Doctoral Thesis

Prime tensor ideals in some triangulated categories of C*-algebras

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Prime tensor ideals
in some triangulated categories
of C*-algebras

a dissertation submitted to
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Abstract

The purpose of this thesis is to inaugurate the application of algebraico-geometric ideas, in the form of Paul Balmer’s tensor triangular geometry [Bal05], to the study of such tensor triangulated categories as arise in relation to the topological K-theory of operator algebras. More precisely, we are interested in Balmer’s spectrum Spc of the equivariant Kasparov category $\mathcal{KK}^G$ associated to a (second countable Hausdorff) locally compact group $G$.

Using work of Ralf Meyer and Ryszard Nest [MN06], we show (Theorem 4.3.7) that if the space Spc($\mathcal{KK}^G$) is covered by the union $\bigcup_H$ Spc($\mathcal{KK}^H$), with $H$ varying among the compact subgroups of $G$, then a strong form of the Baum-Connes conjecture (which is known to hold for all groups with the Haagerup property) holds for $G$.

As a first step towards understanding the spectrum of these tensor triangulated categories, we turn our attention to $\mathcal{T}^G := \langle 1 1 \rangle_{loc}$, the localizing subcategory of $\mathcal{KK}^G$ generated by the tensor unit $1 1 \in \mathcal{T}^G$. If $G = \{1\}$ is the trivial group, then $\mathcal{T}^G$ is better known as the Bootstrap category $\text{Boot} \subseteq \mathcal{KK}$. We prove (Theorem 5.1.11) that the spectrum of its subcategory $\text{Boot}_c$ of compact objects (i.e., separable C*-algebras in the Bootstrap class having finitely generated K-theory) is isomorphic to the Zariski spectrum of the integers: Spc($\text{Boot}_c$) $\cong$ Spec($\mathbb{Z}$).

We conjecture the analog result at least for finite groups, namely, that there is a homeomorphism Spc$(\mathcal{T}_c^G) \cong \text{Spec}(R(G))$, where $R(G) = \text{End}_{\mathcal{T}_c^G}(1)$ is the complex representation ring of the finite group $G$. In order to tackle this conjecture we first provide, speaking quite roughly, an abstract criterion (Theorem 2.5.2) for a ‘continuous generalized support datum’ $(X, \sigma)$ of a certain nice form defined on a compactly generated tensor triangulated category $\mathcal{T}$, to be classifying for the subcategory $\mathcal{T}_c$ of compact objects, i.e. – equivalently – to provide a homeomorphism Spc$(\mathcal{T}_c) \cong X$.

In Section 5.4 we construct such a support $(\text{Spec}(R(G)), \sigma)$ for $\mathcal{T}^G$ when $G$ is finite, but we fail – for the moment – to show that it enjoys all the properties required by our criterion. We can use it though to show that the natural map $\rho : \text{Spec}(\mathcal{T}_c^G) \to \text{Spec}(R(G))$ is a retraction. Along the way to constructing $\sigma$ we define, for $G$ a compact Lie group and $p \in \text{Spec}(R(G))$, a ‘$p$-local’ subcategory $\mathcal{T}_p^G \subseteq \mathcal{T}^G$, and we show that it enjoys several nice properties (see Theorem 5.2.1). Moreover, we use relative homological algebra [MN07] [Mey08] to prove (a new version of) N. Christopher Phillips’s [Phi85] Künneth Theorem for certain important localized K-theory groups (see Theorem 5.3.11).
Riassunto

Scopo di questa tesi è l’introduzione di ragionamenti algebraico-geometrici, in quanto catturati dalla geometria triangolare tensoriale di Paul Balmer \cite{Bal05}, nello studio delle categorie triangolate tensoriali che sorgono in relazione alla $K$-teoria topologica delle algebre d’operatori. Più precisamente, ci interessiamo allo spettro di Balmer della categoria equivariante di Kasparov $KK^G$ associata ad un gruppo localmente compatto $G$.

Grazie a risultati di Ralf Meyer e Ryszard Nest \cite{MN06}, ci è possibile dimostrare (Teorema 4.3.7) che se lo spazio $\operatorname{Spc}(KK^G)$ è ricoperto dall’unione degli spazi $\operatorname{Spc}(KK^H)$, dove $H$ percorre i sottogruppi compatti di $G$, allora $G$ soddisfa una versione forte della congettura di Baum-Connes (che comunque è valida almeno per tutti i gruppi con la proprietà di Haagerup).

Come primo passo verso una comprensione della spettro di tali categorie, ci concentriamo su $T^G := \langle 1 1 \rangle_{\text{loc}}$, la sottocategoria localizzante di $KK^G$ generata dall’unità tensoriale $1 1 \in T^G$. Nel caso del gruppo banale $G = \{1\}$, la categoria $T^G$ è meglio conosciuta sotto il nome di classe Bootstrap\(^1\) $\text{Boot} \subseteq KK$. Dimostriamo quanto segue (Teorema 5.1.11): lo spettro di Balmer della sua sottocategoria $\text{Boot}^c$ degli oggetti compatti (cioè le $C^*$-algebre nella classe Bootstrap aventi gruppi di $K$-teoria di tipo finito) è isomorfo allo spettro di Zariski dell’anello degli interi: $\operatorname{Spc}(\text{Boot}^c) \cong \operatorname{Spec}(\mathbb{Z})$.

Ci sono buone ragioni per ritenere che l’analogo risultato sia valido almeno per i gruppi finiti. Congetturiamo quindi che esista un omeomorfismo $\operatorname{Spc}(T^G_c) \cong \operatorname{Spec}(R(G))$, dove $R(G) = \operatorname{End}_{T^G}(1 1)$ denota l’anello delle rappresentazioni complesse del gruppo finito $G$. Per affrontare questa congettura, incominciamo col dare – in parole approssimative – un criterio astratto (Teorema 2.5.2) per decidere se un ‘dato di supporto generalizzato e continuo’ $(X, \sigma)$ di una certa forma, definito su una categoria triangolata a generazione compatta $T$, si restringa a un dato di supporto classificante per la sottocategoria $T^c$ degli oggetti compatti; vale a dire, un criterio perché $(X, \sigma)$ induca un omeomorfismo $\operatorname{Spc}(T^c) \cong X$.

Nella Sezione 5.4 costruiamo un tale supporto $(\operatorname{Spec}(R(G), \sigma)$ per $T^G$ con $G$ un gruppo finito, senza riuscire però (per il momento) a dimostrare che goda di tutte le proprietà richieste dal criterio summenzionato. Si può comunque mostrare che la funzione naturale $\rho : \operatorname{Spec}(T^G_c) \rightarrow \operatorname{Spec}(R(G))$ è una retrazione. Per costruire $\sigma$ definiamo, se $G$ è un gruppo di Lie compatto e $p$ un ideale primo di $R(G)$, una “localizzazione a $p$” $T^G_p \subseteq T^G$ con numerose proprietà desiderabili (si veda il Teorema 5.2.1). Inoltre utilizziamo l’algebra omologica relativa di \cite{MN07} e \cite{Mey08} per dimostrare una nuova versione del teorema di Künneth, dovuto a N. Christopher Phillips \cite{Phi85}, riguardante certi gruppi di $K$-teoria localizzata (si veda il Teorema 5.3.11).

\(^1\) mi rifiuto categoricamente di tradurre “classe Calzastivale"
Contents

Abstract iii
Riassunto iv

Introduction vii
Reader’s guide x
Notation and conventions xi
Acknowledgements xii

Chapter 1. Preliminaries on triangulated categories 1
  1.1. Definition and first properties 1
    1.1.1. Triangle functors 5
    1.1.2. Coproducts and homotopy colimits 7
  1.2. Localization and calculus of fractions 9
  1.3. Verdier localization 13
    1.3.1. Localizing subcategories 16
  1.4. Bousfield localization 17
    1.4.1. Abstract complementation 17
    1.4.2. Give me a right adjoint... 19
    1.4.3. Complementary pairs 22
    1.4.4. A more classical terminology 23
  1.5. Coherent functors 25
    1.5.1. The universal homological functor 28
  1.6. The Brown representability theorem 30
    1.6.1. Brown representability for the dual 37
  1.7. Compact objects 37
    1.7.1. An adjoint functor theorem 41
    1.7.2. The Neeman localization theorem 42
    1.7.3. A Neeman localization is a nice Bousfield localization 45

Chapter 2. Some tensor triangular geometry 47
  2.1. The axiom(s) and basic properties 47
    2.1.1. The central ring $R_T$ 48
    2.1.2. The graded central ring $R_T^*$ 51
    2.1.3. Tensor triangle functors and tensor ideals 53
    2.1.4. $\alpha$-Compactly generated tensor triangulated categories 54
  2.2. The spectrum of a tensor triangulated category 56
    2.2.1. Local rings and the structure sheaf 60
    2.2.2. The natural comparison $\rho : \text{Spec}(T) \rightarrow \text{Spec}(R_T)$ 61
  2.3. Central localization 63
    2.3.1. Central localization of $\alpha$-compactly generated categories 66
  2.4. Strongly dualizable objects 70
    2.4.1. The $\alpha$-relative case 73
  2.5. Classification in the noetherian case 74
    2.5.1. Preliminaries on supports 75
2.5.2. The proof of Theorem 2.5.2 77

Chapter 3. Relative homological algebra in triangulated categories 83
3.1. Homological functors and ideals 83
3.2. Projective objects and resolutions 85
3.2.1. Enough projective objects 88
3.3. Operations on homological ideals 88
3.4. Relative derived functors 89
3.5. The universal $\mathcal{I}$-exact functor 90
3.6. The ABC spectral sequence 93

Chapter 4. Equivariant Kasparov theory 97
4.1. The representation ring of a compact Lie group 97
4.1.1. Finite cyclic groups 100
4.2. $G$-$C^*$-algebras and Kasparov theory 100
4.2.1. From $R(G)$ to $\text{KK}^G$ 100
4.2.2. The equivariant Kasparov category $\text{KK}^G$ 102
4.2.3. The rich functoriality of $\text{KK}^G$ 104
4.2.4. The tensor triangulated category $\mathcal{T}^G$ 105
4.3. The Baum-Connes Conjecture and the spectrum 106

Chapter 5. Towards a computation of the spectrum of $\mathcal{T}^G_{\mathcal{C}} = \langle \tau^G \mathcal{C} \rangle$ 109
5.1. A trivial example 109
5.1.1. The direct approach 110
5.1.2. The overkill approach 113
5.2. Localization of equivariant $K$-theory 114
5.3. A $G$-equivariant Künneth spectral sequence 115
5.3.1. Phillips’ Künneth Formula 117
5.3.2. The ‘residue field object’ of a prime ideal 118
5.4. The generalized support datum $\sigma$ 119
5.4.1. Surjectivity of $\rho : \text{Spc} \mathcal{T}^G_{\mathcal{C}} \to \text{Spec} R(G)$ 120
5.5. Final remarks and conjectures 122

Appendix A. The Spanier-Whitehead category 125
A.1. Preliminaries on spaces 125
A.2. Admissible categories of algebras 126
A.2.1. Examples 131
A.3. Homotopy of algebras 132
A.4. Inverting the suspension 134
A.5. The triangulation 137
A.6. Localizations of Spanier-Whitehead 146
A.6.1. Equivariant $KK$-theory 147

Bibliography 149
Introduction

During the last thirty years, triangulated categories have spread to many different niches of the mathematical landscape. They are now endemic in algebraic geometry, algebraic topology, algebraic analysis, the representation theory of algebras, the modular representation theory of groups, motivic theory, the theory of operator algebras, etc. In many species, the triangulated category is naturally equipped with a tensor product (i.e., a symmetric monoidal structure). To any such tensor triangulated category one can assign a nice topological space:

**Definition** (P. Balmer [Bal05], see Section 2.2 below). Let $T = (T, \otimes, 1)$ be an essentially small tensor triangulated category. Define its spectrum $\text{Spc}(T)$ to be the set of its thick prime $\otimes$-ideals, i.e., of those proper thick subcategories $\mathcal{P} \subseteq T$ such that $A \otimes B \in T \iff A \in T$ or $B \in T$. It has a natural Zariski topology.

This deceptively simple definition turns out to be quite powerful. To start with, for every object $A \in T$ there is a closed subset $\text{supp}(A)$ of $\text{Spc}(T)$, its support, such that the assignment $\text{supp}$ from objects of $T$ to subsets of the spectrum satisfies a list of natural properties, captured by the concept of support datum (Def. 2.2.7).

This pair $(\text{Spc}(T), \sigma)$ can be characterized as the universal (‘finest’) support datum on $T$. Using $\text{Spc}(T)$, it now becomes possible to apply geometric reasoning to the category $T$: one can use supports to detect vanishing or decomposition properties of objects [Bal05] [Bal07], to construct local-to-global spectral sequences [Bal07] [Bal07b], to glue objects and morphisms defined locally [BF07] [BBC08], and so on. This is tensor triangular geometry. The name ‘geometry’ is even more appropriate if one considers the fact that $\text{Spc}(T)$ is naturally equipped with a sheaf of rings $\mathcal{O}_T$ which makes $\text{Spec}(T) := (\text{Spc}(T), \mathcal{O}_T)$ a locally ringed space.

Another prominent feature of Balmer’s spectrum is that it captures much of the triangular structure of $T$. More precisely, one may classify the radical thick $\otimes$-ideals of $T$ by means of certain subsets of $\text{Spc}(T)$ (note that in many examples all thick $\otimes$-ideals are radical, and that in some cases all thick subcategories are $\otimes$-ideals):

**Theorem** ([Bal05], upgraded in [BKS07]; see Thm. 2.2.11 and Thm. 2.2.15). There is a natural bijection between the set of radical thick $\otimes$-ideals of $T$ and the set of subsets $Y \subseteq \text{Spc}(T)$ such that $Y$ is a union of Zariski closed subsets, each of which has quasi-compact open complement. Conversely, if $(X, \sigma)$ is a support datum on $T$ providing such a classification and where the space $X$ is spectral (in the sense of Hochster [Hoc69], then there is an isomorphism $(X, \sigma) \cong (\text{Spc}(T), \text{supp})$ of support data; in particular, there is a homeomorphism $X \cong \text{Spc}(T)$.

Mining the literature for existing classifications, one can explicitly compute Balmer’s spectrum for various categories. The first example is provided by Thomason’s classification [Ths97] of thick subcategories in algebraic geometry, based on previous work by Hopkins and Neeman.

1. **Example** ([Bal02], cf. [Pet06] for details). Let $X$ be a quasi-compact quasi-separated scheme, and let $D^{\text{perf}}(X)$ be the derived category of perfect complexes
on $X$, equipped with the left derived tensor product; the $\otimes$-unit is $O_X$. Then there is a natural isomorphism of locally ringed spaces $\text{Spec}(\mathcal{D}^{perf}(X)) \cong X$. For an affine scheme $X = \text{Spec}(R)$, this specializes to $\text{Spec}(K^0(R_{\text{proj}})) \cong \text{Spec}(R)$.

In other words, tensor triangular geometry contains algebraic geometry. But there is more.

2. Example ([Bal05], [Pet06]). Let $G$ be a finite group and let $k$ be a field. Let $\text{stmod}(kG)$ be the stable category of finite $kG$-modules, equipped with the tensor product $\otimes_k$ and the tensor unit $k$. Then $\text{Spec}(\text{stmod}(kG)) \cong \text{Proj}(H^*(G, k))$ by the classification in [BCR97] (see also Ex. 2.5.6 below). In this case, tensor triangular geometry recovers the theory of support varieties in modular representation theory.

The spectrum of the ur-triangulated category is also known.

3. Example. Let $\mathcal{F}$ be the stable homotopy category of finite pointed CW-complexes, with the tensor structure given by the smash product. Using the famous classification of the thick subcategories of $\mathcal{F}$ due to Devinatz-Hopkins-Smith ([HS98], but see also [Rav92] [Che06] [Bal08]), one can describe the space $\text{Spec}(\mathcal{F})$ as follows. It has one dense point $\mathcal{F}_{\text{tors}}$, which is the subcategory of torsion spectra. For each prime number $p$, it has a maximal (with respect to reverse inclusion $\equiv$ specialization order) point $\mathcal{P}_{p,\infty}$, which consists of all torsion spectra without $p$-torsion. Between the generic point and $\mathcal{P}_{p,\infty}$ there lies a countable infinite ("chromatic") tower of points $\mathcal{P}_{p,n}$ ($n \geq 1$), where $\mathcal{P}_{p,n}$ is the subcategory of those spectra $X$ whose $p$-localization has vanishing $n$th Morava $K$-theory: $K^{p,n}_*(X_{(p)}) = 0$. Here is a picture, where a point lies above another iff it is contained in its topological closure:

The topology of this space has precisely the following proper non-empty closed subsets: $V(P, f) := \bigcup_{i \in P} \{ \mathcal{P}_{i,f(i)} \}$ where $P$ is a finite set of prime numbers and $f$ is a function $f : P \to \{1, \ldots, \infty\}$.

Note that, in all the above examples, the tensor triangulated category in question appears as the subcategory $\mathcal{T}_c$ of compact objects inside of an ‘ambient’ compactly generated tensor triangulated category $\mathcal{T}$.

Many other tensor triangulated categories await their turn. This thesis is concerned with the equivariant Kasparov category $KK^G$. Its objects are separable $G$-$C^*$-algebras, where $G$ is a second countable locally compact group; its Hom sets are given by Kasparov's $G$-equivariant bivariant $K$-groups $KK^G(A, B)$. The tensor structure is induced by the minimal tensor product of $C^*$-algebras together with the diagonal $G$-action on its factors. We refer to Section 4.2 for a definition and for the main properties of $KK^G$. This category is intimately connected with topological $K$-theory and with the representation theory of $G$.

Remark. The category $KK^G$ is quite large, in the sense that it contains a great variety of complicated objects, but, because of the separability hypothesis built into
the definition and which is needed to make Kasparov’s theory work properly, it has only arbitrary countable coproducts. The lack of arbitrary coproducts leads to some slight difficulties, and it certainly contributed to uglify parts of this thesis. For the sake of exposition, we will pretend for the rest of the introduction that \( KK^G \) has arbitrary coproducts.

The next theorem is an example of the kind of local-to-global thinking that tensor triangular geometry enables. Note that the inclusion \( H \leq G \) of a closed sub-
group induces a restriction \( \otimes \)-triangle functor \( \text{Res}_G^H : KK^G \to KK^H \). By the functor-
oriality of the spectrum, this induces in its turn a map \( \text{Spec}(\text{Res}_G^H) : \text{Spec}(KK^H) \to \text{Spec}(KK^G) \).

**Theorem (Thm. 4.3.7).** Assume that the space \( \text{Spec}(KK^G) \) is covered by the images of \( \text{Spec}(KK^H) \) under \( \text{Spec}(\text{Res}_G^H) \), where \( H \) ranges among the compact subgroups of \( G \). Then the analog of the Baum-Connes conjecture holds for every functor defined on \( KK^G \) and every coefficient algebra \( A \in KK^G \).

Even without knowing what the conclusion means, one may think that it is far too strong a statement to hold for any group \( G \) at all, but in fact it holds for all amenable groups, and even in greater generality (see Section 4.3). Now in order to become famous, it remains only to find a good description of \( \text{Spec}(KK^G) \).

As a first approximation, I thought it reasonable to begin by considering sub-
categories of \( KK^G \) such as the localizing subcategory generated by a nice set of compact objects. After a short while, it seemed even more reasonable to restrict attention to the localizing subcategory of \( KK^G \) generated by the tensor unit, which I denote by \( T^G := \langle 1 \rangle_{\text{loc}} \subseteq KK^G \). By construction, it is a monogenic compactly gen-
erated tensor triangulated category — like the stable homotopy category of spectra, only in a countable sense (see Section 4.2.4). It should be noted that this category is still very interesting, because, as shown in [MN07] (at least if the group \( G \) is compact), every \( A \in KK^G \) has a natural approximation \( LA \to A \) with \( LA \in T^G \) such that \( K^G(LA) \cong K^G(A) \); in other words, \( T^G \) tells you all you want about the \( G \)-equivariant \( K \)-theory of \( C^* \)-algebras (see Prop. 4.2.17).

Now, to be very reasonable, let’s consider the trivial group \( G = \{1\} \). In this case our category \( T^{(1)} = \langle \mathbb{C} \rangle_{\text{loc}} \subseteq KK^G \) is known as the Bootstrap category (of separable \( C^* \)-algebras), so we write \( \text{Boot} := T^{(1)} \). By using the well-known Universal Coefficient Theorem and the Künmeth Theorem of Rosenberg and Schochet, it is a fairly straightforward matter to describe the Balmer spectrum of its compact objects:

**Theorem (Thm. 5.1.11).** \( \text{Spec}(\text{Boot}) \cong \text{Spec}(\mathbb{Z}) \).

Here the ring of integers \( \mathbb{Z} \) appears as the endomorphism ring \( \text{End}_{\text{Boot}}(\mathbb{C}) = \text{End}_{KK}(\mathbb{C}) \) of the tensor unit \( 1 = \mathbb{C} \). More generally for a compact group \( G \), the endomorphism ring of the \( \otimes \)-unit of \( KK^G \), and therefore of its full subcate-
gory \( T^G \), can be identified with the complex representation ring \( R(G) \) (this is the Grothendieck ring of continuous finite dimensional complex \( G \)-modules). Let \( T^G_{\text{c}} \) be the subcategory of compact objects in \( T^G \).

**Conjecture (cf. Conj. 5.5.1).** Let \( G \) be a compact Lie group. Then \( \text{Spec}(T^G_{\text{c}}) \cong \text{Spec}(R(G)) \).

I limit my conjecture to Lie groups because, in this case, it is known that the ring \( R(G) \) is noetherian (even a finitely generated ring), and because the methods I intend to use need this hypothesis. Chapter 5 contains various preliminary results towards a proof of the conjecture in the case of a finite group \( G \) (this is summarized
in Section 5.5). My main tool is the following result, which is also of some independent interest. If \( T \) is a triangulated category with coproducts, call continuous generalized support datum a pair \( (X, \sigma) \) consisting of a topological space \( X \) together with an assignment \( \sigma : \text{obj}(T) \to 2^X \) from the objects of \( T \) to subsets of \( X \), satisfying the axioms of a support datum (except that it need not yield closed subsets of \( X \)) and also satisfying the following continuity condition:

\[
\sigma(\biguplus_i A_i) = \bigcup_i \sigma(A_i).
\]

**Theorem (Thm. 2.5.2).** Let \( T \) be a compactly generated tensor triangulated category. Let \( (X, \sigma) \) be a continuous generalized support datum on \( T \) with the following special form: \( X \) is a Hochster spectral subspace of \( \text{Spec}^h(R) \), the spectrum of homogeneous prime ideals of \( R \), where \( R \) is a noetherian graded subring of the graded endomorphism ring \( \text{End}_T^*(1) \); assume also that \( \sigma(A) = \emptyset \) implies \( A \cong 0 \) for all \( A \in T \). If moreover \( \sigma \) detects objects, that is, if \( \sigma(A) = \emptyset \) implies \( A \cong 0 \) for all \( A \in T \), then \( (X, \sigma) \) is classifying on the subcategory \( T_c \) of compact objects. In particular, there is a canonical homeomorphism \( \text{Spc}(T_c) \cong X \).

Note that the statement in Theorem 2.5.2 is more general, but also more complicated. Its proof is generalized from the classification of thick \( \otimes \)-ideals of [BCR97] (Example 2). The above conjecture was motivated by the computation of \( \text{Spc}(\text{Boot}_c) \) and the picture in Example 3.

**Reader’s guide**

**Chapter 1** presents the basic theory of triangulated categories (except notably for \( t \)-structures), which is the language used throughout the text. We treat Verdier localization, Bousfield localization, Brown representability, and compactly generated categories (relatively to a cardinal number \( \alpha \)). The reader should keep in mind that the triangulated categories we intend to study in later chapters have only countable coproducts. Together with the Appendix, the first chapter is the only truly self-contained part of this thesis.

**Chapter 2** collects some concepts and results of tensor triangular geometry. In particular, the spectrum \( \text{Spc}(T) \) of a tensor triangulated category \( T \) is defined. We treat also, giving full proofs, some as yet unpublished results of Paul Balmer, such as the properties of the (graded) central ring \( R_T \) and the existence of a natural comparison map \( \rho_T : \text{Spc}(T) \to \text{Spec}(R_T) \), as well as the technique of central localization. We only claim some originality in our adaptation of central localization to \( \alpha \)-compactly generated categories (Theorem 2.3.22), and in our criterion for a support datum of a certain form to be classifying (Theorem 2.5.2).

**Chapter 3** treats the theory of relative homological algebra in a triangulated category \( T \), based on the concept of a homological ideal of \( T \). The goal is to enable the reader to understand our use in Chapter 5 of Ralf Meyer’s ABC spectral sequence.

**Chapter 4** collects all we need of topological \( K \)-theory of \( C^* \)-algebras. We discuss the complex representation ring \( R(G) \) of a compact group, and then rush to define the equivariant Kasparov category \( KK^G \), where \( G \) is a second countable locally compact group. We sum up in Theorem 4.2.9 the structural properties of \( KK^G \) that we need. In Subsection 4.2.4, we also define \( T^G \) and note that it is a monogenic \( \aleph_1 \)-compactly generated tensor triangulated category. In the last

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2Assume here that the \( \otimes \)-unit is compact, that the \( \otimes \)-product commutes with coproducts, and that all compact objects are strongly dualizable.
section we briefly present Meyer and Nest’s triangular formulation of the Baum-Connes conjecture with coefficients in a $G$-$C^*$-algebra, and prove our result about covering $\text{Spc}(KK^G)$ (Theorem 4.3.7).

Finally, Chapter 5 is where most of our original contributions are to be found. We begin by giving two proofs that $\text{Spc}(T^G_c) \cong \text{Spec}(\mathbb{Z})$ when $G$ is the trivial group: one using the classical Universal Coefficient and Künneth Theorems of Rosenberg and Schochet together with an explicit computation, and a second one by feeding the classical theorems to the abstract criterion of Theorem 2.5.2. The rest of the chapter is an attempt at using the same criterion for proving that $\text{Spc}(T^G_c) \cong \text{Spec}(R(G))$ for $G$ finite (cf. Conjecture 5.5.1). A few potentially interesting results litter the battlefield.

There is also Appendix A, where we give a little axiomatic framework for producing a triangulated category à la Spanier-Whitehead out of a category of ‘algebras’ with a given action of compact spaces. By localizing this category one may produce a vast array of (tensor) triangulated categories. Starting with the category of $G$-$C^*$-algebras, we construct the tensor triangulated category $KK^G$ in this way. While this particular presentation may be somewhat original, all the ideas used (as well as the homotopies) are traditional and can be found, in one form or another, all over the literature.

The interdependency relation of the chapters is cleverly easy: it is the linear ordering induced by the chapter numbers (with the exception that Chapter 3 makes no reference to Chapter 2). Note however that we tried to facilitate an extemporaneous reading of Chapter 5. The Appendix may be read independently of the rest and fills in some details required in Chapters 2 and 4.

**Notation and conventions**

If forced under threat, we would choose to work in a set theory à la von Neumann-Gödel-Bernays, where the basic objects of discourse are *classes*, and where a *set* is just a class which happens to be the element of some other class. For emphasis, we often say *small set* instead of set. A category $\mathcal{C}$ is usually assumed to be locally small, i.e., to have small Hom sets $\mathcal{C}(A,B)$ for all objects $A, B \in \mathcal{C}$. It is large if it is not locally small. A locally small category $\mathcal{C}$ is said to be small if it has a small set of objects, and essentially small if it has a small set of isomorphism classes of objects, i.e., if there exists a small subcategory $\mathcal{C}' \subseteq \mathcal{C}$ equivalent to $\mathcal{C}$. Since a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is determined by the class of its objects (and by $\mathcal{C}$), we will willingly confuse the category $\mathcal{C}'$ with the class $\mathcal{C}'$. We refer to [Mac98] for other categorical terminology, such as limits and colimits, adjoint functors, additive categories.

If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor, we shall denote by $\text{Ker}(F)$ the full subcategory $\{ A \mid F(A) \cong 0 \} \subseteq \mathcal{C}$. It is actually a strictly full subcategory of $\mathcal{C}$, i.e., it is full and if $A \cong B$ in $\mathcal{C}$ and $B \in \text{Ker}(F)$ then also $A \in \text{Ker}(F)$. The symbol $\text{Im}(F)$ will always denote the essential image of $F$, that is all objects in $\mathcal{D}$ isomorphic to some $F(A)$ with $A \in \mathcal{C}$. Thus $\text{Im}(F) \subseteq \mathcal{D}$ is also strictly full. We will use $\text{ker}(F)$, with lower case ‘k’, for the kernel on morphisms: $\text{ker}(F) := \{ f \in \text{Mor}(\mathcal{C}) \mid F(f) = 0 \}$.

**Warning.** The use we make of $\alpha$-compact object, $\alpha$-compactly generated and similar terms (Def. 1.7.1) for a cardinal number $\alpha$, does not agree with other uses found in the literature of triangulated categories.

**Trivia.** The reader will notice that I sometimes refer to myself as “we”, whereas sometimes we use “I”. We just could not agree on which I should use.
I am profoundly grateful to Prof. Paul Balmer for having been my teacher, advisor and friend throughout my thesis. His good humor, open-mindedness and scientific adroitness are for me a continuous source of inspiration. He has always encouraged and helped me generously, yet trusting me with great liberty in pursuing ideas and choosing problems. Because of his mathematical magnanimity, his ideas permeate this work entirely.

It is a pleasure to thank Professors Max-Albert Knus, Henning Krause and Richard Pink for accepting to read my thesis and be co-examiners. Also, Paul Balmer and Theo Bühler have been especially helpful in trying to improve this text; all remaining mistakes and mathematical or linguistic ungracefulnesses are entirely my responsibility. I wish to thank Prof. Ralf Meyer and Prof. Guido Mislin for their kind readiness to answer my naïve questions on various occasions. For interesting discussions and for generally sharing their mathematical enthusiasm, I am happy to thank my Doktoronkel Baptiste Calmès, Boris Chorny, Jean Fasel, Giordano Favi, and Stefan Gille, as well as my Doktorcousins Theo Bühler and Srdjan Micic. Many other people, too numerous to mention, have shared with me at one time or another their ideas and their love of reason. I thankfully include both of my parents in this list. Last but not least, I wish to thank Coralie Ducrest, my dear friend and life companion; the reasons are bountiful and variegated, but two come to the fore: her unfailing and inspiring love of life, and her mysterious attachment to me.
CHAPTER 1

Preliminaries on triangulated categories

So, few are the mathematicians who can still be sure that no triangulated category is floating in their ink-pot. [Bal05]

***

This first chapter contains the theory of triangulated categories, from the axioms up to the classical results of Verdier and Bousfield localization, Brown representability, compactly generated categories and Neeman’s localization theorem. We give full proofs – except for the first elementary facts and some routine verifications, which are left as exercises – because we need a slight generalization of the material, so that it may be applied also to categories which have only countable coproducts, rather that arbitrary (small) coproducts. This caution is justified, because, as expected, not everything generalizes: in the ‘countable’ version of a compactly generated triangulated category, products are not representable in general (Remark 1.6.7) and Brown representability for the dual doesn’t hold (Remark 1.6.28). Even worse, we shall see in Chapter 2 that, if the category has a tensor product, then it is not true anymore that Brown representability provides an everywhere-defined internal Hom functor (Remark 2.4.12). Nonetheless, the essential still holds, as it was first remarked in [MN06].

This chapter could serve as a compact introduction to triangulated categories (more compact then [Nee01] and covering more of the now-standard tools than [Ver96] – our two book-sized references), except that it has no examples. We refer to e.g. [Wei94, §10] [Kel96] [Kra04] [Büh08] for introductions to the derived category, and to [Mar83] for the homotopy category of spectra (see also [Str03] for a survey of important examples).

Another class of examples, originating in harmonic analysis and operator theory (i.e, \(G\)-C*-algebras) will be summarily discussed in Chapter 4.

1.1. Definition and first properties

In 1962, Puppe [Pup62], motivated by the need for a good stable homotopy category, gave an axiomatic treatment of exact triangles. That paper precedes the introduction of the derived category of chain complexes over a ring in Verdier’s 1963 thesis (which was published much later [Ver77]). Verdier’s axioms for exact triangles give the notion of a “triangulated category”. Algebraic topologists and algebraic geometers have developed several areas of differential homological algebra independently, with different details, nomenclature, and, of course, assignment of credit. The definition of triangulated categories is a case in point. [May05]

A whole little theory can be deduced from these axioms. [Kel08]

***
Let $T$ be an additive category equipped with a self-equivalence $T : T \xrightarrow{\sim} T$. A **triangle** in $(T, T)$ is a diagram of the form

\[ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A). \]

We will occasionally write $[f, g, h]$ to indicate a triangle (1.1.1). In diagrams, a distinguished triangle is often pictured as an actual triangle

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
& \searrow{h} & \downarrow{g} \\
& & C
\end{array}
\]

where the ‘$\circ$’ on the arrow $C \rightarrow A$ (or some other decoration) indicates a morphism $C \rightarrow TA$. A **morphism of triangles** is a commutative diagram

\[ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A) \]

which we may abbreviate to $(a, b, c)$.

**1.1.3. Definition.** (Jean-Louis Verdier, Dieter Puppe). A **triangulated category** $T = (T, T, E)$ consists of an additive category $T$; an additive self-equivalence $T : T \rightarrow T$, called translation or suspension, with quasi-inverse $T^{-1}$; and a collection $E$ of triangles, called the distinguished triangles of $T$. This data is subject to the following axioms:

**TR1** For every object $A \in T$, the triangle $A \xrightarrow{1} A \xrightarrow{0} TA$ is distinguished. For every morphism $f$ in $T$, there exists some distinguished triangle $[f, g, h]$. Every triangle isomorphic to a distinguished triangle is distinguished.

**TR2** (Rotation Axiom). The triangle $[f, g, h]$ is distinguished if and only if the triangle $[g, h, -T(f)]$ is distinguished.

**TR3** (Morphism Axiom). For every commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
& \searrow{b} & \downarrow{c} \\
& & T(a)
\end{array}
\]

with distinguished rows, there exists a (not necessarily unique) morphism $c : C \rightarrow C'$ such that $(a, b, c)$ is a morphism of triangles.

**TR4** (Composition or Octahedron Axiom). For every composition $h = g \circ f$ of morphisms and every choice of distinguished triangles

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g'} F \xrightarrow{f''} TA \\
B \xrightarrow{g} C \xrightarrow{g'} G \xrightarrow{g''} TB \\
A \xrightarrow{h} C \xrightarrow{h'} H \xrightarrow{h''} TC
\end{array}
\]

This data is subject to the following axioms:

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\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
& \searrow{b} & \downarrow{c} \\
& & T(a)
\end{array}
\]

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\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g'} F \xrightarrow{f''} TA \\
B \xrightarrow{g} C \xrightarrow{g'} G \xrightarrow{g''} TB \\
A \xrightarrow{h} C \xrightarrow{h'} H \xrightarrow{h''} TC
\end{array}
\]
containing them, there exist (non-unique) $\ell : F \to H$ and $\ell' : H \to G$ such that the following diagram is an **Octahedron**:

\[
\begin{array}{ccc}
F & \xrightarrow{f'} & G \\
\downarrow{h'} & & \downarrow{g'} \\
A & \xrightarrow{f} & B \\
\uparrow{h} & & \uparrow{g} \\
C & \xrightarrow{h''} & D
\end{array}
\]

By (1.1.4) being an Octahedron we mean all of the following: The cyclicly oriented triangles on the frame of the octahedron are distinguished; the poset oriented triangles are commutative; the two non-cyclic squares containing the center of the octahedron are commutative.

1.1.5. **Remark.** In Verdier’s original definition ([Ver96, II Def. 1.1.1]), the translation $\mathcal{T}$ is an **automorphism** of $\mathcal{T}$. This suffices to cover algebraic examples such as the derived category of a ring, but would leave out important topological ones like the homotopy category of spectra. Nonetheless, there is a simple natural procedure (explained in Section A.4) that can be used to substitute a triangulated category where $\mathcal{T}$ is only a self-equivalence with a triangle equivalent (Def. 1.1.26 below) triangulated category where the translation is an isomorphism. Therefore, in the following, we will nonchalantly pretend that $\mathcal{T}^{-1}\mathcal{T} = \text{id}$ and $\mathcal{T}\mathcal{T}^{-1} = \text{id}$.

1.1.6. **Remark.** Using Rotation (TR2), a distinguished triangle (1.1.1) may be extended on both sides to a sequence

\[
\begin{array}{cccccccc}
\cdots & \xrightarrow{-\mathcal{T}^{-1}(g)} & \xrightarrow{-\mathcal{T}^{-1}(h)} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \mathcal{T}(A) & \xrightarrow{-\mathcal{T}(f)} & \mathcal{T}(B) & \xrightarrow{-\mathcal{T}(g)} & \cdots
\end{array}
\]

where each three consecutive arrows form a distinguished triangle.

In the rest of this section we state the first important consequences of the axioms, leaving the proofs as familiarizing exercises (detailed solutions, and elucidations, can be found in [Büh08, Ch. I]). The Octahedron axiom will not be needed for this: its main *raison d’être* is its role in the localization theory for triangulated categories, see Section 1.3.

1.1.8. **Lemma.** The axioms of a triangulated category are self-dual. More precisely, the triple $(\mathcal{T}, \mathcal{T}, \mathcal{E})$ is a triangulated category if and only if the triple $(\mathcal{T}^{\text{op}}, \mathcal{T}^{-1}, \mathcal{E}^{\text{op}})$ is a triangulated category, where $[f^{\text{op}}, g^{\text{op}}, h^{\text{op}}] \in \mathcal{E}^{\text{op}}$ iff $[f, g, h] \in \mathcal{E}$.

