Delta_{2n}} \omega \circ \pi .
\]

It is straightforward to verify the following identities:

\[
\pi \circ \rho(\sigma) = \rho(\sigma) \circ \sigma^* \pi ,
\]

\[
\omega \circ \rho(\sigma) = \rho(\sigma) \circ \sigma^* \omega ,
\]

\[
\mu \circ \rho(\sigma) = \mu
\]

where \( \sigma \in S_{2n} \) is also understood as a map \( \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) that permutes the variables. Putting these identities together

\[
\tau \circ \rho(\sigma) = \mu \circ \int_{\Delta_{2n}} \omega \circ \pi \circ \rho(\sigma) = \mu \circ \int_{\Delta_{2n}} \sigma^*(\omega \circ \pi)
\]

\[
= \mu \circ \int_{\sigma(\Delta_{2n})} \text{sgn}(\sigma) \omega \circ \pi =: \text{sgn}(\sigma) \tau^\sigma .
\]

\text{sgn}(\sigma) appears because we consider \( \sigma(\Delta_{2n}) \) with its orientation from the embedding in \( \mathbb{R}^{2n} \) which differs by \( \text{sgn}(\sigma) \) from the orientation induced by the map \( \sigma \). Write \( S_{1,2n-1} \) for the \((1,2n-1)\)-shuffles. Then the left hand side of the property (iii) can be written as

\[
- \sum_{\sigma \in S_{1,2n-1}} \text{sgn}(\sigma) \tau \circ \rho(\sigma)(a_0, \ldots, a_{2n}) = - \sum_{\sigma \in S_{1,2n-1}} \mu \circ \int_{\sigma(\Delta_{2n})} \omega \circ \pi(a_0, \ldots, a_{2n})
\]

\[
= - \mu \circ \int_{[0,1] \times \Delta_{2n-1}} \omega \circ \pi(a_0, \ldots, a_{2n})
\]

where \( u_1 \in [0, 1] \) and \((u_2, \ldots, u_{2n}) \in \Delta_{2n-1} \) and \( a_1 = a \). If \( a \in \text{sp}_{2n}(\mathbb{C}) \) then \( a \) is quadratic in \( y \) and only terms of \( \omega \) linear in \( \alpha_{1j} \) contribute. Therefore the only factor in \( \omega \) depending on \( u_1 \) is \( \sum_{i \neq 1} \psi(u_1 - u_i) \alpha_{1i} \). The integral over \( u_1 \) vanishes because

\[
\int_0^1 \psi(u_1 - u_i) \, du_1 = 0 .
\]

If \( a \in \text{so}_m(\mathbb{C}) \), each term gets separately annihilated by \( \pi \). \( \square \)
Corollary 2.12. We consider the algebra $A_{2n|m}^{pol} = A_{2n|m}^{pol} \otimes \mathbb{C} M_r(\mathbb{C})$. Because of Morita invariance

$$\tau^r_{2n|m}(a_0 \otimes M_0, \ldots, a_{2n} \otimes M_{2n}) := \tau_{2n|m}(a_0, \ldots, a_{2n}) \text{tr}(M_0 \cdots M_{2n})$$

defines a cocycle $\tau^r_{2n|m} \in HH^{2n}(A_{2n|m}^{pol})$. It has the invariance properties mentioned in the previous theorem and is obviously also invariant under the adjoint action of $gl(m, \mathbb{C})$ on the matrices $M_j$.

Corollary 2.13. From the proof of the above theorem, it is clear that $\tau_{2n|m}$ and $\tau^r_{2n|m}$ are also cocycles for the non-polynomial algebras $A_{2n|m}$ and $A_{2n|m}^r$ and still satisfy the same invariance properties.

### 2.5 A map from the Hochschild homology to the de Rham Cohomology

Let $M$ be a symplectic manifold and assume that we have chosen a Fedosov connection on the Weyl bundle over $M$. In [16], it was shown that the cocycle $\tau^r_{2n}$ from the last section can be used to construct a trace on the flat sections in the Weyl bundle. A map from the zeroth Hochschild homology of $\Gamma_D(M,W)$ to the 2n-th de Rham cohomology of $M$ with values in $\mathbb{C}[[\epsilon]]$ is constructed. The composition of this map with the integral over $M$ then defines a trace on the algebra $(C^\infty(M), \star)$ which is isomorphic to $\Gamma_D(M,W)$. We will first show that this construction can be used to produce maps from the $k$-th Hochschild homology to the $(2n-k)$-th de Rham cohomology and then give a generalization of this construction to supersymplectic manifolds.

#### 2.5.1 Classical construction

Recall that we defined the Weyl bundle as

$$W(M) = F_G(M) \times_G A_{2n}^r$$

where $G = GL_r(\mathbb{C}) \times Sp_{2n}(\mathbb{C})$. We now write $\tau := \tau^r_{2n}$ for the cocycle in $HH^{2n}(A_{2n}^r)$ that is invariant under the actions of $Sp_{2n}(\mathbb{C})$ and $GL_r(\mathbb{C})$. Then $\tau$ induces a map (for which we still write $\tau$)

$$\tau : F_G(M) \times_G (A_{2n}^r)^{\otimes (2n+1)} \to M \times \mathbb{C}[[\epsilon]].$$

Similarly $\tau$ also induces a map from forms with values in the Weyl bundle to forms on $M$ with values in $\mathbb{C}[[\epsilon]]$. Assume that we have chosen a symplectic connection on $TM$ and
a connection on \( \mathcal{E} \). Let \( A \in \Omega^1(M,W) \) be a 1-form defining a Fedosov connection (see theorem 1.1). We then define the family of maps

\[
\chi_k : \Gamma(M,W)^\otimes(k+1) \to \Omega^{2n-k}(M,\mathbb{C}[[\varepsilon]])
\]

where \( \times \) is the shuffle map, \( (A)_k := (1,A)^k = (1,A,\ldots,A) \) and

\[
(f_0,\ldots,f_k) \times (1,A,\ldots,A) = \sum_{\sigma \in S_{k,2n-k}} \text{sgn}(\sigma)(f_0,g_{\sigma(1)},\ldots,g_{\sigma(2n)})
\]

where \( g_i = f_i \) for \( i \in \{1,\ldots,k\} \) and \( g_i = A \) for \( i \in \{k+1,\ldots,2n\} \). These maps have the following properties:

**Theorem 2.14.** Recall that \( \Gamma_D(M,W) \) is the algebra of flat sections in the Weyl bundle. For \( k = 0,\ldots,2n \), \( \chi_k \) are maps \( HH_k(\Gamma_D(M,W)) \to H_{dR}^{2n-k}(M,\mathbb{C}[[\varepsilon]]) \) which are invariant under the flow of the Heisenberg equation.

**Proof.** First, we compute \( d(\chi_k(f_0,\ldots,f_k)) \). In local coordinates, the derivatives acts by Leibniz’s rule on the arguments \( (f_0,\ldots,f_k,A,\ldots,A) \) of \( \tau \). Recall that the symplectic connection on the Weyl bundle can locally be written in the form \( \nabla = d + \text{ad}(\Gamma) \) where \( \Gamma \) is a 1-form with values in \( \text{sp}_{2n}(\mathbb{C}) \times \text{gl}_r(\mathbb{C}) \subset \mathcal{A}_r^{2n} \). As \( \tau \) is \( \text{sp}_{2n}(\mathbb{C}) \)- and \( \text{gl}_r(\mathbb{C}) \)-invariant (see property ii) in theorem 2.11), the de Rham differentials can be replaced by the covariant derivative:

\[
d(\chi_k(f_0,\ldots,f_k)) = \tau(\nabla((f_0,\ldots,f_k) \times (A)_{2n-k})).
\]

Now, we refer to section 2.2. To apply lemma 2.7 to our situation, we have to use the following "translation": \( \omega \leftrightarrow A, d \leftrightarrow \nabla, d_\omega \leftrightarrow D \). The algebra \( A \) is in our case the algebra \( \Omega^*(M,W) \) with the fibrewise Moyal product and \( a = (f_0,\ldots,f_k) \). Then the lemma states

\[
\nabla(a \times (A)_{(2n-k)}) = Da \times (A)_{(2n-k)} \]

\[
+(-1)^ka \times (A)_{2n+1-k} - (-1)^kb \times (A)_{2n+1-k}.
\]

The first term on the right hand side vanishes because the \( f_i \) are flat sections. Inserted in \( \tau \), the second term vanishes because \( \tau \) is a cocycle. Therefore we find

\[
d \circ \chi_{k+1} = (-1)^k\chi_k \circ b
\]

which shows that \( \chi_k : HH_k(\Gamma_D(M,W)) \to H_{dR}^{2n-k}(M,\mathbb{C}[[\varepsilon]]) \).
Recall that the Heisenberg equation is generated by a Hamiltonian $H_t$ (see section 1.7). Then $\frac{d}{dt} H_t = \nabla H_t + [A, H_t] - \lambda_t = D_A H_t - \lambda_t$, $\frac{d}{dt} f_{i,t} = -[H_t, f_{i,t}]$. We drop the $t$ in the index, write $a = (f_0, \ldots, f_k)$ and simply $D$ instead of $D_A$:

$$\frac{d}{dt} \tau(a \times (A)_{2n-k}) = \tau(-[H,a] \times (A)_{2n-k}) + \tau(a \times (1, DH) \times (A)_{2n-k-1}).$$

The terms containing $\lambda$ cancel because it takes values in the center of the algebra and we work in the normalized Hochschild complex. Using again lemma 2.7, we find

$$\frac{d}{dt} \tau(a \times (1, H) \times (A)_{2n-k-1}) = \tau(Da \times (1, H) \times (A)_{2n-k-1}) + \tau(a \times (1, DH) \times (A)_{2n-k-1}) + (-1)^k \tau(b(a \times (1, H)) \times (A)_{2n-k}).$$

A computation similar to lemma 2.6 leads to

$$b(a \times (1, H)) = ba \times (1, H) + (-1)^{k+1}[H, a].$$

Putting everything together:

$$\frac{d}{dt} \tau(a \times (A)_{2n-k}) = \frac{d}{dt} \tau(a \times (1, H) \times (A)_{2n-k-1}) + (-1)^{k+1} \tau(ba \times (1, H) \times (1, A)_{2n-k}).$$

We see that – passing to the homologies – the maps $\chi_k$ are invariant under the Heisenberg equation.

\[\Box\]

### 2.5.2 Graded construction

The generalization of the above construction to the graded case is straightforward. We assume that $E$ is an oriented bundle and write

$$W(M, E) = F_G(M) \times_G A^r_{2n|m},$$

where $G = \text{GL}_r(\mathbb{C}) \times \text{Sp}_{2n}(\mathbb{C}) \times \text{SO}_m(\mathbb{C})$ and $F_G(M)$ is the bundle of symplectic frames on $TM$, frames on $E$ and oriented orthogonal frames on $E$. It is clear that $\tau := \tau^r_{2n|m}$ induces a map

$$\tau : F_G(M) \times_G (A^r_{2n|m})^\otimes(2n+1) \to M \times \mathbb{C}[[\epsilon]].$$

We choose connections on the bundles $TM, E$ and $E$ and a 1-form $A \in \Omega(M, W)$ defining a super-Fedosov connection. Then the maps

$$\chi_k : \Gamma(M, W)^\otimes(k+1) \to \Omega^{2n-k}(M, \mathbb{C}[[\epsilon]])$$

$$\chi_k(f_0, \ldots, f_k) = \tau((f_0, \ldots, f_k) \times (A)_{2n-k})$$

are well defined. In general, we would have to add the Koszul signs corresponding to the Grassmann degree to the action of the shuffles, but as $A$ has even Grassmann degree, no such sign appears. We state
Theorem 2.15. Write $\Gamma_D(M,W)$ for the algebra of flat sections in the super-Weyl bundle. For $k = 0, \ldots, 2n$, $\chi_k$ are maps from $HH_k(\Gamma_D(M,W)) \to H^{2n-k}_{dR}(M,\mathbb{C}[[\epsilon]])$ which are invariant under the flow of the Heisenberg equation.

Proof. As in the classical case, the relations $\nabla A + A^2 = 0$, $Df_i = \nabla f_i + [A,f_i] = 0$ and $\tau \circ b = 0$ hold. The arguments in the classical proof were completely combinatorial only using the above relations, therefore they apply here without any change. As the flat connection has even Grassmann degree and as the order of the sections $f_i$ is preserved when we take shuffles, there are no additional signs compared to the classical case. □

2.6 Trace on the deformed algebra

As in the previous subsection, we from now on assume that $E$ is oriented. Consider the algebra $\mathcal{A} = (\Gamma_c(M,\Lambda^*E)[[\epsilon]],\star)$ of compactly supported smooth sections. As the quantization map $Q$ maps these functions to flat sections in the Weyl bundle with compact support, the map

$$\text{Tr}_1 : f \in \mathcal{A} \mapsto \int_M \chi_0(Q(f)) \in \mathbb{C}[[\epsilon]]$$

(2.2)

defines a trace on the algebra $\mathcal{A}$. For the trivial algebra $\Gamma_D(V,W)$, $V \subset \mathbb{R}^{2n}$ this trace can be computed explicitly. Recall that $Q(f) = f(x+y,\theta)$. We first compute $\pi(A)_{2n}$ for $A = \omega_{ij}y^i dx^j$ where

$$\begin{pmatrix} \omega_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\pi = \frac{1}{n!} \left( \sum_{1 \leq i < j \leq 2n} \alpha_{ij} du_i \wedge du_j \right)^n = du_1 \wedge \cdots \wedge du_{2n} \sum_{\text{pairs } (i_1,j_1) \cdots (i_n,j_n) \text{ with } i_k < j_k} \alpha_{i_1j_1} \cdots \alpha_{i_nj_n}$$

$$= (-1)^{(n-1)/2} du_1 \wedge \cdots \wedge du_{2n} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \text{id} \otimes \frac{\partial}{\partial y_{\sigma(1)}} \otimes \cdots \otimes \frac{\partial}{\partial y_{\sigma(2n)}}.$$

We then see that

$$\pi(A)_{2n} = (-1)^{(n-1)/2} (2n)! \ du_1 \wedge \cdots \wedge du_{2n} \ dx^1 \wedge \cdots \wedge dx^{2n} 1^{\otimes 2n+1}.$$
2.6. TRACE ON THE DEFORMED ALGEBRA

Therefore

\[
\text{Tr}_1(f) = \int_V \tau(a \times (A)_{2n}) = (-1)^{n(n-1)/2} \mu \circ \omega \text{tr}_{\mathcal{E}_x}(Q(f)) \otimes 1^\otimes 2n \\
= \int_{V \times \mathbb{R}^m} \text{tr}_{\mathcal{E}_x}(f)(x + y, \theta)\big|_{y = 0} dx_1 \cdots dx_{2n} d\theta_1 \cdots d\theta_m.
\]

Thus for the trivial case, the trace is just the integral over the supermanifold of the trace in the coefficient bundle.

As mentioned in section 1.7, the algebra \(\mathcal{A}|_U\) is isomorphic to the trivial algebra \(\Gamma_{D_0}(V, W)\), \(V \subset \mathbb{R}^{2n}\) for a small open neighborhood \(U \subset M\). Recall that the trivialization isomorphism is given by the flow of the Heisenberg equation. As the maps \(\chi_k\) are invariant under this flow, for \(a \in \Gamma_{c,D}(U, W)\) with corresponding trivialized element \(a_0 \in \Gamma_{D_0}(V, W)\), our trace is given by the integral of \(\text{tr}_{\mathcal{E}_x}(a_0)(x, y, \theta)\big|_{y = 0}\) over \(V \times \mathbb{R}^m \subset \mathbb{R}^{2n+m}\).

We want to show that if \(\text{Tr}\) is a trace on \(\mathcal{A}\) then \(\text{Tr} = C \text{Tr}_1\) for a constant \(C \in \mathbb{C}[\varepsilon]\). We first show this locally:

**Lemma 2.16.** Consider the trivial algebra \(\Gamma_{D_0}(U, W)\) for some open subset \(U \subset \mathbb{R}^{2n}\) and assume \(\text{Tr}\) is a trace on this algebra. Then

\[
\text{Tr}(a) = C \int_{\mathbb{R}^{2n+m}} \text{tr}_{\mathcal{E}_x} a(x, y, \theta)\big|_{y = 0} dx_1 \cdots dx_{2n} d\theta_1 \cdots d\theta_m
\]

for any \(a \in \Gamma_{c,D_0}(U, W)\) and some constant \(C \in \mathbb{C}[\varepsilon]\).

**Proof.** We consider terms of the form \(a(x, y, \theta) = f(x + y)\theta_{i_1} \cdots \theta_{i_j} \otimes E_{jk}\) where \(E_{jk}\) is the matrix with coefficients \((E_{jk})_{kl} = \delta_{jl}\delta_{kl}\). For \(j \neq k\), we write \(E_{jk} = [E_{jj}, E_{jk}]\), thus the trace vanishes in this case. For \(j = k\), we see that the trace is independent of \(j\) because \(E_{jj} = E_{\ell\ell} + [E_{j\ell}, E_{\ell j}]\), thus

\[
\text{Tr}(f(x)\theta_{i_1} \cdots \theta_{i_j} \otimes M) = \text{tr}(M) \text{Tr}(f(x)\theta_{i_1} \cdots \theta_{i_j} \otimes E_{jj}).
\]

If \(j < m\), there is an \(i_{j+1} \notin \{i_1, \ldots, i_j\}\) so that \(a = -[a\theta_{i_{j+1}}, \theta_{i_{j+1}}]\) and hence \(\text{Tr}(a) = 0\). In the case \(j = m\), we have \(a(x, y, \theta) = f(x + y)\theta_1 \cdots \theta_m \otimes M\). As \(\partial a/\partial x^i = [a, \omega_{ij}(x^j + y^j)]\), the trace induces a linear functional \(f(x) \mapsto \text{Tr}(f(x + y)\theta_1 \cdots \theta_m \otimes E_{11}) \in \mathbb{C}[\varepsilon]\) that vanishes on derivatives. As each linear functional on \(C_0^\infty(\mathbb{R}^{2n})\) that vanishes on derivatives can be written as a constant times an integral, the claim follows.

**Theorem 2.17.** A trace on \(\mathcal{A}\) is unique up to a multiplicative constant in \(\mathbb{C}[\varepsilon]\).

**Proof.** Assume that we are given a trace \(\text{Tr}\) on \(\mathcal{A}\). As the algebra \(\mathcal{A}\) is locally isomorphic to the trivial algebra, there is a countable open cover \(\{U_i\}_{i=1}^\infty\) of \(M\) so that \(\Gamma_{D_c}(U_i, W)\) is
isomorphic to $\Gamma_{D_0,c}(V_i, W)$ with $V_i \subset \mathbb{R}^{2n}$. For each $U_i$ and its corresponding trivialization, we can according to the above lemma associate a constant $c_i \in \mathbb{C}[[\varepsilon]]$. We show that $c_i = c_j$ if $U_i \cap U_j \neq \emptyset$.

Consider a pair $(i,j)$ so that $U_i \cap U_j \neq \emptyset$. We define the trace $T_r := T - c_i T_1$. By definition, this map vanishes on $U_i$ and therefore on $U_i \cap U_j$. Then according to the above lemma $T_r$ also vanishes on $U_j$ and $c_j = c_i$ follows.

If $M$ is compact, there is another nice proof for the uniqueness of the trace. Assume that $\theta_k, k = 1, \ldots, m$ are orthogonal coordinates on the fibres of the Clifford bundle. We write $\Theta := \theta_1 \cdots \theta_m$ for the superfunction which is a global section of $\Lambda^m E$ corresponding to the orientation on $E$.

**Theorem 2.18.** Assume that $M$ is compact and connected. Then there is a unique trace $T_r : \mathcal{A} \rightarrow \mathbb{C}[[\varepsilon]]$ such that $\text{Tr}(\Theta \otimes 1) = J_M \omega^n$. In other words, the trace is unique up to a multiplicative constant in $\mathbb{C}[[\varepsilon]]$.

**Proof.** We need to show that a trace, for which $\text{Tr}(\Theta \otimes 1) = 0$, vanishes for all sections $f \in \Gamma^\infty(M, \Lambda^* E \otimes \text{End}(E))$. We choose a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of Darboux charts on which the bundles $E$ and $\mathcal{E}$ are trivializable and a corresponding partition of unity $\{\rho_i\}_{i=1}^k$. Using the same arguments as in the proof of the above lemma, we see that the trace locally, i.e. on $U \in \mathcal{U}$ – and therefore also globally – vanishes on sections with Grassmann degree less than $m$. Furthermore the trace acts on a section $f = \phi \Theta \otimes M$ where $\phi \in C^\infty_c(U, \mathbb{C})$ and $M \in \text{End}(\mathbb{C}^r)$ as

$$\text{Tr}(f) = \text{tr}(M) \text{Tr}(\phi \Theta \otimes 1)/r,$$

it thus suffices to consider sections $f = \phi \Theta \otimes 1$. For $\phi, \psi \in C^\infty(M, \mathbb{C})$ it is easy to see that $(\phi \Theta) \ast \psi = (\phi \ast \psi) \Theta$, thus the map

$$T : \phi \mapsto \text{Tr}(\phi \Theta \otimes 1)/r$$

is a trace on $C^\infty(M, \mathbb{C}[[\varepsilon]])$, considered as a subalgebra of $\Gamma^\infty(M, \Lambda^* E \otimes \text{End}(\mathcal{E}))$, that vanishes on constant functions. We write

$$T = \sum_{k=0}^\infty \varepsilon^k T_k$$

so that $T_k : C^\infty(M, \mathbb{C}) \rightarrow \mathbb{C}$. The lowest order term of the trace property $T([\phi, \varphi]) = 0$ then reads $T_0([\phi, \varphi]) = 0$. Because the de Rham cohomology of $M$ in degree $2n$ is one-dimensional, we can write $\phi \omega^n = c \omega^n + d\alpha$ where $c \in \mathbb{C}$ is a constant function and
\( \alpha \in \Omega^{2n-1}(M) \). In local coordinates there are functions \( f_j \) so that \( \alpha = \sum_{j=1}^{2n} (-1)^{j+1} f_j \, dx_1 \wedge \ldots \wedge dx_{2n} \), thus

\[
\phi = c + \sum_{i,j} \frac{\partial (\rho_i f_j)}{\partial x^j} = c + \sum_{i,j,k} \{ \omega_{jk} x^k, \rho_i f_j \}.
\]

This shows that \( \phi \) lies in the kernel of \( T_0 \) and therefore \( T_0 = 0 \). Using the same arguments, we conclude inductively that \( T_k = 0 \) for all \( k > 0 \).

\section*{2.7 Variation for even Grassmann dimension}

If the Grassmann dimension is even, that is we consider \( A^{\text{pol}}_{2n,2m} \), we could define a different metric for the Clifford algebra, that looks closer to the symplectic structure:

\[
g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The corresponding symmetry group is \( \text{So}(m,m) \) \( \cong \) \( \text{So}(2m,\mathbb{C}) \). We introduce the following notation:

\[
(y_1, \ldots, y_{2n}) = (p_1, q_1, \ldots, p_n, q_n),
\]

\[
(\theta_1, \ldots, \theta_{2m}) = (\zeta_1, \eta_1, \ldots, \zeta_m, \eta_m),
\]

\[
\alpha = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i} \right),
\]

\[
g = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial}{\partial \zeta_i} \otimes \frac{\partial}{\partial \eta_i} + \frac{\partial}{\partial \eta_i} \otimes \frac{\partial}{\partial \zeta_i} \right).
\]

With these conventions, the Hochschild cocycle \( \tau \) of degree \( 2n \) has exactly the same formula as in section 2.4:

\[
\tau = \mu \circ \int_{\Delta_{2n}} \omega \circ \pi
\]

with the same definitions of the maps \( \mu \), \( \omega \) and \( \pi \). It is straightforward to check that the redefinition of \( g \) doesn’t change anything in the proofs of the properties of \( \tau \).

\section*{2.8 Weil algebra and Weyl algebra}

In this and the next section, we consider an oriented bundle \( E \to M \) of even rank and we use the above variation of the Clifford structure. We will in these two sections follow section
of \([16]\) and compute the trace of the section \(\Theta := \zeta \eta_1 \cdots \zeta_m \eta_m \in \Gamma(M, \Lambda^{2m} E)[[\varepsilon]]\) for a compact manifold \(M\). We compute the class of \(\chi_0(\Theta)\) using a Chern-Weil homomorphism to the Lie algebra cohomology of the Weyl (Lie) algebra \(A^{N, pol}_{2n,2m}\). We therefore introduce some preliminaries about the Lie algebra cohomology and the Chern-Weil map.

### 2.8.1 Definitions and Notation

Let \(\mathfrak{g}\) be a Lie algebra over a ring \(k\). The **Lie algebra cohomology** of \(\mathfrak{g}\) with values in the Lie algebra module \(M\) is the homology of the Chevalley-Eilenberg complex with cochains \(CC^k(\mathfrak{g}; M) = \text{Hom}(\Lambda^k \mathfrak{g}, M)\) and coboundary map \(d_{\text{Lie}} : CC^k(\mathfrak{g}; M) \rightarrow CC^{k+1}(\mathfrak{g}; M)\) given by

\[
d_{\text{Lie}}(a_0 \wedge \cdots \wedge a_k) = \sum_{i=0}^k (-1)^i a_i (a_0 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} c([a_i, a_j], a_0 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_k) .
\]

We denote the corresponding cohomology by \(HC^*(\mathfrak{g}; M)\). We also consider the homology of the relative cochain complex. For \(\mathfrak{h} \subset \mathfrak{g}\) a Lie subalgebra, the cochains are defined as \(CC^k(\mathfrak{g}, \mathfrak{h}; M) = \text{Hom}(\Lambda^k (\mathfrak{g}/\mathfrak{h}), M)\). These are the cochains that vanish if any argument is in \(\mathfrak{h}\) and they are invariant in the sense that they are annihilated by the Lie derivative

\[
(\mathcal{L}_a c)(a_1, \ldots, a_k) = ac(a_1, \ldots, a_k) + \sum_{i=1}^k (-1)^i c([a, a_i], a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_k)
\]

for any \(a \in \mathfrak{h}\). Let as above \(\mathfrak{g}\) be a Lie algebra with finite dimensional Lie subalgebra \(\mathfrak{h}\) and write \(\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}\) for the canonical projection. The **relative Weil algebra** is then defined as the differential graded commutative algebra \(W(\mathfrak{g}, \mathfrak{h}) = \bigoplus_{i,j \geq 0} W^{i,j}(\mathfrak{g}, \mathfrak{h})\), where \(W^{i,j}(\mathfrak{g}, \mathfrak{h}) = (\Lambda^i (\mathfrak{g}/\mathfrak{h}) \otimes S^j \mathfrak{g})^\mathfrak{h}\) is of degree \(i + 2j\). The differential can be written as a sum \(d = d' + d_{\text{Lie}}\). \(d_{\text{Lie}} : W^{i,j} \rightarrow W^{i+1,j}\) is the Lie differential for the relative Lie algebra cohomology where \((S^j \mathfrak{g})^\mathfrak{h}\) is the module. \(d' : W^{i,j} \rightarrow W^{i-1,j+1}\) is the map

\[
d'(c(v \otimes w_1 \cdots w_{j+1}) := \sum_{k=1}^{j+1} c(\pi(w_k) \wedge v \otimes w_1 \cdots \widehat{w}_k \cdots w_{j+1}) ,
\]

where \(v \in \Lambda^{i-1}(\mathfrak{g}/\mathfrak{h})\) and \(w_k \in \mathfrak{g}\) for \(k = 1, \ldots, j+1\). By the above grading, the differential is of degree 1. We write \(W(\mathfrak{g}) := W(\mathfrak{g}, 0)\) for the absolute Weil algebra.

