An Advanced Frequency-Domain Code for Boiling Water Reactor (BWR) Stability Analysis and Design

A Dissertation Submitted to the ETH ZÜRICH for the degree of Doctor of Technical Sciences

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To my mother
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Abstract

The two-phase flow instability is of interest for the design and operation of many industrial systems such as boiling water reactors (BWRs), chemical reactors, and steam generators. In case of BWRs, the flow instabilities are coupled to the power instabilities via neutronic-thermal hydraulic feedbacks. Since these instabilities produce also local pressure oscillations, the coolant flashing plays a very important role at low pressure.

Many frequency-domain codes have been used for two-phase flow stability analysis of thermal hydraulic industrial systems with particular emphasis to BWRs. Some were ignoring the effect of the local pressure, or the effect of 3D power oscillations, and many were not able to deal with the neutronics-thermal hydraulics problems considering the entire core and all its fuel assemblies.

The new frequency domain tool uses the best available nuclear, thermal hydraulic, algebraic and control theory methods for simulating BWRs and analyzing their stability in either off-line or on-line fashion. The novel code takes all necessary information from plant files via an interface, solves and integrates, for all reactor fuel assemblies divided into a number of segments, the thermal-hydraulic non-homogenous non-equilibrium coupled linear differential equations, and solves the 3D, two-energy-group diffusion equations for the entire core (with spatial expansion of the neutron fluxes in Legendre polynomials). It is important to note that the neutronics equations written in terms of flux harmonics for a discretized system (nodal-modal equations) generate a set of large sparse matrices. The eigenvalue problem associated to the discretized core statics equations is solved by the implementation of the implicit restarted Arnoldi method (IRAM) with implicit shifted QR mechanism.

The results of the steady state are then used for the calculation of the local transfer functions and system transfer matrices. The later are large-dense and complex matrices, (their size for a large reactor could reach 20 Gigabytes) that it is not possible to load into RAM memory of an operating system with 32 bit architecture. A special procedure has been developed within the MATLAB environment to remove this memory limitation, and to invert such large matrices and finally obtain the reactor transfer functions that enable the study of system stability.

Various applications of the present frequency-domain code to a typical BWR fuel assembly, a BWR core, and to a chemical reactor showed a good agreement with reference results.
Sommario

L’instabilità del flusso bifase è d’interesse per la progettazione ed esercizio di molti sistemi industriali come reattori ad acqua bollente, reattori chimici e generatori di vapore. Nel caso dei reattori ad acqua bollente, le instabilità di flusso sono accoppiate ad instabilità di potenza attraverso i feedback tra neutronica e termoidraulica. Siccome queste instabilità causano anche oscillazioni di pressione locali, il flashing del refrigerante gioca un ruolo molto importante a bassa pressione.

Molti codici in dominio di frequenza sono stati usati per analizzare la stabilità dei flussi bifase in sistemi industriali, con particolare attenzione ai reattori ad acqua bollente. In alcuni di questi codici l’effetto della pressione locale o l’effetto delle oscillazioni della distribuzione tridimensionale di potenza sono trascurati, altri codici invece non sono in grado di simulare il problema accoppiato neutronica-termoidraulica esteso all’intero nocciolo, includendo tutte gli elementi di combustibile.

Il nuovo codice in dominio di frequenza si basa sullo stato dell’arte in metodi nucleari, termoidraulici, algebraici e di teoria di controllo per simulare i reattori ad acqua bollente e analizzare la loro stabilità in modo off-line oppure on-line. Il nuovo codice acquisisce dai file dell’impianto attraverso un’interfaccia tutte le informazioni necessarie; per ciascun elemento di combustibile, diviso in un certo numero di nodi assiali, risolve ed integra il sistema di equazioni differenziali corrispondente al modello di termoidraulica in non equilibrio e non omogeneo e risolve per l’intero nocciolo del reattore le equazioni di diffusione a due gruppi in tre dimensioni (con espansione spaziale del flusso basata sui polinomi di Legendre). È importante notare che le equazioni di neutronica formulate in termini di armoniche del flusso neutronico per un sistema discreto (equazioni nodali-modali) generano una serie di matrici sparse di grandi dimensioni. Il problema agli autovalori associato alle equazioni discretizzate di statica del nocciolo e’ risolto implementando il metodo implicito di Arnoldi con il meccanismo QR.

I risultati dello stato stazionario sono poi utilizzati per il calcolo delle funzioni di trasferimento locali e per il sistema di matrici di trasferimento. Queste sono matrici dense a numeri complessi e di grandi dimensioni (possono arrivare fino a 20 Gigabyte), per questo non è possibile memorizzarle nella memoria RAM con un sistema operativo basato sull’architettura a 32 bit. Una procedura particolare è stata sviluppata all’interno dell’ambiente MATLAB, per rimuovere questa limitazione sulla memoria e per consentire di invertire matrici di grandi dimensioni ed ottenere le funzioni di trasferimento necessarie per la valutazione della stabilità del sistema da analizzare.

Il nuovo codice in dominio di frequenza è stato applicato allo studio di un tipico canale bollente, di un intero nocciolo di un reattore ad acqua bollente e di un reattore chimico. I risultati ottenuti sono in buon accordo con quelli di riferimento.
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Chapter 1

Introduction
Boiling Water Reactors (BWR) may undergo flow instabilities at particular operational conditions. In this kind of nuclear power plants (NPP), void oscillations in the core induce power oscillations due to the neutronic feedback. The power oscillations are also intimately coupled to all the other flow variable oscillations. The combined thermal-hydraulic/neutronic behavior is a very important issue in BWR stability analysis which centers on the determination of the operating region where such power-void-flow oscillations may appear and the response of the reactor at these points.

Different kinds of flow and power oscillation phenomena have been observed under low-flow and high-power operational conditions. These oscillations are classified as in-phase (core-wide) or out-of-phase (regional), according to whether the flow rate and the neutron flux oscillate in-phase over the whole core or out-of-phase between individual halves of the core. In the latter case, the power in half of the core increases, while that of the other half decreases. The flow rate exhibits also an asymmetric behavior. Consequently, the average reactor power may remain constant during an oscillation event. Therefore, the core average power monitor system or reactor protection system may not detect even large-amplitude oscillations, Hashimoto (1993).

Normally, stability problems such as these mentioned above may arise during startup or during transients that shift the operating conditions towards the low-flow and high-power region. Figure 1.1 shows the typical power-flow map of a BWR. The red regions are called the exclusion area, since this is the region where the power oscillations may appear.

Historically, as the power density and the number of the fuel bundles in boiling water reactors increased (larger cores with higher power), several plants entered into the exclusion area and
presented power-void-flow oscillations, as shown in the recent compilation by Hänggi et al. (2001), table 1.1. Following these events, there was a renewed interest in nuclear-coupled thermal-hydraulic instabilities. Many codes have been written in both the time and the frequency domain to address the BWR stability issue.

<table>
<thead>
<tr>
<th>Date</th>
<th>Plant</th>
<th>Country, Manufact.</th>
<th>Event (as described by the operator)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30.06.82</td>
<td>Caerso</td>
<td>IT, AMN</td>
<td>Core instability during plant start up</td>
</tr>
<tr>
<td>01.10.83</td>
<td>Caerso</td>
<td>IT, AMN</td>
<td>Core instability during special tests</td>
</tr>
<tr>
<td>17.10.84</td>
<td>S. Maria de Garona</td>
<td>ES, GE</td>
<td>Power oscillations during operation</td>
</tr>
<tr>
<td>23.02.87</td>
<td>TVO 1</td>
<td>FE, ABB</td>
<td>Power oscillations during plant start up</td>
</tr>
<tr>
<td>09.03.88</td>
<td>La Salle 2</td>
<td>US, GE</td>
<td>Core instability with scram caused by neutron flux oscillation</td>
</tr>
<tr>
<td>29.10.88</td>
<td>Vermont Yankee</td>
<td>US, GE</td>
<td>Power oscillations</td>
</tr>
<tr>
<td>26.10.89</td>
<td>Ringhals 1</td>
<td>SE, GE</td>
<td>Instability during power ascent</td>
</tr>
<tr>
<td>08.01.89</td>
<td>Oskarshamn</td>
<td>SE, ABB</td>
<td>Power oscillations</td>
</tr>
<tr>
<td>29.01.91</td>
<td>Cofrentes</td>
<td>ES, GE</td>
<td>Power oscillations due to inadvertent entry in the reactor power-core flow map instability zone “B”</td>
</tr>
<tr>
<td>03.07.91</td>
<td>Isar 1</td>
<td>DE, KWU</td>
<td>Scram due to power oscillations</td>
</tr>
<tr>
<td>15.08.92</td>
<td>WNP</td>
<td>US, GE</td>
<td>Power oscillations</td>
</tr>
<tr>
<td>09.07.93</td>
<td>Perry</td>
<td>US, GE</td>
<td>Entry into a region of core instability</td>
</tr>
<tr>
<td>17.07.96</td>
<td>Forsmark 1</td>
<td>SE, ABB</td>
<td>Local oscillations due to a bad seated fuel assembly</td>
</tr>
<tr>
<td>08.02.98</td>
<td>Oskarshamn 3</td>
<td>SE, ABB</td>
<td>Power oscillations due to a bad combination of core design and control-rod pattern during start up</td>
</tr>
<tr>
<td>25.02.99</td>
<td>Oskarshamn 2</td>
<td>SE, ABB</td>
<td>Power oscillations after a turbine trip with pump runback</td>
</tr>
</tbody>
</table>

Table 1.1 Observed instability events

The time-domain codes that rely on numerical simulation of the phenomena are fairly computer-time consuming, they may suffer from numerical stability problems, and their results need post processing for the stability analysis. There are several such state-of-the-art codes that are routinely used for BWR stability; several are listed and shortly described in a State-of-the-Art (SOAR) report on the subject by D’Auria et al (1997).

On the contrary, the frequency-domain codes (FDCs) are faster but they may need more computer memory. They rely extensively on linearization of the governing equations and closure relationships and algebraic manipulations that necessarily limited in the past their scope. Modern developments (such as the present work), based partly at least on computer-based symbolic manipulation of the equations result in very large sets of equations. Because of this, most of the previous FDCs were limited by the size of the matrices that could be handled and could be loaded in the virtual memory of the computer (RAM); this consequently limited the number of fuel assemblies and nuclear cells that could be considered; see table 1.2.

A common restriction among most previous FDCs is that they assume the water and steam thermal properties to be functions of system pressure only and not of the local temperature and pressure. This greatly simplifies the mathematical derivation of the model, since it decouples the continuity (liquid and vapor) and energy equations from the momentum equation. Another restriction is that most of these codes considered a limited number of channels (fuel assemblies) only and not each individual assembly in the core. From the
neutronics point of view, they are usually based on a very simple model like point kinetics or 1D diffusion theory.

Among the FDCs, only MATSTAB, Hänggi et al (1999, 2001), is able to handle a realistic model of the reactor including all the components and applying the sparse matrix technique in combination with iterative solvers for the large sparse matrices. MATSTAB is a numerically linearized model of the old version of the state-of-the-art time-domain RAMONA code. In MATSTAB, like the other FDCs, the local pressure effect is neglected and therefore the local flashing or condensation is not included in its thermal hydraulic model. The 3D neutronics is based on one and a half groups with the prompt jump approximation. The latter implies neglect of the divergence of the thermal flux and the time derivatives of the fast and thermal fluxes. The 3D flux perturbations are obtained by a simplified solution of the delayed neutron precursor equations.

Since the present and future BWRs (larger core) are subject of the flow instabilities, analytical tools capable of predicting the conditions under which the system becomes unstable are necessary. This requires a realistic model of the reactor, the best available model for thermal hydraulics, and detailed neutronics of the core.

In order to meet the needs of the nuclear industry, a frequency domain code for the design and stability analysis of BWRs has been developed within the framework of the present doctoral dissertation. The guidelines for this work included the following: removing the restrictive assumptions of the previous codes, using state-of-the-art thermal hydraulic models and 3D neutron kinetics, taking advantage of fully analytical techniques (made possible by the use of the MATLAB environment) and advanced numerical methods for the solution of large sparse and large dense complex matrices, including all the requirements for the design of the next generation of BWRs, and getting data directly from the plant simulators normally available in all BWR plants.

<table>
<thead>
<tr>
<th>Code</th>
<th>Model</th>
<th>Thermal hydraulics</th>
<th>Neutronics</th>
<th>Organization</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>NUFREQ-NP</td>
<td>A few</td>
<td>NC NE DF</td>
<td>3-4</td>
<td>PK</td>
<td>RPI  [8]</td>
</tr>
<tr>
<td>LAPUR</td>
<td>1-7</td>
<td>NC EQ SL</td>
<td>PK</td>
<td>ORNL/NRC</td>
<td>[5]</td>
</tr>
<tr>
<td>STAIF</td>
<td>10</td>
<td>NC NE DF</td>
<td>1D</td>
<td>SIEMENS</td>
<td>[5]</td>
</tr>
<tr>
<td>FABLE</td>
<td>24</td>
<td>NC NE SL</td>
<td>PK</td>
<td>GE</td>
<td>[5]</td>
</tr>
<tr>
<td>ODYSY</td>
<td>A few</td>
<td>NC NE DF</td>
<td>1D 1G</td>
<td>GE</td>
<td>[5]</td>
</tr>
<tr>
<td>MATSTAB</td>
<td>Full core</td>
<td>NC NE SL</td>
<td>4</td>
<td>3D 11/2G PJ FDM</td>
<td>FORSKMARK [5]</td>
</tr>
</tbody>
</table>

D : Dimensions  
DF : Drift Flux model  
EQ : Thermal Equilibrium (no subcooled boiling)  
NC : Non Coupled and using the system pressure for the thermal properties of water and steam  
NE : Non thermal Equilibrium (subcooled boiling)  
SL : Velocity ratio approach  
G : Neutron energy groups  
1/2G : One and a half group  
PJ : Prompt Jump approximation  
FDM : Finite Difference method

Table 1.2 Present codes for the BWR stability analysis.
Therefore the main features of this tool have been the following:

- 3D neutronics: two energy groups, six groups of delayed neutrons
- Nodal/modal solution of the time-dependent neutronics equations with $K+1$ order expansion in Legendre polynomials for each nuclear cell and consideration of $M+1$ harmonics for the time-dependent neutron flux, where $K$ and $M$ are user defined parameters;
- Perturbation theory techniques for the reactor dynamics;
- Analytical solution of the fuel rod temperature under both stationary conditions and in transients;
- Non-homogeneous two-phase flow based on the drift-flux approach (four equations for the void fraction ($\alpha$), the volumetric flux of the mixture ($j$), the mixture enthalpy ($h$), and the pressure $p$ (provided by the momentum equation); which constitute a system of non-homogenous, non-linear, coupled differential equations;
- Thermal properties of the water and steam calculated from the steam tables using the local temperature and pressure
- The Drift Flux model parameters calculated locally for each thermal hydraulic cell;
- State-of-the-art closure laws;
- Flashing effects fully included in the conservation equations;
- The two-phase flow region subdivided into a subcooled boiling region (thermally expandable liquid and saturated gas) and a saturated region;
- Non-thermal equilibrium (subcooled boiling) based on the mechanistic model of Lahey (1993);
- Dynamics of the boiling boundary (represented by the Net Vapor Generation point);
- Dynamic distribution of the flow among the parallel-flow bundles constituting the core, including the effects of the 3D power perturbations;
- Very detailed description of modern fuel assemblies with full-length, part-length fuel rods and variable cross sectional flow area, see, for example, figure 1.2 (ESBWR Design description, 2002);
- Large, sparse matrix numerical techniques for the solution of the core statics problem (eigenvalues, fluxes and adjoint fluxes);
- Large, dense complex matrix numerical method for the solution of the transient system equations;
- Interface to nuclear power plant simulators (such as those of Forsmark, Sweden, Leibstadt Switzerland).

The following paragraphs describe briefly the various parts of the present work.

**Neutronics**

In Chapter 2 of the present work, the 3D diffusion equations are solved, analytically, for both core statics and dynamics. The nodal-modal method is used for this task. It consists of neutron flux decomposition into a number of harmonics (modes). In other words, an instantaneous flux at any node of a core discretization, can be considered as a linear combination of different reactor modes, see section 2.5.2 (eq. 2.86). The coefficients of the linear combination of the modes are the stationary solution of the core static equations.
Figure 1.2 Modern fuel assembly, GE12 (ESBWR Design description, 2002). The full- and part-length fuel rods, and water rods are shown in the pictures. Note, that the part-length rods have different lengths; therefore the cross section flow area of the channel is modified at the end of each part-length rod group.

The core static solution (an eigenvalue problem) requires discretization of the multigroup-multidimensional diffusion equations over the core. Among various methods available, the nodal collocation method (NCM) has been chosen. In this technique, fluxes and group transverse leakages are expressed in the form of a truncated tensorial expansion in Legendre polynomials defined over homogeneous parallelepiped nodes (nuclear cells) satisfying current and flux continuity given as node boundary conditions. This discretization algorithm makes possible to increase arbitrarily the order of discretization by varying the order of expansion in Legendre polynomials used as trial functions, Hebert (1987).

After discretization of the 3D diffusion equations, they are written in terms of loss and production matrices. The size of the matrices generated after the discretization of the system of 3D diffusion equations depends on the number of the fuel assemblies, the number of axial nodes and the order of the Legendre polynomials. For example, for a reactor with 700 fuel assemblies and 25 axial nodes, and with the first-order Legendre polynomials, sparse matrices of the size 17500*17500 are obtained. When second-order Legendre polynomials are used, the size of the matrices becomes four times bigger (70000*70000).
The matrix form of the 3D diffusion equations constitutes an eigenvalue problem. The largest eigenvalue of this problem is the multiplication factor \( (K_{\text{eff}}) \) of the core, while the eigenvectors associated to the eigenvalues of the problem are the modes of the flux (flux harmonics). The fundamental mode of the flux provides the stationary flux profile of the core and the core power distribution is obtained from the fundamental mode of the flux. This power distribution is sent to the core thermal hydraulic routines for the calculation of the system thermal-hydraulics state variables. But since the nuclear data depend on the local thermal hydraulics variables (such as the void fraction, fuel temperature, etc.), it is necessary to perform iterations between core statics and thermal hydraulics routines.

A second eigenvalue problem has to be solved to determine the adjoint fluxes. These are necessary for the perturbation theory applications.

The modal neutronics equations are then obtained by expanding the instantaneous flux in terms of its \( M+I \) harmonics (see eq. 2.86) in the set of space-time dependent group diffusion equations. These are later discretized (using NCM), perturbed (applying first-order perturbation theory; Neuhold and Ott, 1985) and Laplace transformed, resulting in mode amplitude perturbations written in terms of external reactivity, nodal void and fuel temperature perturbations. The mode amplitude perturbations are then used for the perturbation of local power. Finally, these will be used in the stability chapter to obtain the system transfer functions.

**Fuel dynamics**

In the third chapter, the conduction equation is solved analytically, in cylindrical coordinates, for a generic segment of a fuel rod, obtaining the temperature profiles in the pellet, gap, and cladding at both steady state and during a transient.

In order to obtain the transient solution, the conduction equation has to be perturbed and Laplace transformed and the appropriate boundary conditions have to be applied. The perturbed form of the conduction equations together with the power perturbation equations are necessary for the coupling of the mode amplitude to the thermal hydraulics equations in the stability chapter.

**Thermal hydraulics**

Chapter four presents the analytical solution for the steady state and transient forms of the thermal hydraulics equations of the system. The stationary power distribution and power perturbations, necessary for this chapter, are obtained in the previous step.

In this thermal hydraulic model, a fuel assembly is assumed to have three regions: single-phase subcooled, two-phase subcooled, and two-phase saturated. In the subcooled region, liquid is thermally expandable and the vapor saturated, while in the saturated region both liquid and vapor are saturated. The thermal properties of the liquid and vapor and their partial derivatives with respect to temperature and pressure are calculated at the local pressure and temperature, by of the steam tables, already integrated in the code. Furthermore flashing is taken into account, as both the space and time variations of the local pressure are considered.
in the energy equation. It is important to note that the use of the local pressure and temperature variations couples the conservation equations. Therefore the momentum equation cannot be integrated separately after integration of the continuity and energy equations, as in the previous frequency domain works.

Each flow channel is divided into a number of axial segments and the conservation equations are solved for each segment. Integration of the equations starts at the core inlet and proceeds up the channel. Knowing the segment inlet conditions, to obtain the segment exit state variables, the non-homogenous coupled differential equations are solved analytically and then integrated over the segment.

The transient solution for a segment requires more effort. The conservation equations have to be perturbed and Laplace transformed and combined with the heat flux and all the closure law perturbations. MATLAB and MAPLE procedures have been written using their symbolic tool boxes to obtain the final equations. This results in a set of non-homogenous coupled linear differential equations, which will be solved and integrated over the segment.

To consider the effects of local discontinuities (pressure drop in spacers and sudden flow area changes) the point transfer matrix technique has been applied. This takes into account all phenomena (flashing, velocity changes etc.) at the discontinuities. Finally, the channel and loop transfer functions linking the variation of the states variables are constructed and sent to the stability chapter.

**Channel and core overall stability**

Methods for the system steady state and stability analysis are described in Chapter 5. The perturbed form of the neutronic, fuel dynamic, and thermal hydraulics equations are here combined to obtain the final system transfer functions.

Using the channel transfer functions with appropriate boundary conditions, it is possible, as a first step, to study the thermal hydraulic behavior (without nuclear feedback) of each fuel assembly separately.

The core flow entering the lower plenum of the BWR is distributed in the larger number of fuel assemblies that are enclosed in fuel boxes and do not communicate radially. In order to study the stability of the overall system, we have developed a very detailed dynamic model for the flow distribution at the core inlet, which involves the 3D core power perturbations and the dynamics of the primary circulation loop. This dynamic flow distribution model allows relating the core nodal state variable perturbations to system external forcing functions. On the other hand, the 3D power dynamics and harmonic perturbations are coupled to the fuel dynamics. The resulting transfer matrices are combined with those of the overall system thermal hydraulics, obtaining the mode amplitude perturbations in terms of the system external forcing functions.

The modal system transfer functions constitute the modal closed loop transfer functions. Starting from the block diagram of the entire system that describes the various feedbacks between its components described above, one can define a forward loop (with a transfer function $G$) and a feedback loop (transfer function $H$), as shown in figure 1.4 below. Knowing that a BWR is a highly non-linear system but it can be described as a linear one in the
neighborhood of stationary operating points, we can investigate the stability of the linear system, figure 1.4, by studying the poles of the system closed-loop transfer function \((G/(1+GH))\) or the roots of the characteristic equations \((1+GH)\).

Furthermore using the poles of the system transfer function, it is possible to calculate the decay ratio and the natural frequency of the system. An alternative way is to examine the behavior of the GH locus in the Nyquist diagram.

**Code validation and applications**

Chapter 6 discusses testing various parts of the analytical tool and computer code and comparison with existing analytical and numerical solutions. For example the 3D power distribution (solution of the neutron diffusion equations) of the code is compared with the analytical solution for a homogenous reactor core and with the numerical solution from a reference code.

The model has finally been used for the stability analysis of an operating BWR. Input nuclear, geometry data and thermal hydraulic parameters have been extracted via an ad-hoc interface from the core simulator files. Then the homogenized nuclear cross sections and diffusion coefficients are generated at the reference reactor state and power reconstructed. Since other thermal hydraulic state variables are also necessary for the stability analysis, the power obtained earlier is sent to the thermal hydraulics routines and state variables are obtained. Because of the fact that the power distribution is function of the void fraction, and of the fuel temperature, some iterations are necessary between power generation and thermal hydraulic state variable calculations. This guarantees that the power and the state variables are consistent. Finally the important parameters are compared with the data available in the reactor simulator files.

The data of the steady state conditions are used for the construction of the local transfer functions and system transfer matrices. These matrices are large, dense and complex. For in the case of the future ESBWR, the size of the matrices is about 25000*25000 and they require a very large amount of virtual memory, RAM, about 20Gbyte. Even if this memory
were available, it would have been very difficult to invert these matrices. In the framework of this doctoral thesis, a procedure has been developed within the MATLAB environment to invert such dense and complex matrices using only 2 Gbyte of virtual memory. However the present processors are still too slow for converting large complex matrices.

Because of the detail and flexibility implemented in the tool, it can be applied not only for the design and stability of BWRs but also for other thermal hydraulics systems (such as conventional boilers). For example, it has been used for the stability analysis of a chemical plant heat exchangers.

In addition, because of the very detailed neutronics, thermal hydraulics and the interface to the reactor simulator, the computer code could be used for core reloading pattern optimization, core peaking factors, control rod worth calculations, hot channel analysis, and other coupled neutronic – thermal hydraulics applications.

The following chapters provide the details of the models and methods used in the present work.
Figure 1.4 Schematic of ESBWR reactor pressure vessel
References:

Chapter 2

Linear neutronics
for
modal, multi-channel analysis
of
out-of-phase instability in BWRs
2.1. Introduction

The core dynamics is governed by the space-time group-diffusion equations. They describe the average reaction rate over an interval of energy referred to as a group, and have the generic form given below, using standard nomenclature, Stacey (1969):

\[ \nabla \cdot D^g(r,t) \nabla \phi^g(r,t) - \left( \Sigma^e_z(r,t) + \Sigma^e_s(r,t) \right) \phi^g(r,t) + \sum_{g' \neq g} \Sigma^g_{g'}(r,t) \phi^{g'}(r,t) \\
+ (1 - \beta) \chi^g_{\beta} \sum_{g' = 1}^{G} V^{g'} \Sigma^{g'}_f(r,t) \phi^{g'}(r,t) + \sum_{l=1}^{M} \lambda_l \chi^g_l C_l(r,t) + Q^g(r,t) = \left( \frac{1}{\psi^g} \right) \phi^g(r,t) \]

\( g = 1 \ldots G \) \hspace{1cm} Eq. 2.1

Where the precursors, satisfy the balance equations for each precursor group,

\[ \beta \sum_{g=1}^{G} V^g \Sigma^g_f(r,t) \phi^g(r,t) - \lambda_l C_l(r,t) = \dot{C}_l(r,t) \]

\( l = 1 \ldots M \) \hspace{1cm} Eq. 2.2

Eqs 2.1 and 2.2 are partial differential equations (PDEs). They can be solved analytically only for the simplest cases. Their approximations have been employed widely to obtain solutions to static or dynamic reactor problems. We have chosen the modal expansion approximation using the “time-synthesis modal expansion technique“, Stacey (1969).

Within the framework of stability analysis of a boiling water reactor with special interest in out-of-phase oscillations, we use the nodal-modal method to solve the 3D neutron diffusion equations. It consists of neutron flux decomposition into a number of harmonics (modes). In other words we can state that an instantaneous flux, at any node of a core discretization, can be considered as a linear combination of different reactor modes, see section 2.5.2 (eq. 2.86). The coefficients of the linear combination of the modes, eq. 2.86, are stationary solution of the core equations.

The core static solution (an eigenvalue problem) requires discretization of the multigroup-multidimensional diffusion equations over the core. Among various methods available, we have chosen the nodal collocation method (NCM). In this technique, fluxes and group transverse leakages are expressed in the form of a truncated tensorial expansion in Legendre polynomials defined over homogeneous parallelepiped nodes (elements), where the node boundary conditions satisfy current and flux continuity. This discretization algorithm makes possible to increase arbitrarily the order of discretization by varying the degree of Legendre polynomials used as trial functions, Hebert (1987).

Later, first-order perturbation theory, Neuhold and Ott (1985), is applied to the space-time dependent neutron diffusion equations (core dynamics equations), where the flux has been expanded in terms of flux harmonics. Then, they are discretized and Laplace transformed to obtain frequency dependent core transfer functions. These are open loop, void and Doppler feedback-to-flux transfer functions. Hence, there is no need to calculate, explicitly, the reactivity. In the stability chapter, the neutronic transfer functions will be combined with core thermal hydraulic transfer functions resulting in the system transfer function.
This chapter is divided into the following sections:

- Two-group energy diffusion equations
- Nodal collocation methods
- Core Statics
- Core dynamics
- Nodal volumetric power
- Modal reactivity feedback
- Neutronics transfer functions

2.2. Two-group energy diffusion equations

Most of state-of-the-art codes have used two-group energy diffusion equations for both statics and dynamics of a BWR core. We will use, also, two-group energy diffusion equations together with six delayed neutron groups. They are obtained from eqs. 2.1 and 2.2, neglecting the external source term (Q) and assuming that the neutrons are born only in the fast group (1) and that there is no up scattering from thermal to fast group.

\[
\frac{1}{\nu_1} \frac{\partial \phi_1}{\partial t} = \nabla \left( D_1 \nabla \phi_1 \right) - \left( \Sigma_{\alpha 1} + \Sigma_{\beta 1} \right) \phi_1 + \left( 1 - \beta \right) \left( \nu_1 \Sigma_{f1} \phi_1 + \nu_2 \Sigma_{f2} \phi_2 \right) + \sum_{l=1}^{6} \lambda_l C_l
\]

Eq. 2.3

\[
\frac{1}{\nu_2} \frac{\partial \phi_2}{\partial t} = \nabla \left( D_2 \nabla \phi_2 \right) - \Sigma_{\alpha 2} \phi_2 + \Sigma_{\beta 2} \phi_1
\]

Eq. 2.4

\[
\frac{\partial C_l}{\partial t} = \beta \left( \nu_1 \Sigma_{f1} \phi_1 + \nu_2 \Sigma_{f2} \phi_2 \right) - \lambda_l C_l \quad l = 1 \ldots 6
\]

Eq. 2.5

The solution of the neutron group-diffusion equations 2.3-5 requires discretization of the flux \( \phi \) and of the transverse leakages (Laplace operators). We will use the NCM for this task.
2.3. Nodal collocation method (NCM)

The nodal collocation method is a technique for discretization of the multidimensional neutron diffusion equation where the solution sought is expressed in the form of a truncated tensorial expansion in Legendre polynomials defined over homogeneous parallelepipeds, Hebert (1987). In this paragraph, we give some features of the NCM and then shortly outline the method. The following three subsections 2.3.1-3 describe in detail the node boundary conditions, the transverse leakages and the “serendipity approximation”.

Important features of NCM:

- It assumes an expansion in Legendre polynomials of the neutron flux and of the transverse leakages over each element (node).
- The transverse leakage matrices are symmetrical, positive definite and diagonally dominant.
- The numerical solution consists of piecewise continuous polynomials defined everywhere in the domain.
- The order of discretization is variable as a function of the degree of the collocation polynomials and can be adapted according to the heterogeneity of the nuclear properties.
- It allows the use of an alternating-direction implicit preconditioning for the numerical solution of the matrix system.
- The linear-order nodal collocation method is identical to the mesh centered finite difference method (MCFD).

In general, the numerical resolution of the static group diffusion equations begins with a discretization step in which differential operators are transformed into finite differences. Here we present the nodal collocation method in the case of one group formalism. Moreover we will limit this study to 3D Cartesian domains composed of an assembly of homogeneous parallelepipeds. Under stationary conditions, for a generic element \( e \), we have:

\[
- \nabla (D \nabla \phi_e) + \Sigma_{\nu} \phi_e = (1 - \beta) \Sigma_{\rho} \phi_e \\
\text{Eq. 2.6}
\]

which can be written also as:

\[
- \frac{\partial}{\partial x} \left( D_1(x,y,z) \frac{\partial \phi_e}{\partial x} \right) - \frac{\partial}{\partial y} \left( D_1(x,y,z) \frac{\partial \phi_e}{\partial y} \right) - \frac{\partial}{\partial z} \left( D_1(x,y,z) \frac{\partial \phi_e}{\partial z} \right) + \Sigma_{\nu} \phi_e = S_e(x,y,z) \\
\text{Eq. 2.7}
\]

Now we assume that nuclear properties are uniform over each parallelepiped in the domain. Furthermore, we will transform the Cartesian coordinates \((x,y,z)\) of the element \(e\) into local coordinates \((u,v,w)\) corresponding to a unitary cube of reference. This is necessary since the Legendre polynomials must be defined over the interval \((-1/2,1/2)\) and to ensure that the resulting reference cube has a unitary volume. The following variable transformation will be used:
\[ u = \frac{1}{\Delta x_e} \left( x - \frac{1}{2} \left( x_{m+\frac{1}{2}} + x_{m+\frac{1}{2}} \right) \right) \]  
\text{Eq. 2.8}

\[ v = \frac{1}{\Delta y_e} \left( y - \frac{1}{2} \left( y_{n+\frac{1}{2}} + y_{n+\frac{1}{2}} \right) \right) \]  
\text{Eq. 2.9}

\[ w = \frac{1}{\Delta z_e} \left( z - \frac{1}{2} \left( z_{\rho+\frac{1}{2}} + z_{\rho+\frac{1}{2}} \right) \right) \]  
\text{Eq. 2.10}

where
\[ \Delta x_e = x_{m+\frac{1}{2}} - x_{m+\frac{1}{2}} \]  
\text{Eq. 2.11}

\[ \Delta y_e = y_{n+\frac{1}{2}} - y_{n+\frac{1}{2}} \]  
\text{Eq. 2.12}

\[ \Delta z_e = z_{\rho+\frac{1}{2}} - z_{\rho+\frac{1}{2}} \]  
\text{Eq. 2.13}

Multiplying by element volume \((V_e)\) we obtain
\[ - \frac{\Delta y_e \Delta z_e}{\Delta x_e} D_{x,e} \frac{\partial^2 \phi_e}{\partial u^2} - \frac{\Delta x_e \Delta z_e}{\Delta y_e} D_{y,e} \frac{\partial^2 \phi_e}{\partial v^2} - \frac{\Delta x_e \Delta y_e}{\Delta z_e} D_{z,e} \frac{\partial^2 \phi_e}{\partial w^2} + V_e \sum \phi_e = V_e S_e (u,v,w) \]  
\text{Eq. 2.14}

As trial functions in each element \((e)\) we use the following truncated expansions of order \(K\):
\[ \phi_e (u,v,w) = \sum_{k_i=0}^{K} \sum_{k_1=0}^{K} \sum_{k_2=0}^{K} \Phi_{k_1,k_2,k_3} P_{k_1} (u) P_{k_2} (v) P_{k_3} (w) \]  
\text{Eq. 2.15}

\[ S_e (u,v,w) = \sum_{k_i=0}^{K} \sum_{k_1=0}^{K} \sum_{k_2=0}^{K} S_{k_1,k_2,k_3} P_{k_1} (u) P_{k_2} (v) P_{k_3} (w) \]  
\text{Eq. 2.16}

where \(P_{k_i}\) is the orthonormal Legendre polynomial of order \(k_i\) in the interval \([-1/2,1/2]\).
For example, in the $u$ direction, the $P_k$ are defined as, Hebert (1987):

$$P_0(u) = 1$$  \hspace{1cm} \text{Eq. 2.17}$$

$$P_1(u) = 2\sqrt{3}u$$  \hspace{1cm} \text{Eq. 2.18}$$

$$P_2(u) = \frac{\sqrt{5}}{2} \left(12u^2 - 1\right)$$  \hspace{1cm} \text{Eq. 2.19}$$

And in general for $k \geq 1$:

$$P_{k+1}(u) = 2\sqrt{\frac{2k+3}{2k+1}} \frac{2k+1}{k+1} u P_k(u) - \sqrt{\frac{2k+3}{2k-1}} \frac{k}{k+1} P_{k-1}(u)$$  \hspace{1cm} \text{Eq. 2.20}$$

Substituting expansions 2.15 and 2.16 into eq. 2.14, multiplying by a weight function:

$$\omega_{l_1,l_2,l_3} = P_i(u)P_j(v)P_k(w)$$  \hspace{1cm} \text{Eq. 2.21}$$

where $l_1 = 0 \ldots K$, $l_2 = 0 \ldots K$, $l_3 = 0 \ldots K$.

and integrating over the reference volume and using orthogonality, we obtain $K^3$ equations:

$$- \Delta y \Delta z F_{e,i}^{k_1,k_2,k_3} - \Delta x \Delta z F_{e,j}^{k_1,k_2,k_3} - \Delta x \Delta y F_{e,z}^{k_1,k_2,k_3} + V_e \Sigma_{e'} \Phi_e^{k_1,k_2,k_3} = V_e \Phi_e^{k_1,k_2,k_3}$$  \hspace{1cm} \text{Eq. 2.22}$$

In (2.22) the leakage terms are,

$$F_{e,x}^{k_1,k_2,k_3} = \frac{D_{x,e}}{\Delta x_e} L_{k_1} \left( \phi_{e,x}^{k_2,k_3} \right)$$  \hspace{1cm} \text{Eq. 2.23}$$

$$F_{e,y}^{k_1,k_2,k_3} = \frac{D_{y,e}}{\Delta y_e} L_{k_2} \left( \phi_{e,y}^{k_1,k_3} \right)$$  \hspace{1cm} \text{Eq. 2.24}$$

$$F_{e,z}^{k_1,k_2,k_3} = \frac{D_{z,e}}{\Delta z_e} L_{k_3} \left( \phi_{e,z}^{k_1,k_2} \right)$$  \hspace{1cm} \text{Eq. 2.25}$$
with the directional fluxes defined as,

$$\phi_{e,x}^{k_1,k_2}(u) = \sum_{k_3=0}^{K} \Phi_{e,x}^{k_1,k_2,k_3} P_{k_3}(u)$$

Eq. 2.26

$$\phi_{e,y}^{k_1,k_2}(v) = \sum_{k_3=0}^{K} \Phi_{e,y}^{k_1,k_2,k_3} P_{k_3}(v)$$

Eq. 2.27

$$\phi_{e,z}^{k_1,k_2}(w) = \sum_{k_3=0}^{K} \Phi_{e,z}^{k_1,k_2,k_3} P_{k_3}(w)$$

Eq. 2.28

and, in general, the leakage function is given as:

$$L_k(f(u)) = \sqrt{2k+1}\left((-1)^{k+1}\left(k(k+1)\frac{d f}{du}\right) -\left(k(k+1)f\left(\frac{1}{2}\right) - \frac{d f}{du}\right)\right)$$

Eq. 2.29

$$+ \sum_{l=0}^{k-2} \left(1+(-1)^{k+1}\right) \sqrt{2l+1} (k(k+1)-l(l+1)) F_l$$

where $f(u)$ is a directional fluxes ($\Phi_{e}$) given by eqs 2.26-8 and $F_l$ (with $l=0...k-2$) is the set of the first k-1 Legendre coefficients ($\Phi_{e,x}^{k_1,k_2,k_3}$) of the directional flux $f(u)$.