1.1.9. **Remark.** The last lemma is quite useful, because it halves the number of proofs by allowing reasoning by duality. Nonetheless, it appears that the ‘right’ triangulation on $\mathcal{T}^{\text{op}}$ should be the **negative triangulation** $(\mathcal{E}^{\text{op}})$, defined by $[f, g, h] \in \mathcal{E}^{\text{op}}$ iff $[-f, -g, -h] \in \mathcal{E}$. (This is required for instance in order for many dualities appearing in nature to be triangle functors, see Def. 1.1.23; cf. Remark 2.4.4. Note also that, if $\mathcal{A}$ is an abelian category, the isomorphism $\mathcal{D}(\mathcal{A}^{\text{op}}) \cong \mathcal{D}(\mathcal{A})^{\text{op}}$ of derived categories becomes a triangle functor if we equip the latter category with the negative triangulation.) Exercise: If $(\mathcal{T}, \mathcal{T}, \mathcal{E})$ is a triangulated category, then so is $(\mathcal{T}, \mathcal{T}, \mathcal{E})$.

1.1.10. **Lemma.** In a distinguished triangle, the composition of two consecutive arrows is zero (i.e., if (1.1.1) is distinguished then $gf = 0$ and $hg = 0$.)
1.1.11. **Proposition.** Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ be a distinguished triangle. Then $(A, f)$ is a weak kernel of $g$: We have $gf = 0$ and, given any $t : X \to B$ such that $gt = 0$, there exists a (non-unique) $\tilde{t} : X \to A$ with $f\tilde{t} = t$. Dually, $(C, g)$ is a weak cokernel of $f$.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \xrightarrow{g} C \xrightarrow{h} TA \\
\downarrow \exists \tilde{t} & & \downarrow 0 \\
X & & \end{array} \quad \begin{array}{ccc}
A & \xrightarrow{f} & B \xrightarrow{g} C \xrightarrow{h} TA \\
\downarrow 0 & & \downarrow \exists \tilde{\epsilon} \\
Y & & \end{array}
\]

\[\Box\]

1.1.12. **Lemma.** Consider an automorphism $(a, b, c)$ of a distinguished triangle:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \xrightarrow{g} C \xrightarrow{h} T(A) \\
\downarrow a & & \downarrow b & \downarrow c \\
A & \xrightarrow{f} & B \xrightarrow{g} C \xrightarrow{h} T(A)
\end{array}
\]

If two out of $\{a, b, c\}$ are nilpotent (with respect to composition), then so is the third.

**Proof.** By the Rotation Axiom, we may assume that $a$ and $c$ are nilpotent. By considering a sufficiently high power $(a, b, c)^N = (a^N, b^N, c^N)$ of the endomorphism, we may assume that $a = 0$ and $c = 0$. Thus $gb = cg = 0$, and by the weak kernel property of $(A, f)$ there exists an $\ell : B \to A$ such that $f\ell = b$. Dually, $bf = fa = 0$ implies the existence of an $m : C \to B$ with $mg = b$. Therefore $b^2 = (mg)(f\ell) = m(gf)\ell = 0$ because of Lemma 1.1.10. \[\Box\]

1.1.13. **Corollary.** (Five Lemma). Consider a morphisms $(a, b, c)$ of distinguished triangles as in (1.1.2). If two out of $\{a, b, c\}$ are isomorphisms, then so is the third, and thus $(a, b, c)$ is an isomorphism of triangles. \[\Box\]

1.1.14. **Corollary.** In (TR1), the distinguished triangle containing a given morphism $f$ is unique up to (non-unique) isomorphism of triangles. \[\Box\]

1.1.15. **Definition.** Inspired by homotopy theory, one calls the third object $C$ in a distinguished triangle (1.1.1) the (mapping) cone of $f$. It is unique up to (non-unique) isomorphism. Abusing notation, we will write cone$(f)$ or $C_f$ for any object in this isomorphism class. \[\Box\]

1.1.16. **Corollary.** A morphism $f$ is invertible if and only if cone$(f) \cong 0$. \[\Box\]

1.1.17. **Definition.** Let $\mathcal{A}$ be some abelian category. An additive functor $H : \mathcal{T} \to \mathcal{A}$ is called **homological** if it sends every distinguished triangle (1.1.1) to an exact sequence

\[(1.1.18) \quad H(A) \xrightarrow{H(f)} H(B) \xrightarrow{H(g)} H(C)\]

in $\mathcal{A}$. Similarly, a functor $H : \mathcal{T}^{\text{op}} \to \mathcal{A}$ is **cohomological** if it sends every distinguished triangle (1.1.1) to an exact sequence

\[H(A) \xleftarrow{H(f)} H(B) \xleftarrow{H(g)} H(C).\]

It is traditional to write $H^n(A) := H_n(A) := H(T^nA)$ for homological and cohomological functors $H, n \in \mathbb{Z}$. We will mostly consider (co)homological functors whose target is the abelian category $\mathbf{Ab}$ of abelian groups.
1.1.19. Remark. Applying a homological functor $H$ to (1.1.7) we obtain a long exact sequence

$$
\cdots \longrightarrow H_1(A) \xrightarrow{H(h)} H(A) \longrightarrow H(B) \longrightarrow H(C) \xrightarrow{H(h)} H_1(A) \longrightarrow \cdots
$$

and similarly for cohomological functors. Thus each distinguished triangle is a mold or matrix for producing long exact sequences out of (co)homological functors.

The last remark is beautifully espoused by the next proposition:

1.1.20. Proposition. Let $A \in \mathcal{T}$ be any object. Then the representable functor $\mathcal{T}(A, -) : \mathcal{T} \to \text{Ab}$ is homological, and the representable functor $\mathcal{T}(-, A) : \mathcal{T}^{\text{op}} \to \text{Ab}$ is cohomological. 

The next and last proposition requires the Octahedron Axiom.

1.1.21. Proposition. (Verdier’s Exercise, [BBD82, Prop. 1.1.11]). Every commutative square $(\star)$ can be completed to a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow \scriptstyle{(\star)} & & \downarrow \\
A' & \longrightarrow & B' \\
\downarrow & & \downarrow \\
A'' & \longrightarrow & B'' \\
\downarrow & & \downarrow \\
TA & \longrightarrow & TB \\
\end{array}
\quad
\begin{array}{ccc}
C & \longrightarrow & TA \\
\downarrow & & \downarrow \\
C' & \longrightarrow & TA' \\
\downarrow & & \downarrow \\
C'' & \longrightarrow & TA'' \\
\downarrow & & \downarrow \\
TC & \longrightarrow & T^2A
\end{array}
$$

where the first three rows and the first three columns are distinguished triangles, and the fourth row (resp. column) is $T$ applied to the first one. Moreover, every square is commutative except for the one marked $(-1)$ which anticommutes (i.e., one way equals minus the other way).

1.1.22. Remark. The axioms of a triangulated category work quite well in practice, although they are not completely satisfactory from a theoretical point of view. For instance, there are some triangles which ‘morally’ should be distinguished because they can be shown to induce long exact sequences on application of all (co)homological functors, but, in some examples, they are not. Many authors abhor the lack of uniqueness of the fill-in morphisms in (TR3), which makes it impossible in general to have functorial cones — this certainly leads to some difficulties, and not a few mistakes. Moreover, the Octahedron Axiom is not universally loved, in any of its formulations (there are some). The axiomatic of triangulated categories is discussed e.g. in [BBD82], [Ive86], [KV87], [Nee91], [Nee01].

1.1.1. Triangle functors. No definition of a mathematical structure is complete without the companion concept of morphism. We will denote by the same letter $T$ the translation of all triangulated categories, as there is no danger of confusion. Let $\mathcal{T}, \mathcal{S}$ be two triangulated categories.

1.1.23. Definition. A triangle functor $F = (F, \varphi) : \mathcal{T} \to \mathcal{S}$ (or triangulated functor, or exact functor), is an additive functor $F : \mathcal{T} \to \mathcal{S}$ together with a specified natural transformation $\varphi : FT \Rightarrow TF$ with the property that, if $[f, g, h]$ is a distinguished triangle of $\mathcal{T}$, then $[Ff, Fg, Fh]$ is a distinguished triangle of $\mathcal{S}$. Triangle functors can be composed in the obvious way. Let $(F_1, \varphi_1), (F_2, \varphi_2) : \mathcal{T} \to \mathcal{S}$ be two parallel triangle functors. A morphism of triangle functors $\alpha : (F, \varphi) \Rightarrow (F', \varphi')$ is...
(\mathcal{F}', \varphi') is a natural transformation \alpha : \mathcal{F} \Rightarrow \mathcal{F}' such that \mathcal{T}\alpha \circ \varphi = \varphi' \circ \alpha \mathcal{T}, i.e., such that the square

\[
\begin{array}{ccc}
FT(A) & \xrightarrow{\varphi} & TF(A) \\
\alpha_{T(A)} & & T(\alpha_A) \\
F'T(A) & \xrightarrow{\varphi'} & TF'(A)
\end{array}
\]

is commutative for all objects \( A \). A triangle functor \((\mathcal{F}, \varphi) : \mathcal{T} \rightarrow \mathcal{S}\) is a triangle equivalence if there is a triangle functor \((\mathcal{G}, \psi) : \mathcal{S} \rightarrow \mathcal{T}\) such that the compositions \((\mathcal{G}\mathcal{F}, \psi\mathcal{F})(\mathcal{G}\varphi)\) and \((\mathcal{F}\mathcal{G}, \varphi\mathcal{G})(\mathcal{F}\psi)\) are isomorphic to the identity triangle functors \((\text{id}_\mathcal{T}, \text{id})\), resp. \((\text{id}_\mathcal{S}, \text{id})\).

1.1.24. Remarks. (a) Let \((\mathcal{F}, \varphi)\) be a triangle functor. Then \varphi is automatically an isomorphism (Exercise: use (TR1)-(TR2) and Corollary 1.1.16).

(b) Let \( \mathcal{H} : \mathcal{T} \rightarrow \mathcal{A} \) be a homological functor. Then the composition

\[
\alpha \colon \mathcal{T}' \xrightarrow{\mathcal{F}} \mathcal{T} \xrightarrow{\mathcal{H}} \mathcal{A} \xrightarrow{\mathcal{G}} \mathcal{A}'
\]

is homological, whenever \( \mathcal{F} \) is a triangle functor and \( \mathcal{G} \) is an exact functor of abelian categories. Moreover, if \( \mathcal{H} \) and \( \mathcal{G} \) are stable (as in Def. 3.1.1), so is \( \mathcal{G}\mathcal{H}\mathcal{F} \), in the obvious way.

(c) We see that a morphism \( \alpha : \mathcal{F} \Rightarrow \mathcal{F}' \) of triangle functors induces for every distinguished triangle \([f, g, h] \) a morphism of distinguished triangles \( \alpha : [\mathcal{F}f, \mathcal{F}g, \mathcal{F}h] \rightarrow [\mathcal{F}'f, \mathcal{F}'g, \mathcal{F}'h] \).

1.1.25. Lemma. ([KV87, §1.4]). A triangle functor is a triangle equivalence if and only if it is an equivalence of the underlying categories. \( \square \)

1.1.26. Definition. ([KV87, §1.6]). Let \((\mathcal{F}, \varphi) : \mathcal{T} \rightarrow \mathcal{S}\) and \((\mathcal{G}, \psi) : \mathcal{S} \rightarrow \mathcal{T}\) be triangle functors, such that \( \mathcal{F} \) is left adjoint to \( \mathcal{G} \). Let \( \eta : \text{id}_\mathcal{T} \Rightarrow \mathcal{G}\mathcal{F} \) and \( \varepsilon : \mathcal{G}\mathcal{F} \Rightarrow \text{id}_\mathcal{S} \) be the unit and counit of the given adjunction. If the following four equivalent conditions are satisfied, we say that \((\mathcal{F}, \mathcal{G}, \eta, \varepsilon)\) is a triangle adjunction:

\begin{align*}
(\text{i}) & \quad \varphi = (\varepsilon\mathcal{T}\mathcal{F}) \circ (\mathcal{F}\mathcal{G}\varepsilon) \circ (\mathcal{F}\mathcal{T}\eta) \\
(\text{ii}) & \quad \psi^{-1} = (\mathcal{G}\mathcal{T}\varepsilon) \circ (\mathcal{G}\varphi\mathcal{G}) \circ (\eta\mathcal{G}) \\
(\text{iii}) & \quad \varepsilon\mathcal{T} = (\mathcal{T}\varepsilon) \circ (\mathcal{G}\varphi) \circ (\mathcal{F}\psi) \\
(\text{iv}) & \quad \eta\mathcal{T} = (\mathcal{F}\varepsilon) \circ (\mathcal{G}\varepsilon) \circ (\eta\mathcal{T}).
\end{align*}

1.1.27. Proposition. Let \((\mathcal{F}, \mathcal{G}, \eta, \varepsilon)\) be an adjunction between two triangulated categories, with unit \( \eta : \text{id} \Rightarrow \mathcal{G}\mathcal{F} \) and counit \( \varepsilon : \mathcal{G}\mathcal{F} \Rightarrow \text{id} \). Then if \((\mathcal{G}, \psi)\) is a triangle functor with \( \mathcal{G} \) as underlying functor, then \((\mathcal{F}, (\varepsilon\mathcal{T}\mathcal{F}) (\mathcal{F}\mathcal{G}\varepsilon) (\mathcal{F}\mathcal{T}\eta))\) is also a triangle functor and the given adjunction is a triangle adjunction. Dually, if \((\mathcal{F}, \varphi)\) is a triangle functor, then \((\mathcal{G}, ((\mathcal{G}\mathcal{T}\varepsilon)(\mathcal{G}\varphi\mathcal{G})(\eta\mathcal{G}))^{-1})\) is a triangle functor making the given adjunction a triangle adjunction.

Proof. This is [KV87, Prop. 1.6]. See [Kel91, §6.7] for a proof. \( \square \)

Whenever we mention adjunctions of triangulated functors, we will always tacitly mean triangle adjunctions. Same with equivalences.

1.1.28. Definition. A triangulated subcategory \( \mathcal{S} \) of a triangulated category \((\mathcal{T}, \mathcal{T}, \mathcal{E})\) is a strictly full additive subcategory of \( \mathcal{T} \) (i.e., it is full and closed with respect to isomorphic objects) such that, whenever \( \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{T}\mathcal{A} \) is a distinguished triangle of \( \mathcal{T} \) and \( \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{S} \), then also \( \mathcal{C} \in \mathcal{S} \) (we say that \( \mathcal{S} \) is closed under taking cones). Then \((\mathcal{S}, \mathcal{T}|\mathcal{S}, \mathcal{E}|\mathcal{S})\) is a triangulated category, and the inclusion functor \( \mathcal{F} : \mathcal{S} \hookrightarrow \mathcal{T} \) is a triangle functor with structure morphism \( \varphi = \text{id} : \mathcal{F}\mathcal{T} = \mathcal{T}\mathcal{F} \).
1.1.2. Coproducts and homotopy colimits. The only limits or colimits that you’re likely to encounter in triangulated categories are (infinite) products and coproducts. If they exist, they are automatically compatible with the triangulation:

1.1.29. Lemma. A product or coproduct of distinguished triangles is distinguished.

Proof. (The claim follows also from Proposition 1.1.27, see [Kel96, Example 8.4].) We need only prove the claim for coproducts, since the same argument in $T^{op}$ yields the result for products. Let $\{A_i \rightarrow B_i \rightarrow C_i \rightarrow T A_i\}_{i \in I}$ be some set of distinguished triangles, and assume that $T$ admits the coproducts $\coprod A_i$, $\coprod B_i$ and $\coprod C_i$. Choosing a distinguished triangle $\coprod A_i \rightarrow \coprod B_i \rightarrow C \rightarrow T \coprod A_i$, we obtain after $|I|$ applications of (TR3) a collection of morphisms $c_i : C_i \rightarrow C$ fitting into the commutative diagram

$$
\begin{array}{ccc}
\coprod A_i & \rightarrow & \coprod B_i \\
\downarrow & & \downarrow \\
\coprod A_i & \rightarrow & \coprod B_i \rightarrow C \rightarrow T \coprod A_i
\end{array}
$$

where the second row and each summand of the first row is a distinguished triangle (we tacitly use that $T$, being an equivalence, commutes with coproducts). For any object $X \in T$, the functor $T(-, X) : T^{op} \rightarrow \text{Ab}$ is cohomological and sends coproducts to products. Therefore, if we apply it to the above diagram we obtain a comparison of long exact sequences, which, by the Five Lemma (in $\text{Ab}$), implies that $T(c, X)$ is invertible. Since $X \in T$ was arbitrary, we conclude by the Yoneda Lemma that $c$ is invertible in $T$. Hence the upper triangle is distinguished by (TR1), being isomorphic to the distinguished triangle in the second row. □

1.1.30. Remark. Direct sums $A \oplus B$ are both products and coproducts. Thus by Lemma 1.1.29, in any triangulated category the direct sum of two distinguished triangles is distinguished. It can be shown that the converse holds too: if the direct sum of two triangles is distinguished, so must be each summand.

1.1.31. Corollary. Let $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow T A$ be distinguished. If $g$ has a right inverse or $f$ a left inverse, then $B \cong A \oplus C$.

Proof. If there is an $s : C \rightarrow B$ with $gs = 1_C$, we may construct the following morphism of distinguished triangles,

$$
\begin{array}{cccc}
A & \rightarrow & A \oplus C & \rightarrow C \\
\downarrow & & \downarrow & \downarrow \\
A & \rightarrow & B & \rightarrow C \\
\downarrow & & \downarrow & \downarrow \\
B & \rightarrow & C & \rightarrow T A
\end{array}
$$

(the first row is distinguished because it is the direct sum of $A = A \rightarrow 0 \rightarrow TA$ and $0 \rightarrow C = C \rightarrow 0$), and we conclude with the Five Lemma that $A \oplus C \cong B$. The other claim is similar. □

À propos splittings, the following observation is often useful:

1.1.32. Lemma. Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$ be a distinguished triangle. Then $h = 0$ $\iff$ $g$ is a split epimorphism $\iff$ $g$ is an epimorphism $\iff$ $f$ is a split monomorphism $\iff$ $f$ is a monomorphism.
1. TRIANGULATED CATEGORIES

Proof. If \( h = 0 \), use that \( g \) is a weak kernel of \( h \) on \( 1_C : C \to C \) to find a right inverse of \( g \). Conversely, if \( g \) is epi, then \( hg = 0 \) (Lemma 1.1.10) implies \( h = 0 \). The rest follows from duality and rotation. \( \Box \)

1.1.33. COROLLARY. In a triangulated category, epimorphisms and monomorphisms are all split. \( \Box \)

1.1.34. COROLLARY. If the underlying category of a triangulated category is abelian, it is semisimple abelian. \( \Box \)

A successful import from stable homotopy is the concept of homotopy colimit ([Fre66b] [Ver96] [BN93]), which we shall use in the following sections. We only treat countable sequences of morphisms, so for the rest of the section we assume that \( T \) is a triangulated category admitting all countable coproducts.

1.1.35. Definition. The homotopy colimit of a sequence of morphisms \( A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \) in \( T \) is the third object \( A \) in any distinguished triangle

\[
\begin{array}{c}
\prod_n A_n \\
\downarrow 1-f \\
\prod_n A_n \\
\end{array} \xrightarrow{A} \prod_n A_n \\
\downarrow \\
\prod_n A_n
\end{array}
\]

containing the morphism \( 1-f \) (or "1-Shift"), which is defined as the unique morphisms whose component at \( A_k \) is \( 1_{A_k} - f_k : A_k \to A_k \oplus A_{k+1} \subset \prod_n A_n \). It is usually denoted by \( \text{hocolim}(A_n, f_n) := A \), and it comes equipped with canonical morphisms \( \{ A_n \to \text{hocolim}(A_n, f_n) \} \) (the components of the second side of the triangle) and \( \{ \text{hocolim}(A_n, f_n) \to A \} \) (the third side).

1.1.36. REMARK. The homotopy colimit of a sequence is subject to the same sort of weak unicity and weak functoriality enjoyed by the cone of a morphism. Using the results obtained so far, a nice little theory of homotopy colimits can be developed (see [Nee01, §1.6-7] for the proofs, or: exercises!). For instance, \( \text{hocolim} \) commutes (non-canonically) with direct sums of sequences. An isomorphism of sequences induces a (non-unique) isomorphism of homotopy colimits. If all the arrows \( f_n \) in the sequence are isomorphisms, one may take \( \text{hocolim}(A_n, f_n) \) to be \( A_0 \) (with the evident canonical maps). On the other hand, if \( f_n = 0 \) for all \( n \) then \( \text{hocolim}(A_n, f_n) \cong 0 \). Another useful fact: the homotopy colimits of any two subsequences of a given sequence are isomorphic (by a subsequence we mean the result of picking a subsequence of \( A_0, A_1, A_2, \ldots \) and filling in by composing the \( f_n \)'s accordingly).

The following is attributed to Freyd (cf. [Ver96, Prop. 1.2.9]):

1.1.37. PROPOSITION. If \( T \) has arbitrary countable coproducts (or countable products), then \( T \) is idempotent complete: if \( e = e^2 : A \to A \) is an idempotent endomorphism, then there exist \( p : A \to B \) and \( i : B \to A \) such that \( ip = e \) and \( pi = 1_B \). Equivalently, there exists an isomorphism \( \phi : A \cong A_1 \oplus A_2 \) making the following square commute:

\[
\begin{array}{ccc}
A & \xrightarrow{e} & A \\
\phi \downarrow & \cong & \downarrow \phi \\
A_1 \oplus A_2 & \xrightarrow{\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)} & A_1 \oplus A_2 \\
\end{array}
\]

(in other words, \( \phi \) identifies \( A \) with \( \text{Im}(e) \oplus \text{Ker}(e) \)).

Sketch of proof. ([Nee01, Prop. 1.6.8].) The equivalence of the two conclusions is clear from Corollary 1.1.31. Given an idempotent \( e = e^2 : A \to A \),
consider the following morphism of sequences:

\[
\begin{array}{cccc}
S_1 & : & A \oplus A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \to & A \oplus A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \to & A \oplus A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \to & \cdots \\
\phi & = & \left( \begin{array}{cc}
1/e - e & -1/e \\
1 - e & 0
\end{array} \right) & \left( \begin{array}{cc}
1/e - e & -1/e \\
1 - e & 0
\end{array} \right) & \left( \begin{array}{cc}
1/e - e & -1/e \\
1 - e & 0
\end{array} \right) & \left( \begin{array}{cc}
1/e - e & -1/e \\
1 - e & 0
\end{array} \right) & \to & \cdots \\
S_2 & : & A \oplus A \begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix} & \to & A \oplus A \begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix} & \to & A \oplus A \begin{pmatrix} e & 0 \\ 0 & 1-e \end{pmatrix} & \to & \cdots 
\end{array}
\]

Since \( S_1 \) decomposes as a direct sum of sequences, its homotopy colimit decomposes accordingly as \( \text{hocolim} S_1 \cong A \oplus 0 = A \). Similarly, \( \text{hocolim} S_2 =: A_1 \oplus A_2 \). Each component of \( \Phi \) is an isomorphism (with itself as inverse), implying that \( \Phi \) induces a (non-unique) isomorphism on homotopy colimits: \( \phi : A \cong A_1 \oplus A_2 \).

1.1.38. Remark. In the absence of countable (co)products, there is a simple procedure (the idempotent completion, or pseudo-abelianization) for formally adding kernels and cokernels of all idempotents morphisms. This works for any additive category, but if the category \( T \) is triangulated, its idempotent completion \( \tilde{T} \) inherits a canonical triangulation in such a way that the canonical fully faithful functor \( T \to \tilde{T} \) is a triangle functor (see [BS01]).

1.2. Localization and calculus of fractions

The material of this section originates from [GZ67].

Let \( C \) be a category, and let \( W \subseteq \text{Mor}(C) \) be a class of morphisms considered as some kind of ‘weak equivalences’: they are possibly not isomorphisms, but we would like them very much to be.

1.2.1. Definition. A localization of \( C \) at \( W \) is a category \( W^{-1}C \) (often also written \( C[W^{-1}] \)) together with a functor \( \ell : C \to W^{-1}C \), such that \( \ell(w) \) is invertible for every \( w \in W \), and such that \( \ell \) is universal among functors inverting \( W \): if \( F : C \to D \) is another functor with the property that \( F(w) \) is invertible for every \( w \in W \), then there exists a unique functor \( \tilde{F} : W^{-1}C \to D \) with \( F = \tilde{F} \circ \ell \).

Thus if a localization of \( C \) at \( W \) exists, it is unique up to unique isomorphism of functors. Indeed, ignoring set-theoretical questions, a model for \( W^{-1}C \) can always be constructed in the following straightforward manner. Simply build the free category on \( \text{obj}(C) \) with the arrows \( \text{Mor}(C) \cup W^{-1} \) (by considering formal compositions of arrows), and then quotient out the (categorical) ideal generated by all the relations which define the composition in \( C \). The localization functor \( \ell \) is simply the identity on objects and sends \( f \in \text{Mor}(C) \) to its equivalence class in \( W^{-1}C \). Thus a general morphism \( A \to B \) of \( W^{-1}C \) is the equivalence class of some finite zig-zag

\[ A = X_0 \to X_1 \sim X_2 \to \cdots \to X_{n-1} \sim X_n = B, \]

where the arrows going in the ‘wrong’ direction (marked with ‘\( \sim \)’) are in \( W \). This construction is unsatisfactory for at least two reasons.

1. Set-theoretical questions often do matter\(^1\). With the above model, it is unclear if \( W^{-1}C \) is locally small if \( C \) is. In general, it is not, so one should start looking for sufficient conditions on \( W \) for this to be the case. Note however that if \( C \) is small, then so is \( W^{-1}C \), since in this case there is only a set of possible zig-zags among any two given objects.

2. More seriously, the zig-zag way to localization is pretty intractable, since the equivalence relation on the zig-zags remains quite implicit and therefore mysterious. With hindsight, it is perhaps not an exaggeration to say

---

\(^1\)See for instance Sections 1.6 and 1.7.
that the goal of much homological and homotopical algebra is to build and study viable models for ‘derived’ and ‘homotopy’ categories $W^{-1}C$.

To begin with, it is certainly a good idea to shorten the zig-zags.

1.2.2. Definition. A class $W$ in $\text{Mor}(C)$ is a right multiplicative system if it has the following properties.

(i) It contains all identities: $1_A \in W$ for all $A \in C$.
(ii) It is closed under composition: $W \circ W \subseteq W$.
(iii) (Right Ore condition). Given a diagram $A \xrightarrow{f} X \xleftarrow{w} B$ with $w \in W$ (a left fraction $w^{-1}f$), there exists a commutative square

$$
\begin{array}{ccc}
Y & \xrightarrow{f'} & B \\
\downarrow{w'} & & \downarrow{w} \\
A & \xrightarrow{f} & X
\end{array}
$$

with $w' \in W$. (Intuitively: it is always possible to flip a left fraction $w^{-1}f : A \to B$ into an equivalent right fraction $f'w'^{-1} : A \to B$.)

(iv) (Right cancellation). Given parallel morphisms $f, g : A \to B$, if there exists $(w : B \to C) \in W$ such that $wf = wg$ then there exists also a $(w' : D \to A) \in W$ such that $fw' = gw'$.

A left multiplicative system is a class $W \subseteq \text{Mor}(C)$ satisfying the dual axioms (i)$^{\text{op}}$=(i), (ii)$^{\text{op}}$=(ii) and (iii)$^{\text{op}}$ (Left Ore condition). Given a diagram $A \xleftarrow{w} X \xrightarrow{f} B$ with $w \in W$ (a right fraction $fw^{-1}$), there exists a commutative square

$$
\begin{array}{ccc}
X & \xleftarrow{w} & B \\
\downarrow{w'} & & \downarrow{w'} \\
A & \xrightarrow{f'} & Y
\end{array}
$$

with $w' \in W$.

(iv)$^{\text{op}}$ (Left cancellation). Given parallel morphisms $f$ and $g$, if there exists $w \in W$ such that $fw = gw$ then there is a $w' \in W$ such that $w'f = w'g$.

A multiplicative system is simultaneously left and right multiplicative. A system $W$ is saturated if it satisfies:

(v) If $f, g, h$ are composable and $hg, gf \in W$, then $g \in W$.

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\sim & & \sim \\
C & \xrightarrow{h} & D
\end{array}
$$

If $W$ is a (left and/or right) multiplicative system, one may alternatively and pertinently say that $W$ admits a calculus of (left and/or right) fractions.

For the rest of this section we work with a right multiplicative system. The dual results for left multiplicative systems, whose formulation we leave to the reader, have dual proofs. For an object $A$, we denote by $W \downarrow A$ the comma category of

\[\text{A multiplicative system is simultaneously left and right multiplicative. A system } W \text{ is saturated if it satisfies:}\
\]

\[\text{(v) If } f, g, h \text{ are composable and } hg, gf \in W, \text{ then } g \in W.\
\]

\[\text{If } W \text{ is a (left and/or right) multiplicative system, one may alternatively and pertinently say that } W \text{ admits a calculus of (left and/or right) fractions.}\
\]

As it is only to be expected, some authors prefer to call left fractions the right ones, and vice versa. Since there is no right answer, we are left with the choice.

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\]
W over A, which has arrows \((w : X \rightarrow A) \in W\) for objects and commutative triangles

\[
\begin{array}{c}
X \\ f
\end{array} \xrightarrow{w} \begin{array}{c}
X' \\ w
\end{array}
\]

for morphisms.

1.2.3. Lemma. Let W be a right multiplicative system in C. Then W ↓ A is cofiltering for every A ∈ C.

Proof. The proof will use each axiom exactly once. First of all, W ↓ A is nonempty because of (i). Given \(w_1 : X_1 \rightarrow A\) and \(w_2 : X_2 \rightarrow A\) in W, there exist by (iii) arrows \(u_1 : Y \rightarrow X_1\) and \(u_2 : Y \rightarrow X_2\) with \(w_1u_1 = w_2u_2\) and (say) \(u_1 \in W\).

By (ii), the composition \(w_1u_1 : Y \rightarrow A\) is in W, so that \((Y,u_1)\) is an object of W ↓ A covering both \((X_1,u_1)\) and \((X_2,u_2)\).

Now let \(f_1,f_2 : (X,u) \cong (Y,v)\) be two parallel arrows of W ↓ A. Since \(v \in W\) and \(vf_1 = u = vf_2\), by (iv) there is a \(w : Z \rightarrow X\) in W with \(f_1w = f_2w\). In other words, there is an arrow \(w : (Z,uw) \rightarrow (X,u)\) in W ↓ A on which \(f_1\) and \(f_2\) coincide. □

1.2.4. Proposition. (Gabriel-Zisman). Let W be a right multiplicative system in C. Then we may construct \(W^{-1}C\) as follows: the objects are those of C, and the Hom sets (classes) are given by the filtering colimit

\[
W^{-1}C(A,B) = \colim_{A \rightarrow X} C(X,B)
\]

indexed by the comma category W ↓ A. Its elements are equivalence classes of right fractions \([fw^{-1}] = [A \xrightarrow{w} X \xrightarrow{f} B]\). Composition is computed by means of the right Ore condition: if \([fu^{-1}] : A \rightarrow B\) and \([gv^{-1}] : B \rightarrow C\) are two morphisms in \(W^{-1}C\), their composite is \([(gf)(w')^{-1}] : A \rightarrow C\) for any commutative diagram

\[
\begin{array}{c}
X \\ f
\end{array} \xleftarrow{u} \begin{array}{c}
A \\ \sim
\end{array} \xrightarrow{f'} \begin{array}{c}
Z \\ v
\end{array} \xrightarrow{g} \begin{array}{c}
C.
\end{array}
\]

The canonical functor \(\ell : C \rightarrow W^{-1}C\) sends \(f : A \rightarrow B\) to the right fraction \(\ell(f) = [f \circ 1^{-1}_A]\).

Proof. We leave it as an exercise to check that the composition as described above is well defined, and that \(\ell\) is a functor enjoying the required universal property. The fact that the colimit (1.2.5) is filtering is given by the previous lemma. □

1.2.6. Remark. Having a calculus of fractions at hand is certainly helpful for understanding a localized category (Problem 2). It may be used to address Problem 1 as well: as is clear from Proposition 1.2.4, in order for \(W^{-1}C\) to be locally small it suffices that, for each \(A \in C\), the cofiltering category \(W ↓ A\) has a small cofinal subcategory (we may say in this case that W is left locally small).

1.2.7. Remark. If W is a (left and/or right) multiplicative system, we obviously have \(W \subseteq \{f \mid \ell(f)\text{ is an iso in }W^{-1}C\}\). Equality holds iff W is saturated (Exercise).

1.2.8. Corollary. If C is an additive category, \(W^{-1}C\) is additive and \(C \rightarrow W^{-1}C\) is an additive functor.
1. TRIANGULATED CATEGORIES

Proof. Indeed, the underlying set of a filtering colimit of abelian groups is the corresponding colimit of the underlying sets, so we can use (1.2.5) to make $W^{-1}C(A, B)$ into an abelian group, and $\ell : C(A, B) \to W^{-1}C(A, B)$ a homomorphism. (Exercise: use Lemma 1.2.3 to deduce the $\mathbb{Z}$-bilinearity of the composition in $W^{-1}C$ from that of $C$.) A direct sum (biproduct) $A \oplus B$ consists of a diagram $A \leftrightarrows A \oplus B \leftrightarrows B$ whose arrows satisfy certain additive equations, so it follows that $\ell$ sends direct sums to direct sums. Finally, it is straightforward to see that $\ell(0)$ is initial and final in $W^{-1}C$, i.e., a zero object. □

We record a little useful remark (for the case when $C$ is additive):

1.2.9. Lemma. Let $W \subseteq \text{Mor}(C)$ be a right multiplicative system. Then

$$\ell(f) = 0 \text{ in } W^{-1}C \iff \exists w \in W \text{ such that } fw = 0 \text{ in } C.$$  

for every $f \in \text{Mor}(C)$.

Proof. If $fw = 0$, then $\ell(f) = 0 \circ \ell(w)^{-1} = 0$. On the other hand, if $\ell(f) = 0$ then there exists in $C$ a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{w} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{0} & B
\end{array}
\]

with $w \in W$, certifying the equivalence of the fractions $A = A \xrightarrow{f} B$ and $A = A \xrightarrow{w} B$. We see that the arrow $X \to B$ is both equal to $fw$ and to zero. □

To conclude, we study the compatibility with coproducts. Let $\alpha$ be a cardinal number. By an $\alpha$-small coproduct we mean a coproduct indexed by a cardinal $\lambda < \alpha$.

1.2.10. Corollary. Let $C$ be a category with arbitrary $\alpha$-small coproducts. Let $W \subseteq \text{Mor}(C)$ be a right multiplicative system such that

(*) if $\{w_i\}_{i \in I} \subseteq W$ and $|I| < \alpha$, then $\coprod_i w_i \in W$.

Then the category $W^{-1}C$ has arbitrary $\alpha$-small coproducts and $\ell : C \to W^{-1}C$ preserves them.

Proof. Let $\{\ell A_i\}_i$ be an $\alpha$-small family of objects of $W^{-1}C$. We have to show that, for any $B$, the canonical map $W^{-1}C(\ell(\coprod_i A_i), \ell B) \to \prod_i W^{-1}C(\ell A_i, \ell B)$ is bijective. By Proposition 1.2.4 we may identify it with the map

$$\Phi : \colim_{(\prod A_i) \to X} C(X, B) \to \prod_i \colim_{X_i \to X} C(X_i, B)$$

sending $[\coprod A_i \xrightarrow{w} X \xrightarrow{f} B]$ to $([A_i \xrightarrow{v_i} X_i \xrightarrow{f \circ j_i} B])_i$, for any commutative diagrams

\[
\begin{array}{ccc}
\prod A_i & \xrightarrow{w} & X \\
\downarrow & & \downarrow \\
A_i & \xrightarrow{v_i} & X_i
\end{array}
\]

obtained for each $i$ by the right Ore condition (Def. 1.2.2(iii)). Injectivity of $\Phi$: Assume that $\Phi([fw^{-1}]) = \Phi([f'w'^{-1}])$. Using ‘primed’ notation in the obvious way,
this means that \([f_{j_i}(v_i)^{-1}] = [(f'_{j'_i})(v'_i)^{-1}]\) for all \(i\), i.e., there are commutative diagrams (on the left hand side)

\[
\begin{array}{ccc}
  X_i & \xrightarrow{f_{j_i}} & Z_i \\
  \downarrow{v_i} & & \downarrow{u} \\
  A_i & \xrightarrow{w} & B \\
  \uparrow{u_i} & & \uparrow{v'_i} \\
  X'_i & \xrightarrow{f'_{j'_i}} & Z_i
\end{array}
\]

with \(u_i \in W\). Since \(\alpha\)-small coproducts exist in \(C\), we may add them all up, getting the commutative diagram on the right hand side. Since \(\prod u_i \in W\) by hypothesis (\(*\)), the latter diagram shows that \([fw^{-1}] = [f'w'^{-1}]\).

\textit{Surjectivity of} \(\Phi\): given a family \([([A_i \xrightarrow{w_i} X_i \xrightarrow{f_i} B])\), by (\(*\)) the diagram

\[
\prod A_i \xleftarrow{\prod w_i} \prod X_i \xrightarrow{(f_i)_i} B
\]

is a fraction mapping under \(\Phi\) to the given family. \(\square\)

### 1.3. Verdier localization

Now we return to the case when \(C =: T\) is a triangulated category. After requiring some modest compatibility, the interaction between the triangulated structure and multiplicative systems will turn out to be most satisfying, as it was discovered by Verdier [Ver96].