---

\(^2\)The variables \(\theta_i, \eta_i\) have to be chosen so that \(\Theta\) corresponds to the orientation of \(E\).
Assume that \( g = h \oplus V \) where \( V \) is an \( h \)-invariant subspace. The projection \( pr: g \to h \) is then \( h \)-equivariant. We identify \( pr \) with an element \( A \in g^* \otimes h \subset W(g) \otimes h \). We can then consider \( A \) as a connection (see [16], [18, 19]) with curvature \( F = dA + \frac{1}{2}[A, A] \in W(g) \otimes h \) where the differential only acts on the Weil algebra part. As \( F^j \in W(g) \otimes S^j h \), we may define for an invariant polynomial \( P \in (S^j h)^* h = W(H, h) \) the map \( \chi_W(P) = \frac{1}{j!}(id \otimes P)(F^j) \). Thus, if we pass to the homology, the projection \( pr \) induces a map

\[
\chi_W: (S^j h)^* h \to H(W(h, h)).
\]

This map is actually independent of the choice of the \( h \)-equivariant projection.

As \( C^i(g, h) = W^{i0}(g, h) \) as a vector space, there is a canonical projection of \( W(g, h) \) to \( C^*(g, h) \). It is easy to see that every cocycle in \( W^{i0} \) is also a cocycle in \( C^i(g, h) \). Furthermore the exact cocycles are the same in both spaces. Therefore the canonical projection passes to the cohomologies. The composition of this homomorphism with the Chern-Weil map leads to the map

\[
\chi: (S^j h)^* h \to HC^*(g, h).
\]

We will use the Lie algebra cohomology and also the Weil algebra in the slightly more general case where \( g \) is a Lie superalgebra with a Lie subalgebra \( h \) of degree zero. Everything still works as said above, we only have to add the Koszul sign whenever we interchange elements.

For the Chern-Weil map \( \chi \) and an invariant polynomial \( P \), we find the following formula:

\[
\chi(P)(v_1, \ldots, v_{2q}) = \frac{1}{q!} \sum_{\sigma \in S_{2q}/S_2^q} \text{sgn}(\sigma)P(C(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, C(v_{\sigma(2q-1)}, v_{\sigma(2q)})),
\]  

where the sum goes over all permutation with \( \sigma(2i-1) < \sigma(2i) \). The sign \( \text{sgn}(\sigma) \) includes the Koszul sign taking into account the permutations of graded terms.

From now on, we consider the algebra \( A := A^{pol}_{2n|2m} \), the Lie algebra \( g := A^{pol}_{2n|2m} = gl_N(A) \) with its Lie subalgebra \( h = sp_{2n}(\mathbb{C}) \times so_{2m}(\mathbb{C}) \times gl_N(\mathbb{C}) \) of elements that are matrices or quadratic in the generators \( p_i, q_i, \zeta_i, \eta_i \). From the explicit from of the product on the Weyl algebra it is easy to see that the vector space generated by the non-quadratic elements \( V = g/h \) is \( h \)-invariant. Therefore the above construction for the Chern-Weil homomorphism can be applied. Actually the subspaces of \( V \) that are generated by the elements of a fixed order are finite dimensional, \( h \)-invariant subspaces and therefore semisimple \( h \)-modules, in particular, \( g \) is a semisimple \( h \)-module.
2.8.2 Lemmata

We restate some lemmata from [16] and sketch the proofs to check that they are still valid in the graded case.

**Lemma 2.19.** For $N \gg n$ and $j > 0$ the Lie algebra cohomology $HC^k(gl_N(A); S^j gl_N(A)^*)$ vanishes for $k = 0, \ldots, 2n - 1$ and is one dimensional for $k = 2n$. In particular, in the case $k = 2n$ and $j = 1$, the map $HH^{2n}(A) \to HC^{2n}(gl_N(A); gl_N(A)^*)$, $\tau \mapsto T$ defined by

$$
\langle a_0 \otimes M_0, T(a_1 \otimes M_1, \ldots, a_{2n} \otimes M_{2n}) \rangle = \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \text{tr}(M_0 M_{\sigma(1)} \cdots M_{\sigma(2n)}) \tau(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(2n)})
$$

is an isomorphism.

**Proof.** It follows from proposition 3.1.1 in [15]. The fact that the Hochschild homology $HH^k(A)$ is concentrated in degree $2n$ (see section 2.3) is used.

Because of Morita invariance for Hochschild homology, we immediately get $HH^{2n}(gl(A)) = HC^{2n}(gl_N(A); gl_N(A)^*)$. It follows that there is a cocycle $T \in HC^{2n}(gl_N(A); gl_N(A)^*)$ that corresponds to the cocycle $\tau$ constructed in section 2.4. $T$ is simply given by antisymmetrizing the $2n$ arguments in $\tau$.

**Lemma 2.20.** For $N \gg n$, $j > 0$ and $k \leq 2n$, we have $HC^k(g, h; S^j g^*) = HC^k(g; S^j g^*)$ which implies that $HC^k(g, h; S^j g^*)$ vanishes for $k = 0, \ldots, 2n - 1$ and is one dimensional in degree $2n$.

**Proof.** We consider the Hochschild-Serre spectral sequence for the pair $(g, h)$ (see [16] Lemma 5.2.). We have

$$
E^{p,q}_2 = HC^q(h) \otimes HC^p(g, h; S^j g^*)
$$

on the other hand, we know that the spectral sequence converges to $H^{p+q}(g; S^j g^*)$ which vanishes for $p + q < 2n$. Using $H^0(h) = \mathbb{C}$ it is then easy to show by induction in $p$ that $H^p(g, h; S^j g^*) = 0$ for $p = 0, \ldots, 2n - 1$ and that $H^{2n}(g, h; S^j g^*)$ is isomorphic to $H^{2n}(g; S^j g^*)$. For the induction step one just considers the $E^{p,0}_2$ for which the spectral sequence is obviously stable.

**Lemma 2.21.** The canonical projection map $H^k(W(g, h)) \to HC^k(g, h)$ is an isomorphism for $k = 0, \ldots, 2n$. 

Proof. We use the filtration $F^p(W(\mathfrak{g}, \mathfrak{h})) = \bigoplus_{j \geq p} W^{i,j}(\mathfrak{g}, \mathfrak{h})$ and consider the corresponding spectral sequence for $W(\mathfrak{g}, \mathfrak{h})$. One finds that $E_1^{p,q} = H^q(\mathfrak{g}, \mathfrak{h}; S^p \mathfrak{g}^*)$ which vanishes due to the previous lemma for $p > 0$ and $q < 2n$. Therefore the terms for $p = 0$ in the spectral sequence are stable and we get the claim.

Lemma 2.22. The cohomology of the Weil algebra $H^q(W(\mathfrak{g}, \mathfrak{h}))$ vanishes for odd $q < 2n$ and is isomorphic to $(S^{q/2} \mathfrak{h}^*)^\mathfrak{h}$ for even $q \leq 2n$.

Proof. This is a well known fact. It can be shown by using a spectral sequence corresponding to the filtration $F^p(W(\mathfrak{g}, \mathfrak{h})) = \bigoplus_{i+j \geq p} W^{i,j}(\mathfrak{g}, \mathfrak{h})$ (which is not the same as in the proof before). For details of the proof see [16], Lemma 5.1, where it is shown in the ungraded case. The proof in the graded case is exactly the same using the Koszul sign convention.

Lemma 2.23. The Chern-Weil map $\chi_W : (S^q \mathfrak{h}^*)^\mathfrak{h} \to H^{2q}(W(\mathfrak{g}, \mathfrak{h}))$ is an isomorphism for $q \leq n$ and $H^k(W(\mathfrak{g}, \mathfrak{h}))$ vanishes for odd $k < 2n$.

Proof. This is shown by an explicit computation of the map $\chi_W$ and using the previous lemma. The proof from [16], proposition 5.1, applies without changes because $\mathfrak{h}$ and therefore the curvature $F$ is even. Thus we don’t need to check any signs.

2.9 Computation of the trace of $\Theta$

We write $\Theta$ for the super-function that can locally be written in the form $\Theta = \zeta_1 \eta_1 \ldots \zeta_n \eta_n$. To compute the trace of $\Theta$, we first find an invariant polynomial $P$ so that $\chi(P)$ is in the cohomology class of $T(\Theta)$. It is then easy to compute the class $\chi(P)$.

In the following lemma we need the fact that an invariant polynomial on a Lie algebra is already defined by its values on the Cartan subalgebra. We write $\mathfrak{h}_c$ for the Cartan subalgebra of $\mathfrak{h}$. It is generated by the elements of the form $q_j p_j$, $\zeta_j \eta_j$, $E_{ij}$ where $E_{ij}$ denotes a matrix with entry 1 at position $(i, j)$ and entry 0 everywhere else.

Lemma 2.24. There is an invariant polynomial $P$ so that $(-1)^n [\chi(P)] = [T(\Theta)]$. It is defined by the formula

$$P(a_1 \otimes M_1, \ldots, a_n \otimes M_n) = \text{tr}(M_1 \cdots M_n) \mu_{n|2m} \int_{[0,1]^n} \omega_n(\Theta \otimes a_1 \otimes \cdots \otimes a_n) du_1 \cdots du_n$$

where $a_i \otimes M_i \in \mathfrak{h}_c$, $i = 1, \ldots, n$ and the maps $\mu_{n|2m}$ and $\omega_n$ are defined as in section 2.4.
Proof. To streamline our notation, we omit the "\( \otimes \)" between "\( pq \)" and "\( M \)" and also just write \( p_i q_i \otimes 1 \) if it doesn’t lead to any confusion. It follows from the bijectivity of Chern-Weil map \( \chi : (S^n h)^* \to H^{2n}(g, h) \) that there exists an invariant polynomial \( P \) such that \( [\chi(P)] = [\Theta(\zeta_1 \eta_1 \ldots \zeta_n \eta_n)] \). As an invariant polynomial is defined by its values on the Cartan subalgebra, it suffices to compute \( P(v_1, \ldots, v_n) \) for \( v_i \in h_c \). Therefore we have to choose elements \( u_i \wedge w_i \), where \( u_i, w_i \in g \), so that \(-v_i = C(u_i \wedge w_i)\) (see formula 2.3). We choose \( u_i = p_i \) and \( w_i = \frac{1}{2} q_i^2 p_i \) so that \( \frac{\partial w_i}{\partial q_i} = v_i \):

\[
\begin{align*}
v_i &= q_j p_j \leadsto w_i = \begin{cases} q_i q_j p_j & i \neq j \\ \frac{1}{2} q_i^2 p_i & i = j \end{cases} \\
v_i &= \eta_j \zeta_j \leadsto w_i = \eta_j \zeta_j q_i \\
v_i &= E_{jj} \leadsto w_i = q_i E_{jj}
\end{align*}
\]

We can assume w.l.o.g. that the \( j = j(i) \) in the above expressions for \( v_i \) are nondecreasing. It is then easy to see that in formula 2.3 only the term with permutation \( \sigma = \text{id} \) contributes and we find

\[
P(v_1, \ldots, v_n) = T(\Theta)(u_1 \wedge w_1 \wedge \cdots \wedge u_n \wedge w_n)
\]

\[
= \mu \circ \int_{\Delta_{2n}} (\omega \circ \pi)(\Theta \otimes p_1 \wedge w_1 \wedge \cdots \wedge p_n \wedge w_n)
\]

\[
= \mu \circ \int_{\Delta_{2n}} \omega(\Theta \otimes 1 \wedge v_1 \wedge \cdots \wedge 1 \wedge v_n) du_1 \wedge \cdots \wedge du_{2n}
\]

As in the last part of the proof of theorem 2.11, we use the fact that the antisymmetrization in the arguments leads to an integral over \([0, 1]^{2n}\) instead of \(\Delta_{2n}\). We finally integrate out the \( u_i \) for \( i \) odd and get the claim.

Note that we have been able to find a \( P \) so that

\[
\chi(P)(u_1, w_1, \ldots, u_n, w_n) = T(\Theta)(u_1, w_1, \ldots, u_n, w_n)
\]

which would actually only be necessary to hold if we pass to the cohomology classes.

\[ \square \]

The above formula allows us to compute the polynomial explicitly.

**Lemma 2.25.** Let \( X = X_1 + X_2 + X_3 \in h \) where \( X_1 \in \text{sp}_{2n}(\mathbb{C}) \), \( X_2 \in \text{so}_{2n}(\mathbb{C}) \) and \( X_3 \in \text{gl}_N(\mathbb{C}) \) then

\[
P(X, \ldots, X) = [\det f(X_1)^{1/2} \det g(X_2)^{1/2} \text{tr}(e^{X_3})]_n
\]

where \( f(x) = \frac{x}{\sinh(x)} \), \( g(x) = \cosh(x) \) and \([F(X)]_n\) denotes the degree \( n \) part in the expansion of \( F(X) \).
2.9. COMPUTATION OF THE TRACE OF $\Theta$

Proof. First, we consider an element in the Cartan subalgebra

$$X = \sum_{i=1}^{n} t_i q_i p_i + \sum_{i=1}^{m} \tau_i \eta_i \zeta_i + \sum_{r=1}^{N} s_r E_{rr} = X_1 + X_2 + X_3.$$ 

To compute $P(X, \ldots, X)$ using the formula from the previous lemma, we remark that at most two derivatives can act on each $X$ and that therefore terms of order at most two in the expansion of the exponential in $\omega_n$ are relevant. In other words, we may replace $\omega_n$ by

$$\prod_{0 \leq i < j \leq 2n} \left( 1 + \varepsilon \psi(u_i - u_j)(\alpha_{ij} + g_{ij}) + \frac{1}{2} \varepsilon^2 \psi(u_i - u_j)^2(\alpha_{ij}^2 + g_{ij}^2) \right).$$

It is convenient to express the terms that appear in the computation of $P(X, \ldots, X)$ by graphs. There are $n$ vertices, each corresponding to an $X$. Each edge between two vertices corresponds to a bidifferential operator $\alpha_{ij}$ or $g_{ij}$. We can think of each $X$ as being either an $X_1$, $X_2$ or $X_3$. It is clear that $X_3$-vertices have no edges. $X_1$- and $X_2$-vertices have at most two edges and therefore the connected subgraphs are linear or cyclic. We first want to rule out the linear case. The integral that appears in such a linear subgraph with $j$ vertices would be

$$\int_{[0,1]^j} \psi(u_1 - u_2) \cdots \psi(u_{j-1} - u_j) du_1 \cdots du_j.$$ 

To compute this integral, we note that $\psi(x) = 2B_1(x)$ where $B_i(x)$ are the $1$-periodic Bernoulli polynomials. They are equal to the Bernoulli polynomials on $[0,1)$, are continuous for $i \geq 2$ and satisfy the convolution identity

$$B_n * B_m = -\frac{n!m!}{(n+m)!} B_{n+m}.$$ 

The above integral is therefore proportional to the zeroth Fourier coefficient of $B_{j-1}$, but the zeroth Fourier coefficients of the Bernoulli polynomials vanish. In the case of the vertices $X_1$ there is a second reason why the linear connected subgraphs vanish which is because all $X_1$-vertices need exactly two derivatives (edges) in order not to be killed by the map $\mu$ where the variables $y$ are set to zero.

As $\alpha_{ii} = g_{ii} = 0$, tadpoles are not allowed. Therefore the connected subgraphs of $X_1$- and $X_2$-vertices are cycles, as we will see actually only cycles of even length as the odd cycles vanish for reasons of symmetry.

So far we didn’t take into account the $\Theta$ in the formula for $P$. If there are no derivatives acting on $\Theta$, it will be mapped to 1 by the Berezin integral. But of course there are additional terms that include derivatives acting on $\Theta$, that is terms containing $g_{0k}$. If we
considered $\Theta$ as an additional vertex, it would still be true that we only have to consider cycles because of the same reason as mentioned above. For every pair of derivatives acting on $\Theta$, we need an additional isolated $X_2$-vertex because otherwise the Berezin integral in $\mu$ would give $0$. We handle this problem by considering this single vertex instead of $\Theta$ as part of the cycle. In other words, we don’t consider $\Theta$ as a new vertex but we consider a second type of cycles of $X_2$-vertices where each vertex could be replaced by $\Theta$. The $X_2$-term of the replaced vertex then guarantees that the Berezin integral in $\mu$ doesn’t vanish. Such a cycle of length $j$ has to be counted as $j$ terms because every vertex could be replaced by $\Theta$.

As we now know all possible types of connected graphs, we compute their contributions to $P(X, \ldots, X)$. We start with a $X_1$-cycle of length $j > 2$. Up to a permutation of the indices, we have to compute

$$
\int_{[0,1]^j} \psi(u_1 - u_2) \cdots \psi(u_{j-1} - u_j) \psi(u_j - u_1) du_1 \cdots du_j \alpha_{12} \cdots \alpha_{\ell-1\ell} \alpha_{\ell 1} (X \otimes \cdots \otimes X).
$$

We see that

$$
\alpha_{12} \cdots \alpha_{\ell-1\ell} \alpha_{\ell 1} (X \otimes \cdots \otimes X) = \frac{1 + (-1)^j}{2^j} \sum_{i=1}^n t_i^j,
$$

in particular only cycles with $j$ even contribute. We compute the integral as above by the convolution identity for the Bernoulli polynomials. Therefore

$$
I_j := \int_{[0,1]^j} \psi(u_1 - u_2) \cdots \psi(u_{j-1} - u_j) \psi(u_j - u_1) du_1 \cdots du_j
$$

$$
= -\frac{(-2)^j}{j!} \int_0^1 B_j(u_1 - u_1) du_1 = -\frac{(-2)^j}{j!} B_j
$$

where $B_j(0) =: B_j$ are the Bernoulli numbers.

If we write $Z_{j}^1$ for the contribution of the cycles of $X_1$-vertices of length $j$, we get $Z_{1}^1 = 0$ for $j$ odd and $Z_{j}^1 = 2^{1-j} I_j \sum_{i=1}^n (\varepsilon t_i)^j$ for $j > 2$ even. In the case $j = 2$, we get an additional factor $\frac{1}{2}$ because the term $(\alpha_{12})^2$ appears in the second order term of the expansion of the exponential in $\omega_n$. The computation for the same cycles made of $X_2$-vertices (not acting on $\Theta$) is the same up to the sign in

$$
g_{12} \cdots g_{\ell-1\ell} g_{\ell 1} (X \otimes \cdots \otimes X) = -\frac{1 + (-1)^j}{2^j} \sum_{i=1}^m \tau_i^j.
$$

and we get $Z_{j}^2 = 0$ for $j$ odd and $Z_{j}^2 = -2^{1-j} I_j \sum_{i=1}^m (\varepsilon \tau_i)^j$ for $j > 2$ even and $Z_{2}^2$ has an additional factor $\frac{1}{2}$. We consider separately the case where a vertex is replaced by $\Theta$. We
denote these contributions by $\tilde{Z}_j^2$. Actually this case is not much different because the derivatives that we have to compute are exactly the same as in the previous case but in the integral $I_j$ the integration over the first variable $u_0 = 0$ is missing. It is easy to see that this doesn’t change the result because the last integral goes over a constant and the length of the interval is 1. The only tricky point about these contributions is that we have to make sure that in the whole graph every variable of $\Theta$ is at most differentiated once. This can be done by adding $\xi_i$ that are commutative variables satisfying $\xi_i^2 = 0$ in the following way: $\tilde{Z}_j^2 = -2^{1-j} I_j \sum_{i=1}^m (\varepsilon \tau_i)^j \xi_i$, also here $\tilde{Z}_j^2$ vanishes for odd $j$ and $\tilde{Z}_2^2$ has an additional factor $\frac{1}{2}$. At the end of the computation, we will have to get rid of the variables $\xi_i$ by setting them to 1. The last contribution comes from $j$ isolated $X_3$ vertices. It is $Z_j^3 = \text{tr}(X_3^j) = \sum_{i=1}^N (s_i)^j$.

We compute $P(X, \ldots, X)$ by summing over all possible configurations of the graph

$$P_n(X, \ldots, X) = \sum_{\ell, k, \kappa} \prod_{j \geq 2} \frac{n!}{\ell_j! (Z_2^j)^{k_j} (\tilde{Z}_2^j)^{\kappa_j} (Z_3)^\kappa} \Big|_{\xi=1}^{\ell_j + k_j + \tilde{k}_j = n}.$$ 

$\ell_j$, $k_j$ and $\tilde{k}_j$ are the multiplicities of the corresponding cycles of length $j$ and $\kappa$ is the number of isolated vertices. The sum then goes over all configurations so that $\kappa + \sum_j j(\ell_j + k_j + \tilde{k}_j) = n$. $G_{\text{aut}}$ is the automorphism group of the graph. It is easy to verify that its cardinality is

$$|G_{\text{aut}}| = \kappa! 2^{\sum_{i=4}^\infty \ell_i + k_i + \tilde{k}_i} \prod_{j \geq 2} \ell_j! k_j! \tilde{k}_j! j^{\ell_i + k_i}.$$ 

$\kappa!$ comes from the fact that a permutation of the isolated vertices is a symmetry. The other three factorial terms say that a permutation of cycles of the same length is also a symmetry. The cycles where no vertex is replaced by $\Theta$ have a cyclic symmetry, therefore the factor $j^{\ell_i + k_i}$. The powers of 2 appear because the cycles of length $\geq 3$ have an additional mirror symmetry. We now consider the generating function

$$S := \sum_{n=0}^\infty \frac{1}{n!} P_n(X, \ldots, X),$$ 

where $P_0 = 1$. which is a formal power series in the variables $t_i, \tau_i$ and $s_i$. This leads to the more natural formula

$$S = \sum_{r=0}^N \exp \left( s_r + \sum_{j \geq 2} \frac{I_j}{2j} \left( \sum_{i=1}^n (\varepsilon t_i)^j - \sum_{i=1}^m (\varepsilon \tau_i)^j (1 + j \xi_i) \right) \right) \Big|_{\xi=1}.$$ 

To compute the above sums over $j$, we use the following identity for the Bernoulli numbers $B_j$:

$$\sum_{k=0}^\infty \frac{B_{2k}}{(2k)!} j^{2k} = \frac{x}{2} \coth \frac{x}{2}.$$
This leads to
\[ f_1(x) := \sum_{j \geq 2} \frac{I_j}{2^j} x^j = 1 - \frac{x}{2} \coth \frac{x}{2}, \]
and as
\[ f_2(x) := \sum_{j \geq 2} \frac{I_j}{2^j} x^j \]
satisfies \( xf'_1(x) = f_2(x) \), one may check by differentiation that
\[ f_2(x) = \log \left( \frac{x/2}{\sinh(x/2)} \right) . \]

This leads to the following explicit formula for \( S \):
\[
S = \sum_{r=1}^{N} e^{sr} \prod_{i=1}^{n} \frac{\varepsilon t_i/2}{\sinh(\varepsilon t_i/2)} \prod_{i=1}^{m} \frac{\sinh(\varepsilon \tau_i/2)}{\varepsilon \tau_i/2} \left. \left( 1 - \xi_i(1 - (\varepsilon \tau_i/2) \coth(\varepsilon \tau_i/2)) \right) \right|_{\xi=1} \\
= \sum_{r=1}^{N} e^{sr} \prod_{i=1}^{n} \frac{\varepsilon t_i/2}{\sinh(\varepsilon t_i/2)} \prod_{i=1}^{m} \cosh(\varepsilon \tau_i/2) 
\]
For \( X \in \mathfrak{h} \), we extend the above polynomial by invariance and get the claim.

We recall from section 1.6 the formula
\[ \Omega = \nabla A + \frac{1}{2} [A, A] + R \]
for the characteristic class \( \Omega \in \Omega^2(M[[\varepsilon]]) \) of the star product. \( A \) is the 1-form from the Fedosov connection. \( R \) is the curvature of the connection \( \nabla \) and can be decomposed into \( R = R_1 + R_2 + \varepsilon R_3 \) where \( R_1 \) (\( R_2 \) resp.) is quadratic in \( y \) (\( \theta \) resp.) and \( R_3 \) takes values in the coefficient bundle. In other words: \( R_{1,2,3} \) are \( \text{sp}_{2n}(\mathbb{C}), \text{so}_{2m}(\mathbb{C}'), \text{gl}_N(\mathbb{C}) \) valued.

**Theorem 2.26.** The trace over the section \( \Theta \) is given by
\[
\text{Tr}(\Theta) = (-\varepsilon)^n \int_M \hat{A}(R_1) \hat{B}(R_2) \text{tr} \left( \exp(R_3 - \Omega/\varepsilon) \right).
\]
where \( \hat{A}(R_1) := \det \left( \frac{R_1/2}{\sinh(R_1/2)} \right)^{1/2} \) is the \( \hat{A} \)-genus and \( \hat{B}(R_2) := \det \left( \cosh(R_2/2) \right)^{1/2} \).

**Proof.** To apply the Chern-Weil map to the polynomial from the previous lemma and evaluate it on \( \Theta \otimes A^\otimes 2n \), we have to compute the curvature \( F(A, A) = [\text{pr} A, \text{pr} A] - \text{pr}([A, A]) \) appearing in the Chern-Weil map. We may assume that \( \text{pr} A = 0 \) as the quadratic part can always be absorbed in the supersymplectic connection. Because the
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Connection $\nabla$ conserves the degree in the Weyl bundle, we also have $\text{pr} \nabla A = 0$. We then get

$$C(A \wedge A) = - \text{pr}(\frac{1}{2}[A, A]) = - \text{pr}(\nabla A + \frac{1}{2}[A, A]) = R - \Omega.$$ 

Inserting this into the formula for the polynomial from the previous lemma immediately gives the claim. Note that we had to assume that the dimension of the coefficient bundle $\mathcal{E}$ is large, i.e. $N \gg n$. If the coefficient bundle is too low dimensional, we just replace the bundle by as many copies of itself as needed: $\mathcal{E} \mapsto \mathcal{E} \oplus \mathcal{E} \oplus \cdots \oplus \mathcal{E}$. \qed
Chapter 3

A Trace for holomorphic differential operators

"Heroes know that things must happen when it is time for them to happen. A quest may not simply be abandoned; unicorns may go unrescued for a long time, but not forever; a happy ending cannot come in the middle of the story."

Prince Lir in "The Last Unicorn" by Peter S. Beagle

3.1 Introduction

Let $E \rightarrow X$ be a holomorphic vector bundle of rank $r$ on an $n$-dimensional compact connected complex manifold $X$. Let $\mathcal{D}_E$ be the sheaf of holomorphic differential operators acting on sections of $E$.