For example, the leakage function in x direction $L_{k_1}(\phi_{e,x}^{k_1,k_2}(u))$ becomes

$$L_{k_1}(\phi_{e,x}^{k_1,k_2}(u)) = \sqrt{2k_1+1}\left((-1)^{k_1+1}\left(k_2(k_1+1)\phi_{e,x}^{k_1,k_2}\left(-\frac{1}{2}\right) + \frac{d \phi_{e,x}^{k_1,k_2}(u)}{du}\right)\right)$$

$$- \left(k_1(k_1+1)\phi_{e,x}^{k_1,k_2}\left(\frac{1}{2}\right) - \frac{d \phi_{e,x}^{k_1,k_2}(u)}{du}\right) + \sum_{l=0}^{k_1-2} \left(1+(-1)^{k_1+1}\right) \sqrt{2l+1} \left(k_1(k_1+1)-l(l+1)\right) \Phi_{e,x}^{k_1,k_2,k_3}$$

finally, The leak functions for the y and z direction are obtained by permutation of the indexes.

The node boundary conditions that have to be applied to the leakage terms, eqs 2.23-5, will be described in the next section.
2.3.1. Conditions at node boundaries

Let attribute numbers $e1$ to $e6$ to the six neighboring elements (nuclear cell) which share a surface with element $e$; see figure 2.1 where the elements $e1$, $e$, and $e2$ are the three consecutive nuclear cell in $x$ direction, the elements $e3$, $e$, and $e4$ are the three consecutive nuclear cell in $y$ direction; and the elements $e5$, $e$, $e6$ are the three consecutive nuclear cell in $z$ direction. In Cartesian coordinates, the boundary conditions (BCs) at the element surfaces are either zero flux or positive albedo. The albedo ($0 \leq \beta(x,y,z) \leq 1$) is defined as the ratio of incoming to outgoing currents at point $(x,y,z)$. In general, the BCs are written as follows, Hebert (1985):

$$D_x (x,y,z) \frac{\partial \phi_e(x,y,z)}{\partial x} \pm \frac{1 - \beta(x,y,z)}{2 + \beta(x,y,z)} \phi_e(x,y,z) = 0$$  Eq. 2.30

$$D_y (x,y,z) \frac{\partial \phi_e(x,y,z)}{\partial y} \pm \frac{1 - \beta(x,y,z)}{2 + \beta(x,y,z)} \phi_e(x,y,z) = 0$$  Eq. 2.31

$$D_z (x,y,z) \frac{\partial \phi_e(x,y,z)}{\partial z} \pm \frac{1 - \beta(x,y,z)}{2 + \beta(x,y,z)} \phi_e(x,y,z) = 0$$  Eq. 2.32

Where the point $(x,y,z)$ is part of the domain boundary and the sign $(\pm)$ refer to the left (-) and the right (+) boundaries with respect to the direction of each axis.

Figure 2.1 The left picture shows the projection of the six neighboring elements ($e1$, $e2$, $e3$, $e4$, $e5$, $e6$) of the element $e$ in the planes $xy$, $zx$, and $zy$. The right picture presents the sharing surfaces of element $e$ with the neighboring elements: $S1 = \{ e1 \cap e \}$; $S2 = \{ e \cap e2 \}$; $S3 = \{ e3 \cap e \}$; $S4 = \{ e \cap e4 \}$; $S5 = \{ e5 \cap e \}$; $S6 = \{ e \cap e6 \}$. 
For the surface located at \( x = x_{e-1/2} \), in the local coordinates, the flux and current continuity conditions, Hebert (1987), can be expressed as:

\[
\phi_e \left( \frac{1}{2}, v, w \right) = \phi_e \left( -\frac{1}{2}, v, w \right) \quad \text{Eq. 2.33}
\]

\[
\frac{D}{\Delta x_e} \frac{\partial}{\partial u} \phi_e \left( \frac{1}{2}, v, w \right) = \frac{D}{\Delta x_e} \frac{\partial}{\partial u} \phi_e \left( -\frac{1}{2}, v, w \right) \quad \text{Eq. 2.34}
\]

### 2.3.2. Leakag e terms

After applying boundary conditions on all six surfaces of element \( e \), we obtain the final expression for the leakage terms eqs. 2.35-7 in the static group diffusion equation, Hebert (1987):

\[
F_{e,x}^{k,k_2,k_3} = \sum_{i=0}^{K-1} \left( A_{e,x}^{k,i,k} \Phi_{e,i}^{k,k_2,k_3} - B_{e,x}^{k,i,k} \Phi_{e}^{k_3,k_2,k_3} + C_{e,x}^{k,i,k} \Phi_{e}^{k,k_3,k_2} \right) \quad \text{Eq. 2.35}
\]

\[
F_{e,y}^{k_1,k,k_3} = \sum_{i=0}^{K-1} \left( A_{e,y}^{k,i,k} \Phi_{e}^{k_1,k,k_3} - B_{e,y}^{k,i,k} \Phi_{e}^{k_1,k,k_3} + C_{e,y}^{k,i,k} \Phi_{e}^{k,k,k_3} \right) \quad \text{Eq. 2.36}
\]

\[
F_{e,z}^{k_1,k_3,k} = \sum_{i=0}^{K-1} \left( A_{e,z}^{k,i,k} \Phi_{e}^{k_1,k_3,k} - B_{e,z}^{k,i,k} \Phi_{e}^{k_1,k_3,k} + C_{e,z}^{k,i,k} \Phi_{e}^{k,k_3,k_1} \right) \quad \text{Eq. 2.37}
\]

the \( A, B \) and \( C \) coefficients are given as

\[
A_{e,\alpha}^{k,i,k} = \frac{(-1)^k}{2K(K+1)} \sqrt{2k+1} \sqrt{2l+1} (K(K+1) - k(k+1))(K(K+1) - l(l+1))W_{e,\alpha}^- \quad \text{Eq. 2.38}
\]

\[
B_{e,\alpha}^{k,i,k} = B_{e,\alpha}^{k,i,k} = \frac{1}{K(K+1)} \left( \frac{D_{u,\alpha}}{\Delta \alpha_e} \left( 1 + (-1)^{k+l} \right)(K(K+1)-k(k+1))(l(l+1)) + \frac{1}{2} \left( K(K+1) - k(k+1) \right) \left( K(K+1) - l(l+1) \right)((-1)^{k+l}W_{e,\alpha}^- + W_{e,\alpha}^+) \right) \quad \text{if } k \geq l \quad \text{Eq. 2.39}
\]

\[
C_{e,\alpha}^{k,i,k} = \frac{(-1)^k}{2K(K+1)} \sqrt{2k+1} \sqrt{2l+1} (K(K+1) - k(k+1))(K(K+1) - l(l+1))W_{e,\alpha}^+ \quad \text{Eq. 2.40}
\]

where \( \alpha = x,y \) or \( z \); \( k=0...K-1 \) and \( l=0...K-1 \).
The $W$’s are MCFD coupling factors associated to each of the surfaces in order to take into account the discontinuities of the diffusion coefficients. The six coupling factors have the following expressions:

$$W_{e,x}^+ = W_{e,x}^- = \frac{2D_{x,e}D_{x,e1}}{(\Delta x_{e}D_{x,e1} + \Delta x_{e1}D_{x,e})}$$  Eq. 2.41

$$W_{e,x}^- = W_{e,x}^+ = \frac{2D_{x,e}D_{x,e2}}{(\Delta x_{e}D_{x,e2} + \Delta x_{e2}D_{x,e})}$$  Eq. 2.42

$$W_{e,y}^- = W_{e,y}^+ = \frac{2D_{y,e}D_{y,e3}}{(\Delta y_{e}D_{y,e3} + \Delta y_{e3}D_{y,e})}$$  Eq. 2.43

$$W_{e,y}^+ = W_{e,y}^- = \frac{2D_{y,e}D_{y,e4}}{(\Delta y_{e}D_{y,e4} + \Delta y_{e4}D_{y,e})}$$  Eq. 2.44

$$W_{e,z}^- = W_{e,z}^+ = \frac{2D_{z,e}D_{z,e5}}{(\Delta z_{e}D_{z,e5} + \Delta z_{e5}D_{z,e})}$$  Eq. 2.45

$$W_{e,z}^+ = W_{e,z}^- = \frac{2D_{z,e}D_{z,e6}}{(\Delta z_{e}D_{z,e6} + \Delta z_{e6}D_{z,e})}$$  Eq. 2.46

**Particular cases:**

- Zero flux boundary condition

$$W_{e,\alpha}^z = \frac{2D_{\alpha,e}}{\Delta \alpha_e}$$  Eq. 2.47

- Positive albedo boundary conditions

$$W_{e,\alpha}^z = \frac{2D_{\alpha,e}\left(1 - \beta_{\alpha}\left(\frac{1}{2}\right)\right)}{2K(K+1)D_{\alpha,e}\left(1 + \beta_{\alpha}\left(\frac{1}{2}\right)\right) + \Delta \alpha_e\left(1 - \beta_{\alpha}\left(\frac{1}{2}\right)\right)}$$  Eq. 2.48

Where + and – indicate surfaces at $e\pm 1/2$.

The leakage terms, eqs 2.35-7, can be plugged into eqs 2.22 to obtain the final system of equations. But the number of unknowns, Legendre coefficients, is large. We apply the “serendipity approximation” to reduce the number of unknowns and therefore the dimension of the system matrices.
2.3.3. Serendipity approximation

Till now, full tensorial expansions involving $K^3$ Legendre coefficients per element (node) have been used. The serendipity approximation, Hebert (1987), is based on the fact that successive Legendre coefficients of the transverse leakage can be computed with a numerical method of decreasing order while keeping the order of discretization constant. Consequently, the total number of unknowns is reduced, since a limited number of Legendre components is required to present the neutron flux over each element.

The 3D nodal collocation method with the serendipity approximation involves $K_i$ Legendre coefficients per element. These coefficients are elements of the set:

$$\{\Phi_{e}^{k_1,k_2,k_3} ; k_1 = 0...K-1; k_2 = 0...K-1-k_1; k_3 = 0...K-1-k_1-k_2\}$$

Eq. 2.49

and $K_i$ is

$$K_i = \frac{1}{6} K(K+1)(K+2)$$

Eq. 2.50

The serendipity approximation is then applied directly in the nodal collocation formalism by replacing equations 2.35-7 with the following ones:

$$F_{e,x}^{k_1,k_2,k_3} = \sum_{l=0}^{K-1-k_1-k_2} \left( A_{e,x}^{k_1,k_2,k_3,l} \Phi_{e}^{l,k_2,k_3} - B_{e,x}^{k_1,k_2,k_3,l} \Phi_{e}^{l,k_2,k_3} + C_{e,x}^{k_1,k_2,k_3,l} \Phi_{e}^{l,k_2,k_3} \right)$$

Eq. 2.51

$$F_{e,y}^{k_1,k_2,k_3} = \sum_{l=0}^{K-1-k_1-k_2} \left( A_{e,y}^{k_1,k_2,k_3,l} \Phi_{e}^{k_1,l,k_3} - B_{e,y}^{k_1,k_2,k_3,l} \Phi_{e}^{k_1,l,k_3} + C_{e,y}^{k_1,k_2,k_3,l} \Phi_{e}^{k_1,l,k_3} \right)$$

Eq. 2.52

$$F_{e,z}^{k_1,k_2,k_3} = \sum_{l=0}^{K-1-k_1-k_2} \left( A_{e,z}^{k_1,k_2,k_3,l} \Phi_{e}^{k_1,k_2,l} - B_{e,z}^{k_1,k_2,k_3,l} \Phi_{e}^{k_1,k_2,l} + C_{e,z}^{k_1,k_2,k_3,l} \Phi_{e}^{k_1,k_2,l} \right)$$

Eq. 2.53
2.4. Core statics

Starting with the time-space two-group energy formulation and eliminating the time derivatives and delayed neutrons, we obtain the statics equations which satisfy the criticality condition. That is an eigenvalue problem and its eigenfunctions correspond to the stationary flux shapes. In our case, from eqs. 2.3 and 2.4 we have:

\[
-\nabla (D_1 \nabla \psi_1) + (\Sigma_{a1} + \Sigma_{s12}) \psi_1 = \frac{1}{\lambda} \left( \nu_1 \Sigma_{f1} \psi_1 + \nu_2 \Sigma_{f2} \psi_2 \right)
\]

Eq. 2.54

\[
-\nabla (D_2 \nabla \psi_2) + \Sigma_{a2} \psi_2 - \Sigma_{s12} \psi_1 = 0
\]

Eq. 2.55

They can be written in matrix form as:

\[
\mathcal{L} \psi = \frac{1}{\lambda} \mathcal{M} \psi
\]

Eq. 2.56

Where the stationary flux shape vector \( \psi \) is,

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

Eq. 2.57

\( \mathcal{L} \) is the leakage operator

\[
\mathcal{L} = \begin{pmatrix}
-\nabla (D_1 \nabla) + \Sigma_{a1} + \Sigma_{s12} & 0 \\
-\Sigma_{s12} & -\nabla (D_2 \nabla) + \Sigma_{a2}
\end{pmatrix}
\]

Eq. 2.58

and \( \mathcal{M} \) the production operator

\[
\mathcal{M} = \begin{pmatrix}
(\nu \Sigma_f)_1 \\
0
\end{pmatrix}
\]

Eq. 2.59

Eq. 2.56 is known as the Lambda modes equation and it is assumed that there is an infinite set of positive eigenvalues, \( \lambda_m \), and their corresponding eigenfunctions, \( \psi_m \), satisfying the continuity and boundary conditions imposed to the neutronic flux in the reactor core. The biggest eigenvalue, \( \lambda_0 \), is associated with the effective multiplication factor of the reactor as \( K_{\text{eff}} = \lambda_0 \), and its corresponding eigenfunction, \( \psi_0 \), is called the fundamental mode and describes the steady-state neutron flux in the core.
Using the spatial discretization technique of section 2.3 we find as an approximation of eq. 2.56, an algebraic generalized eigenvalue problem of the form:

$$L_0 \Psi_m = \frac{1}{\lambda_m} M_0 \Psi_m$$  \hspace{1cm} \text{Eq. 2.60}$$

where $\Psi_m$ is the Legendre expansion coefficients vector, the $L_0$ and $M_0$ matrices are respectively the loss and production matrices obtained by introducing the group leakage terms, eqs 2.51-3), into eq. 2.56 and special ordering of the coefficients $A_{\alpha\alpha}$, $B_{\alpha\alpha}$, $C_{\alpha\alpha}$. The index $\alpha$ indicates the $(x,y,z)$ directions, as before.

$$L_0 = \begin{pmatrix} L_{11} & 0 \\ -L_{21} & L_{22} \end{pmatrix}$$  \hspace{1cm} \text{Eq. 2.61}$$

$$M_0 = \begin{pmatrix} M_{11} & M_{12} \\ 0 & 0 \end{pmatrix}$$  \hspace{1cm} \text{Eq. 2.62}$$

In other words, eq. 2.60 is the discretized form of eq. 2.56. Hence, $\Psi_m$ the eigenfunctions of 2.60 are the discretization of the eigenfunctions of eq. 2.56 $\psi_m$. Note, that if the core has a maximum number of nodes $N (=N_{\text{max}}K_{\text{max}}$; where $N_{\text{max}}$ is the number of the channels and $K_{\text{max}}$ is the number of nodes in each channel), then the dimension of the $\Psi_m$ will be $N_t=2\times K_t\times N$, where the total number of Legendre coefficients per element ($K_t$) is given in 2.50. In the same manner, the dimension of the loss ($L_0$) and production ($M_0$) matrices is $(N_t\times N_t)$.

There are many classical methods to find a few eigenvalues and their corresponding eigenfunctions of the problem 2.60. Almost, all methods require large CPU time. Lehoucq and Sorensen (1995) have developed a very powerful implementation of the Arnoldi process for finding eigenvalues called implicit re-started Arnoldi method (IRAM) which combines Arnoldi factorizations with implicit shifted QR (decomposition) mechanism. Verdú et al. (1999) investigated the performance of various methods. They conclude that the IRAM method is more efficient than classical methods to calculate the neutronic steady-state of a nuclear power reactor and its subcritical modes, such as the power iteration method or subspace iteration method. IRAM is included in the most important libraries of engineering programming languages like C, FORTRAN and MATLAB 6.5 (2002).

The core statics problem starts with reading the nuclear properties matrices (the fission, absorption and removal cross sections, diffusion coefficients and reflector positions) from the Distribution file, provided by the reactor monitoring system. Using sparse matrix techniques, the core loss ($L_0$) and production ($M_0$) matrices are constructed, based on nodal average nuclear properties and boundary conditions. Now, applying large sparse matrix eigenvalue routines (including IRAM ) of MATLAB on eqs 2.60, it is possible to calculate efficiently the eigenvalues (default value is 15) and their corresponding eigenfunctions. As mentioned before, the $\Psi_0$ corresponds to steady-state flux shape of the core and therefore power profile. There are no contributions of the higher harmonics $\Psi_m(m=1,\ldots,M)$ on power; however, they are necessary for the transient analysis, as we will explain later.
2.4.1. Generalized eigenvalue problem

The dimension of the generalized eigenvalue problem, eq. 2.60, can be reduced to half of the initial size \((N_t \times N_t)\). This is needed to improve the computation time and decrease the required virtual memory (RAM). Knowing the structure of the loss and production matrices, we can write the following two equations (using eq. 2.60) for the thermal and epithermal fluxes, Verdú et al. (1999):

\[
\Psi_{2m} = L_{22}^{-1} L_{21} \Psi_{1m} \quad \text{Eq. 2.63}
\]

\[
L_{11}^{-1} \left( M_{11} + M_{12} L_{22}^{-1} L_{21} \right) \Psi_{1m} = \lambda_m \Psi_{1m} \quad \text{Eq. 2.64}
\]

Equation 2.64 gives the dominant eigenvalues and the corresponding epithermal eigenfunctions. This eigenvalue problem cannot be solved in MATLAB environment for even the lowest order Legendre polynomial expansion, because of the memory needed for inverting matrices. Furthermore the direct methods have failed during such calculations. In order to remove these limitations and solve eqs 2.63 and 2.64, we have been using as the basic operation for the iterative eigenvalue calculation methods the product matrix-vector \((z = A \cdot x)\). This, combined with the reverse communication to the ARPAK library (for the application IRAM) allows the computation of the \(M\) (number of required harmonics) largest eigenvalues and epithermal eigenfunctions. This basic operation is needed to remove the memory limitation of the operating system. Let us define matrix \(A\) as follows:

\[
A = L_{11}^{-1} \left( M_{11} + M_{12} L_{22}^{-1} L_{21} \right) \quad \text{Eq. 2.65}
\]

then the algorithm for the basic operation is:

- first guess \(x=\Psi_m\)
- multiplication step: \(y = L_{21} \cdot x\)
- Solve linear system for \(w\): \(L_{22} \cdot w = y\)
- Sum of the products \(s\): \(s = M_{11}x + M_{12}w\)
- Solve linear system \(z\): \(L_{11}z = s\)
The linear systems have been solved by the application of the Stabilized Preconditioned BiConjugate Gradients Method (SPBCGM, our default). Alternatively, it is possible to choose either the Preconditioned Conjugate Gradients Method (PCGM) or the Preconditioned Generalized Minimum Residual Method (PGMRM).

Based on the above algorithm, the IRAM method produces different orders of Lanczos vectors (user defined) for the calculation of the eigenvalues of the matrix \( A \). The Lanczos are given as:

\[
\begin{align*}
\mathbf{z}_0 &= \mathbf{x} \\
\mathbf{z}_1 &= \mathbf{A} \cdot \mathbf{x} \\
\vdots \\
\mathbf{z}_M &= \mathbf{z}_{M-1} \cdot \mathbf{x} = \mathbf{A}^M \cdot \mathbf{x}
\end{align*}
\]

Finally, knowing the eigenfunctions of the epithermal group, the eigenfunctions of the thermal group are obtained from eq. 2.63.

### 2.4.2. Adjoint problem

A group adjoint function is proportional to the gain or loss in reactivity of a reactor due to insertion or removal of one neutron per second in the group at the point \( \mathbf{r}(x,y,z) \). In other words, it measures the importance of the point \( \mathbf{r} \) with respect to the reactivity introduced by an absorber or a fissile material at the point \( \mathbf{r} \), and for this reason, the adjoint function is also called the importance function. An important characteristic is that the adjoint functions are orthogonal to the flux eigenfunctions, Lamarsh (1966).

The space-time dependent diffusion equations are non-self-adjoint. This is made evident by the non symmetric matrices of the algebraic eigenvalue problem 2.60. Consequently, there exists an adjoint problem having the same eigenvalues, Doring and Kalkkuhl (1993):

\[
\mathbf{L}^* \mathbf{\Psi}_m = \frac{1}{\lambda_m^*} \mathbf{M}^* \mathbf{\Psi}_m
\]

where the adjoint loss (\( \mathbf{L}^* \)) and production (\( \mathbf{M}^* \)) matrices are defined as:

\[
\mathbf{L}^* = \begin{pmatrix}
I_{11}^T & -I_{21}^T \\
0 & L_{22}^T
\end{pmatrix}
\]
and

\[
\mathbf{M}' = \begin{pmatrix}
\mathbf{M}_{11}^T & 0 \\
\mathbf{M}_{12}^T & 0
\end{pmatrix}
\]

Eq. 2.68

and the adjoint flux \( \Psi^* \) as

\[
\Psi_m^* = \begin{pmatrix}
\Psi_{1m}^* \\
\Psi_{2m}^*
\end{pmatrix}
\]

Eq. 2.69

The block structure of the discretized adjoint problem 2.66, allows to write the following two equations:

\[
\Psi_{2m}^* = \frac{1}{\lambda_m} \left( \mathbf{L}_{22}^T \right)^{-1} \mathbf{M}_{12} \Psi_{1m}^*
\]

Eq. 2.70

\[
\left( \mathbf{L}_{11}^T \right)^{-1} \left( \mathbf{M}_{11}^T + \mathbf{L}_{21} \left( \mathbf{L}_{22}^T \right)^{-1} \mathbf{M}_{12} \right) \Psi_{1m}^* = \lambda_m \Psi_{1m}^*
\]

Eq. 2.71

On the other hand, from the transpose problem of eq. 2.60, we have

\[
\left( \mathbf{M}_{11}^T + \mathbf{L}_{21} \left( \mathbf{L}_{22}^T \right)^{-1} \mathbf{M}_{12} \right) \left( \mathbf{L}_{11}^T \right)^{-1} \Psi_{1m}^T = \lambda_m \Psi_{1m}^T
\]

Eq. 2.72

The relation between the adjoint (eq. 2.71) and transpose (eq. 2.72) fluxes can be obtained by comparison of the two equations:

\[
\Psi_{1m}^* = \left( \mathbf{L}_{11}^T \right)^{-1} \Psi_{1m}^T
\]

Eq. 2.73

Since the transpose eigenvectors \( \Psi^T \) (found by eq. 2.72) and corresponding eigenvectors of the epithermal group \( \Psi \) form a complete biorthogonal system, they can be calculated from an orthonormal base (say \( \mathbf{X} \)) of \( \Psi \).

In order to obtain the transpose epithermal eigenmodes, we assume that

\[
\Psi_{1m}^T = \mathbf{X} \{ K, N, M \} \cdot \mathbf{U} \{ M, M \}
\]

Eq. 2.74

where the matrix \( \mathbf{U} \) has to be determined. The transpose problem related to the epithermal flux is given as:

\[
\mathbf{A}^T \Psi_{1m}^T = \Psi_{1m}^T \Lambda
\]

Eq. 2.75

where \( \Lambda \{ M, M \} \) is the matrix of eigenvalues (on the diagonal).
Inserting eq. 2.74 into eq. 2.75 and multiplying both sides by $X^T$

$$X^T A^T X \cdot U = U \cdot \Lambda$$  \hspace{1cm} \text{Eq. 2.76}

Hence, the matrix $U$ corresponds to the eigenvectors of the eigenvalue problem (eq. 2.76) with much smaller size \{M,M\} than that of the transpose problem (eq. 2.72). Knowing the $U$ and $X$ matrices, the transpose eigenvectors can be calculated from equation (2.74). Finally, equations 2.73 and 2.70 give the adjoint eigenvectors of the epithermal and thermal groups.

The adjoint eigenmodes calculated earlier (with equations 2.73 and 2.70) have to be brought to the required accuracy. For this task, we use the following steps until convergence.

1) compute the new adjoint epithermal eigenmode ($m$) based on the eigenmodes of the previous step

$$\Psi_{1m}^* = \left(L_{v11}^T\right)^{-1} \left(M_{11}^T \Psi_{1m}^* / \Lambda_{nn} + L_{21}^T \Psi_{2m}^* \right)$$  \hspace{1cm} \text{Eq. 2.77}

2) compute new adjoint thermal eigenmode ($m$)

$$\Psi_{2m}^* = \left(L_{22}^T\right)^{-1} M_{12}^T \Psi_{1m}^* / \Lambda_{nn}$$  \hspace{1cm} \text{Eq. 2.78}

3) check accuracy; and if precision is not reached go to step 1.

4) do the steps 1-3 for the all modes.
2.5. Core dynamics

To study the stability of a BWR core, we have chosen the “time-synthesis modal expansion technique“, Stacey (1969), that allows separation of higher-harmonics effects which may cause out-of-phase oscillations of the reactor power.

We obtain the modal neutronics equations by expanding the instantaneous flux in terms of its $M+1$ harmonics (see eq. 2.86) in the set of space-time dependent group diffusion equations. Then, the equations will be discretized (using NCM), perturbed and Laplace transformed, resulting our core dynamics equations written in term of neutronics transfer functions. Finally, we will give detailed descriptions of the reactivity feedbacks used in these equations. In the next paragraphs we will use the following indices:

- $i$ channel index $(i=1...N_{\text{max}})$
- $k$ node index in a channel $(i=1...K_{\text{max}})$
- $l$ energy or precursor concentration group index
- $m, n$ harmonic mode indices

2.5.1. Space-time dependent diffusion equations

The basic neutronics problem for any transient analysis is the solution of the balance equation for the flux and of the delayed neutron precursor concentrations. Under the assumptions in section 2.2, the two-group diffusion approximation (eqs. 2.3-5) is used. They are given in operator notation, Ginestar (2002) and Stacey (2001), as :

\[
\mathbf{v}^{-1}\phi = -\mathbf{L}\phi + (1 - \beta)\mathbf{M}\phi + \sum_{i=1}^{6} \lambda_i \chi_i C_i
\]

Eq. 2.79

and

\[
\chi \dot{C}_i = \beta_i \mathbf{M}\phi - \lambda_i \chi C_i
\]

Eq. 2.80

Where $\phi$ is the time-dependent flux vector,

\[
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]

Eq. 2.81

$\chi$ the delayed neutron spectrum vector,

\[
\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Eq. 2.82
The $\mathbf{v}^{-1}$ is the matrix of inverse neutron group velocities.

$$
\mathbf{v}^{-1} = \begin{pmatrix}
\frac{1}{v_1} & 0 \\
0 & \frac{1}{v_2}
\end{pmatrix}
$$

Eq. 2.83

Applying the nodal collocation technique of section 2.3, we obtain the discretized approximation of equations 2.79 and 2.80 as follows

$$
\mathbf{v}^{-1}\Phi(t) = -\mathbf{L}(t)\Phi(t) + (1-\beta)\mathbf{M}(t)\Phi(t) + \sum_{i=1}^{6}\lambda_i \chi \mathbf{C}_i(t)
$$

Eq. 2.84

$$
\chi \dot{\mathbf{C}}_i(t) = \beta_i \mathbf{M}(t)\Phi(t) - \lambda_i \chi \mathbf{C}_i(t)
$$

Eq. 2.85

The $\Phi$ and $\mathbf{C}_i$ components are the corresponding coefficients of the Legendre polynomial expansion of the flux ($\phi$) and the $l$-th precursor concentration ($C_l$) in each node, at a given time.

### 2.5.2. Modal expansion and first-order perturbation theory application

For studying the dynamics of harmonics modes, we use the “time-synthesis modal expansion technique”, Stacey (1969). The idea is to choose a set of shapes $\Psi$ from which any instantaneous flux distribution $\Phi$ that occurs during the transient may be constructed by a linear combination. In general, the eigenfunctions $\Psi_m$ of the core-statics eigenvalue problem are used for this task. We will, also, adopt the shapes $\Psi_m$ computed by the application of the nodal collocation technique on the core statics equations, to synthesize the time-dependent Legendre coefficients of flux.

$$
\delta \Phi(t) = \sum_{m=0}^{M} \Psi_m n_m(t)
$$

Eq. 2.86

Where the $n_m$ is time-dependent amplitude of the $m$-th mode. The $M+1$ mode amplitudes are obtained from the solution of the $M+1$ modal core dynamics equations 2.95 coupled with 6 precursor concentration equations 2.96; these are given below.

Using eq. 2.86, the modal expansion for the space and time-dependent neutron flux can be written the following form, Hashimoto (1996).

$$
\Phi(t) = N_0 \Psi_0 + \sum_{m=0}^{M} \Psi_m n_m(t)
$$

Eq. 2.87

where $N_0$ is the steady-state flux amplitude.

Here it is assumed that in stationary conditions the flux consists of only the fundamental mode. This assumption is reasonable because the higher modes decay rapidly. The absolute
magnitude of the flux $N_0$ is determined by the operating power of the reactor and can not be found simply by solving diffusion equations.

The small-perturbation form of eqs 2.84 and 2.85 requires, also, the decomposition of the production and loss operators into steady-state and time-dependent components:

\[ L = L_0 + \delta L \]  
\[ M = M_0 + \delta M \]  
Eq. 2.88  
Eq. 2.89

Substituting eqs 2.87-9 into eq. 2.84 and eq. 2.85, rearranging using eq. 2.60, multiplying the left side of the resulting equations by the adjoint flux and integrating over the whole space and using the biorthogonality relationships:

\[ \langle \Psi'_n, M_0 \Psi_m \rangle = F_n \delta_{nm} \]  
\[ \langle \Psi'_n, M_0 \Psi_m \rangle = Y_n \delta_{nm} \]  
Eq. 2.90  
Eq. 2.91

where $\delta_{nm}$ is the Kronecker sign, we obtain the $n$-th mode perturbation:

\[
\langle \Psi'_n, \sigma^{-1} \Psi_a \rangle \frac{\partial n_n}{\partial t} = -\frac{1}{\lambda_n} \langle \Psi'_n, M_0 \Psi_n \rangle n_n + (1 - \beta) \langle \Psi'_n, M_0 \Psi_n \rangle n_n - \langle \Psi'_n, \delta L \Psi_0 \rangle N_0 \\
+ (1 - \beta) \langle \Psi'_n, \delta M \Psi_0 \rangle N_0 + \sum_{i=1}^{\delta} \lambda_i \langle \Psi'_n, \sigma C_i \rangle + \sum_{m=0}^{M} \langle \Psi'_n, \delta L \Psi_m \rangle n_m + (1 - \beta) \sum_{m=0}^{M} \langle \Psi'_n, \delta M \Psi_n \rangle n_m 
\]

Eq. 2.92

and the precursor concentration perturbation

\[
\frac{\partial}{\partial t} \langle \Psi'_n, \sigma C_i \rangle = \beta_i \langle \Psi'_n, M_0 \Psi_n \rangle n_n + \beta_i \langle \Psi'_n, \delta M \Psi_0 \rangle N_0 + \sum_{m=0}^{M} \beta_i \langle \Psi'_n, \delta M \Psi_m \rangle n_m - \lambda_i \langle \Psi'_n, \sigma C_i \rangle 
\]

Eq. 2.93

Applying first-order perturbation theory, Neuhold and Ott (1985), and dividing both sides of eqs 2.92 and 2.93 by:

\[ F_n = \langle \Psi'_n, M_0 \Psi_n \rangle \]  
Eq. 2.94
the final space-time-modal core dynamics equations become:

\[ \Lambda_n \frac{\partial n_n}{\partial t} = (\rho_n^* - \beta) n_n + \rho_{n,0} N_0 - \rho_{n,0}^* N_0 + \sum_{l=1}^{6} \lambda_n C_{n,l}, \quad n=0...M \]  

\[ \frac{\partial C_{n,l}}{\partial t} = \beta_n n_n + \rho_{n,0}^* N_0 - \lambda_n C_{n,l}, \quad n=0...M, \quad l=1...6 \]  

where

\[ \Lambda_n = \frac{Y_n}{F_n} \]  

Eq. 2.95

Eq. 2.96

Eq. 2.97

Modal reactivity is interpreted as the excitation reactivity of the \( n \)-th mode, which is introduced by a net change in the \( 0 \)-th mode (global mode) reaction rate, Hashimoto (1993):

\[ \rho_{n,0} = \frac{\langle \Psi_n^*, (\delta M - \delta L) \Psi_0 \rangle}{F_n} \]  

Eq. 2.98

The total delayed-neutron reactivity is

\[ \rho_{n,0}^d = \beta \frac{\langle \Psi_n^*, (\delta M \Psi_0) \rangle}{F_n} \]  

Eq. 2.99

with the \( l \)-th group delayed neutron reactivity contribution,

\[ \rho_{n,0}^d_l = \beta_l \frac{\langle \Psi_n^*, (\delta M \Psi_0) \rangle}{F_n} \]  

Eq. 2.100

the modal static reactivity,

\[ \rho_s = 1 - \frac{1}{\lambda_n} \]  

Eq. 2.101

the delayed-neutron concentration,

\[ C_{n,l} = \frac{\langle \Psi_n^*, X C_l \rangle}{F_n} \]  

Eq. 2.102

where \( \langle a, b \rangle \) denotes a scalar product.
2.5.3. Frequency dependent modal equations

Since we are interested in the frequency dependent equations, eqs 2.95 and eqs 2.96 are Laplace transformed, resulting in:

\[ s\Lambda_m \hat{n}_m = (\rho_m - \beta)\hat{n}_m + N_0 \hat{\rho}^d_{m,0} - N_0 \hat{\rho}^d_{m,0} + \sum_{l=1}^{6} \lambda_i \hat{C}_{m,l} \quad m = 0, \ldots, M \quad \text{Eq. 2.103} \]

\[ s\hat{C}_{m,l} = \beta_l \hat{n}_m + N_0 \hat{\rho}^d_{m,0} - \lambda_i \hat{C}_{m,l} \quad m = 0, \ldots, M \quad , \quad l = 1, \ldots, 6 \quad \text{Eq. 2.104} \]

Equations 2.104 are solved for the delayed neutron precursor concentration:

\[ \hat{C}_{m,l} = \frac{\beta_i}{s + \lambda_i} \hat{n}_m + \frac{N_0}{s + \lambda_i} \hat{\rho}^d_{m,0} \quad \text{Eq. 2.105} \]

Substituting eqs 2.105 into eqs 2.103 and rearranging, we find the higher-harmonic amplitude perturbations.

\[ \hat{n}_m = N_0 \frac{\hat{\rho}^d_{m,0} - \hat{\rho}^d_{m,0} + \sum_{i=1}^{6} \frac{\lambda_i}{s + \lambda_i} \hat{\rho}^d_{m,0}}{s\Lambda_m + (\beta - \rho_m) \sum_{i=1}^{6} \frac{\lambda_i}{s + \lambda_i} \beta_i} \quad \text{Eq. 2.106} \]

since the total delayed neutron fraction is

\[ \beta = \sum_{i}^{6} \beta_i \quad \text{Eq. 2.107} \]

eq. 2.106 can be simplified as follows

\[ \hat{n}_m = N_0 \frac{\hat{\rho}^d_{m,0} - \hat{\rho}^d_{m,0} + \sum_{i=1}^{6} \frac{s \hat{\rho}^d_{m,0}}{s + \lambda_i}}{s\Lambda_m - \rho_m + \sum_{i=1}^{6} \frac{s \beta_i}{s + \lambda_i}} \quad \text{Eq. 2.108} \]

The modal frequency-dependent eqs 2.108, later will be used to calculate the core dynamics transfer functions which are needed for the stability analysis.
2.6. Nodal volumetric power

The total nodal power perturbation is written in terms of all the flux harmonics and fission cross section perturbations:

\[ \delta \hat{P}_{i,k} = E \left( \sum_{m=0}^{M} \hat{n}_m \sum_{l=1}^{2} \psi_{i,k,l,m} + N_0 \sum_{l=1}^{2} \delta \hat{\psi}_{i,k,l,m} \right) \]  

Eq. 2.109

Where \( E \) is the conversion factor of fission rate to power-density and \( l \) denotes the neutron energy group. The second term in the right hand side has to be expanded in terms of local void fraction and fuel temperature perturbations. Since we can state that the fission cross section perturbation is:

\[ \psi_{i,k} = \sum_{l=1}^{2} \delta \hat{\psi}_{i,k,l} \]  

Eq. 2.110

and the modal-nodal component of the power perturbation due to the \( m \)-th harmonic amplitude in the \( i \)-th channel and \( k \)-th node

\[ \delta \hat{P}_{i,k,m} = E \sum_{l=1}^{2} \psi_{i,k,l,m} \]  

Eq. 2.111

therefore, the total nodal power perturbation is the superposition of all modal power components and the contribution of the void and fuel temperature perturbations:

\[ \delta \hat{P}_{i,k} = \sum_{m=0}^{M} \delta \hat{P}_{i,k,m} + EN_0 \sum_{l=1}^{2} \left( \frac{\partial \psi_{i,k,l,m}}{\partial \alpha} \right) \psi_{i,k,l,m} + EN_0 \sum_{l=1}^{2} \left( \frac{\partial \psi_{i,k,l,m}}{\partial T_f} \right) \psi_{i,k,l,m} \]  

Eq. 2.112

This total nodal power has to be subdivided into the direct moderator heating (DMH) component \( \delta q_p^* \) and fuel power. For this reason we define the fuel power fraction \( K_p \) (where \( 0 < K_p < 1 \)) so that:

\[ \delta q_p^* = K_p \delta \hat{P}_{i,k} \]  

Eq. 2.113

\[ \delta q_f^* = (1 - K_p) \delta \hat{P}_{i,k} \]  

Eq. 2.114
In the same way we can write the steady-state volumetric power of the $i$-th channel and $k$-th node as:

$$P_{i,k} = EN_0 \frac{2}{\sum_{i=1}^{N_c} \sum_{k=1}^{K_{eq}} \psi_{i,k}}$$  \hspace{1cm} \text{Eq. 2.115}$$

The reactor rated power will be

$$P = \sum_{i=1}^{N_c} \sum_{k=1}^{K_{eq}} P_{i,k}$$  \hspace{1cm} \text{Eq. 2.116}$$

### 2.7. Modal reactivity feedback

In this section, we divide the modal reactivity (eq. 2.98) into two parts, the void ($V$) and Doppler ($D$) reactivities. They are produced by the changes in the void fraction ($\alpha$) and fuel temperature ($T_f$) respectively. Since these changes are local phenomena and vary from one element to the other, they affect the modal reactivity differently. Thus we can state that for a given discretization of the core in nodes, the modal reactivity is the superposition of all void and temperature contributions. Based on these considerations we can write, Hashimoto (1993):

$$\rho_{m,0}^V = \left\langle \Psi_m, \left( \delta \hat{M}^V - \delta \hat{L}^V \right) \Psi_0 \right\rangle F_m$$  \hspace{1cm} \text{Eq. 2.117}$$

$$\rho_{m,0}^D = \left\langle \Psi_m, \left( \delta \hat{M}^D - \delta \hat{L}^D \right) \Psi_0 \right\rangle F_m$$  \hspace{1cm} \text{Eq. 2.118}$$

$$\rho_{m,0} = \rho_{m,0}^V + \rho_{m,0}^D$$  \hspace{1cm} \text{Eq. 2.119}$$

where the production and loss matrices are defined as:

$$\delta \hat{M}^K = \frac{\partial \hat{M}}{\partial x} \delta \hat{x}$$  \hspace{1cm} \text{Eq. 2.120}$$

$$\delta \hat{L}^K = \frac{\partial \hat{L}}{\partial x} \delta \hat{x}$$  \hspace{1cm} \text{Eq. 2.121}$$

and $K=V,D$ and $x=\alpha,T_f$.  