#### 1.3.1. Definition

A saturated multiplicative system \(W \subseteq \text{Mor}(T)\) is said to be \textbf{compatible with the triangulation} if the following conditions are satisfied:

- (vi) If \(w \in W\) then \(T^{\pm 1}(w) \in W\).
- (vii) Every commutative diagram with distinguished rows

\[
\begin{array}{ccc}
  A & \xrightarrow{u} & B \\
  v & & v \\
  A' & \xrightarrow{v} & C' \\
  & & v
\end{array}
\]

and with \(u, v \in W\) can be completed to a morphism of triangles \((u, v, w)\) by some \((w : C \to C') \in W\).

#### 1.3.2. Proposition

Let \(W\) be a multiplicative system in \(T\), compatible with the triangulation. Then the localized category \(W^{-1}T\) inherits a canonical triangulated structure from that of \(T\), so that \((\ell, \text{id}) : T \to W^{-1}T\) is a triangle functor. More precisely, the translation in \(W^{-1}C\) is defined by \(T([fw^{-1}]) := [T(f)T(w)^{-1}]\), and a triangle in \(W^{-1}T\) is distinguished iff it is isomorphic to the image under \(\ell\) of a distinguished triangle in \(T\).

\textbf{Proof.} We already know that \(W^{-1}T\) is additive and \(\ell\) is an additive functor (Cor. 1.2.8). The new translation \(T : W^{-1}T \to W^{-1}T\) is well-defined by (vi). We leave to the reader the rather straightforward verification that the given structure
Consider the full subcategory in terms of objects, because to invert a morphism amounts to killing its cone (Cor. 1.1.16). Consider the full subcategory
\[ \mathcal{J}(W) := \{ \text{cone}(w) \mid w \in W \} \subseteq \mathcal{T}. \]

Then:

1.3.3. Proposition. The triangle functor \( \ell = (\ell, \text{id}) : \mathcal{T} \to W^{-1}\mathcal{T} \) is universal among triangle functors which kill \( \mathcal{J}(W) \). More precisely, \( \ell(\mathcal{J}(W)) \cong 0 \) and, given another triangle functor \( F : \mathcal{T} \to \mathcal{S} \) with \( F(\mathcal{J}(W)) \cong 0 \), there exists a unique triangle functor \( \tilde{F} : W^{-1}\mathcal{T} \to \mathcal{S} \) such that \( \tilde{F} \circ \ell = F \).

Proof. By definition, the image under \( \ell \) of the distinguished triangle \( A \xrightarrow{w} B \to \text{cone}(w) \to TA \) is again distinguished. Since \( \ell(w) \) becomes an isomorphism, we see that \( \ell(\text{cone}(w)) \cong 0 \) for all \( w \in W \). Now let \( \tilde{F} = (\tilde{F}, \varphi) : \mathcal{T} \to \mathcal{S} \) be another triangle functor with the same property. Since \( F(\text{cone}(w)) \cong 0 \) implies that \( F(w) \) is invertible, we obtain by the universal property of localization a unique functor \( \tilde{F} \) such that \( \tilde{F} \ell = F \), and which is just \( F \) on objects. Then \( (\tilde{F}, \varphi : T\tilde{F} \cong \tilde{F}T) \) is a triangle functor by the very definition of the triangulation in \( W^{-1}\mathcal{T} \), and \( (\tilde{F}, \varphi)(\ell, \text{id}) = (F, \varphi) \). □

The subcategories of \( \mathcal{T} \) arising as \( \mathcal{J}(W) \) have a wonderfully simple characterization, as the next theorem shows.

1.3.4. Definition. A subcategory \( \mathcal{J} \subseteq \mathcal{T} \) is thick if it is triangulated and closed under taking direct summands (‘retracts’): \( A \oplus B \in \mathcal{J} \Rightarrow A, B \in \mathcal{J} \). In particular, we always assume a thick subcategory to be strictly full.

1.3.5. Example. If \( F \) is a stable (co)homological functor (Def. 3.1.1) or a triangle functor defined on \( \mathcal{T} \), its object-kernel \( \mathcal{J} = \text{Ker}(F) := \{ A \mid F(A) \cong 0 \} \subseteq \mathcal{T} \) is thick (Exercise). In fact, we’ll see that all thick subcategories arise as the object-kernel of some triangle functor and of some stable homological functor (if we allow the target category to be large).

1.3.6. Example. For every compatible multiplicative system \( W \) of \( \mathcal{T} \), the subcategory \( \mathcal{J}(W) \subseteq \mathcal{T} \) is thick; indeed, we have \( \mathcal{J}(W) = \text{Ker}(\ell : \mathcal{T} \to W^{-1}\mathcal{T}) \). The inclusion ‘\( \subseteq \)’ is clear, so let \( \ell A \cong 0 \). This means that \( \ell \) inverts \( 0 \to A \), so \( 0 \to A \in W \) because \( W \) is by definition saturated, and thus \( A = \text{cone}(0 \to A) \in \mathcal{J}(A) \).

1.3.7. Theorem. If \( W \subseteq \text{Mor}(\mathcal{T}) \) is a multiplicative system compatible with the triangulation, the subcategory \( \mathcal{J}(W) := \{ \text{cone}(w) \mid w \in W \} \subseteq \mathcal{T} \) is thick. Conversely, let \( \mathcal{J} \subseteq \mathcal{T} \) be a thick subcategory; then
\[ W(\mathcal{J}) := \{ w \mid \text{cone}(w) \in \mathcal{J} \} \subseteq \text{Mor}(\mathcal{T}) \]
is a multiplicative system compatible with the triangulation. Moreover, the assignments \( W \mapsto W(\mathcal{J}) \) and \( \mathcal{J} \mapsto W(\mathcal{J}) \) are inclusion-preserving mutually inverse bijections between the class of thick subcategories of \( \mathcal{T} \) and the class of compatible multiplicative systems of \( \mathcal{T} \).

Proof. The moreover part is immediate from the definitions. The first part is Example 1.3.6. Now let \( \mathcal{J} \subseteq \mathcal{T} \) be thick. We first have to show that \( W(\mathcal{J}) \) is a left and right multiplicative system (we check conditions (i)\(^{op}\)–(iv)\(^{op}\) of Definition 1.2.2 – the dual conditions have dual proofs). Axiom (i)\(^{op}\) is clear, since \( \text{cone}(1_A) = 0 \in \mathcal{J} \). Axiom (ii)\(^{op}\): if \( u, v \in W(\mathcal{J}) \) are composable, by the Octahedron Axiom there is a distinguished triangle containing \( \text{cone}(u), \text{cone}(v) \) and \( \text{cone}(vu) \); since \( \mathcal{J} \) is
triangulated and contains the first two objects, it must contain the third one as well, so \( vv \in W(J) \). Left cancellation (iv)\(^{\text{op}} \): let \( f, w \) be given such that \( w \in W(J) \) and \( fw = 0 \); we need to find a \( v \in W(J) \) with \( vf = 0 \). By the weak cokernel property of \( w' : B \to \text{cone}(w) \), we find a \( f' : \) with \( f'w' = f \).

Completing \( f' \) to a distinguished triangle, we see that the component \( v : C \to \text{cone}(f') \) is in \( W(J) \), since \( \text{cone}(v) \cong T \text{cone}(w) \in J \). Moreover, \( vf = (vf')w' = 0w' = 0 \). Before proving axiom (iii)\(^{\text{op}} \), the left Ore condition, let us see that \( W(J) \) is compatible with the triangulation. This is readily checked: axiom (vi) is obvious, as soon as one notices that the translation functor preserves distinguished triangles up to signs, and part (vii) follows from an application of Verdier’s Exercise (Prop. 1.1.21).

Now the left Ore condition can be proved as follows. Given \( A \xrightarrow{w} X \xrightarrow{f} B \) with \( w \in W(J) \), consider the following solid diagram with exact rows:

\[
\begin{array}{cccccc}
T^{-1}\text{cone}(f) & \xrightarrow{f'} & X & \xrightarrow{f} & B & \xrightarrow{w} \text{cone}(f) \\
\downarrow & & \downarrow & & \downarrow & \\
T^{-1}\text{cone}(f) & \xrightarrow{w'f} & A & \xrightarrow{w} \text{cone}(w) & \xrightarrow{v} \text{cone}(f).
\end{array}
\]

Since \( W(J) \) is compatible (part (vii)) and \( 1, w \in W(J) \), we find a fill-in \( v \in W(J) \) so that \( A \to \text{cone}(w'f) \sim B \) is the required left fraction.

The only thing left to check is that \( W(J) \) is saturated (Def. 1.2.2(v)). Let \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \) be such that \( gf \in W(J) \) and \( hg \in W(J) \). By the Octahedron Axiom, we may build a diagram

\[
\begin{array}{ccccc}
  & & C_{gf} & & \\
  & \searrow^{u} & \swarrow & & \nwarrow^{v} \\
C_f & \xrightarrow{w} & C_g & \xrightarrow{v} & C_h \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D
\end{array}
\]

where the two down-pointing triangles are commutative and the other five are distinguished (here \( C_x \) is short for \( \text{cone}(x) \)). From \( C_u = C_{gf} \in J \) and \( C_v = C_{hg} \in J \) we see that \( u, v \in W(J) \). Since, as we have proved, \( W(J) \) is a multiplicative system compatible with translation, \( T(u)v \in W(J) \) as well and therefore \( C_{T(u)v} \in J \). From the two commutative triangles, the bottom-middle distinguished triangle and Lemma 1.1.10 we see that \( T(u)v = 0 \). But then \( uT^{-1}(v) = 0 \), and we deduce from Lemma 1.1.32 and the distinguished triangle

\[
C_f \to T^{-2}C_{T(u)v} \to T^{-1}Ch \xrightarrow{uT^{-1}} TC_f
\]

that \( C_f \) is a direct summand of \( T^{-2}C_{T(u)v} \in J \). But \( J \) is thick, from which it follows that \( C_f \in J \). Finally, by the upper-left distinguished triangle and \( C_f, C_{gf} \in J \) we see that \( C_g \in J \), and therefore \( g \in W(J) \), proving that \( W(J) \) is saturated. \( \square \)
1.3.8. Definition. Let $J \subseteq T$ be a thick subcategory. Then the localization functor $\ell : T \to W^{-1}T$ is called the Verdier quotient of $T$ by $J$ and is denoted by $q : T \to T/J$. It enjoys both the universal property of a localization and that of a ‘quotient on objects’ (Proposition 1.3.3). We will call a sequence

$$J \to T \xrightarrow{q} Q$$

of triangle functors an exact sequence of triangulated categories if, up to equivalence, $j$ is the inclusion of a thick subcategory and $q$ is the corresponding quotient functor.

Given any class $E \subseteq T$ of objects, we may consider the thick subcategory of $T$ generated by $E$:

$$\langle E \rangle := \bigcap \{J \mid J \text{ thick and } E \subseteq J\} \subseteq T.$$ 

It is easy to see that $\langle E \rangle$ can be obtained recursively from $E \cup \{0\}$ by taking cones and direct summands, allowing inductive proofs on the ‘length of an object’.

Thus any family $E$ of objects, such as a (possibly non thick) triangulated subcategory, has a quotient, by which of course we mean the universal triangle functor $q : T \to T/\langle E \rangle$ which kills $E$.

1.3.9. Remark. If one needs the quotient $T/J$ to be locally small, one still has to check that the multiplicative system $W(J)$ is left or right locally small for each given $J$ (see Remark 1.2.6). In Section 1.4 we discuss a situation where the localization $T/J$ can be realized ‘internally’ as a full subcategory of $T$, so that this rather annoying problem completely evaporates.

1.3.1. Localizing subcategories. Let $T$ be a triangulated category with arbitrary $\alpha$-small coproducts, for some cardinal number $\alpha$.

1.3.10. Lemma. For a thick subcategory $J \subseteq T$, the following are equivalent:

(i) $J$ is closed under taking $\alpha$-small coproducts of objects.

(ii) $W(J)$ is closed under taking $\alpha$-small coproducts of morphisms.

Proof. Let $\{A_i \xrightarrow{w_i} B_i \to C_i \to T A_i\}_{i \in I}$ be an $\alpha$-small collection of distinguished triangles in $T$. By Lemma 1.1.29, their coproduct $\bigsqcup A_i \to \bigsqcup B_i \to \bigsqcup C_i \to T \bigsqcup A_i$ is also distinguished. By definition, $w_i \in W(J)$ for all $i \in I$ if and only if $C_i \in J$ for all $i \in I$, and $\bigsqcup w_i \in W(J)$ if and only if $\bigsqcup C_i \in J$. It is now clear that (i) and (ii) state equivalent conditions.

1.3.11. Definition. A subcategory $L \subseteq T$ is $\alpha$-localizing if it is closed under taking $\alpha$-small coproducts (as in condition (i) of the previous lemma).

1.3.12. Remark. If $\alpha > \aleph_0$, an $\alpha$-localizing subcategory $L \subseteq T$ is always thick: let $A \in L$ with $A \cong A_1 \oplus A_2$ in $T$. Let $e : A \to A$ be the idempotent which is the identity on $A_1$ and zero on $A_2$. As in the proof of Proposition 1.1.37, the homotopy colimit of the countable sequence $A \xrightarrow{e} A \xrightarrow{e} A \xrightarrow{e} \cdots$ is isomorphic to $A_1$, and it belongs to $L$, since it is constructed out of copies of $A \in L$ by taking countable coproducts and cones.

1.3.13. Corollary. If $L \subseteq T$ is $\alpha$-localizing, the Verdier quotient $T/L$ has arbitrary $\alpha$-small coproducts and the quotient functor $T \to T/L$ preserves them.

Proof. Corollary 1.2.10 and Lemma 1.3.10.
1.4. Bousfield localization

Roughly following [Nee01, §9], we explore the well-known interesting implications of the following hypothesis: the quotient functor \( T \to T/J \) has a right adjoint. As it turns out, the right adjoint is automatically fully faithful, so that we may identify the localization \( T/J \) with a triangulated subcategory of \( T \). In particular, \( T/J \) is locally small if \( T \) is. This ‘internal’ localization is the original one from homotopy theory, see e.g. [Mar83, §7] for references.

1.4.1. Notation. Let \( \mathcal{E}, \mathcal{E}' \subseteq T \) be classes of objects. We write

\[ \mathcal{E} \perp \mathcal{E}' :\equiv T(\mathcal{E}, \mathcal{E}') = 0 :\equiv T(E, E') \quad \forall E \in \mathcal{E}, \forall E' \in \mathcal{E}' \]

and similarly with single objects, etc., in the obvious way. Moreover, we’ll use the notation

\[ \mathcal{E}^\perp := \{ A \in T \mid \mathcal{E} \perp A \} \quad , \quad \mathcal{E}^\perp := \{ A \in T \mid A \perp \mathcal{E} \} \]

for the right orthogonal, resp. the left orthogonal3 of \( \mathcal{E} \).

1.4.2. Remark. Both \( \mathcal{E}^\perp \) and \( \mathcal{E}^\perp \) are automatically thick triangulated subcategories of \( T \). Moreover, \( \mathcal{E}^\perp \) is closed under the formation of products and \( \mathcal{E}^\perp \) under the formation of coproducts.

Before assuming that we have a right adjoint, we need some preparation.

1.4.1. Abstract complementation. We adapt the beginning of [BBD82], taking care to make precise statements.

1.4.3. Lemma. ([BBD82, Prop. 1.1.9].) Consider the following (solid arrow) diagram, where the rows are distinguished triangles.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & TA \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} & & \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & TA'
\end{array}
\]

Then the following are equivalent:

(i) \( g'bf = 0 \).
(ii) There exists a morphism \( a : A \to A' \) such that \( f'a = bf \).
(iii) There exists a morphism \( c : C \to C' \) such that \( cg = g'b \).
(iv) We may complete the diagram to a morphism of triangles.

If these conditions hold and if moreover \( T(TA, C') = 0 \), then the morphisms \( a \) and \( c \) are unique.

Proof. Obviously (iv) \( \Rightarrow \) (iii) \( \land \) (ii). Each of (ii) or (iii) implies (i) by Lemma 1.1.10, and (i) \( \Rightarrow \) (iv) by Proposition 1.1.11. Thus (i)-(iv) are all equivalent. For the second part, consider the long exact sequence obtained by applying \( T(-, C') \) to the first distinguished triangle:

\[ T(A, C') \xrightarrow{Tf} T(B, C') \xrightarrow{Tg} T(C, C') \xrightarrow{Tc} T(TA, C') \]

If \( T(TA, C') = 0 \), the injectivity of \( Tg \) shows that there is at most one \( c \in T(C, C') \) as in (iii), that is, such that \( g^*(c) = g'b \). The uniqueness of \( a \) is obtained similarly: observe that \( T(TA, C') \cong T(A, T^{-1}C') \) because \( T \) is an equivalence and apply the homological functor \( T(A, -) \) to the first row. \( \square \)

This can be used to make functors out of triangles, an exceptional feat.

3Beware that [Nee01] employs the enantiomeric notation.
1.4.4. Corollary. Let $T$ be a triangulated category. Assume that, for every object $A \in T$, a distinguished triangle

$$\Delta_A : L_A \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} R_A \xrightarrow{\delta_A} T(L_A)$$

is given, so that $\{L_A, T(L_A)\}_{A \in T} \perp \{R_A\}_{A \in T}$. Then:

(i) The assignment $A \mapsto \Delta_A = [\varepsilon_A, \eta_A, \delta_A]$ extends uniquely to a functor

$$\Delta = \left( L \xrightarrow{\varepsilon} \id_T \xrightarrow{\eta} R \xrightarrow{\delta} T(L) \right)$$

from $T$ to the category of distinguished triangles of $T$ and morphisms of triangles. In particular, the assignments $A \mapsto L_A$ and $A \mapsto R_A$ extend uniquely to endofunctors $L, R : T \to T$, such that $\varepsilon := \{\varepsilon_A\}_A : L \to \id_T$ and $\eta := \{\eta_A\}_A : \id_T \to R$ are natural transformations.

(ii) The functorial triangle $\Delta$ depends only on the class $E := \{L_A, T(L_A)\}_{A \in T}$.

Proof. (i) Write $\Delta(X) := \Delta_X, L(X) := L_X, R(X) := R_X$ for $X \in T$. Let $f : A \to B$ be any morphism of $T$, and consider the solid arrow diagram

$$\begin{array}{ccc}
\Delta(A) & : & L(A) \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} R(A) \xrightarrow{\delta_A} TL(A) \\
\downarrow \Delta(f) = & & \downarrow f \\
\Delta(B) & : & L(B) \xrightarrow{\varepsilon_B} B \xrightarrow{\eta_B} R(B) \xrightarrow{\delta_B} TB(L). \\
\end{array}$$

Since $L(A) \perp R(B)$ by hypothesis, the composition $\eta_B \varepsilon_A$ is zero. By Lemma 1.4.3 we may fill in the dotted arrows to form a morphism of triangles, as indicated. Since moreover, also by hypothesis, $TL(A) \perp R(B)$, these arrows are the unique ones making the squares to the left and to the right of $f$ commute. This uniqueness implies that $L(fg) = L(f)L(g)$ and $R(f)r(g)$, for any two composable arrows $f$ and $g$.

(ii) Extend the identity $1_A : A \to A$ to the solid arrow diagram

$$\begin{array}{ccc}
\Delta(A) & : & L(A) \xrightarrow{\varepsilon_A} A \xrightarrow{\eta_A} R(A) \xrightarrow{\delta_A} TL(A) \\
\Phi_A := & & \\
\Delta'(A) & : & L'(A) \xrightarrow{\varepsilon_A'} A \xrightarrow{\eta_A'} R'(A) \xrightarrow{\delta_A'} TL'(A). \\
\end{array}$$

Reasoning as above, we see that there is a unique way of extending this to a natural transformation $\Phi : \Delta \to \Delta'$. By uniqueness again, the similarly defined $\Psi : \Delta' \to \Delta$ must be its inverse. \hfill $\square$

1.4.5. Lemma. In the situation of Corollary 1.4.4, assume that there is a triangulated subcategory $\mathcal{L} \subseteq T$ with $L_A \in \mathcal{L}$ and $R_A \in \mathcal{L}^\perp$ for all $A \in T$. Then the functors $L : T \to T$ and $R : T \to T$ factor as

$$L : T \xrightarrow{\tilde{L}} \mathcal{L} \hookrightarrow T, \quad \text{ resp. } \quad R : T \xrightarrow{\tilde{R}} \mathcal{L}^\perp \hookrightarrow T,$$

where $\tilde{L}$ is right adjoint to the inclusion $\mathcal{L} \hookrightarrow T$ and $\tilde{R}$ is left adjoint to the inclusion $\mathcal{L}^\perp \hookrightarrow T$. Moreover, we can choose $\eta$ for the unit of the adjunction $(\tilde{R}, \mathcal{L}^\perp \hookrightarrow T)$ and $\varepsilon$ for the counit of the adjunction $(\mathcal{L} \hookrightarrow T, \tilde{L})$.

Proof. By hypothesis, the functors $L$ and $R$ factor as claimed. Let’s prove that we have the adjunctions.

Claim: If $A \in \mathcal{L}$ then $\varepsilon_A : L(A) \to A$ is an isomorphism.
Indeed, the diagram of solid arrows

\[
\begin{array}{ccc}
L(A) & \xrightarrow{\varepsilon_A} & A \\
\cong & & \cong \\
\downarrow & & \downarrow \\
A & \rightarrow & A \\
\end{array}
\]

completes by Corollary 1.4.4 to a unique isomorphism of triangles, since both rows satisfy the hypothesis for \( \Delta_A \), and the claim follows immediately.

Now, if \( f : A \rightarrow B \) is a morphism in \( \mathcal{T} \) with \( A \in \mathcal{L} \), we see from the diagram

\[
\begin{array}{ccc}
L(A) & \xrightarrow{\varepsilon_A} & A \\
\cong & & \cong \\
\downarrow & & \downarrow \\
L(B) & \xrightarrow{\varepsilon_B} & B \\
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{f} & \\
\downarrow & & \downarrow \\
R(A) & \rightarrow & R(B) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \rightarrow \\
& & \downarrow \\
T L(A) & \rightarrow & T L(B) \\
\end{array}
\]

that \( f \) lifts uniquely along \( \varepsilon_B \) to a morphism \( L(A) \rightarrow B \). Thus composing with \( \varepsilon \) yields a natural isomorphism \( L(A, L(B)) \cong T(A, B) \), showing that \( \bar{L} : \mathcal{T} \rightarrow \mathcal{L} \) is right adjoint to \( \mathcal{L} \subseteq \mathcal{T} \) and that \( \varepsilon \) is the counit of this adjunction. A dual argument (working in \( \mathcal{T}^{\text{op}} \), with \( (\mathcal{L}^\perp)^{\text{op}} \) in the rôle of \( \mathcal{L} \) and using that \( \mathcal{L} \subseteq (\mathcal{L}^\perp) \)) shows that \( \bar{R} : \mathcal{T} \rightarrow \mathcal{L}^\perp \) is left adjoint to \( \mathcal{L}^\perp \subseteq \mathcal{T} \) and that \( \eta \) is the unit of adjunction. \( \square \)

1.4.6. Remark. The inclusions \( \mathcal{L} \subseteq \mathcal{T} \) and \( \mathcal{L}^\perp \subseteq \mathcal{T} \) are triangle functors, so it follows from Lemma 1.4.5 and Proposition 1.1.27 that their adjoints \( \bar{L} \) and \( \bar{R} \), and therefore the compositions \( L \) and \( R \), are canonically triangle functors.

At this point, the reader may want to jump forward and have a look at Definition 1.4.21 and Proposition 1.4.22, in order to see how nicely the above results relate to (Verdier) localization.

1.4.2. Give me a right adjoint... Let \( J \subseteq \mathcal{T} \) be a thick subcategory of \( \mathcal{T} \), and let \( q : \mathcal{T} \rightarrow \mathcal{T}/J \) be the Verdier quotient functor.

1.4.7. Lemma. Let \( B \in \mathcal{J}^\perp \). Then

\[ q : \mathcal{T}(A, B) \rightarrow \mathcal{T}/J(qA, qB) \]

is an isomorphism for all \( A \in \mathcal{T} \).

Proof. Injectivity: Let \( f : A \rightarrow B \) be such that \( q(f) = 0 \). By Lemma 1.2.9, this means that there exists a \( w : A' \rightarrow A \) with \( \text{cone}(w) \in J \) and such that \( fw = 0 \). Applying \( \mathcal{T}(-, B) \) to \( A' \rightarrow A \rightarrow \text{cone}(w) \rightarrow TA' \) we get an exact sequence

\[
\mathcal{T}(A', B) \xrightarrow{w^*} \mathcal{T}(A, B) \xrightarrow{\text{cone}(w)} \mathcal{T}(\text{cone}(w), B)
\]

Since \( J \perp B \) by hypothesis, the third group vanishes and \( w^* \) is injective. Hence we deduce from \( w^*(f) = fw = 0 \) that \( f = 0 \). (Alternatively, use Lemma 1.4.3.)

Surjectivity: Represent an arbitrary morphism \( g \in \mathcal{T}/J(qA, qB) \) by a fraction \( g = [A \xrightarrow{w} A' \xrightarrow{j} B] \), where \( \text{cone}(w) \in J \). Consider the composition \( fw' \) in the diagram

\[
\begin{array}{ccc}
T^{-1} \text{cone}(w) & \xrightarrow{w'} & A' \\
\downarrow & & \downarrow \\
A & \xrightarrow{w} & \text{cone}(w) \\
\downarrow & & \downarrow \\
B & \xrightarrow{j} & \text{cone}(w) \\
\end{array}
\]

where the first row is an exact triangle. Since \( T^{-1} \text{cone}(w) \in J \) and \( B \in \mathcal{J}^\perp \), we have \( fw' = 0 \) and, by the weak cokernel property of \( (A, w) \), there exists some \( f \) with \( fw = f \). But then \( g = q(f)q(w)^{-1} = q(\bar{f}) \), showing that \( q_{A,B} \) is surjective. \( \square \)
1.4.8. **Hypothesis.** Assume for a while that the Verdier quotient \( q : T \to T/J \) has a right adjoint \( q_r : T/J \to T \).

1.4.9. **Lemma.** \( J \perp q_r(T/J) \).

**Proof.** Let \( A \in J \) and \( B \in T/J \). Then there are isomorphisms
\[
T(A, q_r B) \cong T(J(qA, B) \cong 0,
\]
because of the adjunction and because of \( qA \cong 0 \), respectively. \( \square \)

1.4.10. **Lemma.** Let \( \eta_A : A \to q_r qA \) be the unit of the adjunction \((q, q_r)\). Then \( q(\eta_A) \) is an isomorphism.

**Proof.** ([Nee01, Lemma 9.1.7].) One of the Triangular Identities satisfied by every adjunction tells us that
\[
(1.4.11) \quad \left( \begin{array}{c}
qA \\
q(qA)
\end{array} \right) \xrightarrow{\tilde{q}qA} \left( \begin{array}{c}
q_r qA \\
q_r qB
\end{array} \right) \xrightarrow{\tilde{q}_r qB} \left( \begin{array}{c}
qA \\
qB
\end{array} \right) = 1_{qA}
\]
for all \( A \in T \) (where \( \tilde{q} : q_r \to \text{id}_{T/J} \) is the counit of the adjunction). In other words, \( \tilde{q}qA \) is a left inverse of \( q(\eta_A) \). We are going to show that \( \tilde{q}qA \), and therefore \( q(\eta_A) \), is an isomorphism. Consider the following composition:
\[
T/J(qA, qB) \xrightarrow{\Phi} T(A, q_r qB) \xrightarrow{q} T/J(qA, q_r qB) \xrightarrow{(\tilde{q}qB)} T/J(qA, qB).
\]
Here \( A, B \in T \) are arbitrary; the first map \( \Phi \) is the adjunction isomorphism, which is computed by sending a morphism \( g : qA \to qB \) to \( q_r (g) \circ \eta_A \); the second map is the quotient functor, and is an isomorphism by Lemma 1.4.7, since \( q_r qB \in J/ \) by Lemma 1.4.9. If we compute the total horizontal composition on a \( g \in T/J(qA, qB) \), we obtain
\[
g \mapsto \tilde{q}B \circ q(q_r (g) \circ \eta_A) = \tilde{q}B \circ qq_r (g) \circ q(\eta_A).
\]
But from the following commutative diagram
\[
\begin{array}{ccc}
qA & \xrightarrow{q(qA)} & q_r qA \\
\downarrow{q(\eta_A)} & \xrightarrow{\tilde{q}qA} & \downarrow{q_A} \\
qq_r qA & \xrightarrow{\tilde{q}_r qB} & qB \\
\downarrow{qq_r (g)} & \downarrow{g} & \downarrow{\tilde{q}B} \\
qq_r (g) & \xrightarrow{\tilde{q}_r qB} & qB
\end{array}
\]
(courtesy of (1.4.11) and the naturality of \( \tilde{q} \)) we recognize the latter expression as being equal to \( g \). Hence the triple composition above is just the identity of \( T/J(qA, qB) \). Since \( \Phi \) and \( q \) are isomorphisms, so is \( (\tilde{q}qB) \), and therefore, by the arbitrariness of \( A \) and Yoneda, also \( \tilde{q}qB \). \( \square \)

1.4.12. **Corollary.** \( J = J^- \).

**Proof.** The inclusion ‘\( \subseteq \)’ is immediate from the definitions. Now consider an \( A \in T \) with \( A \perp J^- \). Since \( q_r qA \in J^- \) (Lemma 1.4.9), the unit of adjunction \( \eta_A : A \to q_r qA \) is zero. But in \( T/J \) it becomes invertible (Lemma 1.4.10), therefore \( qA \cong 0 \). Now \( J = \text{Ker}(q) \) because it’s thick, so \( A \in J \). \( \square \)

Let \( j : J^- \to T \) denote the inclusion functor.
1.4.13. Proposition. The composition \( \mathcal{J}^\perp \xrightarrow{j} T \xrightarrow{q} T/J \) is an equivalence with quasi-inverse \( q_r : T/J \to \mathcal{J}^\perp \).

\[
\begin{array}{ccc}
\mathcal{J}^\perp & \xrightarrow{j} & T \\
\downarrow \cong & & \downarrow q \\
T & \xrightarrow{q} & T/J
\end{array}
\]

**Proof.** By Lemma 1.4.7, the composition \( q \circ j \) is fully faithful. Now let \( A \in T \) be any object, and build a distinguished triangle in \( T \) containing the unit \( \eta_A : A \to q_r qA \) of the adjunction \( (q, q_r) \), as follows:

\[
(1.4.14) \quad \hat{A} \xrightarrow{\eta_A} A \xrightarrow{q_r qA} T\hat{A}.
\]

By Lemma 1.4.10, \( q(\eta_A) \) is an isomorphism and therefore \( q\hat{A} \cong 0 \). Thus \( \hat{A} \in \mathcal{J} \), since \( \mathcal{J} \) was assumed to be thick. Assume now that \( A = jA \in J^\perp \). In this case two out of three objects in (1.4.14) belong to the triangulated subcategory \( J^\perp \), and therefore so does the third one, \( \hat{A} \). Hence \( \hat{A} \in J \cap J^\perp \), which implies \( \hat{A} \cong 0 \). In conclusion, we see that \( \eta_A \) restricts to an isomorphism \( \eta : id_{\hat{A}} \cong q_r \circ (qj) \).

On the other hand, for every object \( B = qB \in T/J \) we have \( q_r qB \in J^\perp \) (Lemma 1.4.9 again). Hence we may write \( q_r q = jq_q \) when we understand \( q_r \) as a functor \( T/J \to J^\perp \). By Lemma 1.4.10, applying \( q \) to \( \eta \) gives an isomorphism \( id_{T/J} \to q_r q = qj \circ q_r \). \( \square \)

1.4.15. Theorem. (Bousfield Localization). Let \( \mathcal{J} \) be a thick subcategory of a triangulated category \( T \). Then the following are equivalent:

(i) The quotient functor \( q : T \to T/J \) has a right adjoint.

(ii) The inclusion functor \( i : J \to T \) has a right adjoint.

(iii) For every object \( A \in T \) there is a distinguished triangle

\[
(1.4.16) \quad A \xrightarrow{i} A_j \xrightarrow{\eta} T(A_j) \xrightarrow{\alpha} A_{J^\perp} \to A_{J^\perp} \xrightarrow{\alpha} T(A_{J^\perp})
\]

with \( A_j \in J \) and \( A_{J^\perp} \in J^\perp \).

If the conditions (i)-(iii) are satisfied, the distinguished triangle (1.4.16) is functorial in \( A \in T \) (in the sense of Corollary 1.4.4) and is isomorphic to

\[
(1.4.17) \quad ii_r \xrightarrow{\varepsilon} id_T \xrightarrow{\eta} q_r q \xrightarrow{T(\alpha)} Tii_r,
\]

where \( \varepsilon \) is the counit of the adjunction \( (i, i_r) \) and \( \eta \) is the unit of \( (q, q_r) \).

**Proof.** “(iii)\( \Rightarrow \)(ii)\( \land \)(i)”:

Having chosen for each object \( A \) a triangle as in (1.4.16), we may apply Corollary 1.4.4 and Lemma 1.4.5 with \( L := \mathcal{J} \). Thus the assignment \( A \mapsto A_j \) and \( A \mapsto A_{J^\perp} \) extend to triangle functors \( L, R : T \to T \) factoring as

\[
L : T \xrightarrow{i} J \xrightarrow{\iota} T \quad \text{and} \quad R : T \xrightarrow{j} J^\perp \xrightarrow{\iota} T,
\]

respectively, with adjunctions \( (i, i_r) \) and \( (j, j_r) \). In particular the inclusion \( i : J \hookrightarrow T \) has the right adjoint \( i_r \), so (ii) holds. On the other hand, we still know nothing about the quotient functor \( q \). But \( A_j \in J \), so \( q \) sends the map \( \rho : A \to A_{J^\perp} \) to a natural isomorphism \( q(\rho) : T/J(qB, qA) \cong T/J(qB, qA_{J^\perp}) \). Since \( A_{J^\perp} \in J^\perp \) we also have \( q : T(B, A_{J^\perp}) \cong T/J(B, A_{J^\perp}) \) (Lemma 1.4.7). Combining the two, we get a natural isomorphism \( T/J(qB, A) \cong T(B, A_{J^\perp}) \) \( \text{def} \) \( T(B, R(A)) \) saying that \( R : T \to T \) descends to a functor \( T/J \to T \), right adjoint to \( q : T \to T/J \) as required by (i).
“(i)⇒(iii)”: Consider the distinguished triangle (1.4.14) for an arbitrary object $A \in T$. Then $\tilde{A} \in \text{Ker}(q) = J$ (Lemma 1.4.10) and $q_!qA \in J^\perp$ (Lemma 1.4.9), so (1.4.14) has the desired form (1.4.16).

“(ii)⇒(iii)”: Let $\varepsilon_A : ii_r A \to A$ be the counit of the adjunction $(i, i_r)$, choose a distinguished triangle

$$(1.4.18) \quad ii_r A \xrightarrow{\varepsilon_A} A \xrightarrow{\text{cone}(\varepsilon_A)} Tii_r A.$$

For every $X \in J$, consider the long exact sequence obtained by applying $T(X, -)$ to (1.4.18). The maps induced by $\varepsilon_A$ are isomorphisms, because this is how the adjunction isomorphism $J(-, i_r A) \cong T(i(-), A)$ is obtained. Therefore $T(X, \text{cone}(\varepsilon_A)) = 0$, showing that cone$(\varepsilon_A) \in J^\perp$. Since $ii_r A \in J$, (1.4.18) has the desired form (1.4.17).

Now assume that (i)-(iii) all hold, and write $(q, q_r)$ and $(i, i_r)$ for the two adjunctions. Both rows in

$$ii_r A \xrightarrow{\varepsilon_A} A \xrightarrow{\text{cone}(\varepsilon_A)} Tii_r A$$

$$T^{-1}\text{cone}(\eta_A) \xrightarrow{\eta_A} q_!qA \xrightarrow{\text{cone}(\eta_A)} A$$

are distinguished triangles satisfying (iii), so by uniqueness (Corollary 1.4.4) both are canonically isomorphic to (1.16), to each other and therefore to (1.17).

1.4.19. Remark. Note that the situation is beautifully symmetric. If a quotient functor $q : T \to T/J$ has a right adjoint, then the equivalence of Proposition 1.4.13 identifies it with a left adjoint for the inclusion $j : J^\perp \to T$. Dualizing, this means that the inclusion $(J^\perp)^{\text{op}} \to T^{\text{op}}$ has a right adjoint. But then we may apply “(ii)⇒(i)” in the Theorem to the thick subcategory $(J^\perp)^{\text{op}}$ of $T^{\text{op}}$, and conclude that the quotient

$$(1.4.20) \quad T^{\text{op}} \to T^{\text{op}}/(J^\perp)^{\text{op}}$$

has a right adjoint, i.e., $T \to T/J^\perp$ has a left adjoint. Conversely, if this holds then (1.4.20) has a right adjoint and the pair $(J^\perp)^{\text{op}} \subseteq T^{\text{op}}$ satisfies Hypothesis 1.4.8, so that all the previous results apply to them. These include (using the identification $J = \perp(J^\perp)$ of Corollary 1.4.12 and the fact that dualization exchanges the right and the left orthogonals) that $T \to T/J$ has a right adjoint, and that $J \to T \to T/J^\perp$ is an equivalence with quasi-inverse induced by the left adjoint of $T \to T/J$.

1.4.3. Complementary pairs. I have learned from [MN06] of another way of framing the Bousfield situation, which perhaps best captures the symmetry. Let $T$ be any triangulated category.

1.4.21. Definition. A complementary pair $(\mathcal{L}, \mathcal{R})$ in $T$ is a pair of thick subcategories of $T$, such that

1. $\mathcal{L} \perp \mathcal{R}$, that is: $T(X, Y) = 0$ for all $X \in \mathcal{L}$ and all $Y \in \mathcal{R}$,
2. for every object $A \in T$ there exists a distinguished triangle

$$(L_A \xrightarrow{} A \xrightarrow{} R_A \xrightarrow{} T(L_A))$$
with \( L_A \in \mathcal{L} \) and \( R_A \in \mathcal{R} \).