Global differential operators $D \in \mathcal{D}_E(X) = \Gamma(X, \mathcal{D}_E)$ act on the sheaf (or Dolbeault) cohomology groups $H^j(X, E)$ of $E$ and thus we have algebra homomorphisms

$$H^j : \mathcal{D}_E(X) \rightarrow \text{End}(H^j(X, E)).$$

Since the cohomology of $E$ is finite dimensional, we can consider the Lefschetz number (or supertrace) $L : \mathcal{D}_E(X) \rightarrow \mathbb{C}$,

$$D \mapsto L(D) = \sum_{j=0}^{n} (-1)^j \text{tr}(H^j(D)).$$

If $D = \text{Id}$ is the identity then $L(\text{Id})$ is the holomorphic Euler characteristic of $E$ which is given by the Riemann–Roch–Hirzebruch theorem as the integral over $X$ of a characteristic
class. Our aim is to generalize this formula to the case of a general differential operator $D$ by writing the Lefschetz number as the integral over $X$ of a differential form $\chi_0(D)$ whose value at a point $x \in X$ depends on finitely many derivatives of the coefficients of $D$ at $x$.

The formula for the differential form $\chi_0$ depends on the choice of a connection on the holomorphic vector bundles $T^{1,0}X$ and $E$ and is similar to the formula written in [16] for the canonical trace of the quantum algebra of functions in deformation quantization of symplectic manifolds. Its ingredients are the Hochschild cocycle of [16] and formal differential geometry, see [13], gives a realization of the continuous Hochschild cohomology $HH^*(\mathcal{D}_{n,r}, \mathcal{D}^*_{n,r})$ is one-dimensional, concentrated in degree $2n$ and is generated by a $2n$-cocycle $\tau^r_{2n}: \mathcal{D}^{\otimes (2n+1)} \to \mathbb{C}$ given in [16] by an explicit integral formula. Formal differential geometry, see [13], gives a realization of $\mathcal{D}_E(X)$ as the algebra of horizontal sections for a flat connection $\nabla$ on the bundle of algebras $\hat{\mathcal{D}}_E = J_1E \times_G \mathcal{D}_{n,r} \to X$ with fiber $\mathcal{D}_{n,r}$. Here $J_1E \to X$ denotes the extended frame bundle, whose fiber at $x \in X$ consists of pairs of bases, one of $T^{1,0}_xX$ and one of $E_x$; it is a principal bundle for the group $G = GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$. More generally, let $J_pE$ be the complex manifold of $p$-jets at 0 of local bundle isomorphisms $\mathbb{C}^n \times \mathbb{C}^r \to E$. These manifolds come with holomorphic $G$-equivariant submersions $J_{p+1} \to J_p$ with contractible fibers. The flat connection depends on the choice (unique up to homotopy) of a $G$-equivariant section $\phi: J_1E \to J_\infty E = \lim J_pE$. Such sections can be constructed out of connections on $J_1E$. Upon local trivialization of $J_1E$, the flat connection has the form $\nabla(\hat{D}) = d\hat{D} + [\omega, \hat{D}]$ for some 1-form $\omega$ on $X$ with values in the first order differential operators in $\mathcal{D}_{n,r}$ and the isomorphism $\mathcal{D}_E(X) \to \text{Ker}(\nabla)$ sends $D$ to its Taylor expansion $\hat{D} = \phi_\ast D$ with respect to the local coordinates and trivialization of $E$ given by $\phi$.

With these notations the formula for $\chi_0(D)$ in terms of the horizontal section $\hat{D}$ associated with $D$ is

$$\chi_0(D) = \tau^r_{2n}(\hat{D}, \omega, \ldots, \omega).$$

The multilinear form $\tau^r_{2n}$ on $\mathcal{D}_{n,r}$ is extended to differential forms with values in $\mathcal{D}_{n,r}$ by linearity: if $\omega = \sum \omega_j dx_j$ in terms of local real coordinates $x_j$, $j = 1, \ldots, 2n$,

$$\chi_0(D) = \sum \tau^r_{2n}(\hat{D}, \omega_{j_1}, \ldots, \omega_{j_{2n}}) dx_{j_1} \wedge \cdots \wedge dx_{j_{2n}}.$$ 

The local objects $\hat{D}$ and $\omega$ depend on a choice of a local trivialization of $J_1E$, but the differential form $\chi_0$ is globally defined as a consequence of the fact that $\tau^r_{2n}$ is basic for the action of $G$. We define

$$\text{Tr}_\chi(D) := \frac{1}{(2\pi i)^n} \int_X \chi_0(D).$$

Our main result is
3.1. INTRODUCTION

**Theorem 3.1.** For any \( D \in \mathcal{D}_E(X) \),

\[
L(D) = \text{Tr}_\chi(D).
\]

Moreover, for the identity differential operator, it is known \([15, 29]\) that the class of \( \chi_0(\text{Id}) \) is the component of degree 2\( n \) of the Hirzebruch class \( \text{td}(T_X)\text{ch}(E) \) and thus we recover the Riemann–Roch–Hirzebruch theorem. Also, the direct calculation of \([16]\) shows that \( \chi_0(\text{Id}) \) is the representative of the Hirzebruch class given by the Chern–Weil map in terms of the curvature of the connection on \( T^{1,0}X \oplus E \) canonically associated with \( \phi \).

The proof of the theorem is obtained by showing that the linear functions \( L \) and \( \text{Tr}_\chi \) on the Hochschild 0-th homology

\[
\text{HH}_0(\mathcal{D}_E(X)) = \mathcal{D}_E(X)/[\mathcal{D}_E(X), \mathcal{D}_E(X)]
\]

are equal to a third linear function \( \text{Tr}_{\hat{H}C} \) constructed essentially in \([12, 27]\): a global differential operator \( D \in \mathcal{D}_E(X) \) defines a global 0-cycle in the complex of sheaves \( \mathcal{C}_\bullet(\mathcal{D}_E) \) of Hochschild chains of \( \mathcal{D}_E \), which is quasi-isomorphic to the complex of sheaves \( \mathcal{C}_X[2n] \) of locally constant continuous functions concentrated in degree \(-2n\). Thus there is a map \( \text{Tr}_{\hat{H}C} : \text{HH}_0(\mathcal{D}_E(X)) \to H^0(X, \mathbb{C}_X[2n]) = H^{2n}(X, \mathbb{C}) \simeq \mathbb{C} \).

The statement of theorem 3.1 was conjectured around 2001 by B. Feigin and B. Shoikhet. In the case of curves a formula for \( L(D) \) in terms of residues had been found by A. Beilinson and V. Schechtman (lemma 2.2.3 in \([30]\), see also \([31]\)). A formula for the normalized trace in deformation quantization of a symplectic manifold, analogous to the one of theorem 3.1 was proposed in \([16]\). The proof of that formula is simpler since the space of traces is one-dimensional in that situation, so one just has to check the normalization. The difficulty here is that \( \text{HH}_0(\mathcal{D}_E(X)) \) is not one-dimensional in general. An indirect approach to proving that \( L = \text{Tr}_{\hat{H}C} \), proposed in \([14]\), is to embed \( \mathcal{D}_E(X) \) in a suitable complex of algebras with one-dimensional cohomology and show that both \( L \) and \( \text{Tr}_{\hat{H}C} \) extend to chain maps on this complex. If the Euler characteristic of \( E \) does not vanish one can then deduce from the classical Riemann–Roch–Hirzebruch theorem that \( L = C \cdot \text{Tr}_{\hat{H}C} \) for some \( C \). The rigorous completion of this program presents some technical difficulties, but it should lead to a proof of \( L = \text{Tr}_{\hat{H}C} \) if \( E \) has non-vanishing Euler characteristic. In a recent preprint \([32]\), A. Ramadoss shows that the approach of \([14]\) could be extended to the much more general case where \( X \) admits a vector bundle with non-vanishing Euler characteristic.

Our result gives in particular a different direct proof of the fact that \( L = \text{Tr}_{\hat{H}C} \), without assumptions on \( X \) or \( E \). It does not use the Riemann–Roch–Hirzebruch theorem.
3.2 Hochschild homology of the algebra of differential operators

3.2.1 Hochschild homology

Let $A$ be an algebra over $\mathbb{C}$ with unit 1 and set $\tilde{A} = A/\mathbb{C}1$. We denote $\bar{a}$ the class in $\tilde{A}$ of $a \in A$. As in chapter 2, we will usually consider the normalized Hochschild complex where

$$C_q(A) = A \otimes \tilde{A}^q, \quad q \geq 0,$$

and write $(a_0, \ldots, a_q)$ instead of $a_0 \otimes a_1 \otimes \cdots \otimes a_q$ for an element in $C_q(A)$. For topological algebras one has to take the projective tensor product, as explained in [33], Ch. II.

Let $O_n = \mathbb{C}[\{y_1, \ldots, y_n\}]$ be the algebra of formal powers series in $n$ variables and $D_n = O_n[\partial y_1, \ldots, \partial y_n]$, the algebra of formal differential operators. Let also $O_n^{pol} = \mathbb{C}[y_1, \ldots, y_n]$, $D_n^{pol} = O_n^{pol}[\partial y_1, \ldots, \partial y_n]$ be the subalgebras of polynomial functions and differential operators. As shown by Feigin and Tsygan [15], the Hochschild homology of $D_n^{pol}$ is one-dimensional and concentrated in degree $n$. A representative of a generator of $HH_{2n}(D_n)$ in the normalized Hochschild chain complex is

$$c_{2n} = \sum_{\pi \in S_{2n}} \text{sgn}(\pi) 1 \otimes u_{\pi(1)} \otimes \cdots \otimes u_{\pi(2n)}, \quad u_{2j-1} = \partial y_j, \quad u_{2j} = y_j.$$

Thus there is a unique linear form on Hochschild homology whose value on $c_{2n}$ is one. This linear form is the class of a cocycle in the complex dual to the Hochschild complex. The formula for this cocycle $\tau_{2n}$ is the same as for the cocycle $\tau_{2n}|m$ from section 2.4 if we set $m = 0$ and $\varepsilon = 0$. We only have to identify $D_n^{pol}$ with $A_n^{pol}$ by the following maps on the generators: $y_i \mapsto y_i$ and $\partial y_i \mapsto y_{n+i}$ for $i = 1, \ldots, n$.

It has the following properties.

(i) $\tau_{2n}$ extends to a linear form on $D_n^{\otimes n+1}$ obeying the cocycle condition $\tau_{2n} \circ b = 0$, where $b$ is the Hochschild differential, and the normalization condition: $\tau_{2n}(D_0, \ldots, D_{2n}) = 0$ if $D_j = 1$ for some $j \geq 1$.

(ii) $\tau_{2n}$ is invariant under the action of $GL_n(\mathbb{C})$ on $D_n$ by linear coordinate transformations. Moreover, if $a = \sum a_{jk}y_k \partial y_j + b, a_{jk}, b \in \mathbb{C}$, then

$$\sum_{j=1}^{2n} (-1)^j \tau_{2n}(D_0, \ldots, D_{j-1}, a, D_j, \ldots, D_{2n-1}) = 0.$$

(iii) $\tau_{2n}(c_{2n}) = 1$. 
More generally, let $M_r(A) \simeq M_r(\mathbb{C}) \otimes A$ denote the algebra of $r \times r$ matrices with entries in an associative algebra $A$. Since Hochschild homology is Morita invariant, $HH_\bullet(M_r(\mathcal{D}_n)) \simeq HH_\bullet(\mathcal{D}_n)$ is also one-dimensional and is spanned by $c_{2n}$ where we view $\mathcal{D}_n$ as a subalgebra of $M_r(\mathcal{D}_n)$ via $D \rightarrow \text{Id} \otimes D$. Define a cocycle $\tau_{2n}$ by

$$\tau_{2n}^r(A_0 \otimes D_0, \ldots, A_{2n} \otimes D_{2n}) = \text{tr}(A_0 \cdots A_{2n}) \tau_{2n}(D_0, \ldots, D_{2n}),$$

$A_i \in M_r(\mathbb{C}), D_i \in \mathcal{D}_n$. As a consequence of the properties of $\tau_{2n}, \tau_{2n}$ obeys:

(i) $\tau_{2n}$ is a linear form on $M_r(\mathcal{D}_n)^{\otimes n+1}$ obeying the cocycle condition $\tau_{2n}^r \circ b = 0$ and $\tau_{2n}^r(D_0, \ldots, D_{2n}) = 0$ if, for some $j \geq 1$, $D_j$ is the multiplication by a constant matrix.

(ii) $\tau_{2n}^r$ is invariant under the action of $G = GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$ where $GL_r(\mathbb{C})$ acts on $M_r(\mathcal{D}_n)$ by conjugation. Moreover, if $a = \sum a_{jk} y_k \partial_{y_j} + b$, $a_{jk} \in \mathbb{C}, b \in M_r(\mathbb{C})$ then

$$\sum_{j=1}^{2n} (-1)^j \tau_{2n}^r(D_0, \ldots, D_{j-1}, a, D_j, \ldots, D_{2n-1}) = 0.$$

(iii) $\tau_{2n}^r(c_{2n}) = r$.

Remark 3.2. For any associative algebra $A$ denote by $A_{\text{Lie}}$ the Lie algebra $A$ with bracket $[a, b] = ab - ba$. Then $A_{\text{Lie}}$ acts on $C_p(A)$ via

$$L_a(a_0, \ldots, a_p) = \sum_{j=0}^p (a_0, \ldots, [a, a_j], \ldots, a_p), \quad a \in A_{\text{Lie}}$$

and we have a Cartan formula $L_a = b \circ \iota_a + \iota_a \circ b$ with

$$\iota_a(a_0, \ldots, a_p) = \sum_{j=1}^p (-1)^{j+1}(a_0, \ldots, a_{j-1}, a, a_j, \ldots, a_p).$$

It follows that $A_{\text{Lie}}$ acts trivially on the cohomology. The property (iii) may be rephrased as saying that $\tau_{2n}$ is $G$-basic, namely $G$-invariant and obeying $\tau_{2n}^r \circ \iota_a = 0$, for $a$ in the Lie algebra of $G$ embedded in $\mathcal{D}_{n,r}$ as a Lie algebra of first order operators.

It also follows that the cohomology class of $\tau_{2n}^r$ is invariant under coordinate transformations.
3.3 Hochschild chain complex of the sheaf of differential operators

Let $\mathcal{D}_E$ be the sheaf of differential operators on $E$. In terms of holomorphic coordinates and a local holomorphic trivialization of $E$, a local section of $\mathcal{D}_E$ has the form

$$\sum_I a_I(z_1, \ldots, z_n)\partial^{i_1}_{z_1} \cdots \partial^{i_n}_{z_n}, \quad I = (i_1, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n,$$

with holomorphic matrix-valued coefficients $a_I$, vanishing except for finitely many multi-indices $I$. The sheaf $\mathcal{D}_E$ is a sheaf of locally convex algebras: for any open set $U \subset X$, the locally convex subalgebra $\mathcal{D}_E(U)^{\leq k}$ of operators of order at most $k$ is the space of sections of some vector bundle over $U$ and has the topology of uniform convergence on compact subsets. Then the inductive limit $\mathcal{D}_E(U) = \bigcup_k \mathcal{D}_E(U)^{\leq k}$ with the inductive limit topology is a complete locally convex algebra. This is the topology considered in [12]. Then one has the following result:

**Theorem 3.3.** [12, 27] Every point of $X$ has a coordinate neighborhood $U$ such that $HH_p(\mathcal{D}_E(U)) = 0$ for $p \neq 2n$ and $HH_{2n}(\mathcal{D}_E(U))$ is one-dimensional generated by the class of

$$c_E(U) = \sum_{\pi \in S_{2n}} \text{sgn}(\pi)(1, x_{\pi(1)}, \ldots, x_{\pi(2n)}),$$

where $x_{2j-1} = \partial_{z_j}, x_{2j} = z_j$. Here we identify $x \in \mathcal{D}(U)$ with the multiple of the identity $\text{Id}_r \otimes x \in M_r \otimes \mathcal{D}(U) \simeq \mathcal{D}_E(U)$, with respect to some trivialization of $E$.

**Remark 3.4.** If $E$ is an algebraic vector bundle on a non-singular projective algebraic variety there is no need to give a topology to $\mathcal{D}_E$ and the theorem is true for the algebraic definition of Hochschild homology (with the usual tensor product) provided one understands $\mathcal{D}_E(U)$ as the algebra of holomorphic differential operators on $U$ whose coefficients $a_I$ are rational functions of affine coordinates $z_1, \ldots, z_n$.

3.4 A trace for holomorphic differential operators

Let $X$ be a compact connected complex manifold and $\mathcal{U} = (U_\alpha)$ a finite open cover. Further let $E \to X$ be a holomorphic vector bundle and $\mathcal{D}_E$ the holomorphic differential operators acting on sections of $E$.

Consider the Hochschild complex of sheaves $\mathcal{C}_*(\mathcal{D}_E)$:

$$\cdots \to \mathcal{D}_E \otimes \overline{\mathcal{D}}_E \otimes \mathcal{D}_E \to \mathcal{D}_E \otimes \overline{\mathcal{D}}_E \to \mathcal{D}_E \to 0$$
3.4. A TRACE FOR HOLOMORPHIC DIFFERENTIAL OPERATORS

Let $\mathcal{U} = (U_\alpha)$ be a sufficiently fine open cover of $X$. Let $C^{p,q} = \check{C}^q(\mathcal{U}, C_{-p}(D_E))$, $(q \geq 0, p \leq 0)$ where $\check{C}^q(\mathcal{U}, \mathcal{F}) = \oplus_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$ is the Hochschild–Čech double complex. Global differential operators $D \in D_E(X)$ define cocycles in $C^{0,0}$. The restriction $D|_{U_\alpha}$ of $D$ to a sufficiently small open set is a Hochschild boundary by theorem 3.3. Thus $D|_{U_\alpha} = b\check{\delta}D^{(1)}|_{U_\alpha}$ for some $\check{\delta}D^{(1)} \in C^{-1,0}$. Since $b$ and the Čech differential commute, $(\check{\delta}D^{(1)})_{\alpha \beta} = D^{(1)}_{\alpha} - D^{(1)}_{\beta}$ is a Hochschild cycle for the algebra $D_E(U_\alpha \cap U_\beta)$ because $b\check{\delta}D^{(1)}_{|U_\alpha \cap U_\beta} = D|_{U_\alpha \cap U_\beta} = b\check{\delta}D^{(1)}_{|U_\alpha \cap U_\beta}$. It is thus exact. We can continue in this way and “climb the staircase”, see Fig. 3.1. Assume that we reached $D^{(k)}$ for $k < 2n$. Then $b\check{\delta}D^{(k)} = \check{\delta}bD^{(k)} = \check{\delta}^2D^{(k-1)} = 0$ and therefore $\check{\delta}D^{(k)}$ is exact. So we find $D^{(j)} \in C^{-j,j-1}$, $j = 1, \ldots, 2n$, such that

$$\check{\delta}D^{(j)} = bD^{(j+1)}, \quad j = 1, \ldots, 2n - 1.$$ 

Finally, we get to the point where the Hochschild homology is nontrivial and obtain

$$\check{\delta}D^{(2n)} = s + bD^{(2n+1)}$$

where $s \in C^{2n,-2n}$ has the form

$$s_{\alpha_0, \ldots, \alpha_{2n}} = \lambda_{\alpha_0, \ldots, \alpha_{2n}}(D)c_E(U_{\alpha_0} \cap \cdots \cap U_{\alpha_{2n}})$$

(with $c_E$ from theorem 3.3) for some Čech cocycle $\lambda(D) \in \check{C}^{2n}(\mathcal{U}, \mathbb{C})$ with values in the locally constant sheaf $\mathbb{C}_X$. That $\lambda(D)$ is a cocycle can be seen by applying $\tau_{2n} \circ \check{\delta}$ on both sides of the equation (3.1). Its class

$$[\lambda(D)] \in H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$$
is (up to a global sign) a trace for which we write

$$\text{Tr}_{HC}(D).$$

The fact that this is indeed a trace will be proven in the last section of this chapter where we show that it is identical to the trace on the right-hand side of theorem 3.1.

### 3.5 Formal differential geometry

We recall some notions of formal differential geometry [17, 34, 35], following [13]. Let $W_n = \bigoplus_i \mathcal{O}_n \partial_{y_i}$ be the Lie algebra of formal vector fields and $gl_r(\mathcal{O}_n)$ denote $M_r(\mathcal{O}_n)$ viewed as a Lie algebra with commutator bracket. The Lie algebra $W_n$ acts on $gl_r(\mathcal{O}_n)$ by derivations and we can thus define the semidirect product

$$W_n, r = W_n \ltimes gl_r(\mathcal{O}_n).$$

As $gl_r$ corresponds to linear vector fields on $\mathbb{R}^r$, $W_n, r$ is a Lie subalgebra of $W_{n+r}$. The Lie algebra $W_n, r$ is also embedded in $M_r(D_n)$ (viewed as Lie algebra with commutator bracket) as a Lie subalgebra of first order differential operators. It should be regarded as the Lie algebra of infinitesimal automorphisms of the trivial bundle of rank $r$ over a formal neighborhood of $0 \in \mathbb{C}^n$.

Explicitly a formal vector field $u \in W_n, r$ is given by a formal power series

$$u(y) = \sum_{\alpha \in \mathbb{N}^n} \left( \sum_{i=1}^n a_{\alpha}^i y^\alpha \frac{\partial}{\partial y^i} + b_{\alpha} y^\alpha \right)$$

where $a_{\alpha}^i \in \mathbb{C}$ and $b_{\alpha} \in M_r(\mathbb{C})$.

A local parametrization of $E$ is a holomorphic bundle isomorphism $U \times \mathbb{C}^r \to E|_V$ from the trivial bundle over some neighborhood $U \subset \mathbb{C}^n$ of $0$ to the restriction of $E$ to some open set $V$. Let $J_pE$ be the complex manifold of $p$-jets at $0 \in \mathbb{C}^n$ of local parametrizations.

In local coordinates, that is in a local trivialization of the bundle $E$, a point in $J_pE$ is a map $\phi_z$

$$(\phi_z(y, e))^i = (x^i + \sum_{|\alpha| \leq p} a_{\alpha}^i y^\alpha, \sum_{|\alpha| \leq p \sum_{j=1}^n} (b_{\alpha})^i_j w^\alpha e^j)$$

where $x \in X$, $(y, e) \in \mathbb{C}^n \times \mathbb{C}^r$ are the coordinates of the local parametrization and $a_{\alpha}^i \in \mathbb{C}$, $b_{\alpha} \in \text{GL}_r(\mathbb{C})$. 
In particular, $J_1E$ is the extended frame bundle, whose fiber at $x \in X$ is the space of pairs of bases of the holomorphic tangent space at $x$ and the fiber of $E$ at $x$ respectively. The group $G = GL_n(\mathbb{C}) \times GL_r(\mathbb{C})$ acts freely on the right on each $J_pE$, $p = 1, 2, \ldots$ by linear transformations of $\mathbb{C}^n \times \mathbb{C}^r$ and $J_1E$ is a principal $G$-bundle over $X$. The complex manifolds $J_pE$ form a projective system with surjective $G$-equivariant submersions $J_pE \to J_qE$, $p > q$. We write $J_{\infty}E$ for the projective limit of $J_pE$ for $p \to \infty$. In local coordinates this means that we set $p = \infty$ in the expression for $\phi_z(y,e)$ and consider the sum as a formal power series.

In the language of [13], $J_{\infty}E$ is a holomorphic principal $W_{n,r}$-space. Namely, there is a Lie algebra homomorphism $W_{n,r} \to \mathcal{V}(J_{\infty}E)$ from $W_{n,r}$ to the Lie algebra of holomorphic vector fields on $J_{\infty}E$, which is an isomorphism $W_{n,r} \to T^{1,0}_{\phi}J_{\infty}E$ at each point $\phi \in J_{\infty}E$. To see this isomorphism explicitly, we use again local coordinates. To understand the tangential space of $J_{\infty}E$, we consider a smooth path $\phi_t$ in $J_{\infty}E$. We write $\phi := \phi_0$ and $\dot{\phi} := \frac{d\phi}{dt}|_{t=0}\phi_t$. As $\phi_t(y,e)$ then describes (in a formal sense) a path in $E$, we see that $\dot{\phi}$ is a map

$$\dot{\phi} : (y,e) \in \mathbb{C}^n \times \mathbb{C}^r \mapsto \dot{\phi}(y,e) \in T^{1,0}_{\phi(y,e)}E.$$ 

That shows that an element in the tangent space of $J_{\infty}E$ is a formal holomorphic vector field. We keep the notation $\dot{\phi} \in T^{1,0}_{\phi}(J_{\infty}E)$ for an element in the tangent space. As $\phi \in J_{\infty}E$ is a formal diffeomorphism to a neighborhood of $(x,0) \in E$, the tangent map

$$d_w\phi(w) : T^{1,0}_{\phi(y,0)}(\mathbb{C}^n \times \mathbb{C}^r) \cong \mathbb{C}^{n+r} \to T^{1,0}_{\phi(y,0)}E,$$

where $w := (y,e)$, is a formal isomorphism. We now use the embedding $W_{n,r} \subset W_{n+r} \subset \text{End}(\mathbb{C}^{n+r})$ and invert the above map $d_w\phi$. As

$$(d\phi)^{-1}\dot{\phi}(w) \in T^{1,0}_{\phi(y,0)}(\mathbb{C}^n \times \mathbb{C}^r) \cong \mathbb{C}^{n+r},$$

$(d\phi)^{-1}\dot{\phi}$ defines a formal vector field in $W_{n,r}$. So far we considered $\phi \in J_{\infty}E$ as fixed. If we let $\phi$ vary over $J_{\infty}E$, the composition $d_w\phi u$, where $u \in W_{n,r}$, defines a vector field on $J_{\infty}E$. In other words, $d_w\phi$ is a natural isomorphism from $W_{n,r}$ to a subspace of the holomorphic vector fields on $J_{\infty}E$.