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2.8. Neutronics transfer functions

In order to study core dynamics we have to define the neutronics transfer functions. Starting from eq. 2.108 and recalling that the reactivity may be subdivided into two terms (eq. 2.119), we can rewrite it in the following form.

\[ \hat{n}_m = G_m \left( \hat{\rho}_{ex,m} + \left( \hat{\rho}_{m,0}^d - \sum_{l=1}^{6} \frac{S}{s + \lambda_l} \hat{\rho}_{m,0}^{d,y} \right) + \left( \hat{\rho}_{m,0}^D - \sum_{l=1}^{6} \frac{S}{s + \lambda_l} \hat{\rho}_{m,0}^{D,y} \right) \right) \]  

Eq. 2.122

Where \( \hat{\rho}_{ex,m} \) is an arbitrary external reactivity perturbation and \( G_m \) is the open-loop reactor transfer function of the \( m \)-th mode:

\[ G_m = \frac{N_0}{s \Lambda_m - \rho'_0 + \sum_{l=1}^{6} \frac{s \beta_l}{s + \lambda_l}} \]  

Eq. 2.123

Now let’s expand the second (void reactivity) and the third (Doppler reactivity) terms in equation 2.122, using eq. 2.117 and eq. 2.118:

\[ \hat{\rho}_{m,0}^V - \sum_{l=1}^{6} \frac{S}{s + \lambda_l} \hat{\rho}_{m,0}^{d,y} = \frac{1 - \sum_{l=1}^{6} \frac{s \beta_l}{s + \lambda_l}}{F_m} \left\langle \Psi_m^*, \delta \hat{M}^V \Psi_0 \right\rangle - \frac{1}{F_m} \left\langle \Psi_m^*, \delta \hat{L}^V \Psi_0 \right\rangle \]  

Eq. 2.124

In the same manner, we have:

\[ \hat{\rho}_{m,0}^D - \sum_{l=1}^{6} \frac{S}{s + \lambda_l} \hat{\rho}_{m,0}^{D,y} = \frac{1 - \sum_{l=1}^{6} \frac{s \beta_l}{s + \lambda_l}}{F_m} \left\langle \Psi_m^*, \delta \hat{M}^D \Psi_0 \right\rangle - \frac{1}{F_m} \left\langle \Psi_m^*, \delta \hat{L}^D \Psi_0 \right\rangle \]  

Eq. 2.125

Finally, inserting eq. 2.124 and eq. 2.125 into eq. 2.122 and substituting loss and production matrices with their definition, we obtain a new version of the modal flux amplitude (eq. 2.122).

\[ \hat{n}_m = G_m \left( \hat{\rho}_{ex} + \sum_{i,j}^{N_g} \sum_{k=1}^{K_g} r_{V,i,j,k,m} \delta \hat{a}_{i,k} + \sum_{i,j}^{N_g} \sum_{k=1}^{K_g} r_{D,i,j,k,m} \delta \hat{f}_{i,j,k} \right) \]  

Eq. 2.126

The \( r_{V,i,j,k,m} \) and \( r_{D,i,j,k,m} \) may be considered as the contribution of the local changes (nodal) of the void and the fuel temperature to the reactivity. They are defined as follows:
\[ r_{V,i,k,m} = \frac{1}{F_m} \sum_{l=1}^{2} \psi^{*T}_{i,k,l,m} \left( 1 - \sum_{l=1}^{6} d_l \beta_l \right) \left( \frac{\partial M}{\partial \alpha} - \frac{\partial L}{\partial \alpha} \right)_{i,k,l} \psi_{i,k,l,0} \quad \text{Eq. 2.127} \]

\[ r_{D,i,k,m} = \frac{1}{F_m} \sum_{l=1}^{2} \psi^{*T}_{i,k,l,m} \left( 1 - \sum_{l=1}^{6} d_l \beta_l \right) \left( \frac{\partial M}{\partial T_f} - \frac{\partial L}{\partial T_f} \right)_{i,k,l} \psi_{i,k,l,0} \quad \text{Eq. 2.128} \]

where \( d_l \), \( l=1 \ldots 6 \) are delayed neutron coefficients given as:

\[ d_l = \frac{s}{s + \lambda_l} \]

The neutronics transfer functions \( (G_m, r_{V,i,k,m}, r_{D,i,k,m}) \) defined previously will be combined with core thermal-hydraulics transfer functions to obtain core transfer functions, as shown in the stability chapter.
References:


7. Lamarsh J. R.(1966), Introduction to nuclear reactor theory, Addison-Wesley publishing company, INC.


Chapter 3

Fuel dynamics
3.1. Introduction

The temporal variation of the temperature distribution in the fuel (referred to as fuel dynamics here) depends on the perturbations of the hydrodynamic parameters in the adjoining coolant channel and on the variation of the local volumetric power. In turn, the fuel temperature perturbations affect the heat flux from the fuel rod and the power (via the Doppler effect). The heat flux perturbations affect the hydrodynamic parameters of the coolant, in particular the enthalpy and the void distribution; which influence again in turn the power dynamics via the void-reactivity feedback. Since this coupled problem is central to the stability analysis of a BWR, the fuel rod dynamics has to be treated accurately.

Figure 3.1 shows schematically the fuel rod in a subchannel of the fuel assembly. The power generated in the fuel pellets is transmitted to the surrounding coolant through the gap and the cladding. The driving forces for the heat fluxes inside and at the surface of the fuel rod are the corresponding temperature differences at these locations.

In order to obtain the analytical solution of the governing conduction equation (in cylindrical coordinates \(r, \theta, z\))

\[
\frac{c_p \rho}{\partial t} \frac{\partial T(r, \theta, z, t)}{\partial t} = k \nabla^2 T(r, \theta, z, t) + q''(t)
\]

Eq. 3.1

where the scalar Laplacian operator is

\[
\nabla^2 = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}
\]

Eq. 3.2

for a generic axial fuel segment in a fuel rod, we assume that:

- the thermal properties \((c_p, k, \rho)\) of the fuel and of the cladding and the gap heat transfer coefficient are constant,
- axial heat conduction is negligible,
- heat generation and heat conduction are circumferentially uniform \((\theta\) independent),
- heat generation in the cladding is negligible,
- the heat capacity in the gap is negligible.

The Laplace transform of eq. 3.1, written in terms of a perturbation around a steady state is:

\[
\nabla^2 \delta T(r, \theta, z, s) - \frac{s}{\alpha} \delta T(r, \theta, z, s) + \frac{1}{k} \delta q''(s) = 0
\]

Eq. 3.3

In this chapter, first, we solve eq. 3.1 analytically for a fuel pin under stationary conditions. Then, we give the space-frequency dependent solution of eq. 3.3 for the fuel and cladding.
3.2. Steady state

Using the assumptions made in the previous section, the temperature profile is sought by expanding the Laplacian operator ($\nabla^2$) in the conduction eq. 3.1 for the fuel and the cladding separately, and applying the appropriate boundary conditions.

**Conduction equation in the fuel**

$$\frac{d^2 T_f}{dr^2} + \frac{1}{r} \frac{dT_f}{dr} + \frac{1}{\bar{k}_f} q'' = 0$$

Eq. 3.4

where $\bar{k}_f$ is the average fuel conductivity. The boundary conditions at the center of the pellet are

$$\left. \frac{dT_f}{dr} \right|_{r=0} = 0$$

Eq. 3.5

$$T_f (r=0) = T_{fc}$$

Eq. 3.6

**Conduction equation in the cladding**

$$\frac{d^2 T_c}{dr^2} + \frac{1}{r} \frac{dT_c}{dr} = 0$$

Eq. 3.7
Boundary conditions at the inner \((i)\) and outer \((o)\) surfaces

\[
T_i (r = r_i) = T_{ci} \quad \text{Eq. 3.8}
\]

\[
T_o (r = r_o) = T_{co} \quad \text{Eq. 3.9}
\]

Finally solving differential eqs 3.4 and 3.7, we find the temperature profiles in the fuel and cladding:

\[
T_f (r) = \frac{q^m}{4k_f} \left( r_o^2 - r^2 \right) + T_{fs} \quad \text{Eq. 3.10}
\]

where the fuel average conductivity is given as

\[
\overline{k}_f = \frac{1}{T_{fs} - T_{fs}} \int_{T_{fs}}^{T_{fs}} k_f (T) dT \quad \text{Eq. 3.11}
\]

the temperature profile in the cladding

\[
T_c (r) = \frac{1}{\ln(r_i) - \ln(r_o)} \left( T_{co} \ln(r_o) - T_{ci} \ln(r_i) + (T_{co} - T_{ci}) \ln(r_i) \right) \quad \text{Eq. 3.12}
\]

The constant gap heat transfer coefficient links the outer fuel temperature to the inner cladding temperature:

\[
q^s = H_c (T_{fs} - T_{ci}) \quad \text{Eq. 3.13}
\]

where \(q^s\) is the heat flux at the inner surface of the cladding.
3.3 Fuel rod dynamics

Considering the assumptions made in the introduction, the perturbation of the temperature in a generic segment of the fuel pin (see figure 3.1) can be described by two ordinary differential equations (ODEs), one for the fuel and the other for the cladding, written here in the frequency domain after Laplace transformation of the corresponding time-dependent conduction equations:

\[
\frac{d^2 \delta T_f(r,s)}{dr^2} + \frac{1}{r} \frac{d \delta T_f(r,s)}{dr} - \frac{s}{\alpha_f} \delta T_f(r,s) + \frac{1}{k_f} \delta q''_f(s) = 0 \quad \text{Eq. 3.14}
\]

\[
\frac{d^2 \delta T_c(r,s)}{dr^2} + \frac{1}{r} \frac{d \delta T_c(r,s)}{dr} - \frac{s}{\alpha_c} \delta T_c(r,s) = 0 \quad \text{Eq. 3.15}
\]

Eq. 3.14 and eq. 3.15 are Bessel differential equations. Their general solutions are linear combinations of the modified Bessel functions of the first ($I_0$) and second kind ($K_0$) of zero order, respectively:

\[
\delta T_f(r,s) = \frac{\delta q''_f(s)}{k_f \alpha_f} + C_{i,f} I_0(\varepsilon_f r) + C_{c,f} K_0(\varepsilon_f r) \quad \text{Eq. 3.16}
\]

\[
\delta T_c(r,s) = C_{i,c} I_0(\varepsilon_c r) + C_{c,c} K_0(\varepsilon_c r) \quad \text{Eq. 3.17}
\]

where \(\varepsilon_f = \sqrt{\frac{s}{\alpha_f}}\), \(\varepsilon_c = \sqrt{\frac{s}{\alpha_c}}\) and \(\alpha = \frac{k}{c / \rho}\) is the thermal diffusivity of the material. The two ODEs above and the four following boundary conditions, two for each equation, constitute an initial value problem (IVP).

\[
-k_f \frac{d \delta T_f(r,s)}{dr} \bigg|_{r=r_i} = \frac{r_i}{r_o} \delta q''_f(s) \quad \text{Eq.3.18}
\]

\[
-k_f \frac{d \delta T_f(r,s)}{dr} \bigg|_{r=0} = 0 \quad \text{Eq. 3.19}
\]

\[
-k_c \frac{d \delta T_c(r,s)}{dr} \bigg|_{r=r_i} = \delta q''_c(s) = H_g(\delta T_{fs} - \delta T_{ci}) \quad \text{Eq. 3.20}
\]

\[
-k_c \frac{d \delta T_c(r,s)}{dr} \bigg|_{r=r_o} = \delta q''_c(s) \quad \text{Eq. 3.21}
\]

where \(\delta q''_o\) is the heat flux perturbation at the outer surface of the cladding (the perturbation of the heat flux to the coolant).
The particular solution of the IVP is given as follows; Podowski et al. (1983):

\[
\delta T_{rf}(r_s,s) = \frac{1}{k_f \varepsilon_f} \delta q''_f(s) - \frac{r_f I_0(\varepsilon_f r_s)}{r_s k_f \varepsilon_f I_1(\varepsilon_f r_s)} \delta q''_s(s) \tag{Eq. 3.22}
\]

\[
\delta T_{co}(r_s,s) = Z_1(s) \delta q''_s(s) + Z_2(s) \delta q''_f(s) \tag{Eq. 3.23}
\]

where the coefficients \(C_{1,f}(s), C_{2,f}(s), C_{1,c}(s), C_{2,c}(s), Z_1(s)\) and \(Z_2(s)\) have been obtained using a MATLAB procedure (symbolic toolbox, 2002), written purposely for the IVP, by us:

\[
C_{1,f} = \frac{-r_f \delta q''_s}{r_s k_f \varepsilon_f I_1(\varepsilon_f r_s)} \tag{Eq. 3.24}
\]

\[
C_{2,f} = 0 \tag{Eq. 3.25}
\]

\[
C_{1,c} = -\frac{\delta q''_s K_1(\varepsilon_c r_s) - \delta q''_s K_1(\varepsilon_c r_f)}{I_1(\varepsilon_c r_s) K_1(\varepsilon_c r_o) - I_1(\varepsilon_c r_o) K_1(\varepsilon_c r_f)} \frac{\varepsilon_c k_c}{2} \tag{Eq. 3.26}
\]

\[
C_{2,c} = \frac{\delta q''_s I_1(\varepsilon_c r_s) - \delta q''_s I(\varepsilon_c r_o)}{I_1(\varepsilon_c r_s) K_1(\varepsilon_c r_o) - I_1(\varepsilon_c r_o) K_1(\varepsilon_c r_f)} \frac{\varepsilon_c k_c}{2} \tag{Eq. 3.27}
\]

Since the \(Z_1(s)\) and \(Z_2(s)\) are very long expressions, we have written them in compact form using the following parameters:

\[
Z_1 = \frac{H_g r_s N_1 I_1(\varepsilon_f r_s) N_2}{k_c \varepsilon_c \varepsilon_f D_8} + \frac{N_3}{k_c D_8 \varepsilon_c} \tag{Eq. 3.28}
\]

\[
Z_2 = \frac{H_g r_s N_1 I_1(\varepsilon_f r_s) N_4}{\varepsilon_c \varepsilon_f D_8} \tag{Eq. 3.29}
\]

with

\[
N_1 = I_0(\varepsilon_c r_o) K_1(\varepsilon_c r_o) + K_0(\varepsilon_c r_o) I_1(\varepsilon_c r_o) \tag{Eq. 3.30}
\]

\[
N_2 = k_f \varepsilon_f^2 [K_0(\varepsilon_c r_s) I_1(\varepsilon_c r_f) + I_0(\varepsilon_c r_f) K_1(\varepsilon_c r_f)] \tag{Eq. 3.31}
\]

\[
N_3 = -k_c [I_0(\varepsilon_c r_o) K_1(\varepsilon_c r_f) + K_0(\varepsilon_c r_o) I_1(\varepsilon_c r_f)] \tag{Eq. 3.32}
\]
\[ N_4 = \varepsilon_r D_1 \]  

\[ D_1 = -I_1(\varepsilon_r r_1)K_1(\varepsilon_r r_o) + I_1(\varepsilon_r r_o)K_1(\varepsilon_r r_1) \]  

\[ D_2 = -k_f \varepsilon_r \varepsilon_j r_1 I_1(\varepsilon_f r_1)I_1(\varepsilon_r r_1)K_1(\varepsilon_r r_o) \]  

\[ D_3 = k_f k_r \varepsilon_r \varepsilon_j r_1 I_1(\varepsilon_f r_1)I_1(\varepsilon_r r_o)K_1(\varepsilon_r r_1) \]  

\[ D_4 = -H_g k_r \varepsilon_r r I_0(\varepsilon_f r_1)I_1(\varepsilon_r r_1)K_1(\varepsilon_r r_o) \]  

\[ D_5 = H_g k_r \varepsilon_r r I_0(\varepsilon_f r_1)I_1(\varepsilon_r r_o)K_1(\varepsilon_r r_1) \]  

\[ D_6 = H_g k_r \varepsilon_r r K_0(\varepsilon_r r_1)I_1(\varepsilon_f r_1)I_1(\varepsilon_r r_o) \]  

\[ D_7 = D_1(D_2 + D_3 + D_4 + D_5 + D_6 + D_7) \]  

Equation 3.23 links the outside cladding temperature to the heat flux to the coolant and to the perturbation of the power deposited in the fuel. It will be combined later with the coolant energy equation to yield the perturbations of the coolant enthalpy along the channel.
References:

Chapter 4

Thermal hydraulics
Part I

Segment level development
4.1. General form of conservation equations

Various forms of the conservation equations, together with the original equations for two-phase flow, have been presented in the literature, Yadigaroglu and Lahey, (1976). Starting from the original forms of:

**Vapor continuity equation**

\[
\frac{\partial}{\partial t}\left(\rho_g \langle \alpha \rangle \langle A \rangle \right) + \frac{\partial}{\partial z}\left(\rho_g \langle \alpha \rangle \langle u_g \rangle \langle A \rangle \right) = \delta \omega \quad \text{Eq. 4.1}
\]

**Liquid continuity equation**

\[
\frac{\partial}{\partial t}\left(\rho_l \langle 1-\alpha \rangle \langle A \rangle \right) + \frac{\partial}{\partial z}\left(\rho_l \langle 1-\alpha \rangle \langle u_l \rangle \langle A \rangle \right) = -\delta \omega \quad \text{Eq. 4.2}
\]

**Mixture energy equation (MEE)**

\[
\rho_g \langle \alpha \rangle \frac{\partial \langle e_g \rangle}{\partial t} + \rho_l \langle 1-\alpha \rangle \frac{\partial \langle e_l \rangle}{\partial t} + \rho_g \langle \alpha \rangle \left(\langle u_g \rangle \langle e_g \rangle \right) + \rho_l \langle 1-\alpha \rangle \left(\langle u_l \rangle \langle e_l \rangle \right) = \frac{P_g q^*}{A} - e_{lg} \frac{\delta \omega}{A} + q''
\]

\[
\text{Eq. 4.3}
\]

**Mixture momentum equation**

\[
-\frac{\partial p}{\partial z} = \frac{\partial}{\partial t}\left(\rho_l \langle 1-\alpha \rangle \langle u_l \rangle \right) + \frac{1}{A} \frac{\partial}{\partial z}\left(\rho_l A \langle 1-\alpha \rangle \langle u_l \rangle^2 + \rho_g A \langle \alpha \rangle \langle u_g \rangle^2 \right) + g \langle \rho \rangle \sin \theta + \frac{\tau_u P_f}{A}
\]

\[
\text{Eq. 4.4}
\]

where \(\delta \omega\) is the mass exchange rate between phases and \(e_k (k=l \text{ or } g)\) is the phase total specific energy (kinetic + potential + specific enthalpy).

and assuming that:

- kinetic and potential energy are negligible in the MEE.
- the cross sectional flow area is constant over the segment
- vertical direction \(\theta = \pi / 2\)
- volumetric vapor generation rate \(\Gamma = \delta \omega / A\)
Recasting the conservation equations in terms of the volumetric fluxes,

\[ \langle j_s \rangle = \langle \alpha \rangle \langle u_s \rangle_g \]  
\[ \langle j_l \rangle = \langle j \rangle - \langle \alpha \rangle \langle u_s \rangle_g \]  
\[ \langle j \rangle = \langle j_s \rangle + \langle j_l \rangle \]

and from dropping now on the <> sign from the equations and assuming that all the variables are averaged over the cross section, the above conservation equations can be manipulated and recast into a new form in terms of volumetric fluxes, Zuber and Staub (1967).

**Vapor continuity equation**

\[
\frac{\partial \alpha}{\partial t} = \frac{\Gamma_g}{\rho_g} - \alpha \frac{\partial \rho_g}{\partial t} - j_g \frac{\partial \rho_g}{\partial z} - \frac{\partial j_g}{\partial z}
\]

**Mixture continuity equation**

\[
\frac{\partial j}{\partial z} = \Gamma_g \left( \frac{1}{\rho_g} - \frac{1}{\rho_l} \right) - \alpha \frac{\partial \rho_g}{\partial t} - j_g \frac{\partial \rho_g}{\partial z} - \frac{1}{\rho_l} \frac{\partial \rho_l}{\partial t} - \frac{j_l}{\rho_l} \frac{\partial \rho_l}{\partial z}
\]

This obtained by substituting eq. 4.8 into the liquid continuity equation, written in terms of volumetric fluxes.

**Mixture energy equation**

\[
\alpha \rho_g \frac{\partial h_g}{\partial t} + (1-\alpha) \rho_l \frac{\partial h_l}{\partial t} + \rho_g j_g \frac{\partial h_g}{\partial z} + \rho_l j_l \frac{\partial h_l}{\partial z} = \frac{P_g q''}{A} + \frac{\partial p}{\partial t} - h_g \Gamma_g + q''
\]

**Mixture momentum equation**

\[-\frac{\partial p}{\partial z} = \frac{\partial (\rho_l j_l + \rho_g j_g)}{\partial t} + \frac{\partial}{\partial z} \left( \rho_l \frac{j_l^2}{1-\alpha} + \rho_g \frac{j_g^2}{\alpha} \right) + g \rho + f \frac{m^2}{2 \rho D_n} \Phi_{LO}^2 \]

In this new set of equations are also considered space-time variations of the steam-water properties. Furthermore, the mixture is allowed to be in non thermal equilibrium. Due to the presence of these partial derivatives and the time variation of the pressure in the energy equation, the conservation equations written in terms of volumetric fluxes, are coupled and can not be integrated over the segment individually. The development of a method for the integration of these coupled equations is one the highlights of this work. Later in this chapter, we will discuss their integration over a constant cross section flow area.
4.2. Conservation equations for a segment

The conservation equations, determined previously, are applied to the single-phase, two-phase subcooled and two-phase saturated regions in this section. It is important to note that local pressure drops are not considered in the momentum equation, since it is assumed that the local discontinuities (spacers and/or area changes) appear only at the edge of a segment. In this chapter, the equations will be developed and later integrated over a segment of a flow channel.

4.2.1. Single-phase subcooled region \((j, h, p)\)

In this region, water properties are calculated based on the local pressure and temperature. The conservation equations allow the liquid thermal expansion. We use the mixture volumetric flux \((j=j_l)\), liquid enthalpy, and pressure as the state variables.

**Mixture continuity equation (Eq. 4.9)**

\[
\frac{\partial j}{\partial z} = -\frac{1}{\rho_l} \frac{\partial h_l}{\partial t} \frac{\partial h_l}{\partial z} - \frac{j}{\rho_l} \rho_l \frac{\partial h_l}{\partial z}
\]

Eq. 4.12

**Liquid energy equation (Eq. 4.10)**

\[
\rho_l \frac{\partial h_l}{\partial t} + \rho_l j \frac{\partial h_l}{\partial z} = \frac{P_h q''}{A} + \frac{\partial p}{\partial t} + q''
\]

Eq. 4.13

**Mixture (liquid) momentum equation (Eq. 4.11)**

\[
-\frac{\partial p}{\partial z} = \frac{\partial (\rho_l j)}{\partial t} + \frac{\partial}{\partial z} \left( \rho_l j^2 \right) + g \rho + f \frac{m^2}{2 \rho_l D_n}
\]

Eq. 4.14

\(\rho_l''\) stands for the liquid density variation with respect to the local liquid temperature, see section 4.3.5.
4.2.2. Two-phase subcooled region \((\alpha, j, h_l, p)\)

Because of the presence of subcooled liquid and saturated gas, the thermal properties will be calculated properly for each phase using the local temperature and pressure. Furthermore, the conservation equations, automatically, account for subcooled boiling (non thermal equilibrium) and liquid thermal expansion, flashing and variation of the gas thermal properties. The state variables are void fraction, mixture volumetric flux, liquid enthalpy, and pressure.

**Vapor continuity equation**

\[
\frac{\partial \alpha}{\partial t} = \frac{\Gamma_{g,\text{sub}}}{\rho_g} - \frac{\alpha}{\rho_g} \frac{\partial p}{\partial t} - \frac{j_g}{\rho_g} \frac{\partial p}{\partial z} - (\alpha c_0) \frac{\partial j}{\partial z} - (C_o, j + V_g) \frac{\partial \alpha}{\partial z} \quad \text{Eq. 4.15}
\]

**Mixture continuity equation**

\[
\frac{\partial j}{\partial z} = \Gamma_{g,\text{sub}} \left( \frac{1}{\rho_g} - \frac{1}{\rho_l} \right) - \frac{\alpha}{\rho_g} \frac{\partial p}{\partial t} - \frac{j_g}{\rho_g} \frac{\partial p}{\partial z} - \frac{1 - \alpha}{\rho_l} \frac{\partial h_l}{\partial t} - \frac{j_l}{\rho_l} \frac{\partial h_l}{\partial z} \quad \text{Eq. 4.16}
\]

**Liquid energy equation**

\[
(1 - \alpha) \rho_l \frac{\partial h_l}{\partial t} + \rho_l j_l \frac{\partial h_l}{\partial z} = \frac{P_l q^*}{A} + q^w - h_l \Gamma_{g,\text{sub}} - (\alpha \rho_g h_g' - 1) \frac{\partial p}{\partial t} - \rho_g j_g h_g' \frac{\partial p}{\partial z} \quad \text{Eq. 4.17}
\]

**Mixture momentum equation**

\[
- \frac{\partial p}{\partial z} = \frac{\partial (\rho_l j_l + \rho_g j_g)}{\partial t} + \frac{\partial}{\partial z} \left( \frac{\rho_l j_l^2}{1 - \alpha} + \rho_g j_g^2 \right) + g \rho + f \frac{m^2}{2 \rho_l D_h} \Phi_{LO}^2 \quad \text{Eq. 4.18}
\]

\(\rho_l^*\) stands for the liquid density variation with respect to the local liquid temperature and \(\rho_g^*, h_g'\) denote the gas density and enthalpy variation with respect to the local pressure, see section 4.3.5. The subcooled volumetric vapor generation rate equation \(\Gamma_{g,\text{sub}}\) is given in section 4.3.1.
4.2.3. Two-phase saturated region \((\alpha, j, p)\)

In this region, both phases are saturated and their thermal properties follow the saturation line. Moreover, the conservation equations take into account flashing, and the variation of the liquid and gas thermal properties with pressure. The state variables are void fraction, mixture volumetric flux, and pressure.

**Vapor continuity equation**

\[
\frac{\partial \alpha}{\partial t} = \frac{\Gamma_{g,sat}}{\rho_g} - \frac{\alpha}{\rho_g} \frac{\partial \rho_g'}{\partial t} - \frac{j_g}{\rho_g} \frac{\partial \rho_g'}{\partial z} - \left( \alpha C_0 \right) \frac{\partial j}{\partial z} - \left( \frac{C_0 j + V_g}{\rho_g} \right) \frac{\partial \alpha}{\partial z}
\]

Eq. 4.19

**Mixture continuity equation**

\[
\frac{\partial j}{\partial z} = \Gamma_{g,sat} \left( \frac{1}{\rho_g} - \frac{1}{\rho_f} \right) - \left( \frac{\alpha}{\rho_g} \rho_g' + \frac{1-\alpha}{\rho_f} \rho_f' \right) \frac{\partial \rho_g'}{\partial t} - \left( \frac{j_g}{\rho_g} \rho_g' + \frac{j_i}{\rho_f} \rho_f' \right) \frac{\partial \rho_g'}{\partial z}
\]

Eq. 4.20

**Mixture energy equation:**

This reduces to an equation for the volumetric vapor generation rate \(\Gamma_{g,sat}\) in the saturation region, see section 4.3.1.

**Mixture momentum equation**

\[
-\frac{\partial \rho}{\partial z} = \frac{\partial (\rho_f j_i + \rho_g j_g)}{\partial t} + \left( \rho_f j_i^2 + \rho_g j_g^2 \right) \frac{\partial}{\partial z} \left( \frac{1}{\rho_f} \rho_f' + \frac{1}{\rho_g} \rho_g' \right) + g \rho + f \frac{\dot{m}^2}{2 \rho_f D_f} \Phi_{lo}^2
\]

Eq. 4.21

\(\rho_g', \rho_f', h_i', h_f'\) represent the liquid and gas density and enthalpy variations with respect to the local pressure, see section 4.3.5.
4.3. Closure relationships

The conservation equations given in the previous sections cannot be solved alone, since the number of unknowns in a segment is larger than the number of equations. Therefore, it is necessary to close the system with additional equations.

4.3.1. Volumetric vapor generation rate

In the two-phase subcooled region, we use the mechanistic model, Lahey and Moody (1993), for the volumetric vapor generation rate due to subcooled boiling plus some additional terms for the flashing effect. This flashing effect is obtained from the energy conservation equation in the saturated region and is applied also to the subcooled region. However, for our convenience we split the total volumetric vapor generation rate in three different parts: flashing, subcooling, heat and term.

- **Flashing term:**

  \[
  \Gamma_{\text{flash}} \left( \alpha, j, \rho, \frac{\partial p}{\partial t}, \frac{\partial p}{\partial z} \right) = \frac{1}{h_{fg}} \left( \frac{\partial p}{\partial t} \left( 1 - \alpha \rho \frac{h_f}{h_g} - (1 - \alpha) \rho \frac{h_f}{h_j} \right) - \frac{\partial p}{\partial z} \left( \rho \frac{j-f}{j-g} h_g' + \rho \frac{j-j}{j-j} h_j' \right) \right)
  \]  
  \text{Eq. 4.22}

- **Subcooled vapor generation term:**

  \[
  \Gamma_{\text{sub}} \left( \alpha, h, p, h_j, q_V \right) = \frac{P_j q_V (h_i - h_j)}{A \left( h_j - h_i \right) \left( h_{fg} + \frac{\rho_j}{\rho_g} \left( h_j - h_i \right) \right)} - \frac{H_0}{v_{fg}} \alpha \left( T_f - T_i \right)
  \]  
  \text{Eq. 4.23}

- **Vapor generation rate due to the total heat input:**

  \[
  \Gamma_{\text{heat}} \left( p, q_V, q_T \right) = \frac{1}{h_{fg}} \left( \frac{q_T P_j}{A} + q_T \right)
  \]  
  \text{Eq. 4.24}

For each region, the appropriate sum of the terms given above is used.

- **Volumetric vapor generation rate in the subcooled boiling region**

  \[
  \Gamma_{g,\text{sub}} = \Gamma_{\text{sub}} + \Gamma_{\text{flash}}
  \]  
  \text{Eq. 4.25}

- **Volumetric vapor generation rate in the saturated region**

  \[
  \Gamma_{g,\text{sat}} = \Gamma_{\text{heat}} + \Gamma_{\text{flash}}
  \]  
  \text{Eq. 4.26}
### 4.3.2. Phase friction factor and Phase Reynolds Number

Zigrang and Sylvester (1985) is one of the most recommended phase friction. It is used in many state of the art codes like RELAP, Cheval-Lellouche (1999). This correlation approximates accurately Moody’s diagram for all kinds of pipes (smooth or with different degrees of roughness ($\varepsilon$)).

\[
f_k = f(Re_k) = \left(-2\log_{10}\left(\frac{\varepsilon}{3.7D_h} + \frac{2.51}{Re_k}\left(1.14 - 2\log_{10}\left(\frac{\varepsilon}{D_h} + \frac{21.52}{Re_k^{0.8}}\right)\right)\right)\right)^{-2} \quad \text{Eq. 4.27}
\]

where the phase ($k=l, g$) Reynolds Number is

\[
Re_k = Re_{k}(j_k) = \frac{\rho_k j_k D_k}{\mu_k} \quad \text{Eq. 4.28}
\]

### 4.3.3. Friction multipliers

In the two-phase region we use only two multipliers as default. Chisholm’s multiplier for frictional pressure drop in a segment and Homogeneous multiplier for local pressure drop.

#### 4.3.3.1 Chisholm multiplier

The Modified Chisholm multiplier(1974) is used in both two-phase (subcooled and saturated) regions as the friction multiplier.

\[
\Phi_{ch}^2 = 1 + \left(\Gamma_{ch}^2 - 1\right)x + a_{ch}(1-x)x \quad \text{Eq. 4.29}
\]

where

\[
\Gamma_{ch}(f_l, f_g) = \left(\frac{\rho_l f_g}{\rho_g f_l}\right)^{0.5} \quad \text{Eq. 4.30}
\]

\[
a_{ch}(Re_f, \Gamma_{ch}) = \frac{C_1(\Gamma_{ch}^2 - 1)}{(1 + 10^{-5} Re_f)(1 + C_2 \Gamma_{ch}^2)} \quad \text{Eq. 4.31}
\]

\[
x(\alpha, j, h, p) = \frac{\rho_g j_g}{\rho_l j_l + \rho_g j_g} \quad \text{Eq. 4.32}
\]

\[
C_1 = 0.27 \quad \text{Eq. 4.33}
\]

\[
C_2 = 0.007
\]
As all the terms in the Chisholm multiplier are indirectly functions of void fraction and of the mixture volumetric flux and the quality in general is considered function of the liquid enthalpy and pressure:

\[ \Phi_{LO}^2 = \Phi_{ch}^2 (\alpha, j, h_l, p) \quad \text{Eq. 4.34} \]

### 4.3.3.2 Homogenous multiplier

This multiplier will be applied only at the edge of segments for local pressure drop calculation.

\[ \Phi_{ihm}^2 (\alpha, j, h_l, p) = 1 + x \left( \frac{\rho_\alpha}{\rho_g} - 1 \right) \quad \text{Eq. 4.35} \]

### 4.3.4. Heat transfer coefficient

Originally Chen (1966) divided the heat transfer coefficient into two parts, a nucleate boiling part and a forced convection part. This correlation was originally used only in the saturated region.

\[ H_{chen} = H_{nb} + H_{fc} \quad \text{Eq. 4.36} \]

\[ q^* = (H_{nb} + H_{fc})(T_{co} - T_f) = q_{nb}^* + q_{fc}^* \quad \text{Eq. 4.37} \]

Later on, Collier (1972) modified Chen’s correlation and extended its field of applicability to the subcooled region. Therefore we use Collier correlation, in its general form (in our case, however, \( T_l \) is the liquid temperature given by the subcooled boiling model, rather than the thermal-equilibrium bulk liquid temperature).

\[ q^* = C_{nb}(p)S(\alpha, j)(T_{co} - T_f(p)) + C_{fc}(\alpha, j)F(\alpha)(T_{co} - T_f) = q_{nb}^* + q_{fc}^* \quad \text{Eq. 4.38} \]

where

\[ C_{fc}(\alpha, j) = 0.023 \text{Re}_{l}^{0.8} \text{Pr}_{l}^{0.4} \frac{k_t}{D_h} \quad \text{Eq. 4.39} \]

and

\[ C_{nb}(p) = 0.0012 \left( \frac{k_f^{0.79} C_f^{0.45} \rho_f^{0.49}}{\sigma_f^{0.5} \mu_f^{0.29} \rho_g^{0.24} \mu_{fg}^{0.24}} \right) \left( \frac{h_{fg}}{T_{sat} v_{fg}} \right)^{0.75} (T_{co} - T_f(p)) \quad \text{Eq. 4.40} \]
the suppression factor, Todreas and Kazimi (1993), fitted by

\[
S(\alpha, f) = \frac{1}{1 + 2.53 \times 10^{-6} \cdot \text{Re}^{1.17}} \tag{4.41}
\]

\[
\text{Re} = \text{Re}_j F^{1.25} \tag{4.42}
\]

with the factor \( F \), Todreas and Kazimi (1993), fitted by

\[
F = 2.35 \left( 0.213 + \frac{1}{X_{tt}} \right)^{0.736} \quad \text{for } 1/X_{tt} > 0.1 \tag{4.43}
\]

\[
F = 1 \quad \text{for } 1/X_{tt} < 0.1 \tag{4.44}
\]

where \( X_{tt} \) is the Martinelli parameter, given as

\[
X_{tt} = \left( \frac{1 - x}{x} \right)^{0.9} \left( \frac{\rho_g}{\rho_f} \right)^{0.5} \left( \frac{\mu_f}{\mu_g} \right)^{0.1} \tag{4.45}
\]

Finally, for the following treatment, we note that \( H_{\text{chen}} = H_{\text{chen}}(\alpha, j, h_f, p, T_{co}) \).
4.3.5. Water properties

The thermodynamic properties of the steam and water, and their partial derivatives are calculated at the local pressure and temperature, using the steam tables, Schmidt 1982. In the subcooled region, the liquid is thermally expandable and the gas saturated. Moreover, the variation of the liquid density depends on its local enthalpy (local temperature), while the gas density variation is related only to the pressure.

\[
\frac{\partial \rho_l(h)}{\partial t} = \frac{\partial \rho_l}{\partial h_l} \frac{\partial h_l}{\partial t} = \rho_l^\prime \frac{\partial h_l}{\partial t} \quad \text{Eq. 4.46}
\]

\[
\frac{\partial \rho_l(h)}{\partial z} = \frac{\partial \rho_l}{\partial h_l} \frac{\partial h_l}{\partial z} = \rho_l^\prime \frac{\partial h_l}{\partial z} \quad \text{Eq. 4.47}
\]

where the double prime denotes the derivative along a constant pressure. In the saturated region, both phases follow the saturation line (subscript \(f\) and \(g\))

\[
\frac{\partial \rho_f(p)}{\partial z} = \frac{\partial \rho_f}{\partial p} \frac{\partial p}{\partial z} = \rho_f^\prime \frac{\partial p}{\partial z} \quad \text{Eq. 4.48}
\]

\[
\frac{\partial \rho_f(p)}{\partial t} = \frac{\partial \rho_f}{\partial p} \frac{\partial p}{\partial t} = \rho_f^\prime \frac{\partial p}{\partial t} \quad \text{Eq. 4.49}
\]

\[
\frac{\partial h_f(p)}{\partial z} = \frac{\partial h_f}{\partial p} \frac{\partial p}{\partial z} = h_f^\prime \frac{\partial p}{\partial z} \quad \text{Eq. 4.50}
\]

\[
\frac{\partial h_f(p)}{\partial t} = \frac{\partial h_f}{\partial p} \frac{\partial p}{\partial t} = h_f^\prime \frac{\partial p}{\partial t} \quad \text{Eq. 4.51}
\]

\[
\frac{\partial \rho_g(p)}{\partial z} = \frac{\partial \rho_g}{\partial p} \frac{\partial p}{\partial z} = \rho_g^\prime \frac{\partial p}{\partial z} \quad \text{Eq. 4.52}
\]

\[
\frac{\partial \rho_g(p)}{\partial t} = \frac{\partial \rho_g}{\partial p} \frac{\partial p}{\partial t} = \rho_g^\prime \frac{\partial p}{\partial t} \quad \text{Eq. 4.53}
\]

\[
\frac{\partial h_g(p)}{\partial z} = \frac{\partial h_g}{\partial p} \frac{\partial p}{\partial z} = h_g^\prime \frac{\partial p}{\partial z} \quad \text{Eq. 4.54}
\]

\[
\frac{\partial h_g(p)}{\partial t} = \frac{\partial h_g}{\partial p} \frac{\partial p}{\partial t} = h_g^\prime \frac{\partial p}{\partial t} \quad \text{Eq. 4.55}
\]

where, again, the prime denotes the derivatives with respect to pressure along the saturation line.
4.3.6. Enthalpy at the net vapor generation (NVG) point

In our subcooled model, we neglect the wall void age region. The net vapor generation point \((Z_d)\) is determined by the use of Saha and Zuber (1974) model. The correlation distinguishes between high and low mass flux regions, using the Peclet number.

\[
Pe = \frac{RePr \dot{m} D_h c_l}{k_l}
\]

Eq. 4.56

For \(Pe < 70,000\) (low mass flux):

\[
h_{ld}(p, q'') = h_f - 0.0022 \frac{q'' D_h c_l}{k_l}
\]

Eq. 4.57

For \(Pe > 70,000\) (high mass flux):

\[
h_{ld}(j, p, q'') = h_f - 154 \frac{q''}{\dot{m}}
\]

Eq. 4.58

At low mass flux, the Stanton number may already have the value of \(St=0.0065\); the bubbles may detach from the wall when \(Nu=455\) is reached. On the contrary at high mass flux, the Nusselt number may already have the value of \(Nu=455\), NVG will occur when the Stanton number reaches the value of \(St=0.0065\).