We choose to call a triangle as in (2) a **gluing triangle** for \( A \) and \((\mathcal{L}, \mathcal{R})\).

1.4.22. **Proposition.** ([MN06, Prop. 2.9]). If \((\mathcal{L}, \mathcal{R})\) is a complementary pair in \( T \), the following self-dual statements hold true:

(i) \( \mathcal{L} = \perp \mathcal{R} \) and \( \mathcal{R} = \perp \mathcal{L} \). In particular, \( \mathcal{L} \) and \( \mathcal{R} \) determine each other, \( \mathcal{L} \) is closed under taking coproducts, and \( \mathcal{R} \) is closed under taking products in \( T \).

(ii) For each object \( A \), the gluing triangle is uniquely determined up to unique isomorphism. It follows that the assignment \( A \mapsto L_A \) determines a functor \( L : T \to \mathcal{L} \), and \( A \mapsto R_A \) a functor \( R : T \to \mathcal{R} \).

(iii) The functor \( L \) is right adjoint to the inclusion \( \mathcal{L} \hookrightarrow T \), and \( R \) is left adjoint to the inclusion \( \mathcal{R} \hookrightarrow T \). In particular, \( L \) and \( R \) define triangle functors.

(iv) The functor \( L \) descends to a triangle equivalence \( T/\mathcal{R} \sim T \), quasi-inverse to \( L : T \to T/\mathcal{R} \); dually, \( R \) descends to a triangle equivalence with quasi-inverse \( R : T \to T/\mathcal{L} \).

(v) The triangle functor \( T/\mathcal{L} \sim \mathcal{R} \hookrightarrow T \) is right adjoint to the localization \( T \to T/\mathcal{R} \), and \( T/\mathcal{R} \sim \mathcal{L} \hookrightarrow T \) is left adjoint to \( T \to T/\mathcal{R} \).

**Proof.** We have already proved everything (only with slightly different notation). Note that half of the assertions follow by duality, since in \( T^{\text{op}} \) the roles of \( \mathcal{L} \) and \( \mathcal{R} \) become exchanged. For a start, the hypotheses (1) and (2) of the complementary pair together with Lemma 1.4.5 yield (ii) and (iii). The implication ‘(iii) ⇒ (i)’ in Theorem 1.4.15 (with \( J := \mathcal{L} \)) provides \( q : T \to T/\mathcal{L} \) with a right adjoint, given on objects by \( A \mapsto R_A \), i.e. by \( R \). This proves point (v). Thus the thick subcategory \( \mathcal{L} \) satisfies Hypothesis 1.4.8, and we get: (i) by Corollary 1.4.12, and (iv) by Proposition 1.4.13.

1.4.23. **Remark.** Meyer and Nest give a no-prisoners blitzkrieg proof in [MN06]. In particular they avoid discussing calculus of fractions and Verdier localization, by showing directly that \( L : T \to \text{Im}(L) \) and \( R : T \to \text{Im}(R) \) satisfy the universal property of the localizations \( T \to T/\mathcal{R} \) and \( T \to T/\mathcal{L} \), respectively. Still, they need the Octahedron Axiom in the form of Verdier’s Exercise (Prop. 1.1.21).

1.4.4. **A more classical terminology.** Algebraic topologists are surely more familiar with the following terminology, which we briefly discuss for completeness:

1.4.24. **Definition.** A **(Bousfield) localization functor** is a pair \((R, \rho)\) where \( R : T \to T \) is an additive functor\(^4\) and \( \rho : \text{id}_T \to R \) a natural transformation, such that \( R(\rho) : R \to R^2 \) is an isomorphism and moreover \( R(\rho) = \rho_R \).

---

\(^4\)Needless to say, a Localization functor is usually denoted by \( L \), not by \( R \); of course we are trying hard not to confuse the reader by introducing mutually inconsistent sets of notations...
1.4.25. Remark. Given a localization functor \((R, \rho)\), an object \(A\) is traditionally called \(R\)-acyclic iff \(R(A) \cong 0\); it is \(R\)-local iff \(\rho_A : A \to R(A)\) is an isomorphism. Note that the full subcategory of \(R\)-acyclic objects is \(\text{Ker}(R)\), and that of \(R\)-local ones is \(\text{Im}(R)\) (indeed, \(R\)-local objects obviously are in the essential image of \(R\); conversely, given an isomorphism \(f : A \to R(B)\) we see from the commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & R(B) \\
\rho_A \downarrow & \cong & \rho_{R(B)} \downarrow \\
R(A) & \xrightarrow{R(f)} & R^2(B)
\end{array}
\]

that \(A\) is \(R\)-local). In particular they are both thick triangulated subcategories of \(T\) (\(\text{Im}(R)\) because of the “idempotency” of \(R\).)

We want to show that Bousfield localization functors are nothing but incognito complementary pairs of subcategories.

1.4.26. Lemma. ([BIK07, §3]). Let \(R : T \to T\) be a functor and let \(\rho : \text{id}_T \to R\) be a natural transformation. Then the following conditions are equivalent:

(i) \((R, \rho)\) is a Bousfield localization functor.

(ii) There exists an adjunction \(F : T \rightleftarrows S : G\) such that: (a) the right adjoint \(G\) is fully faithful, (b) \(R = GF\), and (c) \(\rho : \text{id}_T \to GF\) is the unit of adjunction.

In particular, a Bousfield localization functor is automatically and canonically a triangle functor (cf. Prop. 1.1.27).

Proof. (i)⇒(ii): Let \(G : S := \text{Im}(R) \hookrightarrow T\) be the inclusion of \(R\)-local objects (cf. the above remark) and let \(F : T \to S\) be the restriction of \(R\) to its essential image. Then it is easily checked that the two assignments

\[
\begin{align*}
S(FA, B) &\to T(A, GB) \\
T(A, GB) &\to S(FA, B)
\end{align*}
\]

\(f \mapsto G(f) \circ \rho_A\)

\(\rho_B^{-1} \circ F(g) \mapsto g\)

define mutually inverse natural bijections, for \(A \in T\) and \(B \in \text{Im}(R)\).

(ii)⇒(i): Let \(\rho : \text{id}_T \to GF\) and \(\lambda : FG \to \text{id}_S\) denote the unit, resp. the counit, of the adjunction, and set \(R := GF\). The triangular identities of the adjunction state that

\[
\begin{align*}
(F & E \rho) \quad \quad (G \rho G) \\
&\cong \quad \quad (G \lambda F) \quad \quad \text{id}_F \\
&\cong \quad \quad \text{id}_G
\end{align*}
\]

Since \(G\) is fully faithful, \(\lambda\) is invertible ([Mac98, Thm. IV.3.1]), and by (1.4.27) so is \(F\rho\) and thus \(R\rho = GF\rho\). The computation

\[
R\rho = GF\rho \quad \text{(1.4.27)} \quad (G\lambda F)^{-1} \quad \text{(1.4.28)} \quad \rho_{GF} = \rho_R
\]

completes the proof that \((R, \rho)\) is a Bousfield localization. \(\square\)

Now we go full circle, proving the equivalence of the three terminologies.

1.4.29. Proposition. For a triangulated category \(T\), the following three sets of data are equivalent:

(i) A Bousfield localization functor \((R, \rho)\).

(ii) A complementary pair \((L, R)\) of subcategories (Def. 1.4.21), with a choice of the morphism \(A \to R_A\) in the gluing triangle of each object \(A \in T\).

(iii) A Verdier localization functor \(q : T \to S\) with a right adjoint \(q_r\).
1.5. COHERENT FUNCTORS 25

Proof. (iii)⇒(ii): If a Verdier quotient $q : T \to S$ has a right adjoint, the pair $(\ker(q), \ker(q)^\perp)$ is complementary by the implication ‘(i)⇒(iii)’ in Theorem 1.4.15.

(ii)⇒(i): Any such choice of $A \to R_A$ extends uniquely by Lemma 1.4.5 to a functor $R : T \to T$ together with a natural transformation $\rho : \text{id}_T \to R$, such that $R$ factors as $T \xrightarrow{\tilde{R}} L^\perp \hookrightarrow T$, where $\tilde{R}$ is left adjoint to the inclusion $L^\perp \hookrightarrow T$, and where $\rho$ is the unit of this adjunction. But then the conditions in Lemma 1.4.26(ii) are satisfied (with $S := L^\perp$, $F := \tilde{R}$ and $G := (L^\perp \hookrightarrow T)$) and therefore $(R, \rho)$ is a Bousfield localization.

(i)⇒(iii): The functor part of a localization functor $(R, \rho)$ factors as

$$
\begin{array}{ccc}
T & \xrightarrow{R} & \text{Im}(R) \\
q \downarrow & & \downarrow \rho \\
T/\ker(R) & \xrightarrow{\tilde{R}} & T
\end{array}
$$

The induced functor $\tilde{R}$ is an equivalence: It is essentially surjective by definition; any morphism $f : R(A) \to R(B)$ in $\text{Im}(R) \subseteq T$ comes via $\tilde{R}$ from the fraction $[p_B^{-1} \circ f \circ A]$, since $f$ is also a morphism in $T$; on the other hand, if $0 = \tilde{R}([f \circ s^{-1}]) \overset{\text{Def.}}{=} R(f)R(s)^{-1}$, then also $f = 0$. Hence $\tilde{R}$ is fully faithful as well. By (the proof of) Lemma 1.4.26, we know that the functor $R$ has the inclusion $\text{Im}(R) \hookrightarrow T$ for a right adjoint. We may now transport the inclusion $\text{Im}(R) \hookrightarrow T$ along the equivalence $\tilde{R}$ to define a (fully faithful) right adjoint of the Verdier quotient functor $q$. □

1.4.30. Remark. Note that the localizing pair corresponding to a Bousfield localization $(R, \rho)$ is the pair $(\ker(R), \text{Im}(R))$ with $R$-acyclics on the left hand side and $R$-local objects on the right hand side.

1.5. Coherent functors

We briefly review some classical results of Peter Freyd [Fre66]. We refer the reader to [Bel00] and to [Nee01, §5] for exhaustive and somewhat complementary treatments of the Freyd category and its dual concept. See [Ver96, II.3] for another definition of the Freyd category, which uses images rather than quotients. Moreover, both definitions can be given as certain additive quotients of the category of morphisms (see loc. cit.).

Let $C$ be a (locally small, but not necessarily small) additive category.

1.5.1. Definition. A functor $F : C^{\text{op}} \to \text{Ab}$ is coherent (or finitely presentable) if it is a quotient of representable functors, i.e., if there is an exact sequence

$$
C(-, A_1) \xrightarrow{F} C(-, A_0) \xrightarrow{F} 0
$$

of functors $C^{\text{op}} \to \text{Ab}$, for some $A_0, A_1 \in C$. We remind the reader that the (possibly very large) category $\text{Add}(C^{\text{op}}, \text{Ab})$ of additive functors $C^{\text{op}} \to \text{Ab}$ is abelian, where exactness (and therefore kernels, cokernels, etc.) is seen objectwise in $\text{Ab}$. The Freyd category of $C$, denoted $\text{Coh}(C)$ (or often $A(C)$), is the full subcategory in $\text{Add}(C^{\text{op}}, \text{Ab})$ of coherent functors. The additive Yoneda functor equips it with a natural fully faithful embedding $h : C \to \text{Coh}(C)$.

1.5.2. Lemma. An object of $\text{Coh}(C)$ is projective if and only if it is a direct summand of a representable functor. Thus $\text{Coh}(C)$ has enough projectives.
Proof. Consider a representable functor $h(A) = \mathcal{C}(-, A)$, and let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\text{Add}(\mathcal{C}^{\text{op}}, \text{Ab})$ (i.e., it is objectwise exact). If we apply $\text{Add}(\mathcal{C}^{\text{op}}, \text{Ab})(h(A), -)$ to this sequence, by Yoneda we obtain $0 \to F'(A) \to F(A) \to F''(A) \to 0$, which is exact. This shows that $h(A)$ is projective in $\text{Add}(\mathcal{C}^{\text{op}}, \text{Ab})$, so a fortiori it is projective in $\text{Coh}(\mathcal{C})$. By the very definition of coherent functors, we see that representable functors already provide enough projectives. Conversely, let $P \in \text{Coh}(\mathcal{C})$ be projective with presentation $h(A_1) \to h(A_0) \xrightarrow{\sigma} P \to 0$. By the projectivity of $P$, there exists a splitting $\sigma : P \to h(A_0)$ of $\pi$, whereby we deduce that $P$ is a direct summand of $h(A_0)$.

1.5.3. Remark. Although $\text{Add}(\mathcal{C}^{\text{op}}, \text{Ab})$ is obscenely huge\(^5\), its full subcategory $\text{Coh}(\mathcal{C})$ is locally small. In fact, if $\varphi : F \to G$ is a morphism of coherent functors, there exist epimorphisms $\pi : h(A) \to F$ and $\rho : h(B) \to G$ for some $A, B \in \mathcal{C}$; since the representable $h(A)$ is projective, the morphism $\varphi \pi$ lifts along $\rho$ to some $\tilde{\varphi} : h(A) \to h(B)$. It follows by Yoneda that $\text{Coh}(\mathcal{C})(F, G) = \text{Nat}(F, G)$ is a quotient of the small set $\text{Nat}(h(A), h(B)) \cong \mathcal{C}(A, B)$, so it is itself small.

1.5.4. Lemma. Every morphism in $\text{Coh}(\mathcal{C})$ has a cokernel. Moreover, $(\text{Coh}(\mathcal{C}), h)$ is universal for this property: Every additive functor $f : \mathcal{C} \to \mathcal{A}$ to an additive category with cokernels extends uniquely (up to unique isomorphism) along $h$ to a right exact functor $\tilde{f} : \text{Coh}(\mathcal{C}) \to \mathcal{A}$.

Proof. Let $\varphi : F \to G$ be a morphism in $\text{Coh}(\mathcal{C})$. Having chosen presentations for $F$ and $G$, we may, in the usual way, use the projectivity of representable functors to lift $\varphi$ to a commutative diagram

\[
\begin{CD}
 h(A_1) @>>> h(A_0) @>>> F @>>> 0 \\
 @VVV @VVV @VV{\varphi}V \\
 h(B_1) @>>> h(B_0) @>>> G @>>> 0.
\end{CD}
\]

Then the obvious sequence

\[
\begin{CD}
 h(B_1 \oplus A_0) @>>> h(B_0) @>>> \text{Coker}(\varphi) @>>> 0
\end{CD}
\]

is exact, showing that $\text{Coker}(\varphi)$ is coherent (the cokernel is taken in the abelian category $\text{Add}(\mathcal{T}^{\text{op}}, \text{Ab})$, of course). Therefore $\text{Coh}(\mathcal{C})$ has cokernels. Given an additive functor $f : \mathcal{C} \to \mathcal{A}$ to a category with cokernels, the only way to extend $f$ to a right exact $f : \text{Coh}(\mathcal{C}) \to \mathcal{A}$ is via $f(F) := \text{Coker}(f(a) : f(A_1) \to f(A_0))$, where $h(a) : h(A_1) \to h(A_0) \to F \to 0$ is some presentation of $F$. The rest follows from the universal property of cokernels.

In particular, for an additive functor $f : \mathcal{C} \to \mathcal{C}'$ there is an essentially unique right exact functor $\text{Coh}(f) : \text{Coh}(\mathcal{C}) \to \text{Coh}(\mathcal{C}')$ making the following diagram commutative (up to isomorphism):

\[
\begin{CD}
 \mathcal{C} @>{f}>> \mathcal{C}' \\
 @VV{h}V @VV{h}V \\
 \text{Coh}(\mathcal{C}) @>{\text{Coh}(f)}>> \text{Coh}(\mathcal{C}').
\end{CD}
\]

Thus $\text{Coh}$ defines a pseudofunctor from additive categories to categories with cokernels and right exact functors.

1.5.5. Lemma. Let $\mathcal{C}$ be an additive category.

(i) If $\mathcal{C}$ has weak kernels, its Freyd category $\text{Coh}(\mathcal{C})$ is abelian.

---

\(^5\)This is not a technical term: don’t google it!
(ii) If \( C \) has \( \alpha \)-indexed coproducts (for some cardinal \( \alpha \)), then \( \text{Coh}(C) \) also has \( \alpha \)-indexed coproducts and the Yoneda embedding \( h : C \to \text{Coh}(C) \) preserves them.

**Proof.** (i) By definition, \( \text{Coh}(C) \) is a full (!) subcategory of abelian category \( \text{Add}(\text{C}^{\text{op}}, \text{Ab}) \), and we have seen that it is closed under the formation of cokernels. In order to prove it abelian, it’s enough to show that it has kernels. These can be constructed as follows, if weak kernels are available in \( C \). Let the commutative square

\[
\begin{array}{ccc}
A_1 & \xrightarrow{a} & A_0 \\
\downarrow{f_1} & & \downarrow{f_0} \\
B_1 & \xrightarrow{b} & B_0
\end{array}
\]

in \( C \) represent a morphism \( \varphi : \text{Coker}(h(a)) \to \text{Coker}(h(b)) \) of \( \text{Coh}(C) \). Choose weak kernels \( W_0 \to A_0 \) and \( W_1 \to A_1 \) of \( f_0 \) and \( bf_1 \), respectively. By the ‘versal’ property of weak kernels, we may fill in the diagram with a morphism \( w : W_1 \to W_0 \). We leave it to the reader to check that \( \text{Coker}(h(w)) \) verifies the universal property of the kernel of \( \varphi \).

(ii) Let \( \{ F_i \}_{i \in I} \) be some family of coherent functors with presentations

\[
h(B_i) \xrightarrow{h(f_i)} h(A_i) \xrightarrow{f_i} F_i \xrightarrow{i} 0,
\]

and assume that the coproducts \( \coprod_i X_i \) and \( \coprod_i Y_i \) exist in \( C \). Then the cokernel of

\[
h(\coprod_i B_i) \xrightarrow{h(\coprod_i f_i)} h(\coprod_i A_i)
\]

verifies the universal property of the coproduct of \( \{ F_i \}_{i} \) in \( \text{Coh}(C) \). In particular the coproduct of a family \( \{ h(A_i) \}_{i} \) of representable functors is given by \( h(\coprod_i A_i) \), so that \( h \) sends coproducts of \( C \) to coproducts of \( \text{Coh}(C) \).

The next summarizing proposition states roughly that \( \text{Coh}(-) \) is the inverse of the operation \( \text{Proj}(-) \) of considering projective objects.

1.5.6. Proposition.  (i) If \( C \) is an additive category with weak kernels, the category \( \text{Coh}(C) \) is abelian and its projective objects are precisely the direct summands of representable functors. Thus \( C \simeq \text{Im}(h) = \text{Proj}(\text{Coh}(C)) \) if \( C \) is idempotent complete.

(ii) Conversely, if \( \mathcal{A} \) is an abelian category with enough projectives the inclusion \( \text{Proj}(\mathcal{A}) \subseteq \mathcal{A} \) extends to an equivalence \( \text{Coh}(\text{Proj}(\mathcal{A})) \simeq \mathcal{A} \).

**Proof.** We have already proved part (i). If the abelian category \( \mathcal{A} \) has enough projectives, every \( A \in \mathcal{A} \) has a presentation \( P_1 \to P_0 \to A \to 0 \) with \( P_1, P_0 \in \text{Proj}(\mathcal{A}) \), which can be used to define \( f(A) := \text{Coker}(hP_1 \to hP_0) \in \text{Coh}(\text{Proj}(\mathcal{A})) \).

It is readily checked that \( f \) extends to a well-defined functor with \( fi = h \), as depicted below \( (i : \text{Proj}(\mathcal{A}) \subseteq \mathcal{A} \) being the inclusion).

\[
\begin{array}{ccc}
\text{Proj}(\mathcal{A}) & \xrightarrow{i} & \mathcal{A} \\
\downarrow{h} & & \downarrow{f} \\
\text{Coh}(\text{Proj}(\mathcal{A})) & \xrightarrow{i} &
\end{array}
\]

By the universal property of \( \text{Coh} \), the inclusion \( i \) extends to a right exact \( \tilde{i} \) such that \( \tilde{ih} = i \) and therefore \( if \cong \text{id} \). By construction of \( f \) we have \( f\tilde{i} \cong \text{id} \). □
1. TRIANGULATED CATEGORIES

1.5.1. The universal homological functor. Every triangulated category $\mathcal{T}$ has weak cokernels and weak kernels, given for a morphism $f : A \to B$ by any cone $B \to \text{cone}(f)$, respectively any ‘homotopy fiber’ $T^{-1}\text{cone}(f) \to A$. We may thus apply the previous results, as well as their duals. To be more precise, the dual of the Freyd category, say $\mathbf{Coh}'(\mathcal{T})$, is the full subcategory of $\text{Add}(\mathcal{T}^{\text{op}}, \mathbf{Ab})$ of those functors $F$ having a corepresentation by representables, i.e. an exact sequence $0 \to F \to h(A_1) \to h(A_0)$.

Write $h' : \mathcal{T} \to \mathbf{Coh}'(\mathcal{T})$ for the restriction of the Yoneda embedding. Clearly $(\mathbf{Coh}'(\mathcal{T}), h')$ enjoy results dual to those proved so far (in the same generality). In the following, we play the two constructions against each other in order to get a nice symmetrical statement. But first we need:

1.5.7. Lemma. If $H : \mathcal{T} \to \mathcal{A}$ is a homological functor, the induced functor $\tilde{H} : \mathbf{Coh}(\mathcal{T}) \to \mathcal{A}$ is exact.

Proof. We already know that $\tilde{H}$ is right exact. The following proof of “$H$ homological $\Rightarrow \tilde{H}$ left exact” is taken from [Nee01]. First we prove that $\tilde{H}$ is exact on sequences of the form $0 \to F' \to h(A) \to F \to 0$. Since $F'$ is coherent, we may cover it with a $h(A') \to F'$. The composite $h(A') \to F' \to h(A)$ has the form $h(a)$ for some arrow $a : A' \to A$ of $\mathcal{T}$, which we may complete to a distinguished triangle $A'' \xrightarrow{a'} A' \xrightarrow{a} A \to TA''$. Then $h(A'') \xrightarrow{h(a'' \cdot a)} h(A') \xrightarrow{h(a)} h(A)$ is exact, from which we deduce that $F' = \text{Coker}(h(a'))$. Since $\tilde{H}$ is right exact, $\tilde{H}(F') = \text{Coker}(\tilde{H}h(a'))$ and $\tilde{H}(F) = \text{Coker}(\tilde{H}h(a))$. Using that $\tilde{H}h = H$, we obtain a commutative diagram

\[
\begin{array}{ccc}
H(A'') & \xrightarrow{H(a'')} & H(A') \\
\downarrow & & \downarrow \\
\tilde{H}(F') & & \tilde{H}(F) \\
\end{array}
\]

The long row is exact because $H$ is homological, so we conclude that $0 \to \tilde{H}(F') \to \tilde{H}(h(A)) \to \tilde{H}(F) \to 0$ is exact, as wished.

Consider now a general exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathbf{Coh}(\mathcal{T})$. Construct a commutative diagram in $\mathbf{Coh}(\mathcal{T})$ with exact rows and columns:

\[
\begin{array}{cccc}
0 & 0 & 0 & \\
0 & \xrightarrow{h(A')} & \xrightarrow{h(A')} & 0 \\
0 & \xrightarrow{h(A' \oplus A'')} & \xrightarrow{h(A'')} & 0 \\
0 & \xrightarrow{\pi'} & \pi & \xrightarrow{\pi''} \\
0 & \xrightarrow{F'} & \xrightarrow{F} & \xrightarrow{F''} & 0 \\
0 & 0 & 0 &
\end{array}
\]

(choose representable covers $\pi' : h(A') \to F'$ and $\pi'' : h(A'') \to F''$; use the projectivity of $h(A'')$ to lift $\pi''$ along $F \to F''$, thereby obtaining the morphism of short exact sequences $(\pi', \pi, \pi'')$; then apply the Snake Lemma to this morphism$^6$).

$^6$This is how one proves the Horseshoe Lemma in abelian categories.
Now unleash upon it the right exact functor $\tilde{H}$ and get:

$$
\begin{array}{c}
0 \\
\downarrow \\
\tilde{H}(K') & \beta & \tilde{H}(K) & \tilde{H}(K'') & \rightarrow \\
\alpha & \downarrow & \rightarrow & \rightarrow \\
0 & H(A') & H(A' \oplus A'') & H(A') & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
\tilde{H}(F') & \gamma & \tilde{H}(F) & \tilde{H}(F'') & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & \\
\end{array}
$$

The columns are exact because of the special case of the theorem that we have already proved. The central row is exact because it’s the image of a split short exact sequence. Now note that $\alpha$ is mono, being the composition of two monos. But then $\beta$ must be mono too, so the first row is exact after all. Finally, the Snake Lemma applied to the morphism of exact sequences from the first to the second row implies that $\gamma$ is mono. Therefore $\tilde{H}$ is an exact functor. □

1.5.8. Example. The Yoneda functor $T \to \text{Add}(T^{\text{op}}, \text{Ab})$ is homological, because, by Proposition 1.1.20, it sends every distinguished triangle $A \to B \to C \to TA$ to an exact sequence $T(-, A) \to T(-, B) \to T(-, C)$ (we have already used this fact in the above proof). The restrictions $h : T \to \text{Coh}(T)$ and $h' : T \to \text{Coh}'(T)$ are also homological functors, because the two Freyd categories are abelian by Proposition 1.5.6 and its dual.

1.5.9. Proposition. $\text{Coh}(T) \simeq \text{Coh}'(T)$ along the induced functors $\tilde{h}'$ and $\tilde{h}$.

Proof. The category $\text{Coh}(T)$ has kernels (resp., $\text{Coh}'(T)$ has cokernels) because $T$ has weak kernels (resp. weak cokernels). Therefore, by the universal property of $\text{Coh}'$ (resp. of $\text{Coh}$) the functor $h$ (resp. $h'$) induces an essentially unique left exact functor $\tilde{h} : \text{Coh}'(T) \to \text{Coh}(T)$ with $h\tilde{h}' = h$ (resp. an essentially unique right exact functor $\tilde{h}' : \text{Coh}(T) \to \text{Coh}(T)$ with $\tilde{h}'h = h'$).

By the previous lemma (and its dual), $\tilde{h}$ and $\tilde{h}'$ are both left and right exact, so it follows from uniqueness that they must be mutually quasi-inverse equivalences. □

For triangulated categories, the universal property of the Freyd category takes the following form.

1.5.10. Theorem. The Yoneda embedding $h : T \to \text{Coh}(T)$ is the universal homological functor: it is homological, and if $H : T \to A$ is any homological functor to some abelian category $A$, then there exists a unique (up to unique isomorphism)
exact functor $\tilde{H} : \text{Coh}(T) \to A$ such that $\tilde{H} \circ h = H$.

\[
\begin{array}{ccc}
T & \xrightarrow{h} & \text{Coh}(T) \\
\downarrow H & & \downarrow \tilde{H} \\
A & \xleftarrow{\text{Im}(h)} & \text{Coh}(T)
\end{array}
\]

Moreover, the Freyd category $\text{Coh}(T)$ is a Frobenius abelian category: it has enough projectives and enough injectives, and they coincide. They also coincide with $\text{Im}(h) \simeq T$ iff $T$ is idempotent complete.

**Proof.** The first part of the theorem is just Lemma 1.5.4, upgraded by Lemma 1.5.7. The moreover part follows immediately from Proposition 1.5.6(i), together with its dual and Proposition 1.5.9. \qed

1.5.11. **Remark.** The translation functor $T : T \to T$ induces in an obvious way a translation $T : \text{Coh}(T) \to \text{Coh}(T)$ on the Freyd category, making it a stable abelian category (i.e., an abelian category equipped with a self-equivalence) and making the Yoneda functor a stable homological functor (i.e., one equipped with an isomorphism $H \circ T \simeq T \circ H$). In Theorem 1.5.10, we may substitute ‘homological’ with ‘stable homological’ and ‘exact’ with ‘stable exact’.

## 1.6. The Brown representability theorem

1.6.1. **Warning.** The Brown representability we concern ourselves with is usually referred to in the literature as “Brown representability for cohomology”. Confusingly, a different phenomenon (which will not be covered here) concerning the representability of homological functors on compact objects in a compactly generated category is also called Brown representability ("for homology"). Neither is a special case of the other, and the second kind of representability can also be stated in terms of cohomology functors. We refer the reader to [CKN01] for explanations of these and related ideas, for a brief history, and for a comprehensive guide to the literature.

For the following we fix an infinite regular cardinal $\alpha$ with $\alpha > \aleph_0$. By definition (assuming the axiom of choice – which *of course* we do\(^7\)), the regularity of $\alpha$ means the following: if $S = \coprod_{i \in I} S_i$ is a coproduct of sets such that $|I| < \alpha$ and $|S_i| < \alpha$ for every $i \in I$, then also $|S| < \alpha$. For instance every successor cardinal number is regular. Thus $\aleph_1$, the next cardinal number after $\aleph_0 = |\mathbb{N}|$, is regular.

1.6.2. **Definition.** A set $S$ whose cardinality is strictly less than $\alpha$, in symbols $|S| < \alpha$, will be said to be $\alpha$-small. By an $\alpha$-small coproduct in a category we mean a coproduct indexed by an $\alpha$-small set. Let $T$ be a triangulated category with $\alpha$-small coproducts. An $\alpha$-localizing subcategory of $T$ is a triangulated subcategory which is closed under the formation of $\alpha$-small coproducts. We will write $(\mathcal{E})_{\text{loc}}$ for the $\alpha$-localizing subcategory generated by a subclass $\mathcal{E} \subseteq T$. Finally, a cohomological functor $H : T^{\text{op}} \to \text{Ab}$ is $\alpha$-decent on $\mathcal{S}_0$, where $\mathcal{S}_0 \subseteq T$ is some set of objects, if the following two conditions are met:

- **(D1)** $H$ sends $\alpha$-small coproducts of $T$ to (\(\alpha\)-small) products of abelian groups.
- **(D2)** $|H(A)| < \alpha$ for every object $A \in \mathcal{S}_0$.

The use of specifying a set $\mathcal{S}_0$ will become clear in a moment.

---

\(^7\)Try and construct a functor out of a universal property without Choosing; or to invert a fully faithful essentially surjective functor... (for instance).
1.6.3. Remark. By Proposition 1.1.37 (and $\alpha > \aleph_0$), a triangulated category $\mathcal{T}$ with $\alpha$-small coproducts is idempotent complete: every idempotent $e = e^2 : A \to A$ gives rise to a splitting $A \cong \text{Im}(e) \oplus \text{Ker}(e)$. Also, every $\alpha$-localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ is thick by Remark 1.3.12.

1.6.4. Remark. (a) For our future applications to Kasparov theory we will need only the case $\alpha = \aleph_1$. Note that an $\aleph_1$-small coproduct is just a countable coproduct, i.e., one indexed by a (finite or infinite) countable set.

(b) The above terminology was chosen so that one recovers the usual notions of a (small) set, (small) coproduct, localizing subcategory, decent homological functor, and so on, by a simple dirty trick: just choose $\alpha$ to be the “cardinality of a proper class” (or, alternatively, the cardinality of the fixed universe we’re working in). Then $\alpha$-small is the same as “small”. In this way the following $\alpha$-relative results are genuine — although very modest, of course — generalizations of their usual absolute counterparts. Unfortunately this leads our cardinality notation to clash with that of other authors ([MN06], [Nee01]). Note that for the absolute case, condition (D2), and therefore also (Go) below, become void.

In order to state the main result of this section, we still need the following

1.6.5. Definition. ([Kra02].) Let $\mathcal{T}$ be a triangulated category with all $\alpha$-small coproducts. An $\alpha$-small set of objects $S_0 \subseteq \mathcal{T}$ with $T(S_0) = S_0$ $\alpha$-perfectly generates $\mathcal{T}$ if the following three hypotheses hold:

(G1) $S_0$ detects objects: for all $A \in \mathcal{T}$, $\mathcal{S}_0 \perp A \Rightarrow A \cong 0$.

(G2) For every countable set of morphisms $\{f_i : A_i \to B_i\}_i$, if the induced maps $\mathcal{T}(S, A) \to \mathcal{T}(S, B)$ are surjective for all $i$ and all $S \in S_0$, then also

$$(\prod_i f_i)_* : \mathcal{T}(S, \prod_i A_i) \to \mathcal{T}(S, \prod_i B_i)$$

is surjective for all $S \in S_0$.

(Go) $|\mathcal{T}(S, A)| < \alpha$ for all $S \in S_0$ and $A \in \mathcal{T}$ (that is, the representable cohomological functor $\mathcal{T}(\_ , A)$ is $\alpha$-decent on $S_0$ for every object $A \in \mathcal{T}$).

A triangulated category $\mathcal{T}$ with $\alpha$-small coproducts will be said to be $\alpha$-perfectly generated if it is $\alpha$-perfectly generated by some $\alpha$-small set of objects.

The goal of this section is to prove the following ($\alpha$-relative form of a) strong version, due to Krause [Kra02, Thm. A], of the Brown Representability Theorem for cohomology functors. In the next section we will derive from it the ($\alpha$-variant of) Neeman’s Brown Representability Theorem for compactly generated triangulated categories.

1.6.6. Theorem. (Brown Representability.) Let $\mathcal{T}$ be an $\alpha$-perfectly generated triangulated category. For every functor $H : \mathcal{T}^{\text{op}} \to \text{Ab}$ the following are equivalent:

(i) $H$ is representable.

(ii) $H$ is cohomological and $\alpha$-decent on some set $S_0$ of perfect generators.

1.6.7. Remark. In the absolute case (that is, if $\alpha$ is the cardinality of a proper class, or alternatively, if it is the — strongly inaccessible — cardinality of your Grothendieck universe), then the Brown representability theorem can be used to represent arbitrary small products in $\mathcal{T}$. Indeed, in this case for every small family $\{X_i\}$ of objects the cohomological functor $\prod_i \mathcal{T}(\_ , X_i) : \mathcal{T}^{\text{op}} \to \text{Ab}$ is decent on some generating set and is therefore representable. On the other hand, if there exists a cardinal $\beta$ with $\beta < \alpha$ but $2^\beta \geq \alpha$ (e.g. if $\alpha = \aleph_1$), then $|\prod_\beta \mathcal{T}(A, A)| \geq 2^\beta \geq \alpha$ for every nonzero $A \in \mathcal{T}$, showing that the coproduct of $\beta$ copies of $A$ doesn’t exist in $\mathcal{T}$. This and similar annoying problems (cf. Remarks 1.6.28 and 2.4.12) reveal the net drawback of working with a category having only $\alpha$-small coproducts for
some small $\alpha$, rather than with a genuinely compactly generated category having all small coproducts.

**Proof of Theorem 1.6.6.** Clearly (i) implies (ii), because of Axiom (Ga). To prove that (ii) implies (i), we closely follow [Kra02]. Let’s begin with a couple of lemmata. For any additive category $T$, let $\text{Coh}(T)$ be, as in the previous section, the Freyd category of coherent functors on $T$, and let $h : T \to \text{Coh}(T)$ denote the Yoneda embedding. Given a subset $S_0 \subseteq T$ of objects, we will write $S := \text{Add}_\alpha(S_0)$ for the smallest subcategory of $T$ containing $S_0$ and closed under taking $\alpha$-coproduct and direct summands. Let $j : S \to T$ be the inclusion functor. Then $j$ induces a right exact functor $j^* := \text{Coh}(j) : \text{Coh}(S) \to \text{Coh}(T)$ via $S(-, A) \mapsto T(-, A)$.

1.6.8. Lemma. Let $T$ be an additive category with $\alpha$-small coproducts and weak kernels, and let $S_0$ be an $\alpha$-small set of objects satisfying (Ga); write $S := \text{Add}_\alpha(S_0)$. Then

(i) $S$ has weak kernels and $\text{Coh}(S)$ is abelian.

(ii) The restriction $F \mapsto F|_S$ induces an exact functor $j_* : \text{Coh}(T) \to \text{Coh}(S)$.

(iii) $j_*$ is right adjoint to the functor $j^* : \text{Coh}(S) \to \text{Coh}(T)$.

(iv) $j_* \circ j^* \cong \text{id}_{\text{Coh}(S)}$ and $j_*$ induces equivalence $\text{Coh}(T)/\text{Ker}(j_*) \to \text{Coh}(S)$.

**Proof.** Let $A \in T$. We assumed that $|S_0| < \alpha$ and that $|T(S, A)| < \alpha$ for all $S \in S_0$. Hence we may build in $T$ the coproducts $A_S := \coprod_{T(S, A)} S$ and also $\tilde{A} := \coprod_{S \in S_0} A_S$. There is a morphism $p_A := \{p_{f, S \to A, S} : \tilde{A} \to A\}$ having the property that $p_\alpha : T(\tilde{S}, \tilde{A}) \to T(S, A)$ is surjective for all $S \in S_0$, and thus for all $S \in S$. We have constructed an “$S$-approximation” of $A$.

(i): By Lemma 1.5.5(i), we need only show that every $f : S \to S'$ in $S$ has a weak kernel. To find one, choose a weak kernel $C \to S$ in $T$ and compose it with an $S$-approximation $\tilde{C} \to C$.

(ii): Restriction is certainly exact, so we only have to check that if $F \in \text{Coh}(T)$ then $F|_S$ is coherent on $S$, and it is enough to see it for a representable $F = T(-, A)$.

Now choose an $S$-approximation $\tilde{A} \to A$, a weak kernel $K \to \tilde{A}$ for it and an $S$-approximation $\tilde{K} \to K$ in order to obtain an exact sequence $b(\tilde{K})|_S \to h(\tilde{A})|_S \to h(A)|_S \to 0$, showing that $F|_S \in \text{Coh}(S)$.