We show that this is indeed a Lie algebra isomorphism. We consider two formal vector fields $u, v \in W_{n,r}$ and their vector fields $d_w\phi u$ and $d_w\phi v$ on $J_{\infty}E$. We write $v = \sum_{i=1}^{n+r} v_i \frac{\partial}{\partial w^i}$ where the coefficients $v_i$ are formal power series in $w$. Note that $d_w\phi v$ corresponds to the section

$$d_w\phi v = \frac{\partial \phi^i}{\partial w^j} v^j \frac{\partial}{\partial x^i},$$

where the summation convention is assumed. If we understand this vector field as an infinitesimal change of the coordinates defined by the parametrization $\phi$, we see that the
derivative of $d_w \phi v$ in the direction of $d_w \phi u$ is given by
\[
\frac{\partial}{\partial w^j} \left( \frac{\partial \phi^i}{\partial w^k} u^k \right) v^j \frac{\partial}{\partial x^i} = \frac{\partial^2 \phi^i}{\partial w^j \partial w^k} u^k v^j \frac{\partial}{\partial x^i} + \frac{\partial \phi^i}{\partial w^j} \frac{\partial u^k}{\partial w^j} v^j \frac{\partial}{\partial x^i}.
\]
The first part is symmetric in $u$ and $v$ and therefore cancels when we consider the Lie bracket, thus
\[
[d_w \phi u, d_w \phi v] = d_w \phi [v, u] = -d_w \phi [u, v].
\]
The inverse map defines a holomorphic 1-form $\Omega_{MC} \in \Omega^{1,0}(J_\infty E, W_{n,r})$ with values in $W_{n,r}$ and the homomorphism property is equivalent to the Maurer-Cartan equation
\[
d\Omega_{MC} + \frac{1}{2}[\Omega_{MC}, \Omega_{MC}] = 0.
\]
Because $\Omega_{MC}^{-1}$ is for every $\phi \in J_\infty E$ an isomorphism $W_{n,r} \rightarrow T^\phi J_\infty E$, the above equation is equivalent to show that
\[
d\Omega_{MC}(\Omega_{MC}^{-1} u, \Omega_{MC}^{-1} v) + [u, v] = 0
\]
for all $u, v \in W_{n,r}$. We now use the Cartan formula for a 1-form $\Omega_{MC}$ and vector fields $\phi$ and $\psi$ in the form
\[
d\Omega_{MC}(\phi, \psi) = -\Omega_{MC}([\phi, \psi]) + \phi \Omega_{MC}(\psi) - \psi \Omega_{MC}(\phi).
\]
Because of the homomorphism property, this leads to
\[
d\Omega_{MC}(\Omega_{MC}^{-1} u, \Omega_{MC}^{-1} v) = -[u, v] + \phi(v) - \psi(u).
\]
As $u$ and $v$ are constant w.r.t. $x \in X$, the last two terms vanish.
The fibers of the bundle $J_\infty E/G \rightarrow J_1 E/G = X$ are contractible and therefore there exists a smooth section (unique up to homotopy) $\phi : X \rightarrow J_\infty E/G$ or, equivalently, a smooth $G$-equivariant section $\tilde{\phi} : J_1 E \rightarrow J_\infty E$. The Maurer-Cartan form $\Omega_{MC}$ pulls back to a $G$-equivariant 1-form $\omega := \tilde{\phi}^* \Omega_{MC}$ on $J_1 E$ obeying the Maurer-Cartan equation. This induces a flat connection on the associated bundle
\[
\tilde{D}_E = J_1 E \times_G M_r(D_n) \rightarrow X.
\]
The horizontal sections are in one-to-one correspondence with global differential operators: to $D \in D_E(X)$ there corresponds the horizontal section $\tilde{D}$. Its value at $x \in X$ is the Taylor expansion at 0 of $D$ with respect to the coordinates and the trivialization defined by $\phi$ at the point $x$. Conversely, every horizontal section comes from a differential operator. In explicit terms, let us choose a local trivialization of $J_1 E = U \times G$ over $U \subset X$. Then the
3.6. HOCHSCHILD AND DE RHAM COHOMOLOGY

restriction of \( \phi \) to \( U \) is given by a map \( \phi^U: U \to J_\infty E|_U \) and \( \omega = \phi^*\Omega_{MC} \) is a \( W_{n,r} \)-valued 1-form on \( U \). The Taylor expansion \( \hat{D} \) is given on \( U \) by a map \( U \to M_r(D_n), x \mapsto \hat{D}_x \) obeying

\[
d\hat{D} + [\omega, \hat{D}] = 0.
\]

A change of trivialization is given by a gauge transformation \( g: U \to G \). The section changes as \( \hat{D}_x \mapsto g_x \cdot \hat{D}_x \) and \( \omega \) as \( \omega \mapsto g \cdot \omega - dgg^{-1} \) and \( dgg^{-1} \) is a 1-form with values in the Lie algebra of \( G \), embedded in \( M_r(D_n) \) as the Lie algebra of first order operators of the form \( \sum a_{jk}y_k \frac{\partial}{\partial y_j} + b, a_{jk} \in \mathbb{C}, b \in M_r(\mathbb{C}) \).

**Proposition 3.5.** Let \( \Omega^* \) be the complex of sheaves of complex-valued smooth differential forms on \( X \) with de Rham differential and let \( \mathcal{C}(D_E) \) be the complex of sheaves of Hochschild chains of \( D_E \). There is a homomorphism of complexes of sheaves

\[
\chi_*: \mathcal{C}_*(D_E) \to \Omega^{2n-*}
\]

depending on a choice of section of \( J_\infty E/G \to X \), inducing an isomorphism of the cohomology sheaves. The map \( \chi_0: D_E(X) \to \Omega^{2n}(X) \) on global differential operators is the map appearing in theorem 3.1 and \( \chi_{2n} \) maps \( (D_0, \ldots, D_{2n}) \in \mathcal{C}_{2n}(U) \) to the function \( \tau_{2n}(\hat{D}_0, \ldots, \hat{D}_{2n}) \) on the open set \( U \).

### 3.6 Hochschild and de Rham cohomology

This section is dedicated to the construction of \( \chi_* \) and the proof of Proposition 3.5, which are very similar to the ones for the maps \( \chi_k \) in the section 2.5.

We construct a homomorphism of complexes of sheaves

\[
\chi_*: \mathcal{C}_*(D_E) \to \Omega^{2n-*}
\]

from the Hochschild chain complex of the sheaf \( D_E \) to the sheaf of smooth de Rham forms. It is based on formal geometry and thus depends on a choice of section of \( J_\infty E/G \) (but the map induced on homology is canonical). The map \( \chi_0: D_E(X) \to \Omega^{2n}(X) \) on global differential operators is the one appearing in theorem 3.1.

To do this, we apply the constructions from section 2.2 to the smooth de Rham complex \( A = \Omega(U, \hat{D}_E) \) with values in the vector bundle \( \hat{D}_E \) on some open subset \( U \subset X \). Let \( \hat{D} \in A^0 \) denote the horizontal section corresponding to a differential operator \( D \in D_E(U) \). Locally, upon trivialization of \( T^{1,0}X \) and \( E \), the condition of horizontality is \( d\hat{D} + [\omega, \hat{D}] = 0 \) for some Maurer-Cartan element \( \omega \).
Proposition 3.6. Let $U$ be a sufficiently small open neighborhood of any point in $X$. Let $D_0, \ldots, D_p \in \mathcal{D}_E(U)$ be differential operators on $U$ and $\hat{D}_0, \ldots, \hat{D}_p \in A^0$ be the corresponding horizontal sections of $\hat{\mathcal{D}}_E$ on $U$. Then the differential $(2n-p)$-forms on $U$

$$\chi_p(D_0, \ldots, D_p) = \tau_{2n}^r \left( \text{sh}_{p,2n-p}(\hat{D}_0, \hat{D}_1, \ldots, \hat{D}_p, \omega, \ldots, \omega) \right),$$

are well-defined (i.e., independent of the trivialization of $J_1E$), continuous, and obey the relations

$$d \circ \chi_p = (-1)^{p-1} \chi_{p-1} \circ b.$$ 

Proof. If we change the trivialization of the extended frame bundle $J_1E$, then $\hat{D}, \omega$ change by the action of an element of $G$, under which $\tau_{2n}^r$ is invariant, and the shift of $\omega$ by a one-form with values in the Lie algebra of $G$ embedded in $A$. By property (ii) of $\tau_{2n}^r$, see 3.2.1, the right-hand side of (3.3) is unaffected by such a shift. The continuity is clear: since $\tau_{2n}^r$ depends non-trivially only on finitely many Taylor coefficients of its arguments, the $C^\ell$-norms on compact subsets of $\chi_p(D_0, \ldots, D_p)$ are estimated by $C^\ell$-norms of the coefficients of $D_0, \ldots, D_p$ which by analyticity are in turn controlled by sup norms on (slightly larger) compact subsets.

In the notation of Proposition 2.4,

$$\chi_p(D_0, \ldots, D_p) = \tau_{2n}^r \left( (\hat{D}_0, \ldots, \hat{D}_p) \times (\omega)_{2n-p} \right).$$

The cochain $a = (\hat{D}_0, \ldots, \hat{D}_p)$ obeys $d_\omega(a) = 0$, thus the homomorphism property of Proposition 2.4 reduces to

$$\delta \sum_{k \geq 0} (-1)^k a \times (\omega)_k = \sum_{k \geq 0} (-1)^k b a \times (\omega)_k, \quad \delta = b \pm d.$$ 

The component of Hochschild degree $2n$ is

$$(-1)^{p-1}b(a \times (\omega)_{2n-p+1}) + (-1)^p(-1)^p d(a \times (\omega)_{2n-p}) = (-1)^{p-1}b a \times (\omega)_{2n-p+1}.$$ 

If we apply $\tau_{2n}^r$, the first term vanishes and we obtain the claim. 

Since the expression on the right-hand side of (3.3) is local, it is clear that $\chi_p$ are compatible with the inclusion of open sets and thus define maps of sheaves. Moreover, by the normalization of $\tau_{2n}^r$, we see that $\chi_{2n}(c_E(U)) = r$, where $c_E(U)$ is the generator of theorem 3.3. Thus $\chi_\ast$ induces a non-trivial map on homology. By theorem 3.3 this map is an isomorphism. This concludes the proof of proposition 3.5.
3.7 The two traces are equal

Here we replace the open cover considered in section 3.4 by a refinement which is the open star of a triangulation of $X$. Then the Hochschild–Čech cocycle $(D^{(j)})$ defines a cocycle, still denoted by $(D^{(j)})$ for the refinement. Choose a smooth finite triangulation $|K| \to X$ of $X$ with underlying simplicial complex $K$. The open star of the triangulation is the open cover $U = (U_\alpha)_{\alpha \in K_0}$ of $X$ labeled by the set of vertices of the triangulation, such that $U_\alpha$ is the complement of the simplices not containing $\alpha$.

Let $C$ be the cell decomposition of $X$ dual to the above triangulation. Its cells are in one-to-one correspondence with the simplices of the triangulation. We denote $C_{\alpha_0,\ldots,\alpha_p}$ the $(2n-p)$-cell corresponding to the simplex $\sigma_{\alpha_0,\ldots,\alpha_p}$ with vertices $\alpha_0,\ldots,\alpha_p$. We orient the dual cells by the condition that $C_{\alpha_0,\ldots,\alpha_p} \cdot \sigma_{\alpha_0,\ldots,\alpha_p} = 1$ on the intersection index (see appendix A.2).

Proposition 3.7. Let $s = s(D)$ be the cocycle (3.2). Then

$$
\text{Tr}_\chi(D) = \sum_{\alpha_0 < \cdots < \alpha_{2n}} \int_{C_{\alpha_0,\ldots,\alpha_{2n}}} \chi_{2n}(s_{\alpha_0,\ldots,\alpha_{2n}}),
$$

where $\chi_{2n}$ is defined in proposition 3.6 for the open set $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{2n}}$.

Proof. We first prove by induction that for all $p = 0,\ldots, 2n-1$,

$$
\text{Tr}_\chi(D) = \sum_{\alpha_0 < \cdots < \alpha_p} \int_{C_{\alpha_0,\ldots,\alpha_p}} \chi_p(bD^{(p+1)}_{\alpha_0,\ldots,\alpha_p}),
$$

(3.4)

and then deduce the claim by doing a further induction step. For $p = 0$ eq. (3.4) follows from

$$
\text{Tr}_\chi(D) = \sum_{\alpha} \int_{C(\alpha)} \chi_0(D|_{U_\alpha}),
$$

and $D|_{U_\alpha} = bD^{(1)}_\alpha$. Assume that the claim is proved up to some $p < 2n - 1$. Then, by proposition 3.5 and Stokes’ theorem (the signs are discussed in the appendix, see (A.7)),

$$
\text{Tr}_\chi(D) = \sum_{\alpha} \int_{C(\alpha)} \chi_0(D|_{U_\alpha}),
$$

and $D|_{U_\alpha} = bD^{(1)}_\alpha$. Assume that the claim is proved up to some $p < 2n - 1$. Then, by proposition 3.5 and Stokes’ theorem (the signs are discussed in the appendix, see (A.7)),
we get
\[
\text{Tr}_\chi(D) = \sum_{\alpha_0 < \cdots < \alpha_p} \int_{C_{\alpha_0 \cdots \alpha_p}} \chi_p(bD^{(p+1)}_{\alpha_0 \cdots \alpha_p})
\]
\[
= (-1)^p \sum_{\alpha_0 < \cdots < \alpha_p} \int_{C_{\alpha_0 \cdots \alpha_p}} d\chi_{p+1}(D^{(p+1)}_{\alpha_0 \cdots \alpha_p})
\]
\[
= (-1)^p (-1)^p \sum_{\beta, \alpha_0 < \cdots < \alpha_p} \int_{C_{\beta, \alpha_0 \cdots \alpha_p}} \chi_{p+1}(D^{(p+1)}_{\alpha_0 \cdots \alpha_p})
\]
\[
= \sum_{\alpha_0 < \cdots < \alpha_{p+1}} \int_{C_{\alpha_0 \cdots \alpha_{p+1}}} \chi_{p+1}(\delta D^{(p+1)}_{\alpha_0 \cdots \alpha_{p+1}}).
\]

Since \(\delta D^{(p+1)} = bD^{(p+2)}\) if \(p < 2n - 1\) the induction step is complete.

Now we do this step once more for \(p = 2n - 1\). The calculation is the same but the conclusion is different since \(\delta D^{(2n)} = s + bD^{(2n+1)}\). We obtain
\[
\text{Tr}_\chi(D) = \sum_{\alpha_0 < \cdots < \alpha_{2n}} \int_{C_{(\alpha_0 \cdots \alpha_{2n})}} \chi_{2n}(s + bD^{(2n+1)})_{\alpha_0 \cdots \alpha_{2n}}.
\]

Moreover, \(\chi_{2n}\) coincides with \(\tau_{2n}\) composed with the Taylor expansion and thus is a cocycle, i.e., it vanishes on exact chains such as \(bD^{(2n+1)}\).

The integral over the 0-dimensional cycle \(C_{\alpha_0 \cdots \alpha_{2n}}\) is the evaluation of the integrand times the sign of the orientation which is the sign \(\epsilon(\alpha_0, \ldots, \alpha_{2n})\) of the orientation of \(\sigma_{\alpha_0 \cdots \alpha_{2n}}\) relative to the orientation of \(X\).

**Corollary 3.8.**
\[
\text{Tr}_\chi(D) = r \sum_{\alpha_0 < \cdots < \alpha_{2n}} \lambda_{\alpha_0 \cdots \alpha_{2n}}(D) \epsilon(\alpha_0, \ldots, \alpha_{2n}).
\]
Chapter 4

Lefschetz number

"The battle of Helm's Deep is over; the battle for Middle Earth is about to begin."

Gandalf in "The Lord of the Rings: The Two Towers" by J. R. R. Tolkien

Let $X$ be an $n$-dimensional compact connected complex manifold and $E \rightarrow X$ a holomorphic vector bundle of rank $r$. We denote by $\mathcal{D}_E(X)$ the holomorphic differential operators acting on sections of $E$. As these operators commute with the Dolbeault differential $\bar{\partial}$, they naturally act on the Dolbeault (or sheaf) cohomology groups $H^j(X, E)$. Thus we have an algebra homomorphism

$$H^j : \mathcal{D}_E(X) \rightarrow \text{End}(H^j(X, E)).$$

Since the cohomology of $E$ is finite dimensional, we can consider the Lefschetz number (or supertrace) $L : \mathcal{D}_E(X) \rightarrow \mathbb{C}$,

$$D \mapsto L(D) = \sum_{j=0}^{n} (-1)^j \text{tr}(H^j(D)),$$

which is a trace on the algebra $\mathcal{D}_E(X)$. The intention of this chapter is to find a local formula for this trace from which we will see that it is proportional to the Hochschild-Čech trace (see section 3.4). Together with proposition 3.7 this will prove theorem 3.1.

4.1 Idea

First we remark that $L(D)$ can be computed as a supertrace over the space $\Omega^{(0, \bullet)}(M, E)$ instead of its cohomology if $D$ is "attenuated" by the heat kernel operator $e^{-t\Delta_3}$ where
$\Delta_{\bar{\partial}} = \bar{\partial}^* \partial + \partial \bar{\partial}^*$:

$$L(D) = \sum_{j=0}^{n} (-1)^j \text{tr}_{\Omega^{(0,j)}(X,E)} \left( D e^{-t \Delta_{\bar{\partial}}} \right) =: \text{Str} \left( D e^{-t \Delta_{\bar{\partial}}} \right).$$

This supertrace is independent of $t$ (see proposition 4.1). The idea is to rewrite this expression by inserting a partition of unity $\{ \rho_\alpha \}$ that corresponds to an open finite cover $\{ U_\alpha \}$ of $X$ (i.e. $\text{supp} \rho_\alpha \subset U_\alpha$) for which $2n+2$-fold intersections are empty. We then let $D$ ”climb the staircase” as in section 3.4 and use the properties of the supertrace so that all operators, except one, only appear in commutators. The first step of this calculation explicitly looks as follows:

$$\text{Str} \left( D e^{-t \Delta_{\bar{\partial}}} \right) = \sum_\alpha \text{Str} (\rho_\alpha D_\alpha e^{-t \Delta_{\bar{\partial}}}) = \sum_\alpha \text{Str} (\rho_\alpha [D_{\alpha}^1, D_{\alpha}^2] e^{-t \Delta_{\bar{\partial}}})$$

$$= \sum_\alpha \left( \text{Str} ([D_\alpha^2, \rho_\alpha] D_\alpha^1 e^{-t \Delta_{\bar{\partial}}}) + \int_0^t ds \text{Str} (\rho_\alpha D_\alpha^1 e^{-(t-s) \Delta_{\bar{\partial}}} D_\alpha^2 e^{-s \Delta_{\bar{\partial}}}) ds \right)$$

$$= \sum_\alpha \left( \text{Str} ([D_\alpha^2, \rho_\alpha] D_\alpha^1 e^{-t \Delta_{\bar{\partial}}}) - \int_0^t \text{Str} ([\bar{\partial}, \rho_\alpha] D_\alpha^1 e^{-(t-s) \Delta_{\bar{\partial}}} [\bar{\partial}^*, D_\alpha^2] e^{-s \Delta_{\bar{\partial}}}) ds \right).$$

We used the properties of the supertrace and the relations $[\bar{\partial}, D_\alpha^i] = [\bar{\partial}, \Delta_{\bar{\partial}}] = [\bar{\partial}^*, \Delta_{\bar{\partial}}] = 0$. If we continue the calculation, the operators $D_\alpha^i$, $i \geq 1$ should only appear in the form $[\bar{\partial}^*, D_\alpha^i]$. The reason for this will become clear in subsection 4.3.3, where we evaluate the standard Hochschild cycle of dimension $2n$ in the expression. To continue the calculation, we have to find ”combinatorial” rules to describe the terms. This can be done in an elegant way using a Dolbeault-Hochschild double complex (see section 4.3). The main ingredient for this construction is a JLO-type cocycle which we construct in section 4.2.

Actually the terms in the above computation are not well defined as the operators $D_\alpha^i$ are only defined on $U_\alpha$ but the heat kernel $e^{-t \Delta_{\bar{\partial}}}$ extends the support of the operator in the supertrace to the whole manifold. Therefore we have to use an approximation of the heat kernel which has a support in some small neighborhood of the diagonal.

### 4.2 A JLO-type cocycle and approximation of the heat kernel

Let $(\Omega^{(0,\bullet)}(X,E), \bar{\partial})$ be the Dolbeault complex with values in the holomorphic vector bundle $E$. We fix Hermitian metrics on $T_X$ and on $E$. These metrics induce an $L^2$ inner
product $\langle , \rangle$ on the Dolbeault complex and a self-adjoint positive semidefinite Laplace operator $\Delta_{\bar{\partial}} = \bar{\partial}\partial^* + \partial\partial^*$ with discrete spectrum. By Hodge theory, the cohomology group $H^j(X, E)$ is isomorphic to the space of harmonic forms $\text{Ker}(\Delta_{\bar{\partial}})$. Moreover we have the following standard fact.

**Proposition 4.1.** For any $D \in \mathcal{D}_E(X)$ and $t > 0$, $De^{-t\Delta_{\bar{\partial}}}$ is a trace class operator on the Hilbert space of square integrable Dolbeault forms. The expression

$$
\sum_{j=0}^{n} (-1)^j \text{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar{\partial}}})
$$

is independent of $t$ and is equal to $L(D)$.

**Proof.** We refer, e.g., to [11] for the trace class property. As the heat kernel depends smoothly on $t$ for $t > 0$, the trace is also smooth in $t$. The independence of $t$ is checked by differentiation:

$$
\frac{d}{dt} \text{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar{\partial}}}) = -\text{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar{\partial}}}(\partial\partial^* + \partial^*\partial))
$$

$$
= -\text{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar{\partial}}\partial\partial^*}) - \text{tr}_{\Omega^{(0,j-1)}(X,E)}(\partial De^{-t\Delta_{\bar{\partial}}\partial^*})
$$

$$
= -\text{tr}_{\Omega^{(0,j)}(X,E)}(De^{-t\Delta_{\bar{\partial}}\partial\partial^*}) - \text{tr}_{\Omega^{(0,j-1)}(X,E)}(De^{-t\Delta_{\bar{\partial}}\partial\partial^*}).
$$

Here we use the fact that $\partial$ commutes both with $D$ (since $D$ is holomorphic) and with the Laplacian. Taking the sum with alternating signs yields the claim.

Thus we can evaluate the sum in the limit $t \to \infty$. Since 0 is an isolated eigenvalue of the positive semidefinite operator $\Delta_{\bar{\partial}}$, we obtain the alternating sum of traces on harmonic forms, namely $L(D)$.

Let $U$ be an open subset of $X$ and $A = \mathcal{D}_E(U)$ be the algebra of holomorphic differential operators on the restriction of $E$ to $U$. $A$ is the countable strict inductive limit over $j$ of the Fréchet spaces $\mathcal{D}_E^j(U)$ of holomorphic differential operators of order at most $j$. The seminorms on $\mathcal{D}_E^j(U)$ are given by the supremum norms over the compact sets $K \subset U$. $A$ is therefore an LF-Space, i.e. it has the natural topology of a complete locally convex space.

The Dolbeault complex $(M_c(U) = \Omega^{\bullet,0}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{D}_E(U), \partial\partial^*, \bar{\partial})$ with compact support and values in $\mathcal{D}_E$ is a locally convex differential graded algebra and an $A$-bimodule. In local coordinates it is the graded algebra generated by $M_c(C^\infty_c(U))$ of degree 0, $dz_i$ of degree 1 and $\partial z_i$ of degree zero. The algebra $M_c(U)$ is the inductive limit over $j$ and $K$ of the locally convex subalgebras $M_{K,j}$ of operators of order at most $j$ and with compact
support \( K \subset U \). The space \( M_{K,j} \) is the space of sections \( x \to D_x \) of some vector bundle on \( U \) with support in \( K \), and has the topology defined by the system of (semi)norms given by the \( C^\ell \)-norms for \( \ell \geq 0 \).

For \( f : (0,1] \to \mathbb{C} \) a function with asymptotic expansion \( f(t) = \sum_{n=-\infty}^{\infty} a_n t^n \) for \( t \to 0 \), we denote the part of non-positive powers by \( [f(t)]_\leq := \sum_{n=0}^{\infty} a_n t^n \).

Now we are ready to state the following proposition introducing the cocycle \( \Psi \):

**Proposition 4.2.** Let \( U \subset X \) be an open subset, \( A = \mathcal{D}_E(U) \) and \( M_c = M_c(U) \) be the Dolbeault complex with values in \( A \) and compact support. Let \( K^N_s \) be an approximation of the heat kernel with support in some small neighborhood of the diagonal and \( K^N_s \) the corresponding operator (see subsection A.1.1 for details). Suppose that \( D_0 \in M^p_c, D_1, \ldots, D_p \in A \). Then

\[
\Psi_p(D_0, \ldots, D_p) = (-1)^{\frac{d(p+1)}{2}} \left[ \int \Delta_p \text{Str}(D_0 K^N_{s_0}[\bar{\partial}^* D_1] K^N_{s_1} \cdots [\bar{\partial}^* D_p] K^N_{s_p}) ds_1 \cdots ds_p \right].
\]

(4.1)

where \( \text{Str} \) denotes the alternating sum of traces over the Hilbert space of square integrable sections of \( \Lambda(T^{0,1}U)^* \otimes E|_U \). \( \Psi_p \) is well defined, independent of \( N \) for \( N \geq n - p + \lfloor \frac{d}{2} \rfloor \) where \( d \) is the sum of the degrees of the operators \( D_0, [\bar{\partial}^*, D_1], \ldots, [\bar{\partial}^*, D_p] \), and defines a continuous cocycle in the Hochschild-Čech double complex:

\[
\Psi = \sum_p \Psi_p \in \bigoplus_{p=0}^{n} \text{Hom}(M^p_c \otimes \bar{A}^{\otimes p}, \mathbb{C})[t^{-1}] \simeq C^0(A, M^*_c)[t^{-1}].
\]

**Remark 4.3.** Explicitly, the cocycle condition for \( \Psi \) reads as follows:

\[
\Psi_p(b(D_0, \ldots, D_{p+1})) = \Psi_{p+1}([\bar{\partial}, D_0], D_1, \ldots, D_{p+1})
\]

for \( p = 0 \ldots n - 1 \), where \( b \) denotes the Hochschild differential.

**Remark 4.4.** For any \( D_0 \in M^0_c \) we have

\[
\Psi_0(D_0) := [\text{Str}(D_0 K^N)]_\leq = [\text{Str}(D_0 K_t)]_\leq.
\]

This follows directly from the estimates about the approximated heat kernel in part i) of theorem A.1. This remark will be generalized for \( \Psi_p, p = 0, \ldots, 2n \) in remark A.11.

**Remark 4.5.** The formula for our cocycle \( \Psi \) is in fact identical to the formula for an entire cyclic cocycle constructed in [28] where a \( \theta \)-summable Fredholm module over a \( C^* \)-algebra was considered. The module is endowed with a certain self-adjoint operator that corresponds to the Dirac operator \( \bar{\partial} + \partial^* \) in our case.
To prove the above theorem, one first needs to say more about the approximated heat kernels and show that the integral and the asymptotic expansion in the definition of $\Psi_p$ are well defined. This is done explicitly in the appendix A.1 where the following theorem is proven:

**Proposition 4.6.** Let $n$ be the complex dimension of $X$. Take the operators $D_0, D_1, \ldots, D_p$ as in proposition 4.2. We write $d$ for the sum of the degrees of the differential operators $D_0, [\bar{\partial}^*, D_1], \ldots, [\bar{\partial}^*, D_p]$ which are defined on a small (see remark below) open set $U \subset X$. Recall that the approximated heat kernel depends on the constants $N$ and $\varepsilon$. Then for $N$ big enough and $\varepsilon$ small enough, $\Psi_p(D_0, \ldots, D_p)$ is well defined and a polynomial in $t^{-1}$ of degree $n - p + \lfloor \frac{d}{2} \rfloor$. More precisely, for $N \geq n - p + \lfloor \frac{d}{2} \rfloor$ and $\varepsilon < \frac{1}{p+1} \text{dist}(X \setminus U, \text{supp}(D_0))$, where $\text{dist}$ means the geodesic distance, it is independent of $N$ and $\varepsilon > 0$. Furthermore, $\Psi_p(D_0, \ldots, D_p)$ depends continuously on $D_0, D_1, \ldots, D_p$.