4.3.7. Drift flux fit parameters

There is a large number of correlations to evaluate the drift velocity \((V_{ji})\) and the distribution parameter \((C_0)\); see Coddington and Macian (2002). We have included two of the most important correlations. They are the Chexal-Lellouche (1997) and the Dix (1971); (see Coddington and Macian 2002) correlations, see appendix A2. Furthermore, there is also the possibility to consider the drift flux model parameters variable or constant over the channel. The latter case allows to reduce the drift flux model to the homogenous model, if one assumes \(C_0=1\) and \(V_{ji}=0\).

4.4. Steady-state

The conservation equations written for a segment, in the pervious paragraphs, provide the steady-state in each region. They consist of a set of coupled linear differential equations (three equations in the single-phase and the two-phase saturated regions, and four equations in the two-phase subcooled region) with the same number of unknowns and forcing functions (right hand side of equations). The steady-state solution will be found iteratively based on segment-average coefficients and forcing functions. The analytical solution for the steady-state is given in section 4.4.5.
4.4.1. Steady State in the single-phase subcooled region in terms of \((j, h, p)\)

In this region, the liquid volumetric flux is the same as that of the mixture, since the gas term is equal to zero. It is important to note that the mixture mass flux is constant at steady-state, but the mixture volumetric flux varies linearly with enthalpy. This links the mixture continuity equation to the energy equation. Finally, since the momentum equation is a function of the mixture volumetric flux and depends on liquid density variations, it is coupled to the energy and continuity equations. Therefore they should be solved together. The steady-state equations are

\[
\frac{dj}{dz} + \left( \frac{j}{\rho_l} \rho_j^* \right) \frac{dh_l}{dz} = 0 \tag{Eq. 4.59}
\]

\[
\dot{m} \frac{dh_l}{dz} = q'' P_h + q''_r \tag{Eq. 4.60}
\]

\[
(2j \rho_l) \frac{dj}{dz} + (j^2 \rho_l^*) \frac{dh_l}{dz} + \frac{dp}{dz} = -\left( g \rho_l + f \frac{\dot{m}^2}{2 \rho_l D_h} \right) \tag{Eq. 4.61}
\]

4.4.2. Steady State in the two-phase subcooled region \((\alpha, j, h, p)\)

In the two-phase region the equations are more complex, but the coupling comes from the same reasoning as in the case of single phase, plus from the void related terms. As we mentioned earlier, the vapor and liquid properties are functions of the local pressure and temperature. In order to obtain a set of linear differential equations with constant coefficients, we have substituted the total vapor generation rate with its components (subcooling and flashing terms, see eq. 4.25) in the conservation equations (section 4.2.2) and grouped the coefficients of the same state variable.

**Vapor continuity equation:**

\[
\left( C_0 j + V_g \right) \frac{d\alpha}{dz} + (\alpha C_0) \frac{dj}{dz} + \left( \frac{j_g}{\rho_g^*} \right) \frac{dh_g}{dz} + \frac{1}{\rho_g h_{fg}^*} \left( \rho_g j_g h_g^* + \rho_f j_f h_f^* \right) \frac{dp}{dz} = \frac{\Gamma_{\text{sub}}}{\rho_g} \tag{Eq. 4.63}
\]

**Mixture continuity equation:**

\[
\frac{dj}{dz} + \left( \frac{j_l}{\rho_l} \rho_l^* \right) \frac{dh_l}{dz} + \left( \frac{j_g}{\rho_g} \rho_g^* + \left( \frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \frac{1}{h_{fg}^*} \left( \rho_g j_g h_g^* + \rho_f j_f h_f^* \right) \frac{dp}{dz} = \left( \frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \frac{\Gamma_{\text{sub}}}{\rho_l} \tag{Eq. 4.64}
\]
Liquid energy equation:

\[
(\rho_l j_l) \frac{dh_l}{dz} + \left( \rho_g j_g h'_g - \frac{h_{fg}}{h_{fg}} (\rho_g j_g h'_g + \rho_f j_f h'_f) \right) \frac{dp}{dz} = \frac{q''_h}{A} + q''_p - h_{fg} \Gamma_{sub}
\]

Eq. 4.65

Momentum equation:

\[
\left( -\frac{2 \rho_l j_l (C_0 j + V_{gl})}{(1 - \alpha)} + \frac{\rho_l j_l^2}{(1 - \alpha)} + \rho_g \left( C_0 j + V_{gl} \right)^2 \right) \frac{d\alpha}{dz} + \left( \frac{2 \rho_l j_l (1 - \alpha C_0)}{(1 - \alpha)} + 2 \rho_g j_g C_0 \right) \frac{dj}{dz} + \left( \frac{j_l^2}{(1 - \alpha) \rho_l^*} \right) \frac{dh_l}{dz} + \left( \frac{j_g^2}{\alpha \rho_g^*} + 1 \right) \frac{dp}{dz} = -\left( g \rho + \frac{\dot{m}^2}{2 \rho_l D_h} \Phi_{ch,0} \right)
\]

Eq. 4.66

where \( \Gamma_{sub} \) was defined in eq. (4.23).

4.4.3. Steady state in the saturated region \((\alpha, j, p)\)

Substituting the volumetric vapor generation rate with its expression (eq. 4.26) in the conservation equations (section 4.2.3) and grouping terms, we obtain a set of linear differential equations with coefficients calculated at the local pressure. In this region, all the liquid and gas properties follow the saturation line. Moreover, the enthalpy and density variations have been expressed in terms of pressure variations.

Vapor continuity equation:

\[
(C_0 j + V_{gl}) \frac{d\alpha}{dz} + \alpha C_0 \frac{dj}{dz} + \left( \frac{j_g}{\rho_g} \rho_g^* + 1 \right) \left( \rho_g j_g h'_g + \rho_f j_f h'_f \right) \frac{dp}{dz} = \frac{\Gamma_{heat}}{\rho_g}
\]

Eq. 4.67

Mixture continuity equation:

\[
\frac{dj}{dz} + \left( \frac{j_g}{\rho_g} \rho_g^* + \frac{j_f}{\rho_f} \rho_f^* + \frac{1}{h_{fg}} \left( \frac{1}{\rho_g} - \frac{1}{\rho_f} \right) \left( \rho_g j_g h'_g + \rho_f j_f h'_f \right) \right) \frac{dp}{dz} = \left( \frac{1}{\rho_g} - \frac{1}{\rho_f} \right) \Gamma_{heat}
\]

Eq. 4.68
Momentum equation:

\[
\left\{ -\frac{2\rho_f j_l (C_{0l} + V_{g})}{(1-\alpha)} + \rho_f j_l^2 \left(\frac{C_{0l} + V_{g}}{1-\alpha}\right)^2 + \rho_g \left(\frac{C_{0l} + V_{g}}{1-\alpha}\right)^2 \right\} \frac{d\alpha}{dz} + \left\{ \frac{2\rho_f j_l (1-\alpha C_{0l})}{(1-\alpha)} + 2\rho_g j_g C_{0l} \right\} \frac{dj}{dz} + \left( \frac{j_l^2}{1-\alpha} \rho_f' + \frac{j_g^2}{\alpha} \rho_g' + 1 \right) \frac{dp}{dz} \right\} = - \left\{ g \rho + f \frac{m^2}{2\rho_f D_h} \Phi_{c,o}^2 \right\}
\]

where \( \Gamma_{\text{heat}} \) was given in eq. (4.24).

### 4.4.4. Boiling boundary detection

Checks for the NVG and the equilibrium boiling boundaries are performed in every segment in the subcooled region. After the location of the corresponding points, the detection stops. These points divide the respective segments, where they are situated, into two parts. The two boundaries are sought, assuming linear profiles for the liquid enthalpy and the saturation enthalpy.

- **Net vapor generation point:**

It is defined as the intersection of two curves, the enthalpy of the subcooled liquid and enthalpy at NVG criterion, Saha and Zuber (1974); see section 4.3.6. If a segment includes the boiling incipience, then we can state that

\[
h_l(z_d) = h_l^{in} + \frac{h_l^{out} - h_l^{in}}{\Delta z} (z_d - z_{in}) \quad \text{Eq. 4.70}
\]

\[
h_{ld}(z_d) = h_{ld}^{in} + \frac{h_{ld}^{out} - h_{ld}^{in}}{\Delta z} (z_d - z_{in}) \quad \text{Eq. 4.71}
\]

\[
\frac{\Delta z_{d,l}}{\Delta z} = \frac{h_l^{in} - h_{ld}^{in}}{\Delta h_{ld} - \Delta h_l} \quad \text{Eq. 4.72}
\]

Therefore, the enthalpy at the NVG point is given as

\[
h_{ld}(z_d) = h_{ld}^{in} \frac{\Delta h_{ld} - h_{ld}^{in} \Delta h_l}{\Delta h_{ld} - \Delta h_l} \quad \text{Eq. 4.73}
\]

where \( h_{ld}^{in} \) and \( h_{ld}^{out} \) are calculated with the heat flux and \( h_f \) at the inlet and exit of the segment, respectively.
- Equilibrium Boiling Boundary (EBB):

In the same way, the EBB is defined as the intersection of the liquid enthalpy curve with the curve of the local saturation enthalpy (obtained from the local pressure), see figure 4.2. For the segment that includes EBB, we can state

\[
\frac{\Delta z_{eq}}{\Delta z} = \frac{h_{f}^{in} - h_{l}^{in}}{\Delta h_{f} - \Delta h_{l}} \tag{Eq. 4.74}
\]

and determine the liquid enthalpy at the stationary EBB:

\[
h_{l}(z_{eq}) = h_{l}^{in} + h_{f}^{out} - h_{f}^{in} \frac{\Delta z_{eq}}{\Delta z} \tag{Eq. 4.75}
\]

After determination of \(\Delta z_{eq}\), the conservation equations of two-phase saturated region will be integrated over the rest of the segment.
4.4.5. Nodal steady state solution

The steady-state linear differential equations given in sections 4.4.1, 4.4.2 and 4.4.3 are sets of coupled equations. We apply the same method as core simulators use to obtain the stationary solution of these coupled linear differential equations with constant coefficients. In our development, the constant coefficients are calculated using nodal averaged state variables. This choice has been done in compatibility with the transient solution which requires the nodal averaged state variables. Since the average values are unknown, an iterative procedure is necessary. Finally the results of iteration \( (\frac{da}{dz}, \frac{dj}{dz}, \frac{dp}{dz} \text{ and } \frac{dh}{dz}) \) are integrated over the segment to obtain the state variables at the segment exit. The local drift flux parameters \((C_0, V_g)\) are updated by the actual average state variables during the iterations.

- Steady-state solution for the saturated two-phase region

For example, we rewrite the equations of section 4.4.3 in the following form, where the \( a_i \) correspond to the coefficients term by term.

\[
\begin{align*}
\frac{da}{dz} + a_2(\alpha) \frac{dj}{dz} + a_3(\alpha) \frac{dp}{dz} &= f_1(q^*, q^*_r) \\
0 + \frac{dj}{dz} + a_6(\alpha, j) \frac{dp}{dz} &= f_2(q^*, q^*_r) \\
a_7(\alpha, j) \frac{da}{dz} + a_8(\alpha, j) \frac{dj}{dz} + a_9(\alpha, j) \frac{dp}{dz} &= f_3(\alpha, j)
\end{align*}
\]

In matrix notation, these can be expressed as

\[
\frac{dx}{dz} = A_0^{-1} f_0
\]

with

\[
x\{3,1\} = [\alpha \ j \ p]^T
\]

\[
A_0\{3,3\} = \begin{pmatrix} a_1(j) & a_2(\alpha) & a_3(\alpha, j) \\ 0 & 1 & a_6(\alpha, j) \\ a_7(\alpha, j) & a_8(\alpha, j) & a_9(\alpha, j) \end{pmatrix}_o
\]

\[
f_0\{3,1\} = [f_1(q^*, q^*_r) \ f_2(q^*, q^*_r) \ f_3(\alpha, j)]_0
\]

Where \( A_0 \) is the matrix of coefficients and \( f_0 \) the vector of forcing functions. In general, the dimensions of \( A_0 \) and \( f_0 \) are related to the number of equations and therefore to the region. The solution \( (dx/dz) \) is achieved when state variables at the segment exit satisfy the convergence criterion. For a given solution, the conservation equations are integrated as:
4.5. Transient

The perturbed forms of the conservation equations are obtained by their linearization and Laplace transformation, neglecting second order terms of perturbations. After substituting the heat flux and vapor generation perturbations in the perturbed conservation equations, the final form of the equations is obtained; they constitute a system of non-homogenous, coupled, linear differential equations.

4.5.1. Single-phase region \((\delta_j, \delta_h, \delta_p)\)

The time-dependent version of the single-phase region conservation equations was given in section 4.2.1. Here we present only the initial (if it is necessary for better comprehension) and final perturbed form of these, which will be used later for calculation of the analytical solution.

- Liquid continuity (eq. 4.12)

\[
\frac{d\delta_j}{dz} + \left( j \frac{\rho_j^s}{\rho_j} \right) \frac{d\delta h_i}{dz} + \left( \frac{\rho_i^s}{\rho_j} d h_i + s \frac{\rho_i^s}{\rho_j} \right) \delta h_i = 0
\]

Eq. 4.84

Since we need our conservation equation in matrix form, from now on they will be written as

\[
t_{1,11} \frac{d\delta j}{dz} + t_{1,12} \frac{d\delta h_i}{dz} + t_{2,11} \delta j + t_{2,12} \delta h_i = 0
\]

Eq. 4.85

where the coefficients of eq. 4.84 and 4.85 correspond term by term. Note, the subscripts of the \(t\) coefficients correspond to the elements of the matrices \(T_1\) and \(T_2\) in section 4.5.9.

- Liquid energy conservation equation (eq. 4.13)

After having substituted the heat flux perturbation with its expression in single-phase region.

\[
\left( \rho_j \right) \frac{d\delta h_i}{dz} + \left( \rho_j \frac{d h_i}{dz} + \frac{P_h \ TF_j^q}{A \ HF_{DEN}} \right) \delta j + \left( \rho_j s + \frac{P_h \ TF_j^q}{A \ HF_{DEN}c_1} \right) \delta h_i + \left( \frac{P_h \ TF_j^q}{A \ HF_{DEN}c_1} - s \right) \delta p
\]

\[
+ \left( \frac{P_h \ TF_j^q}{A \ HF_{DEN}} \right) \delta q'' - \delta q'' = 0
\]

Eq. 4.86
where the TF’s are transfer function and the superscript/subscripts denote the output/input perturbations. The $HF_{DEN}$ is a coefficient that is given in section 4.5.7. In compact form:

$$t_{1,22} \frac{d\delta h_i}{dz} + t_{2,22} \delta \gamma + t_{2,22} \delta h_i + t_{2,25} \delta p + f_{q_2} = 0$$

Eq. 4.87

- **Momentum conservation equation (eq. 4.14)**

$$\frac{2\rho_j}{dz} \frac{d\delta j}{dz} + (j^2 \rho_j^*) \frac{d\delta h_i}{dz} + \frac{d\delta p}{dz} + \left( \frac{\partial (dp)}{\partial m} \frac{\delta m}{\partial f} + \frac{\partial (dp)}{\partial f} \frac{df}{dj} \right) \delta j$$

$$+ \left( \frac{d\delta h_i}{dj} + \frac{\partial (dp)}{\partial \rho_l} \frac{\partial \rho_l}{\partial \rho} \right) \rho_j^* \delta h_i = 0$$

Eq. 4.88

The friction factor perturbations are given in section 4.5.5. Equation 4.88 is written in compact form as

$$t_{1,31} \frac{d\delta \gamma}{dz} + t_{1,32} \frac{d\delta h_i}{dz} + t_{1,33} \frac{d\delta p}{dz} + t_{2,31} \delta \gamma + t_{2,32} \delta h_i = 0$$

Eq. 4.89

### 4.5.2. Subcooled two-phase region ($\delta \alpha$, $\delta \gamma$, $\delta h_i$, $\delta p$)

Because of the complexity of the equation in this section, we give here not only the original linearized and Laplace transformed but also the final forms (after introduction of heat flux and vapor generation perturbations). This last set of equations will be used, later, for the calculation of eigenvalues and eigenvectors of a segment, see section 4.5.9.

- **Vapor continuity (eq. 4.15)**

Continuity of vapor, before introduction of vapor generation rate perturbation:

$$\left( C_{g,j} + V_g \right) \frac{d\delta \alpha}{dz} + C_{g,j} \frac{d\delta j}{dz} + \rho_g' \frac{d\delta p}{dz} + \left( \delta \alpha \right) \left( C_{g,j} + V_g \right) \rho_g' \delta p - \frac{1}{\rho_g} \delta \Gamma_{g,nub} = 0$$

Eq. 4.90
after the necessary substitutions

\[
\frac{d\delta \alpha}{dz} + t_{1,12} \frac{d\delta j}{dz} + t_{1,14} \frac{d\delta p}{dz} + t_{2,11} \delta \alpha + t_{2,12} \delta j + t_{2,13} \delta h + t_{2,14} \delta p + f q_l = 0
\]

Eq. 4.91

where

\[
t_{1,11} = C_0 j + V_g \\
t_{1,12} = C_0 \alpha \\
t_{1,13} = 0 \\
t_{1,24} = j_g \frac{\rho'_{g}}{\rho_g} \frac{TF^{T_{F_g,ab}}}{\rho_g} \\
t_{2,11} = s + \frac{d}{dz} C_0 + \frac{\rho'_{g}}{\rho_g} \frac{dp}{dz} (C_0 j + V_g) - \frac{1}{\rho_g} \left( \frac{TF^{T_{F_g,ab}}}{HF_{DEN}} - \frac{TF^{T_{F_q,ab}}}{HF_{DEN}} \right) \\
t_{2,12} = \left( \frac{\rho'_{g}}{\rho_g} \frac{dp}{dz} \alpha + \frac{d\alpha}{dz} \right) C_0 - \frac{1}{\rho_g} \left( \frac{TF^{T_{F_g,ab}}}{HF_{DEN}} \right) \\
t_{2,13} = -\frac{1}{\rho_g} \left( \frac{TF^{T_{F_{h_0},ab}}}{HF_{DEN}c_1} \right) \\
t_{2,14} = \left( s\alpha + \frac{\Gamma_{g,sub}}{\rho_g} \right) \frac{\rho'_{g}}{\rho_g} \frac{1}{\rho_g} \left( \frac{TF^{T_{F_q,ab}}}{HF_{DEN}} \right) \\
f q_l = \frac{1}{\rho_g} \left( \frac{TF^{T_{F_q,ab}}}{HF_{DEN}} \right) \delta q'' - \frac{TF^{T_{F_h,ab}}}{HF_{DEN}} \delta h_d \\
\]

Eq. 4.92

Eq. 4.93

Eq. 4.94

Eq. 4.95

Eq. 4.96

Eq. 4.97

Eq. 4.98

Eq. 4.99

Eq. 4.100

and the TF coefficients are given in sections 4.5.4 and 4.5.7.
Mixture continuity (eq. 4.16)

The original perturbed equation:

\[
\frac{d \delta j}{dz} + \frac{\rho_j^*}{\rho_j} \frac{d \delta h_l}{dz} + \frac{\rho_g^*}{\rho_g} \frac{d \delta p}{dz} + \left( \frac{\rho_g^*}{\rho_g} \frac{dp}{dz} - \frac{\rho_j^*}{\rho_j} \frac{dh_l}{dz} \right) \left( C_0 j + V_{gg} \right) \delta \alpha \\
+ \left( \frac{\rho_g^*}{\rho_g} \frac{dp}{dz} \left( C_0 \alpha + \frac{\rho_j^*}{\rho_j} \frac{dh_l}{dz} \left( 1 - \alpha C_0 \right) \right) \delta j + \left( \frac{s \left( 1 - \alpha \right) - \Gamma_{g,sub}}{\rho_j} \right) \frac{\rho_j^*}{\rho_j} \delta h_l \\
+ \left( s \alpha + \frac{\Gamma_{g,sub}}{\rho_g} \right) \frac{\rho_g^*}{\rho_g} \delta p - \frac{\Delta \rho}{\rho_j \rho_g} \delta \Gamma_{g,sub} = 0
\]

Eq. 4.101

after introduction of the vapor generation rate perturbation, becomes

\[
t_{2,22} \frac{d \delta j}{dz} + t_{2,23} \frac{d \delta h_l}{dz} + t_{2,24} \frac{d \delta p}{dz} + t_{2,21} \delta \alpha + t_{2,22} \delta j + t_{2,23} \delta h_l + t_{2,24} \delta p + f q_2 = 0
\]

Eq. 4.102

where

\[
t_{2,21} = 0
\]

Eq. 4.103

\[
t_{2,22} = 1
\]

Eq. 4.104

\[
t_{2,23} = j \frac{\rho_j^*}{\rho_j}
\]

Eq. 4.105

\[
t_{2,24} = j \frac{\rho_k^*}{\rho_k} - \frac{\Delta \rho}{\rho_g \rho_j} \frac{TF_{\Gamma_{g,sub}}}{dp/dz}
\]

Eq. 4.106

\[
t_{2,21} = \left( \frac{\rho_g^*}{\rho_g} \frac{dp}{dz} - \frac{\rho_j^*}{\rho_j} \frac{dh_l}{dz} \right) \left( C_0 j + V_{gg} \right) - \frac{\Delta \rho}{\rho_g \rho_j} \left( \frac{TF_{\Gamma_{g,sub}}}{HF_{DEN}} - \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} \right)
\]

Eq. 4.107

\[
t_{2,22} = \frac{\rho_g^*}{\rho_g} \frac{dp}{dz} C_0 \alpha + \frac{\rho_j^*}{\rho_j} \frac{dh_l}{dz} \left( 1 - C_0 \alpha \right) - \frac{\Delta \rho}{\rho_g \rho_j} \left( \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} - \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} \right)
\]

Eq. 4.108

\[
t_{2,23} = \left( s \left( 1 - \alpha \right) - \Gamma_{g,sub} \right) \frac{\rho_j^*}{\rho_j} \frac{\rho_j^*}{\rho_j} \frac{\rho_j^*}{\rho_j} \left( \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} - \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} \right)
\]

Eq. 4.109

\[
t_{2,24} = \left( s \alpha + \frac{\Gamma_{g,sub}}{\rho_g} \right) \frac{\rho_g^*}{\rho_g} - \frac{\Delta \rho}{\rho_g \rho_j} \left( \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} - \frac{TF_{\Gamma_{g,sub}}^{\Gamma_{f}}}{HF_{DEN}} \right)
\]

Eq. 4.110
\[
f q_2 = \frac{\Delta \rho}{\rho_g \rho_l} \left( \frac{\left( T F_{q}^{\gamma, e, ab} T F_{q, a}^{\gamma} T F_{q, a}^{T_{u}} \right)}{H F_{D E N}} \right) \delta q'' - T F_{h_{d}}^{\gamma, e, ab} \delta h_d \tag{Eq. 4.111}
\]

Where the TF coefficients are given in sections 4.5.4 and 4.5.7.

- Liquid energy conservation equation (eq. 4.17)

\[
\rho_i j_i \frac{d \delta h_i}{dz} + \rho_g j_g h_g' \frac{d \delta p}{dz} + \left( \rho_g h_g' \frac{dp}{dz} - \rho_l \frac{dh_l}{dz} \right) (C_o j + V_{ji}) \delta \alpha \\
+ \left( \rho_l (1 - C_o \alpha) \frac{dh_l}{dz} + \rho_g h_g' \alpha C_o \frac{dp}{dz} \right) \delta j + \left( s \rho_l (1 - \alpha) - \Gamma_{g, sub} \right) \delta h_l \\
+ \left( s (\alpha \rho_l h_g' (1 - 1) + h_g' \Gamma_{g, sub}) \right) \delta p + h_g \alpha \Gamma_{g, sub} - \frac{P_h}{A} \delta q'' - \delta q'' = 0
\tag{Eq. 4.112}
\]

Substituting the volumetric vapor generation rate and heat flux perturbations with their expressions given in sections 4.5.4 and 4.5.7,

\[
t_{1,33} \frac{d \delta h_i}{dz} + t_{1,34} \frac{d \delta p}{dz} + t_{2,31} \delta \alpha + t_{2,32} \delta j + t_{2,33} \delta h_l + t_{2,34} \delta p + f_{q_3} = 0
\tag{Eq. 4.113}
\]

where

\[
t_{1,31} = 0
\tag{Eq. 4.114}
\]

\[
t_{1,32} = 0
\tag{Eq. 4.115}
\]

\[
t_{1,33} = \rho_i j_i
\tag{Eq. 4.116}
\]

\[
t_{1,34} = \rho_g j_g h_g' + h_g TF_{q, a}^{\gamma, e, ab}
\tag{Eq. 4.117}
\]

\[
t_{2,31} = \left( \rho_g h_g' \frac{dp}{dz} - \rho_l \frac{dh_l}{dz} \right) (C_o j + V_{ji}) + h_g \left( T F_{q, a}^{\gamma, e, ab} - T F_{q, a}^{T_{u}} \frac{T F_{q, a}^{\gamma}}{H F_{D E N}} \right) + \frac{P_h}{A} T F_{a}^{q'} \tag{Eq. 4.118}
\]

\[
t_{2,32} = \rho_l (1 - C_o \alpha) \frac{dh_l}{dz} + \rho_g h_g' C_o \alpha \frac{dp}{dz} + h_g \left( T F_{q, a}^{\gamma, e, ab} - T F_{q, a}^{T_{u}} \frac{T F_{q, a}^{T_{u}}}{H F_{D E N}} \right) + \frac{P_h}{A} T F_{a}^{q'} \tag{Eq. 4.119}
\]

\[
t_{2,33} = \rho_l (1 - \alpha) s - \Gamma_{g, sub} + h_g \left( T F_{h_{d}}^{\gamma, e, ab} - T F_{h_{d}}^{\gamma, e, ab} \frac{T F_{h_{d}}^{T_{u}}}{H F_{D E N}} \Gamma_{l}^{c_l} \right) + \frac{P_h}{A} T F_{h_{d}}^{q'} \tag{Eq. 4.120}
\]
\[ t_{2,34} = s\left( \alpha \rho_g' h_g' + h_{lg} TF_{dp/dy}^{T_{x,y}} - 1 \right) + h_{lg} \left( T_F^{T_{x,y}} - T_F^{T_{x,y}} / HF_{DEN} \right) + h_{cg} \Gamma + P_b \left( T_F^{T_{x,y}} / HF_{DEN} \right) \]

Eq. 4.121

\[ f_{q_3} = \left( -h_{lg} T_F^{T_{x,y}} / HF_{DEN} + P_b \left( T_F^{T_{y,x}} / HF_{DEN} \right) \delta q' - \delta q'' + h_{lg} T^{T_{x,y}} \delta h_{ad} \right) \]

Eq. 4.122

- **Momentum conservation equation (eq. 4.18)**

The momentum equation requires friction factor (section 4.5.5), friction multiplier (section 4.5.6) and mass flux perturbations (section 4.7.2). Inserting them in the perturbed form of the mixture momentum conservation equation, we obtain

\[ t_{1,41} \frac{d\delta \alpha}{dz} + t_{1,42} \frac{d\delta j}{dz} + t_{1,43} \frac{d\delta h_t}{dz} + t_{1,44} \frac{d\delta \phi}{dz} + t_{2,41} \delta \alpha + t_{2,42} \delta j + t_{2,43} \delta h_t + t_{2,44} \delta \phi = 0 \]

Eq. 4.123

where

\[ t_{1,41} = \rho_l \left( -2 \frac{j_i}{(1 - \alpha)} (C_0j + V_{gy}) + \frac{j_i^2}{(1 - \alpha)^2} \right) + \rho_g \left( C_0j + V_{gy} \right) \]

Eq. 4.124

\[ t_{1,42} = 2 \left( \rho_l \frac{j_i}{(1 - \alpha)} \right) \left( 1 - C_0 \alpha \right) + \rho_g j_i C_0 \]

Eq. 4.125

\[ t_{1,43} = \frac{j_i^2}{(1 - \alpha)} \rho_g' \]

Eq. 4.126

\[ t_{1,44} = 1 + \frac{j_i^2}{\alpha} \rho_g' \]

Eq. 4.127

\[ t_{2,41} = -s \Delta \rho \left( C_0j + V_{gy} \right) + \frac{\partial \left( dp \right)}{\partial \alpha} \frac{\partial \left( dp \right)}{\partial f} \frac{\partial \left( dp \right)}{\partial f} \frac{\partial \left( dp \right)}{\partial m} \frac{\partial \left( dp \right)}{\partial \alpha} \frac{\partial \Phi_{LO}^2}{\partial \alpha} \]

Eq. 4.128

\[ t_{2,42} = s \left( \rho_l - \Delta \rho \alpha C_0 \right) + \frac{\partial \left( dp \right)}{\partial f} \frac{\partial \left( dp \right)}{\partial j} \frac{\partial \Phi_{LO}^2}{\partial \alpha} \frac{\partial \Phi_{LO}^2}{\partial j} \]

Eq. 4.129
\[ t_{2,43} = \left( j_s + \frac{\partial \left( \frac{dp}{dz} \right)_{gr}}{\partial \rho_t} + \frac{\partial \left( \frac{dp}{dz} \right)_{fr}}{\partial \rho_t} \right) \rho_t + \frac{\partial \left( \frac{dp}{dz} \right)_{fr}}{\partial \Phi_{1,o}} \frac{\partial \Phi_{1,o}^2}{\partial h_t} \]  
Eq. 4.130

\[ t_{2,44} = \left( j_s + \frac{\partial \left( \frac{dp}{dz} \right)_{gr}}{d \rho_g} \right) \rho_g' + \frac{\partial \left( \frac{dp}{dz} \right)_{fr}}{\partial \Phi_{1,o}} \frac{\partial \Phi_{1,o}^2}{\partial p} \]  
Eq. 4.131

\[ f q_a = 0 \]  
Eq. 4.132

### 4.5.3. Saturated two-phase region \((\delta \alpha, \delta j, \delta p)\)

As in the previous section, we give the original perturbed and final form of the conservation equations. In this region vapor and liquid are saturated and their variations depend only on local pressure.

- **Vapor continuity (eq. 4.19):** The vapor continuity equation

\[
(C_0 j + V_g) \frac{d \delta \alpha}{dz} + C_0 \alpha \frac{d \delta j}{dz} + \rho_g' \frac{d \delta p}{dz} + \left( s + C_0 \frac{dj}{dz} + \rho_g' \frac{dp}{dz} (C_0 j + V_g) \right) \delta \alpha \\
+ C_0 \left( \frac{d \alpha}{dz} + \rho_g' \frac{dp}{dz} \alpha \right) \delta j + \left( \Gamma_{g,\text{sat}} + s \alpha \right) \rho_g' \delta p - \frac{1}{\rho_g} \Gamma_{g,\text{sat}} = 0
\]  
Eq. 4.133

needs the volumetric vapor generation rate perturbation and becomes

\[
t_{1,11} \frac{d \delta \alpha}{dz} + t_{1,12} \frac{d \delta j}{dz} + t_{1,13} \frac{d \delta p}{dz} + t_{2,11} \delta \alpha + t_{2,12} \delta j + t_{2,13} \delta p + f q_1 = 0
\]  
Eq. 4.134

where

\[
t_{1,11} = C_0 j + V_g
\]  
Eq. 4.135

\[
t_{1,12} = C_0 \alpha
\]  
Eq. 4.136

\[
t_{1,13} = j_s \rho_g' \frac{\Gamma_{g,\text{sat}}}{\rho_g}
\]  
Eq. 4.137
\[ t_{2,11} = s + C_0 \frac{d j}{dz} + \left( C_0 j + V_{fg} \right) \frac{\rho_f'}{\rho_f} \frac{dp}{dz} - \frac{1}{\rho_g} \left( \frac{TF_{g,sw}^\Gamma - TF_{q,sw}^\Gamma}{HF_{DEN}} \right) \]

Eq. 4.138

\[ t_{2,12} = \alpha C_0 \frac{\rho_f'}{\rho_g} \frac{dp}{dz} + C_0 \frac{d \alpha}{dz} - \frac{1}{\rho_g} \left( \frac{TF_{j,sw}^\Gamma}{HF_{DEN}} \right) \]

Eq. 4.139

\[ t_{2,13} = \left( s \alpha + \frac{\Gamma_{g,sw}}{\rho_g} \right) \frac{\rho_f'}{\rho_f} \frac{\Gamma_{l,sw}}{\rho_l} + \frac{1}{\rho_g} \left( \frac{TF_{q,sw}^\Gamma h_f'}{c_f} + TF_{q,sw}^\Gamma \right) \]

Eq. 4.140

\[ f q_1 = \frac{1}{\rho_g} \left( \frac{TF_{q,sw}^\Gamma + TF_{q,sw}^\Gamma}{HF_{DEN}} \right) \delta q'' - TF_{q,sw}^\Gamma \delta q'' \]

Eq. 4.141

See sections 4.5.4 and 4.5.7 for the TF coefficients.

- **Mixture continuity (eq. 4.20)**

Inserting the perturbation of the volumetric vapor generation rate (section 4.5.4) in the following mixture continuity equation,

\[
\frac{d \delta j}{dz} + \left( j_f \frac{\rho_f'}{\rho_f} + j_g \frac{\rho_g'}{\rho_g} \right) \frac{d \delta p}{dz} + \left( \frac{\rho_f'}{\rho_f} - \frac{\rho_f'}{\rho_f} \right) \frac{dp}{dz} \left( C_0 j + V_{fg} \right) \delta \alpha + \left( C_0 \rho_f' + \left( 1 - \alpha C_0 \right) \frac{\rho_f'}{\rho_f} \right) \frac{dp}{dz} j
\]

\[
+ \left( s \alpha \frac{\rho_f'}{\rho_g} + s \left( 1 - \alpha \right) \frac{\rho_f'}{\rho_f} - \frac{\rho_f'}{\rho_f} \frac{\rho_g'}{\rho_g^2} \right) \Gamma_{g,sw} \delta p - \frac{\Delta \rho}{\rho_f \rho_g} \delta \Gamma_{g,sw} = 0
\]

Eq. 4.142

leads to

\[ t_{1,22} \frac{d \delta j}{dz} + t_{1,23} \frac{d \delta p}{dz} + t_{1,21} \delta \alpha + t_{2,22} \delta j + t_{2,23} \delta p + f q_2 = 0 \]

Eq. 4.143

where

\[ t_{1,21} = 0 \]

Eq. 4.144

\[ t_{1,22} = 1 \]

Eq. 4.145

\[ t_{1,23} = j_f \frac{\rho_f'}{\rho_f} + j_g \frac{\rho_g'}{\rho_g} \frac{\Delta \rho}{\rho_f \rho_g} \frac{TF_{q,sw}^\Gamma}{dp/dz} \]

Eq. 4.146
\[ t_{2,21} = \left( \frac{\rho'_f - \rho'_g}{\rho_f - \rho_g} \right) \frac{dp}{dz} (C_0 j + V_{gy}) - \frac{\Delta \rho}{\rho_g \rho_f} \left( \frac{TF_{q,sw}^{T_f}}{HF_{DEN}} - \frac{TF_{q,sw}^{T_f} \Gamma_{\alpha}^{T_f}}{HF_{DEN}} \right) \]  
Eq. 4.147

\[ t_{2,22} = \left( \frac{\rho'_f - \rho'_g}{\rho_f - \rho_g} \right) C_0 \alpha \frac{dp}{dz} - \frac{\Delta \rho}{\rho_g \rho_f} \left( \frac{TF_{q,sw}^{T_f}}{HF_{DEN}} - \frac{TF_{q,sw}^{T_f} \Gamma_{\alpha}^{T_f}}{HF_{DEN}} \right) \]  
Eq. 4.148

\[ t_{2,23} = \delta \left( \frac{\rho'_f - \alpha \left( \frac{\rho'_f - \rho'_g}{\rho_f - \rho_g} \right) \rho_f}{\rho_f} \right) \left( \frac{\rho'_f - \rho'_g}{\rho_f - \rho_g} \right) \Gamma_{g,sat} - \frac{\Delta \rho}{\rho_g \rho_f} \left( \frac{TF_{q,sw}^{T_f}}{HF_{DEN}} \right) \left( \frac{TF_{q,sw}^{T_f} h'_f}{c_f} + TF_{q}^{T_f} \right) \]  
Eq. 4.149

\[ f_{q_2} = \frac{\Delta \rho}{\rho_g \rho_f} \left( \frac{TF_{q,sw}^{T_f} \Gamma_{\alpha}^{T_f}}{HF_{DEN}} \right) \delta q'' - TF_{q,sw}^{T_f} \delta q'' \]  
Eq. 4.150

The TF with subscript \( T_{co} \) or \( q'' \) are given in section 4.5.7.