(iii): Let $F \in \text{Coh}(S)$ and $G \in \text{Coh}(T)$. If $F = S(-, S)$ we have by Yoneda

$$\text{Coh}(T)(j^*F, G) = \text{Nat}(T(-, jS), G)$$

$$= G(jS) = G|_S(S) \cong \text{Nat}(S(-, S), G|_S)$$

$$= \text{Coh}(S)(F, j_*G).$$

This extends to general $F$ because $j^*$ is right exact.

(iv): As before, we extend the identity $f_*j^*S(-, S) = S(-, S)$ for representables to an isomorphism $f_*j^* \cong \text{id}_{\text{Coh}(S)}$ using the right-exactness of $j^*$. Now $\text{Coh}(T)/\text{Ker}(j^*)$ is the quotient in the sense of abelian categories ([Gab62, §III]). The equivalence (1.6.9) follows from (i)-(iii) by [Gab62, Prop. III.5, p. 374].

The next lemma motivates Axiom (G2) of a perfectly generating set.

1.6.10. Lemma. Let $T, S_0$ and $S$ be as in the previous lemma. Then the following are equivalent:

(i) The homological functor

$$s := j_* \circ h : T \to \text{Coh}(T) \to \text{Coh}(S), \quad A \mapsto T(-, A)|_S$$

...
preserves countable coproducts.
(ii) Hypothesis (G2) holds for $S_0$.

Proof. First notice that $s$ is homological, being the composition of a homological functor followed by an exact one (Lemma 1.6.8(ii)). Since $h$ automatically preserves coproducts (Lemma 1.5.5(ii)), $s$ preserves countable coproducts as soon as $j_*$ preserves them. Actually, if and only if $j_*$ preserves them: if $F = \prod_i F_i$ is a countable coproduct in $\text{Coh}(T)$, recall from the proof of Lemma 1.5.5 that it can be presented by

$$T(-, \prod_i A_i) \xrightarrow{\text{h}} T(-, \prod_i B_i) \xrightarrow{\text{f}} F \xrightarrow{\text{g}} 0,$$

where each $\varphi_i : A_i \rightarrow B_i$ presents $F_i$. Now consider the following commutative diagram in $\text{Coh}(S)$:

$$\begin{array}{ccc}
\prod_i T(-, A_i)|_S & \rightarrow & \prod_i T(-, B_i)|_S \rightarrow \prod_i F|_S \rightarrow 0 \\
\downarrow & & \downarrow \\
T(-, \prod_i A_i)|_S & \rightarrow & T(-, \prod_i B_i)|_S \rightarrow F|_S \rightarrow 0
\end{array}$$

The rows are exact because $j_*$ is exact, by Lemma 1.6.8(ii). If the coproduct $F$ is not preserved by $j_*$, the third vertical map is not an isomorphism. But then one of the other two is also not an isomorphism, and we have found a coproduct in the image of $h$ which is not preserved by $j_*$. Thus, as claimed, (i) holds if $j_*$ preserves countable coproducts. By Lemma 1.6.8(iv), $j_*$ factors as

$$\text{Coh}(T) \xrightarrow{j_*} \text{Coh}(S) \xrightarrow{q} \text{Coh}(T)/\text{Ker}(j_*),$$

where $q$ is a quotient functor of abelian categories. Since the second factor is an equivalence, $j_*$ preserves countable coproducts iff $q$ does. But the latter is equivalent with $\text{Ker}(q) = \text{Ker}(j^*)$ being closed under the formation of countable coproducts of coherent functors. Let $F = \prod_i F_i$ be such a coproduct, presented as above. Now note that $F_i \in \text{Ker}(j_*)$ iff $T(-, \varphi_i)|_S$ is surjective, and $F \in \text{Ker}(j_*)$ iff $T(-, \prod_i \varphi_i)|_S$ is surjective. Since it suffices to check surjectivity on $S_0$, it follows that $T$ satisfies axiom (G2) if and only if $\text{Ker}(j_*)$ is closed under countable coproducts, iff (i) holds.

Proof of Theorem 1.6.6. Now let $T$ be the triangulated category in the theorem, and let $S_0$ be an $\alpha$-small perfectly generating set of objects. With $S$ and $j$ as above, we see that the hypotheses of the two previous lemmata are satisfied.

1.6.11. Construction. We proceed now to recursively build a countable sequence

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots$$

in $T$, together with a collection of maps $\{\pi_n : T(-, X_n) \rightarrow H\}_{n \geq 0}$ such that $\pi_{n+1} \circ T(-, \varphi_n) = \pi_n$, where $H : T^{\text{op}} \rightarrow \text{Ab}$ is the given cohomological functor that we’re going to represent. Consider the set $I := \coprod_{S \in S_0} H(S)$; an element thereof is a pair $(x, S_x)$ with $x \in H(S_x)$, for some $S_x \in S_0$. Clearly $I$ is $\alpha$-small, because of the $\alpha$-decency of $H$ on $S_0$ – Axiom (D2) –, the $\alpha$-smallness of $S_0$ and the regularity of $\alpha$. Therefore the coproduct

$$X_0 := \coprod_{(x, S_x) \in I} S_x$$
surjectivity of \( \pi \) is an epimorphism in \( \text{Add} \). Thus we may define
\[
\pi_0 : T(\text{–}, X_0) \to H
\]
to be the map corresponding to the diagonal element \((x)_{(x,S_0)} \in H(X_0)\) via the Yoneda isomorphism \( H(X_0) \cong \text{Nat}(T(\text{–}, X_0), H) \).

Assume now that we have already constructed the sequence up to \( X_n \), as well as the map \( \pi_n : T(\text{–}, X_n) \to H \). Define
\[
K_n := \ker(\pi_n), \quad I_n := \prod_{S \in S_0} K_n(S).
\]
As a subset of \( T(S, X_n) \), each \( K_n(S) \) is \( \alpha \)-small by Axiom (Go). It follows, as for \( I_0 \), that \( I_n \) is \( \alpha \)-small, and we may define
\[
Y_n := \prod_{(f : S \to X_n) \in I_n} S_f
\]
in \( T \). Now we define a morphism
\[
\lambda_n := (f)_{f \in I_n} : Y_n \to X_n
\]
simply by the universal property of the coproduct \( Y_n \), and we complete it to a distinguished triangle
\[
\begin{array}{ccc}
Y_n & \xrightarrow{\lambda_n} & X_n \\
\downarrow{\varphi_n} & & \downarrow{\varphi_n} \\
X_{n+1} & \xrightarrow{\rho_n} & T Y_n
\end{array}
\]
In order to complete the construction, we are left only with defining \( \pi_{n+1} \). By applying the cohomological functor \( H \) to (1.6.12), we obtain the exact sequence
\[
H(T Y_n) \xrightarrow{H(\rho_n)} H(X_{n+1}) \xrightarrow{H(\varphi_n)} H(X_n) \xrightarrow{H(\lambda_n)} H(Y_n).
\]
By construction, the composition \( H(\lambda_n) \circ \pi_n(1_{X_n}) \) vanishes. [Indeed, consider the commutative diagram
\[
\begin{array}{ccc}
1_{X_n} & \xrightarrow{f^*} & T(X_n, X_n) \\
\downarrow{\pi_n} & & \downarrow{H(f)} \\
T(S_f, X_n) & \xrightarrow{\pi_n, S_f} & H(S_f) \xrightarrow{\text{pr}_f} \prod_{f \in I_n} S_f
\end{array}
\]
The \( f \)-component of the first row is \( \pi_n, S_f \circ f^* \), and its image of \( 1_{X_n} \), is just \( \pi_n, S_f(f) \), which is zero since \( f \in I_n \). Therefore \( \pi_n(1_{X_n}) \) comes from an element of \( H(X_{n+1}) \) via \( H(\varphi_n) \). By Yoneda again, this means that there is a factorization
\[
\begin{array}{ccc}
T(\text{–}, X_n) & \xrightarrow{T(-, \varphi_n)} & T(\text{–}, X_{n+1}) \\
\downarrow{\pi_n} & & \downarrow{\pi_{n+1}} \\
& \downarrow{H} &
\end{array}
\]
as wished.

1.6.14. Remark. The reason for defining \( X_0 \) and \( \pi_0 \) as above, is that \( \pi_0|_S \) is an epimorphism in \( \text{Add}(S, \text{Ab}) \). To see this, note that it suffices to show the surjectivity of \( \pi_0 : T(S, X_0) \to H(S) \) for every \( S \in S_0 \). But for each \( x \in H(S) \), there is a canonical morphism \( S \to \prod_{(y,S_y) \in I_n} S_y = X_0 \) which is mapped to \( x \) by \( \pi_0 \). Thus \( \pi_0 \) is epi, and we see \( \pi_n|_S \) is also epi for every \( n \), because of (1.6.13).
1.6.15. Lemma. There is an exact sequence

\begin{equation}
0 \longrightarrow \prod_n T(-, X_n) \xrightarrow{1 - \varphi_n} \prod_n T(-, X_n) \xrightarrow{\pi_n} H|_S \longrightarrow 0
\end{equation}

in \text{Coh}(S) (in particular, \(H|_S \in \text{Coh}(S)\)).

**Proof.** To start with, we claim that the sequence

\begin{equation}
T(-, Y_n)|_S \xrightarrow{T(-, \lambda_n)|_S} T(-, X_n)|_S \xrightarrow{\pi_n|_S} H|_S \longrightarrow 0
\end{equation}

is exact for every \(n \geq 0\) (so that \(H|_S \in \text{Coh}(S)\)). Indeed, we have already seen that \(\pi_n|_S\) is surjective (Remark 1.6.14), and exactness in the middle holds because the first map factors as

\[T(-, Y_n)|_S \twoheadrightarrow K_n|_S \twoheadrightarrow T(-, X_n)|_S,\]

by the construction of \(\lambda_n\). Thus (1.6.17) is exact. Moreover, we have a morphism

\[
\begin{array}{ccc}
0 & \longrightarrow & K_n|_S \\
\downarrow & & \downarrow \\
0 & \longrightarrow & K_{n+1}|_S \\
\end{array}
\]

of short exact sequences. From (1.6.12) we see that \(\varphi_n \circ \lambda_n = 0\); hence we may set the first vertical map to zero and deduce the existence of the dotted arrow, which is a splitting of the lower exact sequence. We may assemble this data into the commutative diagram

\[
\begin{array}{ccc}
T(-, X_1)|_S & \xrightarrow{T(-, \varphi_1)|_S} & T(-, X_2)|_S \\
\downarrow \scriptstyle{\pi_1|_S} & & \downarrow \scriptstyle{\pi_2|_S} \\
H|_S \oplus K_1|_S & \xrightarrow{1 \circ 0} & H|_S \oplus K_2|_S \\
\end{array}
\]

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\downarrow \scriptstyle{\pi_3|_S} & & \downarrow \scriptstyle{\pi_3|_S} \\
H|_S \oplus K_3|_S & \xrightarrow{1 \circ 0} & H|_S \oplus K_4|_S \\
\end{array}
\]

in \text{Coh}(S). Taking colimits in \text{Coh}(S), we see that \(H|_S \cong \text{colim}_n T(-, \varphi_n)\), from which we deduce the exactness of (1.6.16). \(\square\)

1.6.18. Lemma. \(X := \text{hocolim}(X_n, \varphi_n)\) represents \(H\) on \(T' := (\mathcal{S}_0)_{\text{loc}} \subseteq T\).

**Proof.** Recall that the homotopy colimit is defined by a distinguished triangle

\begin{equation}
\prod_n X_n \xrightarrow{1 - \varphi} \prod_n X_n \longrightarrow X \longrightarrow T(\prod_n X_n)
\end{equation}

which exists because \(T\) has countable coproducts; the map \(1 - \varphi\) is the one whose \(n\)th component is \(1 - \varphi_n\). By applying \(H\) to (1.6.19) we get an exact sequence

\[
H(X) \longrightarrow H(\prod_n X_n) \xrightarrow{H(1 - \varphi)} H(\prod_n X_n).
\]

Using Yoneda and (D1), we see that the sequence \((\pi_n)_n\) defines an element of the middle group which vanishes under \(H(1 - \varphi)\), since \(H(\varphi_n)(\pi_{n+1}) = \pi_n\) by construction. Thus it comes from an element of \(H(X)\), i.e., a map

\[\pi : T(-, X) \to H.\]

We claim that \(\pi|_S\) is an isomorphism, which would prove the lemma.

Recall that by Lemma 1.6.10 the functor \(A \mapsto T(-, A)|_S, T \mapsto \text{Coh}(S)\), is homological and preserves coproducts. Thus if we apply it to (1.6.19) we get the
first exact row in the following diagram in \text{Coh}(\mathcal{S}):\n(1.6.20)\n\begin{array}{c}
\prod_n s(X_n) \xrightarrow{1-\varphi} \prod_n s(X_n) \xrightarrow{} s(X) \xrightarrow{} \prod_n s(TX_n) \xrightarrow{1-\varphi} \prod_n s(TX_n)
\end{array}
\xrightarrow{\pi|\mathcal{S}}
\begin{array}{c}
\prod_n s(X_n) \xrightarrow{1-\varphi} \prod_n s(X_n) \xrightarrow{(\pi_n)_{n\in\mathcal{S}}} \mathcal{H}|\mathcal{S} \xrightarrow{} 0
\end{array}

The lower row is (1.6.16), and therefore it’s also exact. The upper right $1-\varphi$ is a monomorphism, because we assumed that $T(S) = S$. We conclude that $\pi$ is an isomorphism, as we have claimed. \hfill \square


\textbf{Proof.} Fix an arbitrary object $B \in T$. By Axiom (Ga), the cohomological functor $T(-,B)$ is $\alpha$-decent on $S_0$, so we may apply Construction 1.6.11 to it. Thus there is an object $X' \in T'$ and a morphism
\[ \pi' : T(-,X') \to T(-,B) \]
of functors $T^{\text{op}} \to \text{Ab}$. By Yoneda, $\pi'$ is induced by a morphism $p : X' \to B$, which we may complete to a distinguished triangle:
\[ X' \xrightarrow{p} B \xrightarrow{} C \xrightarrow{} TX'. \]

Let $S \in \mathcal{S}$, and consider the following exact sequence.
\[ T(S,X') \xrightarrow{\pi'_S} T(S,B) \xrightarrow{} T(S,C) \xrightarrow{} T(T^{-1}S,X') \xrightarrow{\pi'_{T^{-1}S}} T(T^{-1}S,B) \]
The leftmost and rightmost maps are isomorphisms by Lemma 1.6.18, thus $T(S,C) = 0$ for all $S \in \mathcal{S}$. We use Axiom (G1) to conclude that $C \cong 0$ and therefore $B \cong X' \in T'$.

Finally, Theorem 1.6.6 is an immediate consequence of Lemmata 1.6.18 and 1.6.21.

The following statements are immediate consequences of the proof of the theorem.

1.6.22. Corollary. If $T$ is $\alpha$-perfectly generated with generating set $S_0$, then $\langle S_0 \rangle_{\text{loc}} = T$. In words, the smallest $\alpha$-localizing subcategory containing $S_0$ is equal to the whole category $T$.

1.6.23. Corollary. In the situation of Theorem 1.6.6, the representing object for $H$ is isomorphic to a countable homotopy colimit of $\alpha$-small coproducts of objects from an $\alpha$-perfectly generating set $S_0$. In particular, for every $A \in T$ there exists a distinguished triangle
\[ \prod_{i \in I} S_i \xrightarrow{} \prod_{i \in I} S_i \xrightarrow{} A \xrightarrow{} T(\prod_{i \in I} S_i) \]
with $|I| < \alpha$ and $S_i \in S_0$ for all $i \in I$.

1.6.24. Corollary. Let $\alpha : H \to H'$ be a natural transformation between two cohomological functors $H, H' : T^{\text{op}} \to \text{Ab}$ which send $\alpha$-coproducts of $T$ to products. If the component $\alpha_S : H(S) \to H'(S)$ is an isomorphism for every $S$ in a set $\mathcal{S}_0$ of $\alpha$-perfect generators, then $\alpha$ is an isomorphism on the whole category $T$. 

1.7. COMPACT OBJECTS

Proof. Let $A \in T$, and choose a distinguished triangle as in Corollary 1.6.23. By the hypotheses on $H$ and $H'$, $\alpha$ induces the following morphism of exact sequences.

$$
\begin{array}{cccccc}
\prod H(S_i) & \longrightarrow & \prod H(S_i) & \longrightarrow & H(A) & \longrightarrow \prod H(TS_i) & \longrightarrow & \prod H(TS_i) \\
\prod \alpha_{S_i} & \downarrow & \prod \alpha_{S_i} & \downarrow & \alpha_A & \downarrow & \prod \alpha_{TS_i} & \downarrow \prod \alpha_{TS_i} \\
\prod H'(S_i) & \longrightarrow & \prod H'(S_i) & \longrightarrow & H'(A) & \longrightarrow \prod H'(TS_i) & \longrightarrow & \prod H'(TS_i)
\end{array}
$$

Since each $\alpha_{TS_i}$ and each $\alpha_{S_i}$ is an isomorphism, the Five Lemma assures us that $\alpha_A$ is also an isomorphism. □

1.6.1. Brown representability for the dual. If one wishes to represent homological functors $H : T \to \text{Ab}$, instead of cohomological ones, there is a quite similar theorem ([Nee98]). Unfortunately this works only in the absolute case; more precisely, only when the cardinal $\alpha$ is strongly inaccessible: $\beta < \alpha \Rightarrow 2^\beta < \alpha$. We state the result for completeness, although we won’t use it.

1.6.25. Definition. A small set $S_0$ of objects in $T$ is a set of symmetric generators for $T$ if it satisfies Axiom (G1) (it detects objects) and if moreover

(G3) There exists a small set $R_0 \subseteq T$ such that, for any morphism $f : A \to B$ in $T$, the map $f_* : T(S, A) \to T(S, B)$ is surjective for all $S \in S_0$ iff $f^* : T(B, R) \to T(A, R)$ is injective for all $R \in R_0$.

It is readily checked that (G3) implies (G2), and clearly $T$ has a set of symmetric generators iff $T^{\text{op}}$ does. Since we’re in the absolute case, arbitrary small products exist in $T$ (Remark 1.6.7), i.e., arbitrary small coproducts exist in the dual category $T^{\text{op}}$. Therefore we may apply Theorem 1.6.6 to $T^{\text{op}}$ and obtain

1.6.26. Theorem (Brown Representability for the dual). Let $T$ be a triangulated category with all small coproducts and with a small set of symmetric generators. Then a functor $H : T \to \text{Ab}$ is representable in $T$ iff it is homological and it preserves products. □

1.6.27. Exercise. If $S_0$ is a set of compact objects (i.e., for each $S \in S_0$ the representable functor $T(S, -)$ commutes with all coproducts) satisfying (G1), then it is also a set of symmetric generators for $T$. (Hint: consider for each $S \in S_0$ the functor $T^{\text{op}} \to \text{Ab}$, $A \mapsto \text{Hom}(T(S, A), \mathbb{Q}/\mathbb{Z})$.)

1.6.28. Remark. In the general $\alpha$-relative case, one cannot use Brown Representability to represent products (Remark 1.6.7) so that it is not even clear how one should go about formulating an $\alpha$-relative version of Theorem 1.6.26.

1.7. Compact objects

As before, let $\alpha$ be an uncountable regular cardinal number, and let $T$ be a triangulated category where all $\alpha$-small coproducts exist. Again, we recover the usual ‘absolute’ concepts and results by choosing for $\alpha$ the cardinality of a proper class (or more conveniently, by dropping every mention of $\alpha$).

1.7.1. Definition. An object $A \in T$ is said to be $\alpha$-compact if the functor $T(A, -) : T \to \text{Ab}$ commutes with $\alpha$-small coproducts, and if $|T(A, B)| < \alpha$ for every $B \in T$. The triangulated category $T$ is $\alpha$-compactly generated if there exists some $\alpha$-small set $\mathcal{G}_0$ of $\alpha$-compact objects with $T(\mathcal{G}_0) = \mathcal{G}_0$, and which generates the category: $T(\mathcal{G}_0, A) = 0 \Rightarrow A = 0$. If $T$ is $\alpha$-compactly generated, we will write $T_c$ for the full subcategory of $\alpha$-compact objects.
1.7.2. Lemma. A set of $\alpha$-compact generators is also a set of $\alpha$-perfect generators.

Proof. That’s immediate from the definitions. \hfill\Box

Therefore we may apply the previous results in order to obtain Brown Representability in compactly generated categories.

1.7.3. Theorem (Brown Representability). Let $\mathcal{T}$ be an $\alpha$-compactly generated triangulated category, and let $\mathcal{G}_0$ be some set of $\alpha$-compact generators of $\mathcal{T}$. Then

(i) A functor $H : \mathcal{T}^{op} \to \text{Ab}$ is representable in $\mathcal{T}$ if and only if it is cohomological and $\alpha$-decent on $\mathcal{G}_0$.

(ii) Every object $A \in \mathcal{T}$ is a homotopy colimit of $\alpha$-small coproducts of $\alpha$-compact objects in $\mathcal{G}_0$. In particular, there exists a distinguished triangle

\begin{equation}
X \longrightarrow X \longrightarrow A \longrightarrow TX
\end{equation}

where $X$ is an $\alpha$-small coproduct of objects in $\mathcal{G}_0$.

Proof. Part (i) is Theorem 1.6.6 and the above Lemma 1.7.2; part (ii) is Corollaries 1.6.22 and 1.6.23. \hfill\Box

1.7.5. Corollary. Let $\mathcal{T}$ be a triangulated category with $\alpha$-coproducts. An $\alpha$-small set $\mathcal{E}$ of $\alpha$-compact objects is a set of generators for $\mathcal{T}$ if and only if $\mathcal{T} = \langle \mathcal{E} \rangle_{\text{loc}}$.

Proof. If $\mathcal{E}$ is a set of generators, it is immediate from Theorem 1.7.3 (ii) that $\langle \mathcal{E} \rangle_{\text{loc}} = \mathcal{T}$. Conversely, assume the latter equality. Consider an object $A \in \mathcal{T}$ such that $\mathcal{E} \perp A$. Since the left orthogonal subcategory $\downarrow \{A\}$ is automatically localizing, and since it contains $\mathcal{E}$, it must contain $\langle \mathcal{E} \rangle_{\text{loc}}$, which is the whole category by hypothesis. Hence $A \in \mathcal{T} \perp A$, which implies $A \cong 0$. We have proved that $\mathcal{E}$ generates $\mathcal{T}$. \hfill\Box

1.7.6. Lemma. Let $\mathcal{T}$ have $\alpha$-coproducts. Then $\mathcal{T}_{\alpha} \subset \mathcal{T}$ is a thick subcategory of $\mathcal{T}$.

Proof. Clearly it is closed under translations. It is also closed under taking direct summands: if $A$ is a direct summand of $A'$, then $\mathcal{T}(A,B)$ is a direct summand of the group $\mathcal{T}(A',B)$. Therefore if the latter is $\alpha$-small so is the first, and if $\mathcal{T}(A',\_)$ commutes with $\alpha$-coproducts, so does $\mathcal{T}(A,\_)$.

Let $A \to B \to C \to TA$ be a distinguished triangle with $A,B \in \mathcal{T}_{\alpha}$. We see from the exact sequence

\[ T(B,D) \longrightarrow T(C,D) \longrightarrow T(TA,D), \]

with $D \in \mathcal{T}$ arbitrary, that $|T(C,D)| \leq |T(B,D)| \times |T(TA,D)| < \alpha \times \alpha = \alpha$. From the Five Lemma and the same (long) exact sequence we see that if $\mathcal{T}(T^nA,\_)$ and $\mathcal{T}(T^nB,\_)$ commute with $\alpha$-coproducts ($n = 0,1$), then so does $\mathcal{T}(C,\_)$. Hence $\mathcal{T}_{\alpha}$ is also closed under taking cones, and is therefore a thick subcategory. \hfill\Box

1.7.7. Lemma. Let $C \in \mathcal{T}$ be such that $\mathcal{T}(C,\_)$ commutes with countable coproducts, and let $(A_n,f_n)$ be a sequence in $\mathcal{T}$. Then the canonical map

\[ \text{colim} \mathcal{T}(C,A_n) \xrightarrow{\cong} \mathcal{T}(C,\text{hocolim}(A_n,f_n)) \]

is an isomorphism.

Proof. ([Nee92a, Lemma 1.5].) By definition, $A := \text{hocolim}(A_n,f_n)$ fits into a distinguished triangle $\prod A_n \xrightarrow{1-f} \prod A_n \xrightarrow{f} A \xrightarrow{h} \prod TA_n$. Apply $\mathcal{T}(C,\_)$ to it and get the first exact row in the commutative diagram of abelian groups

\begin{align*}
\mathcal{T}(C,A) & \xrightarrow{h} \mathcal{T}(C,\prod TA_n) \xrightarrow{(1-f)_*} \mathcal{T}(C,\prod TA_n) \\
& \cong \mathcal{T}(C,\text{hocolim}(A_n,f_n)) \\
\mathcal{T}(C,A) & \xrightarrow{\cong} \bigoplus \mathcal{T}(C,TA_n) \xrightarrow{1-f_*} \bigoplus \mathcal{T}(C,TA_n).
\end{align*}
By the hypothesis on $C$ the canonical vertical maps in the middle and right hand side are isomorphisms and thus the bottom sequence is also exact. The map $1 - Tf$, sending $(a_0, a_1, \ldots)$ to $(a_0, a_1 - Tf_0(a_0), a_2 - Tf_1(a_1), \ldots)$, is injective, implying that $h_\ast = 0$. Therefore the first row in the diagram

$$
\begin{array}{c}
T(C, \coprod A_n) \xrightarrow{(1-f)} T(C, \coprod A_n) \\
\cong \quad \cong \quad \text{can}
\end{array}
\xrightarrow{\bigoplus} T(C, A) \longrightarrow 0
$$

is exact; the second row is exact by the definition of $\text{colim}(T(C, A_n))$. We conclude with the Five Lemma that can is an isomorphism. 

**1.7.8. Proposition.** Let $T$ be $\alpha$-compactly generated, and let $E \subseteq T_\ast$ be an $\alpha$-small set of $\alpha$-compact objects. Then $T = \langle E \rangle_{\text{loc}}$ if and only if $T_\ast = \langle E \rangle$ (and iff $E^\perp \cong 0$, by Corollary 1.7.5.)

**Proof.** Assume that $T = \langle E \rangle_{\text{loc}}$. By Corollary 1.7.5, we may choose $E$ as a set of generators. Then by Theorem 1.7.3(ii) every object $A \in T$ is a homotopy colimit of a sequence $X_0 \to X_1 \to \cdots$ where each $X_n$ is an $\alpha$-small coproduct of objects in $E$. By Lemma 1.7.7, the identity $1_A : A \to A$ factors via the canonical map $X_n \to A$, for some $n \in \mathbb{N}$. Therefore $A$ is a direct summand of an $\alpha$-small coproduct $\coprod E_i$ with each $E_i \in E$. If $A$ is $\alpha$-compact, $T(A, -)$ commutes with such coproducts, and we see that the inclusion $A \to \coprod E_i$ must factor through a finite subsum. Hence $A \in \langle E \rangle$ and thus $T_\ast \subseteq \langle E \rangle$. The other inclusion $\langle E \rangle \subseteq T_\ast$ follows from Lemma 1.7.6. Conversely, assume now that $T_\ast = \langle E \rangle$. Since $T$ is $\alpha$-compactly generated, there exists some $\alpha$-small set $G_0 \subseteq T_\ast$ of generators. But then $T = \langle G_0 \rangle_{\text{loc}} \subseteq \langle \langle E \rangle \rangle_{\text{loc}} = \langle E \rangle_{\text{loc}}$ by Corollary 1.7.5. 

The next proposition gives a characterization of $\alpha$-compactly generated categories.

**1.7.9. Proposition.** Let $T$ be a triangulated category with $\alpha$-coproducts. Then $T$ is $\alpha$-compactly generated if and only if $T = \langle T_\ast \rangle_{\text{loc}}$ and $T_\ast$ is essentially $\alpha$-small (i.e., it has an $\alpha$-small set of isomorphism classes of objects).

**Proof.** If $T$ is $\alpha$-compactly generated, choose a set $G_0 \subseteq T_\ast$ of $\alpha$-compact generators. By Theorem 1.7.3(ii) the inclusions $\langle G_0 \rangle_{\text{loc}} \subseteq \langle T_\ast \rangle_{\text{loc}} \subseteq T$ are equalities. Let’s verify that $T_\ast$ is essentially $\alpha$-small. By Proposition 1.7.8, $T = \langle G_0 \rangle_{\text{loc}}$ implies that $T_\ast = \langle G_0 \rangle$. Equivalently, all $\alpha$-compact objects can be obtained in countably many steps, where one starts with $G_0$ and at each step one takes translations of, direct summands of, and cones of morphisms between, already constructed objects. For every pair $A, B$ of $\alpha$-compact objects we have $|T(A, B)| < \alpha$ and therefore there are less then $\alpha$ (isomorphism classes of) cones of morphisms from $A$ to $B$. Setting $B := A$, we see that $T(A, A)$ is $\alpha$-small; in particular the ring $\text{End}_T(A)$ has less than $\alpha$ idempotent elements, hence $A$ has less then $\alpha$ direct summands. Since the initial set $G_0$ is $\alpha$-small, and since $\alpha$ is regular, we see that the number of non-isomorphic compact objects that can be constructed is strictly less then $\alpha$. In other words, $T_\ast$ is essentially $\alpha$-small. 

Conversely, let $T = \langle T_\ast \rangle_{\text{loc}}$. Let $G_0$ be a set of representatives for the isomorphism classes of objects of $T_\ast$ (hence $G_0$ is $\alpha$-small by hypothesis), and consider an object $A \in G_0$. The category

$$
\downarrow(T^\ast A) = \{ B \in T \mid T_\ast(B, A) = 0 \}
$$

is a localizing subcategory of $T$ and it contains $G_0$ by the hypothesis on $A$. Therefore it contains the smallest such $\alpha$-localizing subcategory, which is $T$. In particular,
A ∈  \perp (T^* A) and therefore T(A, A) = 0, which implies that A ∼= 0. This proves that \mathcal{G}_0 generates \mathcal{T}, and is therefore an \alpha-small set of \alpha-compact generators. \hfill \square

1.7.10. Notation. For a class \mathcal{E} \subseteq \mathcal{T} of objects in a triangulated category, we will use the short-hand

\[ T^* \mathcal{E} := \{ T^n A \mid A \in \mathcal{E}, n \in \mathbb{Z} \}, \]

as well as \( T^* A := T^* \{ A \} \) for one object \( A \in \mathcal{T} \).

1.7.11. Corollary. Let \( \mathcal{T} \) be a triangulated category with \( \alpha \)-small coproducts. Let \( \mathcal{S} \subseteq \mathcal{T} \) be a set of \( \alpha \)-compact objects. Then \( \langle \mathcal{S} \rangle_{\text{loc}} \) is \( \alpha \)-compactly generated with \( \mathcal{S} \) a generating set, and its \( \alpha \)-compact objects are given by

\[ (\langle \mathcal{S} \rangle_{\text{loc}})_c = \langle \mathcal{S} \rangle = \langle \mathcal{S} \rangle_{\text{loc}} \cap \mathcal{T}_c. \]

It follows that the assignments

\[ \mathcal{L} \mapsto \mathcal{L}_c \quad \text{and} \quad \mathcal{C} \mapsto \langle \mathcal{C} \rangle_{\text{loc}} \]

define mutually inverse isomorphisms between the lattice of \( \alpha \)-compactly generated \( \alpha \)-localizing subcategories \( \mathcal{L} \subseteq \mathcal{T} \) and the lattice of essentially \( \alpha \)-small thick subcategories \( \mathcal{C} \subseteq \mathcal{T}_c. \)

1.7.14. Warning. Whenever we say “\( \mathcal{L} \) is a compactly generated localizing subcategory of \( \mathcal{T} \)” , we mean that the compact generators of \( \mathcal{L} \) are – as in the corollary – compact objects in \( \mathcal{T} \), not only in \( \mathcal{L} \). See Subsection 1.7.2, where it is shown that for such a pair \( \mathcal{L} \subseteq \mathcal{T} \) the quotient \( \mathcal{T} / \mathcal{L} \) is again compactly generated.

Proof. We have seen above that \( \langle \mathcal{S} \rangle_{\text{loc}} \) is compactly generated with generator set \( \mathcal{S} \). The first equality in (1.7.12) is Proposition 1.7.8. Note that the inclusion \( \langle \mathcal{S} \rangle \subseteq \langle \mathcal{S} \rangle_{\text{loc}} \cap \mathcal{T}_c \) is obvious from the definitions. The remaining inclusion \( \mathcal{T}_c \cap \langle \mathcal{S} \rangle_{\text{loc}} \subseteq \langle \mathcal{S} \rangle_{\text{loc}} \) is also clear: every \( \alpha \)-compact object of \( \mathcal{T} \) is an \( \alpha \)-compact object in each of the localizing subcategories to which it belongs. Therefore (1.7.12) is true. In particular, if a thick subcategory \( \mathcal{C} \subseteq \mathcal{T}_c \) is essentially small, by choosing \( \mathcal{S} \) to be a skeleton of \( \mathcal{C} \) we see that \( \mathcal{C} \mapsto \langle \mathcal{C} \rangle_{\text{loc}} \mapsto ((\langle \mathcal{C} \rangle_{\text{loc}})_c \) is the identity. On the other hand, if a localizing subcategory \( \mathcal{L} \) is compactly generated, then \( \langle \mathcal{L} \rangle_{\text{loc}} = \mathcal{L} \) by Proposition 1.7.8. Thus the functions (1.7.13) are mutually inverse bijections, and since they clearly preserve inclusions, they are lattice isomorphisms. \hfill \square

Next we see that, by invoking Brown representability, it is possible to construct a complementary pair (Def. 1.4.21) of subcategories out of any set of compact objects (cf. \cite[§2]{BCR97}).

1.7.15. Proposition. Let \( \mathcal{T} \) be a triangulated category with \( \alpha \)-small coproducts, and let \( \mathcal{S} \subseteq \mathcal{T}_c \) be a \( \alpha \)-small set of \( \alpha \)-compact objects of \( \mathcal{T} \). Then \( \langle \mathcal{S} \rangle_{\text{loc}}, (T^* \mathcal{S})^\perp \) is a complementary pair of subcategories out of any set of compact objects (cf. \cite[§2]{BCR97}).

Proof. The moreover part is readily verified. The subcategory \( \langle \mathcal{S} \rangle_{\text{loc}} \) is localizing by definition. Since every \( S \in \mathcal{S} \) is compact, the right orthogonal category \( (T^* \mathcal{S})^\perp \) is localizing, and therefore so is their intersection \( (T^* \mathcal{S})^\perp = \bigcap_{S \in \mathcal{S}} (T^* S)^\perp \).

Clearly \( \langle \mathcal{S} \rangle_{\text{loc}} \perp (T^* \mathcal{S})^\perp \), since \( (T^* \mathcal{S})^\perp = \langle \langle \mathcal{S} \rangle_{\text{loc}} \rangle^\perp \).

It remains to construct the gluing triangle for an object \( A \in \mathcal{T} \). Let \( \mathcal{L} := \langle \mathcal{S} \rangle_{\text{loc}}. \) Consider the restriction \( T(-, A)|_{\mathcal{L}} : \mathcal{L} \to \text{Ab} \) of the representable functor \( T(-, A) \).

It is a decent cohomological functor, and so by Theorem 1.7.3 it is representable: there exists an object \( L_A \in \mathcal{L} \) and an isomorphism

\[ \mathcal{L}(B, L_A) \cong \mathcal{T}(B, A), \]

(1.7.16)
natural in \( B \in \mathcal{L} \). In other words, the inclusion \( \mathcal{L} \to \mathcal{T} \) has a right adjoint \( A \mapsto L_A \).

Choosing \( B = L_A \), we see that to the identity \( L_A \to L_A \) there corresponds via (1.7.16) a morphism \( \varepsilon_A : L_A \to A \) (i.e., the counit of the adjunction), which is the final morphism to \( A \) from an object of \( \mathcal{L} \). In particular, \( \varepsilon_A \) is an isomorphism if \( A \in \mathcal{L} \). Complete \( \varepsilon_A \) to a distinguished triangle:

\[
L_A \xrightarrow{\varepsilon_A} A \xrightarrow{} R_A \xrightarrow{} T(L_A).
\]

For every object \( B \in \mathcal{T} \) it gives rise a the long exact sequence

\[
\cdots \to T(B, L_A) \xrightarrow{\varepsilon_A} T(B, A) \xrightarrow{} T(B, R_A) \to \cdots
\]

Let \( B \in \mathcal{L} \). Then \( (\varepsilon_A)_* \) is the isomorphism of (1.7.16), and consequently \( T(B, R_A) = 0 \). This proves that \( R_A \in \mathcal{L}^\perp_\mathcal{T} \), as wished.

1.7.17. Notation. In the situation of Proposition 1.7.15, we will write \( \mathcal{C}_\mathcal{S} := (\mathcal{S}) \) and \( \mathcal{T}_\mathcal{S} := (\mathcal{S})_{\text{loc}} \) (so that \( (\mathcal{T}_\mathcal{S})_e = \mathcal{C}_\mathcal{S} \)), as well as use the notation

\[
L_\mathcal{S}(A) \xrightarrow{\varepsilon_\mathcal{S}(A)} A \xrightarrow{\eta_\mathcal{S}(A)} R_\mathcal{S}(A) \xrightarrow{} T L_\mathcal{S}(A)
\]

for the functorial gluing triangle for the complementary pair \((\mathcal{T}_\mathcal{S}, \mathcal{T}_\mathcal{S}^\perp)\). If \( \mathcal{C} \) is an essentially \( \alpha \)-small thick subcategory of \( \alpha \)-compact objects of \( \mathcal{T} \), we will write \( L_\mathcal{C} \) and \( R_\mathcal{C} \) instead of \( L_\mathcal{S} \) and \( R_\mathcal{S} \), where \( \mathcal{S} \) is some set of representatives of the isomorphism classes (a ‘skeleton’) of \( \mathcal{C} \).