The following remark follows from the proof of the above proposition:

**Remark 4.7.** If we replace in the formula for $\Psi_p$ one of the approximated heat kernels $k^N$ by the exact heat kernel $k$, the $dx$-integral still only goes over a compact set. Thus we can choose $\varepsilon$ small enough so that the formula for $\Psi_p$ is still well defined. From the above proof it is also clear that this procedure doesn’t change the value of $\Psi_p$.

The above proposition and remark allow us to prove the cocycle property for $\Psi$ using the main theorem of topological quantum mechanics and Stokes’ theorem. We therefore first rewrite the expression for $\Psi_p$.

**Cocycle property**

**Lemma 4.8.** The formula for $\Psi_p$ can also be written in the following symmetric form:

$$\Psi_p(D_0, \ldots, D_p) = \left[ \int_{t \Delta_p} \text{Str}(D_0 K_s^N \cdots D_p K_s^N) \right]^{-1}, \quad N \geq n - p + \lfloor \frac{d}{2} \rfloor$$

where $K^N_s = K^N_s(1 + \bar{\partial}^* ds)$ and $\bar{\partial}^*$ and $ds$ anticommute.

**Proof.** First, it has to be understood that only the $p$-form part with respect to $ds$ contributes to the integral. This implies that we get all relevant terms by taking $K^N_s = K^N_s \bar{\partial}^* ds_i$ for all $i$, except one. Doing so, we get $p + 1$ different terms:

$$(-1)^{(p-1)/2+i} \sum_{i=0}^{p} D_0 K^N_{s_0} \bar{\partial}^* \cdots D_i K^N_{s_i} \cdots D_p K^N_{s_p} \bar{\partial}^* ds_1 \wedge \cdots \wedge ds_p$$
The sign comes from commuting the $ds$ with the $\bar{\partial}^*$ and from the fact $ds_0 \wedge \ldots \wedge ds_i \wedge ds_{p} = (-1)^i ds_1 \wedge \ldots \wedge ds_p$.

On the other hand consider formula (4.1). First remark that $\bar{\partial}^*$ commutes with $K_s = e^{-s\Delta_\bar{\partial}}$ because $[\bar{\partial}^*, \Delta_\bar{\partial}] = 0$. In the formula (4.1) this is still true for $K_s^N_i$ because we can replace the approximated heat kernel by the full (see remark 4.7), then commute and replace back. Therefore and because $(\partial^*)^2 = 0$, most of the terms vanish if we expand the commutators in 4.1. To be more concrete, denote terms of the form $\bar{\partial}^*D_i K_s^N_i$ by $A_i$ for $i \geq 1$, terms of the form $D_i \bar{\partial}^* K_s^N_i$ by $B_i$ for $i \geq 1$, and $A_0 = D_0 K_s^N_0$. Because of the above argument, a term vanishes if a $B$ is followed by an $A$. Thus the remaining terms are

$$(-1)^{p(p+1)/2} \sum_{i=0}^p (-1)^{p-i} A_0 \ldots A_i B_{i+1} \ldots B_p$$

\[ \square \]

**Lemma 4.9** (Topological Quantum Mechanics). Define $K_s = e^{-s\Delta_\bar{\partial} + \bar{\partial}^* ds}$. Then $d_s K_s + [\bar{\partial}, K_s] = 0$.

The bracket on the right hand side is a superbracket with respect to the grading where the forms $d\bar{z}$ and $ds$ have degree 1. We assume that $ds$ anticommutes with $\bar{\partial}$ and $\bar{\partial}^*$.

**Proof.** $K_s = e^{-s\Delta_\bar{\partial} + \bar{\partial}^* ds}$. As $[\bar{\partial}, \Delta_\bar{\partial}] = 0$ and $[\bar{\partial}, \bar{\partial}^*] = \Delta_\bar{\partial}$ we find $[\bar{\partial}, K_s] = e^{-s\Delta_\bar{\partial}} [\bar{\partial}, \bar{\partial}^* ds] = e^{-s\Delta_\bar{\partial}} ds \Delta_\bar{\partial} = -d_s K_s$.

\[ \square \]

**Lemma 4.10.** $\Psi_p$ is a cocycle in the sense of prop 4.2:

$$\Psi_p(b(D_0, \ldots, D_{p+1})) = \Psi_{p+1}([\bar{\partial}, D_0], D_1, \ldots, D_{p+1}).$$

**Proof.** In short, the proof is an application of Stokes’ theorem and uses lemma 4.9 and remark 4.7.

The definition of the Hochschild differential and Stokes’ theorem tell us that (we suppress the bracket $[\ldots]_-$ in the notation):

$$\Psi_p(b(D_0, \ldots, D_{p+1})) = \sum_{j=0}^{p+1} (-1)^j \int_{\partial t \Delta_p} \text{Str}(D_0 K_{s_0}^N \cdots D_{p+1} K_{s_{p+1}}^N)$$

$$= \int_{\partial t \Delta_p} \text{Str}(D_0 K_{s_0}^N \cdots D_{p+1} K_{s_{p+1}}^N) = \int_{t \Delta_p} d_s \text{Str}(D_0 K_{s_0}^N \cdots D_{p+1} K_{s_{p+1}}^N)$$
By Leibniz’s rule the differential acts on each $K^N_s$ separately. By remark A.11 we can replace the respective $K^N_s$ by $K_s$, use lemma 4.9 and replace back $K_s$ by $K^N_s$, i.e. we are allowed to write $dK^N_s = -[ar{\partial}, K^N_s]$ in the above expression. Furthermore we use $[\bar{\partial}, D_j] = 0$ for $j \neq 0$ and $\text{Str}([\bar{\partial}, D_0 Y]) = \text{Str}((-1)^p D_0 [\bar{\partial}, Y] + [\bar{\partial}, D_0]Y) = 0$ for $Y = K_{s_0} D_1 \cdots D_{p+1} K_{s_{p+1}}$:

$$\int_{t \Delta_p} d_s \text{Str}(D_0 K^N_{s_0} \cdots D_{p+1} K^N_{s_{p+1}}) = (-1)^{p+1} \int_{t \Delta_p} \text{Str}(D_0 [\bar{\partial}, K^N_{s_0} D_1 \cdots D_{p+1} K^N_{s_{p+1}}])$$

$$= \int_{t \Delta_p} \text{Str}([\bar{\partial}, D_0]K^N_{s_0} D_1 \cdots D_{p+1} K^N_{s_{p+1}}) = \Psi_{p+1}([\bar{\partial}, D_0], D_1, \ldots, D_{p+1})$$

□

This finally proves proposition 4.2.

4.3 Construction of the cocycle $\sigma$

We introduce our main technical tool, a cocycle in a double complex associated to an open set $U$. In the following two subsections we will first describe its properties, then explain the construction by heat kernel methods and finally apply it to prove that the Lefschetz number is proportional to the Hochschild-Čech trace (see section 3.4).

4.3.1 Properties of the $\sigma$-cocycle

Let $U$ be a sufficiently small open neighborhood of an arbitrary point in $X$. Let $A = D_E(U)$ and let $B = C^\infty(U)$ be the algebra of smooth complex valued functions on $U$. Consider further $C_p(A) = A \otimes \bar{A}^p$ with Hochschild differential $b$ of degree $-1$ and $C_p(B) = B \otimes \bar{B}^p$ with differential $s$ of degree $+1$ given by

$$s(\rho_0 \otimes \cdots \otimes \rho_p) = 1 \otimes \rho_0 \otimes \cdots \otimes \rho_p. $$

Let $C^c_p(B)$ be the subcomplex spanned by $(\rho_0, \ldots, \rho_p)$ with compact common support $\cap_{i} \text{Supp}(\rho_i)$. Remember that we denote by $[f(t)]_\infty = a_{-N} t^{-N} + \cdots + a_1 t^{-1} + a_0$ the singular part of an asymptotic Laurent series $f(t) \sim \sum_{j \geq -N} a_j t^j$.

**Proposition 4.11.** Let $U \subset X$ be an open set. Let $A = D_E(U)$, $B = C^\infty(U)$. There exist linear maps

$$\sigma_p : C_p(A) \otimes C^c_p(B) \to \mathbb{C}[t^{-1}]$$

such that the coefficients of $\sigma_p(D_0, \ldots, D_p, \rho_0, \ldots, \rho_p)$ are continuous in $(D_0, \ldots, D_p)$ and
(i) Let $C^p(B)$ is the subcomplex spanned by $(\rho_0, \ldots, \rho_p)$ with empty common support $\cap_{i=0}^p \text{supp}(\rho_i)$. Then $\sigma_p$ vanishes on $C_p(A) \otimes C^p(B)$.

(ii) For any $D \in C^p(A)$ and $\rho \in C^p(B)$,

$$\sigma_p(bD \otimes \rho) = \sigma_{p+1}(D \otimes s\rho), \quad p \geq 0,$$

(iii) $\sigma_0(D, \rho) = \left[ \sum_{j=0}^{2n} (-1)^j \text{tr}_{\Omega^0(U, E)}(D \rho e^{-t\Delta}) \right]_0$.

(iv) Suppose that $U$ is some coordinate neighborhood of a point and let $c_E(U)$ be the cocycle appearing in theorem 3.3. Assume further that $\rho_0, \ldots, \rho_{2n} \in C_c(U)$ are functions such that the metrics are flat $^1$ on some neighborhood of $\cap_{i=0}^{2n} \text{supp}(\rho_i)$. Then

$$\sigma_{2n}(c_E(U); \rho_0, \ldots, \rho_{2n}) = \frac{r}{(2\pi i)^n} \int_U \rho_0 \rho_1 \cdots \rho_{2n},$$

where $r$ is the rank of $E$.

4.3.2 Construction

Remember from section 2.1, paragraph ”Hochschild homology with values in modules”, that may consider a module which is itself a finite complex with differential $d_M$. Recall that we wrote $\delta = d_H + (-1)^p d_M: C^{p,q} \to C^{p+1,q} \oplus C^{p,q+1}$ for the total differential of this double complex.

Let now $\rho_0, \ldots, \rho_p \in C^\infty(U)$. View $C^\infty(U)$ as a subalgebra of $M = \Omega^0(\mathcal{U}) \otimes_{\mathcal{O}(\mathcal{U})} \mathcal{D}_E(U)$ embedded as $C^\infty(U) \otimes \text{id}$. Since $C^0(A, M) = M$, we may consider $\rho_i$ as a 0-cochain and define

$$Z^p(\rho_0, \ldots, \rho_p) = \rho_0 \cup \delta \rho_1 \cup \cdots \cup \delta \rho_p \in C^p(A, M),$$

where the cup product is defined using the product $M \otimes_A M \to M$. Clearly

$$\delta Z^p(\rho_0, \ldots, \rho_p) = Z^{p+1}(1, \rho_0, \ldots, \rho_p). \quad (4.2)$$

If $\cap \text{supp}(\rho_i)$ is compact, then $Z^p(\rho_0, \ldots, \rho_p)$ takes values in differential operators with compact support and therefore is a cochain in $C^p(A, M_c)$.

Let $\cup: C^\bullet(A, M^\ast_c) \otimes C^\bullet(A, M_c) \to C^\bullet(A, A^\ast)$ be the cup product associated with the map $M^\ast_c \otimes_A M_c \to A^\ast$ sending $\varphi \otimes x$ to the linear form $a \mapsto \varphi(xa)$. We set

$$\sigma_p(\rho_0, \ldots, \rho_p) = \Psi \cup Z^p(\rho_0, \ldots, \rho_p) \in C^p(A, A^\ast)[t^{-1}].$$

$^1$We can always choose our metric so that its curvature vanishes for small neighborhoods of discrete points.
4.3.3 Proof of Proposition 4.11

Claim (ii) follows from the fact that \( \Psi \) is a cocycle and equation (4.2). To prove the remaining claims let us write \( \sigma_p \) more explicitly:

\[
\sigma_p(D_0, \ldots, D_p; \rho_0, \ldots, \rho_p)
\]

\[
= \sum_{j=0}^{p} (-1)^{j(p-j)} \Psi_j(Z_{p-j}^p(D_{j+1}, \ldots, D_p; \rho_0, \ldots, \rho_p)D_0, D_1, \ldots, D_j).
\]

The component \( Z_{p-j}^p \) in \( \text{Hom}(\tilde{A}^{\otimes p-j}, M_i^j) \) of \( Z^p \) is given by

\[
Z_{p-j}^p(D_{j+1}, \ldots, D_p; \rho_0, \ldots, \rho_p) = \sum_{\pi \in S_{p-j}} \text{sgn}(\pi) \rho_{\pi(1)} \cdots B_{\pi(p)}(\rho_p),
\]

where \( B_i(\rho) = [D_{j+i}, \rho] \) for \( i = 1, \ldots, p-j \), and \( B_i(\rho) = [\partial, \rho] \) for \( i = p-j+1, \ldots, p \). From these expressions it is clear that (i) and (iii) hold. Let us turn to (iv). We need to evaluate \( \sigma_{2n}(\psi(U); \rho_0, \ldots, \rho_{2n}) \). By multiplying \( \rho_0 \) by a partition of unity we may assume that the support of \( \rho_0 \) is contained in a small coordinate neighborhood of a point. We have to compute a sum of \((2n)!\) terms of the form (4.3) where \( D_0 = 1 \) and the remaining \( D_k \) are partial derivatives \( \partial_{z_i} \) or operators of multiplication by \( z_i \). The arguments \( D_k \) occurring in \( Z_{2n-j}^{2n} \) appear in the combination \([D_k, \rho]\) which vanishes if \( D_k = z_i \). Therefore the only non-vanishing terms in the sum (4.3) have \( j \geq n \) and \( D_{j+1}, \ldots, D_{2n} \) are all derivatives \( \partial_{z_i} \). On the other hand, if \( j > n \) then \( Z_{2n-j}^{2n} \) vanishes since a product of more than \( n \) \((0,1)\)-forms is zero. Thus only the term with \( j = n \) survives and we have (setting \( \partial_i = \partial_{z_i} \))

\[
Z_{2n}^{2n}(\partial_1, \ldots, \partial_n; \rho_0, \ldots, \rho_{2n}) = \rho_0 \frac{\partial \rho_1}{\partial z_1} \cdots \frac{\partial \rho_n}{\partial z_n} \cdots \frac{\partial \rho_{2n}}{\partial z_n} + \cdots
\]

where the dots denote the remaining shuffles. Therefore

\[
\sigma_{2n}(\psi(U); \rho_0, \ldots, \rho_{2n}) = (-1)^{n(n+1)/2} \sum_{\pi \in S_n} \text{sgn}(\pi) \Psi_n(B, z_{\pi(1)} \cdots, z_{\pi(n)}).
\]

The sign comes from commuting the \( z_i \) and the \( \partial_j \). \( B \) is the multiplication operator

\[
B = \sum_{\pi \in S_{2n}} \text{sgn}(\pi) \rho_0 \frac{\partial \rho_{\pi(1)}}{\partial z_1} \cdots \frac{\partial \rho_{\pi(n)}}{\partial z_n} \frac{\partial \rho_{\pi(n+1)}}{\partial z_{n+1}} \cdots \frac{\partial \rho_{\pi(2n)}}{\partial z_{2n}} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.
\]

Note that since \( B \) is the operator of multiplication by a \((0,n)\)-form, the only trace appearing in the alternating sum defining \( \Psi_n \) is the trace over \( \Omega^{0,n} \) and it comes with a sign \((-1)^n\). Let us calculate \( \Psi_n(B, z_1, \ldots, z_n) \). The calculation for all other permutations is similar and gives the same contribution to the sum over \( S_n \).

\[
\Psi_n(B, z_1, \ldots, z_n) = (-1)^{n+n+n(n+1)/2} \int_{t\Delta_n} tr_{\Omega^{0,n}}(BK_{s_1}[\partial^* z_1]K_{s_1} \cdots [\partial^* z_n]K_{s_n})ds_1 \cdots ds_n.
\]
The operators $B, [\bar{\partial}^*, z_i]$ are differential operators of zeroth order. It follows from proposition A.6 that only the leading term as $t \to 0$ contributes to the calculation of the trace. Thus we can replace the heat kernel by the standard heat kernel on $\mathbb{C}^n$. In this case

$$\bar{\partial} = \sum d\bar{z}_i \frac{\partial}{\partial \bar{z}_i}, \quad \bar{\partial}^* = -\sum \frac{\partial}{\partial z_i} \iota_{\frac{d}{dt}}, \quad \Delta_B = -\sum_{j=1}^n \frac{\partial^2}{\partial z_j \partial \bar{z}_j},$$

where $\iota$ denotes interior multiplication. Thus $\Delta_B$ is $-4$ times the standard Laplacian and the heat kernel is

$$k_t(z, z') = \frac{1}{(\pi t)^n} e^{-\frac{|z - z'|^2}{t}}.$$

Now $[\bar{\partial}^*, z_i] = -\iota_{\partial/\partial z_i}$, which commutes with the heat kernel. The heat kernels combine to $k_{s_0} \cdots k_{s_n} = k_t$, since $\sum s_i = t$ on $t\Delta_n$. Let us write $B = b(z)d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$. Then we obtain

$$\Psi_n(B, z_1, \ldots, z_n) = \int_U b(z) \text{tr}_{\bar{\partial}^*} k_t(z, z) |dz| \int_{t\Delta_n} ds_1 \cdots ds_n$$

$$= \frac{r}{n! \pi^n} \int_U b(z) |dz|.$$

The standard volume form $|dz|$ is

$$|dz| = (-2i)^{-n} d\bar{z}_1 \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n$$

$$= (-2i)^{-n} (-1)^{n(n-1)/2} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

Thus $b(z) |dz| = (-2i)^{-n} (-1)^{n(n-1)/2} \rho_0 d\rho_1 \cdots d\rho_{2n}$. Inserting this in the formula (4.4) gives the formula that had to be proved.

### 4.3.4 A local formula for the Lefschetz number

We will now show, that the Lefschetz number and the Hochschild-Čech trace are proportional.

Here it is useful to consider the open cover $(U_\alpha)$ which is the open star of the triangulation of $X$ explained in section 3.7. Remind that the open star of the triangulation is the open cover $\mathcal{U} = (U_\alpha)_{\alpha \in K_0}$ of $X$ labeled by the set of vertices of the triangulation, such that $U_\alpha$ is the complement of the simplices not containing $\alpha$. By construction, for all $\alpha_0 < \cdots < \alpha_p$,

(a) $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is empty or contractible

(b) If $p > 2n$ then $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is empty.
4.3. CONSTRUCTION OF THE COCYCLE $\sigma$

For convenience, let us also fix a total ordering on the set of vertices of the triangulation.

**Lemma 4.12.** Let $(\rho_\alpha)$ be a partition of unity subordinate to the open covering $(U_\alpha)$. Let $D \in D_E(X)$ and $s \in C^{2n}(U, C_{2n}(D_E))$ be the cocycle (3.2) in section 3.4. Then

$$\sum_{p=0}^{2n} (-1)^p \text{tr}(H^p(D)) = \sum_{\alpha_0 < \cdots < \alpha_{2n}} \sigma_{2n}(s_{\alpha_0, \ldots, \alpha_{2n}}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_{2n}}).$$

Here we use the abbreviation

$$\rho_{\alpha_0, \ldots, \alpha_q} = \sum_{\pi \in S_{q+1}} \text{sgn}(\pi) \rho_{\alpha_{\pi(0)}} \otimes \cdots \otimes \rho_{\alpha_{\pi(q)}}.$$

**Proof.** Out of $D$ we construct the cochains $D^{(j)}$ obeying (3.4).

$$L(D) = \sum_{j=1}^{n} (-1)^j \left[ \text{tr}_{\Omega^{(0,j)}}(X,E)(D e^{-t\Delta}) \right]_-. $$

$$= \sum_{\alpha} \sum_{j=0}^{n} (-1)^j \left[ \text{tr}_{\Omega^{(0,j)}}(X,E)(\rho_\alpha D e^{-t\Delta}) \right]_-. $$

$$= \sum_{\alpha} \sigma_0(D; \rho_\alpha), \quad D_\alpha = D|_{U_\alpha} \in D_E(U_\alpha).$$

Now $D_\alpha = bD^{(1)}_\alpha$ and proposition 4.11 (ii) implies

$$L(D) = \sum_{\alpha} \sigma_1(D^{(1)}_\alpha; 1, \rho_\alpha)$$

$$= \sum_{\alpha, \beta} \sigma_1(D^{(1)}_\alpha; \rho_\beta, \rho_\alpha)$$

$$= \sum_{\alpha \neq \beta} \sigma_1(D^{(1)}_\alpha; \rho_\beta, \rho_\alpha) + \sum_{\beta} \sigma_1(D^{(1)}_\beta; \rho_\beta, \rho_\beta)$$

$$= \sum_{\alpha \neq \beta} \sigma_1(D^{(1)}_\alpha - D^{(1)}_\beta; \rho_\beta, \rho_\alpha).$$

In the last step we have replaced the last occurrence of $\rho_\beta$ by $-\sum_{\alpha \neq \beta} \rho_\alpha \mod \mathbb{C}1$. We see that $\tilde{\delta}D^{(1)}_{\beta,\alpha}$ appears. Thus we can iterate the procedure. At the $q$-th step we obtain similarly for $q < 2n$,

$$\sum_{\alpha_0 < \cdots < \alpha_q} \sigma_q(\tilde{\delta}D^{(q)}_{\alpha_0, \ldots, \alpha_q}; \rho_{\alpha_0, \ldots, \alpha_q}) = \sum_{\alpha_0 < \cdots < \alpha_q} \sigma_q(bD^{(q+1)}_{\alpha_0, \ldots, \alpha_q}; \rho_{\alpha_0, \ldots, \alpha_q})$$

$$= \sum_{\alpha_0 < \cdots < \alpha_q} \sigma_{q+1}(D^{(q+1)}_{\alpha_0, \ldots, \alpha_q}; 1 \otimes \rho_{\alpha_0, \ldots, \alpha_q})$$

$$= \sum_{\alpha_0 < \cdots < \alpha_{q+1}} \sigma_{q+1}(\tilde{\delta}D^{(q+1)}_{\alpha_0, \ldots, \alpha_{q+1}}; \rho_{\alpha_0, \ldots, \alpha_{q+1}}).$$
If \( q = 2n \) we have an additional term containing \( s \) and we obtain

\[
L(D) = \sum_{\alpha_0 < \cdots < \alpha_{2n}} \sigma_{2n}(s_{\alpha_0, \ldots, \alpha_{2n}}; \rho_{\alpha_0, \ldots, \alpha_{2n}})
+ \sum_{\alpha_0 < \cdots < \alpha_{2n+1}} \sigma_{2n+1}(\delta D^{(2n+1)}; \rho_{\alpha_0, \ldots, \alpha_{2n+1}}).
\]

Since there are no non-empty \((2n+2)-\)fold intersections, \((\rho_{\alpha_0, \ldots, \rho_{2n+1}})\) belongs to \( C^\sigma(B) \) and therefore, by proposition 4.11, (i), the second term vanishes.

Let us now choose the Hermitian metrics so that they are flat on the disjoint closed sets \( \cap_{j=0}^{2n} \text{supp}(\rho_{\alpha_i}), \alpha_0 < \cdots < \alpha_{2n} \). By proposition 4.11, (iv), we have

\[
\sum_{p=0}^{2n} (-1)^p \text{tr}(H^p(D)) = (2n + 1)! \frac{r^n}{(2\pi i)^n} \sum_{\alpha_0 < \cdots < \alpha_{2n}} \lambda_{\alpha_0, \ldots, \alpha_{2n}}(D) \int_X \rho_{\alpha_0} d\rho_{\alpha_1} \cdots d\rho_{\alpha_{2n}}.
\]

Now the common support of the functions \( \rho_{\alpha_i} \) in each summand is contained in a simplex \( \sigma_{\alpha_0, \ldots, \alpha_{2n}} \). Moreover each of the functions vanishes on the corresponding face and \( \sum_{i=0}^{2n} \rho_{\alpha_i} = 1 \) on some neighborhood of the simplex. Therefore the integral may be evaluated as follows.

**Lemma 4.13.** Let \( H_p \in \mathbb{R}^{p+1} \) be the hyperplane \( \sum_{i=0}^p t_i = 1 \) and \( \Delta_p = H_p \cap [0,1]^{p+1} \) the standard simplex, with (standard) orientation given by the ordered basis \( \partial t_1, \ldots, \partial t_p \). Let \( \rho_0, \ldots, \rho_p \) be smooth functions defined on some open neighborhood \( U \subset H_p \) of \( \Delta_p \) such that \( \rho_0 + \cdots + \rho_p = 1 \) and \( \rho_i(t) = 0 \) if \( t_i \leq 0 \). Then

\[
\int_{\Delta_p} \rho_0 d\rho_1 \cdots d\rho_p = \frac{1}{(p+1)!}.
\]

**Proof.** We prove by induction in \( p \) the more general formula

\[
\int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_p = \frac{k!}{(p+k)!}, \quad k = 0, 1, 2, \ldots
\]

This formula trivially holds for \( p = 0 \). By Stokes’ formula,

\[
\int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_p = - \int_{\Delta_p} \rho_0^k d\rho_1 \cdots d\rho_{p-1} d\rho_0
= (-1)^p \frac{1}{k+1} \int_{\Delta_p} d(\rho_0^{k+1} d\rho_1 \cdots d\rho_{p-1})
= (-1)^p \frac{1}{k+1} \int_{\partial\Delta_p} \rho_0^{k+1} d\rho_1 \cdots d\rho_{p-1}
\]

This is the desired result.

\[\square\]
Since $\rho_j$ vanishes on the $j$th face of $\Delta_p$, only the $p$th face (where $t_p = 0$) contributes. This face is $\Delta_{p-1}$ and the restriction of $\rho_0, \ldots, \rho_{p-1}$ obey the assumptions of the lemma. Taking into account the sign $(-1)^p$ relating the orientation of $\Delta_{p-1}$ to the induced orientation, we obtain
\[
\int_{\Delta_p} \rho_k^0 d\rho_1 \cdots d\rho_p = \frac{1}{k+1} \int_{\Delta_{p-1}} \rho_k^{k+1} d\rho_1 \cdots d\rho_{p-1},
\]
proving the induction step.

**Corollary 4.14.** Let $\epsilon(\alpha_0, \ldots, \alpha_{2n}) \in \{-1, 1\}$ be the orientation of the simplex $\sigma_{\alpha_0, \ldots, \alpha_{2n}}$ relative to the canonical orientation of $X$. Then
\[
L(D) = \frac{r}{(2\pi i)^n} \sum_{\alpha_0 < \cdots < \alpha_{2n}} \lambda_{\alpha_0, \ldots, \alpha_{2n}}(D) \epsilon(\alpha_0, \ldots, \alpha_{2n})
\]
Appendix A

Additions

A.1 Heat kernel estimates

In this section we prove the proposition 4.6 which completes the proof of proposition 4.2 for our JLO-type cocycle Ψ. In the first subsection, it is shown that the integrand in the definition of the JLO-cocyle is smooth for \( s \in [0, 1]^{p+1} \setminus \{0\} \). In the second subsection, we apply this result to compute the asymptotic expansion. In particular it will follow from this computation that \( \Psi_p \) is well defined and continuous in the operators \( D_0, \ldots, D_p \).