- **Momentum conservation equation (eq. 4.21)**

The mixture momentum equation is identical as that of the two-phase subcooled region, except for the partial derivative of the liquid density, which will become function of local pressure now:

\[ t_{1,31} \frac{d\delta \alpha}{dz} + t_{1,32} \frac{d \delta j}{dz} + t_{1,33} \frac{d \delta p}{dz} + t_{2,31} \delta \alpha + t_{2,32} \delta j + t_{2,33} \delta p = 0 \]  
Eq. 4.151

where

\[ t_{1,31} = \rho_f \left( -2 \frac{j_i}{(1 - \alpha)} (C_0 j + V_{gy}) + \frac{j_i^2}{(1 - \alpha)^2} \right) + \rho_g \left( C_0 j + V_{gy} \right)^2 \]  
Eq. 4.152

\[ t_{1,32} = 2 \left( \rho_f \left( \frac{j_i}{1 - \alpha} (1 - C_0 \alpha) + \rho_g j_g C_0 \right) \right) \]  
Eq. 4.153

\[ t_{1,33} = 1 + \frac{j_i^2}{(1 - \alpha) \rho_f} + \frac{j_i^2}{\alpha \rho_g} \]  
Eq. 4.154

\[ t_{2,31} = -s \Delta \rho \left( C_0 j + V_{gy} \right) + \frac{\partial}{\partial \alpha} \left( \frac{dp}{dz} \right)_{gr} + \frac{\partial}{\partial f} \frac{dp}{dz} + \frac{\partial}{\partial m} \frac{dp}{dz} + \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \Phi_{L,0}} \frac{dp}{dz} \]  
Eq. 4.155
\[
t_{2,32} = s \left( \rho_f - \Delta \rho \alpha C_0 \right) + \frac{\partial (dp)}{\partial j} \frac{\partial f}{\partial j} + \frac{\partial (dp)}{\partial m} \frac{\partial m}{\partial j} + \frac{\partial (dp)}{\partial \Phi_{LO}} \frac{\partial \Phi_{LO}^2}{\partial j} \quad \text{Eq. 4.156}
\]

\[
t_{2,43} = \left( j_s + \frac{\partial (dp)}{\partial \rho_f} \right) \frac{\partial f}{\partial s} + \left( j_g + \frac{\partial (dp)}{\partial \rho_f} \right) \frac{\partial g}{\partial s} + \left( \frac{d}{dp} \right) \frac{\partial \rho_f}{\partial \Phi_{LO}} \frac{\partial \Phi_{LO}^2}{\partial p} \quad \text{Eq. 4.157}
\]

fq3 = 0 \quad \text{Eq. 4.158}

### 4.5.4. Volumetric vapor generation rate perturbation

The time dependent equations are given in section 4.3.1. After their linearization and Laplace transformation, the perturbed form of the vapor generation rate is obtained.

- **Subcooled region**

The total vapor generation rate perturbation is the sum of two terms; the subcooled boiling and flashing contributions:

\[
\delta \Gamma_{g,sub} = \delta \Gamma_{sub} + \delta \Gamma_{flash} \quad \text{Eq. 4.159}
\]

- **Lahey’s subcooled vapor generation term (eq. 4.23)**

After linearization and Laplace transformation of the subcooling term of the vapor generation rate and substitution of the perturbation of the NVG enthalpy (see section 4.5.8), we obtain

\[
\delta \Gamma_{sub} = TF_{\alpha}^{\Gamma_{sub}} \delta \alpha + TF_{j}^{\Gamma_{sub}} \delta j + TF_{h_l}^{\Gamma_{sub}} \delta h_l + TF_{p}^{\Gamma_{sub}} \delta p + TF_{q}^{\Gamma_{sub}} \delta q + TF_{h_d}^{\Gamma_{sub}} \delta h_d \quad \text{Eq. 4.160}
\]

where

\[
TF_{\alpha}^{\Gamma_{sub}} = \frac{\partial \Gamma_{sub}}{\partial \alpha} \quad \text{Eq. 4.161}
\]

\[
TF_{j}^{\Gamma_{sub}} = \frac{\partial \Gamma_{sub}}{\partial j} \quad \text{Eq. 4.162}
\]

\[
TF_{h_l}^{\Gamma_{sub}} = \frac{\partial \Gamma_{sub}}{\partial h_l} \quad \text{Eq. 4.163}
\]
\[ TF_{p}^{\Gamma_{\text{sub}}} = \frac{\partial \Gamma_{\text{sub}}}{\partial h_{g}} h'_{g} + \frac{\partial \Gamma_{\text{sub}}}{\partial h_{f}} h'_{f} \quad \text{Eq. 4.164} \]

\[ TF_{q}^{\Gamma_{\text{sub}}} = \frac{\partial \Gamma_{\text{sub}}}{\partial q^*} \quad \text{Eq. 4.165} \]

\[ TF_{h_{d}}^{\Gamma_{\text{sub}}} = \frac{\partial \Gamma_{\text{sub}}}{\partial h_{d}} \quad \text{Eq. 4.166} \]

**Flashing term (eq. 4.22)**

\[ \Delta \Gamma_{\text{flash}} = TF_{dp/dz}^{\Gamma_{\text{flash}}} \frac{d \delta p}{dz} + TF_{\alpha}^{\Gamma_{\text{flash}}} \delta \alpha + TF_{j}^{\Gamma_{\text{flash}}} \delta j + TF_{p}^{\Gamma_{\text{flash}}} \delta p \quad \text{Eq. 4.167} \]

Where TF are the transfer functions of the flashing term, obtained from its linearization:

\[ TF_{dp/dz}^{\Gamma_{\text{flash}}} = \text{LIN}^{\Gamma_{\text{flash}}} \quad \text{Eq. 4.168} \]

\[ TF_{\alpha}^{\Gamma_{\text{flash}}} = \text{LIN}_{\alpha} \quad \text{Eq. 4.169} \]

\[ TF_{j}^{\Gamma_{\text{flash}}} = \text{LIN}_{j} \quad \text{Eq. 4.170} \]

\[ TF_{p}^{\Gamma_{\text{flash}}} = \text{LIN}_{p}^{\Gamma_{\text{flash}}} + s \text{LIN}_{dp/dt}^{\Gamma_{\text{flash}}} \quad \text{Eq. 4.171} \]

Thus, the total perturbation of the vapor generation rate in two-phase subcooled region is the sum of the above terms

\[ \Delta \Gamma_{g,\text{sub}} = TF_{dp/dz}^{\Gamma_{\text{e,sub}}} \frac{\partial \delta p}{\partial z} + TF_{\alpha}^{\Gamma_{\text{e,sub}}} \delta \alpha + TF_{j}^{\Gamma_{\text{e,sub}}} \delta j + TF_{p}^{\Gamma_{\text{e,sub}}} \delta p + TF_{h_{d}}^{\Gamma_{\text{e,sub}}} \delta h_{d} + TF_{q}^{\Gamma_{\text{e,sub}}} \delta q^* + TF_{h_{d}}^{\Gamma_{\text{e,sub}}} \delta h_{d} \quad \text{Eq. 4.172} \]

where

\[ TF_{dp/dz}^{\Gamma_{\text{e,sub}}} = \text{LIN}_{dp/dz}^{\Gamma_{\text{e,sub}}} \quad \text{Eq. 4.173} \]

\[ TF_{\alpha}^{\Gamma_{\text{e,sub}}} = TF_{\alpha}^{\Gamma_{\text{e,sub}}} + \text{LIN}_{\alpha}^{\Gamma_{\text{e,sub}}} \quad \text{Eq. 4.174} \]

\[ TF_{j}^{\Gamma_{\text{e,sub}}} = TF_{j}^{\Gamma_{\text{e,sub}}} + \text{LIN}_{j}^{\Gamma_{\text{e,sub}}} \quad \text{Eq. 4.175} \]
\[ T_{F_{h_i}}^{V_{g, sat}} = T_{F_{h_i}}^{V_{sat}} \]  
Eq. 4.176

\[ T_{F_{p}}^{V_{g, sat}} = T_{F_{p}}^{V_{sat}} + T_{F_{p}}^{V_{flash}} \]  
Eq. 4.177

\[ T_{F_{q}}^{V_{g, sat}} = T_{F_{q}}^{V_{sat}} \]  
Eq. 4.178

\[ T_{F_{h_i}}^{V_{g, sat}} = T_{F_{h_i}}^{V_{sat}} \]  
Eq. 4.179

- **Saturated region (eq. 4.26)**

Perturbation of the total vapor generation rate is considered as the sum of two terms, the heat input, and flashing components:

\[ \delta \Gamma_{g, sat} = \delta \Gamma_{heat} + \delta \Gamma_{flash} \]  
Eq. 4.180

**Heat input term (eq. 4.24)**

\[ \delta \Gamma_{heat} = \frac{\partial \Gamma_{heat}}{\partial \left( \frac{1}{h_{fg}} \right)} \delta \left( \frac{1}{h_{fg}} \right) + \frac{\partial \Gamma_{heat}}{\partial q''} \delta q'' + \frac{\partial \Gamma_{heat}}{\partial q'''} \delta q''' \]  
Eq. 4.181

\[ \frac{\partial \Gamma_{heat}}{\partial \left( \frac{1}{h_{fg}} \right)} = q''p_h + q''_f \]  
Eq. 4.182

\[ \delta \left( \frac{1}{h_{fg}} \right) = -\frac{h_f' - h_f'}{h_{fg}^2} \delta p \]  
Eq. 4.183

\[ \frac{\partial \Gamma_{heat}}{\partial q''} = \frac{P_h}{h_{fg}A} \]  
Eq. 4.184

\[ \frac{\partial \Gamma_{heat}}{\partial q'''} = \frac{1}{h_{fg}} \]  
Eq. 4.185

Summing the heat and the flashing terms leads to:

\[ \delta \Gamma_{g, sat} = T_{F_{dp/dz}}^{V_{g, sat}} \frac{\partial \delta p}{\partial z} + T_{F_{\alpha}}^{V_{sat}} \delta \alpha + T_{F_{j}}^{V_{sat}} \delta j + T_{F_{p}}^{V_{sat}} \delta p + T_{F_{q}}^{V_{sat}} \delta q'' + T_{F_{q''}}^{V_{sat}} \delta q''' \]  
Eq. 4.186

where

\[ T_{F_{dp/dz}}^{V_{g, sat}} = LIN_{dp/dz}^{V_{g, sat}} \]  
Eq. 4.187

\[ T_{F_{\alpha}}^{V_{sat}} = LIN_{\alpha}^{V_{sat}} \]  
Eq. 4.188
\[ TF_{j_{\text{q,act}}}^{\Gamma_{\text{heat}}} = LIN_{j_{\text{q,act}}}^{\Gamma_{\text{heat}}} \]

Eq. 4.189

\[ TF_{p_{\text{q,act}}}^{\Gamma_{\text{heat}}} = - \frac{h'_g - h'_f}{h'_{fg}} \frac{\partial \Gamma_{\text{heat}}}{\partial (1/h'_{fg})} + TF_{p_{\text{q,act}}}^{\Gamma_{\text{heat}}} \]

Eq. 4.190

\[ TF_{q_{\text{q,act}}}^{\Gamma_{\text{heat}}} = \frac{\partial \Gamma_{\text{heat}}}{\partial q''} \]

Eq. 4.191

\[ TF_{q''_{\text{q,act}}}^{\Gamma_{\text{heat}}} = \frac{\partial \Gamma_{\text{heat}}}{\partial q''} \]

Eq. 4.192

### 4.5.5. Friction factor perturbation

In our transient analysis, friction factor is considered only function of Reynolds number.

\[ \delta f_k = \frac{\partial f_k}{\partial Re_k} \delta Re_k \]

Eq. 4.193

Since Reynolds number depends on state variables (void fraction and volumetric flux),

\[ \delta Re_k = \frac{\partial Re_k}{\partial \alpha} \delta \alpha + \frac{\partial Re_k}{\partial j} \delta j \]

Eq. 4.194

the friction factor is also related to same variables. Perturbation of the friction factor for a phase is written as:

\[ \delta f_k = \frac{\partial f_k}{\partial Re_k} \frac{\partial Re_k}{\partial \alpha} \delta \alpha + \frac{\partial f_k}{\partial Re_k} \frac{\partial Re_k}{\partial j} \delta j \]

Eq. 4.195

### 4.5.6 Friction multiplier perturbation

Perturbed forms of the multipliers are the needed for the pressure drop calculation. They are used in the two-phase mixture momentum conservation equation.
4.5.6.1 Chisholm multiplier perturbation

The Chisholm correlation (see 4.3.3.1) has been perturbed with respect to the state variables \((\alpha, j, h, p)\), using the phase Reynolds number and friction factors as follows:

\[
\delta \Gamma_{\text{Chis}} = \frac{\partial \Gamma_{\text{Chis}}}{\partial f_g} \delta f_g + \frac{\partial \Gamma_{\text{Chis}}}{\partial f_l} \delta f_l \quad \text{Eq. 4.196}
\]

\[
\delta a_{\text{Chis}} = \frac{\partial a_{\text{Chis}}}{\partial \Gamma_{\text{Chis}}} \delta \Gamma_{\text{Chis}} + \frac{\partial a_{\text{Chis}}}{\partial \text{Re}_l} \delta \text{Re}_l \quad \text{Eq. 4.197}
\]

and the quality perturbation is

\[
\delta x = \frac{\partial x}{\partial \alpha} \delta \alpha + \frac{\partial x}{\partial j} \delta j + \frac{\partial x}{\partial h} \delta h + \frac{\partial x}{\partial p} \delta p \quad \text{Eq. 4.198}
\]

where

\[
\frac{\partial x}{\partial \alpha} = \frac{\partial x}{\partial j_l} \frac{\partial j_l}{\partial \alpha} + \frac{\partial x}{\partial j_g} \frac{\partial j_g}{\partial \alpha} \quad \text{Eq. 4.199}
\]

\[
\frac{\partial x}{\partial j} = \frac{\partial x}{\partial j_l} \frac{\partial j_l}{\partial j} + \frac{\partial x}{\partial j_g} \frac{\partial j_g}{\partial j} \quad \text{Eq. 4.200}
\]

\[
\frac{\partial j_l}{\partial \alpha} = -\frac{\partial j_g}{\partial \alpha} \quad \text{Eq. 4.201}
\]

\[
\frac{\partial j_l}{\partial j} = (1 - C_o \alpha) \quad \text{Eq. 4.202}
\]

\[
\frac{\partial j_g}{\partial \alpha} = C_o j + V_{\text{ij}} \quad \text{Eq. 4.203}
\]

\[
\frac{\partial j_g}{\partial j} = C_o \alpha \quad \text{Eq. 4.204}
\]
In the subcooled region

\[
\frac{\partial x}{\partial h_i} = \frac{\partial x}{\partial \rho_i} \rho_i''
\]

Eq. 4.205

\[
\frac{\partial x}{\partial p} = \frac{\partial x}{\partial \rho_g} \rho_g'
\]

Eq. 4.206

In the saturated region, the quality depends only on pressure and the partial derivative with respect to enthalpy is set to zero and pressure dependency is given as:

\[
\frac{\partial x}{\partial p} = \frac{\partial x}{\partial \rho_f} \rho_f' + \frac{\partial x}{\partial \rho_g} \rho_g'
\]

Eq. 4.207

Finally the general perturbed form of the friction multiplier, in compact form, is given as:

\[
\partial \Phi^2_{lo} = \frac{\partial \Phi^2_{ch}}{\partial \alpha} \delta \alpha + \frac{\partial \Phi^2_{lo}}{\partial j} \delta j + \frac{\partial \Phi^2_{lo}}{\partial h_i} \delta h_i + \frac{\partial \Phi^2_{lo}}{\partial p} \delta p
\]

Eq. 4.208

where

\[
\frac{\partial \Phi^2_{ch}}{\partial \alpha} = \frac{\partial \Phi_{ch}}{\partial \alpha} \frac{\partial x}{\partial \alpha} + \frac{\partial \Phi_{ch}}{\partial G_{ch}} \left( \frac{\partial G_{ch}}{\partial \rho_f} \frac{\partial \text{Re}_f}{\partial \alpha} + \frac{\partial G_{ch}}{\partial \rho_i} \frac{\partial \text{Re}_i}{\partial \alpha} \right)
\]

\[
+ \frac{\partial \Phi_{ch}}{\partial G_{ch}} \left( \frac{\partial G_{ch}}{\partial \rho_f} \frac{\partial \text{Re}_f}{\partial \alpha} + \frac{\partial G_{ch}}{\partial \rho_i} \frac{\partial \text{Re}_i}{\partial \alpha} \right) + \frac{\partial a_{ch}}{\partial G_{ch}} \frac{\partial \text{Re}_i}{\partial \alpha}
\]

Eq. 4.209

\[
\frac{\partial \Phi^2_{ch}}{\partial j} = \frac{\partial \Phi_{ch}}{\partial j} \frac{\partial x}{\partial j} + \frac{\partial \Phi_{ch}}{\partial G_{ch}} \left( \frac{\partial G_{ch}}{\partial \rho_f} \frac{\partial \text{Re}_f}{\partial j} + \frac{\partial G_{ch}}{\partial \rho_i} \frac{\partial \text{Re}_i}{\partial j} \right)
\]

\[
+ \frac{\partial \Phi_{ch}}{\partial G_{ch}} \left( \frac{\partial G_{ch}}{\partial \rho_f} \frac{\partial \text{Re}_f}{\partial j} + \frac{\partial G_{ch}}{\partial \rho_i} \frac{\partial \text{Re}_i}{\partial j} \right) + \frac{\partial a_{ch}}{\partial G_{ch}} \frac{\partial \text{Re}_i}{\partial j}
\]

Eq. 4.210

\[
\frac{\partial \Phi^2_{lo}}{\partial h_i} = \frac{\partial \Phi_{ch}}{\partial h_i} \frac{\partial x}{\partial h_i}
\]

Eq. 4.211

\[
\frac{\partial \Phi^2_{lo}}{\partial p} = \frac{\partial \Phi_{ch}}{\partial p} \frac{\partial x}{\partial p}
\]

Eq. 4.212
4.5.6.2 Homogenous multiplier perturbation

This multiplier is used only for local pressure drop calculation. It is considered function of quality.

\[ \delta \Phi_{HM}^2 = \frac{\partial \Phi_{HM}^2}{\partial x} \delta x \]  
Eq. 4.213

Substituting the quality perturbation eq. 4.198 into Eq. 4.213 we obtain the general expression of the homogenous multiplier perturbation

\[ \delta \Phi_{HM}^2 = \frac{\partial \Phi_{HM}^2}{\partial \alpha} \delta \alpha + \frac{\partial \Phi_{HM}^2}{\partial j} \delta j + \frac{\partial \Phi_{HM}^2}{\partial h_j} \delta h_j + \frac{\partial \Phi_{HM}^2}{\partial p} \delta p \]  
Eq. 4.214

where

\[ \frac{\partial \Phi_{HM}^2}{\partial \alpha} = \frac{\partial \Phi_{HM}^2}{\partial x} \frac{\partial x}{\partial \alpha} \]  
Eq. 4.215

\[ \frac{\partial \Phi_{HM}^2}{\partial j} = \frac{\partial \Phi_{HM}^2}{\partial x} \frac{\partial x}{\partial j} \]  
Eq. 4.216

\[ \frac{\partial \Phi_{HM}^2}{\partial h_j} = \frac{\partial \Phi_{HM}^2}{\partial x} \frac{\partial x}{\partial h_j} \]  
Eq. 4.217

\[ \frac{\partial \Phi_{HM}^2}{\partial p} = \frac{\partial \Phi_{HM}^2}{\partial x} \frac{\partial x}{\partial p} \]  
Eq. 4.218

4.5.7. Heat flux perturbation and its coupling to cladding temperature perturbation

From linearization and Laplace transformation of heat flux given in section 4.3.4,

\[ \delta q^* = T F_a^{q*} \delta \alpha + T F_j^{q*} \delta j + T F_{l_i}^{q*} \delta l_i + T F_p^{q*} \delta p + T F_{v_o}^{q*} \delta v_o \]  
Eq. 4.219

where the TF correspond to the transfer functions of the heat flux perturbation related to the subscript variable. They are, simply, partial derivatives of the heat flux, which were calculated with a Matlab procedure, written for that purpose.

\[ T F_a^{q*} = \frac{\partial q^*}{\partial \alpha} \]  
Eq. 4.220

\[ T F_j^{q*} = \frac{\partial q^*}{\partial j} \]  
Eq. 4.221
This perturbed form will be combined now with the cladding temperature perturbation (obtained in the fuel dynamics chapter).

\[ \delta T_{co} = TF_{q}^{T_{co}} \delta q^" + TF_{p}^{T_{co}} \delta q^" \]

Equations (4.219) and (4.225) are solved for heat flux and cladding temperature. The results are:

\[ \delta q^" = - \frac{\left( TF_{q}^{T_{co}} \delta q^" + TF_{p}^{T_{co}} \delta T_{co} + TF_{q}^{T_{co}} \delta p + TF_{q}^{T_{co}} \delta \bar{g} + TF_{p}^{T_{co}} \delta \alpha \right)}{HF_{DEN}} \]

and

\[ \delta T_{co} = - \frac{\left( TF_{q}^{T_{co}} \delta q^" + TF_{p}^{T_{co}} \left( TF_{q}^{T_{co}} \delta T_{co} + TF_{p}^{T_{co}} \delta p + TF_{q}^{T_{co}} \delta \bar{g} + TF_{p}^{T_{co}} \delta \alpha \right) \right)}{HF_{DEN}} \]

where

\[ HF_{DEN} = TF_{q}^{T_{co}} TF_{p}^{T_{co}} - 1 \]

This last form of the heat flux perturbation will be used in the transient computations.

### 4.5.8. Enthalpy at NVG point

It is in general function of void fraction, mixture volumetric flux, local pressure and heat flux at NVG.

\[ \delta h_{d} = \frac{\partial h_{d}}{\partial \bar{j}} \delta \bar{j}_{d} + \frac{\partial h_{d}}{\partial \bar{p}} \delta \bar{p}_{d} + \frac{\partial h_{d}}{\partial q^"} \delta q_{d}^" \]

where the coefficients are its partial derivatives.
4.5.9. Nodal transient solution

In general, the perturbed form of the conservation equations can be written as

\[
C_1 \frac{\partial \delta y}{\partial t} + C_2 \frac{\partial \delta y}{\partial z} + C_3 \delta y + F \delta q = 0
\]

Eq. 4.230

After Laplace transformation they result in:

\[
C_2 \frac{\partial \delta y}{\partial z} + (C_1 s + C_3) \delta y + F \delta q = 0
\]

Eq. 4.231

The solution of this set of non-homogenous coupled linear differential equations is obtained by application of the following procedure:

\[
T_1 = C_2
\]

Eq. 4.232

\[
T_2 = C_1 s + C_3
\]

Eq. 4.233

\[
T = -T_1^{-1} T_2
\]

Eq. 4.234

\[
H = -T_1^{-1} F \delta q
\]

Eq. 4.235

Substituting (4.232) to (4.235) in (4.231), leads to

\[
\frac{\partial \delta y}{\partial z} = T \delta y + H
\]

Eq. 4.236

or

\[
\delta y' = T \delta y + H
\]

Eq. 4.237

Let \(X_{nn}\) be the matrix of the eigenvectors (\(n\) by \(n\) matrix) corresponding to the \((n)\) eigenvalues of the matrix \(T\{n,n\}\), then from matrix algebra we can write that

\[
\delta y = X_{nn} \delta w
\]

Eq. 4.238

\[
\delta y' = X_{nn} \delta w'
\]

Eq. 4.239

where \(\delta w\) and \(\delta w'\) are projections of the \(\delta y\) and \(\delta y'\) in the eigenspace of the \(T\) matrix.

Substituting them in eq. 4.237

\[
X_{nn} \delta w' = TX_{nn} \delta w + H
\]

Eq. 4.240

and multiplying both sides of eq. 4.240 by \(X_{nn}^{-1}\), we obtain
\[ \delta \mathbf{w}' = X_{nn}^{-1} T_{nn} \delta \mathbf{w} + X_{nn}^{-1} \mathbf{H} \]  
Eq. 4.241

or

\[ \delta \mathbf{w}' = \Lambda \delta \mathbf{w} + X_{nn}^{-1} \mathbf{H} \]  
Eq. 4.242

where

\[ \Lambda = X_{nn}^{-1} T_{nn} X_{nn} \]  
Eq. 4.243

The ODEs of eq. 4.237 are now decoupled eq. 4.242 and their analytical solutions exist. The solution for the i-th component of the vector \( \delta \mathbf{w} \) is given as

\[ \delta \omega_i' = \lambda_i \delta \omega_i + r_i \]  
Eq. 4.244

\[ \delta \omega_i = e^{\lambda_i z} \left( \int e^{-\lambda_i z} r_i \, dz + c_i \right) \]  
Eq. 4.245

This could be written in tensorial form:

\[
\delta \mathbf{w}(z) = \begin{pmatrix}
e^{\lambda_1 z} & 0 & & \vdots \\
0 & \ddots & & \vdots \\
& \ddots & \ddots & \vdots \\
0 & & \ddots & e^{\lambda_n z}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
\vdots \\
c_n
\end{pmatrix}
- \Lambda^{-1} X_{nn}^{-1} \mathbf{H}
\]
Eq. 4.246

where the vector \( \mathbf{c} \) contains the integration coefficients. It is obtained using the segment inlet perturbations \( \delta \mathbf{w}(0) \).

\[ \mathbf{c} = \delta \mathbf{w}(0) + \Lambda^{-1} X_{nn}^{-1} \mathbf{H} \]  
Eq. 4.247

Substituting \( \mathbf{c} \) in eq. 4.246 and multiplying both sides by \( X_{nn} \) leads to

\[
\delta \mathbf{y}(z) = X_{nn} \begin{pmatrix}
e^{\lambda_1 z} & 0 & & \vdots \\
0 & \ddots & & \vdots \\
& \ddots & \ddots & \vdots \\
0 & & \ddots & e^{\lambda_n z}
\end{pmatrix}
\begin{pmatrix}
\mathbf{X}_{nn}^{-1} \delta \mathbf{y}(0) + \Lambda^{-1} X_{nn}^{-1} \mathbf{H} \\
\vdots \\
\mathbf{X}_{nn}^{-1} \delta \mathbf{y}(0) + \Lambda^{-1} X_{nn}^{-1} \mathbf{H}
\end{pmatrix}
- \mathbf{X}_{nn} \Lambda^{-1} X_{nn}^{-1} \mathbf{H}
\]
Eq. 4.248

In the presence of abrupt changes in flow area and local pressure drop, \( \delta \mathbf{y}(0) \) should be corrected by the use of a point transfer matrix \( \mathbf{P}_{nn} \). This will be discussed in detail, later.
\[ \delta y_i^i(0) = P_m^i \delta y_{i,\text{out}}^{i-1} \]  
Eq. 4.249

Thus, eq. 4.248 calculated at the exit of the i-th segment could be written as follows:

\[ \delta y_{i,\text{out}}^{i}(\Delta z) = F_A^i P_m^i \delta y_{i,\text{out}}^{i-1} + F_A^i T_I^i H^i - T_I^i H^i \]  
Eq. 4.250

where

\[ T^{-1} = X_{nn} \Lambda^{-1} X_{nn}^{-1} \]  
Eq. 4.251

\[ F_A = \begin{pmatrix} e^{\lambda_i \Delta z} & 0 \\ 0 & e^{\lambda_i \Delta z} \end{pmatrix} \]  
Eq. 4.252
Part II

Channel level development
4.6. Channel steady-state

The equations developed in the first part for a segment will be used to obtain the steady-state of the entire channel. The channel will be divided in segments with constant cross section flow area, in such a way that any flow area changes take place at the segment boundaries. Thus, the effects of discontinuities due to flow area changes (contraction and expansion) and local resistance (spacers) will be treated at the edge of each segment. Later, the solution of the conservation equations will be applied to each segment.

In the presence of subcooled liquid, there are three distinct regions in the thermal hydraulic channel. They are separated by the NVG \((z_d)\) and equilibrium boiling \((z_{eq})\) positions. Since the analytical solution of the conservation equations is different in each region, these boundaries should be sought first.

The steady-state conservation equations, written in the first part of this chapter, are solved in each segment. The vector solution, space variation of the state variables \((d\mathbf{x}/dz)\), requires some iteration (usually 2 or 3). An iterative method is used to obtain the correct segment average values \((\mathbf{x})\) and local drift flux parameters \((C_0, V_gj)\), see paragraph 4.4.5.

The iteration will be interrupted, when the error on the segment exit values at step \(i\) is less than \(\mathbf{\epsilon}\) (vector of the absolute errors):

\[
\left| \mathbf{x}^i - \mathbf{x}^{i-1} \right| \leq \mathbf{\epsilon}
\]

Eq. 4.253

Otherwise we update the nodal average data and repeat the same procedure till the needed accuracy is achieved.

The following figures show the structure of the program that is applied to select the region and, thus, use the right set of equations. The loops between connectors (points 1, 2, 3, 4, 5, 6 figure 4.3) present the subroutines which calculate the solution and their integral over the segment length.

The detection of the boiling boundaries takes place between connectors 1-2 (for the NVG, \(z_d\)) and 3-4 (for the equilibrium boiling, \(z_{eq}\)). The determination method and criteria are given in detail in section 4.4.4. In order to subdivide the channel in different regions, we have used two indices \(I_{zd}\) and \(I_{zeq}\). Zero indices correspond to the absence of two-phase region. The values different from zero indicate the indices of the first segment in the two-phase subcooled and in the two-phase saturated region, see figure 4.3.

Routine selection for the steady-state calculations is shown in figure 4.4. The left flow chart (between connectors 1 and 2) presents the logic used in the single-phase region and the first node in the two-phase subcooled region. The SS_SP and SS_TPSUB are the programs which provide the stationary solutions in the single and two-phase subcooled regions. Correspondingly, the middle and the right flow chart are applied to the two-phase subcooled and saturated regions. The SS_TPSAT provides the steady-state solution in the two-phase saturated region. The segments where the boiling boundaries appear, are subdivided into two parts and we call appropriate programs for each subsegment.
Figure 4.3 Region selection in the steady-state calculation.

Figure 4.4 Logic for the selection of the steady state routines and segment subdivision.
4.7. Local pressure drop

In a BWR fuel assembly, the geometrical discontinuities may be due to the spacers or abrupt changes in the flow area. Their positions and pressure drop K factors are given. Here, we assume that they appear only at the nodes (edges of segments). This may require a redistribution of K factors and/or use of variable segment lengths. Our code is designed for both methods. However it is worthwhile standardizing these points in the channel, using some appropriate method.

Spacers and flow area changes are not necessarily in the same place, but to address the more general case, we assume that they exist at the \textit{i-th node}, see figure 4.5.

![Figure 4.5 Spacers and flow area changes are shown for the general geometrical discontinuity](image)

In order to couple the nodal averaged thermal hydraulics transfer functions to those of the neutronics (see chapter 2), one should write these transfer functions in terms of channel inlet state variable perturbations (see chapter 5). This requires all the nodal transfer functions and those related to the discontinuities. As mentioned in section 4.5.9, the segment inlet state variable perturbations are related to those at the exit of the previous segment, using a point transfer matrix. The analytical solution of the coupled neutronics and thermal hydraulics problem eliminates the intermediate dependency on the local perturbations in the equations used for this purpose (see chapter 5). Therefore the needed transfer functions of the channel will be provided below in matrix notation and, in a form compatible with these we will write the equations at the discontinuities in matrix form also. Later, we will give a full detailed discussion on their determination.
4.7.1. Steady state treatment

We distinguish two cases: first, discontinuities in the saturated region and second, in the subcooled region. Referring to figure 4.5, we give appropriate equations in each case.

- Saturated region

In this region due to local pressure drop (spacers and / or flow area change), there is an increment of quality; some liquid flashes to vapor and thus, the void fraction and the mixture volumetric flux are increased. Furthermore as the flow area changes, the mixture mass flux varies accordingly. Based on these considerations we can relate the state variables at the inlet of the $i$-th segment (subscript $i$) to those at the exit of $i-1$-th (subscript $e$).

$$h_i = h_e$$  \hspace{1cm} \text{Eq. 4.254}

where

$$h = h_f + x h_{fg}$$  \hspace{1cm} \text{Eq. 4.255}

$$A_i \dot{m}_i = A_e \dot{m}_e$$  \hspace{1cm} \text{Eq. 4.256}

$$p_i = p_e - \left(K_{SPC} + K_{AREA}\right) \frac{\dot{m}_e^2}{2 \rho_{f,e} \phi_{HM,e}}$$  \hspace{1cm} \text{Eq. 4.257}

First, we calculate the pressure at the segment inlet (after the local discontinuity) using eq. 4.257. Knowing $p_i$, we compute local water and steam properties. Then, the quality is obtained from the equations 4.254 and 4.258.

$$x_i = \frac{h_i - h_{f,j}}{h_{g,j} - h_{f,j}}$$  \hspace{1cm} \text{Eq. 4.258}

The mixture mass flux is calculated by eq. 4.256. Finally the volumetric flux and void fraction at segment inlet are computed as follows

$$j_i = \dot{m}_i \left(\frac{1-x_i}{\rho_{f,j}} + \frac{x_i}{\rho_{g,j}}\right)$$  \hspace{1cm} \text{Eq. 4.259}

$$\alpha_i = \frac{\dot{m}_i x_i}{\rho_{g,j} \left(C_{u,j} + V_g\right)}$$  \hspace{1cm} \text{Eq. 4.260}
- Subcooled region

In the subcooled region (section 4.2.2) the water properties depend on liquid temperature (thermally expandable) and gas properties follow the local pressure (at saturation conditions). Thus, we assume that the local losses affect mainly gas properties (gas enthalpy, density, etc...), and the liquid enthalpy does not change

\[ h_{i,j} = h_{i,e} \]  

Eq. 4.261

The other considerations already made for the saturated region remain valid. Therefore, we apply again equations 4.254, 4.256, 4.257, 4.259, and 4.260 (using liquid subcooled properties) but the quality, in this case, is calculated from the equations 4.254 and 4.261 as follows

\[ x_i = \frac{h_i - h_{i,j}}{h_{e,i} - h_{e,j}} \]  

Eq. 4.262

- Single-phase region

We do not expect any significant effect of the pressure drop on the other flow variables, therefore, we use only equations 4.261 and 4.257 with \( \Phi_{iHM,e}^2 = 1 \) and \( A_{i,j} = A_{e,j} \).

4.7.2. Transient treatment

Although the conditions are the same as at steady-state, the problem of the determination of phase shift and magnitude changes of state variables at segment inlet is more complex during the transient. We will deal with this problem by means of the point transfer matrix concept, which has been successfully applied in the thermal hydraulic analysis of single-phase pipeline systems.