1.7.19. Proposition. Let \( \mathcal{S}, \mathcal{S}' \) be two sets of compact objects.

(i) If \( \mathcal{S}' \subseteq \mathcal{S} \), then \( L_{\mathcal{S}'}(\eta_\mathcal{S}) : L_{\mathcal{S}'} \circ L_\mathcal{S} \to L_{\mathcal{S}'} \) is an isomorphism.

(ii) \( L_\mathcal{S} \cong L_{\mathcal{S}'} \) iff \( \mathcal{C}_\mathcal{S} = \mathcal{C}_{\mathcal{S}'} \) iff \( \mathcal{T}_\mathcal{S} = \mathcal{T}_{\mathcal{S}'} \).

Proof. (i) For every object \( A \), consider the gluing triangle for \( \mathcal{S} \). Then \( R_\mathcal{S}(A) \in \mathcal{S}^\perp \subseteq \mathcal{S}'^\perp \). Thus \( L_{\mathcal{S}'}(R_\mathcal{S}A) = 0 \) and \( L_{\mathcal{S}'} \eta_\mathcal{S}(A) \) is an isomorphism.

(ii) The second equivalence is Corollary 1.7.11. By Proposition 1.4.22, the category \( \mathcal{T}_\mathcal{S} \) is equivalent to the image of \( L_\mathcal{S} \) (because \( A \in \mathcal{T}_\mathcal{S} \) iff \( \varepsilon(A) : L_\mathcal{S} \to A \) is an isomorphism), and similarly for \( \mathcal{S}' \). Thus \( L_{\mathcal{S}'} \cong L_{\mathcal{S}'} \) iff \( \mathcal{T}_\mathcal{S} \cong \mathcal{T}_{\mathcal{S}'} \). Since the latter subcategories are strictly full, they are equivalent iff they are equal.

1.7.20. An adjoint functor theorem.

1.7.20. Proposition. Let \( \mathcal{T} \) be \( \alpha \)-compactly generated, and let \( F : \mathcal{T} \to \mathcal{L} \) be a triangle functor. Assume that

(i) \( F \) sends \( \alpha \)-coproducts of \( \mathcal{T} \) to coproducts in \( \mathcal{L} \), and

(ii) \( |\mathcal{L}(F(S), B)| < \alpha \) for all \( S \in \mathcal{S}_0 \) and \( B \in \mathcal{L} \), where \( \mathcal{S}_0 \) is some \( \alpha \)-perfectly generating set for \( \mathcal{T} \).

Then \( F \) has a right adjoint \( G : \mathcal{L} \to \mathcal{T} \).

Proof. Let \( B \in \mathcal{T} \). The two conditions guarantee that the cohomological functor \( \mathcal{L}(F(-), B) : \mathcal{T}^\text{op} \to \text{Ab} \) is \( \alpha \)-decent on \( \mathcal{S}_0 \). By Brown Representability (Theorem 1.6.6), there is an object \( G(B) \in \mathcal{T} \) and a natural isomorphism

\[
\mathcal{L}(F(A), B) \cong \mathcal{T}(A, G(B))
\]

natural in \( A \in \mathcal{T} \). Using this, one may easily extend \( B \mapsto G(B) \) to a functor \( \mathcal{L} \to \mathcal{T} \) in such a way that the above isomorphism becomes an isomorphism of bifunctors \( \mathcal{T}^\text{op} \times \mathcal{L} \to \text{Ab} \), as required.
1.7.2. The Neeman localization theorem. The following result is Neeman’s generalization to triangulated categories ([Nee92a, Thm. 2.1]) of Thomason-Trobaugh’s localization theorem in the algebraic $K$-theory of schemes ([ITT90]). More precisely, Thomason had proved Theorem 1.7.21 in the special case when $T$ is $\text{D}^{\text{perf}}(X)$, the derived category of perfect complexes on a quasi-compact quasi-separated scheme $X$, and $L$ is $\text{D}^{\text{perf}}_Z(X)$, the localizing subcategory of complexes acyclic outside the closed subset $Z \subset X$.

1.7.21. THEOREM. Let $T$ be an $\alpha$-compactly generated triangulated category. Let $L_0 \subseteq T_c$ be some (necessarily essentially $\alpha$-small) subset of $\alpha$-compact objects, and let $L := (L_0)_{\text{loc}}$ be the $\alpha$-localizing subcategory of $T$ generated by $L_0$. Consider the resulting diagram of inclusions and quotient functors.

$$
\begin{array}{ccc}
L & \rightarrow & T \\
\downarrow & & \downarrow F \\
L_c & \rightarrow & T_c \\
\end{array}
$$

Then the following hold true:

(i) The induced functor $F$ is fully faithful.

(ii) The image of $F$ consists of $\alpha$-compact objects of $T/L$.

(iii) $F(T_c/L_c)$ is a cofinal subcategory of $(T/L)_c$: for every $A \in (T/L)_c$ there are some $A' \in (T/L)_c$ and $B \in T_c/L_c$ such that $A \oplus A' \cong F(B)$.

For the following, we substitute $L_0$ with an $\alpha$-small set of representatives of the isomorphism classes.

Proof. First of all, note that $L$ is $\alpha$-compactly generated (1.7.5), and that the quotient functor $q : T \rightarrow T/L$ preserves coproducts. Now we could try to apply Proposition 1.7.20 to find a right adjoint $q_\star : T/L \rightarrow T$ for $q$. This won’t do, because we need to know that the object $q_\star(B) = q_\star B$, for $B \in T$, is defined by a homotopy colimit having a certain specific form, a fact which unfortunately is not automatic from Construction 1.6.11. What we need is:

1.7.22. LEMMA. For every $B \in T$, there is a sequence $B = Y_0 \rightarrow Y_1 \rightarrow \cdots$ such that, for all $n$, cone$(g_n) \in L$. Moreover, $B' := \text{hocolim}(Y_n, g_n) \in L^\perp$ and the cone of the canonical map $B \rightarrow B'$ lies in $L$.

Proof. We begin by recursively constructing the sequence. Let $Y_0 := B \in T$. Assume that it’s already done up to $\cdots \rightarrow Y_{n-1} \rightarrow Y_n$, with cone$(g_{n-1}) \in L$. Let

$$L_n := \prod_{(u, C_u \rightarrow Y_n) \in T(L_0, Y_n)} C_u, \quad \ell_n := (u)_n : L_n \rightarrow Y_n.$$ 

That is, we assemble all morphisms from $L_0$ to $Y_n$ into a single one. (Note that the coproduct is indexed on an $\alpha$-small set, since $L_0$ is an $\alpha$-small collection of $\alpha$-compact objects.) Now complete $\ell_n$ to a distinguished triangle:

$$L_n \xrightarrow{\ell_n} Y_n \xrightarrow{g_n} Y_n \rightarrow TL_n.$$ 

Thus cone$(g_n) = TL_n \in L$, and our construction is complete.

Claim 1: $B' := \text{hocolim}(Y_n, g_n) \in L^\perp$.

It’s enough to show that $L_0 \perp B'$, so consider a $v : C \rightarrow B'$ with $C \in L_0$. But $C$ is $\alpha$-small, so $v$ must factor through some $Y_n \rightarrow B'$, by Lemma 1.7.7. By construction, since $C \in L_0$, the composition $C \rightarrow Y_n \rightarrow Y_{n+1}$ vanishes and therefore $v = 0$, proving Claim 1.

Claim 2: The induced morphism $q : B \rightarrow B'$ has its cone in $L$. 

Since cone\( (g_n) \in \mathcal{L} \) for all \( n \geq 0 \), the quotient functor \( q : \mathcal{T} \to \mathcal{T}/\mathcal{L} \) sends \( B' \) to the homotopy colimit of a sequence of isomorphisms. But it is easy to see that, in this case, the canonical maps from each term to the homotopy colimit are all isomorphisms (cf. Remark 1.1.36).

1.7.23. Remark. It follows in particular from the lemma that every \( B \in \mathcal{T} \) fits in a distinguished triangle \( T^{-1} \text{cone}(g) \to B \xrightarrow{g} B' \to \text{cone}(g) \) with \( T^{-1} \text{cone}(g) \in \mathcal{L} \) and \( B' \in \mathcal{L}^+ \). Therefore we find ourselves in a Bousfield localization: \( (\mathcal{L}, \mathcal{L}^+) \) is a complementary pair in \( \mathcal{T} \), the quotient functor \( q \) has a right adjoint \( q_r : \mathcal{T}/\mathcal{L} \to \mathcal{T} \), and so on. Moreover, by the uniqueness of gluing triangles, we may identify \( g : B = Y_0 \to \text{hocolim} Y_n = B' \) with the unit \( \eta_B : B \to q_r B \) of the adjunction. But even more is true: this Bousfield localization is smashing, by which we mean:

1.7.24. Lemma. \( \mathcal{L}^+ \) is closed under the formation of \( \alpha \)-small coproducts.

Proof. Thanks to \( \mathcal{L}^+ = \mathcal{L}_0^+ = \bigcap_{A \in \mathcal{C}_0} \{A\}^+ \), it suffices to show that each right orthogonal \( \{A\}^+ \) is \( \alpha \)-localizing. But this is clear, since \( \mathcal{T}(A, -) \) commutes with \( \alpha \)-small coproducts if \( A \) is \( \alpha \)-compact.

1.7.25. Lemma. Let \( f : A \to B \) be a morphism in \( \mathcal{T} \) with \( A \in \mathcal{L}_c \), and let \( s' : B' \to B \) be a morphism with \( \text{cone}(s) \in \mathcal{L} \). Then there exist \( g : A' \to B' \) and \( A' \to A \) such that \( ft = sg \) and \( \text{cone}(t) \in \mathcal{L}_c \).

\[
\begin{array}{ccc}
A' & \xrightarrow{3t} & A \\
\downarrow{g} & & \downarrow{f} \\
B' & \xrightarrow{s} & B \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{cone}(t) & \in & \mathcal{L}_c \\
\downarrow & & \downarrow \\
\text{cone}(s) & \in & \mathcal{L} \end{array}
\]

Proof. Since \( \text{cone}(s) \in \mathcal{L} \) and \( \mathcal{L} \) is \( \alpha \)-compactly generated by \( \mathcal{L}_0 \), we know that \( \text{cone}(s) \) is isomorphic to a homotopy colimit \( X_0 \to X_1 \to \cdots \), where each \( X_n \) is an \( \alpha \)-small coproduct of generators from \( \mathcal{L}_0 \) (Theorem 1.7.3(ii)). The compactness of \( A \) implies by Lemma 1.7.7 that \( s'f : A \to \text{cone}(s) \) factors through some stage \( X_n \). Indeed, since it’s compact, trough some finite subsum \( X_n' \subseteq X_n \). Thus we have a commutative (solid) square

\[
\begin{array}{ccc}
A' & \xrightarrow{3t} & A \\
\downarrow{g} & & \downarrow{f} \\
B' & \xrightarrow{s} & B \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{cone}(t) & \xrightarrow{t} & \text{cone}(s) \\
\downarrow & & \downarrow \\
\text{cone}(s') & \xrightarrow{t} & \text{cone}(s') \end{array}
\]

with \( X_n' \in \langle \mathcal{L}_0 \rangle = \mathcal{L}_c \), and by completing it to a morphism of distinguished triangle we get the desired maps \( t \) and \( g \).

1.7.26. Remark. Neeman’s original [Nee92a, Lemma 2.3] is slightly different: the hypothesis is that \( \text{cone}(s) \) be a finite extension of coproducts of objects in \( \mathcal{L}_0 \), and in the conclusion \( \text{cone}(t) \) belongs to \( \langle \mathcal{L}_0 \rangle \). Then the proof doesn’t have to appeal to Brown representability.

1.7.27. Lemma. The canonical map

\[ \Phi : \mathcal{T}/\mathcal{L}_c(A, B) \to \mathcal{T}/\mathcal{L}(A, B) \]

is an isomorphism for \( A \in \mathcal{T}_c \) and any \( B \in \mathcal{T} \). In particular, the induced functor \( F : \mathcal{T}_c/\mathcal{L}_c \to (\mathcal{T}/\mathcal{L})_c \) is fully faithful.
Proof. Consider the commutative diagram
\[
\begin{array}{c}
\mathcal{T}(A, B) \xrightarrow{q} \mathcal{T}/\mathcal{L}(A, B) \\
\downarrow q' \quad \downarrow \phi \quad \Rightarrow \quad \downarrow (\eta_B), \\
\mathcal{T}/\mathcal{L}_c(A, B) \xrightarrow{} \mathcal{T}(A, q_r B),
\end{array}
\]
where the isomorphism is that of the adjunction \((q, q_r)\). By Lemma 1.7.22, we may assume that \(\eta_B : B \rightarrow q_r B\) is the canonical morphism \(Y_0 \rightarrow hocolim(Y_n, g_n) = q_r B\) of a sequence \(B = Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \ldots\) such that \(\text{cone}(g_n) \in \mathcal{L}\) for all \(n\).

Surjectivity: We show the surjectivity of the dotted map. Let \(f : A \rightarrow q_r B\) be given. Since \(A\) is \(\alpha\)-compact, \(f\) factors as \(A \xrightarrow{f'} Y_n \xrightarrow{\phi} q_r B\) (Lemma 1.7.7). Moreover, the composition \(g_n^0 : B \rightarrow Y_n\) of the transition maps also has its cone in \(\mathcal{L}\) (by the Octahedron Axiom and induction). Thus we may apply Lemma 1.7.25 in order to obtain the commutative square
\[
\begin{array}{c}
Z \xrightarrow{t} A \\
\downarrow h \quad \downarrow f' \\
B \xrightarrow{g_n^0} Y_n
\end{array}
\]
with cone\((t)\) \(\in \mathcal{L}_c\). Hence we have found a fraction \([ht^{-1}] \in \mathcal{T}/\mathcal{L}_c(A, B)\) with \(\eta_B \circ \Phi([ht^{-1}]) = f\).

Injectivity: As an instant of reflection shows, it suffices to prove that if \(q(f) = 0\) for a \(f \in \mathcal{T}(A, B)\), then \(q'(f) = 0\) already. But if \(q(f) = 0\), then \(\eta_B f = 0\), and since \(A\) is \(\alpha\)-compact some composition \(A \xrightarrow{f} B \xrightarrow{g_n^0} Y_n\) is already 0. Now complete \(g_n^0\) to a distinguished triangle
\[
\begin{array}{c}
L \xrightarrow{f} B \xrightarrow{g_n^0} Y_n \xrightarrow{} \mathcal{T} L.
\end{array}
\]
Since \(g_n^0 f = 0\), there exists a \(f' : A \rightarrow L\) with \(\ell f' = f\). Using that \(L \in \mathcal{L}\), we may apply Lemma 1.7.25 again to get a commutative square
\[
\begin{array}{c}
W \xrightarrow{s} A \\
\downarrow k \quad \downarrow f' \\
0 \xrightarrow{} L
\end{array}
\]
with cone\((s)\) \(\in \mathcal{L}_0\). Thus \(q'\) sends \(s\) to an isomorphism and \(f'\) to 0. Therefore \(q'(f') = 0\) as well as \(q'(f) = q'(\ell)q'(f') = 0\).

1.7.28. Lemma. \(q(\mathcal{T}_c) \subseteq (\mathcal{T}/\mathcal{L})_c\).

Proof. If \(A\) is \(\alpha\)-compact and \(\{B_n\}_n\) is an \(\alpha\)-small family in \(\mathcal{T}/\mathcal{L}\), we may compute
\[
\begin{align*}
\mathcal{T}/\mathcal{L}(qA, \bigsqcup B_n) & \cong \mathcal{T}(A, \bigsqcup B_n) \\
& \cong \mathcal{T}(A, q_r \bigsqcup B_n) \\
& \cong \bigsqcup \mathcal{T}(A, q_r B_n) \\
& \cong \bigsqcup \mathcal{T}/\mathcal{L}(qA, B_n),
\end{align*}
\]
showing that \(\mathcal{T}/\mathcal{L}(qA, -)\) commutes with \(\alpha\)-small coproducts. We still have to show that \(|\mathcal{T}/\mathcal{L}(qA, B)| < \alpha\) for every \(B \in \mathcal{T}/\mathcal{L}\). Now Lemma 1.7.27 gives us a
canonical isomorphism
\[ T/L(qA, B) \cong T/L_c(qA, B), \]
where each element on the right is represented by a fraction \( A \overset{s}{\leftarrow} C \overset{f}{\rightarrow} B \) with \( C \in L_0 \). Since \( L_c \) is essentially \( \alpha \)-small, we may choose these \( C \) from an \( \alpha \)-small skeleton \( \tilde{L}_c \subseteq L_c \). Since each object \( C \) is \( \alpha \)-compact and the cardinal \( \alpha \) is regular, we deduce that there are at most
\[ \sum_{C \in \tilde{L}_c} |T(C, A \oplus B)| < \alpha \]
equivalence classes of such fractions. □

We are only left with part (iii), which follows from:

1.7.29. Lemma. \( \langle q(T_c) \rangle = (T/L)_c \).

Proof. Construct an \( \alpha \)-small generating set \( G_0 \) for \( T \) by picking an object in each isomorphism class of \( T_c \). Since \( \langle G_0 \rangle_\alpha = T \) and since \( q : T \to T/L \) preserves \( \alpha \)-coproducts, we have \( \langle q(G_0) \rangle_{\text{loc}} = q(T) = T/L \). Moreover, the objects of \( q(G_0) \) are \( \alpha \)-compact in \( T/L \) (Lemma 1.7.28). It follows that \( T/L \) is also \( \alpha \)-compactly generated (an interesting fact that we store in Corollary 1.7.30 below). Hence \( (T/L)_c = (\langle q(G_0) \rangle_{\text{loc}})_c = \langle G_0 \rangle \) (Proposition 1.7.8), from which we see that \( \langle q(T_c) \rangle = (T/L)_c \). □

The proof of the \( \alpha \)-version of Neeman’s Localization Theorem is now complete. □

1.7.3. A Neeman localization is a nice Bousfield localization. The proof of Theorem 1.7.21 shows that the situation there described is the right kind of localization one should consider if wishing to stay inside the world of \( \alpha \)-compactly generated categories:

1.7.30. Corollary. If \( T \) is \( \alpha \)-compactly generated and \( \mathcal{L} \subseteq T \) is a compactly generated \( \alpha \)-localizing subcategory, then \( T/L \) is also \( \alpha \)-compactly generated. A generating set is provided by the canonical image of any generating set for \( T \). □

(It may also be a good idea to recall at this point the lattice isomorphism of Corollary 1.7.11). We may call a localization functor \( q : T \to T/L \) as in Theorem 1.7.21 (i.e., where \( T \) is \( \alpha \)-compactly generated and \( \mathcal{L} \subseteq T \) is an \( \alpha \)-compactly generated \( \alpha \)-localizing subcategory) a Neeman \( \alpha \)-localization functor.

Actually, in the course of the proof we have also seen that a Neeman localization functor has a right adjoint. It is even a \textbf{smashing} Bousfield localization, by which topologists mean that the right adjoint functor \( q_r \) (and thus the Bousfield localization functor \( R = q_r q : T \to T \)) preserves coproducts.

1.7.31. Theorem. Every Neeman \( \alpha \)-localization functor \( q : T \to T/L \) has a right adjoint which commutes with \( \alpha \)-small coproducts.

Proof. See Remark 1.7.23 and Lemma 1.7.24. It remains to see that the statement of the lemma, i.e., that \( L^\perp \) is \( \alpha \)-localizing, implies that \( q_r \) commutes with \( \alpha \)-small coproducts. But this is clear from the symmetry of Bousfield localization: there is an equivalence \( L^\perp \simeq T/L \) identifying the inclusion \( L^\perp \hookrightarrow T \) with the right adjoint \( q_r \), and the inclusion commutes with coproducts because \( L^\perp \) is localizing. □
CHAPTER 2

Some tensor triangular geometry

This second chapter ended up being a rather haphazard collection of facts about tensor triangulated categories that will be needed later. Because of this bias, it should not be used as an introduction to Paul Balmer’s geometric theory of tensor triangulated categories, or tensor triangular geometry for short. For this, and for examples and applications, we refer to the original works [Bal05] [Bal02] [Bal07] [BF07] [BBC08].

Section 2.1 explains the (light) axiomatisation used in the present work for the interaction between the triangulation and the tensor product, and derives from it some elementary properties of the endomorphism ring of the tensor unit. In Section 2.2 we introduce the main tool of tensor triangular geometry, namely the spectrum $\text{Spc}(\mathcal{T})$ of a tensor triangulated category $\mathcal{T}$, and we give a rapid overview of some aspects of the geometric theory. In particular, we recall the important result that the radical thick tensor ideals of $\mathcal{T}$ are classified by certain subsets of the spectrum (Theorem 2.2.11), and that, vice-versa, a space providing such a classification must be homeomorphic to the spectrum (Theorem 2.2.15). In Section 2.3 we present yet another idea of Balmer’s, central localization, and we proceed to adapt it to compactly generated tensor triangulated categories. After a brief discussion of strongly dualizable objects in closed tensor categories (Section 2.4), we conclude in Section 2.5 by proving a result which, in favourable situations and under a noetherian hypothesis, allows one to “compute” the spectrum $\text{Spc}(\mathcal{T}_{sd})$ of the category $\mathcal{T}_{sd}$ of strongly dualizable objects in a compactly generated tensor triangulated category $\mathcal{T}$, by means of the above-mentioned classification (see Theorem 2.5.2).

2.1. The axiom(s) and basic properties

2.1.1. Definition. By a tensor triangulated category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ we shall understand a triangulated category $\mathcal{T}$ together with a ‘compatible’ unital symmetric monoidal structure, or tensor category structure, $(\mathcal{T}, \otimes, \mathbb{1}, \alpha, \rho, \lambda, \gamma)$. We refer to [Mac98, XI] for the definition and elementary facts about tensor categories, with which we assume some familiarity. The natural associativity isomorphism $\alpha : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$, left unit $\lambda : \mathbb{1} \otimes A \cong A$, right unit $\rho : A \otimes \mathbb{1} \cong A$ and commutativity isomorphism $\gamma : A \otimes B \cong B \otimes A$ are part of the given structure, but will be omitted from the notation whenever possible. The only compatibility axiom needed for all basic results of triangular geometry is a rather modest one:

$\otimes \Delta$ The functor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a triangle functor in each variable.

In view of Subsection 2.1.2, where we shall consider the natural grading given by the translation, we want to assume a stronger axiom:

$\otimes \Delta^+$ The translation functor is defined by tensoring with an object:

$$\mathcal{T} = S \otimes (-) : \mathcal{T} \to \mathcal{T}.$$ We require of this object $S$ the property that the symmetry isomorphism $\gamma_{S,S} : S \otimes S \to S \otimes S$ is equal to $(-1) \cdot 1_{S \otimes S}$. 

47
2.1.2. Remark. Clearly $(\otimes \Delta^+) \implies (\otimes \Delta)$: for every object $A \in \mathcal{T}$, the pair $((-) \otimes A, r)$ is a triangle functor, with

$$r_B : T(B) \otimes A = (S \otimes A) \otimes A \xrightarrow{\alpha^{-1}} S \otimes (B \otimes A) = T(B \otimes A)$$

given simply by the associativity. Similarly, the isomorphism

$$\ell_A : A \otimes (S \otimes B) \xrightarrow{\alpha} (A \otimes S) \otimes B \xrightarrow{\gamma_1} (S \otimes A) \otimes B \xrightarrow{\alpha^{-1}} S \otimes (A \otimes B)$$

provides a triangle functor $(A \otimes (-), \ell)$ in a natural way. It is also easy to check that the natural isomorphisms $r_{A,B} : T(A) \otimes B \to T(A \otimes B)$ and $\ell_{A,B} : A \otimes T(B) \to T(A \otimes B)$ satisfy the following compatibility conditions (see the proof of Thm A.5.3 (c)): the two triangles

\begin{equation}
\label{eq:2.1.3}
\begin{array}{ccc}
\mathbb{I} \otimes T(A) & \xrightarrow{\lambda_{T(A)}} & T(A) \xrightarrow{\rho_{T(A)}} T(A) \otimes \mathbb{I} \\
\ell_{1,A} \downarrow & & \downarrow r_{1,A} \\
T(1 \otimes A) & \xrightarrow{T(\lambda_A)} & T(A) \xleftarrow{T(\rho_A)} T(A \otimes 1)
\end{array}
\end{equation}

are commutative, and the next square anticommutes:

\begin{equation}
\label{eq:2.1.4}
\begin{array}{ccc}
T(A) \otimes T(B) & \xrightarrow{r_{A,T(B)}} & T(A \otimes T(B)) \\
\ell_{T(A),B} \downarrow & & \downarrow T(\ell_{A,B}) \\
T(T(A) \otimes B) & \xrightarrow{T(\ell_{A,B})} & T^2(A \otimes B).
\end{array}
\end{equation}

Next we review the basic properties of the endomorphism ring of the $\otimes$-unit, $\text{End}(\mathbb{I})$, and of its graded version $\text{End}^\ast(\mathbb{I}) := \bigoplus \mathcal{T}(\mathbb{I}, T^n \mathbb{I})$. We shall observe that the axioms of a tensor structure imply that $\text{End}(\mathbb{I})$ is commutative, and make $\mathcal{T}$ canonically into an $\text{End}(\mathbb{I})$-linear category. Using $(\otimes \Delta^+)$, this can be extended to a grading-preserving action of $\text{End}^\ast(\mathbb{I})$ onto the graded Hom sets $\text{Hom}(A, T^n B)$, which turns out to be natural in a graded way. Moreover, $\text{End}^\ast(\mathbb{I})$ turns out to be graded commutative (but note that, by [Sua04], conditions (2.1.3) and (2.1.4) would already suffice for the latter to be true).

2.1.1. The central ring $R_T$.

2.1.5. Definition. Following [Bal08], we call the endomorphism ring of the tensor unit

$$R_T := \text{End}_\mathcal{T}(\mathbb{I})$$

the central ring of the tensor triangulated category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{I})$. The central ring owes its prominence to the fact that it acts on every other Hom set via the tensor product. Namely, given $r \in \text{End}_\mathcal{T}(\mathbb{I})$ and any morphism $f : A \to B$ in $\mathcal{T}$, we define the canonical left action of $r$ on $f$, written $r \cdot f$, as the map defined by the commutative diagram

$$\begin{array}{ccc}
\mathbb{I} \otimes A & \xrightarrow{r \otimes f} & \mathbb{I} \otimes B \\
\lambda_A \downarrow & \cong & \lambda_B \\
A & \xrightarrow{r \cdot f} & B.
\end{array}$$

In the special case when $f = 1_A$ we shall also write $r|_{A} := r \cdot 1_A : A \to A$.

The similar definition given by tensoring with $\mathbb{I}$ on the right and by applying the right unit isomorphism $\rho$ yields a canonical right action, which actually turns out to be the same. The next proposition collects some elementary facts in this vein.
2.1.6. Proposition. (i) The central ring $R_T$ is commutative.
(ii) The canonical left and right actions of $R_T$ on $\text{Hom}(A,B)$ are well-defined and coincide.
(iii) Composition in $T$ is $R_T$-bilinear: if $f$ and $g$ are composable, then
\[ r \cdot (f \circ g) = (r \cdot f) \circ g = f \circ (r \cdot g) \]
for all $r \in R_T$ (we may say that the elements of $R_T$ are central morphisms). In particular, given a commutative square (1) and an $r \in R_T$ we get the commutative square (2):

\[ \begin{array}{c}
A \xrightarrow{f} B \\
A' \xrightarrow{f'} B'
\end{array} \quad \begin{array}{c}
A \xrightarrow{f} B \\
A' \xrightarrow{f'} B'
\end{array} \]

(iv) The two ring structures on $R_T$ given by composition and by its canonical action on itself coincide.

Proof. (i): Let $r, s \in R_T$ and consider the following diagram:

The central square commutes because $\otimes$ is a bifunctor; the four triangles commute by an axiom of tensor categories (saying $\lambda_{1,1} = \rho_{1,1}$); and the four remaining squares by naturality of $\lambda$ and $\rho$. Hence we read $rs = sr$ off the perimeter. Part (iv) is also clear from this same picture.

(ii): In the diagram

the two triangles commute because of another axiom of tensor categories (the one saying that the switch $\gamma$ interchanges $\lambda$ and $\rho$), and the square commutes by naturality of $\gamma$. Therefore the two ways of defining the dotted arrow yield the same result: $r \cdot f = f \cdot r$. Now we show that the left canonical action is indeed an action.
Let $r, s \in R_T$, and consider the diagram

$$
\begin{array}{ccc}
A & \xleftarrow{\lambda} & \mathbb{I} \otimes A \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
(\mathbb{I} \otimes \mathbb{I}) \otimes A & \xrightarrow{(r \otimes s) \otimes f} & (\mathbb{I} \otimes \mathbb{I}) \otimes B \\
\downarrow{\lambda \otimes A} & & \downarrow{\lambda \otimes B} \\
\mathbb{I} \otimes (\mathbb{I} \otimes A) & \xrightarrow{r \otimes (s \otimes f)} & \mathbb{I} \otimes (\mathbb{I} \otimes B) \\
\end{array}
$$

The central square commutes by naturality of the associativity $\alpha$, and the two triangles commute by yet another axiom of monoidal categories ([Mac98, (4) p. 252]) together with $\lambda_{1,1} = \rho_{1,1}$, which we have already used above. By definition, the upper composition from $A$ to $B$ is $r \cdot (s \cdot f)$, and the lower one is $(r \cdot s) \cdot f = (r \circ s) \cdot f$, so the two are equal. The additivity of the action, that is $r \cdot (f + g) = r \cdot f + r \cdot g$ and $(r + s) \cdot f = r \cdot f + s \cdot f$, is a consequence of the additivity of $\circ$.

Finally, the identities required in part (iii) are obtained from $r \otimes (fg) = (r \otimes f)(1 \otimes g) = (1 \otimes f)(r \otimes g)$ by comparison along $\lambda$. □

2.1.7. REMARK. Proposition 2.1.6 (iii) says precisely that $\mathcal{T}$ is an $R_T$-linear category. In particular, we see that each $r \in R_T$ defines a natural transformation

$$r_{(-)}: \text{id}_T \rightarrow \text{id}_T,$$

and that the assignment $r \mapsto r_{(-)}$ is a ring homomorphism $R_T \rightarrow \text{End}(\text{id}_T)$ to the endomorphism ring of the identity functor of $\mathcal{T}$.

Next we see that the canonical action of $R_T$ commutes with the translation, so that $R_T$ also acts on morphisms of triangles (here we use $(\otimes \Delta^+)$).

2.1.8. LEMMA. We have $T(r \cdot f) = r \cdot T(f)$ for all $r \in R_T$ and all $f \in T(A,B)$. (That is, $r_{(-)}$ is an endomorphism of the triangle functor $(\text{id}_T, \text{id}_T)$).

PROOF. Build the following diagram using the arrows $\ell, r$ of Remark 2.1.2:

$$
\begin{array}{ccc}
T(A) & \xrightarrow{\ell \otimes r} & T(B) \\
\downarrow{T(\lambda \otimes)} & & \downarrow{T(\lambda_B)} \\
T(\mathbb{I} \otimes A) & \xrightarrow{T(r \otimes f)} & T(\mathbb{I} \otimes B) \\
\end{array}
$$

The two triangles commute by (2.1.3), and the square by naturality of $\ell$. The top path from left to right is $r \cdot Tf$, while the bottom one is $T(r \cdot f)$. □

2.1.9. COROLLARY. Letting an $r \in R_T$ act on a morphism of triangles $(a,b,c)$ defines another morphism or triangles $(r \cdot a, r \cdot b, r \cdot c)$:

$$
(2.1.10) \hspace{1cm} A \xrightarrow{r_{-a}} B \xrightarrow{r_{-b}} C \xrightarrow{T(r_{-c})} TA
$$

In particular, $R_T$ acts on each distinguished triangle $A \rightarrow B \rightarrow C \rightarrow TA$ via the automorphisms $(r_{[-A]}, r_{[-B]}, r_{[-C]})$.

PROOF. By the naturality of the action we can build a commutative diagram like (2.1.10), except that it displays $r \cdot Ta$ where we’d like to see $T(r \cdot a)$. But this is corrected by Lemma 2.1.8. □
2.1.2. The graded central ring \( R_T \). The translation automorphism \( T : T \to T \) allows us to construct a graded version of \( T \). Namely, consider the \( \mathbb{Z} \)-graded abelian groups

\[
T^*(A, B) := T(A, T^* B) := \bigoplus_{n \in \mathbb{Z}} T(A, T^n B)
\]

as the Hom sets of a category \( T^* \), where composition is defined by

\[
T(B, T^n C) \times T(A, T^m B) \to T(A, T^{n+m} C)
\]

\[
(B \overset{g}{\to} T^m C, A \overset{f}{\to} T^n B) \mapsto g \circ f := (A \overset{f}{\to} T^n (B) \overset{T^m g}{\to} T^{n+m} (C)).
\]

Clearly this composition is associative, and the category \( T^* \) contains \( T \) as the subcategory of degree-0 morphisms. We sometimes write \( |f| \) to indicate the degree, that is \( |f| = n \) means \( f \in T^n \).

2.1.11. Definition. We call the graded endomorphism ring

\[
R_T^* := T^*(\mathbb{I}, \mathbb{I}) = \bigoplus_n T(\mathbb{I}, T^n \mathbb{I})
\]

the graded central ring of \( T \).

We have seen in the previous subsection that the tensor product makes \( T \) an \( R_T \)-linear category. Similarly, it makes \( T \) ‘graded \( R_T^* \)-linear’ in a sense made precise by the next proposition. Recall that by \((\otimes \Delta +)\) the translation functor is \( T = S \otimes (\cdot) \). Define \( S^n \) recursively by \( S^0 := \mathbb{I} \), \( S^1 := S \) and \( S^n := S \otimes S^{n-1} \) \((n \geq 0)\).

(i) There is a canonical left action of \( R_T \), extending that of \( R_T \) on \( T(A, B) \) and turning \( T^*(A, B) \) into a graded left \( R_T^* \)-module. The action is uniquely determined by setting

\[
r \cdot f := \left( A \overset{\lambda^{-1}}{\to} \mathbb{I} \otimes A \overset{\rho \odot \lambda}{\to} (S^n \otimes \mathbb{I}) \otimes (S^m \otimes B) \cong S^{n+m} \otimes B \right)
\]

for \( r : \mathbb{I} \to T^n \mathbb{I} \) and \( f : A \to T^m B \) with \( n, m \geq 0 \), where the last isomorphism is any combination of the structural maps \( \alpha, \lambda, \rho \). Similarly, \( T^*(A, B) \) can be canonically made into a graded right \( R_T^* \)-module by

\[
f \cdot r := \left( A \overset{-r}{\cong} A \otimes \mathbb{I} \overset{\otimes \gamma}{\to} S^m \otimes B \otimes S^n \overset{1 \otimes \gamma}{\cong} S^m \otimes S^n \otimes B \cong S^{n+m} \otimes B \right).
\]

(ii) When \( A = B = \mathbb{I} \), both actions coincide with composition in \( T^*(\mathbb{I}, \mathbb{I}) \).
(iii) The left and right actions coincide up to the sign rule: \( r \cdot f = (-1)^{|r||f|} f \cdot r \).

In particular (with (ii)) the graded central ring \( R_T^* \) is graded commutative.
(iv) Composition in \( T^* \) is ‘graded \( R_T^* \)-bilinear’, in the sense that

\[
r \cdot (g \circ f) = (r \cdot g) \circ f = (-1)^{|r||g|} g \circ (r \cdot f) \quad \text{and}
\]

\[
(g \circ f) \cdot r = g \circ (f \cdot r) = (-1)^{|r||f|} (g \cdot r) \circ f
\]

for all homogeneous \( r \in R_T^*, f \in T^*(A, B) \) and \( g \in T^*(B, C) \).

Proof. Recall that we have \( \gamma_{S,S} = -1_{S \otimes S} \) by Axiom \((\otimes, \Delta +)\). Using an axiom of symmetric monoidal categories, one may derive from this by induction that

\[
(S^{n+m} = S^n \otimes S^m \overset{\gamma}{\to} S^m \otimes S^n = S^{n+m}) = (-1)^{nm} \cdot 1_{S^{n+m}}.
\]

As the reader already suspects, we are going to use the coherence theorem. More precisely one may proceed as follows. By the coherence theorem (see e.g. [JS93, §1]) there exists a tensor equivalence \( \Phi : T \simeq T' \) where \( T' \) is a strict tensor category, i.e., one where the isomorphisms \( \alpha, \rho \) and \( \lambda \) are identities. Now \( T' := \Phi(S) \otimes - \) is still an
endoequivalence, but we would like it to be an automorphism. This can be achieved by the well-known Spanier-Whitehead trick (see Construction A.4.1) coupled with Proposition A.5.25, which say: There exists a tensor equivalence \( \Psi : T' \simeq T'' \) such that the functor \( \Psi \Phi(S) \otimes - : T' \to T'' \) is an automorphism. It is also immediate from the proof of Proposition A.5.25 that \( T'' \) is again strict. Clearly, it is enough to prove the claims in the proposition for the pair \( (T', \Psi \Phi(S) \otimes -) \) instead of \( (T, S \otimes -) \), since we may transport the results along the tensor equivalence \( \Psi \Phi \); thus we may assume that \( T = (T, \otimes, 1, \gamma) \) is strict, and that \( T = S \otimes - \) is an automorphism.

We are going to generalize (2.1.15) using the coherence theorem for symmetric monoidal categories, simplified by the fact that we assumed \( T \) strict.

2.1.16. Lemma ([Mac98, Thm. XI.1.1]). For \( \sigma, \tau \in S_n \), there is exactly one isomorphism \( M^\sigma_n \Rightarrow M^n_\tau \) that can be constructed by using \( \times, \otimes \), identity maps and occurrences of \( \gamma \).