A.1.1 Heat kernel approximation

In this subsection, we show some estimates for the approximated heat kernel and use them to prove that the integrand \( f(s) \) in the formula for \( \Psi_p \) (see formula (4.1)) is smooth for \( s \in [0, 1]^{p+1} \setminus \{0\} \).

We recall from [11] the notions of a generalized Laplacian and the corresponding heat kernel. A generalized Laplacian \( H \) acting on sections of a vector bundle \( \mathcal{E} \) over a \( d \)-dimensional Riemannian manifold \( (X, g) \) is a second-order differential operator, which in local coordinates can be written as \( H = -\sum_{i,j=1}^{d} g^{ij} \partial_i \partial_j + \text{first order terms} \). It is easy to verify that \( \Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} \) is 4 times such a Laplacian if we set \( \mathcal{E} = E \otimes \Lambda^* T^{(0,1)} X \). Therefore we may directly use the results about the heat kernel from [11] considering \( X \) as a smooth \( 2n \)-dimensional Riemannian manifold.

We write \( \mathfrak{D}_\mathcal{E}(X) \) for the space of smooth differential operators acting on smooth sections \( \Gamma(X, \mathcal{E}) = \bigoplus \Omega^{0,j}(X) \). \( \Gamma(X, \mathcal{E}) \) is a locally convex space where the norms are the \( C^k \)-norms. These norms can be constructed by choosing a finite open cover of coordinate
neighborhoods of $X$. We then consider a cover of $X$ of compact sets that are slightly smaller than the previous open sets. The $C^k$-norms are then defined by the sum of the $C^k$-norms on the compact sets and with respect to the corresponding coordinates. Furthermore we can assume that the $C^k$-norms on $\Gamma(X, E)$ are increasing, i.e. $\|\phi\|_k \leq \|\phi\|_\ell$ for $k \leq \ell$.

The spaces $\mathcal{D}^j_{\mathbb{E}}(X) \subset \mathcal{D}_{\mathbb{E}}(X)$ of differential operators of order $j$ are spaces of sections of a certain hermitian vector bundle over $X$, and so one can define increasing $C^k$-norms on them in a similar way as above. Then $\mathcal{D}_{\mathbb{E}}(X)$ is an LF-space which is the strict inductive limit of $\mathcal{D}^j_{\mathbb{E}}(X)$, see, e.g., [37].

For two vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over the manifold $X$, we denote by $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ the exterior product which is a vector bundle over $X \times X$. The heat kernel $k_t(x,y)$ is a family of sections $k_t \in \Gamma(X \times X, \mathcal{E} \boxtimes \mathcal{E}^*)$ defined for $t > 0$ which is $C^1$ with respect to $t$ and $C^2$ with respect to $x$ and solves the equation

$$\partial_t k_t(x,y) + \Delta \bar{\partial} k_t(x,y) = 0$$

with initial condition $\lim_{t \to 0} k_t(x,y) = \delta(x-y)$ where $\delta$ is the Dirac distribution with respect to the Riemannian density on $X$. The heat kernel exists and is unique. There is an approximation to the heat kernel $k^N_t(x,y)$ of the form

$$k^N_t(x,y) = (\pi t)^{-n} e^{-d(x,y)^2/4t} \sum_{i=0}^N t^i \psi_i(x,y)$$

where $d(x,y)$ is the geodesic distance and $\psi_i(x,y)$ are linear maps $\mathcal{E}_y \to \mathcal{E}_x$ depending smoothly on $(x,y)$ and with support in the set where $d(x,y) \leq \varepsilon$ for some fixed $\varepsilon$ which can be chosen arbitrarily small. Furthermore the leading term $\psi_0(x,y)$ is the parallel transport from $\mathcal{E}_y$ to $\mathcal{E}_x$ along the unique geodesic from $y$ to $x$ and with respect to a connection that is associated to our (generalized) Laplacian (for details concerning the connection see [11], proposition 2.5). The maps $\psi_i, i > 0$ can be chosen so that the following theorem holds:

**Theorem A.1.** Let here $\| \cdot \|_\ell$ be $C^\ell$-norms for sections in the bundle $\mathcal{E} \boxtimes \mathcal{E}^*$.

(i) $k^N_t$ approximates the heat kernel $k_t$ in the sense that

$$\|\partial^m_t (k_t - k^N_t)\|_\ell = O(t^{N-n-\ell/2-m}) \quad \text{for } t \to 0.$$

(ii) $k^N_t$ is an approximate solution of the heat equation such that the remainder $r^N_t(x,y) := (\partial_t + \Delta \bar{\partial})k^N_t(x,y)$ satisfies the estimates

$$\|\partial^k_t r^N_t\|_\ell < C t^{N-n/2-k-\ell/2}$$

for some constant $C$ depending on $\ell$ and $k$. 
Proof. See [11], theorem 2.23 or 2.30 for part (i) and theorem 2.20 for part (ii). □

We define the operator $K_t$ on smooth sections $\varphi \in \Gamma(X, E)$ by

$$(K_t \varphi)(x) = \int_X k_t(x, y)\varphi(y)|dy|_g$$

(A.1)

where $|dy|_g$ is the Riemannian density on $X$. $\varphi_t := K_t \varphi$ is a solution of the heat equation $\partial_t \varphi_t + \Delta_\varphi \varphi_t = 0$ with initial condition $\lim_{t \to 0} \varphi_t = \varphi$. In the same way, we also define the operators $K^N_t$ that correspond to the approximated heat kernel $k^N_t$.

The operator $K_t$ satisfies the following estimates:

**Lemma A.2.** We write $\|\cdot\|_\ell$, $\ell \geq 0$ for the $C^\ell$-norms on $\Gamma(X, E)$ resp. $\mathcal{D}_E(X)$. Fix an $\delta > 0$ small enough and assume $s, s' \in [\delta, 1]$ and $t \in [0, 1]$. Then for each $\ell$ and each of the following inequalities there is a constant $C$ so that

$$(i) \quad \|K^N_s \varphi - \varphi\|_\ell \leq C \|\varphi\|_{\ell+1} \sqrt{t}$$

$$(ii) \quad \|K^N_s \varphi - K^N_t \varphi\|_\ell \leq C \|\varphi\|_0 |s - s'|$$

$$(iii) \quad \|K^N_s \varphi\|_\ell \leq C \|\varphi\|_0$$

$$(iv) \quad \|D K^N_0 \varphi\|_\ell = \|D \varphi\|_\ell \leq C \|D\|_\ell \|\varphi\|_{\ell+d}$$

for every differential operator $D \in \mathcal{D}_E(X)$ of degree $d$.

Proof. (i) The proof is essentially the same as for the first part of theorem 2.29 in [11]. We consider the formula (A.1) for $K^N_t \varphi$, change to exponential coordinates for $y (y \mapsto \exp_x y)$ and write $\varphi(x, y) := \varphi(\exp_x y)$ and $\psi_j(x, y) = \psi_j(x, \exp_x y)$, in the latter case with a slight abuse of the notation. We may assume that $\varepsilon$ in the definition of $k^N_t$ is smaller than the injectivity radius of the exponential map, so that the previous change to exponential coordinates in well defined. The substitution $y = \sqrt{t}v$ leads to

$$(K^N_t \varphi - \varphi)(x) = \frac{1}{\pi^n} \int_{T_x X} e^{-\|v\|^2} \left( \sum_{j=0}^N t^j \psi_j(x, \sqrt{t}v)\varphi(x, \sqrt{t}v)\rho(x, \sqrt{t}v) - \varphi(x, 0) \right) dv$$

where we used $\frac{1}{\pi^n} \int_{T_x X} e^{-\|v\|^2} = 1$, and $\rho(x, y) := \sqrt{\det(\exp_x y)}$ is the factor coming from the Riemannian density. As $\psi_j(x, y) = 0$ for $\|y\| > \varepsilon$, it is a compactly supported function on $TX$. For $j > 0$, it is therefore clear – by taking the supremum over $y$ – that $t^j \psi_j(x, y)\varphi(x, y)\rho(x, y)$ is bounded by a constant times $\sqrt{t}\|\varphi\|_0$. For $j = 0$, we write $f(x, y) = \psi_0(x, y)\varphi(x, y)\rho(x, y)$. As $f(x, 0) = \varphi(x, 0)$, we get by the mean value theorem for the $t^0$-term

$$\frac{1}{\pi^n} \int_{T_x X} ve^{-\|v\|^2} \partial_y f(x, \sqrt{t}v)\sqrt{td}v$$

for
for some $t' \in [0, t]$. This expression is bounded by a constant times $\sqrt{t} \| \varphi \|_1$ and the claim follows in the case $\ell = 0$. For $\ell > 0$, we use the same arguments, but the function $f(x, y)$ is replaced by $\partial_x^\ell f(x, y)$ where $|\alpha| \leq \ell$.

(ii) $K_s^N$ is an integral operator with kernel depending $C^1$ on $s$ for $s > 0$. Therefore the mean value theorem tells us that

$$|\partial_x^\ell K_s^N(x, y) - \partial_x^\ell K_s^{N'}(x, y)| \leq \sup_{s \in [\delta, 1]} |\partial_x^\ell K_s^N(x, y)| |s - s'|$$

from which the claim follows.

(iii) Is obvious as the kernel is smooth in $x$ for all $s \in [\delta, 1]$.

(iv) Also obvious.

By iterating the above lemma, we find the following estimate:

**Lemma A.3.** Let $D_i \in \mathcal{D}_\varepsilon(X)$ be differential operators of degree $d_i$, $i = 1, \ldots, m$. Fix a $\delta > 0$ and a set $I \subset \{1, \ldots, m\}$. Let $s_i \in [0, 1], s'_i = 0$ for $i \in I$ and $s_i, s'_i \in [\delta, 1]$ for $i \notin I$. Then there is a constant $C$ and an $L \leq \ell + m + \sum_{i=1}^m d_i$ so that

$$\|D_1 K_{s_1}^N D_2 K_{s_2}^N \cdots D_m K_{s_m}^N \varphi - D_1 K_{s'_1}^N D_2 K_{s'_2}^N \cdots D_m K_{s'_m}^N \varphi\|_\ell \leq C \|\varphi\|_L \left( \sum_{i \in I} \sqrt{s_i} + \sum_{i \notin I} |s_i - s'_i| \right) \prod_{j=1}^k \|D_j\|_L$$

**Proof.** Using the triangle inequality and lemma A.2, we find

$$\|DK_s^N \varphi_1 - DK_{s'}^N \varphi_2\|_\ell \leq \|(DK_s^N - DK_{s'}^N) \varphi_1\|_\ell + \|DK_{s'}^N(\varphi_1 - \varphi_2)\|_\ell$$

$$\begin{cases}
\forall s' \leq C \sqrt{s} \|\varphi_1\|_{\ell + d} \|D\|_\ell + C \|\varphi_1 - \varphi_2\|_{\ell + d} \|D\|_\ell \\
\forall s' > \delta \leq C |s - s'| \|D\|_\ell \|\varphi_1\|_0 + C \|\varphi_1 - \varphi_2\|_0 \|D\|_\ell.
\end{cases}$$

The proof is straightforward by induction on $m$. 

**Lemma A.4.** The function $f(s)$, which is the integrand in the definition of $\Psi_p$, is continuous for $s \in [0, 1]^{p+1} \setminus \{0\}$. In particular the integral over $t \Delta_p$ in the definition of $\Psi_p$ (see proposition 4.2) is well defined for $t \in (0, 1]$.

**Proof.** An operator $D \in \mathcal{D}_\varepsilon(X)$ with continuous kernel $D(x, y) \in \Gamma(X \times X, \mathcal{E} \boxtimes \mathcal{E}^*)$ is of trace class, and the supertrace can be written as

$$\text{Str}(D) = \sum_{k=0}^{n} (-1)^k \text{Tr}_{\Omega^0, k(X, E)}(D)$$

$$= \sum_{k=0}^{n} (-1)^k \int_X \text{tr}_{E_x \otimes \Lambda^0, k(T_x X)} D(x, x) dx|_g.$$
A.1. HEAT KERNEL ESTIMATES

For the integral over $t \Delta_p$ in the definition of $\Psi_p$ to be convergent, it is sufficient to show that the function $f(s) := \text{Str}(D_0 K^N_{s_0} [\bar{\partial}^* , D_1] K^N_{s_1} \cdots [\bar{\partial}^*, D_{p}] K^N_{s_p})$ is continuous in $s$ for $s \in t \Delta_p$. As the heat kernel $k^N_{s_0}$ is $C^1$ with respect to $s_i$ for $s_i > 0$, this is clear except for points on the boundary of $t \Delta_p$. For a point $s' \in t \partial \Delta_p$, let $I$ be the subset of $\{1, \ldots, n\}$ so that $s'_i = 0 \Leftrightarrow i \in I$ and take a $\delta > 0$ so that $s'_i > \delta \forall i \notin I$. As there is at least one $i \notin I$ and as the trace is cyclic, we can w.l.o.g. assume that $p \not\in I$. To simplify the notation, we set $A_{s_0 \ldots s_{p-1}} = D_0 K^N_{s_0} [\bar{\partial}^*, D_1] K^N_{s_1} \cdots [\bar{\partial}^* , D_{p-1}] K^N_{s_{p-1}}$ and $B_s = [\bar{\partial}^* , D_{p}] K^N_{s_p}$. We write the supertrace as

$$\text{Str}(D_0 K^N_{s_0} \cdots [\bar{\partial}^*, D_{p}] K^N_{s_p}) = \sum_{k=0}^{p} (-1)^k \int_{X \times X} \langle \psi_k \mid A_{s_0 \ldots s_{p-1}}(x,y) B_s(y,x) \mid \psi_k \rangle \, dx \, dy$$

where $\{\psi_k\}$ for fixed $k$ and $i = 1 \ldots i_k$ is a basis for $E \otimes \Lambda^0 \Lambda \left( T_x X \right)$. Now we consider $A_{s_0 \ldots s_{p-1}}$ as operator acting on $\varphi_{s_p} := B_s(\cdot, x)v$ where $x \in X$ and $v \in E_x$ are considered as parameter. Then we get by the triangle inequality and lemma A.3 that

$$\|A_{s_0 \ldots s_{p-1}} \varphi_{s_p} - A_{s_0' \ldots s_{p-1}'} \varphi_{s_p'}\| \leq \| (A_{s_0 \ldots s_{p-1}} - A_{s_0' \ldots s_{p-1}'}) \varphi_{s_p}\| + \| A_{s_0' \ldots s_{p-1}'} (\varphi_{s_p} - \varphi_{s_p'})\| \leq \bar{C} \left( \sum_{i \in I \setminus \{p\}} \sqrt{s_i} + \sum_{i \not\in I \setminus \{p\}} \|s'_i - s_i\| \right) \|\varphi_{s_p}\|_L + \bar{C} \|\varphi_{s_p} - \varphi_{s_p'}\|_L \|D_0\|_L \prod_{j=1}^{p-1} \|D_j\|_L$$

where $\bar{C} = C \|D_0\|_L \prod_{j=1}^{p-1} \|[\bar{\partial}^*, D_j]\|_L$. We use the mean value theorem and find

$$\|\varphi_s - \varphi_{s'}\|_L \leq \|s - s'\| \sup_{(x,v) \in E, \|v\| \leq 1} \|B_v(\cdot, x)v\|_{L+1} \leq C \|s - s'\| \|D_{p}\|_{L+1}.$$ 

As the integral of the trace goes over a compact set, we have shown that $f$ is continuous in $s$ for $s \in t \partial \Delta$ and

$$|f(s) - f(s')| \leq C \left( \sum_{i \in I} \sqrt{s_i} + \sum_{i \not\in I} \|s'_i - s_i\| \right) \prod_{j=0}^{p} \|D_j\|_{L+2}$$  \hspace{1cm} (A.2)

Proposition A.5. The function $f(s)$ (see lemma A.4) is $k$-times continuously differentiable for $s \in [0, 1]^{p+1} \setminus \{0\}$ and $N = N_k$ large enough.

Proof. The proof works in exactly the same way as in the previous lemmata (A.2, A.3, A.4). We generalize the estimates in lemma A.2 by adding time derivatives: Fix an $\delta > 0$ and
assume \( s, s' \in [\delta, 1] \) and \( t \in [0, 1] \). Then for each \( \ell, m \) and each of the following inequalities, there is a constant \( C \) so that

\[
(i) \quad \| \partial_t^m K_t^N \varphi - (-\Delta_\theta)^m \varphi \|_\ell \leq C \| \varphi \|_{2m+\ell+1} \sqrt{t}
\]

\[
(ii) \quad \| \partial_s^m K_s^N \varphi - \partial_s^m K_s^N \varphi \|_\ell \leq C \| \varphi \|_0 |s - s'|
\]

\[
(iii) \quad \| \partial_s^m K_s^N \varphi \|_\ell \leq C \| \varphi \|_0
\]

\[
(iv) \quad \|(-\Delta_\theta)^m DK_t^N \varphi \|_\ell = \|(-\Delta_\theta)^m D \varphi \|_\ell \leq C \| \varphi \|_{\ell+2m}
\]

where \( \| \cdot \|_\ell, \ell \geq 0 \) are \( C^\ell \)-norms on \( \Gamma(X, \mathcal{E}) \), \( \mathcal{D}_\mathcal{E}(X) \) respectively. We only prove the first estimate as the others are easy to show (see lemma A.2). Recall from theorem A.1 that the remainder \( r_t^N = (\partial_t + \Delta_\theta)k_t^N \) satisfies \( \| \partial_t^k r_t^N \|_\ell \leq Ct^{N-k-(n+\ell)/2} \). By the iterated application of \( \partial_t k_t^N = -\Delta_\theta k_t^N + r_t^N \), we find

\[
\partial_t^m k_t^N = (-\Delta_\theta)^m k_t^N + \sum_{j=0}^{m-1} (-\Delta_\theta)^{m-1-j} \partial_t^j r_t^N
\]

and hence the estimate

\[
\| \partial_t^m K_t^N \varphi - (-\Delta_\theta)^m K_t^N \varphi \|_\ell \leq \sum_{j=0}^{m-1} \| \Delta_\theta^{m-1-j} \partial_t^j r_t^N \varphi \|_\ell
\]

\[
\leq \sum_{j=0}^{m-1} \| \Delta_\theta^{m-1-j} \|_\ell \| \partial_t^j r_t^N \|_{\ell+2(m-1-j)} \| \varphi \|_0 \leq C \| \varphi \|_0 t^{N-m+1-(n+\ell)/2}.
\]

We require \( N \) to be large enough, namely \( N \geq \frac{n+\ell-1}{2} + m \). On the other hand we have the estimate

\[
\|(-\Delta_\theta)^m K_t^N \varphi - (-\Delta_\theta)^m \varphi \|_\ell \leq C \| \Delta_\theta^m \|_\ell \| K_t^N \varphi - \varphi \|_{\ell+2m} \leq C \| \varphi \|_{2m+\ell+1} \sqrt{t}
\]

The estimate (i) then follows by the triangle inequality.

Using the above estimates, it is now straightforward to generalize lemma A.3 to

\[
\| D_t \partial_{s_1}^{m_1} K_{s_1}^{N_1} D_s \partial_{s_2}^{m_2} K_{s_2}^{N_2} \cdots D_s \partial_{s_p}^{m_p} K_{s_p}^{N_p} \varphi - D_t \partial_{s_1}^{m_1} K_{s_1}^{N_1} D_s \partial_{s_2}^{m_2} K_{s_2}^{N_2} \cdots D_s \partial_{s_p}^{m_p} K_{s_p}^{N_p} \varphi \|_\ell \leq C \| \varphi \|_L \left( \sum_{i \in I} \sqrt{s_i} + \sum_{i \in I} |s_i - s'_i| \right)
\]

which is true for some \( L \leq \ell + \sum_i (d_i + 2m_i) + 1 \). Then we see as in lemma A.4 that the partial derivatives of \( f(s) \) up to degree \( k \) are continuous.
A.1.2 Computation of $\Psi_p$ and power counting

In this subsection we explain an algorithm to compute $\Psi_p$ which will lead to the result summarized in the following proposition:

**Proposition A.6.** Let $n$ be the complex dimension of $X$. Take the operators $D_0, D_1, \ldots, D_p$ as in proposition 4.2. We write $d$ for the sum of the degrees of the differential operators $D_0, [\bar{\partial}^* D_1], \ldots, [\bar{\partial}^* D_p]$ which are defined on a small (see remark below) open set $U \subset X$. Recall that the approximated heat kernel depends on the constants $N$ and $\varepsilon$. Then for $N$ big enough and $\varepsilon$ small enough, $\Psi_p(D_0, \ldots, D_p)$ is well defined and a polynomial in $t^{-1}$ of degree $n - p + \lfloor \frac{d}{2} \rfloor$. More precisely, for $N \geq n - p + \lfloor \frac{d}{2} \rfloor$ and $\varepsilon < \frac{1}{p+1} \text{dist}(X \setminus U, \text{supp}(D_0))$, where $\text{dist}$ means the geodesic distance, it is independent of $N$ and $\varepsilon > 0$. Furthermore, $\Psi_p(D_0, \ldots, D_p)$ depends continuously on $D_0, D_1, \ldots, D_p$.

**Remark A.7.** The set $U$ in the above proposition has to be small in the sense that lemma A.9 holds for any compact subset $K \subset U$. For larger $U$ the above proposition would still be true with the exception that the upper bound for $\varepsilon$ would need a more careful definition.

The main idea of the computation is to "commute" the operators $[\bar{\partial}^*, D_i]$ in the formula (4.1) for $\Psi_p$ to the left and to use a saddle point approximation for the heat kernel integrals. As a preparation for this computation, we formulate the following lemmata:

**Lemma A.8.** Let $U \subset X$ be a (small) open subset of $X$ so that the exponential map w.r.t. any point in $U$ and restricted to the preimage of $U$ is a diffeomorphism. Assume that $x_1, x_2 \in U$, then (in local coordinates) there is a smooth matrix valued function $a(x_1, x_2)$ so that
\[
\frac{\partial}{\partial x_2} d(x_1, x_2)^2 = a(x_1, x_2) \frac{\partial}{\partial x_1} d(x_1, x_2)^2.
\]

**Proof.** We construct such a map explicitly: We introduce the coordinates $(x, \xi) = (x_1, \log x_2^{x_1})$ and $(y, \eta) = (x_2, \log x_2^{x_1})$. Obviously $|\xi| = d(x_1, x_2) = |\eta|$. Therefore we find
\[
\frac{\partial}{\partial x_2} d(x_1, x_2)^2 = \frac{\partial \xi}{\partial x_2} \frac{\partial |\xi|^2}{\partial \xi} = \frac{\partial \xi}{\partial x_2} \frac{\partial |\xi|^2}{\partial \eta} = \frac{\partial \eta}{\partial x_2} \frac{\partial |\xi|^2}{\partial \eta} \frac{\partial |\xi|^2}{\partial \xi} d(x_1, x_2)^2.
\]

As the exponential coordinates are smooth coordinates, the lemma follows. 

**Lemma A.9.** Let $K \subset X$ be a (small) compact subset so that the exponential map w.r.t. any point in $K$ and restricted to the preimage of $K$ is injective. Take $x_1, x_2, x_3 \in K$ and $s_1, s_2 \in (0, 1]$, then for fixed $x_1, x_3, s_1, s_2$ the function
\[
f(x_2) = \frac{d(x_1, x_2)^2}{s_1} + \frac{d(x_2, x_3)^2}{s_2}
\]
has a unique minimum in the point $\bar{x}$ that lies on the geodesic through $x_1$ and $x_3$ and satisfies $d(x_1, \bar{x})/s_1 = d(x_3, \bar{x})/s_2$. We choose exponential coordinates $\xi = \log_{\bar{x}} x_2$ and expand $f$ in the point $\bar{x}$. This leads to the following expressions for $f$:

$$f(x_2) = \frac{d(x_1, x_2)^2}{s_1 + s_2} + \left( \frac{1}{s_1} + \frac{1}{s_2} \right) G_{ij}(s_1, s_2, x_1, x_2(\xi), x_3)\xi^i \xi^j$$

$$= \frac{d(x_1, x_3)^2}{s_1 + s_2} + \left( \frac{1}{s_1} + \frac{1}{s_2} \right) G_{ij}(s_1, s_2, x_1, \bar{x}, x_3)\xi^i \xi^j + G_{ijk}(s_1, s_2, x_1, x_2(\xi), x_3)\xi^i \xi^j \xi^k$$

for smooth functions $G_{ij}$ and $G_{ijk}$. The matrix $G_{ij}(s_1, s_2, x_1, x_2, x_3)$ defined and bounded on $([0, 1]^2 \setminus \{0\}) \times K^3$ is positive definite and there is a constant $C > 0$ so that the smallest eigenvalue of the matrix is greater than $C$ for all $x_1, x_2, x_3 \in K$ and $s_1, s_2 \in [0, 1]$. Furthermore $G_{ij}$ is homogeneous of degree 0 in $s_1, s_2$ and we have $G_{ij}(s_1, s_2, x_1, \bar{x}, x_3) \to \delta_{ij}$ for $|x_1 - x_3| \to 0$.

**Proof.** If $x_2$ is not on the geodesic between $x_1$ and $x_3$, it is easy to see that there is always a point on the geodesic for which one term of $f$ has the same value and the other one is smaller. For $x_2$ on the geodesic we have $d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3)$ from which $d(x_1, \bar{x})/s_1 = d(x_3, \bar{x})/s_2$ follows. The critical point $\bar{x}$ of the smooth function $s_1 s_2 f(x_2)$ is a smooth function of $x_1, x_3, s_1, s_2$, homogeneous of degree zero in $s_1, s_2$, as long as the Hessian is nondegenerate, which is the case if $K$ is small enough and $s_1/(s_1 + s_2) \in (-\varepsilon_1, 1 + \varepsilon_1)$ for some $\varepsilon_1 > 0$. The expansion of $f$ is just the Taylor expansion (with remainder) in the point $x_2 = \bar{x}$. This gives

$$G_{ij}(s_1, s_2, x_1, x_2(\xi), x_3) = \frac{1}{s_1 + s_2} \int_0^1 (1 - u) \frac{\partial^2}{\partial \eta^i \partial \eta^j} (s_2 d(x_1, \exp_{\bar{x}}(\eta))^2 + s_1 d(\exp_{\bar{x}}(\eta), x_3)^2)|_{\eta = u} du.$$

From this expression we see that $G_{ij}$ is homogeneous in $s$ and is smooth for $s_1/(s_1 + s_2) \in (-\varepsilon_1, 1 + \varepsilon_1)$, $x - 1, x_3 \in K$. In particular it is a bounded continuous function on $([0, 1]^2 \setminus \{0\}) \times K^2$.