- Saturated region

By perturbing equations 4.254, 4.256, 4.257 follows:

\[ \delta h_i = \delta h_e \]  

Eq. 4.263

\[ A_i \delta m_i = A_e \delta m_e \]  

Eq. 4.264

\[ \delta p_i = \delta p_e - \delta \left( (K_{SPC} + K_{AREA}) \frac{m_i^2}{2 \rho_{f,e}} \Phi_{iHM,e}^2 \right) \]  

Eq. 4.265
The mixture enthalpy perturbation written in terms of void fraction, mixture volumetric flux, and pressure perturbations is

\[
\delta h = \frac{\partial h}{\partial \alpha} \delta \alpha + \frac{\partial h}{\partial j} \delta j + \frac{\partial h}{\partial p} \delta p
\]

Eq. 4.266

the partial derivatives

\[
\frac{\partial h}{\partial \alpha} = \frac{\partial h}{\partial x} \frac{\partial x}{\partial \alpha}
\]

Eq. 4.267

\[
\frac{\partial h}{\partial j} = \frac{\partial h}{\partial x} \frac{\partial x}{\partial j}
\]

Eq. 4.268

\[
\frac{\partial h}{\partial p} = h' + xh' + \frac{\partial h}{\partial x} \frac{\partial x}{\partial p}
\]

Eq. 4.269

and the mixture mass flux

\[
\dot{m}(j_i, j_g, p) = \rho_i j_i + \rho_g j_g
\]

Eq. 4.270

is perturbed to obtain

\[
\delta \dot{m} = \frac{\partial \dot{m}}{\partial \alpha} \delta \alpha + \frac{\partial \dot{m}}{\partial j} \delta j + \frac{\partial \dot{m}}{\partial p} \delta p
\]

Eq. 4.271

where

\[
\frac{\partial \dot{m}}{\partial \alpha} = \frac{\partial \dot{m}}{\partial j_i} \frac{\partial j_i}{\partial \alpha} + \frac{\partial \dot{m}}{\partial j_g} \frac{\partial j_g}{\partial \alpha}
\]

Eq. 4.272

\[
\frac{\partial \dot{m}}{\partial j} = \frac{\partial \dot{m}}{\partial j_i} \frac{\partial j_i}{\partial j} + \frac{\partial \dot{m}}{\partial j_g} \frac{\partial j_g}{\partial j}
\]

Eq. 4.273

\[
\frac{\partial \dot{m}}{\partial p} = \frac{\partial \dot{m}}{\partial \rho_f} \rho'_f + \frac{\partial \dot{m}}{\partial \rho_g} \rho'_g
\]

Eq. 4.274

Inserting the expression of mass flux perturbation (eq. 271), homogeneous multiplier (see section 4.5.6.2) and mixture enthalpy (eq. 4.266) in equations 4.263, 4.264 and 4.265, we obtain three equations in terms of the unknown inlet state variable perturbations (\(\delta \alpha_i, \delta j_i, \delta p_i\)). They are written in matrix form as
\[
\begin{pmatrix}
\frac{\partial h}{\partial \alpha_i} & \frac{\partial h}{\partial j_i} & \frac{\partial h}{\partial p_i} \\
A_i \frac{\partial m}{\partial \alpha_i} & A_i \frac{\partial m}{\partial j_i} & A_i \frac{\partial m}{\partial p_i} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\delta \alpha_i \\
\delta j_i \\
\delta p_i
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial h}{\partial \alpha_e} & \frac{\partial h}{\partial j_e} & \frac{\partial h}{\partial p_e} \\
A_e \frac{\partial m}{\partial \alpha_e} & A_e \frac{\partial m}{\partial j_e} & A_e \frac{\partial m}{\partial p_e} \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
\delta \alpha_e \\
\delta j_e \\
\delta p_e
\end{pmatrix}
\] Eq. 4.275

where

\[
\frac{\partial P_L}{\partial \alpha_e} = -(K_{SPC} + K_{AREA}) \frac{\dot{m}_e^2}{2\rho_{f,e}} \Phi_{HM,e}^2 \left(1 \frac{\partial m}{\partial \alpha_e} + \frac{\partial \Phi_{HM,e}^2}{\partial \alpha} \right)
\] Eq. 4.276

\[
\frac{\partial P_L}{\partial j_e} = -(K_{SPC} + K_{AREA}) \frac{\dot{m}_e^2}{2\rho_{f,e}} \Phi_{HM,e}^2 \left(2 \frac{\partial m}{\partial j_e} + \frac{\partial \Phi_{HM,e}^2}{\partial j} \right)
\] Eq. 4.277

\[
\frac{\partial P_L}{\partial p_e} = -(K_{SPC} + K_{AREA}) \frac{\dot{m}_e^2}{2\rho_{f,e}} \Phi_{HM,e}^2 \left(2 \frac{\partial m}{\partial p_e} + \frac{\partial \Phi_{HM,e}^2}{\partial p} \right) - \frac{\rho'_f}{\rho_{f,e}}
\] Eq. 4.278

The vector \((\delta \alpha_e, \delta j_e, \delta p_e)\) in the right hand side is the solution of transient equations, at the exit of the \(i-1\)-th segment. Solving equation (4.275) for the perturbations at the inlet of the \(i\)-th segment \((\delta \alpha_i, \delta j_i, \delta p_i)\):

\[
\begin{pmatrix}
\delta \alpha_i \\
\delta j_i \\
\delta p_i
\end{pmatrix} = \begin{pmatrix}
\frac{\partial h}{\partial \alpha_i} & \frac{\partial h}{\partial j_i} & \frac{\partial h}{\partial p_i} \\
A_i \frac{\partial m}{\partial \alpha_i} & A_i \frac{\partial m}{\partial j_i} & A_i \frac{\partial m}{\partial p_i} \\
0 & 0 & 1
\end{pmatrix}^{-1}\begin{pmatrix}
\frac{\partial h}{\partial \alpha_e} & \frac{\partial h}{\partial j_e} & \frac{\partial h}{\partial p_e} \\
A_e \frac{\partial m}{\partial \alpha_e} & A_e \frac{\partial m}{\partial j_e} & A_e \frac{\partial m}{\partial p_e} \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
\delta \alpha_e \\
\delta j_e \\
\delta p_e
\end{pmatrix}
\] Eq. 4.279

where all the partial derivatives have been calculated using steady-state data. This is given in matrix notation as

\[
\delta \mathbf{x}_i = \mathbf{P} \delta \mathbf{x}_e
\] Eq. 4.280

The point transfer matrix \(\mathbf{P}\), provides the correct relations including the flashing effect due to local losses between the exit of segment \(i-1\) and the inlet of segment \(i\).
- Two-phase subcooled region

Following the same procedure as in the saturated region, one can determine the point transfer matrix. The difference is the constant liquid enthalpy assumption, made earlier.

\[ \delta h_i = \delta h_e \] Eq. 4.281

\[ A_i \delta \dot{m}_i = A_e \delta \dot{m}_e \] Eq. 4.282

\[ \delta h_{i,j} = \delta h_{i,x} \] Eq. 4.283

\[ \delta p_i = \delta p_e - \delta \left( K_{SPC} + K_{AREA} \frac{\dot{m}_e^2}{2 \rho_{f,e}} \Phi_{HM,e}^2 \right) \] Eq. 4.284

In the subcooled boiling region, the mixture enthalpy, the quality, the homogenous multiplier (4.5.6.2), and the mass flux perturbations are also function of the liquid enthalpy perturbations.

\[ \delta x = \frac{\partial x}{\partial \alpha} \delta \alpha + \frac{\partial x}{\partial j} \delta j + \frac{\partial x}{\partial h_i} \delta h_i + \frac{\partial x}{\partial p} \delta p \] Eq. 4.285

\[ \delta h = \frac{\partial h}{\partial \alpha} \delta \alpha + \frac{\partial h}{\partial j} \delta j + \frac{\partial h}{\partial h_i} \delta h_i + \frac{\partial h}{\partial p} \delta p \] Eq. 4.286

\[ \delta \dot{m} = \frac{\partial \dot{m}}{\partial \alpha} \delta \alpha + \frac{\partial \dot{m}}{\partial j} \delta j + \frac{\partial \dot{m}}{\partial h_i} \delta h_i + \frac{\partial \dot{m}}{\partial p} \delta p \] Eq. 4.287

Here, we give only the enthalpy and pressure partial derivatives. The other terms were already defined

\[ \frac{\partial x}{\partial h_i} = \frac{\partial x}{\partial \rho_i} \rho_i^" \] Eq. 4.288

\[ \frac{\partial x}{\partial p} = \frac{\partial x}{\partial \rho_g} \rho_g' \] Eq. 4.289

\[ \frac{\partial h}{\partial h_i} = (1-x) + \frac{\partial h}{\partial x} \frac{\partial x}{\partial \rho_i} \rho_i^" \] Eq. 4.290

\[ \frac{\partial h}{\partial p} = x h_g' + \frac{\partial h}{\partial x} \frac{\partial x}{\partial \rho_g} \rho_g' \] Eq. 4.291

\[ \frac{\partial \dot{m}}{\partial h_i} = \frac{\partial \dot{m}}{\partial \rho_i} \rho_i^" \] Eq. 4.292
\[
\frac{\partial m}{\partial p} = \frac{\partial m}{\partial \rho_g} \rho'_g \quad \text{Eq. 4.293}
\]

\[
\frac{\partial P_L}{\partial h_i} = - (K_{SPC} + K_{AREA}) \frac{m^2}{2\rho_i,e} \Phi_{IM,e}^2 \left( \frac{2}{m_e} \frac{\partial m}{\partial h_i} + \frac{1}{\Phi_{IM,e}^2} \frac{\partial \Phi_{IM,e}^2}{\partial h_i} \right) \left( - \frac{\rho'_g}{\rho_i,e} \right) \quad \text{Eq. 4.294}
\]

\[
\frac{\partial P_L}{\partial p} = - (K_{SPC} + K_{AREA}) \frac{m^2}{2\rho_i,e} \Phi_{IM,e}^2 \left( \frac{2}{m_e} \frac{\partial m}{\partial p} + \frac{1}{\Phi_{IM,e}^2} \frac{\partial \Phi_{IM,e}^2}{\partial p} \right) \quad \text{Eq. 4.295}
\]

Inserting the above perturbations into the equations 4.281, 4.282, 4.283 and 4.284 gives

\[
\begin{pmatrix}
\frac{\partial x}{\partial \alpha_e} & \frac{\partial x}{\partial j} & \frac{\partial x}{\partial h_i} & \frac{\partial x}{\partial p} \\
A_e \frac{\partial m}{\partial \alpha_e} & A_e \frac{\partial m}{\partial j} & A_e \frac{\partial m}{\partial h_i} & A_e \frac{\partial m}{\partial p} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\delta x_i \\
\delta j_i \\
\delta h_{i,i} \\
\delta p_i
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial x}{\partial \alpha_e} & \frac{\partial x}{\partial j} & \frac{\partial x}{\partial h_i} & \frac{\partial x}{\partial p} \\
A_e \frac{\partial m}{\partial \alpha_e} & A_e \frac{\partial m}{\partial j} & A_e \frac{\partial m}{\partial h_i} & A_e \frac{\partial m}{\partial p} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\delta \alpha_e \\
\delta j_e \\
\delta h_{i,e} \\
\delta p_e
\end{pmatrix}
\]

\text{Eq. 4.296}

Equation 4.296 can be easily solved by inverting the matrix on the left hand side

\[
\begin{pmatrix}
\delta x_i \\
\delta j_i \\
\delta h_{i,i} \\
\delta p_i
\end{pmatrix}
= \left( \begin{pmatrix}
\frac{\partial x}{\partial \alpha_e} & \frac{\partial x}{\partial j} & \frac{\partial x}{\partial h_i} & \frac{\partial x}{\partial p} \\
A_e \frac{\partial m}{\partial \alpha_e} & A_e \frac{\partial m}{\partial j} & A_e \frac{\partial m}{\partial h_i} & A_e \frac{\partial m}{\partial p} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \right)^{-1}
\begin{pmatrix}
\frac{\partial x}{\partial \alpha_e} & \frac{\partial x}{\partial j} & \frac{\partial x}{\partial h_i} & \frac{\partial x}{\partial p} \\
A_e \frac{\partial m}{\partial \alpha_e} & A_e \frac{\partial m}{\partial j} & A_e \frac{\partial m}{\partial h_i} & A_e \frac{\partial m}{\partial p} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\delta \alpha_e \\
\delta j_e \\
\delta h_{i,e} \\
\delta p_e
\end{pmatrix}
\]

\text{Eq. 4.297}

or in compact form

\[
\delta \mathbf{x}_i = \mathbf{P} \delta \mathbf{x}_i 
\]

\text{Eq.4.298}

the above equation is similar to the one we found for the saturated region. Obviously, the \(\delta \mathbf{x}_i\), \(\delta \mathbf{x}_e\), and \(\mathbf{P}\) have different dimensions. Equation 4.296 fully includes the flashing and liquid thermal expansion effects.
- Single-phase subcooled region

In the single-phase subcooled region, the quality and homogenous multiplier are zero and one, respectively. Therefore the equations at the discontinuities are given as:

\[ A_t \frac{\delta j_i}{\delta x} = A_e \frac{\delta j_e}{\delta x} \quad \text{Eq. 4.299} \]

\[ \frac{\delta h_{i,d}}{\delta x} = \frac{\delta h_{i,e}}{\delta x} \quad \text{Eq. 4.300} \]

\[ \delta p_i = \delta p_e - \delta \left( K_{SPC} + K_{AREA} \frac{m_e^2}{2 \rho_{i,e}} \right) \quad \text{Eq. 4.301} \]

Matrix presentation of the above equations is following.

\[
\begin{pmatrix}
A_t & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\delta j_i \\
\delta h_{i,d} \\
\delta p_i
\end{pmatrix} = \begin{pmatrix}
A_e & 0 & 0 \\
0 & 1 & 0 \\
\frac{\partial P_i}{\partial j_e} & \frac{\partial P_i}{\partial h_{i,e}} & 1
\end{pmatrix} \begin{pmatrix}
\delta j_e \\
\delta h_{i,e} \\
\delta p_e
\end{pmatrix} \quad \text{Eq. 4.302}
\]

Where the mass flux partial derivatives are the same as the two-phase subcooled region defined above. The solution of equation (4.302) is obtained by multiplying both sides by the inverse coefficients matrix of unknowns.

\[
\begin{pmatrix}
\delta j_i \\
\delta h_{i,d} \\
\delta p_i
\end{pmatrix} = \begin{pmatrix}
A_t & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
A_e & 0 & 0 \\
0 & 1 & 0 \\
\frac{\partial P_i}{\partial j_e} & \frac{\partial P_i}{\partial h_{i,e}} & 1
\end{pmatrix} \begin{pmatrix}
\delta j_e \\
\delta h_{i,e} \\
\delta p_e
\end{pmatrix} \quad \text{Eq. 4.303}
\]

This could be written in the same form that we have written for two-phase regions.

\[ \delta x_i = P \delta x_e \quad \text{Eq. 4.304} \]

where \( P \) point transfer matrix is

\[
P = \begin{pmatrix}
\frac{A_e}{A_t} & 0 & 0 \\
0 & 1 & 0 \\
\frac{\partial P_i}{\partial j_e} & \frac{\partial P_i}{\partial h_{i,e}} & 1
\end{pmatrix} \quad \text{Eq. 4.305}
\]
4.8. Boiling Boundary Perturbation

Single-phase length in a heated channel depends upon where the NVG point appears. The NVG position in turn must be subject to the variation of thermal hydrodynamic parameters, such as mixture volumetric flux (flow velocity), heat flux and pressure. The instantaneous position of the NVG point \( z_d(t) \)

\[ z_d(t) = z_{d0} + \delta z_d(t) \]  

Eq. 4.306

is obtained from

\[ h_t(z_d(t), t) = h_{ld}(z_d(t), t) \]  

Eq. 4.307

By first-order Taylor expansions between \( z_{d0} \) and \( z_d(t) \) of eq. 4.307

\[ h_t(z_{d0}, t) + \frac{\partial h_t}{\partial z} \bigg|_{z_{d0}} \delta z_d = h_{ld}(z_{d0}, t) + \frac{\partial h_{ld}}{\partial z} \bigg|_{z_{d0}} \delta z_d \]  

Eq. 4.308

Writing now

\[ h_t(z_{d0}, t) = h^0_t(z_{d0}) + \delta h_t(z_{d0}, t) \]  

Eq. 4.309

and

\[ h_{ld}(z_{d0}, t) = h^0_{ld}(z_{d0}) + \delta h_{ld}(z_{d0}, t) \]  

Eq. 4.310

recognizing that at steady-state

\[ h^0_t(z_{d0}) = h^0_{ld}(z_{d0}) \]  

Eq. 4.311

approximating \( \frac{\partial h_t}{\partial z} \bigg|_{z_{d0}} \) by \( \frac{dh^0_t}{dz} \bigg|_{z_{d0}} \) and \( \frac{\partial h_{ld}}{\partial z} \bigg|_{z_{d0}} \) by \( \frac{dh^0_{ld}}{dz} \bigg|_{z_{d0}} \) and solving for \( \delta z_d \) we obtain

\[ \delta z_d = \frac{\delta h_t(z_{d0}, t) - \delta h_{ld}(z_{d0}, t)}{\frac{dh^0_{ld}}{dz} \bigg|_{z_{d0}} - \frac{dh^0_t}{dz} \bigg|_{z_{d0}}} \]  

Eq. 4.312

As \( h_{ld} = h_{id}(\dot{m}, q^*, p) \) and neither \( \dot{m} \) nor \( q^* \) vary at steady-state in the segment under consideration, neglecting the week dependence on \( p \), and further noting that

\[ \frac{dh^0_t}{dz} \bigg|_{z_{d0}} = \frac{1}{\dot{m}_0} \left( \frac{q^*_{z_d,0} P_h}{A} + q^*_{r,z_d,0} \right) \]  

Eq. 4.313
Eq. 4.312 can be written as

$$\delta z_d = \left( \frac{dh^0}{dz} \right)_{z_{d0}}^{-1} \left( -\delta h_i \left( z_{d0}, t \right) + \delta h_{ld} \left( z_{d0}, t \right) \right)$$ Eq. 4.314

Finally, expanding $\delta h_{ld}$ linearly in eq. 4.314 and Laplace transforming it, results in

$$\delta z_d = \left( \frac{dh^0}{dz} \right)_{z_{d0}}^{-1} \left( -\delta h_i \left( z_{d0} \right) + \frac{\partial h_{ld}}{\partial j} \left|_{(q_d^0, w)} \right. \delta j + \frac{\partial h_{ld}}{\partial p} \left|_{(q_d^0, w)} \right. \delta p + \frac{\partial h_{ld}}{\partial q^*_d} \left|_{(q_d^0, w)} \right. \delta q^*_d \right)$$ Eq. 4.315

Figure 4.6 Variation of the liquid enthalpy and of the liquid enthalpy at NVG around the NVG point
Trigonometry of the shaded triangle provides eq. 4.312.

The results obtained for the NVG position perturbation can be extended to the case of the equilibrium boiling boundary perturbation. Therefore we rewrite equation eq. 4.314 as follows

$$\delta z_{eq} = \frac{\delta h_i \left( z_{eq0}, t \right) - \delta h_f \left( z_{eq0}, t \right)}{\frac{dh_i^0}{dz} \left|_{z_{eq0}} \right. - \frac{dh_f^0}{dz} \left|_{z_{eq0}} \right.}$$ Eq. 4.316
4.9. Channel transfer function

The exact analytical solutions for a segment (eq. 4.250) were written in terms of the perturbations of the previous segment. In this section, using matrix notation, we link the segment exit perturbation to the channel external forcing functions \( \delta y_1 = [\delta j_{in} \; \delta h_{lin} \; \delta p_{in}]^T \) and volumetric nodal power vector \( H_i \). Eq. 4.250 was given as

\[
\delta y_{out} (\Delta z) = F_j P_j \delta y_{out}^{i-1} + F_j T_j^{-1} H_j - T_j^{-1} H_j
\]

By successive backward substitution of the perturbations, we have obtained the following expression for each region for a channel.

- Single-phase region

\[
\delta y_{N1sub+1} = \Theta_{1\phi,i} \delta y_1 + \sum_{i=1}^{N1sub} \Lambda_{1\phi,i} H_i
\]

Eq. 4.317

where

\[
\Lambda_i = \begin{cases} 
\Theta_{1\phi,i} (F_{1\phi,i} - I) T_{1\phi,i}^{-1} & \text{if } 1 \leq i \leq N1sub \\
(F_{1\phi,N1sub} - I) T_{1\phi,N1sub}^{-1} & \text{if } i = N1sub
\end{cases}
\]

Eq. 4.318

and

\[
\Theta_{1\phi,i} = \prod_{k=1}^{N1sub} F_{1\phi,k} P_{1\phi,k}
\]

Eq. 4.319

- Two-phase subcooled region

Similarly to eq. 4.317 we have

\[
\delta y_{N2sub+1} = \Theta_{2\phi,i} \delta y_{N1sub+1} + \sum_{i=1}^{N2sub} \Lambda_{2\phi,i} H_{2\phi,i}
\]

Eq. 4.320

where \( \Theta_{2\phi,i} \) and \( \Lambda_{2\phi,i} \) have the same expressions as those for single-phase region but written for the two-phase subcooled region.

- Two-phase saturated region

Likewise the single-phase and two-phase subcooled we can write

\[
\delta y_{exit} = \Theta_{2\phi,sat,i} \delta y_{N2sub+1} + \sum_{i=1}^{N2sat} \Lambda_{2\phi,sat,i} H_{2\phi,sat,i}
\]

Eq. 4.321
In the absence of power perturbation, the above equations reduce to

\[ \delta y_{N1sub+1} = \Theta_{1\phi,1} \delta y_1 \]  
Eq. 4.322

\[ \delta y_{N2sub+1} = \Theta_{2\phi sub,1} \delta y_{N1sub+1} = \Theta_{2\phi sub,1} \Theta_{1\phi,1} \delta y_1 \]  
Eq. 4.323

\[ \delta y_{exit} = \Theta_{2\phi sat,1} \delta y_{N2sub+1} = \Theta_{2\phi sat,1} \Theta_{2\phi sub,1} \Theta_{1\phi,1} \delta y_1 \]  
Eq. 4.324

The total two-phase region (subcooled and saturated) transfer functions are obtained by inserting eq. 4.320 into eq. 4.321.

\[ \delta y_{exit} = \Theta_{2\phi sat,1} \Theta_{2\phi sub,1} \delta y_{N1sub+1} + \sum_{j=1}^{N2sub+N2sat} \Theta_j \Lambda_j H_j \]  
Eq. 4.325

for 1 \( \leq j \leq N2sub \) and 1 \( \leq i \leq N2sub \)

\[ i=j : H_j = H_{2\phi sub,i} ; \Lambda_j = \Lambda_{2\phi sub,i} ; \Theta_j = \Theta_{2\phi sat,1} \]

for \( N2sub < j \leq N2sat \) and 1 \( \leq i \leq N2sat \)

\[ i=j- N2sub : H_j = H_{2\phi sat,i} ; \Lambda_j = \Lambda_{2\phi sat,i} ; \Theta_j = I \]

where I is the identity matrix.

Finally, we can link the channel exit perturbation to those of the channel inlet and power perturbations by inserting eq. 4.317 into eq. 4.315.

\[ \delta y_{exit} = \Theta_{2\phi sat,1} \Theta_{2\phi sub,1} \delta y_{N1sub+1} + \sum_{j=1}^{K_{all}} \Theta_j \Lambda_j H_j \]  
Eq. 4.326

for 1 \( \leq j \leq N1sub \) and 1 \( \leq i \leq N1sub \),

\[ i=j : H_j = H_{1\phi sub,i} ; \Lambda_j = \Lambda_{1\phi sub,i} ; \Theta_j = \Theta_{2\phi sat,1} \Theta_{2\phi sub,1} \]

for \( N1sub < j \leq N1sub+N2sub \) and 1 \( \leq i \leq N2sub \)

\[ i=j- N1sub : H_j = H_{2\phi sub,i} ; \Lambda_j = \Lambda_{2\phi sub,i} ; \Theta_j = \Theta_{2\phi sat,1} \]

for \( N1sub+N2sub < j \leq N1sub+N2sub+N2sat \) and 1 \( \leq i \leq N2sat \)

\[ i=j- N2sub+N2sub : H_j = H_{2\phi sat,i} ; \Lambda_j = \Lambda_{2\phi sat,i} ; \Theta_j = I \]

In accordance with Leibnitz’s rule for the integration of the momentum equation from the channel inlet to the moving boiling boundary position and over the whole two-phase region we have (Yadigaroglu, 1975; Peng et al., 1985):
Single-phase region

\[ \delta \Delta p_{1,\phi, zd} = \delta \Delta p_{1,\phi, zd0} + \frac{dp}{dz}_{\phi, zd0} \delta z_d \quad \text{Eq. 4.327} \]

Two-phase region

\[ \delta \Delta p_{2,\phi, zd} = \delta \Delta p_{2,\phi, zd0} - \frac{dp}{dz}_{2,\phi, zd0} \delta z_d \quad \text{Eq. 4.328} \]

where \( \delta \Delta p_{1,\phi, zd0} \) and \( \delta \Delta p_{2,\phi, zd0} \) are obtained from equations (4.317) and (4.326). Furthermore, the void fraction perturbation at the void departure point (Peng, 1985) are given as

\[ \delta \alpha_{zd0} = - \frac{d\alpha}{dz}_{2,\phi, zd0} \delta z_d \quad \text{Eq. 4.329} \]
References:

3. Chexal-Lelluche (1999), Pressure drop technology for design and analysis, EPRI SIMENS.
5. Collier J. (1972), Convective boiling and condensation, McGraw-Hill
9. Schmidt E. (1982), Properties of water and steam in SI-Units (0-800 C, 0-1000 bar); Springer-Verlag.
Chapter 5

Channel and overall core stability
5.1. Introduction

A boiling channel or the entire reactor system can be described in terms of a linear time
invariant system of equations (LTIS), using the linearized equations in terms of system state-
variables derived in the previous chapters. The stability of the autonomous system can be
studied by means of the existing control theories combined with sparse matrix techniques.
In this chapter, we will use standard nomenclature, Weaver (1968). This helps to understand
the multi-input multi-output (MIMO) channel or core models. In general, we use:

\( \mathbf{A} \): system matrix
\( \mathbf{B} \): control matrix
\( \mathbf{C} \): output matrix
\( \mathbf{e} \): input signal to forward loop
\( \mathbf{G} \): forward-loop transfer function; or system transfer function
\( \mathbf{H} \): feedback loop transfer function
\( s \): complex variable
\( \mathbf{u} \): control vector
\( \mathbf{W} \): system characteristic function
\( \mathbf{x} \): state-variable input vector
\( \mathbf{y} \): state-variable output vector
\( \rho_e \): external reactivity
\( \rho_f \): feedback reactivity matrix

Stability of a MIMO system requires the determination of the system characteristic function.
This can be defined in two different ways: the first approach consists in writing the LTIS in
matrix form as:

\[
\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \quad \text{Eq. 5.1}
\]

\[
\mathbf{y} = \mathbf{C} \mathbf{x} \quad \text{Eq. 5.2}
\]

A forced response of the MIMO system (Eq. 5.1) may be sought by the Sylvester expansion
(Weaver 1968) of the system matrix (\( \mathbf{A} \)). This theorem provides means for evaluating
functions expressed as infinite power series of a matrix.

\[
f(\mathbf{A}) = \sum_{i=1}^{\infty} c_i \mathbf{A}^i \quad \text{Eq. 5.3}
\]

Assuming that the system matrix has \( n \) distinct eigenvalues, then Sylvester’s theorem states
that the function \( f \) can be written as:

\[
f(\mathbf{A}) = \sum_{i=1}^{n} f(\lambda_i) \mathbf{F}(\mathbf{A}) \quad \text{Eq. 5.4}
\]
where
\[
F(\lambda_i) = \prod_{j=1, j\neq i}^{n} \left( \frac{A - \lambda_i I}{\lambda_i - \lambda_j} \right)
\]  
Eq. 5.5

This expansion is applied to the state transition matrix (\(\Phi\)), resulting in:
\[
\Phi(t) = e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} F(\lambda_i)
\]  
Eq. 5.6

Moreover, the time-dependent solution of the system (eq. 5.1) is obtained as:
\[
x(t) = \Phi(t)x(t_0) + \int_{t_0}^{t} \Phi(t-\tau)Bu(\tau) d\tau
\]  
Eq. 5.7

Finally, from the Laplace transform of the state transition, the system transfer function can readily be written:
\[
G(s) = \frac{y(s)}{u(s)} = C^T(sI - A)^{-1}B = C^T\Phi(s)B
\]  
Eq. 5.8

The system characteristic equation, in this case, is defined as
\[
W(s) = \det(sI - A) = 0
\]  
Eq. 5.9

The second method is the feedback analysis of the closed-loop transfer function of the linear system.

\[\text{Figure 5.1 Block diagram of a feedback control system}\]
For the block diagram of a feedback control system presented in figure 5.1, the following three equations can be obtained:

\[ \rho_f(s) = H(s)y(s) \]  
\[ e(s) = u(s) - \rho_f(s) \]  
\[ y(s) = G(s)e(s) \]

Substituting eq. 5.10 into eq. 5.11 and this into eq. 5.12, we obtain the closed-loop transfer function of the system in terms of forward and feedback loop transfer functions.

\[ \frac{y(s)}{u(s)} = \frac{G(s)}{1 + G(s)H(s)} \]

The characteristic equation of the system is given as:

\[ W(s) = 1 + G(s)H(s) = 0 \]

Depending on the method applied, a LTIS can be studied by analyzing the roots (of eq. 5.9) or zeros of the characteristic equation (eq. 5.14).

Before focusing on the dynamic response of the system, it is important to say some words regarding the steady state and the transfer function generation. Computation of the steady state starts with reading the input data from Distribution and Master files normally available to the plant operator, see figure 5.2. The appropriate data will be loaded to the modules for core statics (see figure 5.3), core thermal hydraulics (see figure 5.4), core-external thermal hydraulics and fuel (see figure 5.6) steady states. After this, the system stationary data will be made available to the system transfer functions module, which prepares all the nodal transfer functions and non-frequency dependent coefficients. These steps are presented in flowchart fashion below.

The dimensions of the various vectors and matrices used are given in curly brackets next to the corresponding symbols; the following system dimensions are used:

- Number of fuel assembly layers in y-direction in the core \( I_{\text{max}} \)
- Number of fuel assembly layers in x-direction in the core \( J_{\text{max}} \)
- Number of nodes in one channel \( K_{\text{max}} \)
- Size of 3D matrices \( CD = I_{\text{max}} \times J_{\text{max}} \times K_{\text{max}} \)
- Number of channels in core \( N_{\text{max}} \)
- Type of channels \( N_{\text{T}_{\text{max}}} \)
- Number of nodes in core-external path \( N_{\text{EX}} \)
Distribution file
Sys. SS data

Core discretization CD={I_{max},J_{max},K_{max}}
Neutron flux (g=1,2) \Phi_g\{CD\}
Delayed neutrons \beta, \lambda
Cross sections: (y=f_g, a_g, r), \Sigma_y \{CD\},
\Delta \Sigma_y/\Delta \alpha \{CD\},
\Delta \Sigma_y/\Delta T_f \{CD\}
Diffusion coeff. D_g \{CD\}
Reflector position
Void profile \alpha \{CD\}
Power profile P \{CD\}
Inlet flow of core channels J \{CD\}
Rated power
Pressure (dome, UP, LP)
Water level in dome
Feed water: h, flow rate, position
Pump velocity
Core inlet mass flow rate
Core bypass mass flow rate

Master file
(Sys. geometry)

User inputs file

Legendre order \ KL_{max}  
Modes \ M_{max}  
fuel & Clad Radii \ RF, RC

Store-Data

Figure 5.2 Read inputs block diagram.
Core statics solution

Data-Import
Core discretization $CD=\{I_{\text{max}},J_{\text{max}},K_{\text{max}}\}$
Number of Channels $N_{\text{max}}$
Delayed neutrons $\beta, \lambda$
$\Sigma_y \{CD\}$, where $y=f_g a_g, r$
$\Delta \Sigma_y / \Delta \alpha \{CD\}$,
$\Delta \Sigma_y / \Delta T_f \{CD\}$,
Diffusion coeff. $D_{g}\{CD\}$,
Reflector position
Legendre order $K_{L_{\text{max}}}$
Modes $M_{\text{max}}$

Testing-data
Neutron flux, $\phi_g \{CD\}$
Power profile $P \{CD\}$
Rated power

NO

Hebert discretization $L, M$
Find Eig. V. & Eig. F.
Calculate and compare SS Power Profile

YES

Testing-data Done?

Hebert discretization $L, M$
Find Eig. V. & Eig. F.
Calculate and compare SS Power Profile

$K_t$, $m=0..M_{\text{max}}$, $\lambda_m$,
$\Phi_{g;m}\{NK=N_{\text{max}}*K_t\}$,
$L\{NK,NK\}$,
$M\{NK,NK\}$,
$\Delta L / \Delta \alpha \{NK,NK\}$
$\Delta M / \Delta T_f \{NK,NK\}$

stop

Figure 5.3 Core static block diagram.

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Steady-state core thermal-hydraulics

<table>
<thead>
<tr>
<th>Power profile $P$ {CD}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel inlet flows $J(I_{\text{max}}, J_{\text{max}})$</td>
</tr>
<tr>
<td>Flow of bypasses</td>
</tr>
<tr>
<td>Rated power</td>
</tr>
<tr>
<td>Pressure (UP, LP)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Core discretization {CD}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of channels $N_{\text{max}}$</td>
</tr>
<tr>
<td>Length of channels $L_{\text{max}}$</td>
</tr>
<tr>
<td>Type of channels $NT_{\text{max}}$</td>
</tr>
<tr>
<td>Channel identifiers $IDC {NT_{\text{max}}}$</td>
</tr>
<tr>
<td>Channel geometries $A {NT_{\text{max}}, K_{\text{max}}}, D_h {NT_{\text{max}}, K_{\text{max}}}, P_h {NT_{\text{max}}, K_{\text{max}}}, N_{\text{rod}} {NT_{\text{max}}}, Spacer {NT_{\text{max}}, (zsp, kspc)}$, $Kin {NT_{\text{max}}}, Kout {NT_{\text{max}}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Testing-data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Void profile $\alpha$ {CD}</td>
</tr>
<tr>
<td>Channels $\Delta p$</td>
</tr>
</tbody>
</table>

| CD \{I_{\text{max}}, J_{\text{max}}, K_{\text{max}}\} |
| W&S prop. \{CD\} |
| $\alpha$ \{CD\} |
| $T_1, h_1, h_g, T_g, T_{co}$ \{CD\} |
| $f, f_g, \phi_{eb}, \phi_{HM}p$ \{CD\} |
| $C_0, V_g, J_l, J_g$ \{CD\} |
| $Z_{bb}$ \{N_{\text{max}}, 2\} |

Figure 5.4 Steady-state core thermal hydraulic block diagram.
Figure 5.5 Steady-state core-external path thermal hydraulic block diagram.
Figure 5.6 Steady-state fuel rod radial temperature block diagram.

SS fuel-temperature

- Power profile $P\{\text{CD}\}$
- Clad. temp. $T_{co}\{\text{CD}\}$
- Fuel, gap, cladding prop. $\{\text{NT}_{\text{max}}\}$

Core discretization $\{\text{CD}\}$
- Type of channels $\{\text{NT}_{\text{max}}\}$
- Identifier of channels $\text{IDC} \{\text{NT}_{\text{max}}\}$
- Fuel, gap, cladding geom. $\{\text{NT}_{\text{max}}\}$
- Radii in fuel $\text{RF}$
- Radii in cladding $\text{RC}$

Calculate:
- Clad. radial temp. distrib. $T_C$
- Average fuel conductivity $k_f$
- Fuel radial temp. distrib. $T_F$
- Average fuel temperature $T_f$

<table>
<thead>
<tr>
<th>$T_C$</th>
<th>${\text{RC},{\text{CD}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_f$</td>
<td>${\text{CD}}$</td>
</tr>
<tr>
<td>$T_F$</td>
<td>${\text{RF},{\text{CD}}}$</td>
</tr>
<tr>
<td>$T_f$</td>
<td>${\text{CD}}$</td>
</tr>
</tbody>
</table>

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Figure 5.7 System transfer function module block diagram.

The steady-state data generated in the block diagrams (figures, 5.2- 5.7), will be employed in the next paragraphs, producing the channel and system transfer functions needed for the stability analysis.
This chapter includes:
- Stability criteria
- Single channel stability and transfer functions
- Parallel channels transfer functions and flow redistribution
- System transfer functions
5.2. Stability criteria

The stability of the LTIS is determined by the location of the poles of the closed loop transfer function in the complex plane. These poles are the zeros (roots) of the system characteristic equation (5.9 or 5.14). It is important to note that the zeros are those values of $s$ for which the transfer function vanishes and poles (singular points) are the $s$ for which transfer function becomes infinity. A LTIS is stable if the poles lie on the left half of the complex plane. In terms of matrix notation, if all the eigenvalues of the system matrix ($A$) have negative real parts, the stability is guaranteed for any input perturbation.

Several definitions for the stability of a dynamical system are available, but in the framework of reactor analysis, we consider only stability in sense of Liapunov and asymptotic instability. Assuming that the equilibrium state is located at the origin, $X(0)=0$, and letting $S(r)$ be a spherical region of radius $r>0$ about the origin, where $S(r)$ consists of points $X$ satisfying $\|X\| < r$, we shortly explain the two definitions (Weaver 1968).

**Stability in the sense of Liapunov**: states that the origin is stable if for every positive number $r$ there exists a positive number $R$ such that solution starting in $S(R)$ remains in $S(r)$ as times goes to infinity, figure 5.8 left plot.

$$\forall r \in \mathbb{R} \ \exists R \in \mathbb{R} \Rightarrow S(R) \subset S(r)$$

**Asymptotic stability**: the origin is asymptotically stable if for every positive number $r$ there exists a positive number $R$ such that a solution starting in $S(R)$ remains in $S(r)$ and tends toward the origin as time goes to infinity, figure 5.8 right plot.

$$\forall r \in \mathbb{R} \ \exists R \in \mathbb{R} \Rightarrow S(R) \subset S(r) ; S(R) \to \mathbf{0}$$

If a LTIS has poles on the imaginary axis and in left half of complex domain, it is stable in the sense of Liapunov. This system becomes asymptotically stable, if the poles on the imaginary axis are removed.
In reality, it is not always possible to define the zeros and poles of the system. Therefore, we need other criteria to study the stability of very large systems, like a nuclear reactor. Hence, graphical methods become more practical and suitable. In these methods, the complex variable $s$ is substituted by $j\omega$, which allows plotting the transfer functions in the frequency plane. For a rigorous analysis, the following methods have been recommended, Harrer (1963):

- Initial and final values theorems
- Bode analysis ($\zeta_n$, $\omega_n$)
- Nyquist plane
- Nichols chart
- Decay Ratio

**Initial and final values theorems**: both, the value that a system reaches after infinite time and its value at zero time value can be determined from the transfer function. The following results are special applications of the limit theorems of Laplace transforms, Harrer (1963).

\[
\lim_{t \to \infty} F(t) = \lim_{s \to 0} G(s) \quad \text{Eq. 5.15}
\]

\[
\lim_{t \to 0} F(t) = \lim_{s \to \infty} G(s) \quad \text{Eq. 5.16}
\]

Where $F(t)$ is the inverse Laplace transform of the transfer function $G(s)$. The results of the limits are the response for a unit step input signal with initial conditions, zero.

**Bode analysis**: the system performance information can be obtained by a phase and amplitude plot versus frequency. In this plot the amplitude (db or logarithmic scale) and phase (rad/s) of a given transfer function are calculated using following formula.

\[
M = 20 \log_{10} |G(j\omega)| \quad \text{Eq. 5.17}
\]

\[
\varphi = \angle G(j\omega) \quad \text{Eq. 5.18}
\]

Note, we have been using, for better comprehension, degree units for the phase and non-logarithmic scale for the amplitude.