Call this the canonical isomorphism \( M^\sigma_n \Rightarrow M^n_\tau \).

2.1.17. Lemma. Let \( M^\sigma_n \Rightarrow M^n_\tau \) be the canonical isomorphism. Then its evaluation at \( S \) in all variables is equal to \( \text{sign}(\sigma^{-1}) \cdot 1_{S^n} : S^n \to S^n \).

Proof. This can be easily proved using \( \gamma_{S,S} = -1_{S^2} \) by recursion on the construction of the canonical isomorphism.

Now consider slightly more general functors \( F : T \times \cdots \times T \to T \), which we call \( S^n \)-words (in \( n \) variables), of the following form:

\[
F : (A_1, \ldots, A_n) \mapsto S^{m_1} \otimes A_{\sigma(1)} \otimes S^{m_2} \otimes \cdots \otimes A_{\sigma(n)} \otimes S^{m_{n+1}},
\]

where \( m_i \geq 0 \) for all \( i \) and \( \sum_i m_i = m \). Thus an \( S^n \)-word has \( n \) variable tensor factors and \( m \) constant factors equal to \( S \). We call an isomorphism \( \theta : F \Rightarrow G \) between \( S^n \)-words a \( \gamma \)-path if it is constructed out of \( \gamma \) and identities, just like the canonical morphisms. Note however that \( \gamma \)-paths are not unique, as we see from \( \gamma = -1 \neq 1 : S^2 \to S^2 \). As for canonical morphisms, each \( \gamma \)-path \( F \Rightarrow F \) induces a permutation of its factors by viewing every \( \gamma_{X,Y} \) in it as a transposition of its two arguments.

2.1.18. Corollary. Let \( F, G \) be two \( S^n \)-words with \( G = S^m \otimes M^n_\tau \), and let \( \theta_1, \theta_2 : F \Rightarrow G \) be two \( \gamma \)-paths. Then \( \text{sign}(\sigma_1) \cdot \theta_1 = \text{sign}(\sigma_2) \cdot \theta_2 \), where \( \sigma_i \) denotes the permutation induced on the \( S \)-factors by \( \theta_i \).

In other words, this says that any two ways of using \( \gamma \) to bring the \( S \)-factors to the front of an \( S^m \)-word differ only by the product of the signatures of the induced permutations on the \( S \)-factors.

Proof. By freeing the \( S \)-factors, we may regard each \( \theta_i \) as being the canonical morphism \( \tilde{F} := M^{m+n}_\sigma \Rightarrow M^{m+n}_{\sigma_1} \otimes M^n_\tau \). Build the following commutative triangle of canonical morphisms (on the left):

\[
\begin{array}{ccc}
M^{m+n}_{\sigma_1} \otimes M^n_\tau & \xrightarrow{\text{can}} & M^{m+n}_{\sigma_1} \\
\downarrow{\text{can = can} \otimes \text{id}} & & \\
M^{m+n}_\sigma \otimes M^n_\tau & \xrightarrow{\text{can}} & M^{m+n}_\sigma \\
\end{array}
\]

\[
\begin{array}{ccc}
S^m \otimes M^n_\tau & \xrightarrow{\text{sign}(\sigma_2)^{-1} \cdot 1_{S^n}} & S^m \otimes M^n_\tau \\
\downarrow{\theta_2} & & \\
S^m \otimes M^n_\tau & \xrightarrow{\theta_1} & S^m \otimes M^n_\tau \\
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{F} & \xrightarrow{\text{can}} & M^{m+n}_{\sigma_1} \otimes M^n_\tau \\
\downarrow{\text{can}} & & \\
M^{m+n}_\sigma \otimes M^n_\tau & \xrightarrow{\text{can}} & M^{m+n}_\sigma \\
\end{array}
\]
2.1. THE AXIOM(S) AND BASIC PROPERTIES

By reinserting the $S$-factors at their places, we obtain by Lemma 2.1.17 the commutative diagram on the right. This proves the claim.

Armed with this corollary, we now study the grading-preserving pairing in $T^*$

$\boxtimes : T^n(A, B) \otimes T^{n'}(A', B') \rightarrow T^{n+n'}(A \otimes A', B \otimes B')$

defined as follows: if $n, n' \geq 0$, set

$f \boxtimes f' := (A \otimes A' f \circ f') S^n \otimes B \otimes S^{n'} \otimes B' \overset{1 \otimes \gamma \otimes 1}{\rightarrow} S^{n+n'} \otimes B \otimes B' = S^{n+n'} \otimes B \otimes B'$,

and for general $n, n' \in \mathbb{Z}$ reduce to the first case by applying a high enough power of the automorphism $T: T \rightarrow T$.

Claim: The pairing $\boxtimes$ is associative and natural (for degree-0 morphisms).

Proof of Claim. Naturality is obvious, because $\otimes$ is a bifunctor and $\gamma$ is natural. Associativity follows easily from the observation that, for any $B, B', B'' \in T$, the square (where we put $\cdot$ instead of $\otimes$ for space)

$\begin{array}{ccc}
S^n \cdot B \cdot S^{n'} \cdot B' \cdot S^{n''} \cdot B'' & \rightarrow & S^n \cdot B \cdot S^{n'} \cdot S^{n''} \cdot B'. B'' \\
1 \otimes 1 \otimes 1 \downarrow & & \downarrow 1 \otimes 1 \otimes 1 \\
S^n \cdot S^{n'} \cdot B \cdot B' \cdot S^{n''} \cdot B'' & \rightarrow & S^n \cdot S^{n'} \cdot S^{n''} \cdot B' \cdot B''
\end{array}$

is commutative by Corollary 2.1.18.

End of the proof of Prop. 2.1.12. Since $T$ is strict, we see that the definition of $\boxtimes$ subsumes that of the left and right actions as defined in (2.1.13) and (2.1.14) (for the left action, note that the property $\gamma \circ \rho = \lambda$ becomes $\gamma_{\mathbb{Z}, X} = 1_X$ in strict tensor categories). Therefore part (i) follows from the Claim, once we know part (ii). Note also that, by the very definition of $\boxtimes$, in order to check the identities in (ii)-(iv) we may assume that the degrees of the morphisms are non-negative.

Given $f : \mathbb{1} \rightarrow S^n \otimes \mathbb{1} = S^n$ and $f' : \mathbb{1} \rightarrow S^{n'} \otimes \mathbb{1} = S^{n'}$, we see from the commutativity of the diagram

that $\boxtimes$ specializes to the composition in $T^*(\mathbb{1}, \mathbb{1})$. Thus part (ii) is also proved as soon as we settle part (iii). At this point we leave the verification of (iii) and (iv) to the reader; this involves the straightforward construction of some diagrams with lots of identity arrows like the one above (only bigger), where everything commutes up to a sign. By Corollary 2.1.18 one only has to keep track of the rearrangement of the prefixed powers of $S$ in order to see the correct signs appear.

2.1.3. Tensor triangle functors and tensor ideals.

2.1.19. Definition. A morphism of tensor triangulated categories, or tensor triangle functor (or $\otimes$-$\Delta$-functor), is simply a triangle functor $F = (F, \varphi)$ which is also a tensor functor: there are isomorphisms $\mathbb{1} \cong F(\mathbb{1})$ and $F(A) \otimes F(B) \cong F(A \otimes B)$ satisfying suitable coherence conditions (see [Mac98, XI.2]). We may
compose morphisms in the evident way. We say that \( F \) is strict if it is strict as a tensor functor, i.e., if the two structure isomorphisms are identities.

2.1.20. **Definition.** A tensor ideal (or \( \otimes \)-ideal) \( \mathcal{J} \) in a tensor triangulated category \( \mathcal{T} \) is a triangulated subcategory which is closed under tensor products with arbitrary objects: if \( A \in \mathcal{J} \) and \( X \in \mathcal{T} \) then \( A \otimes X \in \mathcal{J} \).

2.1.21. **Lemma.** Let \( \mathcal{J} \subseteq \mathcal{T} \) be a thick tensor ideal. Then the quotient category canonically inherits from \( \mathcal{T} \) a tensor structure making it a tensor triangulated category, and the quotient functor \( q : \mathcal{T} \to \mathcal{T}/\mathcal{J} \) becomes a strict tensor triangle functor.

**Proof.** Let \((w : A \to B), (v : C \to D) \in W(\mathcal{J})\). By constructing an Octahedron on the composition \( w \otimes v = (B \otimes v) \circ (w \otimes C) \), we find a distinguished triangle containing the three objects \( \text{cone}(w \otimes C) = \text{cone}(w) \otimes C, \text{cone}(B \otimes v) = B \otimes \text{cone}(v) \) and \( \text{cone}(w \otimes v) \) (note how we have used Axiom \((\otimes \Delta)\)). Since \( \mathcal{J} \) is a \( \otimes \)-ideal, the first two objects are in \( \mathcal{J} \), and therefore so is the third, showing that \( w \otimes v \in W(\mathcal{J}) \). Therefore we may set

\[
[fw^{-1}] \otimes [gv^{-1}] := [(f \otimes g)(w \otimes v)^{-1}]
\]

for any fractions \([fw^{-1}], [gv^{-1}] \in \text{Mor}(\mathcal{T}/\mathcal{J})\). Clearly, this is the only way to define a functor \( \otimes : \mathcal{T}/\mathcal{J} \times \mathcal{T}/\mathcal{J} \to \mathcal{T}/\mathcal{J} \) which makes

\[
\begin{array}{ccc}
\mathcal{T} \times \mathcal{T} & \xrightarrow{\otimes} & \mathcal{T} \\
q \times q & \downarrow & \downarrow q \\
\mathcal{T}/\mathcal{J} \times \mathcal{T}/\mathcal{J} & \xrightarrow{\otimes} & \mathcal{T}/\mathcal{J}
\end{array}
\]

commute. It is now straightforward to check that the new \( \otimes \) is indeed a functor. Then the structural isomorphisms \( \alpha, \lambda, \rho, \gamma \) (and \( r, \ell \)) define on the quotient analogous isomorphisms – whose naturality is easily checked –, which obviously will still make the necessary diagrams commute. The translation for \( \mathcal{T}/\mathcal{J} \) is \( qS \otimes (-) \), and \( qS \otimes qS = -1_{qS \otimes qS} \), so that the quotient also satisfies \((\otimes \Delta+)\). The canonical functor \( q \) is a strict morphism by construction. \( \square \)

We conclude with a nice remark on the interaction of ideals with complementary pairs (Def. 1.4.21).

2.1.22. **Lemma.** Let \((\mathcal{L}, \mathcal{R})\) be a complementary pair in a \( \otimes \)-\( \Delta \)-category \( \mathcal{T} \), and assume that \( \mathcal{L} \) and \( \mathcal{R} \) are \( \otimes \)-ideals. Then the gluing triangle for any object \( A \in \mathcal{T} \) is obtained by tensoring with \( A \) the gluing triangle

\[
L(1) \longrightarrow 1 \longrightarrow R(1) \longrightarrow TL(1)
\]

of the tensor unit. Moreover, the canonical maps \( L(1) \to 1 \) and \( 1 \to R(1) \) induce isomorphisms \( L(1) \otimes L(1) \cong L(1) \) and \( R(1) \otimes R(1) \cong R(1) \).

**Proof.** Simply tensor (2.1.23) with \( A \), and note that \( A \otimes L(1) \in \mathcal{L} \) and \( A \otimes R(1) \in \mathcal{R} \) because \( \mathcal{L} \) and \( \mathcal{R} \) are \( \otimes \)-ideals Then conclude by uniqueness of the gluing triangle for \( A \). The moreover part follows similarly by tensoring (2.1.23) with \( L(1) \), resp. \( R(1) \), and using the fact that \( \mathcal{L} \otimes \mathcal{R} \subseteq \mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{L}^\perp \simeq 0 \). \( \square \)

2.1.4. **\( \alpha \)-Compactly generated tensor triangulated categories.** In considering \( \otimes \)-\( \Delta \)-triangulated categories which also happen to be \( \alpha \)-compactly generated (Def. 1.7.1), we will always tacitly assume the following reasonable hypothesis:

\((\otimes 1)\) The tensor unit is \( \alpha \)-compact, and the tensor product commutes with \( \alpha \)-small coproducts.
2.1.24. Remark. Note that in case $T$ is a closed tensor category (see Section 2.4), then by definition for each object $A \in T$ the functor $- \otimes A$ has a right adjoint $\text{Hom}(A, -)$, so that it automatically commutes with all coproducts. On the other hand, one would still have to assume the compacity of $I$ (as shown for instance by localizing the homotopy category of spectra at an appropriate homology theory, see [HPS97, Ex. 1.2.3 (g)]).

For the rest of this section, let $T$ be an $\alpha$-compactly generated $\otimes$-$\Delta$-category.

2.1.25. Lemma. If $J \subseteq T_e$ is an $\otimes$-$\Delta$-ideal of $T_e$, then $\langle J \rangle_{\text{loc}}$ is an ($\alpha$-localizing)
$\otimes$-$\Delta$-ideal of $T$.

Proof. Recall that $\langle J \rangle_{\text{loc}}$ is $\alpha$-compactly generated with (a skeleton of) $J$
as generating set (Cor. 1.7.11), so Brown representability applies to this category: given an $A \in \langle J \rangle_{\text{loc}}$, we may find a distinguished triangle $\coprod A_i \to \coprod A_i \to A \to T(\coprod A_i)$ with all $A_i \in J$. If $C \in T_e$ is any compact objects, by our axiom $(\otimes \Delta)$
we obtain a distinguished triangle $\coprod C \otimes A_i \to \coprod C \otimes A_i \to C \otimes A \to T(\coprod C \otimes A_i)$.

Since $J$ is an ideal, each $C \otimes A_i$ belongs to $J$, and thus $\coprod C \otimes A_i \in \langle J \rangle_{\text{loc}}$. Since the latter category is triangulated, we get $C \otimes A \in \langle J \rangle_{\text{loc}}$. This shows that $\langle J \rangle_{\text{loc}}$
is closed at least under tensoring with $\alpha$-compact objects. But a general $B \in T$ fits
into some distinguished triangle $\coprod B_i \to \coprod B_i \to B \to T(\coprod B_i)$ with $B_i \in T_e$, and
by tensoring it with $A \in \langle J \rangle_{\text{loc}}$ we obtain, by what we just proved, a distinguished triangle $\coprod A \otimes B_i \to \coprod A \otimes B_i \to A \otimes B \to T(\cdots)$ where the first two vertices
belong in $\langle J \rangle_{\text{loc}}$. Hence so does the third vertex, that is $A \otimes B$. \hfill $\Box$

2.1.26. Corollary. If $T$ is such that $T_e \otimes T_e \subseteq T$, then the lattice isomorphism
of Corollary 1.7.11 restricts to thick $\otimes$-ideals of $T_e$ and $\alpha$-compactly generated
loc-$\Delta$-ideals of $T$.

Proof. Recall that the bijection sends a thick subcategory $J \subseteq T_e$ to $\langle J \rangle_{\text{loc}}$,
and it sends an $\alpha$-compactly generated localizing subcategory $L \subseteq T$ to the thick
subcategory $L_e$ of its $\alpha$-compact objects. The first assignment sends ideals to ideals by Lemma 2.1.25. On the other hand, if $L$ is a $\otimes$-ideal of $T$ we have
$T_e \otimes L_e \subseteq T_e \cap L = L_e$ by the hypothesis. \hfill $\Box$

Let’s return for a moment to the general case.

2.1.27. Lemma. Let $(S, \otimes, I)$ be any $\otimes$-$\Delta$-category. Let $\mathcal{E}, F \subseteq S$ be two classes
of objects. Then $\langle \mathcal{E} \rangle \otimes F \subseteq \mathcal{E} \otimes F$.

Proof. Let $B \in F$. By axiom $(\otimes \Delta)$, $(\cdot) \otimes B$ is a triangle functor, from which
we see that $\{X \in S \mid X \otimes B \in \langle \mathcal{E} \otimes B \rangle\} =: \mathcal{C}$ is a thick triangulated subcategory of $S$ containing $\mathcal{E}$, hence containing $\langle \mathcal{E} \rangle$. (Indeed, let $X$ be a direct summand of $X_3$ and let $X_1 \to X_2 \to X_3 \to TX_1$ be a distinguished triangle with $X_{1,2} \in \mathcal{C}$; then $X_1 \otimes B \to X_2 \otimes B \to X_3 \otimes B \to T(X_1 \otimes B)$ is distinguished with the first
two vertices in $\mathcal{C}$, and $X \otimes B$ is a direct summand of the third vertex $X_3 \otimes B$; hence $X \otimes B \in \langle \mathcal{E} \otimes B \rangle$, because the latter subcategory is triangulated and thick
by definition, showing that $X \in \mathcal{C}$.) Therefore $\langle \mathcal{E} \rangle \otimes B \subseteq \langle \mathcal{E} \otimes B \rangle \subseteq \langle \mathcal{E} \otimes F \rangle$ for
all $B \in F$, that is $\langle \mathcal{E} \rangle \otimes F \subseteq \langle \mathcal{E} \otimes F \rangle$. \hfill $\Box$

2.1.28. Proposition. Let $(S, \otimes, I)$ be any $\otimes$-$\Delta$-category, and consider two
classes of objects $\mathcal{E}, F \subseteq T$. Then $\langle \mathcal{E} \rangle \otimes \langle F \rangle \subseteq \langle \mathcal{E} \otimes F \rangle$. In particular, $\langle I \rangle \otimes \langle I \rangle \subseteq \langle I \rangle$.

Proof. Let $A \in \langle \mathcal{E} \rangle$. Since $A \otimes (\cdot)$ is a triangle functor, we see as above that the subcategory $\{Y \in S \mid A \otimes Y \in \langle \mathcal{E} \otimes F \rangle\}$ is a thick triangulated subcategory
of $S$ containing $F$, hence containing $\langle F \rangle$; therefore the intersection
\[
\bigcap_{A \in \langle \mathcal{E} \rangle} \{ Y \in S | A \otimes Y \in \langle \mathcal{E} \otimes F \rangle \} = \{ Y \in S | \mathcal{E} \otimes Y \subseteq \langle \mathcal{E} \otimes F \rangle \}
\]
also contains $\langle F \rangle$. Therefore $\langle \mathcal{E} \rangle \otimes \langle F \rangle \subseteq \langle \mathcal{E} \otimes F \rangle$, from which we derive with Lemma 2.1.27 the inclusion $\langle \mathcal{E} \rangle \otimes \langle F \rangle \subseteq \langle \mathcal{E} \otimes F \rangle$. □

2.1.29. COROLLARY. Let $\mathcal{T}$ be an $\alpha$-compactly generated $\otimes$-$\Delta$-category, and assume that there is some generating $\alpha$-small set $\mathcal{G}_0 \subseteq \mathcal{T}$ such that $\mathcal{G}_0 \otimes \mathcal{G}_0 \subseteq \mathcal{G}_0$. Then $\mathcal{T} \otimes \mathcal{T} \subseteq \mathcal{T}$.

Proof. By Proposition 1.7.8 we know that $\mathcal{T} = (\mathcal{G}_0)$, so the statement follows immediately from Proposition 2.1.28 for $\mathcal{E} := \mathcal{F} := \mathcal{G}_0$.

A very amenable kind of tensor triangulated category is the following:

2.1.30. DEFINITION. An $\alpha$-compactly generated $\otimes$-$\Delta$-category is monogenic (if it satisfies $\langle \otimes \rangle \otimes T \subseteq \langle \otimes \rangle$ and if the countable set $T^* \mathbb{1} = T^* \{ \mathbb{1} \} := \{ T^n \mathbb{1} | n \in \mathbb{Z} \}$ provides a set of $\alpha$-compact generators for it.

Since $T^* \mathbb{1} \otimes T^* \mathbb{1} \subseteq T^* \mathbb{1}$ and $\langle T^* \mathbb{1} \rangle = \langle \mathbb{1} \rangle$, we obtain by Corollary 2.1.29:

2.1.31. LEMMA. Let $\mathcal{T}$ be a monogenic $\alpha$-compactly generated $\otimes$-$\Delta$-category. Then $\mathcal{T} \otimes \mathcal{T} \subseteq \mathcal{T}$.

2.1.32. COROLLARY. In a monogenic $\alpha$-compactly generated $\otimes$-$\Delta$-category $\mathcal{T}$, every thick subcategory of $\mathcal{T}$ is automatically a thick $\otimes$-$\Delta$-ideal of $\mathcal{T}_\circ$. Similarly, an $\alpha$-localizing subcategory is automatically an $(\alpha$-localizing) $\otimes$-$\Delta$-ideal of $\mathcal{T}$.

Proof. Let $\mathcal{C} \subseteq \mathcal{T}_\circ$ be a thick subcategory. Then $\mathcal{T}_\circ \otimes \mathcal{C} = \langle \mathbb{1} \rangle \otimes \mathcal{C} \subseteq \langle \mathbb{1} \otimes \mathcal{C} \rangle = \mathcal{C}$ by Lemma 2.1.27. This proves the first statement. Every $\alpha$-localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ has the form $\langle \mathcal{C} \rangle_{\text{loc}} \subseteq \mathcal{T}$ where $\mathcal{C}$ is the thick subcategory $\mathcal{L} \subseteq \mathcal{T}$ (Cor. 1.7.11), so the second claim follows from the first and Lemma 2.1.25.

2.1.33. REMARK. As we shall see, a genuine compactly generated category can always be equipped with an internal Hom functor $\text{Hom}$; one may use this to show that $\mathcal{T}_\circ \otimes \mathcal{T}_\circ \subseteq \mathcal{T}_\circ$ (cf. [HPS97, Thm. 2.1.3 (d)]). On the other hand, we shall prove in Subsection 2.4.1 that the hypothesis $\mathcal{T}_\circ \otimes \mathcal{T}_\circ \subseteq \mathcal{T}_\circ$ guarantees that the internal Hom functor $\text{Hom}(B, C)$ for $\mathcal{T}$ is defined at least if $B \in \mathcal{T}_\circ$ (and $C$ arbitrary), and therefore it is everywhere defined on the $\otimes$-$\Delta$-subcategory $\mathcal{T}_\circ$.

2.2. The spectrum of a tensor triangulated category

In this section, we give a concise presentation of some core ideas of tensor triangular geometry. For the proofs we refer to the original papers [Bal02] [Bal05] [Bal07]. Some of Balmer’s work initially assumed a noetherian hypothesis which was later removed in [BKS07].

The organizing concept of triangular geometry is the spectrum of a $\otimes$-$\Delta$-category.

2.2.1. DEFINITION. (Paul Balmer). Let $\mathcal{T}$ be an essentially small tensor triangulated category. A prime ideal in $\mathcal{T}$ is a proper and thick (Def. 1.3.4) subcategory $\mathcal{P} \subseteq \mathcal{T}$ which is a tensor ideal, i.e., such that $A \otimes B \in \mathcal{P}$ whenever one of $A$ or $B$ is in $\mathcal{T}$, and which is prime, i.e., such that if $A \otimes B \in \mathcal{P}$, then either $A \in \mathcal{P}$ or $B \in \mathcal{P}$. The spectrum of $\mathcal{T}$, denoted by $\text{Spc}(\mathcal{T})$, is the set of all its prime ideals. It comes equipped with the Zariski topology, which is given by the following basis of open sets:

$$U(A) := \{ \mathcal{P} | A \in \mathcal{P} \} \subseteq \text{Spc}(\mathcal{T}) \quad A \in \mathcal{T}.$$
The closed complement
\[ \text{supp}(A) := \{ \mathcal{P} \mid A \not\in \mathcal{P} \} = \text{Spc}(T) \setminus U(A) \]
is called the **support** of the object \( A \).

2.2.2. **Remarks.** (a) A point \( \mathcal{P} \) belongs to the support of \( A \) iff \( A \not\simeq 0 \) in the localization \( T/\mathcal{P} \); that is, the support of \( A \) contains exactly those points of the spectrum ‘at which’ \( A \) doesn’t vanish.

(b) We assumed that \( T \) is locally small so that its spectrum is a small topological space, as we all like them to be. Of course the definition still makes sense in general. Similar remarks can be made throughout the following. However, the ‘nice’ categories where the spectrum has its most useful applications so far are all essentially small. Indeed, in all the examples in my knowledge where the spectrum has been computed explicitly, \( T \) appears as the subcategory of compact objects in some ambient compactly generated category.

The following proposition shows that the spectrum deserves its name:

2.2.3. **Proposition.** The topological space \( \text{Spc}(T) \) is spectral in the sense of Hochster [Hoc69], i.e., it has the following properties: (i) it is quasi-compact, (ii) the quasi-compact open subsets form an open basis (in particular, they are closed under taking finite intersections), and (iii) every non-empty irreducible closed subset has a unique generic point (in particular \( \text{Spc}(T) \) is \( T_0 \): two points with the same closure must be the same point).

**Proof.** Property (i) is [Bal05, Cor. 2.15], (ii) is [Bal05, Rem. 2.7], and (iii) is [Bal05, Prop. 2.18] (see also [BKS07] for more remarks on spectral spaces in this context). \( \square \)

2.2.4. **Remark.** We recall that Melvin Hochster [Hoc69] has shown that spectral spaces, and spectral maps thereof (those for which the preimage of every quasi-compact open is again quasi-compact open), are exactly the topological spaces and maps which occur in the image of the Zariski spectrum functor \( R \mapsto \text{Spec}(R) \) for commutative rings. This nice fact is not overly useful though, as it can be shown that there is no functor, right inverse to the Zariski spectrum, which realizes such rings and homomorphisms, unless one is ready to severely restrict the relevant categories (see loc. cit. for details).

The spectrum is a contravariant functor from locally small \( \otimes \Delta \)-categories to spectral spaces:

2.2.5. **Proposition.** Every morphism \( F : T \to S \) of \( \otimes \Delta \)-categories induces a continuous spectral map \( \text{Spc}(F) : \text{Spc}(S) \to \text{Spc}(T) \) defined simply by
\[
\mathcal{P} \mapsto \text{Spc}(F)(\mathcal{P}) := F^{-1}\mathcal{P} := \{ A \in T \mid F(A) \in \mathcal{P} \}.
\]
If \( F, G \) are composable morphisms, then \( \text{Spc}(G \circ F) = \text{Spc}(F) \circ \text{Spc}(G) \), and \( \text{Spc}(\text{id}_T) = \text{id}_{\text{Spc}(T)} \). Moreover,
\[
(2.2.6) \quad \text{Spc}(F)^{-1}(\text{supp}_T(A)) = \text{supp}_S(FA)
\]
for every object \( A \in T \).

**Proof.** See [Bal05, Prop. 3.6]. To see that \( \text{Spc}(F) \) is a spectral map (which is not stated in loc. cit.), one only has to know that the quasi-compact open subsets of the spectrum are precisely those of the form \( U(A) \) for some object \( A \) ([Bal05, Prop. 2.14]); then it follows from (2.2.6) and \( U(A) = \text{supp}(A)^c \) that \( \text{Spc}(F)^{-1}U(A) = U(FA) \), showing that the preimage of a quasi-compact open is again quasi-compact, as required. \( \square \)
Similarly to the Zariski spectrum, Balmer’s spectrum behaves nicely with respect to quotients by ideals (see [Bal05, Prop. 3.11]).

Next we explain the universal property which earns the spectrum \( \text{Spc}(T) \) the honorific “the”. It will also become clear that Definition 2.2.1 is not a naïve aping of the construction of the Zariski spectrum of a ring, but on the contrary it has compelling conceptual justifications of its own.

**2.2.7. Definition.** A **support datum** for \( T \) is a pair \((X, \sigma)\), where \( X \) is a topological space, and \( \sigma \) is a function assigning to every object of \( T \) a closed subset of \( X \). This is subject to the following five axioms.

(SD1) \( \sigma(0) = \emptyset \) and \( \sigma(\mathbb{1}) = X \).
(SD2) \( \sigma(A \oplus B) = \sigma(A) \cup \sigma(B) \).
(SD3) \( \sigma(TA) = \sigma(A) \).
(SD4) \( \sigma(C) \subseteq \sigma(A) \cup \sigma(B) \) if there is a distinguished triangle \( A \to B \to C \to TA \).
(SD5) \( \sigma(A \otimes B) = \sigma(A) \cap \sigma(B) \).

A **morphism of support data** \( f : (X, \sigma) \to (X', \sigma') \) (on the same category \( T \)) is a continuous map \( f : X \to X' \) such that \( f^{-1}(\sigma'(A)) = \sigma(A) \) for all \( A \in T \).

**2.2.8. Theorem.** (Universal Property). The spectrum \( (\text{Spc}(T), \text{supp}) \) of Definition 2.2.1 is the universal (final) support datum on \( T \): It is a support datum, and for every other support datum \( (X, \sigma) \) on \( T \) there exists a unique continuous map \( f : X \to \text{Spc}(T) \) such that \( \sigma(A) = f^{-1}(\text{supp}A) \) for all \( A \in T \). Moreover, \( f \) is explicitly given by

\[
 f(x) = \{ A \in T \mid x \notin \sigma(A) \} \in \text{Spc}(T).
\]

**Proof.** [Bal05, Theorem 3.2]. □

Later on we will be handling ‘supports’ on ‘big’ categories for which some of the properties of Def. 2.2.7 may fail. Accordingly, we enlarge our vocabulary a little:

**2.2.9. Definition.** Let \( \sigma : \text{obj}(T) \to 2^X \) be an assignment from the objects of \( T \) to (possibly non closed) subsets of \( X \). We call \((X, \sigma)\) a **pre-support datum** if it satisfies (SD1)-(SD4). We say that \((X, \sigma)\) is a **generalized support datum** if it satisfies (SD1)-(SD5) (and hence it is a true support datum if it has closed images). We say that \((X, \sigma)\) is **continuous** if for every coproduct \( \coprod A_i \in T \) the equality \( \sigma(\coprod A_i) = \bigcup_i \sigma(A_i) \) holds. For any pre-support datum \((X, \sigma)\) on \( T \), we will write \( \text{Ker}(\sigma) = \{ A \in T \mid \sigma(A) = \emptyset \} \).

It will be useful to note some obvious general facts.

**2.2.10. Lemma.** Fix a pre-support datum \((X, \sigma)\) on \( T \).

(i) For every subset \( Y \subseteq X \), the full subcategory \( \mathcal{T}_{\sigma,Y} := \{ A \mid \sigma(A) \subseteq Y \} \subseteq T \)

is a thick subcategory of \( T \). If \((X, \sigma)\) is continuous, then \( \mathcal{T}_{\sigma,Y} \) is localizing. If \((X, \sigma)\) is a generalized support datum, then \( \mathcal{T}_{\sigma,Y} \) is a thick \( \otimes \)-ideal. (In particular this holds for \( \text{Ker}(\sigma) = \mathcal{T}_{\sigma,0} \).

(ii) The pre-support datum \((X, \sigma)\) induces a pre-support datum \((X, \overline{\sigma})\) on the Verdier quotient \( T/\text{Ker}(\sigma) \) by setting \( \overline{\sigma}(qA) := \sigma(A) \). By construction, \( \overline{\sigma} \)

detects objects: \( \overline{\sigma}(A) = 0 \implies A \cong 0 \) for any \( A \in T/\text{Ker}(\sigma) \).

(iii) If \((X, \sigma)\) is a generalized support datum (or a support datum, or continuous) on \( T \), then so is \((X, \overline{\sigma})\) on \( T/\text{Ker}(\sigma) \).

□

A thick \( \otimes \)-ideal \( \mathcal{J} \) of \( T \) is called **radical** if \( A^{\otimes n} \in \mathcal{J} \) for some \( n \in \mathbb{N} \) implies \( A \in \mathcal{J} \). As stated in the next fundamental theorem, the spectrum classifies thick
radical ideals by means of suitable subsets: a $Y \subseteq \text{Spc}(T)$ is a Thomason subset if it has the form $Y = \bigcup_i Z_i$, where each $Z_i$ has quasi-compact open complement.¹

2.2.11. **Theorem.** There is an inclusion preserving bijection between the set of radical thick $\otimes$-ideals of $T$ and the set of Thomason subsets of $\text{Spc}(T)$, given by

\begin{align*}
Y & \mapsto T_Y := \{ A \in T \mid \text{supp}(A) \in Y \} \\
J & \mapsto \text{supp}(J) := \bigcup_{A \in J} \text{supp}(A).
\end{align*}

for a Thomason subset $Y \subseteq \text{Spc}(T)$ and a radical ideal $J \subseteq T$.

**Proof.** [Bal05, Theorem 4.10].

Conversely, a support datum that classifies radical ideals must be isomorphic to $\text{Spc}(T)$. More precisely:

2.2.14. **Definition.** A support datum $(X, \sigma)$ on $T$ is said to be classifying if:

(i) the space $X$ is spectral (see Proposition 2.2.3), and

(ii) the assignments (2.2.12) and (2.2.13) (defined using $\sigma$ instead of $\text{supp}$) are mutually inverse bijections between Thomason subsets of $X$ and radical ideals of $T$.

2.2.15. **Theorem.** Let $(X, \sigma)$ be a classifying support datum on $T$. Then the unique morphism $f : (X, \sigma) \to (\text{Spc}T, \text{supp})$ induced by the universal property is an isomorphism.

**Proof.** This is [Bal05, Theorem 5.2], up to a noetherian hypothesis that was eliminated in [BKS07].

2.2.16. **Remarks.** (a) Theorem 2.2.15 gives another conceptual characterization of the spectrum: it is (up to unique isomorphism of support data) the unique classifying support datum.

(b) It happens sometimes that all thick $\otimes$-ideals of $T$ are automatically radical, for instance when $T$ is rigid (see Def. 2.4.11) as we now show. By Lemma 2.4.8 every object $A$ is a direct summand of $A \otimes A' \otimes A$. Therefore, if $J$ is any thick $\otimes$-ideal of $T$, and $A^\otimes n \in J$ for some $n \geq 1$, we see that $A^\otimes \lfloor n/2 \rfloor$ is a summand of $A^\otimes n \otimes (\cdots) \in J$ and is therefore in $J$. Recursively, we reduce the problem to the case when $A \otimes A \in J$. But then $A$ is a summand of $A^\otimes 2 \otimes A' \in J$ and therefore is in $J$, showing that $J$ is radical.

(c) If the space $X$ is noetherian (all open subsets are compact), such as the spectrum of a noetherian commutative ring, or the underlying space of an algebraic variety, then a Thomason subset $Y \subseteq X$ is simply a union of closed subsets, that is, a specialization closed subset: $x \in Y \Rightarrow [x] \subseteq Y$.

As in algebraic geometry, the support can be used to detect the vanishing and decomposition properties of objects and morphisms of $T$. We only quote:

2.2.17. **Proposition** ([Bal05, Cor. 2.4]). Let $A \in T$ be any object. Then $A \cong 0$ in $T/P$ for all $P \in \text{Spc}(T)$, if and only if $A$ is $\otimes$-nilpotent.

Such results are particularly strong, once again, for rigid categories, where $A^\otimes n \cong 0$ iff $A \cong 0$. Using the spectrum in the same spirit leads to a systematic way of producing local-to-global spectral sequences, with applications for instance to the algebraic $K$-theory of schemes ([Bal07] [Bal07b]).

¹The name ‘Thomason subset’ is justified by [Ths97, Theorem 3.15], the precursor of Theorem 2.2.11. Conceptually, a Thomason subset of a spectral space $X$ is just an open set in the Hochster dual topology of $X$ ([Hoc69]).
2.2.1. **Local rings and the structure sheaf.** The spectrum \( \text{Spc}(T) \) of a tensor triangulated category comes naturally equipped with a sheaf of rings, which is constructed as follows. For a closed subset \( Z \subseteq \text{Spc}(T) \), denote by
\[
T_Z = \{ A \mid \text{supp}(A) \subseteq Z \} \subseteq T
\]
the full subcategory of objects supported on \( Z \), as we did above. This is a thick \( \otimes \)-ideal, so the Verdier quotient \( T/T_Z \) inherits a tensor triangulated structure from that of \( T \), where the tensor unit, again written \( 1 \otimes 1 \), is the image of \( 1 \otimes 1 \in T \). In particular, the ring \( R_{T/T_Z} = \text{End}_{T/T_Z}(1 \otimes 1) \) is commutative.

2.2.18. **Definition.** The **structure sheaf** \( \mathcal{O}_T \) of a \( \otimes \)-triangulated category \((T, \otimes, 1)\) is defined to be the sheafification of the presheaf of rings
\[
U \mapsto R_{T/T_U} = \text{End}_{T/T_U}(1 \otimes 1)
\]
(where \( U^c = \text{Spc}(T) \setminus U \)). This is a sheaf of commutative rings, which turns the spectrum into a ringed space, denoted by
\[
\text{Spec}(T) := (\text{Spc}(T), \mathcal{O}_T).
\]
In the rest of this section we are going to prove that the stalks of the structure sheaf are always local rings.

2.2.19. **Lemma.** The stalk of \( \mathcal{O}_T \) at a point \( P \in \text{Spc}(T) \) is given by
\[
\mathcal{O}_{T,P} = \varinjlim_{U \ni P} R_{T/T_U} \cong R_{T/P}.
\]

**Proof.** Notice that \( P = \bigcup_{U \ni P} T_U \). Indeed, let \( U \) be an open containing \( P \). If \( A \in T_U \) then \( \text{supp}(A) \) and \( U \) are disjoint; in particular \( P \notin \text{supp}(A) \) and therefore \( A \in P \). Conversely, if \( A \in P \) then \( P \) is contained in the open subset \( \text{supp}(A) \), so \( A \in T_{\text{supp}(A)} \). Now it follows from this equality that for any two objects \( A, B \in T \), the group \( T/P(A, B) \), as defined via right fractions, is (canonically isomorphic to) the colimit \( \varinjlim_{U \ni P} T/T_U(A, B) \). In particular, the case \( A = B = 1 \) proves the claim. (Note that the identification is a ring isomorphism, since the transition maps in the colimit are given by functors.) \( \square \)

2.2.20. **Lemma.** Let \( f \in R_T = \text{End}_T(1) \). Then
\[
\langle (\text{cone}(f))_\otimes \rangle = \{ A \in T \mid f|A \text{ is nilpotent} \}
\]
where \( (X)_\otimes \) is used to denote the thick \( \otimes \)-ideal in \( T \) generated by the object \( X \). In particular, \( \langle (\text{cone}(f^n))_\otimes \rangle = \langle (\text{cone}(f))_\otimes \rangle \) for all \( n \geq 1 \).