For a Euclidean metric it is an application of the law of cosines to show that $G_{ij} = \delta_{ij}$. By rescaling $x_i \mapsto \lambda x_i$ and taking into account that $d(\lambda x_i, \lambda x_j)/\lambda \to |x_i - x_j|$ for $\lambda \to 0$, we see that $G_{ij}(s_1, s_2, x_1, x_2, x_3) \to \delta_{ij}$ if $|x_1 - x_2| + |x_2 - x_3| \to 0$. Therefore also $G_{ij}(s_1, s_2, x_1, \bar{x}, x_3) \to \delta_{ij}$ for $|x_1 - x_3| \to 0$. As $K$ is small, we are still close to the Euclidean case and therefore $G_{ij} - \delta_{ij}$ is small, from which the existence of $C$ follows.

**Lemma A.10.** *(Asymptotic expansion under the integral)* We write $|f(t)|$, for the asymptotic expansion of the function $f$ in the variable $t$ in $t = 0$. In the following cases we are allowed to interchange the asymptotic expansion and the integration:
(i) Let \( f : [0, 1]^{p+1} \sim \{0\} \to \mathbb{C} \) be a smooth function and assume that there is an \( n \in \mathbb{N} \) so that \( F(s, t) := t^n f(st) \) can be continued to a function in \( C^\infty(\Delta_p \times [0, 1]) \). Then

\[
\left[ \int_{\Delta_p} f(st) ds \right]_t = \int_{\Delta_p} [f(st)]_t ds.
\]

(ii) Let \( K, G_{ij}, G_{ijk}, x_1, x_2(\xi) \) be as in Lemma A.9. We abbreviate \( G_{ij} := G_{ij}(s_1, s_2, x_1, x_3), G_{ij}(\xi) := G_{ij}(s_1, s_2, x_1, x_2(\xi), x_3) \) and \( G_{ijk}(\xi) := G_{ijk}(s_1, s_2, x_1, x_2(\xi), x_3) \). Let \( H : \mathbb{R}^{2n} \to \mathbb{C} \) be a smooth function with support in a small neighborhood of the origin. Then

\[
\left[ \int_{\mathbb{R}^{2n}} H(\sqrt{t}\xi)e^{-aG_{ij}(\sqrt{t}\xi)\xi^i\xi^j} d\xi \right]_{\sqrt{t}} = \int_{\mathbb{R}^{2n}} [H(\sqrt{t}\xi)e^{-aG_{ijk}(\sqrt{t}\xi)\xi^i\xi^j\xi^k}]_{\sqrt{t}} e^{-aG_{ij}\xi^i\xi^j} d\xi
\]

where \( a \) is any positive constant.

**Proof.** (i) As \( [f(st)]_t = t^{-n}[F(st)]_t \), it suffices to show that we can interchange the integral and the asymptotic expansion for \( F \). Because \( F \) is smooth, its asymptotic expansion is given by the Taylor series and we have to show that in the following expression the limit and the integral are interchangeable:

\[
\lim_{t \to 0} \int_{\Delta_p} F(s, t) - \sum_{k=0}^\ell \frac{\partial^k F(s, t)}{t^{\ell+1}} ds.
\]

This is true because the integrand is dominated by \( \sup_{t \in [0, 1]} |\partial_{t}^{\ell+1} F(s, t)|/(\ell + 1)! \).

(ii) As in part (i), we consider the remainder of the Taylor expansion:

\[
\frac{(\partial/\partial \sqrt{t})^m}{m!} H(\sqrt{t}\xi)e^{-aG_{ij}(\sqrt{t}\xi)\xi^i\xi^j} = \sum_{|\alpha|=m} \xi^{\alpha}_{\alpha} \frac{\partial^{\alpha}}{\alpha!} H(\eta)e^{-aG_{ij}(\eta)\xi^i\xi^j} \bigg|_{\eta=\sqrt{t}\xi} e^{-aG_{ij}\xi^i\xi^j}.
\]

As \( H \) has compact support, we can estimate this by

\[
\|H(\eta)G_{ijk}(\eta)\eta^i\|_m (1 + \|\xi\|^2)^m \xi^{\alpha}_{\alpha} e^{-aG_{ij}\xi^i\xi^j} > C\|\xi\|^2,
\]

According to lemma A.9, there is a constant \( C \), so that

\[
G_{ij}\xi^i\xi^j + G_{ijk}(\sqrt{t}\xi)\xi^i\xi^j\xi^k \sqrt{t} = G_{ij}(\sqrt{t}\xi)\xi^i\xi^j > C\|\xi\|^2,
\]

for all \( \xi \) such that \( \sqrt{t}\xi \) is in the support of \( H \). Thus it follows again by the dominated convergence theorem that the asymptotic expansion and the integral commute. \( \square \)

---

\(^1\)By “\( C^\infty \) on a closed set” we mean that every derivative exists in the interior and extends continuously to the boundary.
Proof of the proposition. We write Greek letters for multi-indices in $\mathbb{N}^{2n}_0$ and Latin letters for indices in $\mathbb{N}_0$.

We consider again the function $f(s) := \text{Str}(D_0K^N_{s_0}[(\bar{\partial})^*, D_1]K^N_{s_1} \cdots [\bar{\partial}^*, D_p]K^N_{s_p})$. As we are going to show, the asymptotic expansion of $f(st)$ w.r.t. $t$ in $t = 0$ exists, has lowest order $-n - \lfloor \frac{p}{2} \rfloor$ and the coefficients are smooth functions of $s \in \Delta_p$. Therefore the function $F(s,t) := t^{p + \frac{p}{2}} f(st)$ is smooth\(^2\) and we can apply lemma A.10 (i):

$$
\Psi_p(D_0, \ldots, D_p) = (-1)^{p(p+1)/2} \int_{\Delta_p} [t^p f(st)] - ds.
$$

To compute the asymptotic expansion of $f(st)$, we consider the kernel

$$(D_0K^N_{s_0}[(\bar{\partial})^*, D_1]K^N_{s_1} \cdots [\bar{\partial}^*, D_p]K^N_{s_p})(x_0, x_{p+1}).$$

Recall that $D_0$ has compact support $K \subset U \subset X$ where $U$ is an open set (see also proposition 4.2). As $K^N_{s_1}(x_i, x_{i+1})$ vanishes for $d(x_i, x_{i+1}) > \varepsilon$, there is an $\varepsilon > 0$ so that $(p+1)\varepsilon$ is smaller than the geodesic distance between $K$ and $X \setminus U$. Then in the above kernel only the values of terms inside a compact subset $K_{\varepsilon}$ of $U$ play a role and therefore it is well defined. We assume that $K_{\varepsilon}$ is small enough to apply lemma A.9.

We want to "move" the operators $[\bar{\partial}^*, D_1]$ to the left. First just consider a term $K^N_{s_1}D_0K^N_{s_2}$. We may assume that $D$ in local coordinates has the form $\rho(x)\bar{\partial}^\alpha$ where $\text{supp} \rho \subset K_{\varepsilon}$. Explicitly, the above term is given by the integral

$$
\int_{X} \sum_{0 \leq i, j \leq N} s_1^i s_2^j \psi_i(x_1, x_2) e^{-d(x_1, x_2)^2/s_1} \rho(x_2) \bar{\partial}^\alpha \psi_j(x_2, x_3) e^{-d(x_2, x_3)^2/s_2} \psi_{x_2} \psi_{x_3} \left( \frac{e^{-d(x_2, x_3)^2/s_2}}{(\pi s_2)^n} \right) |dx_2|_g.
$$

We write $|dx_2|_g = \sigma(x_2)dx_2$ and integrate by parts to bring the $\bar{\partial}_{x_2}$-operator to the left. Then we make repeatedly use of lemma A.8 to "replace" the $x_2$-derivatives by $x_1$-derivatives, i.e. we use an identity of the form

$$
\bar{\partial}_{x_2}^\alpha e^{-d(x_1, x_2)^2/s_1} = \sum_{\beta + \gamma = \alpha} h_{\beta, \gamma}(x_1, x_2) \bar{\partial}_{x_1}^\gamma e^{-d(x_1, x_2)^2/s_1},
$$

which holds for some smooth functions $h_{\beta, \gamma}$. Writing down again the integral, we find an expression of the form

$$
\int_{X} \sum_{0 \leq i, j \leq N} \sum_{\alpha \leq \alpha} s_1^i s_2^j H_{i, j, \alpha'}(x_1, x_2, x_3) \bar{\partial}_{x_1}^\alpha e^{-d(x_1, x_2)^2/s_1 - d(x_2, x_3)^2/s_2} \left( \frac{1}{(\pi s_1)^n (\pi s_2)^n} \right) dx_2
$$

\(^2\)From proposition A.5 follows that $F$ is smooth for $(s, t) \in \Delta_p \times (0, 1]$. The existence of the asymptotic expansion shows that its derivatives can be continued to $t = 0$. Hence $F \in C^\infty(\Delta_p \times [0, 1])$.  

where $H_{i,j,o'}$ are smooth functions. If we apply the above procedure to shift all derivatives in the expression $D_0K_{s_0}^N \ldots [\tilde{D}_p, D_p]K_{s_p}^N$ to the left, we get

$$\int_{X^p} \sum_{|\gamma| \leq N} \sum_{|\alpha| \leq d} s^3 H_{\gamma,\alpha}(x_0, \ldots, x_p) \partial_{x_0}^{\sum_{j=0}^{p} d(x_j, x_{j+1})^2/s_j} e^{-\sum_{j=0}^{p} d(x_j, x_{j+1})^2/s_j} (\pi s_0)^n \ldots (\pi s_p)^n dx_1 \ldots dx_p. \quad (A.3)$$

We omitted the terms for which $|\gamma| := \sum_{j=1}^{p-1} \gamma_j > N$, but we will see later that they would only produce (irrelevant) terms of higher order in $t$. We rewrite the exponent in the above expression using lemma A.9 repeatedly:

$$\sum_{j=0}^{p} \frac{d(x_j, x_{j+1})^2}{s_j} \frac{d(x_0, x_{p+1})^2}{s_0 + \cdots + s_p} + \sum_{\ell=1}^{p} \left( \frac{1}{s_0 + \cdots + s_{\ell-1}} + \frac{1}{s_{\ell}} \right) G_{ij}(s_0 + \cdots + s_{\ell-1}, s_{\ell}, x_0, x_{\ell}, x_{\ell+1}) \xi_{ij} \xi_{\ell}$$

where $\xi_{\ell} = \ln_{x_\ell} x_\ell$, $\bar{x}_{\ell} = \bar{x}_\ell(x_{\ell-1}, x_{\ell+1})$. Now we change to the variables $\xi^i$ in the integral and rescale $\xi^i \mapsto \sqrt{t} \xi^i$ as well as $s_i \mapsto ts_i$ so that $(s_0, \ldots, s_p) \in \Delta_p$. We temporarily forget the last term in the exponent and suppress the arguments of $G_{ij}$:

$$t^{p-n} \int_{(T_0 X)^p} \sum_{|\gamma| \leq N} \sum_{|\alpha| \leq d} s^3 H_{\gamma,\alpha}(x_0, \ldots, x_{p+1}) \partial_{x_0}^{\sum_{\ell=1}^{p} \left( \frac{1}{s_0 + \cdots + s_{\ell-1}} + \frac{1}{s_{\ell}} \right) G_{ij}(s_0 + \cdots + s_{\ell-1}, s_{\ell}, x_0, x_{\ell}, x_{\ell+1}) \xi_{ij} \xi_{\ell}} e^{-\sum_{\ell=1}^{p} \left( \frac{1}{s_0 + \cdots + s_{\ell-1}} + \frac{1}{s_{\ell}} \right) G_{ij}(s_0 + \cdots + s_{\ell-1}, s_{\ell}, x_0, x_{\ell}, x_{\ell+1}) \xi_{ij} \xi_{\ell}} (\pi s_0)^n \ldots (\pi s_p)^n d\xi_1 \ldots d\xi_p \quad (A.4)$$

where the Jacobi determinant has been absorbed in $H_{\gamma,\alpha}$. Due to lemma A.10 ii) we are allowed to expand asymptotically w.r.t. $\sqrt{t}$ under the integral. Keep in mind that $x_\ell = \exp_{x_\ell}(\sqrt{t} \xi_{\ell})$ so that the arguments of $H_{\gamma,\alpha}$ as well as of $G_{ij}$ depend on $\sqrt{t}$. In the expansion of the exponent, there will be singular terms in $s$, namely powers of the factor

$$\frac{1}{s_0 + \cdots + s_{\ell-1}} + \frac{1}{s_{\ell}} = \frac{s_0 + \cdots + s_{\ell}}{(s_0 + \cdots + s_{\ell-1})s_{\ell}},$$

but as these factors only appear paired with $\xi_{ij} \xi_{\ell}$, the singularities cancel as we see in the following computation. After the expansion we have to compute integrals of the form

$$\int_{T_0 X} \xi_{\ell}^\beta e^{-\sum_{\ell=1}^{p} \left( \frac{1}{s_0 + \cdots + s_{\ell-1}} + \frac{1}{s_{\ell}} \right) G_{ij}(s_0 + \cdots + s_{\ell-1}, s_{\ell}, x_0, x_{\ell}, x_{\ell+1}) \xi_{ij} \xi_{\ell}} d\xi_{\ell} = C_{\beta}(s, x) \left( \frac{(s_0 + \cdots + s_{\ell-1})s_{\ell}}{(s_0 + \cdots + s_{\ell})} \right)^{|\beta|+n}$$

where $C_{\beta}(s, x)$ is a smooth function, homogeneous of degree 0 in $s$, vanishing unless $|\beta| = \sum \beta_i$ is even. Terms with $|\beta|$ even correspond to even terms in the asymptotic expansion in powers of $\sqrt{t}$. Therefore we actually have an asymptotic series in $t$. 

\section{A.1. HEAT KERNEL ESTIMATES}

where $H_{i,j,o'}$ are smooth functions.
We repeat the above steps for \( \xi_2, \ldots, \xi_p \). As
\[
\prod_{\ell=1}^{p} \frac{(s_0 + s_1 + \cdots + s_{\ell-1})s_\ell}{s_0 + s_1 + \cdots + s_\ell} = \frac{s_0s_1 \cdots s_p}{s_0 + s_1 + \cdots + s_p},
\]
the singularities from the denominator in equation (A.4) disappear, and we get
\[
(D_0K_{t,s_0}^N \cdots D_pK_{t,s_p}^N)(x_0, x_{p+1}) = t^{p-n} \sum_{|\alpha| \leq d} \sum_{k=0}^{N} t^k f_k(s, x_0, x_{p+1}) \partial_x^\alpha e^{-d(x_0, x_{p+1})^2} + O(t^{p-n+N+1})
\]
for smooth functions \( f_k : \Delta_p \times K \times K_\varepsilon \rightarrow \mathbb{C} \). Remember that
\[
f(s, t) = \int_K (D_0K_{t,s_0}^N \cdots D_pK_{t,s_p}^N)(x_0, x_0) dx_0.
\]
The integral over \( x_0 \in K \) and the asymptotic expansion commute for the same reason as in lemma A.10. We see in the above formula that the negative powers in \( t \) are only produced by the derivative \( \partial_x^\alpha \). As \( \lim_{x_p \to x_0} \partial_x^\alpha d(x_0, x_p)^2 = 0 \) for \( |\alpha| = 1 \), we need at least two derivatives to get a negative power in \( t \). Thus the negative power is at most \( \lfloor |\alpha|/2 \rfloor \).

In formula (A.3), the coefficient functions of the operators \( D_0, D_1, \ldots, D_p \) have been absorbed in the function \( H_{\gamma,\alpha} \). It is easy to check that they enter linearly and with derivatives of order at most \( d \), which is the sum of the degrees of the differential operators, in this function. After formula (A.4) when we do the expansion, we get an additional derivative for every order of \( \sqrt{t} \). Therefore the coefficients of \( \Psi_p \) only depend on finitely many derivatives of the operators \( D_0, D_1, \ldots, D_p \) restricted to the compact set \( K_\varepsilon \) that was mentioned in the beginning of the proof. This means that we can estimate \( \Psi_p \) by a product of \( C^k \)-norms over the compact set \( K_\varepsilon \) of the operators \( D_i \). As the operators \( D_1, \ldots, D_p \) are holomorphic and actually defined on an open set containing \( K_\varepsilon \), we can use the Cauchy integral formula to estimate their \( C^k \)-norms by the sup-norms over a compact set that is slightly bigger than \( K_\varepsilon \). This shows that \( \Psi_p \) is continuous in the operators \( D_0, \ldots, D_p \).

**Remark A.11.** If we replace in the formula for \( \Psi_p \) one of the approximated heat kernels \( k^N \) by the exact heat kernel \( k \), the \( dx \)-integral still only goes over a compact set. Thus we can choose \( \varepsilon \) small enough so that the formula for \( \Psi_p \) is still well defined. From the above proof it is also clear that this procedure doesn’t change the value of \( \Psi_p \).

### A.2 Triangulations and signs

We give some informations about the signs used in section 3.7.
Let $T$ be a smooth finite triangulation of the oriented $d$-dimensional manifold $X$. Let $\sigma_{\alpha_0, \ldots, \alpha_p} \subset X$ denote the simplex with vertices $\alpha_0, \ldots, \alpha_p$. It is the image of the standard oriented simplex $\Delta_p = \{ t \in [0, 1]^{p+1} \mid \sum t_i = 1 \}$ sending the $i$-th vertex with $t_i = 1$ to $\alpha_i$ and thus comes with an orientation, for which

$$\partial \sigma_{\alpha_0, \ldots, \alpha_p} = \sum_{j=0}^{p} (-1)^j \sigma_{\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_p}. \quad (A.5)$$

The cells of the dual cell decomposition $T^*$ of $X$ (see [25]) are in one-to-one correspondence with the simplices of the triangulation. The $(d-p)$-cell $C_{\alpha_0, \ldots, \alpha_p}$ intersects only the $p$-simplex $\sigma_{\alpha_0, \ldots, \alpha_p}$ and meets it transversally in exactly one interior point. Let us orient the cells by the condition that the intersection index is one:

$$C_{\alpha_0, \ldots, \alpha_p} \cdot \sigma_{\alpha_0, \ldots, \alpha_p} = 1 \quad (A.6)$$

This means in particular that the top-dimensional cells $C_a$ have the same orientation as $X$. With this convention both $C_{\alpha_0, \ldots, \alpha_p}$ and $\sigma_{\alpha_0, \ldots, \alpha_p}$ change their orientation under permutation of the indices according to the sign of the permutation.

If $c_p$ is a $p$-cell of $T^*$ and $c'_{d-p+1}$ is a $(d-p+1)$-cell of $T$, we have

$$\partial c_p \cdot c'_{d-p+1} = (-1)^p c_p \cdot \partial c'_{d-p+1}.$$  

By combining this equation with (A.5) and (A.6) we obtain the formula for the boundary of dual cells:

$$\partial C_{\alpha_0, \ldots, \alpha_p} = (-1)^{d+p} \sum_{\beta} C_{\beta, \alpha_0, \ldots, \alpha_p}, \quad (A.7)$$

with summation over all $\beta$ such that $\beta, \alpha_0, \ldots, \alpha_p$ are the vertices of a simplex of the triangulation.

### A.3 Add-on to section 2.8 (Weil algebra and Weyl algebra)

Recall from section 2.8 the shortcuts $A = A_{2n|m}^{pol}$, $g = gl_N(A) = A_{2n|m}^{N, pol}$, $\mathfrak{h} = sp_{2n}(\mathbb{C}) \times so_{2n}(\mathbb{C}) \times gl_N(\mathbb{C}) \subset g$. We introduce the Lie subalgebra $W_{n,m,N} \subset g$ with generators $f(q)p_i \otimes 1$, $g(\theta)\zeta_i \otimes 1$, $h(q) \otimes M$, where $f$, $g$ and $h$ are polynomials and $M \in gl_N(\mathbb{C})$. From now on we omit "\otimes 1" and "$\otimes M" in expressions like the above generators when it doesn’t lead to any confusion. We define the Lie subalgebra $\mathfrak{h}_1 := \mathfrak{h} \cap W_{n,m,N}$. Recall from section 2.8 that we write $\chi : (S\mathfrak{h})^{*\mathfrak{h}} \rightarrow HC(g, \mathfrak{h})$ for the Chern-Weil map. We also
write $\chi$ for its restriction to $\mathfrak{h}_1$. Furthermore we write $i$ for the obvious restriction maps in the below diagram. We show in this section that the map $\chi \circ i = i \circ \chi$ in the following commutative diagram is bijective:

$$
\begin{array}{ccc}
S^n(\mathfrak{h})^* & \xrightarrow{i} & S^n(\mathfrak{h}_1)^* \\
\downarrow{\chi} & & \downarrow{\chi} \\
HC^{2n}(\mathfrak{g}, \mathfrak{h}) & \xrightarrow{i} & HC^{2n}(W_{n,m,N}, \mathfrak{h}_1)
\end{array}
$$

We have already shown that the Chern-Weil map on the left hand side is bijective. As an invariant polynomial is given by its restriction to the Cartan subalgebra and as $\mathfrak{h}_1$ contains the Cartan subalgebra of $\mathfrak{h}$, the restriction $i : S^n(\mathfrak{h})^* \rightarrow S^n(\mathfrak{h}_1)^*$ is injective. It remains to show that the Chern-Weil map on the right hand side is an isomorphism.

**Lemma A.12.** $CC^n(W_{n,m,N}, \mathfrak{h}_1)$ vanishes for odd $1 \leq q \leq 2n - 1$ and the Chern-Weil map $\chi : (S^q\mathfrak{h}_1)^* \rightarrow H^{2q}(W_{n,m,N}, \mathfrak{h}_1)$ is an isomorphism.

**Proof.** This can be proven using invariant theory (see [36], chapter 2, section 2). $W_{n,m,N}$ is linearly generated by elements of the form $f_1(q) \otimes 1$, $f_2(\theta) \zeta \otimes 1$ and $f_3(q) \otimes M$ where $f_{1,2,3}$ are polynomials and $M \in gl_N(\mathbb{C})$. As we consider the $gl_n(\mathbb{C}) \times gl_m(\mathbb{C}) \times gl_N(\mathbb{C})$-basic part, the quadratic terms of $f_1$ and $f_2$ must vanish as well as the constant part of $f_3$. Invariant theory tells us that every $q (\zeta)$ has to be paired with a $q (\eta)$ and we have to take traces over the matrices $M$. We can translate this into the language of graphs. We represent every $q (\theta)$ by an incoming arrow, every $p (\zeta)$ by an outgoing arrow and the matrices $M$ by an incoming and an outgoing arrow. To distinguish the different terms, the $p$-$q$-edges are drawn in plain style, the $\theta$-$\zeta$-edges in wiggly style and the $M$-edges are dashed. The generators of $W_{n,m,N}/gl_{n,m,N}$ are then represented by the graphs

$$
\begin{array}{c}
\begin{array}{c}
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\end{array}
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\begin{array}{c}
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\bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array}
\end{array}
\end{array}
$$

...
A basis element in $\mathcal{C}C^*(W_{n,m,N}, h_1)$ can be represented as a graph consisting of the above element. We denote the number of the above elements appearing in a graph by $p_0, p_2, p_3, \ldots, q_0, q_2, q_3, \ldots, r_1, r_2, r_3, \ldots$, where $p, q, r$ correspond to the above three types of elements and the index indicates the number of incoming edges. As incoming and outgoing edges have to be paired, the number of incoming edges has to be equal to the number of outgoing edges. This leads to the equations

\[
\begin{align*}
p_0 &= p_2 + 2p_3 + 3p_4 + \cdots + r_2 + 2r_3 + 3r_4 + \ldots \\
q_0 &= q_2 + 2q_3 + 3q_4 + \ldots
\end{align*}
\]

We easily see that the elements of type $p_0$ can only be paired with the elements of type $p_i, i \geq 2$ and $r_i, i \geq 2$. As the elements in $\mathcal{C}C^*(W_{n,m,N}, h_1)$ are antisymmetric with respect to interchanging terms of type $p_0$ but the terms of type $p_i, i \geq 2$ and $r_i, i \geq 1$ are symmetric with respect to the incoming edges, we can connect at most one $p_0$ to one of these elements. With the same reasoning but interchanging the words symmetric and antisymmetric, we see that at most one element of type $q_0$ can be connected to a element of type $q_i, i \geq 2$. This leads to

\[
\begin{align*}
p_0 &\leq p_2 + p_3 + p_4 + \cdots + r_2 + r_3 + r_4 + \ldots \\
q_0 &\leq q_2 + q_3 + q_4 + \ldots
\end{align*}
\]

We easily conclude that $p_0 = p_2 + r_2, q_0 = q_2$ and $p_i = q_i = r_i = 0$ for $i \geq 3$. Therefore only the following graph elements are allowed:

Thus the terms in $\mathcal{C}C^*(W_{n,m,N}, h_1)$ are combinations of hedgehogs of the following types:
It particularly follows that only even cochains are possible as every hedgehog has an even number of vertices. We are going to show that the Chern-Weil map produces exactly the same type of diagrams. We recall that $pr : W_{n,m,N} \to \mathfrak{h}_1$ is the (orthogonal) projection, $C(v \wedge w) = [pr(v), pr(w)] - pr([v, w])$ the corresponding curvature and the Chern-Weil map $\chi : (S^q\mathfrak{h}_1)^* \to H^{2q}(W_{n,m,N}, \mathfrak{h}_1)$ can then be written as

$$\chi(P)(v_1, \ldots, v_{2q}) = \frac{1}{q!} \sum_{\sigma \in S_{2q}/S_2^q} \text{sgn}(\sigma) P(C(v_{\sigma(1)}, v_{\sigma(12)}), \ldots, C(v_{\sigma(2q-1)}, v_{\sigma(2q)})).$$  \hspace{1cm} (A.8)

where the sum goes over all permutations with $\sigma(2i-1) < \sigma(2i)$. The sign $\text{sgn}(\sigma)$ includes the Koszul sign taking into account the permutations of graded terms. It is easy to see that only the following terms give contributions in $\chi$:

$$C(p_i \otimes 1, f(q)p_j \otimes 1) = \frac{\partial f}{\partial q_i} p_j,$$

$$C(\zeta_i \otimes 1, f(\eta)\zeta_j \otimes 1) = -\frac{\partial f}{\partial \eta_i} \zeta_j,$$

$$C(p_i \otimes 1, q_j^i \otimes M) = \delta^i_j \otimes M,$$

where $f$ are quadratic polynomials. We therefore see that the terms that don’t vanish when inserted in the curvature $C$ correspond to the following graphs:

As $P$ is an $gl_{n,m,N}$-invariant polynomial, we have to build graphs of the above subgraphs where all arrows are paired. Is easy to see that we get the same hedgehogs as above and therefore the Chern-Weil map is an isomorphism.
A.4 Combinatorial proofs

A.4.1 A chain map from Hochschild homology to de Rham cohomology

Recall from section 2.5 that the Hochschild cocycle $\tau$ on the Weyl algebra has been used to construct maps

$$
\chi_k : \Gamma(M, W)^{(k+1)} \to \Omega^{2n-k}(M, \mathbb{C}[[\varepsilon]])
$$

$$
\chi_k(f_0, \ldots, f_k) = \tau_{2n}(\langle f_0, \ldots, f_k \rangle \times (A)_{2n-k}),
$$

and we applied the construction with the shift by a Maurer-Cartan element from section 2.2 to show the homology morphism property for $\chi$, that is

$$
d \circ \chi_{k+1} = (-1)^k \chi_k \circ b.
$$

We give here a second proof of this property by straightforward combinatorial arguments. Take $k \in \{1, \ldots, 2n\}$. Suppose that $f_0, \ldots, f_k \in \Gamma_D(M, W)$ are flat sections, that is $\nabla f + [A, f] = 0$. We write $a_0 = f_0, \ldots, a_k = f_k, a_{k+1} = \cdots = a_{2n} = A$ and define

$$
\chi_k(f_0, \ldots, f_k) = \sum_{\sigma \in S_{2n,k}} \text{sgn}(\sigma) \tau_{2n}(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(2n)}),
$$

where $S_{2n,k} \subset S_{2n}$ are the shuffles with $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(2n)$. Then the following theorem holds:

**Theorem A.13.** For $k = 0, \ldots, 2n - 1$

$$
d \chi_{k+1}(f_0, \ldots, f_{k+1}) = (-1)^k \delta \chi_k(f_0, \ldots, f_{k+1})
$$

where $d$ is the de Rham differential and $\delta$ the Hochschild differential.