The poles of $G$ may be calculated by a fit procedure, Podowski et al (1983). Let $s_1$ and $s_2$ be the complex conjugate poles of the system transfer function, then they could be written, Weaver (1963), as:

\[
s_1 = -\omega_n \left( \zeta + j\sqrt{1-\zeta^2} \right) \quad \text{Eq. 5.19}
\]

\[
s_2 = -\omega_n \left( \zeta - j\sqrt{1-\zeta^2} \right) \quad \text{Eq. 5.20}
\]

Where $\omega_n$ and $\zeta_n$ are the natural frequency and damping factor respectively. The system is stable if the damping factor is positive. A large system may have a few complex conjugate poles; in this case, the pole with largest damping factor is of interest to us. This guarantees fast return to the desired steady-state conditions.
Nyquist stability criterion and chart: this is a method for determining whether a complex function has zeros in the right half of the complex plane, Weaver (1963). The Nyquist criterion states that an unstable system is one described by a function, which encircles the -1 point or passes through it. The stability gain margin (SGM) is represented by the distance (d) at which the curve of a stable system passes the -1 point, Harrer (1963). Expressed in decibels:

\[ SGM = 20 \log_{10} \left( \frac{1}{d} \right) \]

Eq. 5.21

Another parameter is the stability phase margin. It is defined as the phase angle through which the GH locus must be rotated in order that the unity magnitude point \( |GH| = 1 \) passes through the -1 point in the GH plane.

Nichols chart: this approach uses the fact that the graph of a complex quantity (forward loop transfer function, G) where amplitude is plotted over phase in rectangular coordinates, the curved coordinates then give the response of a loop closed around the unity with unity feedback. This allows tying together the Bode analysis and the Nyquist criterion, Harrer (1963).

Decay Ratio (DR): this technique is one of the most important linear stability indicators, NEA (2001), frequently used in the nuclear industry and defined as the ratio of the least damped power oscillations with the natural frequency of the reactor, Askari and Hennig, (2001). Recalling the that the poles of the system transfer function can be written in terms of the damping factor and the natural frequency, DR can be expressed as, NEA (2001):

\[ DR = \exp \left( 2\pi \frac{\Re(s_i)}{\Im(s_i)} \right) \]

Eq. 5.22
5.3. Single channel stability and transfer functions

In a boiling water reactor usually, several types of fuel assemblies are used with different structure and fuel compositions. Therefore, study of the thermal hydraulic behavior of each fuel assembly type could be of great interest for owners or nuclear industries. Assuming that under stationary or transient conditions the core pressure drop (consequently channel) is imposed by the core-external pressure difference, we can write

\[ \Delta p_{ch} = \Delta p_i = \Delta p_i^{1\phi} + \Delta p_i^{2\phi} = 0 \]  
Eq. 5.23

and

\[ \Delta p_{ext} = \delta p_{in} - \delta p_{out} = 0 \]  
Eq. 5.24

where \( \Delta p_i^{1\phi} \) and \( \Delta p_i^{2\phi} \) are the perturbations of the pressure drop over the single and the two-phase regions of the channel, and \( \delta p_{in} \) and \( \delta p_{out} \) are the pressure perturbations at core inlet and exit, respectively.

We study now the thermal hydraulic stability of a single channel (say \( i\)-th) in parallel, but uncoupled (see, next section) with the others. Let define the following channel transfer functions

\[ \delta p_i^{1\phi} = G_i^{-1} \delta j_i^{in} \]  
Eq. 5.25

\[ \delta p_i^{2\phi} = H_i \delta j_i^{in} \]  
Eq. 5.26

and write equation 5.23 in the following form

\[ \delta p_{ch} = \left( \frac{\delta p_i^{1\phi}}{\delta j_i^{in}} + \frac{\delta p_i^{2\phi}}{\delta j_i^{in}} \right) \delta j_i^{in} \]  
Eq. 5.27

where \( G_i \) and \( H_i \) are forward and feedback loop transfer functions of the channel \( i \) in the core, respectively, see section 4.9. The closed loop block diagram of channel \( i \) is shown in figure 5.9.
The closed loop transfer function is obtained by the introduction of equations 5.25 and 5.26 into equation 5.27.

\[
\frac{\delta j_{in}^i}{\delta \Delta p_{ch}} = \frac{1}{G_i^{-1} + H_i}
\]  
Eq. 5.28

This can be written as:

\[
\frac{\delta j_{in}^i}{\delta \Delta p_{ch}} = \frac{G_i}{1 + G_i H_i}
\]  
Eq. 5.29

Using equations 5.25 and 5.26, we can calculate other interesting transfer functions:

\[
\frac{\delta \Delta p_{i1}^{1\phi}}{\delta \Delta p_{ch}} = \frac{1}{1 + G_i H_i}
\]  
Eq. 5.30

\[
\frac{\delta \Delta p_{i2}^{2\phi}}{\delta \Delta p_{ch}} = \frac{G_i H_i}{1 + G_i H_i}
\]  
Eq. 5.31

A comparison has been done between eq. 5.31 and the equivalent transfer function in NUFREQ-N (Podowski et al, 1983) and NUFREQ-NP (Peng et al, 1985). We found that the expressions are the same.

Similarly, we can find the closed loop transfer function of the inlet enthalpy perturbation to total channel pressure perturbation.

\[
\frac{\delta h_{in}^{iH}}{\delta \Delta p_{ch}} = \frac{G_i^{iH}}{1 + G_i^{iH} H_i^{H}}
\]  
Eq. 5.32
Where

\[
G_i^h = \frac{\delta h_i^{in}}{\delta \Delta p_i^{1b}} \quad \text{Eq. 5.33}
\]

\[
H_i^h = \frac{\delta \Delta p_i^{2b}}{\delta h_i^{in}} \quad \text{Eq. 5.34}
\]

### 5.4. Parallel channel transfer functions and flow redistribution

There are several methods to handle the transient flow redistribution in parallel channel arrays only connected at plena, via their coupled momentum-energy equations. In this work, we make the following assumptions:

- Pressure levels in the plena are prescribed by the core-external path.
- The flow direction is in positive direction of z-axis. Thus, flow reversal is not allowed.
- The total mass flow rate at core inlet is equal to the sum of the mass flow rates of the channels.

From the last consideration, we have

\[
\delta M_{in} = \sum_{i=1}^{N_{max}} \delta M_i^{in} \quad \text{Eq. 5.35}
\]

this in terms of volumetric flux can be written as follows

\[
\delta j_{in} = \sum_{i=1}^{N_{max}} a_i \delta j_i^{in} \quad \text{Eq. 5.36}
\]

where \( a_i \) is the area fraction of the channel \( i \)

\[
a_i = \frac{A_i}{\sum_k N_k A_k} \quad \text{Eq. 5.37}
\]

and

\[
N_{max} = \sum_k N_k \quad \text{Eq. 5.38}
\]

The inlet volumetric flux of the channels is affected by the perturbations of power input (volumetric power in the fuel and DMH) and the inlet lower plenum state-variables (\( \delta h_{in}, \delta p_{in}, \delta j_{in} \)). Therefore, volumetric fluxes of the channels \( \delta j_i \) are connected, via very complex mechanisms, to the power and plenum inlet perturbations.
The following section describes in detail the necessary procedures for obtaining the parallel channel transfer functions.

Figure 5.10 shows how channel \( i \) is coupled to the rest of the core, lower and upper plenum. Note, that the lower plenum inlet perturbations are obtained from the core-external path equations. These will be related to the channel inlet perturbations. Furthermore, the nodal volumetric power (generated in the fuel) and the direct moderator heating couple this thermal hydraulic system to the neutronics.

\[
\Delta \bar{p}_{ch,i} = \Delta p
\]

Eq. 5.39

In addition, the inlet enthalpy and pressure of the channels are the same as those at the inlet of the lower plenum.
\[ \delta h_i^{in} = \delta h_i^{in} \quad \text{Eq. 5.40} \]
\[ \delta p_i^{in} = \delta p_i^{in} \quad \text{Eq. 5.41} \]

Recalling that the total pressure drop \( \delta\Delta p_{ch,i} \) of channel \( i \) (section 4.9), can be written as

\[ \delta\Delta p_{ch,i} = P_i^h \delta j_i^{in} + P_i^p \delta h_i^{in} + P_i^p \delta p_i^{in} + \sum_{k=1}^{K_{\text{in}}} \left( P_{i,k}^q \delta q_{i,k}^{in} + P_{i,k}^r \delta q_{i,k}^{in} \right) \quad \text{Eq. 5.42} \]

and that the nodal volumetric power in the fuel and DMH are fractions of the total nodal volumetric power

\[ \delta q_{i,k}^m = K_p \delta P_{i,k} \quad \text{Eq. 5.43} \]
\[ \delta q_{i,k}^{m,i} = \left(1 - K_p\right) \delta P_{i,k} \quad \text{Eq. 5.44} \]

inserting equations 5.39-41, 5.43 and 5.44 into eq. 5.42 results in

\[ \delta\Delta p = P_i^h \delta j_i^{in} + P_i^p \delta h_i^{in} + P_i^p \delta p_i^{in} + \sum_{k=1}^{K_{\text{in}}} \left( K_p P_{i,k}^q + \left(1 - K_p\right) P_{i,k}^r \right) \delta P_{i,k} \quad \text{Eq. 5.45} \]

We now solve eq. 5.45 for \( \delta j_i^{in} \)

\[ \delta j_i^{in} = \left( P_i^r \right)^{-1} \left( \delta\Delta p - \left( P_i^p \delta h_i^{in} + P_i^p \delta p_i^{in} + \sum_{k=1}^{K_{\text{in}}} \left( K_p P_{i,k}^q + \left(1 - K_p\right) P_{i,k}^r \right) \delta P_{i,k} \right) \right) \quad \text{Eq. 5.46} \]

and insert eq. 5.46 into eq. 5.36 to obtain the total volumetric flux as a sum of the weighted volumetric fluxes of the channels.

\[ \delta j_i = \sum_{l=1}^{N_{\text{in}}} a_i \left( P_i^l \right)^{-1} \left( \delta\Delta p - \left( P_i^h \delta h_i^{in} + P_i^p \delta p_i^{in} + \sum_{k=1}^{K_{\text{in}}} \left( K_p P_{i,k}^q + \left(1 - K_p\right) P_{i,k}^r \right) \delta P_{i,k} \right) \right) \quad \text{Eq. 5.47} \]

Defining, for convenience

\[ F^l = \sum_{i=1}^{N_{\text{in}}} a_i \left( P_i^l \right)^{-1} \quad \text{Eq. 5.48} \]

and substituting it into eq. 5.47, and solving the resulting equation for the total pressure drop, we obtain

\[ \delta\Delta p = \left( F^l \right)^{-1} \delta j_i + \left( F^l \right)^{-1} \sum_{l=1}^{N_{\text{in}}} a_i \left( P_i^l \right)^{-1} \left( P_i^h \delta h_i^{in} + P_i^p \delta p_i^{in} + \sum_{k=1}^{K_{\text{in}}} \left( K_p P_{i,k}^q + \left(1 - K_p\right) P_{i,k}^r \right) \delta P_{i,k} \right) \quad \text{Eq. 5.49} \]
Inserting eq. 5.49 into 5.46 and rearranging it, we obtain the channel inlet volumetric flux perturbations:

\[ \delta j_i^{in} = (P_i^j)^{-1} \left( (F_i^j)^{-1} \delta j_i + \left( (F_i^j)^{-1} \sum_{l=1}^{N_{\text{max}}} a_i \left( P_i^l \right)^{-1} P_i^h \right) - P_i^h \right) \delta h_i^{in} \]

\[ + \left( (F_i^j)^{-1} \left( \sum_{l=1}^{N_{\text{max}}} a_i \left( P_i^l \right)^{-1} P_i^p \right) - P_i^p \right) \delta p_i^{in} \]

\[ + (F_i^j)^{-1} \sum_{l=1}^{N_{\text{max}}} a_i \left( P_i^l \right)^{-1} \sum_{k=1}^{K_{\text{max}}} \left( K_p P_i^{p_{l,k}} + (1 - K_p) P_i^{p_{l,k}} \right) \delta P_{l,k} - \sum_{k=1}^{K_{\text{max}}} \left( K_p P_i^{p_{l,k}} + (1 - K_p) P_i^{p_{l,k}} \right) \delta P_{l,k} \]

\[ = \left( (P_i^j)^{-1} \sum_{l=1}^{N_{\text{max}}} \sum_{k=1}^{K_{\text{max}}} \left( F_i^q P_i^{q_{l,k}} \delta P_{l,k} - \left( P_i^j \right)^{-1} P_i^p \right) \delta P_{l,k} \right) \]

\[ \text{Eq. 5.50} \]

Now, we define certain intermediate constants for some sub expressions in eq. 5.50.

\[ F^h = \left( \sum_{l=1}^{N_{\text{max}}} a_i \left( P_i^l \right)^{-1} P_i^h \right) \]

\[ F^p = \left( \sum_{l=1}^{N_{\text{max}}} a_i \left( P_i^l \right)^{-1} P_i^p \right) \]

\[ D_{l,k}^q = K_p P_{l,k}^q \]

\[ D_{l,k}^y = (1 - K_p) P_{l,k}^y \]

\[ D_{l,k}^{qy} = D_{l,k}^q + D_{l,k}^y \]

\[ V_i^{j} = \left( P_i^{j} / F_i^{j} \right)^{-1} \]

\[ F_{l,k}^q = a_i V_i^{j} D_{l,k}^q \]

\[ F_{l,k}^y = a_i V_i^{j} D_{l,k}^y \]

\[ F_{l,k}^{qy} = F_{l,k}^q + F_{l,k}^y \]

\[ \text{Eq. 5.51} \]

\[ \text{Eq. 5.52} \]

\[ \text{Eq. 5.53} \]

\[ \text{Eq. 5.54} \]

\[ \text{Eq. 5.55} \]

\[ \text{Eq. 5.56} \]

\[ \text{Eq. 5.57} \]

\[ \text{Eq. 5.58} \]

\[ \text{Eq. 5.59} \]

Rewriting eq. 5.50, using the above constants results in

\[ \delta j_i^{in} = V_i^{j} / \delta j_i + \left( V_i^{j} / F_i^{h} - \left( P_i^{j} \right)^{-1} P_i^h \right) \delta h_i^{in} + \left( V_i^{j} / F_i^{p} - \left( P_i^{j} \right)^{-1} P_i^p \right) \delta p_i^{in} \]

\[ + \left( P_i^{j} \right)^{-1} \sum_{l=1}^{N_{\text{max}}} \sum_{k=1}^{K_{\text{max}}} F_{l,k}^{qy} \delta P_{l,k} - \left( P_i^{j} \right)^{-1} \sum_{k=1}^{K_{\text{max}}} D_{l,k}^{qy} \delta P_{l,k} \]

\[ \text{Eq. 5.60} \]
Finally, we define the following parameters:

$$V_i^h = \left( V_i^h F^h - (P_i^h)^{-1} P_i^h \right)$$  \hspace{1cm} \text{Eq. 5.61}

$$V_i^p = \left( V_i^p F^p - (P_i^p)^{-1} P_i^p \right)$$  \hspace{1cm} \text{Eq. 5.62}

and inserting them into eq. 5.60, gives the compact form of the volumetric flux perturbation of channel $i$.

$$\delta j_{in} = V_i^h \delta j_{in} + V_i^h \delta h_{in} + V_i^p \delta p_{in} + (P_i^p)^{-1} \sum_{k=1}^{K_{max}} \sum_{l=1}^{N_{max}} D_{i,k,l} \delta P_{i,l} - (P_i^p)^{-1} \sum_{k=1}^{K_{max}} \sum_{l=1}^{N_{max}} F_{i,k,l} \delta P_{i,k}$$  \hspace{1cm} \text{Eq. 5.63}

This equation states that the volumetric flux perturbation at channel inlet is a function of lower plenum inlet perturbations, but also depends on the power perturbations in all the core nodes. In the absence of power perturbations, the volumetric flux perturbation at channel $i$ inlet is function of only the lower plenum volumetric flux, enthalpy and pressure perturbations. Otherwise, the total nodal power perturbations are also function of the lower inlet perturbations. This coupling problem will be discussed and solved in section 5.6.

### 5.4.2. Inlet channel perturbations in matrix format

The transient flow redistribution is governed by the lower plenum input perturbations and the nodal power perturbations (in the fuel and the moderator). This can be written in matrix format. For this purpose, we define the following vectors and matrices.

**Perturbation vector at lower plenum inlet**

$$\mathbf{x}_{LPin} \{3,1\} = \begin{pmatrix} \delta j_{in} \\ \delta h_{in} \\ \delta p_{in} \end{pmatrix}^T$$  \hspace{1cm} \text{Eq. 5.64}

**Perturbation vector at the inlet of channel $i$**

$$\mathbf{x}_i \{3,1\} = \begin{pmatrix} \delta j_i^{in} \\ \delta h_i^{in} \\ \delta p_i^{in} \end{pmatrix}^T$$  \hspace{1cm} \text{Eq. 5.65}

**Total nodal power perturbation for the nodes of channel $i$**

$$\mathbf{\delta P}_i \{K_{max},1\} = \begin{pmatrix} \delta P_{i,1} \\ \cdots \\ \delta P_{i,k} \\ \cdots \\ \delta P_{i,K_{max}} \end{pmatrix}^T$$  \hspace{1cm} \text{Eq. 5.66}

Matrix of coefficients $\mathbf{V}_i$ in eqs. 5.40, 5.41, 5.56, 5.61, 5.62; $D_i$ in eqs. 5.53-5.55; $F_i$ in eqs. 5.57-5.59
Finally, we can state that the channel inlet perturbations are obtained as superposition of a very large number of perturbations, see figure 5.11. Eq. 5.63, in matrix notation can be expressed in very simple manner:

\[
x_i^{in} = V_i x_{LPin} + D_i \delta P_i + \sum_{l} F_l \delta P_l
\]

Figure 5.11 Transient flow redistribution block diagram for channel \( i \). It shows that the channel-inlet volumetric flux perturbation is function of inlet lower plenum and 3D nodal power perturbations.
Since for all the channels of the core we can write equations similar to eq. 5.70 then the subscript \( i \) can vary from 1 to \( N_{max} \). Therefore the core flow distribution involves \( N_{max} \) equations and \( N_{max} \times K_{max} + 3 \) variables. This problem for the time being cannot be handled since we, still, have to couple the core thermal hydraulics, to core external thermal hydraulics and neutronics, see section 5.6.

### 5.5. System thermal hydraulics

#### 5.5.1 Channel exit perturbations

The thermal hydraulics of a single channel \( i \) can be expressed in matrix form in terms of input perturbations (inlet and power). For this task, we use the results obtained in the previous section and those in the channel thermal hydraulics, part II. We define for a channel \( i \) the matrix \( C_i \) and the vectors \( y_i \) and \( y_{ex,i} \) as follows:

\[
C_i \{3 + 4K_{max}, K_{max} + 3\} = \begin{bmatrix}
C_{i,11} \{3, K_{max}\} & C_{i,12} \{3,3\} \\
C_{i,21} \{4K_{max}, K_{max}\} & C_{i,22} \{4K_{max},3\}
\end{bmatrix}
\]

Eq. 5.71

\[
y_{ex,i} \{3,1\} = (\delta\alpha_{ex,i}, \delta j_{ex,i}, \delta\Delta p_{ch,i})^T
\]

Eq. 5.72

\[
y_i \{4K_{max},1\} = (\overline{\delta\alpha_i}, \overline{\delta j_i}, \overline{\delta T_{i,j}}, \overline{\delta p_i})^T
\]

Eq. 5.73

where for the \( i \)-th channel

\( C_{i,11}, C_{i,12}, C_{i,21}, C_{i,22} \) matrices of the nodal channel transfer functions

\( \delta\alpha_{ex,i} \) void fraction perturbation at the channel exit

\( \delta j_{ex,i} \) volumetric flux at the channel exit

\( \delta\Delta p_{ch,i} \) total pressure drop perturbation defined as the sum of the \( \delta\Delta p_i^{\phi} \) and the \( \delta\Delta p_i^{\delta} \)

and the vectors

\( \overline{\delta\alpha_i} \{K_{max},1\} \) nodal average void fraction perturbation

\( \overline{\delta j_i} \{K_{max},1\} \) nodal average volumetric flux perturbation

\( \overline{\delta T_{i,j}} \{K_{max},1\} \) nodal average liquid temperature perturbation

\( \overline{\delta p_i} \{K_{max},1\} \) nodal average local pressure perturbation.

Then the channel dynamics can be written as:

\[
\begin{pmatrix}
y_{ex,i} \{3,1\} \\
y_i \{4K_{max},1\}
\end{pmatrix} = C_i \begin{pmatrix}
\delta P_i \{K_{max},1\} \\
x_i^{in} \{3,1\}
\end{pmatrix}
\]

Eq. 5.74
This is equivalent to the following two equations:

\[ y_{ex,i} = C_{i,1}\delta P_i + C_{i,12}x_{in} \]  

Eq. 5.75

\[ y_i = C_{i,21}\delta P_i + C_{i,22}x_{in} \]  

Eq. 5.76

where the already calculated channel inlet perturbations \( x_{in} \) are given by eq. 5.70, which involves also the nodal power perturbations of the core and the perturbation at the inlet of the lower plenum.

### 5.5.2 Core exit perturbations

Using the vector \( y_{ex,i} \), we reconstruct the vapor and liquid mass flow rate perturbation \( \delta M_{f,j} \) and \( \delta M_{g,j} \) at the channel exit from the known perturbations of \( \alpha, j, \) and \( p \). For this purpose, we introduce the matrix \( m_j \), which contains the partial derivatives of the vapor and liquid mass flow rates at the exit of the channel, with respect to the variables \( \alpha, j, \) and \( p \) (see chapter 4, part II).

\[
m_j = \begin{pmatrix}
\frac{\partial \hat{m}_g}{\partial \alpha} & \frac{\partial \hat{m}_g}{\partial j} & \frac{\partial \hat{m}_g}{\partial p} \\
\frac{\partial \hat{m}_f}{\partial \alpha} & \frac{\partial \hat{m}_f}{\partial j} & \frac{\partial \hat{m}_f}{\partial p} \\
0 & 0 & 1
\end{pmatrix}
\]  

Eq. 5.77

Thus the exit perturbations are expressed in terms of mass flow rates.

\[
y'_{ex,i} \{3,1\} = A_m y_{ex,i}
\]  

Eq. 5.78

where \( y'_{ex,i} \{3,1\} = (\delta M_{g,j} \delta M_{f,j} \delta \Delta p_{ch,j})^\top \). Knowing the channel exit \( y'_{ex,i} \) one can construct the total mass flow rates of vapor and liquid at the exit of the core by summing over all channels.

\[
y_{ex} = N \sum_{i=1}^{N_{max}} y'_{ex,i}
\]  

Eq. 5.79

Where \( N \) is the matrix,

\[
N = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/N_{max}
\end{pmatrix}
\]  

Eq. 5.80
and

\[ y_{ex}^{3,1} = \begin{pmatrix} \delta \dot{M}_g \\ \delta \dot{M}_f \\ \delta \Delta p_{ch} \end{pmatrix} \]

Eq. 5.81

We can now express the core exit perturbations in terms of lower plenum and power perturbations. This can be done by insertion of eq. 5.70 into eq. 5.75 and the results into eq. 5.78 and finally summing on all channels via eq. 5.79.

\[ y_{ex} = \sum_{i}^{N_{max}} W_i \delta P_i + M x_{LPin} \]

Eq. 5.82

Where the \( W_i \) and \( M \) matrices are given by,

\[ W_i^{3,K_{max}} = N A_i \left( C_{i,11} + C_{i,12} D_i \right) + \left( N \sum_{l}^{N_{max}} A_l m_l C_{l,12} \right) F_i \]

Eq. 5.83

\[ M^{3,3} = N \sum_{l}^{N_{eq}} A_l m_l C_{l,12} V_i \]

Eq. 5.84

Figure 5.12 shows a graphical interpretation of equations 5.75, 5.78 and 5.82.
The core exit perturbations given by eq. 5.82 are coupled to the thermal hydraulics of the core-external path with the condition that the total pressure drop over the core and core-external path is equal zero \( (\delta \Delta p_{\text{core}} + \delta \Delta p_{\text{ex.core}} = 0) \). The thermal hydraulics of the path external to the core can be presented as:

\[
x_{LPin} = E_Y \{3,3\} y_{ex} \{3,1\} + E_U \{3,4\} u \{4,1\}
\]

Eq. 5.85

where \( E_Y \) and \( E_U \) are the transfer matrices of the core-external path, and \( u \) is the vector of externally imposed perturbations (feed water mass flow rate, feed water enthalpy, pump velocity, reference pressure), see figure 5.13.

\[
u = \begin{pmatrix} \delta M_{fw} \\ \delta h_{fw} \\ \delta v_p \\ \delta p_{ref} \end{pmatrix}
\]

Eq. 5.86

Figure 5.13 Core-external path components and input perturbations are shown on the left. This is presented as a multi-input multi-output linear system on the right.
The coupling can be done by inserting eq. 5.82 into eq. 5.85 and solving for $x_{LPin}$.

$$x_{LPin} = L^{-1} \left( E_Y \left( \sum_{i} W_i \delta P_i \right) + E_U u \right)$$  \hspace{1cm} Eq. 5.87

Where the matrix $L$ is given by

$$L = I - E_Y M$$  \hspace{1cm} Eq. 5.88

Finally, in order to couple thermal hydraulics to neutronics we should express $y_i$ in terms of power and external perturbations. Therefore, we inserting eq. 5.87 into eq. 5.70 and then from the resulting expression into eq. 5.76. This results in the nodal-average perturbations of channel $i$ in terms of power and external perturbations.

$$y_i = \sum_j S_{ij} \delta P_j + E_i u$$  \hspace{1cm} Eq. 5.89

Where the $S_{ij}$ and $E_i$ are,

$$S_{ij} (4K_{\max}, K_{\max}) = C_{i21} + C_{i22} \left( D_i + V_i L^{-1} E_i W_i + F_i \right) \quad \text{if } j = i \quad Eq. 5.90$$

$$S_{ij} (4K_{\max}, K_{\max}) = C_{i22} \left( V L^{-1} E W_j + F_j \right) \quad \text{if } j \neq i \quad Eq. 5.91$$

$$E_i (4K_{\max}, 4) = C_{i22} V L^{-1} E_U \quad Eq. 5.92$$

Eq. 5.89 can also be written in matrix form,

$$y \{4N_{\max} K_{\max}, 1\} = S \{4N_{\max} K_{\max}, N_{\max} K_{\max}\} \delta P \{N_{\max} K_{\max}, 1\} + E \{4N_{\max} K_{\max}, 4\} u \{4,1\} \quad Eq. 5.93$$

This corresponds to a thermal hydraulic system which can be excited with 3D power ($\delta P$) and external ($u$) perturbations, see figure 5.14.

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5.6. Coupling system thermal hydraulics to neutronics

In order to determine the open loop and closed loop system transfer functions, the channel-wise 1D thermal hydraulics has to be coupled to the 3D modal nodal power and modal flux amplitude perturbations. The coupling consists of inserting the expression of the fuel temperature perturbation in the neutronics equations and combining these with the system thermal hydraulics equations obtained in the previous section. This results in a new set of mode amplitude perturbations, written in terms of only system external perturbations. Stability of the resulting dynamical system can be determined by studying the resulting transfer function (see stability criteria section), which includes now void and Doppler reactivity feedbacks.

The main results obtained previously will be repeated first; these are the set of equations that govern system thermal hydraulics, fuel dynamics, modal power and amplitudes of the modal fluxes.

Channel thermal hydraulics (nodal-average perturbations of channel $i$; eq. 5.89)

$$y_i = \sum_{j}^{N_{ch}} S_j \delta P_j + E_i u$$  \hspace{1cm} \text{Eq. 5.94}

Core thermal hydraulics (see figure 5.14, eq. 5.93) in matrix notation:

$$y = SD\delta P + EU$$  \hspace{1cm} \text{Eq. 5.95}

Nodal-average perturbations of fuel temperature (see fuel dynamics chapter), for channel $i$, node $k$:

$$\overline{\delta T}_{i,k} = T_{i,k}^{\alpha} \overline{\delta \alpha}_{i,k} + T_{i,k}^{j} \overline{\delta j}_{i,k} + T_{i,k}^{T} \overline{\delta T}_{i,k} + T_{i,k}^{p} \overline{\delta p}_{i,k} + T_{i,k}^{q} \overline{\delta q_{i,k}}$$  \hspace{1cm} \text{Eq. 5.96}

Modal nodal power perturbations ($M+1$ equations for each node, eq. 2.111), for channel $i$, node $k$ and mode $m$, for two neutronics groups ($l=1,2$):

$$\delta P_{i,k,m} = En m \sum_{l=1}^{2} \psi_{i,k,l,m} \overline{\delta \alpha}_{i,k} + \overline{\delta \alpha}_{i,k} + \overline{\delta \alpha}_{i,k} + \overline{\delta q_{i,k}}$$  \hspace{1cm} \text{Eq. 5.97}

The nodal power perturbation (eq. 2.112) is obtained by summing the modal contributions (see figure 5.16):

$$\delta P_{i,k} = \sum_{m=0}^{M} \delta P_{i,k,m} + EN_{0} \overline{\delta \alpha}_{i,k} + \overline{\delta \alpha}_{i,k} + \overline{\delta \alpha}_{i,k} + \overline{\delta q_{i,k}}$$  \hspace{1cm} \text{Eq. 5.98}

The nodal fuel power perturbation is a fixed fraction $K_p$ of the power perturbation,
\[ \delta q_{\text{DMH}}^{m} = K_p \delta P_{i,k} \quad \text{Eq. 5.99} \]

and the remaining fraction provides the direct moderator heating perturbation (DMH), (see figure 5.16):

\[ \delta q_{\text{DMH}}^{m} = (1 - K_p) \delta P_{i,k} \quad \text{Eq. 5.100} \]

The modes amplitude perturbations \((M+1)\) equations, see figure 5.15 and eq. 2.126, are:

\[ n_m = G_m \left( \rho_m + \sum_{l=1}^{6} r_{V,l,k,m} \delta \alpha_{l,k} + \sum_{l=1}^{6} r_{D,l,k,m} \delta T_{f,l,k} \right) \quad ; \quad m=0,M \quad \text{Eq. 5.101} \]

Forward loop transfer functions, which relate the mode amplitude perturbation to the external reactivity perturbations, see eq. 2.109:

\[ G_m = \frac{N_0}{s \Lambda_m - \rho_m + \sum_{l=1}^{6} \frac{s \beta_l}{s + \lambda_l}} \quad ; \quad m=0,M \quad \text{Eq. 5.102} \]

The listed equations above form different components of our dynamical system. These will be combined to obtain the feedback loop, open loop and closed loop transfer functions that are needed for the stability analysis. For this task, we first insert eq. 5.99 into eq. 5.96 and then the resulting perturbation of average fuel temperature into eq. 5.98 to get the modal nodal power perturbation:

\[ \delta P_{i,k} = E \left( \sum_{m=0}^{M} \left( \sum_{l=1}^{6} \psi_{l,k,m} \right) n_m + N_0 \left( \sum_{l=1}^{2} T_{i,k}^{m} \frac{\partial \Sigma_f}{\partial T_f} + \left( K_{p,i}^{m} \delta P_{i,k} \right) \right) \right) \delta \alpha_{l,k} \]

\[ + \left( N_0 \sum_{l=1}^{2} \frac{\partial \Sigma_f}{\partial T_f} \psi_{l,k,0} \left( T_{i,k}^{m} \delta \alpha_{l,k} + T_{i,k}^{m} \delta T_{f,l,k} + T_{i,k}^{m} \delta \bar{P}_{l,k} \right) \right) \]

\[ + \left( N_0 \sum_{l=1}^{2} \frac{\partial \Sigma_f}{\partial T_f} \psi_{l,k,0} \left( T_{i,k}^{m} K_{p,i}^{m} \delta P_{i,k} \right) \right) \quad ; \quad m=0,M \quad \text{Eq. 5.103} \]

Eq. 5.103 is rearranged to obtain an expression for the total nodal power perturbation of channel \(i\), node \(k\)

\[ \delta P_{i,k} = \sum_{m=0}^{M} Q_{i,k}^{m} n_m + Q_{i,k}^{m} \delta \alpha_{l,k} + Q_{i,k}^{m} \delta j_{l,k} + Q_{i,k}^{m} \delta T_{f,l,k} + Q_{i,k}^{m} \delta \bar{P}_{l,k} \quad \text{Eq. 5.104} \]

where

\[ Q_{i,k}^{m} = E \left( 1 - Q_{i,k}^{m} \right)^{-1} \sum_{l=1}^{2} \psi_{l,k,j,m} \psi_{l,k,m} \quad \text{Eq. 5.105} \]
\[
Q_{i,k}^q = E \left(1 - Q_{i,k}^q\right)^{-1} N_0 \left( \sum_{l=1}^{2} \left( T_{i,k}^q \left( \frac{\partial \Sigma_f}{\partial T_f} \right)_{j,l}, \psi_{i,k,l,0} \right) \right) \quad \text{Eq. 5.106}
\]

\[
Q_{i,k}^l = E \left(1 - Q_{i,k}^l\right)^{-1} N_0 \left( \sum_{l=1}^{2} \left( \frac{\partial \Sigma_f}{\partial T_f} \right)_{j,l}, \psi_{i,k,l,0} \right) T_{i,k}^l \quad \text{Eq. 5.107}
\]

\[
Q_{i,k}^t = E \left(1 - Q_{i,k}^t\right)^{-1} N_0 \left( \sum_{l=1}^{2} \left( \frac{\partial \Sigma_f}{\partial T_f} \right)_{j,l}, \psi_{i,k,l,0} \right) T_{i,k}^t \quad \text{Eq. 5.108}
\]

\[
Q_{i,k}^p = E \left(1 - Q_{i,k}^p\right)^{-1} N_0 \left( \sum_{l=1}^{2} \left( \frac{\partial \Sigma_f}{\partial T_f} \right)_{j,l}, \psi_{i,k,l,0} \right) T_{i,k}^p \quad \text{Eq. 5.109}
\]

\[
Q_{i,k}^w = E N_0 \left( \sum_{l=1}^{2} \left( \frac{\partial \Sigma_f}{\partial T_f} \right)_{j,l}, \psi_{i,k,l,0} \right) T_{i,k}^w K_p \quad \text{Eq. 5.110}
\]

Eq. 5.104 can be written in matrix notation for channel \(i\):

\[
\delta P_{i} \{K_{\text{max}}, 1\} = Q^q_{i} \{K_{\text{max}}, M + 1\} n \{M + 1, 1\} + Q^l_{i} \{K_{\text{max}}, 4K_{\text{max}}\} y_i \{4K_{\text{max}}, 1\} \quad \text{Eq. 5.111}
\]

where \(n\) is the vector of the modal amplitude perturbations.