**Proof.** We will prove that

$$
d \chi_{k+1} + (-1)^k \sum_{\sigma \in S_{2n+1,k+1}} \text{sgn}(\sigma) \delta \tau_{2n}(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(2n+1)})
$$

$$
= (-1)^k \delta \chi_k(f_0, \ldots, f_{k+1})
$$

which proves the claim as $\delta \tau_{2n} = 0$. In the second term we set $a_0 = f_0, \ldots, a_{k+1} = f_{k+1}, a_{k+2} = \cdots = a_{2n+1} = A$. As a shortcut we write $T_1 + (-1)^k T_2 = (-1)^k T_3$ for the above identity. First, we rewrite $T_1$. As $\tau_{2n}$ is $\text{sp}(2n)$-invariant, we can replace the
So far we have shown that all summands in the Hochschild differential of the terms $a_i$ by $\nabla a_i$ and use the identities $\nabla f = -[A, f]$ and $\nabla A = -A^2 + \Omega$. 

We can forget about $\Omega$ as it lies in the center of the algebra. Therefore we get

$$T_1 = - \sum_{\rho \in S_{2n,k+1}} \sum_{l=0}^{2n} \text{sgn}(\rho)(-1)^{B(\rho,l)} \tau_{2n}(a_0, a_{\rho(1)}, \ldots, a'_{\rho(l)}, \ldots, a_{\rho(2n)})$$

where $a'_j = [A, a_j]$ for $j \leq k + 1$ and $a'_j = A^2$ for $j \geq k + 2$, and $(-1)^{B(\rho,l)}$ is the sign coming from commuting the one forms $A$ (resp. $\nabla$) with the $A$’s in $a_{\rho(1)}, \ldots, a_{\rho(l-1)}$. More explicitly we can write $B(\rho, l) = \max\{\rho(1), \ldots, \rho(l-1), k+1\} - k - 1$ which is the total degree of the terms $a_0, \ldots, a_{\rho(l-1)}$. We get similar looking sums by writing $T_2$ and $T_3$ more explicitly:

$$T_2 = \sum_{\sigma \in S_{2n+1,k+1}} \sum_{j=0}^{2n+1} (-1)^j \text{sgn}(\sigma) \tau_{2n}(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(j)}a_{\sigma(j+1)}, \ldots, a_{\sigma(2n+1)})$$

$$T_3 = \sum_{\rho \in S_{2n,k}} \sum_{l=0}^{k+1} (-1)^l \text{sgn}(\rho) \tau_{2n}(a_0, a_{\rho(1)}, \ldots, a_{\rho(l')}a_{\rho(l'+1)}, \ldots, a_{\rho(2n+1)})$$

The relationship between $\rho, l$ and $\sigma, l'$ in the above formula for $T_3$ will be explained in the last paragraph. I cheated a bit by not writing the last term in the Hochschild differential explicitly: coming from commuting the one forms $A$.

Furthermore we can see that $T_2$ contains all possible summands of this type, all together $\binom{2n+1}{k+1}(2n+2)$ terms. $T_1$ consists of all terms where $a_{\sigma(j)} = A$ or $a_{\sigma(j+1)} = A$, all together $\binom{2n+1}{k+1}(2n+k+3)$ terms. And $T_3$ of all terms where $a_{\sigma(j)} \neq A$ and $a_{\sigma(j+1)} \neq A$, $\binom{2n}{k}(k+2)$ terms. So the only thing that remains to show is that every term appears with two different signs.

To make the notation more compact we introduce the following definition of a permutation:

$$\pi_{m}(s) = \begin{cases} 
  s & s < m \\
  2n + 1 & s = m \\
  s - 1 & s > m 
\end{cases}$$

and we remark that $\text{sgn}(\pi_m) = (-1)^{2n+1-m}$. Furthermore if $\rho \in S_{2n}$ we write $\tilde{\rho}$ for the permutation in $S_{2n+1}$ with $\tilde{\rho}|_{1..2n} = \rho$ and $\tilde{\rho}(2n+1) = 2n+1$. Obviously $\text{sgn}(\tilde{\rho}) = \text{sgn}(\rho)$. 

APPENDIX A. ADDITIONS
A.5. THE COCYCLE $\sigma$

Now, we start by comparing the terms in $T_1$ to the corresponding terms in $T_2$. A term in $T_1$ is determined by the pair $(\rho, l)$. On the other hand the corresponding term in $T_2$ is determined by the pair $(\sigma, j)$. To see the relative signs of these terms, we write down explicitly the map $(\rho, l) \mapsto (\sigma, j)$.

1. If $\rho(l) \geq k + 2$. In this case we have $B(\rho, l) = \rho(l) - k - 2$ and we find that $(\sigma, j) = (\pi_{\rho(l)+1} \circ \tilde{\rho} \circ \pi_{l+1}, l)$. Collecting all signs, we get the relative sign $(-1)^{k+1}$. So the terms cancel.

2. If $\rho(l) \leq k + 1$ there are two terms: (a) $A a_{\rho(l)}$ and (b) $-a_{\rho(l)} A$.
   
   (a) We find $(\sigma, j) = ((\pi_{\max\{\rho(1), \rho(l), k+1\}})^{-1} \circ \tilde{\rho} \circ \pi_l, l)$. The relative sign is again $(-1)^{k+1}$.
   
   (b) We find $(\sigma, j) = ((\pi_{\max\{\rho(1), \rho(l), k+1\}})^{-1} \circ \tilde{\rho} \circ \pi_{l+1}, l)$. The relative sign is again $(-1)^{k+1}$.

Finally, we have to compare the terms in $T_3$ to their corresponding terms in $T_2$. In the above formula for $T_3$ we set $l' = \rho^{-1}(l)$ for $l = 1 \ldots k$ and $l' = 0$ for $l = 0$, and we define $\sigma_\rho = (\pi_{l+1})^{-1} \circ \rho \circ \pi_0$. With $j = l'$ we can directly compare the summands in $T_3$ to $T_2$ and get the relative sign 1. Therefore we have shown that $T_1 + (-1)^k T_2 = (-1)^k T_3$. \boxed{}

A.5 The cocycle $\sigma$

In section 4.3, a cocycle $\sigma$ has been constructed as a cup product of the cocycles $\Psi$ and $Z$ and has then been used to carry out a ”climbing the staircase” argument. In this section, we basically do the same computation but only use the properties of the cocycle $\Psi$. The rest of the computation is done ”by hand”, that is without using any tools from algebra.

For $p \in \mathbb{N}$ and $j \in \{0, \ldots, p\}$ we define

$$Z^p_j(D_1, \ldots, D_k; \rho_0, \ldots, \rho_p) = (-1)^{jp} \rho_0 \text{Sh}_{k,j}(\text{ad}(D_1), \ldots, \text{ad}(D_k))(\rho_1, \ldots, \rho_p)$$

where $k = p - j$ and $\rho_i, i = 0 \ldots p$ are smooth functions on $M$. $\text{Sh}(\ldots)$ is a shortcut for the following expression:

$$\mu \circ \sum_{\text{sh} \in S_{k,j}} \text{sgn}(\text{sh}) \text{sh}(D_1, \ldots, D_k; \underbrace{\partial, \ldots, \partial}_{j \text{ factors}})(\rho_1, \ldots, \rho_p)$$

\footnote{For the "special" case $l = k + 1$ it is $\sigma_\rho = (\pi_{k+1})^{-1} \circ \rho \circ \pi_0$.}
we consider two following terms and rewrite them as follows:

Now, we write ad(\sum on the right is zero if sh(ad(\sum = id the term (\rho
convention we write the left hand side of the above equation a s
Proof. To reduce the redundancy of the notation, we try to omit every part of the terms that is not essential for understanding the calculations. Particularly, we usually suppress the term (\rho_2, \ldots, \rho_{p+1}) and also the global sign \((-1)^{ip}\) in the definition of Z_j^p. With this
convention we write the left hand side of the above equation as

\begin{align*}
(-1)^i & D_{j+1} Z_j^p(D_{j+2}, \ldots, D_{p+1}; \rho_1, \ldots, \rho_{p+1}) + (-1)^{i+1} Z_j^p(b(D_{j+1}, \ldots, D_{p+1}); \rho_1, \ldots, \rho_{p+1}) \\
& + (-1)^{p+1} Z_j^p(D_{j+1}, \ldots, D_p; \rho_1, \ldots, \rho_{p+1}) D_{p+1} = [\bar{\partial}, Z_{j-1}^p(D_{j+1}, \ldots, D_{p+1}; \rho_1, \ldots, \rho_{p+1})] \\
& + Z_{j+1}^p(D_{j+1}, \ldots, D_{p+1})(1, \rho_1, \ldots, \rho_{p+1}),
\end{align*}

where

\[\bar{b}(D_0, \ldots, D_k) := \sum_{j=0}^{k-1} (-1)^j (D_0, \ldots, D_j D_{j+1}, \ldots, D_k)\]
denotes the differential of the bar-complex.

Proof. To reduce the redundancy of the notation, we try to omit every part of the terms that is not essential for understanding the calculations. Particularly, we usually suppress the term (\rho_2, \ldots, \rho_{p+1}) and also the global sign (-1)^{ip} in the definition of Z_j^p. With this
convention we write the left hand side of the above equation as

\begin{align*}
(-1)^i [D_{j+1}, \rho_1] & \text{Sh}(\text{ad}(D_{j+2}), \ldots, \text{ad}(D_{p+1})) + (-1)^i \rho_1 D_{j+1} \text{Sh}(\text{ad}(D_{j+2}), \ldots, \text{ad}(D_{p+1})) \\
& + (-1)^{i+1} \rho_1 \text{Sh}(\text{ad}(D_{j+1} D_{j+2}), \ldots, \text{ad}(D_{p+1})) + \ldots \\
& + (-1)^{p+1} \rho_1 \text{Sh}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))) + \ldots \\
& + (-1)^p \rho_1 \text{Sh}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_p) D_{p+1}) D_{p+1}
\end{align*}

Now, we write ad(D_i D_{i+1}) = l(D_i) ad(D_{i+1}) + ad(D_i) r(D_{i+1}) in the above terms. Then we consider two following terms and rewrite them as follows:

\begin{align*}
A_k &= (-1)^{k-1} \left( \text{Sh}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{k-1}) r(D_k), \ldots, \text{ad}(D_{p+1})) \\
& - \text{Sh}(\text{ad}(D_{j+1}), \ldots, l(D_k) \text{ad}(D_{k+1}), \ldots, \text{ad}(D_{p+1}))) \right) \\
& = (-1)^{k-1} \mu \circ \sum_{sh \in S_{p-j}} \text{sgn}(sh) \sum_{sh(k-j-1) < i < sh(k-j)} \text{ad}(D_k)_i \text{sh}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_k), \ldots, \text{ad}(D_{p+1}))
\end{align*}

ad(D_k)_i means that ad(D_k) acts on the i-th factor in the tensor product. Note that the sum on the right is zero if sh(k - j - 1) + 1 = sh(k - j). In particular for j = 0 only sh = id is possible and therefore A_k vanishes for every k. Now, we interchange \(\bar{\partial}\) and \(D_{j+1}\) and get

\begin{align*}
& = (-1)^{k-1} \mu \circ \sum_{sh \in S_{p-j}} \text{sgn}(sh) \sum_{sh(k-j-1) < i < sh(k-j)} (-1)^{i+k+j} \text{ad}(\bar{\partial})_i \text{sh}_i(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))
\end{align*}
A.5. \textsc{The cocycle }$\sigma$

The shuffle $sh_i \in S_{p-j+1,j-1}$ is defined by:

\[
sh_i(1) = sh(1), \ldots, sh_i(k-j-1) = sh(k-j-1), sh_i(k-j) = i
\]

\[
sh_i(k-j+1) = sh(k-j), \ldots, sh_i(p+1-j) = sh(p-j).
\]

The sign in front of $\text{ad}(\bar{\partial})_i$ comes from commuting $\bar{\partial}$ through $(i-1)-(k-j-1)$ other $\bar{\partial}$'s. Now we calculate the relative sign of $sh \in S_{p-j,j}$ and $sh_i \in S_{p-j+1,j-1}$. We visualize $sh$ as follows:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

On the other hand $sh_i$ is visualized as

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

where the light gray square is the new element $(k-j+1)$ that is mapped to the position $i$. To pass from $sh$ to $sh_i$, we have to apply the following cycle:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

On the right of the position $i$ this cycle maps black squares to black squares and on the left of it white to white ones. Counting the squares gives

\[
n_{br} = \#(\text{black squares on the right of } i) = (p+1-j) - (k-j)
\]

\[
n_{wl} = \#(\text{white squares on the left of } i) = (i-1) - (k-j-1)
\]

The relative sign of the shuffles is given by the sign of the cycle which is $(-1)^{n_{br}+n_{wl}} = (-1)^{p+j+1}$. Now, we are ready to write $A_k$ as sum over the $sh \in S_{p-k,j+1}$:

\[
A_k = (-1)^{p} \mu \circ \sum_{sh \in S_{p-j+1,j-1}} \text{sgn}(sh) \text{ad}(\bar{\partial})_{sh(k-j)} \text{sh(}D_{j+1}, \cdots, \text{ad}(D_{p+1})\text{)}
\]

In the next step, we calculate the sum $\sum_{k=j+1}^{p+1} A_k$. Therefore we remark that one can obviously write this sum in the same form as the above formula for $A_k$, but $\text{ad}(\bar{\partial})$ acts on any term in the tensor product where a $D_k, k = j+1, \ldots, p+1$ appears. As $\bar{\partial}^2 = 0$, $\text{ad}(\bar{\partial})$
acts like zero on the other terms. So we can let it act on all terms in the tensor product which is the same as to write the commutator of \(\tilde{\partial}\) with the whole term:

\[
\sum_{k=j+1}^{p+1} A_k = (-1)^p[\tilde{\partial}, Sh_{p-j+1,j-1}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))]
\]

So far, we have shown that the left hand side of the lemma can be written as

\[
(-1)^j[D_{j+1}, \rho_1] Sh_{p-j,j}(\text{ad}(D_{j+2}), \ldots, \text{ad}(D_{p+1})) + (-1)^p \rho_1[\tilde{\partial}, Sh_{p-j+1,j-1}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))]
\]

\[
= (-1)^p[\tilde{\partial}, \rho_1 Sh_{p-j+1,j-1}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))]
\]

\[
+ (-1)^{p+1}[\tilde{\partial}, \rho_1] Sh_{p-j+1,j-1}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))
\]

\[
+ (-1)^j[D_{j+1}, \rho_1] Sh_{p-j,j}(\text{ad}(D_{j+2}), \ldots, \text{ad}(D_{p+1}))
\]

It is almost what we want. We just have to rewrite the last two terms:

\[
(-1)^j \sum_{sh \in S_{p-j,j}, \hat{sh}(1)=1} \text{sgn}(sh) \hat{sh}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))(\rho_1, \ldots, \rho_{p+1})
\]

\[
+ (-1)^{p+1} \sum_{\tilde{sh} \in S_{p-j+1,j}, \tilde{sh}(p-j+2)=1} \text{sgn}(sh) \tilde{sh}(\text{ad}(D_{j+1}, \ldots, \text{ad}(D_{p+1}))(\rho_1, \ldots, \rho_{p+1})
\]

Again, we have to consider the relative signs of the shuffles. For \(sh \in S_{p-j,j}\), \(\hat{sh}\) is given by

\[
\hat{sh}(1) = 1, \ \hat{sh}(i) = sh(i-1) + 1
\]

Essentially \(sh\) and \(\hat{sh}\) are the same shuffles, so their relative sign is 1. \(\tilde{sh}\) is given by

\[
\tilde{sh}(1) = sh(1) + 1, \ldots, \tilde{sh}(p-j+2) = sh(p-j+2) + 1, \ldots, \tilde{sh}(p+1) = sh(p) + 1
\]

Obviously \(\tilde{sh} = \hat{sh} \circ \sigma\) where \(\sigma\) is the cycle \((1, 2, \ldots, p-j+2)\). Therefore the relative sign is \((-1)^{p+j+1}\). So the above terms can be unified to

\[
(-1)^j \sum_{sh \in S_{p-j+1,j}} \text{sgn}(sh) sh(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))(\rho_1, \ldots, \rho_{p+1})
\]

\[
= (-1)^j Sh_{p-j+1,j}(\text{ad}(D_{j+1}), \ldots, \text{ad}(D_{p+1}))(\rho_1, \ldots, \rho_{p+1})
\]

Remembering that we had a global sign \((-1)^{jp}\) in the definition of \(Z_j^p\) we can now see that we got exactly the terms on the right hand side of the identity in the above lemma. \(\square\)
Let again $D_0, \ldots, D_{p+1}$ be holomorphic differential operators and $\rho_0, \ldots, \rho_{p+1}$ smooth functions on $M$. Then we define

$$\sigma_p(D_0, \ldots, D_p; \rho_0, \ldots, \rho_p) = \sum_{j=0}^{p} \Psi_j(Z_j^p(D_{j+1}, \ldots, D_p; \rho_0, \ldots, \rho_p)D_0, D_1, \ldots, D_j)$$

**Theorem A.15.** $\sigma_p$ satisfies the identity

$$\sigma_p(b(D_0, \ldots, D_{p+1}); \rho_1, \ldots, \rho_{p+1}) = \sigma_{p+1}(D_0, \ldots, D_{p+1}; 1, \rho_1, \ldots, \rho_{p+1})$$

**Proof.** Applying the definition of the Hochschild differential we get

$$\sigma_p(b(D_0, \ldots, D_{p+1}); \rho_1, \ldots, \rho_{p+1}) =$$

$$= \sum_{j=0}^{p} \left\{ \Psi_j(Z_j^p(D_{j+2}, \ldots, D_{p+1})D_0D_1, \ldots, D_{j+1})$$

$$- \Psi_j(Z_j^p(D_{j+2}, \ldots, D_{p+1})D_0, D_1D_2, \ldots, D_{j+1})$$

$$\pm \ldots$$

$$+ (-1)^j \Psi_j(Z_j^p(D_{j+2}, \ldots, D_{p+1})D_0, \ldots, D_jD_{j+1})$$

$$+ (-1)^{j+1} \Psi_j(Z_j^p(D_{j+1}D_{j+2}, \ldots, D_{p+1})D_0, \ldots, D_j)$$

$$\pm \ldots$$

$$+ (-1)^p \Psi_j(Z_j^p(D_{j+1}, \ldots, D_{p-1}, D_pD_{p+1})D_0, \ldots, D_j)$$

$$+ (-1)^{p+1} \Psi_j(Z_j^p(D_{j+1}, \ldots, D_p)D_{p+1}D_0, \ldots, D_j) \right\}$$

$$= \sum_{j=0}^{p} \left\{ \Psi_j(b(Z_j^p(D_{j+2}, \ldots, D_{p+1})D_0, D_1, \ldots, D_{j+1}))$$

$$+ (-1)^j \Psi_j(D_{j+1}Z_j^p(D_{j+2}, \ldots, D_{p+1})D_0, \ldots, D_j)$$

$$+ (-1)^{j+1} \Psi_j(Z_j^p(b(D_{j+1}, \ldots, D_{p+1}))D_0, \ldots, D_j)$$

$$+ (-1)^{p+1} \Psi_j(Z_j^p(D_{j+1}, \ldots, D_p)D_{p+1}D_0, D_1, \ldots, D_j) \right\}$$

On the first term in the "\{ \ldots \}" we apply the cocycle property of $\Psi$ and on the other three terms lemma A.14. The terms with $[\partial, \cdot]$ cancel because they always appear twice with different signs and we immediately get the right hand side of the theorem.

**Lemma A.16.** Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of $M$ and $\{\rho_\alpha\}_{\alpha \in I}$ a corresponding partition of unity. We write $D_{\alpha_1, \ldots, \alpha_n}^q$ for an object that defines for every $(\alpha_1, \ldots, \alpha_n) \in I^n$ a holomorphic operator in $\mathcal{D}(U_{\alpha_1} \cap \cdots \cap U_{\alpha_n})$ and which is antisymmetric in the $\alpha$’s, i.e. $D_{\alpha_1, \ldots, \alpha_n}^{(p)} = \text{sgn}(\sigma)D_{\alpha_1, \ldots, \alpha_n}^{(p)}$ for every $\sigma \in S_q$. Then the following equation holds

$$\sum_{\alpha_1, \ldots, \alpha_p \in I \text{ \ and \ pairwise \ } \neq} \sigma_p(D_{\alpha_1, \ldots, \alpha_p}^{(p)}; 1, \rho_{\alpha_1}, \ldots, \rho_{\alpha_p}) = \sum_{\alpha_0, \ldots, \alpha_p \in I \text{ \ and \ pairwise \ } \neq} \sigma_p(\delta D_{\alpha_0, \ldots, \alpha_p}^{(p)}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_p})$$
where $\delta$ is the Čech differential, i.e.

$$\delta D^{(p)}_{\alpha_0, \ldots, \alpha_p} = \sum_{i=0}^{p} (-1)^i D^{(p)}_{\alpha_0, \ldots, \alpha_i, \ldots, \alpha_p}$$

**Proof.** We start with the left hand side of the lemma and replace 1 by $\sum_{\alpha_i \in I} \rho_{\alpha_i}$. Then we write

$$\sum_{\alpha_0 \in I} \sum_{\alpha_1, \ldots, \alpha_p \in I \atop \text{pairwise } \neq} \sum_{\alpha_0, \ldots, \alpha_p \in I \atop \text{pairwise } \neq} + \sum_{j=1}^{p} \sum_{\alpha_1, \ldots, \alpha_p \in I \atop \text{pairwise } \neq, \alpha_j = \alpha_j} \sigma_p(D^{(p)}_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_p})$$

We replace $\rho_{\alpha_j} = 1 - \sum_{\alpha_j', \neq \alpha_j} \rho_{\alpha_j'}$. As there are derivatives acting on the $\rho_{\alpha_1}, \ldots, \rho_{\alpha_p}$ in $\sigma_p$ (see definition of $Z^p_j$) the 1 disappears and we get (using $\alpha_j' \mapsto \alpha_j$)

$$\sum_{\alpha_0, \ldots, \alpha_p \in I \atop \text{pairwise } \neq} \sigma_p(D^{(p)}_{\alpha_1, \ldots, \alpha_p}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_p}) - \sum_{j=1}^{p} \sum_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p \in I \atop \text{pairwise } \neq, \alpha_j = \alpha_j} \sigma_p(D^{(p)}_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_j})$$

In the last sum the terms for which $\alpha_j$ is equal to one of the other $\alpha$’s are zero because of the antisymmetry of $\sigma_p(\ldots; \rho_0, \rho_1, \ldots, \rho_p)$ in $\rho_1, \ldots, \rho_p$. By the antisymmetry of $D$ with respect to the $\alpha$’s we get

$$D^{(p)}_{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_p} = (-1)^{j-1} D^{(p)}_{\alpha_0, \ldots, \alpha_j, \ldots, \alpha_p}$$

which gives exactly the signs of the Čech differential. \hfill \Box

Now, we are ready for our "climb the staircase" argument (see section 3.4):

$$\text{Str}(De^{-i\Delta \delta}) = \sigma_0(D; 1) = \sum \sigma_0(D_{\alpha_0}; \rho_{\alpha_0}) = \sum \sigma_0(bD^{(1)}_{\alpha_0}; \rho_{\alpha_0}) = \sum \sigma_1(D^{(1)}_{\alpha_1}; 1, \rho_{\alpha_1})$$

$$= \sum \sigma_1(\delta D^{(1)}_{\alpha_0, \alpha_1}; \rho_{\alpha_0}, \rho_{\alpha_1}) = \sum \sigma_1(bD^{(2)}_{\alpha_0, \alpha_1}; \rho_{\alpha_0}, \rho_{\alpha_1}) = \ldots$$

$$= \sum \sigma_j(\delta D^{(j)}_{\alpha_0, \ldots, \alpha_j}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_j}) = \sum \sigma_j(bD^{(j+1)}_{\alpha_0, \ldots, \alpha_j}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_j})$$

$$= \sum \sigma_{j+1}(D^{(j+1)}_{\alpha_1, \ldots, \alpha_{j+1}; 1, \rho_{\alpha_1}, \ldots, \rho_{\alpha_{j+1}}}) = \sum \sigma_{j+1}(\delta D^{(j+1)}_{\alpha_0, \ldots, \alpha_{j+1}; \rho_{\alpha_0}, \ldots, \rho_{\alpha_{j+1}}})$$

The summation in the above equations always goes over all appearing $\alpha$’s. Now, we explain the steps in the calculation:

(1) Here we use the fact that $\delta^2 = 0$ and that the Hochschild homology $HH_j(D(U))$ is trivial for $j = 0, \ldots, 2n - 1$. We construct $D^{(j)}$ in a way that $bD^{(j)} = \delta D^{(j-1)}$. From this follows $b\delta D^{(j)} = \delta bD^{(j)} = \delta^2 D^{(j-1)} = 0$. Therefore there is an element $D^{(j+1)}$ so that $bD^{(j+1)} = \delta D^{(j)}$.\hfill
(2) This step is theorem A.15.

(3) This step is lemma A.16.
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A.5. **THE COCYCLE $\sigma$**


*Hat’s dir nicht gefall’n,*  
*bohr’ dir doch ein Loch ins Knie*  
*manchen kann man’s recht machen,*  
*aber allen eben nie.*

*The Fraggles*
Curriculum Vitae

May 5th 1976  Born in Zürich, Switzerland

1983-1989  Primary School in Regensdorf ZH, Switzerland

1989-1996  High School in Oerlikon ZH, Switzerland

1996-2001  Diploma in Theoretical Physics, ETH Zürich, Switzerland

2001-2008  Doctoral studies of Mathematics at ETH Zürich under the supervision of Prof. Dr. Giovanni Felder. Teaching assistant and organizer of the assistant group at the Department of Mathematics, ETH Zürich.