This equation is coupled to the channel thermal hydraulics by inserting the vector \(y_i\) of the nodal-average perturbations of channel \(i\), eq. 5.94:

\[
\delta P_{i} = Q^v_{i} n + Q^v_{i} \left( \sum_{j=0}^{N_{\text{max}}} S_j \delta P_j + E_i u \right) \quad \text{Eq. 5.112}
\]

We can group the power perturbation terms in eq. 5.112:

\[
\delta P_{i} - Q^v_{i} \sum_{j=0}^{N_{\text{max}}} S_j \delta P_j = Q^v_{i} n + Q^v_{i} E_i u \quad \text{Eq. 5.113}
\]

The channel power perturbations (eqs 5.112 and 5.113) depend on all harmonics, power of the other channels and the external perturbations. Eq. 5.113 can be written for the whole core in matrix form as:

\[
\left( 1 - Q^v S \right) \delta P = Q^v n + Q^v E_i u \quad \text{Eq. 5.114}
\]

where the matrices defined above have the following dimensions:

\[
\delta P \{N_{\text{max}} K_{\text{max}}, 1\}
\]
Finally, the 3D power perturbations are obtained from eq. 5.114. They are only functions of mode amplitudes and external perturbations; all couplings between fuel, thermal hydraulics, flow distribution in the core and neutronics have been considered:

$$\delta P = \left( I - Q^T S \right)^{-1} \left( Q^T n + Q^T E u \right)$$

Eq. 5.115

On the other hand, we substitute the fuel temperature perturbation in eq. 5.101 with its expression (eq. 5.96) and group the void terms. In the resulting equation, we plug eq. 5.99.

$$n_m = G_m \left( \rho_m + \sum_{i=1}^{N_m} \sum_{k=1}^{K_m} \left( r_{Y,j,k,m} + r_{D,j,k,m} T_{i,k}^a \right) \delta \alpha_{i,k} \right.$$  
$$\left. + \sum_{i=1}^{N_m} \sum_{k=1}^{K_m} \left( T_{i,k}^j \delta j_{i,k} + T_{i,k}^p \delta p_{i,k} + T_{i,k}^{q_m} K_p \delta \rho_{i,k} \right) \right) ; \quad m=0,M \quad \text{Eq. 5.116}$$

The M+1 equations 5.116 in vector form become:

$$n_m = G_m \left( \rho_m + \sum_{i=1}^{N_m} \left( R_{i,j,k,m}^{a_m} \right)^T \delta \alpha_{i,m} + \left( R_{i,j,k,m}^{m} \right)^T \delta j_{i,m} + \left( R_{i,j,k,m}^{p_m} \right)^T \delta p_{i,m} + \sum_{i=1}^{N_m} \left( R_{i,j,k,m}^{q_m} \right)^T \delta \rho_{i,m} \right) \right) ; \quad m=0,M \quad \text{Eq. 5.117}$$

where

$$R_{i,j,k,m}^{a_m} = r_{Y,j,k,m} + r_{D,j,k,m} T_{i,k}^a$$  \quad \text{Eq. 5.118}

$$R_{i,j,k,m}^{j_m} = r_{D,j,k,m} T_{i,k}^j$$  \quad \text{Eq. 5.119}

$$R_{i,j,k,m}^{m_m} = r_{D,j,k,m} T_{i,k}^m$$  \quad \text{Eq. 5.120}

$$R_{i,j,k,m}^{p_m} = r_{D,j,k,m} T_{i,k}^p$$  \quad \text{Eq. 5.121}

$$R_{i,j,k,m}^{q_m} = r_{D,j,k,m} T_{i,k}^{q_m} K_p$$  \quad \text{Eq. 5.122}
Equation 5.117 is written in compact form as

\[ n_m = G_m \left( \rho_{ex} + \left( R^{q_m} \right)^T \delta P + \sum_{i=1}^{N_{max}} \left( R_{i}^{y_m} \right)^T \gamma_i \right) \quad ; \quad m=0,M \quad \text{Eq. 5.123} \]

where

\[ \left( R_{jm}^{m} \right)^T = \begin{pmatrix} \left( R_{i}^{y_m} \{ K_{max},1 \} \right)^T \\ R_{jm}^{m} \{ K_{max},1 \} \\ R_{jm}^{m} \{ K_{max},1 \} \\ R_{jm}^{m} \{ K_{max},1 \} \end{pmatrix} \quad \text{Eq. 5.124} \]

\[ \left( R_{jm}^{m} \right)^T = \begin{pmatrix} \left( R_{1}^{q_m} \{ K_{max},1 \} \right)^T \\ \vdots \\ \left( R_{N_{max}}^{q_m} \{ K_{max},1 \} \right)^T \end{pmatrix} \quad \text{Eq. 5.125} \]

Inserting the channel state-variable perturbations into eq. 5.123 and rearranging it, we obtain

\[ n_m = G_m \left( \rho_{ex} + \left( R^{y_m} \right)^T S + \left( R^{q_m} \right)^T \delta P + \left( R^{y_m} \right)^T E \ u \right) \quad ; \quad m=0,M \quad \text{Eq. 5.126} \]

where

\[ \left( R^{jm} \right)^T = \begin{pmatrix} \left( R^{y_m} \right)^T \\ \vdots \\ \left( R^{y_m} \right)^T \end{pmatrix} \quad \text{Eq. 5.127} \]

Using the expression of 3D power perturbation (eq. 5.115) and doing the necessary matrix manipulations, we get

\[ n_m = G_m \left( \rho_{ex} + \left( R^{y_m} \right)^T S + \left( R^{q_m} \right)^T \left( I - Q^S \right)^{-1} Q^m \right. \]

\[ + \left( \left( R^{y_m} \right)^T S + \left( R^{q_m} \right)^T \left( I - Q^S \right)^{-1} Q^m + \left( R^{y_m} \right)^T E \ u \right) \quad ; \quad m=0,M \quad \text{Eq. 5.128} \]

The M+1 equations 5.128, can be written in the following form

\[ n_m = G_m \left( \rho_{ex} + H_m \ n + K_m \ u \right) \quad ; \quad m=0,M \quad \text{Eq. 5.129} \]

where

\[ H_m \{ 1,M+1 \} = \begin{bmatrix} H_m^0 & \cdots & H_m^m & \cdots & H_m^M \end{bmatrix} \quad \text{Eq. 5.130} \]
\[ K_m \{1, 4\} = \begin{bmatrix} K^1_m & \cdots & K^m_m & \cdots & K^4_m \end{bmatrix} \]  
Eq. 5.131

and

\[ H^m_m = \left( \left( R^{m^2} \right)^T S + \left( R^{m^2} \right)^T \right) \left( I - Q^T S \right)^{-1} Q^{m^2} \]  
Eq. 5.132

\[ K^m_m = \left( \left( R^{m^2} \right)^T S + \left( R^{m^2} \right)^T \right) \left( I - Q^T S \right)^{-1} Q^T + \left( R^{m^2} \right)^T E^m \]  
Eq. 5.133

Now we present the system of equations 5.129 in matrix notation:

\[ (I - G \cdot H) n = G \cdot \rho_{ex} + G \cdot K \cdot u \]  
Eq. 5.134

where the forward loop transfer matrix is,

\[ G \{M + 1, M + 1\} = \begin{bmatrix} G_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & G_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & G_M \end{bmatrix} \]  
Eq. 5.135

the feedback loop transfer matrix,

\[ H \{M + 1, M + 1\} = \begin{bmatrix} H^0_0 & \cdots & H^m_0 & \cdots & H^M_0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & H^0_m & \cdots & H^M_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ H^0_M & \cdots & H^m_M & \cdots & H^M_M \end{bmatrix} \]  
Eq. 5.136

and the control transfer matrix

\[ K \{M + 1, 4\} = \begin{bmatrix} K^1_M & \cdots & K^m_M & \cdots & K^4_M \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & K^1_m & \cdots & K^4_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ K^1_M & \cdots & K^m_M & \cdots & K^4_M \end{bmatrix} \]  
Eq. 5.137

Equation 5.134 can be easily solved by inverting the matrix \((I - G \cdot H)\).

\[ n = \left( I - G \cdot H \right)^{-1} G \cdot \rho_{ex} + \left( I - G \cdot H \right)^{-1} K \cdot u \]  
Eq. 5.138
This is exactly the expression that we looked for. Eq. 5.138 determines the perturbations of the amplitude of the flux harmonics ($\theta$-$M$) in terms of system imposed perturbations. It is written in matrix format, where it is difficult to see the external perturbations to the mode amplitude transfer functions. Therefore, we define the system characteristic matrix $A$ as follows:

$$A\{M+1,M+1\} = I - GH$$

Eq. 5.139

Consequently, the inverse of the system characteristic matrix,

$$A^{-1}\{M+1,M+1\} = \frac{A^*}{|A|}$$

Eq. 5.140

where $A^*$ is the adjoint of $A$. Now, inserting eq. 5.140 into eq. 5.138 we obtain

$$n = \frac{1}{|A|} A^*GP_{ex} + \frac{1}{|A|} A^*GKu$$

Eq. 5.141

From which the components of the mode amplitude perturbation vector become obvious

$$n_m = \left(|A|^{-1} \sum_{i=0}^{M} G_i a_{m,i}^*\right)\rho_{ex} + \left(|A|^{-1} \sum_{i=0}^{M} K_i G_i a_{m,i}^*\right)\delta M_{fw} + \left(|A|^{-1} \sum_{i=0}^{M} K_i^2 G_i a_{m,i}^*\right)\delta h_{fw} + \left(|A|^{-1} \sum_{i=0}^{M} K_i^2 G_i a_{m,i}^*\right)\delta \nu_p + \left(|A|^{-1} \sum_{i=0}^{M} K_i^2 G_i a_{m,i}^*\right)\delta p_{ref}$$

; $m=0,M$

Eq. 5.142

The M+1 equations 5.142 present mode perturbations in terms of imposed perturbations to our dynamical system and their closed loop transfer functions. We study the stability of the each mode by calculation of the mode decay ratio and its resonance frequency, and use Nyquist, Bode and Nichols diagrams; see section 5.2. Figure 5.18 shows the simplified block diagram of equations 5.142.
Figure 5.15 Harmonics perturbations

Figure 5.16 Modal nodal power perturbation
Figure 5.17 Nodal void and fuel temperature averages

Figure 5.18 Simplified block diagram for our thermonuclear dynamical system (MIMO).
References:

Chapter 6

Code validation and applications
6.1. Introduction

The previous chapters have provided the various parts and models used in this development. Here, different steps of validation of the tool and some of its applications are presented.

For the thermal-hydraulics part of the development, the validation has been done first at the elementary level by checking the heat and mass balances, in order to make sure that the conservation equations are correctly integrated over a segment or over a thermal hydraulic channel. Second, the results obtained from the tool have been compared with some reference data, with analytical solutions when available or with the results of other codes.

In order to display the capabilities of the code, it has been applied on two industrial plants, a chemical and a nuclear reactor. The results were in good agreement with reference results or those calculated by the other codes.

6.2. Thermal hydraulic validation

Validation of the thermal hydraulics parts of the code, started with the comparison of the steam table routines, written fully in MATLAB, with those from reference data, as shown in the two examples of figures 6.1 and 6.2. After this step, the results of a single-channel model, with uniform heat input, have been compared with the analytical solution, where the pressure drop and mass fluxes, liquid and mixture enthalpies have been checked. Furthermore, a comparison of void fractions calculated by the code and those calculated using the relation between quality and void of the homogenous model and that of drift flux model for the case with constant drift flux parameters has be done, see figure 6.3.

![Figure 6.1 Comparison between calculated liquid densities and reference data](image)
Figure 6.2 Comparison between calculated gas densities and reference data

Figure 6.3 Comparison between the void fraction obtained from the code and reference void computations. Test conditions: Inlet pressure 13 bar, Mass flux 1.88 kg/m²/s, Inlet temperature 462.87 K, Heat input 0.138 MW.
6.3. Neutronic validation

Since the 3D neutron distribution plays an important role in the stability analysis of BWRs, the solution of the core statics has been tested in different ways. First, the results obtained from the code (where the corresponding routines were re-written in MATLAB language) were compared with those calculated at the Polytechnic University of Valencia (UPV) earlier. Table 6.1 presents the first four eigenvalues of one of the Ringhals reactor cycles (with 728 fuel assemblies). The agreement is excellent; any discrepancies in the last significant digits can be attributed to differences in the computer environments.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>ETHZ</th>
<th>UPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>1.001771</td>
<td>1.001770</td>
</tr>
<tr>
<td>Second</td>
<td>0.994937</td>
<td>0.994936</td>
</tr>
<tr>
<td>Third</td>
<td>0.992972</td>
<td>0.992973</td>
</tr>
<tr>
<td>Fourth</td>
<td>0.991048</td>
<td>0.991045</td>
</tr>
</tbody>
</table>

Table 6.1 Comparison between eigenvalues calculated in this work and by UPV

Furthermore, not only the eigenvalues but also eigenvectors associated to the eigenvalues have shown good agreement. Figure 6.4 provides comparison between the UPV and our thermal flux profile for the first mode. Since the eigenvectors are very large vectors, we present here the eigenvector amplitude of the first 260 nodes only.

![Figure 6.4 Comparison between the first eigenmode calculated by UPV and by our tools](image)

As second method for testing the core statics solution, a homogenous core has been modeled with a reference code (widely used for reactor physics applications) and with our tools. The reference code uses the finite difference method for solving multi-group neutron diffusion equations with the same boundary conditions like our code. As for the cylindrical geometry
used the analytical solution of the core statics is known, this can also be used as the last step of the neutronic code validation.

The homogenous core has been modeled using cross sections of a single fuel assembly type and reflectors (top, lateral, and bottom). The core was made up of 121 bundles had the quasi-cylindrical shape of a typical PWR. Zero flux boundary conditions have been applied at the external surfaces of the reflectors. Table 6.2 shows how an increase of the Legendre polynomial order of our solutions improves the core first eigenvalue ($K_{\text{eff}}$).

<table>
<thead>
<tr>
<th>Legendre Polynomial order</th>
<th>Eigenvalue</th>
<th>Difference in pcm</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>1.00241</td>
<td>239</td>
</tr>
<tr>
<td>Second</td>
<td>1.00097</td>
<td>95</td>
</tr>
<tr>
<td>Third</td>
<td>1.00049</td>
<td>47</td>
</tr>
<tr>
<td>Reference</td>
<td>1.00002</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.2 Eigenvalue comparisons

It is important to note that even a 239 pcm difference between the reference eigenvalue and that from the code can be considered very good agreement.

Furthermore, the normalized axial power profile of the reference code computation, of our code and the analytical solution have been compared and are presented in figure 6.5. The agreement is excellent between the results of the two computations.

![Figure 6.5 Normalized axial power shapes calculated by three methods.](image)

The normalized radial power distribution on the symmetry line were also compared with the reference code data and again good agreement has been observed; see figure 6.6.
6.4. Chemical Reactor

The code has been applied next for the design of a chemical reactor. This device was similar to a large gas-to-boiling-water heat exchanger with parallel tubes, and worked at low pressure, around 10 bar at the water side. The liquid on the tube side was used for removal of the heat generated due to an exothermic chemical reaction on the primary side. This heat removal process was controlled by the adjustment of the system pressure on the tube side.

The stability behavior of the reactor under nominal conditions and with changes of the system important parameters, such as temperature and mass flow rate of the primary side, inlet orifices and system pressure was investigated.

For this task, it was necessary to change slightly the fuel model of the code (since it was designed for nuclear fuel) and to adapt it to the simulation of the dynamics of the heat exchanger tube wall. After appropriate change in the heat transfer model, the thermal hydraulic part of the code was used for the stability analysis of the reactor. Steady-state operation results were compared to those from a reference computation with another sophisticated code performed at Imperial College in London. They were in good agreement. This has shown that the complex thermal hydraulic model (non homogenous coupled differential equations) of the code works well not only at high pressure (nuclear reactors) but also at low pressure.

The reactor was shown to be stable by stability analysis based on the Nyquist and Bode diagrams at nominal conditions, see figures 6.7 and 6.8. The Nyquist plot of the open-loop system transfer function does not cross RE = -1 line, as shown in figure 6.7. The Bode plot of the inlet-flow-rate to pressure-drop-across-the-channel transfer function shows that the pressure perturbation versus inlet velocity does not touch or even approach a zero amplitude, indicative of instability.

Figure 6.6 Normalized radial power shapes calculated by two codes, at core symmetry line.
Figure 6.7 Nyquist diagram, stable case at nominal conditions

Figure 6.8 Bode plot of the inlet-flow-rate to pressure-drop-across-the-channel transfer function at nominal conditions. Top: phase of the transfer function; bottom: amplitude. (There is no discontinuity in the phase diagram as the phases of +180 and -180 degrees are identical)
Various parameters have then been varied and the stability of the system at each case has been investigated. Below are presented Nyquist and Bode plots related to an unstable situation finally reached by reduction of the inlet orificing, figures 6.9 and 6.10.

Figure 6.9 Nyquist plot, unstable case reached by reduction of the inlet orificing at nominal Conditions.

Figure 6.10 Bode plot of the inlet-flow-rate to pressure-drop-across-the-channel transfer function at nominal conditions but with reduced inlet orificing. Top: phase of the transfer function; bottom: amplitude.
6.5. Nuclear Reactor

To test the performance of the code in nuclear applications, the Forsmark reactors have been selected. Forsmark is a three-unit Swedish nuclear power station. Each unit is a large BWR. The main parameters of the three units are given in Table 6.3.

<table>
<thead>
<tr>
<th>Unit</th>
<th>Power [MW]</th>
<th>Flow rate [kg/s]</th>
<th># Assemblies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2700</td>
<td>11000</td>
<td>676</td>
</tr>
<tr>
<td>2</td>
<td>2700</td>
<td>11000</td>
<td>676</td>
</tr>
<tr>
<td>3</td>
<td>3020</td>
<td>11000</td>
<td>648</td>
</tr>
</tbody>
</table>

Table 6.3 Forsmark plant main data

In the next sections, results obtained from the steady state solution and transients related to the following cases are provided:

- Forsmark unit one, at beginning of the cycle 19
- Forsmark unit two, at middle of the cycle 17

6.5.1 Steady state

The steady state operating conditions of the reactor have been obtained by reading the geometry, thermal hydraulic and nuclear data from reactor files and simulating the reactor state based on the models discussed in the previous chapters. Among the enormous amount of data available at the end of the steady state calculation, some that represent different levels of detail of our code have been selected for presentation in the following sections. Results that were also available in the reactor data files have been compared with the calculated ones, and reasonable agreement between them has been observed.

**Core nodalization**

Solution of core statics equations requires a core nodalization. Normally, reactor core is divided in various radial nodes (one homogenized assembly per node, in x and y directions). The axial planes are then obtained by subdivision of fuel assemblies in some axial nodes (typically 25 for BWRs). The number and size of meshes in each direction (x,y,z) are read from reactor files. This nodalization usually provides accurate profile of power and thermal-hydraulic state variables. Figure 6.11 shows the radial (a) and axial (b) nodalization for Forsmark reactor with 676 fuel assemblies. Note, in this figure the position of reflectors are not presented, since the albedo boundary conditions has been used at the external surfaces of the core.
Results of Forsmark unit one, at the beginning of Cycle 19

Various flux harmonics of the core of this unit have been calculated by our core statics routines. The first three harmonics (fundamental, first, and second modes) are presented below in terms of radial profiles of axially averaged bundle fluxes.

Figure 6.12 Fast flux, fundamental mode
The flux harmonics (Figs. 6.12 to 6.14) represent the shape of the time-dependent variation of the flux mode perturbations. The total fast or thermal flux perturbation is then defined as a superposition of these perturbations. Normally, the shape of the fundamental mode depends on the reactor core loading, while the first and second harmonic shapes are sinusoidal. The latter shapes are distinguished, mainly, from each other by the position of the symmetry line,
which is about 90° rotated. However under stationary conditions, higher harmonics of the flux do not exist and the only contribution to the power profile comes from the fundamental mode. The stationary fast and thermal fluxes are finally used for obtaining the 3D power profile. This can be plotted at each core plane or for each fuel assembly. Figures 6.15 to 6.18 show the radial power distributions at various axial levels. The different planes represent equally-spaced elevations from the bottom of the core.

Figure 6.15 Power distribution at plane 5

Figure 6.16 Power distribution at plane 10
The shape of the reactor power depends on various factors such as the type of fuel assembly, the axial enrichment variation, or the axial change in composition (due, e.g. to the use of part-length fuel rods), presence of control rods, etc. The assemblies are normally arranged in a way that assembly design limits are not violated during any operational conditions. In the
above plots, the anti-symmetric loading pattern of the assemblies with the special position of the assemblies with low power (turquoise and sky blue colors at the symmetry lines in figures 6.17 and 6.18) determine the power shape.

In addition to the 3D power distribution for this case, the radial distribution of the channel-average void fraction and the total assembly power distribution are provided in figures 6.19 and 6.20, respectively.
**Forsmark unit two, at middle of cycle 17**

In order to present some other available data at the end of the steady-state calculation (not shown for the previous case), results of Forsmark Unit 2 are used in the following plots. The Unit 2 has the same number of fuel assemblies, and the same operational conditions (core power and mass flow rate) but a different loading pattern. This difference could be observed by the comparison of the power average distribution of the two units; see figures 6.20 and 6.21. The main difference is due to the position of the assemblies with lower power (light green and turquoise colors) in the central part of the core.

![Figure 6.21 Total assembly power distribution](image1)

Since core assemblies have different power, the total core flow is distributed among them such that they satisfy the pressure boundary condition over the core. Figure 6.22 shows the flow distribution among the assemblies.

![Figure 6.22 Assembly mass flow rate distribution](image2)
Another interesting result to present is the 3D void distribution. Since the pressure boundary condition and inlet temperature are the same for all the core fuel assemblies, the position of net vapor generation depends only on the assembly power and mass flow rate. The axial void distribution at two different axial, vertical planes is shown below. The axial planes 6 and 12 correspond to the columns of the rows J = 6 and J = 12 of the radial plane, respectively.

Figure 6.23 Axial assembly void fraction profile, at axial plane 6

Figure 6.24 Axial assembly void fraction profiles, at axial plane 12
6.5.2 Transient results

Transient analysis starts with setting up the system matrices using the steady state data and then combining them and finally solving them for the system state variable perturbations and transfer functions.

Here we present some of the results of the Forsmark Unit 2 case, at the middle of Cycle 17. For this test case, the fundamental and first and second harmonic perturbations have been taken into account. Figures 6.25 and 6.26 present the fundamental mode and the first harmonic perturbations due to an imposed external reactivity perturbation, respectively. The second harmonic is not presented since it was similar to the first harmonic.

A study of the reactor harmonics showed that the fundamental mode had a dominant contribution, with respect to the other harmonics, in the reactor power oscillation. Therefore we could conclude that the reactor could have been subject to global oscillations. This has been confirmed by superimposing different power harmonics perturbations.

Figure 6.25 Axially averaged fundamental mode perturbations
Any other external perturbation can be used for perturbing the system. The feed water mass flow rate perturbation has been used for a second example. The plots below show axial power perturbations due to imposed external perturbations (mass flow rate of the feed water). Studying the phase of the perturbations, it can be concluded that the reactor is subject to global type of oscillations as, the phases of the local power oscillations are the same at each level of the core (see figure 6.27) and also are the same as those of the axially averaged radial power oscillations, see figure 6.28. In case of out-of-phase oscillation we would have expected the phase change to be about 180° in the two halves of each core level (in figure 6.27) and the two halves of the core in figure 6.28.
Figure 6.27 Axial power perturbations, at axial plane 15

Figure 6.27 shows the power oscillation at axial plane 15. As it can be seen from this plot, the power oscillation changes phase in the axial direction, because of the coupling effect with thermal-hydraulics variables, which vary axially. The power oscillation phases are constant at each axial level because of the neutron flux effect.
Besides obtaining the state variable perturbations, one can also use the system transfer function for the stability analysis. The plot below shows the open loop system transfer function. System results to be stable since any of the modes cross $RE = -1$ line.

![Figure 6.29 Locus GH, open loop transfer function for modes 0, 1 and 2](image)

The decay ratio (DR) and the resonance frequency (RF) of the system have also been investigated, by interpolating the system transfer function and finding its poles, their associated resonance frequency and damping factor. The results of the two cases of the Forsmark reactor are given in Table 6.4. The reference DRs and RFs are calculated from measured signals in the plant.

<table>
<thead>
<tr>
<th>Case</th>
<th>RF</th>
<th>RF-ref</th>
<th>DR</th>
<th>DR-ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forsmark 1 boe 19</td>
<td>0.5</td>
<td>0.52</td>
<td>0.51</td>
<td>0.61</td>
</tr>
<tr>
<td>Forsmark 2 moc 17</td>
<td>0.5</td>
<td>0.45</td>
<td>0.55</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Table 6.4 Comparison between decay ratio and resonance frequency; the calculated and reference data.

Discrepancies between the calculated and the reference data can be due to several reasons. First, because of the fact that the reference data are calculated based on parametric or non-parametric methods for the interpolation of the power signals. The goodness of the reference data depend not only on the method used for the interpolation but also the order of the
polynomials used. Various studies on this topic showed that reference data may have even large uncertainties. Second, a higher number of frequencies for approximating the system transfer function could improve the results but this is much more time consuming.

6.6. Conclusions

The code produced in this work for the stability analysis can deal with thermal-hydraulic and/or nuclear systems of large size. It can provide very detailed thermal hydraulic results, not only on for high pressure systems but also for systems that work at low pressure.

The graphical interfaces and movie features of the tool help to comprehend and interpret the stability characteristics of the examined systems.

Various tests have been conducted to check the reliability of the results. They are accurate when the inputs are sufficiently accurate for the detailed computations performed in the code. For example, the core statics routines give very accurate results when compared with the accurate results of reference codes.

Interfaces to core simulators have been developed and tested, but it is important to note that for any new reactor the tool and new interface should be calibrated and verified.

Many other features and components to the code could be added in order to expand the field of applications. For example by calculating the reactor peaking factor and some other important design limit parameters, the code could be used for core loading pattern optimization.
## A1. Nomenclature

Note, index b refers to bulk parameters in the table.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MATLAB</th>
<th>units</th>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>(c, c_p, c_v)</td>
<td>J/K, J/K/kg</td>
<td>(\text{Heat capacity, Specific heat capacity constant pressure, Heat capacity at volume constant})</td>
<td></td>
</tr>
<tr>
<td>f</td>
<td>(f)</td>
<td></td>
<td>(64/Re \quad \text{Re} &lt; 2000) (0.316 \times \text{Re}^{0.25}) (4000 &lt; \text{Re} &lt; 3 \times 10^4) (0.184 \times \text{Re}^{0.2}) (3 \times 10^4 &lt; \text{Re} &lt; 10^6)</td>
<td>(\text{Darcy-Weisbach friction factor})</td>
</tr>
<tr>
<td>h</td>
<td>(h)</td>
<td>J/kg</td>
<td>(c_p \Delta T)</td>
<td>(\text{Specific enthalpy})</td>
</tr>
<tr>
<td>k</td>
<td>(k)</td>
<td>Wm(^{-1})K(^{-1})</td>
<td></td>
<td>(\text{Thermal conductivity})</td>
</tr>
<tr>
<td>(m)</td>
<td>(\text{dotm})</td>
<td>kg m(^{-2}) (\text{s}^{-1})</td>
<td>(\rho u)</td>
<td>(\text{Mass flux})</td>
</tr>
<tr>
<td>p</td>
<td>(p)</td>
<td>Pa (N/m(^2))</td>
<td></td>
<td>(\text{Pressure})</td>
</tr>
<tr>
<td>(q'')</td>
<td>(q_{ii})</td>
<td>Wm(^{-2})</td>
<td>((k_y)p/A)</td>
<td>(\text{Heat flux})</td>
</tr>
<tr>
<td>(q'')</td>
<td>(q_{iii})</td>
<td>Wm(^{-3})</td>
<td>((k_y)p/v)</td>
<td>(\text{Volumetric power, related to the fraction in the fuel})</td>
</tr>
<tr>
<td>(q'')</td>
<td>(q_{iiig})</td>
<td>Wm(^{-3})</td>
<td>((1-k_y)p/v)</td>
<td>(\text{Volumetric power, related to the gamma rays fraction})</td>
</tr>
<tr>
<td>r</td>
<td>(r)</td>
<td>m (r)</td>
<td></td>
<td>(\text{Radius})</td>
</tr>
<tr>
<td>s</td>
<td>(s)</td>
<td></td>
<td>(s = \sigma + j\omega)</td>
<td>(\text{Complex variable of the s-plane } s \in \mathbb{C})</td>
</tr>
<tr>
<td>x</td>
<td>(x)</td>
<td></td>
<td></td>
<td>(\text{Quality})</td>
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<tr>
<td>A</td>
<td>(A)</td>
<td>m(^2)</td>
<td></td>
<td>(\text{Area})</td>
</tr>
<tr>
<td>(A_c)</td>
<td>(A_c)</td>
<td>m(^2)</td>
<td>(p_{ch}^2 - \pi R_{co}^2)</td>
<td>(\text{Cross section flow area})</td>
</tr>
<tr>
<td>Bi</td>
<td>(Bi)</td>
<td></td>
<td>(R_{\text{cond}} / R_{\text{conv}} = \text{Lh} / k)</td>
<td>(\text{Biot number: relates conduction and convection resistances.})</td>
</tr>
</tbody>
</table>

Bi<<1 \(\text{small } \Delta T \text{ in solid}\)  
Bi>>1 \(\text{large } \Delta T \text{ in solid}\)
<table>
<thead>
<tr>
<th>Bo</th>
<th>$\frac{q^*}{mh_{th}}$</th>
<th>Boiling number: relates heat input to evaporation heat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_m$</td>
<td>#/cm$^3$</td>
<td>Delayed neutron precursor concentration of type m</td>
</tr>
<tr>
<td>$D_h$</td>
<td>$D_h$</td>
<td>m</td>
</tr>
<tr>
<td>$D_{co}$</td>
<td>$D_{co}$</td>
<td>m</td>
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<tr>
<td>$D^#$</td>
<td></td>
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</tr>
<tr>
<td>$F_o$</td>
<td>$F_o$</td>
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</tr>
<tr>
<td>$H_b$</td>
<td>$H_b$</td>
<td>Wm$^{-2}$K$^{-1}$</td>
</tr>
<tr>
<td>$H_g$</td>
<td>$H_g$</td>
<td>Wm$^{-2}$K$^{-1}$</td>
</tr>
<tr>
<td>$M$</td>
<td>$M$</td>
<td>kg</td>
</tr>
<tr>
<td>$\dot{M}$</td>
<td>$\dot{M}$</td>
<td>kg s$^{-1}$</td>
</tr>
<tr>
<td>Nu</td>
<td>Nu</td>
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</tr>
<tr>
<td>$P_w$</td>
<td>$P_w$</td>
<td>M</td>
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<tr>
<td>$P_{ch}$</td>
<td>$P_{ch}$</td>
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<td>$P$</td>
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<td>W m$^{-3}$</td>
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<td>$Q^g$</td>
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<tr>
<td>Symbol</td>
<td>Description</td>
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<tr>
<td>R, r</td>
<td>Radius</td>
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<tr>
<td>m</td>
<td>s</td>
<td>constant</td>
</tr>
<tr>
<td>Reynolds number: characterizes motion of fluid bulk</td>
<td></td>
<td></td>
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<tr>
<td>Suppression factor</td>
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<td></td>
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<tr>
<td>Nu = ( \frac{q^*}{mc_\ell \Delta T} )</td>
<td>Stanton number: relates heat input to heat transport.</td>
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</tr>
<tr>
<td>Temperature</td>
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<td>Volume</td>
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<td>Xtt</td>
<td>Martinelli parameter</td>
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<td>m/s</td>
<td>Thermal diffusivity</td>
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<tr>
<td>Void fraction</td>
<td></td>
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<tr>
<td>Delayed neutron precursor fraction of type m</td>
<td></td>
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<tr>
<td>Fraction of the neutrons produced by decay precursor</td>
<td></td>
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<tr>
<td>Fraction of the neutrons produced directly by fission</td>
<td></td>
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<tr>
<td>Infinitesimal value</td>
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<tr>
<td>Phase difference</td>
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<tr>
<td>Neutron flux in group g</td>
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<tr>
<td>Decay constant of delayed neutron precursor of type m</td>
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<tr>
<td>Average number of neutrons produced in a fission induced by group g'</td>
<td></td>
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<tr>
<td>Density</td>
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<td>Fuel</td>
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<td>Vapor</td>
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<td>Saturation conditions</td>
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<td>Symbol</td>
<td>Description</td>
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<tr>
<td>( \nu )</td>
<td>( \mathrm{cm/s} )</td>
<td>Average speed of neutrons in group ( g )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( \omega )</td>
<td>( s^{-1} )</td>
</tr>
<tr>
<td>( \psi )</td>
<td>\psi</td>
<td>Lambda modes of static neutron balance</td>
</tr>
<tr>
<td>( \Phi )</td>
<td>\Phi</td>
<td>Legendre coefficients</td>
</tr>
<tr>
<td>( \Sigma_a )</td>
<td>\Sigma_a</td>
<td>cm(^{-1})</td>
</tr>
<tr>
<td>( \Sigma_f )</td>
<td>\Sigma_f</td>
<td>cm(^{-1})</td>
</tr>
<tr>
<td>( \Sigma_s )</td>
<td>\Sigma_s</td>
<td>cm(^{-1})</td>
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<td>( \langle \cdot \rangle )</td>
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</tbody>
</table>
A2. Drift flux fit parameters

The functional form of the fitting parameters ($C_0$ and $V_{gj}$) and their first derivatives should be continuous and should restrict void fraction to the range between zero and one. For physical systems, phase velocities and void fraction should be continuous except at an interface between a two-phase mixture and pure vapor or pure liquid or at geometric discontinuities. The behavior of $C_0$ and $V_{gj}$ is important as the void fraction approaches zero or one, and as the pressure approaches the critical pressure.

- **Critical pressure limit:** above the critical pressure the phases can not be distinguished, therefore, flow must be homogeneous and phase velocities equal.

\[
\lim_{p \to p_{cr}} V_{gj} = 0 \quad \text{Eq. 1}
\]

\[
\lim_{p \to p_{cr}} C_0 = 1 \quad \text{Eq. 2}
\]

\[u_g = u_l \quad \text{for} \quad p \geq p_{cr}\]

- **Zero pressure limit:** Since the specific volume goes to infinity, when the pressure tends to zero, any voidage causes the void fraction go to one and the mean vapor velocity to become infinite.

\[
\lim_{p \to 0} \frac{1}{\rho_g} \to \infty \quad \text{Eq. 3}
\]

thus,

\[
\lim_{p \to 0} \frac{\rho_g}{\rho_l} = 0 \quad \text{Eq. 4}
\]

in this situation vapor velocity must reduce to:

\[
\lim_{\alpha \to 1} V_g = \lim_{p \to 0} \frac{C_0 \langle j_l \rangle + V_{gj}}{1 - C_0} \to \infty \quad \text{Eq. 5}
\]

This leads $C_0=0$.

- **Limit as the void fraction goes to one:**

\[
\lim_{\alpha \to 1} V_{gj} = 0 \quad \text{Eq. 6}
\]

\[
\lim_{\alpha \to 1} C_0 = 1 \quad \text{Eq. 7}
\]

- **Limit as the void fraction goes to zero:**

\[
\lim_{\alpha \to 0} V_{gj} \geq 0 \quad \text{Eq. 8}
\]

\[
\lim_{\alpha \to 0} C_0 = 0 \quad \text{Eq. 9}
\]
Chexal-Lellouche Void model description (1997, EPRI)

The void fraction model presented by Chexal-Lellouch covers full range of pressure, flows and void fractions. It has been tested against several sets of experimental data covering a wide range of initial conditions and geometries. It also has been used in several state of the art codes like RAMONA 5, RELAP 5, and RETRAN-3D...

The fitting parameters, $C_0$ and $V_{g_i}$, for a cocurrent-upward flow are also functions of void fraction. Thus void fraction has to be calculated iteratively.

**Distribution parameter:**

\[
C_0 = \frac{L}{K_0 + (1 - K_0)(\alpha)^r} \quad \text{Eq. 10}
\]

\[
L = \frac{1 - EXP(-C_p(\alpha))}{1 - EXP(-C_p)} \quad \text{Eq. 11}
\]

\[
C_p = \frac{4p_{crit}^2}{p(p_{crit} - p)} \quad \text{Eq. 12}
\]

\[
K_0 = B_1 + (1 - B_1) \sqrt[4]{\frac{\rho_g}{\rho_l}} \quad \text{Eq. 13}
\]

\[
B_1 = \min(0.8, A_1) \quad \text{Eq. 14}
\]

\[
A_1 = \frac{1}{1 + EXP\left(-\frac{Re_{max}}{60000}\right)} \quad \text{Eq. 15}
\]

\[
Re_{max} = \max\left(Re_l, Re_g\right) \quad \text{Eq. 16}
\]

\[
1 + 1.57 \frac{\rho_g}{\rho_l} \quad \text{Eq. 17}
\]
Drift velocity:

\[ V_{sy} = 1.414 \sqrt{\frac{(\rho_i - \rho_g)g^2\sigma}{\rho_i^2}} C_1 C_2 C_3 C_4 \]  \hspace{1cm} \text{Eq. 18}

Where \( g \) is acceleration of gravity and \( C_1, C_2, C_3, C_4 \), are given

\[ C_1 = (1 - \langle \alpha \rangle)^{\rho_i} \]  \hspace{1cm} \text{Eq. 19}

\[ C_2 = 1 - \text{EXP} \left( -\frac{C_5}{1 - C_5} \right) \hspace{1cm} \text{if} \hspace{1cm} C_5 < 1 \text{ and } (\rho_i / \rho_g) \geq 18 \]  \hspace{1cm} \text{Eq. 20}

\[ C_2 = 1 \hspace{1cm} \text{if} \hspace{1cm} C_5 \geq 1 \text{ and } (\rho_i / \rho_g) \geq 18 \]  \hspace{1cm} \text{Eq. 21}

\[ C_2 = 0.4757 \left( \frac{\rho_i}{\rho_g} \right)^{0.7} \hspace{1cm} \text{if} \hspace{1cm} (\rho_i / \rho_g) < 18 \]  \hspace{1cm} \text{Eq. 22}

\[ C_3 = \max \left\{ 0.5, \text{EXP} \left( \frac{\text{Re}_l}{300000} \right) \right\} \]  \hspace{1cm} \text{Eq. 23}

\[ C_4 = 1 \hspace{1cm} \text{if} \hspace{1cm} C_7 \geq 1 \]  \hspace{1cm} \text{Eq. 24}

\[ C_4 = \frac{1}{1 - \text{EXP}(-C_8)} \hspace{1cm} \text{if} \hspace{1cm} C_7 < 1 \]  \hspace{1cm} \text{Eq. 25}

\[ C_5 = \sqrt{\frac{150}{\rho_g \rho_i}} \]  \hspace{1cm} \text{Eq. 26}

\[ C_7 = \left( \frac{D_2}{D_h} \right)^{0.6} \]  \hspace{1cm} \text{Eq. 27}

\[ C_8 = \frac{C_7}{1 - C_7} \]  \hspace{1cm} \text{Eq. 28}

\[ D_2 = 0.09144 \]  \hspace{1cm} \text{Eq. 29}
**Dix Correlation (1971)**

The Dix model was derived for the analysis of BWR under operating conditions; see in Coddington and Macian (2002).

**Distribution parameter:**

\[
C_0 = \frac{j_g}{j} \left( 1 + \left( \frac{j}{j_g} - 1 \right) \left( \frac{\rho_g}{\rho_l} \right)^{0.1} \right)
\]

Eq. 30

**Drift velocity:**

\[
V_{gw} = 2.94 \sqrt{\frac{(\rho_l - \rho_g) g \sigma}{\rho_g^2}}
\]

Eq. 31

**References:**

Chexal-Lelluche (1997), Void Fraction technology for design and analysis, EPRI SIEMENS, TR106326.

CURRICULUM VITAE

PERSONAL DATA

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EDUCATION

Doctorate in Technical Sciences
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Swiss Federal Institute of Technology Zürich, Switzerland
April 2008

2001 M.Sc. in Nuclear Engineering
University of Rome “La Sapienza”, Italy

PROFESSIONAL EXPERIENCE

Since June 2006 Colenco Power Engineering:

- Participated in various plant design review projects, developed different reactor models for the study of reactor core parameters (such as power peaking factors, critical boron concentration, burnup, Xenon equilibrium, reactivities coefficients and shutdown margins) at various reactor states, using diffusion and Monte-Carlo methods.
- Since December 2006 Colenco radiation protection responsible (Strahlenschutz Verantwortlicher, licensed by Swiss and German Authorities).
- Since September 2006 responsible for scientific software selection and acquisition.

2001-2006 Doctoral thesis: Developed a frequency domain code for the Boiling Water Reactor stability analysis and design. The code includes: 3D
Neutron kinetics, analytical solution of the fuel rod dynamics, non-thermal equilibrium (subcooled boiling) non-homogeneous two phase flow (drift-flux) model, Flashing effects, dynamics of the boiling boundary, Dynamic distribution of the flow among parallel bundles.
Prof. G. Yadigaroglu, ETH Zürich

November -December 2003  Collaborated on the development of a 3D code for the core static calculations. Polytechnic University of Valencia, Spain

February -March 2002  Worked on the core simulator (POLCA) data and studied interface to the reactor. Forsmark nuclear power plant, Sweden.

June-September 2000  Masters Project part II: “the High performance time series analysis code (HPTSAC2000)”. Studied reactor on line monitoring system and developed HPTSAC2000 code.
Prof. D’Auria University of Pisa (Italy); Dr. Dieter Hennig Paul Scherrer Institut, PSI

October 1999 – April 2000  Masters Project part I: “A flashing simulator for natural circulation heated system (BWR)”. Prof. D’Auria University of Pisa (Italy), Prof. Rizwan Udin, University of Illinois (USA).

September 1999  Studied stability of boiling water reactors using system code RELAP; Prof. D’Auria, University of Pisa (Italy)

---

SPECIALIZATION/EXPERTISE

Nuclear Field:

- Stability of BWRs and performance (forced and natural circulation)
- Reactor physics and core design
- Reactor thermal hydraulics
- Fuel dynamics

System modelling:

- Two-phase flow
- Heat transfer and thermodynamics
- Stability and control
- Advanced signal analysis in time and frequency domain, linear and non-linear auto-regressive models
- Algebra and numerical methods of very large system (dense and sparse matrices). Used in most of engineering fields like, Nuclear, Mechanical, Electrical, and Civil Engineering.
- Programming and simulation (2D, 3D advanced presentations) in MATLAB environment.