**Proof.** Write \( \mathcal{C} := \{ A \in T \mid f|A \text{ is nilpotent} \} \) for the category on the right hand side. It is easy to see that \( \mathcal{C} \) is a thick tensor ideal: it is triangulated because, if \( A \to B \to C \to TA \) is a distinguished triangle with \( A, B \in \mathcal{C} \), then \( (f|A, f|B, f|C) \) is an automorphism of this triangle (Corollary 2.1.9) with two nilpotent components \( f|A \) and \( f|B \), and thus \( f|C \) is also nilpotent (Lemma 1.1.12), showing that \( C \in T \). It is clearly thick; if \( A \) is a direct summand of \( A' \), then \( f|A \) restricts to \( f|A \) by naturality of the \( R_T \)-action and we see that if the first map is nilpotent so must be the second. Finally, \( \mathcal{C} \) is a tensor ideal because \( f|A \otimes 1_B = (f|A) \otimes 1_B \). Now consider a distinguished triangle \( 1 \to C_f \to T \to T \) containing \( f \). By Corollary 2.1.9 there is a morphism of distinguished triangles:

\[
\begin{array}{c}
\begin{array}{ccc}
1 & \xrightarrow{f} & 1 \\
\downarrow f & & \downarrow f \\
1 & \xrightarrow{g} & C_f & \xrightarrow{h} & T(1)
\end{array}
\end{array}
\]
Since \((f|_{C_f})g = gf = 0\) and \(h(f|_{C_f}) = T(f)h = 0\), we can substitute 0 for the two vertical \(f\)'s and the one \(T(f)\) in (2.2.21) and still have a morphism of triangles. Now we apply Lemma 1.1.12 to conclude that \(f|_{C_f}\) is nilpotent, so that \(C_f \in \mathcal{C}\). Since the latter is a thick tensor ideal, this implies \((C_f)\subseteq \mathcal{C}\), as wished.

On the other hand, for every object \(A\) the morphism \(f|_A\) has cone \(C_f \oplus A\) (by our axiom \((\otimes \Delta)\)), and therefore \(f|_A\) becomes an isomorphism in \(T/(C_f)\). Assume now that \(f|_A\) is nilpotent. Then its image in the quotient is a nilpotent automorphism of \(A \in T/(C_f)\). It follows that \(1_A\) is nilpotent and therefore zero in \(T/(C_f)\). Since \((C_f)\) is thick, \(A\) belongs to \((C_f)\), proving the other inclusion \(\mathcal{C} \subseteq (C_f)\).

2.2.22. **Lemma.** Let \(\mathcal{P} \in \text{Spec}(\mathcal{T})\) and let \(A, B \in T/\mathcal{P}\). If \(A \otimes B \cong 0\), then either \(A \cong 0\) or \(B \cong 0\).

**Proof.** The thick \(\otimes\)-ideal \(0 = \{A \mid A \cong 0\} \subseteq T/\mathcal{P}\) is prime because \(\mathcal{P}\) is prime in \(T\); if we denote by \(q : T \to T/\mathcal{P}\) the localization functor, then \(0 \cong qA \otimes qB = q(A \otimes B)\) means \(A \otimes B \in \mathcal{P}\) because \(\mathcal{P}\) is thick, and thus either \(A \in \mathcal{P}\) or \(B \in \mathcal{P}\) because \(\mathcal{P}\) is prime.  

2.2.23. **Theorem.** \(\text{Spec}(\mathcal{T})\) is a locally ringed space.

**Proof.** We have to prove that each ring \(\mathcal{O}_{\mathcal{T}, \mathcal{P}}\) is local, or equivalently, that the set of non-invertible elements thereof is closed under sum and is therefore an ideal. By contraposition and by Lemma 2.2.19 we may consider elements \(f, g \in \text{End}_{T/\mathcal{P}}(\mathbb{1})\) such that \(f + g\) is invertible in \(\text{End}_{T/\mathcal{P}}(\mathbb{1})\), and will have to prove that either \(f\) or \(g\) is also invertible. Consider the morphism

\[ h := (f + g)|_{\text{cone}(f) \otimes \text{cone}(g)} : \text{cone}(f) \otimes \text{cone}(g) \to \text{cone}(f) \otimes \text{cone}(g). \]

It is an isomorphism, since \(\text{cone}(h) \cong \text{cone}(f + g) \otimes \text{cone}(f) \otimes \text{cone}(g)\) and since \(\text{cone}(f + g) = 0\) by hypothesis. By Lemma 2.2.20, both \(f|_{\text{cone}(f) \otimes \text{cone}(g)}\) and \(g|_{\text{cone}(f) \otimes \text{cone}(g)}\) are nilpotent endomorphisms, and therefore, by the binomial formula, their sum \(h\) is also nilpotent. Thus \(h\) is a nilpotent automorphism, which implies that \(\text{cone}(f) \otimes \text{cone}(g) \cong 0\). It follows from Lemma 2.2.22 that either \(\text{cone}(f)\) or \(\text{cone}(g)\) is zero, i.e., as wished, that either \(f\) or \(g\) is invertible in \(\text{End}_{T/\mathcal{P}}(\mathbb{1})\).  

2.2.24. **Remark.** If \(F : \mathcal{T} \to \mathcal{S}\) is a morphism of \(\otimes\)-\(\Delta\)-categories, the induced map \(\text{Spc}(F) : \text{Spc}(\mathcal{S}) \to \text{Spc}(\mathcal{T})\) can be naturally extended to a morphism of locally ringed spaces \((\text{Spec}(F), \text{Spec}(F)^2) : \text{Spec}(\mathcal{S}) \to \text{Spec}(\mathcal{T})\). Indeed, the sheaf homomorphism \(\text{Spec}(F)^2\) can be obtained in the obvious way from the functors

\[ \mathcal{T}/U = \mathcal{S}/\mathcal{S}(\text{Spec}(F)^{-1}U)^c \]

induced by \(F\) for all open \(U \subseteq \text{Spec}(\mathcal{T})\). One has to check that on stalks it gives a local homomorphisms of local rings, and indeed for each \(P \in \text{Spec}(\mathcal{S})\) the ring homomorphisms \(\text{Spec}(F)^{P} : \text{R}_{\mathcal{T}/F^{-1}P} \to \text{R}_{\mathcal{S}/P}\) induced by \(F\) is local, since \(f \in \mathfrak{m}_{F^{-1}P} \Leftrightarrow f\) is not invertible in \(\text{R}_{\mathcal{T}/F^{-1}P} \Leftrightarrow \text{cone}(f) \not\in F^{-1}P \Leftrightarrow \text{cone}(F(f)) \not\in \mathcal{P} \Leftrightarrow f \in \mathfrak{m}_{P}\).

Thus Balmer’s spectrum can be seen as a functor from the category of (essentially small) \(\otimes\)-\(\Delta\)-categories to that of locally ringed spaces, i.e., to ‘geometry’.

2.2.2. **The natural comparison** \(\rho : \text{Spec}(\mathcal{T}) \to \text{Spec}(\mathcal{R}_{\mathcal{T}})\). So far we have learned two ways of assigning, functorially, a spectral space to a \(\otimes\)-\(\Delta\)-category \(\mathcal{T}\): a first way is by the Zariski spectrum \(\text{Spec}(\mathcal{R}_{\mathcal{T}})\), another by Balmer’s spectrum \(\text{Spc}(\mathcal{T})\). It is only natural to compare the two. In a very recent preprint [Bal08], Paul Balmer addresses this question by studying the following construction:
2.2.25. **Proposition.** There is a continuous map $\rho_T : \text{Spc}(T) \to \text{Spec}(R_T)$, which is defined by

$$\rho_T(P) := \{ f | \text{cone}(f) \not\in P \} \subseteq R_T \quad (P \in \text{Spc}(T)).$$

This map is spectral, and it is natural: it defines a natural transformation $\rho : \text{Spc} \to \text{Spec}$ of functors from $\otimes$-$\Delta$-categories to spectral spaces.

**Proof.** Let $q : T \to T/P$ be the quotient functor, and consider its restriction $q : R_T = \text{End}_T(1) \to \text{End}_T/P(1) = R_T/P$. By Theorem 2.2.23, $R_T/P$ is a local ring. Its maximal ideal $m_P$ is the set of non-units. But $\rho_T(P)$ is the preimage of $m_P$ along the ring homomorphism $q$:

$\rho_T(P) \overset{\text{Def.}}{=} \{ f | \text{cone}(f) \not\in \rho(T) \}$

$= \{ f | \text{cone}(q(f)) = q(\text{cone}(f)) \not\in 0 \text{ in } T/P \}$

$= q^{-1}(m_P),$

and therefore it is a prime ideal of $R_T$. To see that $\rho_T$ is continuous and spectral, it is enough to check that the preimage of a Zariski basic open $D_f := \{ p | f \not\in p \} \subseteq \text{Spec}R$ is quasi-compact open. Indeed, by definition

$\rho^{-1}_T D_f = \{ P | f \not\in \rho(T) \}$

$= \{ P | \text{cone}(f) \in P \}$

$= U(\text{cone}(f))$

and we know that the latter is a typical quasi-compact open subset of $\text{Spc}T$. Now let $F : T \to S$ be a morphism of $\otimes$-$\Delta$-categories. We have to show that the square

$$\begin{array}{ccc}
\text{Spc}S & \xrightarrow{\rho_S} & \text{Spc}T \\
\rho_S \downarrow & & \rho_T \downarrow \\
\text{Spec}R_S & \xrightarrow{\rho_T} & \text{Spec}R_T
\end{array}$$

is commutative. Once again, it’s enough to unfold the definitions and to use that $F$ is a triangle functor, in order to see that

$$\text{Spec } F \circ \rho_S(P) = F^{-1} \{ f \in R_S | \text{cone}(f) \not\in P \}$$

$= \{ g \in R_T | F(\text{cone } g) = \text{cone } F(g) \not\in P \}$

$= \rho_T(F^{-1}P)$

$= \rho_T \circ \text{Spec } F(P)$

for every thick $\otimes$-prime $P$ of $S$. \qed

2.2.27. **Remark.** Since we know that $\text{Spec}(T) = (\text{Spc}(T), O_T)$ is a locally ringed space (Thm. 2.2.23), a more conceptual way to construct $\rho$ is via the universal property of the Zariski spectrum, which says that it is contravariantly adjoint to the global section functor: there is a natural isomorphism

$$\Phi : \text{Hom}_{\text{Rings}}(R, O_X(X)) \cong \text{Hom}_{\text{R}}(X, \text{Spec}(R))$$

for $R$ a commutative ring and $X$ a locally ringed space (Exercise). Given a ring homomorphism $\varphi : R \to O_X(X)$, the corresponding map $\Phi(\varphi) : X \to \text{Spec}(R)$ is (at the space level) the one sending $x \in X$ to the preimage of the maximal ideal $m_x \subseteq O_X(x)$ along $\varphi \circ \text{res} : O_X(X) \to O_X(x)$. Unfolding the definitions, one sees that $\rho_T$ is obtained as $\Phi$ of the canonical homomorphism $R_T \to O_T(\text{Spc } T)$. In particular, $\rho_T$ can be canonically upgraded to a morphism of locally ringed spaces.

Using $R_T^\ast$ instead of $R_T$ one may define a graded version of $\rho_T$. 

2.2.28. Definition. Given a graded commutative ring \( R \), we denote by \( \text{Spec}^h(R) \) the set of homogeneous prime ideals in \( R \), equipped with the Zariski topology (the closed subsets have the form \( \mathcal{V}(I) = \{ p \in \text{Spec}^h(R) \mid p \supseteq I \} \) for a homogeneous ideal \( I \); recall that an ideal is homogeneous if it is generated by its homogeneous elements). This yields a functor \( \text{Spec}^h \) from graded commutative rings to spectral spaces.

One may then define a comparison map

\[ \rho_T^* : \text{Spc}(\mathcal{T}) \longrightarrow \text{Spec}^h(R_T^*) \]

\[ \mathcal{P} \mapsto \rho_T^*(\mathcal{P}) := \langle f \in R_T^* \mid f \text{ homogeneous and cone } f \notin \mathcal{P} \rangle \]

so that one has a commutative diagram

\[
\begin{array}{ccc}
\text{Spc}(\mathcal{T}) & \xrightarrow{\rho_T^*} & \text{Spec}^h(R_T^*) \\
\downarrow & & \downarrow (-)^0 \\
\text{Spec}(R_T) & & \text{Spec}(R_T)
\end{array}
\]

where \((-)^0\) indicates the restriction to degree-0 elements. Since we shall make no use of it, we leave it to the reader to formulate and prove the analog for \( \rho_T^* \) of Proposition 2.2.25.

2.3. Central localization

In this section we present another idea of [Bal08]. As we have seen in Subsection 2.1.1, the central ring \( R_T \) acts nicely on the whole category \( \mathcal{T} \). In particular, each Hom set \( \mathcal{T}(A,B) \) is an \( R_T \)-module, so we may localize it with respect to any multiplicative subset \( S \subseteq R_T \). It turns out that this simple procedure yields a new \( \otimes' \)-\( \Delta \)-category.

We begin by the following construction, which is the obvious generalization of the localization of an \( R \)-algebra (or an \( R \)-module) at a multiplicative system of central elements.\(^2\)

2.3.1. Lemma. Let \( R \) be a commutative ring with 1, and let \( \mathcal{C} \) be an \( R \)-linear category (the Hom sets are \( R \)-modules and composition is \( R \)-bilinear). Let \( S \subseteq R \) be a multiplicative system (\( 1 \in S \) and \( S \cdot S \subseteq S \)).

(i) Then the composition in \( \mathcal{C} \) descends to an associative bilinear map

\[ S^{-1}\mathcal{C}(B,C) \times S^{-1}\mathcal{C}(A,B) \rightarrow S^{-1}\mathcal{C}(A,C) \]

\[ \left( \frac{g}{t}, \frac{f}{s} \right) \mapsto \frac{g}{t} \circ \frac{f}{s} := \frac{g \circ f}{ts} \]

thus defining an \( S^{-1}R \)-linear category \( S^{-1}\mathcal{C} \) with the same objects as \( \mathcal{C} \), and with Hom sets given by the localized \( R \)-modules \( S^{-1}\mathcal{C}(A,B) \). There is a canonical \( R \)-linear functor

\[ \text{loc} : \mathcal{C} \rightarrow S^{-1}\mathcal{C} \]

which sends an \( f \in \mathcal{C}(A,B) \) to the fraction \( \frac{f}{1} \in S^{-1}\mathcal{C}(A,B) \).

(ii) Moreover, the canonical functor \( \text{loc} : \mathcal{C} \rightarrow S^{-1}\mathcal{C} \) has the usual universal property of localizations: If \( F : \mathcal{C} \rightarrow \mathcal{D} \) is another \( R \)-linear functor to an \( S^{-1}R \)-linear category (i.e., all \( s \in S \) act invertibly on each Hom set of \( \mathcal{D} \)), then there exists a unique \( S^{-1}R \)-linear functor \( F : S^{-1}\mathcal{C} \rightarrow \mathcal{D} \) such that \( F \circ \text{loc} = F \).

\(^2\)By contrast, localization à la Gabriel-Zisman is a non-obvious generalization of the same.
Proof. All claims require only straightforward verifications, which we omit. We only comment that the associativity of the composition in \( S^{-1}C \) is a consequence of the bilinearity (‘centrality’) of the \( R \)-action on \( C \). Given three composable fractions \( \frac{a}{b} \), \( \frac{c}{d} \) and \( \frac{e}{f} \), we may multiply with \( uts \) the element \( \frac{(a, c)}{(b, d)} \) of \( S^{-1}C \). By bilinearity we obtain \( (h_2)f - h(gf) = 0 \) in \( C \), and hence \( \frac{a}{b} = \frac{a}{b} \) in \( S^{-1}C \). \( \square \)

If \( S \) is a multiplicative set in the central ring \( R_T \) of a tensor triangulated category \( T \), we call \( S^{-1}T \) the central localization of \( T \) at \( S \). Reassuringly, \( S^{-1}T \) is also a Verdier verdier quotient of \( T \) by the evident thick \( \otimes \)-ideal:

2.3.2. Theorem (Central localization). Let \( S \subseteq R_T \) be a multiplicative system in the central ring of a tensor triangulated category \( T \). Consider the thick tensor ideal 

\[ J_S := (\text{cone}(s) \mid s \in S)_{\otimes} \]

of \( T \) generated by the cones of morphisms in \( S \), and let \( q : T \to T/J_S \) be the Verdier quotient. Then \( q \) induces a canonical \( R_T \)-linear isomorphism \( \overline{\pi} : S^{-1}T \cong T/J_T \) such that \( \overline{\pi} \circ \text{loc} = q \). In particular, the central localization \( S^{-1}T \) is a \( \otimes \)-\( \Delta \)-category with \( R_S^{-1}T = S^{-1}R_T \), and \( \text{loc} \) is a \( \otimes \)-\( \Delta \)-functor.

In order to prove the theorem we need two preliminary facts.

2.3.3. Lemma. \( J_S = \{ A \in T \mid \exists s \in S \text{ such that } s|_A = 0 \} \).

Proof. Write \( J'_S \) for the class on the right hand side, and note that, since \( S \) is a multiplicative set, it is the same as \( J'_S = \{ A \mid \exists s \in S \text{ such that } s|_A \text{ nilpotent} \} \). Thus by Lemma 2.2.20 we have \( J'_S = \bigcup_{s \in S} (A)_{\otimes} \subseteq J \). To prove the other inclusion \( J_S \subseteq J'_S \) it suffices to see that \( J'_S \) is a thick \( \otimes \)-ideal. It is a triangulated subcategory by Lemma 1.1.12, and it is thick and a \( \otimes \)-ideal because \( \otimes \) is \( R_T \)-bilinear (as in the proof of Lemma 2.2.20).

2.3.4. Lemma. Let \( g \in T(B, X) \). Then \( \text{cone}(g) \in J_S \) if and only if there exist \( k, \ell : X \to B \) and an \( s \in S \) such that \( kg = s|_B \) and \( \ell g = s|_A \).

Proof. Consider a distinguished triangle containing \( g \) and its endomorphism induced by any \( s \in S \) (Cor. 2.1.9):

\[
\begin{tikzcd}
B \arrow[r, g] \arrow[dd, s|_B] & X \arrow[rd, s|_X] \arrow[r, u, mapsto, cone(g)] & cone(g) \arrow[r, v] \arrow[dd, s|_{cone(g)}] & TB \arrow[dd, T(s|_B)] \\
\downarrow{k} \downarrow{\ell} \downarrow{s|_X} & \downarrow{s|_{cone(g)}} & \downarrow{T(s|_B)} \downarrow{\text{cone}(g)} \downarrow{TB} \downarrow{u} \downarrow{TB}
\end{tikzcd}
\]

If \( \text{cone}(g) \in J_S \), by Lemma 2.3.3 we may choose an \( s \in S \) with \( s|_{\text{cone}(g)} = 0 \). Then \( us|_X = s|_{\text{cone}(g)} \) \( u = 0 \) and (with a rotation) \( s|_B T^{-1}(v) = T^{-1}(vs|_{\text{cone}(g)}) = 0 \), so that we find \( \ell \) and \( k \) with the desired properties by the weak kernel and cokernel properties. If on the other hand such \( k, \ell \) exist for some \( s \in S \), then \( s|_{\text{cone}(g)} \) \( u = 0 \) and \( vs|_{\text{cone}(g)} = 0 \). Thus we may substitute \( 0 \) for \( s|_B \) and \( s|_X \) in the (solid) diagram, and conclude with Lemma 1.1.12 that \( s|_{\text{cone}(g)} \) is nilpotent. But the powers of \( s \) are again in \( S \), so \( \text{cone}(g) \in J_S \) by Lemma 2.3.3.

Proof of Theorem 2.3.2. For each \( s \in S \) and each \( A \in T \) we have \( \text{cone}(s|_A) = \text{cone}(s) \otimes A \in J_S \) and therefore \( s|_A \) becomes invertible in \( T/J_S \). Hence by Lemma 2.3.1 the quotient functor \( q : T \to T/J_S \) induces a (unique) \( S^{-1}R \)-linear functor \( \overline{\pi} : S^{-1}T \to T/J_S \), which is the identity on objects and sends \( \frac{A}{s} \in S^{-1}T(A, B) \) to the (say) left fraction \( [(s|_B)^{-1} \circ f] \in T/J_S(A, B) \). We
show that \( \overline{\eta} \) is fully-faithful. \textit{Fall:} Let \( [A \xrightarrow{f} X \xleftarrow{g} B] \in T/J_S(A,B), \) so that \( \text{cone}(g) \in J_S. \)

\[
\begin{array}{c}
A \xrightarrow{f} X \xleftarrow{g} B \\
\downarrow k \quad \ \quad \downarrow s_{|B} \\
\quad \quad B
\end{array}
\]

By Lemma 2.3.4 there exist \( s \in S \) and \( k : X \to B \) with \( kg = s |_B \). Therefore \( [fg^{-1}] = [(s |_B)^{-1}(kf)] = \overline{\eta}((k|_B)) \) comes from \( S^{-1}T \). \textit{Faithful:} Now let \( \frac{f}{s} \in S^{-1}T(A,B) \). If \( [(s |_B)^{-1}f] = 0 \), by definition there exists in \( T \) a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow h \quad & \quad & \downarrow s_{|B} \\
B & \xleftarrow{g} & B
\end{array}
\]

with \( \text{cone}(h) \in T_S \), and therefore \( \text{cone}(g) \in J_S \) by thickness. As before, by Lemma 2.3.4 there are \( t \in S \) and \( k : X \to B \) with \( kg = t |_B \). Hence

\[
t \cdot f = t |_B \circ f = (kg) f = k(gf) = 0
\]

for a \( t \in S \), showing that \( f \) vanishes in \( S^{-1}T \). \hfill \Box

2.3.5. \textbf{Remark.} As in the previous subsection, there is a graded version of Theorem 2.3.2. We don’t go into that, as we shall not need it. On the other hand, we are going to use the following (minimalist) graded version of Lemma 2.3.1.

Let \( T \) be a \( \otimes \Delta \)-category. Recall the \( \mathbb{Z} \)-graded category \( T^\ast \) from Subsection 2.1.2. By Proposition 2.1.12, its Hom sets \( T(A,T^*B) \) are graded (left, say, to fix ideas) modules over the graded central ring \( R_T \), which is graded commutative. We want to localize this category at homogeneous prime ideals \( p \in \text{Spec}^h(R_T^\ast) \). For this we use:

2.3.6. \textbf{Lemma.} Let \( R \) be a graded commutative ring, and let \( C \) be a ‘graded \( R \)-linear category’, as in Prop. 2.1.12 (the Hom sets are graded \( R \)-modules and composition is graded bilinear: \( r \cdot (g \circ f) = (r \cdot g) \circ f = (-1)^{|r||g|} g \circ (r \cdot f) \) for homogeneous morphisms). Let \( S \) be a multiplicative set in \( R \) consisting of homogeneous elements of even degree. Then there is a \( \mathbb{Z} \)-graded category \( S^{-1}C \) with the same objects as \( C \), with Hom sets the localized graded modules \( S^{-1}C(A,B) \), and with composition induced from that of \( C \) just as in Lemma 2.3.1 (i).

\textbf{Proof.} By the formula for graded bilinearity, we see that homogeneous elements \( r \in R \) of even degree are central, i.e. they satisfy \( r \cdot (f \circ g) = (r \cdot f) \circ g = f \circ (r \cdot g) \).

It is now clear that the formula \( \left( \frac{f}{s} \right) \circ \left( \frac{g}{t} \right) := \left( \frac{fsg}{st} \right) \) for composing fractions gives a well-defined associative composition for \( S^{-1}C \), just as in Lemma 2.3.1 \hfill \Box

For a homogeneous prime \( p \in \text{Spec}^h(R) \), we consider the multiplicative subset of \( R \)

\[
(2.3.7) \quad S_p := (R \setminus p) \cap \bigoplus_n R^{2n}.
\]

If \( T \) is a tensor triangulated category, by Proposition 2.1.12 we may apply Lemma 2.3.6 to the category \( C := T^\ast \) for the canonical graded action of \( R := R_T^\ast \), and
get a localized graded category $S^{-1}_p T^*$ at each graded homogeneous prime $p \in \text{Spec}^h(R_T)$.

We note:

2.3.8. Proposition. Let $M$ be a graded module over a graded commutative ring $R$. Then $M \cong 0$ iff $S^{-1}_p M \cong 0$ for all $p \in \text{Spec}^h(R)$.

Proof. One shows first by Zorn that every homogeneous ideal is contained in a maximal, and therefore prime, homogenous one. Then the classical proof as for the ungraded case applies, homogenized ([AM69, Prop. 3.8]).

2.3.1. Central localization of $\alpha$-compactly generated categories. As witnessed by its title, the goal of this subsection is to adapt the procedure of central localization to $\alpha$-compactly generated categories so that, by starting out with one, we still get one. The straightforward approach fortunately turns out to work just fine.

Fix an $\alpha$-compactly generated tensor triangulated category $T$, and let $S \subseteq R_T$ be a multiplicative system of its central ring. Consider the thick $\otimes$-ideal

$$\mathcal{J}_S := \langle \text{cone}(s) \mid s \in S \rangle_{\otimes} \subseteq T_c,$$

as well as its $\alpha$-localizing closure

$$\mathcal{L}_S := \langle \mathcal{J}_S \rangle_{\text{loc}} \subseteq T.$$

Notice that $\mathcal{L}_S$ is again a (thick) $\otimes$-ideal, thanks to Lemma 2.1.25. What we now do is apply central localization by $S$ on the category $T_c$ of $\alpha$-compact objects, and see what happens to the big category $T$. By Theorem 2.3.2, the quotient functor $q_c : T_c \rightarrow T_c/\mathcal{J}_S$ induces the evident isomorphism

$$(2.3.9) \quad q_c : S^{-1}T_c \cong T_c/\mathcal{J}_S$$

of $R_T$-linear categories sending $\frac{[A \otimes B]}{s} \mapsto [f(s|_A)^{-1}] = [s|_B]^{-1} f = s^{-1} q_c(f)$. By construction, the category $\mathcal{L}_S$ is $\alpha$-compactly generated with $\alpha$-compact objects $(\mathcal{L}_S)_c = \mathcal{J}_S \subseteq T_c$. Therefore Neeman’s Localization Theorem kicks in, showing in particular that the induced functor $F$ in the diagram below is a fully faithful embedding (Lemma 1.7.27).

$$\begin{array}{ccc}
\mathcal{L}_S \xrightarrow{\mathcal{J}_S} & T \xrightarrow{q} & T/\mathcal{L}_S \\
\parallel & \parallel \uparrow & \parallel \\
\mathcal{J}_S \xrightarrow{q_c} & T_c \xrightarrow{\mathcal{J}_S} & T_c/\mathcal{J}_S
\end{array}$$

Recall now that a Neeman localization is a Bousfield localization (Theorem 1.7.31), so that all the results of Section 1.4 apply: The functor $q$ has a right adjoint $q_r : T/\mathcal{L}_S \rightarrow T$, the inclusion functor $i : \mathcal{L}_S \hookrightarrow T$ has a right adjoint $i_r : T \rightarrow \mathcal{L}$, and the unit $\eta : id \rightarrow q_r q = R_S$ and counit $\varepsilon : L_S := ii_r \rightarrow id$ of the respective adjunctions assemble into a functorial distinguished triangle

$$(2.3.10) \quad \xymatrix{ L_S(A) \ar[r]^-{\varepsilon_A} & A \ar[r]^-{\eta_A} & R_S(A) \ar[r] & TL_S(A) }$$

which is the gluing triangle for the complementary pair $\mathcal{L}_S, \text{Im}(R_S))$. Moreover, $\text{Im}(R_S) = \mathcal{L}_S^\perp$ is $\alpha$-localizing (which is equivalent to the fact that $q_r$ commutes with $\alpha$-coproducts).

2.3.11. Lemma. The quotient functor $q : T \rightarrow T/\mathcal{L}_S$ is $R_T$-linear and it inverts all endomorphisms of the form $s|_A$ ($s \in S, A \in T$).
Proof. Let \( s \in S \) and \( A \in T \). Then \( \text{cone}(s|_A) = \text{cone}(s) \otimes A \in \mathcal{L}_S \), because \( \text{cone}(s) \in \mathcal{J}_S \subseteq \mathcal{L}_S \) by definition and \( \mathcal{L}_S \) is a \( \otimes \)-ideal. \( \square \)

In particular, by the universal property of central localization (Lemma 2.3.1(ii)) the functor \( q : T \rightarrow T/L_S \) factors as

\[
\begin{array}{ccc}
T & \xrightarrow{\eta} & T/L_S \\
\downarrow & & \downarrow \\
S^{-1}T & \xrightarrow{\sim} & T_c/J_S
\end{array}
\]

We clearly have a commutative square

(2.3.12) \[
\begin{array}{ccc}
S^{-1}T & \xrightarrow{\eta} & T/L_S \\
\downarrow & & \downarrow \\
S^{-1}T_c & \xrightarrow{\eta} & T_c/J_S
\end{array}
\]

where every functor is the identity on objects. The next result and its corollary ought to be compared with [BIK07, Thm. 4.7].

2.3.13. Proposition. The canonical functor \( \eta \) restricts to an isomorphism

\[
\bar{\eta} : S^{-1}T(C,B) \xrightarrow{\sim} T(L_S(C,B)).
\]

of \( S^{-1}R_T \)-modules for all \( C \) \( \alpha \)-compact and \( B \in T \) arbitrary.

Proof. Fix a \( C \in T_c \). We may view

(2.3.14) \[
\bar{\eta} : S^{-1}T(C,-) \rightarrow T(L_S(C,-))
\]

as a morphism of homological functors to \( S^{-1}R_T \)-modules, both of which commute with \( \alpha \)-small coproducts. Moreover, \( \bar{\eta} \) is an isomorphism on compact objects, as we see from (2.3.12). It follows that (2.3.14) is an isomorphism on the \( \alpha \)-localizing subcategory generated by \( T_c \), which is equal to the whole category \( T \). \( \square \)

2.3.15. Corollary. Let \( C,B \in T \) with \( C \) \( \alpha \)-compact. Then \( \eta_B : B \rightarrow q_r q(B) = R_S(B) \) induces an isomorphism

\[
\beta : S^{-1}T(C,B) \xrightarrow{\sim} T(C,R_S(B))
\]

of \( R_T \)-modules.

Proof. Since \( \eta \) is natural, the following square commutes for all \( f : C \rightarrow B \),

\[
\begin{array}{ccc}
C & \xrightarrow{\eta_B} & q_r q(C) \\
\downarrow & & \downarrow \\
B & \xrightarrow{\eta_B} & q_r q(B)
\end{array}
\]

showing that the next (solid) diagram is commutative.
The functor \( q_r \) is fully faithful by the general properties of Bousfield localization, while \((\eta_C)^*\) is an isomorphism by the general properties of adjunctions (i.e., by the universal property of the unit). By the compactness of \( C \) and by Proposition 2.3.13, \( q \) induces the isomorphism \( \eta \). Composing this isomorphism with the other two, we see that \( \beta \), the factorization of \((\eta_B)^*\) through \( \text{loc} \), is an isomorphism as claimed. \( \square \)

2.3.16. Lemma. The endofunctors \( L_S \) and \( R_S \) are \( RT \)-linear: we have equality

\[
R_S(r \cdot f) = r \cdot R_S(f) : R_S(A) \to R_S(B).
\]

for every \( r \in RT \), \((f : A \to B) \in \text{Mor}(T)\), and similarly with \( L_S \). In particular, setting \( f := 1_A \) we obtain \( R_S(r \upharpoonright_A) = r \upharpoonright_{R_S(A)}; R_S(A) \to R_S(A) \).

Proof. Consider the (solid arrow) diagram with distinguished rows

\[
\begin{array}{cccc}
L_S A & \to & A & \to & R_S(A) & \to & TL_S A \\
\downarrow & & \downarrow r f & & \downarrow & & \downarrow \\
L_S B & \to & B & \to & R_S B & \to & TL_S B.
\end{array}
\]

We may complete it to a morphism of distinguished triangles in two ways. First, by using the functoriality of the distinguished triangle:

\[
\begin{array}{cccc}
L_S A & \to & A & \to & R_S(A) & \to & TL_S A \\
\downarrow L_S(r \cdot f) & & \downarrow r f & & \downarrow R_S(r \cdot f) & & \downarrow TL_S(r \cdot f) \\
L_S B & \to & B & \to & R_S B & \to & TL_S B.
\end{array}
\]

A second way is to use on \( f \) the functoriality of the triangle, and then to apply the canonical \( RT \)-action (and Corollary 2.1.9):

\[
\begin{array}{cccc}
L_S A & \to & A & \to & R_S(A) & \to & TL_S A \\
\downarrow r \cdot L_S(f) & & \downarrow r f & & \downarrow r \cdot R_S(f) & & \downarrow r(TL_S(f)) + T(r \cdot L_S(f)) \\
L_S B & \to & B & \to & R_S B & \to & TL_S B.
\end{array}
\]

But \( TL_S A \in L_S \), while \( R_S B \in L_S^+ \). Thus \( T(TL_S A, R_S B) = 0 \), and Lemma 1.4.3 shows that the two morphisms of triangles must coincide. \( \square \)

The following is a useful converse of Lemma 2.3.11.

2.3.17. Lemma. If \( A \in T \) is such that \( s \upharpoonright_A \) is invertible for all \( s \in S \), then \( \eta_A : A \to R_S(A) \) is an isomorphism. In particular, \( A \in \text{Im}(R_S) \).
Moreover, the following hold true:

The hypothesis on \( A \) implies that \( \text{loc} \) is an isomorphism. By Corollary 2.3.15, the map \( \beta \) is an isomorphism on compact objects. Hence their composition (\( \eta_A \)_*) is a morphism of cohomological functors both of which send coproducts to products (since they are representable), and such that it is an isomorphism at each \( C \in \mathcal{T}_c \). By Corollary 1.6.24, \( (\eta_A)_* \) is an isomorphism at every object. By Yoneda, \( \eta_A \) is an isomorphism in \( \mathcal{T} \), showing that \( A \in \text{Im}(R_S) \).

(Note that the middleman \( S^{-1}\mathcal{T}(-, A) \), however, is cohomological but does not send coproducts to products in general.) □

**2.3.18. Corollary.** The subcategory \( \text{Im}(R_S) = \mathcal{L}_S^1 \subseteq \mathcal{T} \) is a \( \otimes \)-ideal.

**Proof.** Consider an object in the essential image of \( R_S \), which we may assume to be of the form \( R_S(A) \), and let \( B \in \mathcal{T} \) be arbitrary. By Lemma 2.3.11, \( q \) inverts all morphisms \( s|_A \) with \( s \in S \). By Lemma 2.3.16, \( R_S \) is \( R_T \)-linear, so (using that also \( - \otimes B \) is \( R_T \)-linear) we may compute

\[
\sigma|_{R_S(A)\otimes B} = (\sigma|_{R_S(A)}) \otimes B = R_S(s|_A) \otimes B
\]

and see that every \( s \in S \) acts invertibly on \( R_S(A) \otimes B \). We conclude with Lemma 2.3.17 that \( R_S(A) \otimes B \in \text{Im}(R_S) \), as wished. □

So both \( \mathcal{L}_S \) and \( \mathcal{L}_S^1 \) are \( \otimes \)-ideals. Hence Lemma 2.1.22 yields

**2.3.20. Corollary.** The gluing triangle \((2.3.10)\) of each object \( A \in \mathcal{T} \) is obtained by tensoring \( A \) with the gluing triangle

\[
\mathcal{L}_S(\mathbb{1}) \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} R_S(\mathbb{1}) \xrightarrow{\eta} \mathcal{T}\mathcal{L}_S(\mathbb{1})
\]

of the tensor unit. Moreover,

\[
\mathcal{L}_S(\mathbb{1}) \otimes \mathcal{L}_S(\mathbb{1}) \cong \mathcal{L}_S(\mathbb{1}) \quad \text{and} \quad R_S(\mathbb{1}) \otimes R_S(\mathbb{1}) \cong R_S(\mathbb{1})
\]

along \( \varepsilon \), resp. \( \eta \). □

We now integrate the results of this subsection into one big fat theorem, for ease of reference.

**2.3.22. Theorem.** Let \( \mathcal{T} \) be an \( \alpha \)-localizing compactly generated \( \otimes \)-\( \Delta \)-category, and let \( S \) be a multiplicative subset of the central ring \( R_T \). Write

\[
\mathcal{J}_S := \langle \text{cone}(s) \mid s \in S \rangle_{\otimes} \subseteq \mathcal{T}_c \quad \text{and} \quad \mathcal{L}_S := \langle \mathcal{J}_S \rangle_{\text{loc}} \subseteq \mathcal{T}.
\]

Then the pair

\[
(\mathcal{L}_S, \mathcal{T}_S := \mathcal{L}_S^1)
\]

is a complementary pair (Def. 1.4.21) of localizing \( \otimes \)-ideals in \( \mathcal{T} \). In particular, the gluing triangle for an object \( A \in \mathcal{T} \) is obtained by tensoring \( A \) with the gluing triangle for the \( \otimes \)-unit

\[
\mathcal{L}_S(\mathbb{1}) \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} R_S(\mathbb{1}) \xrightarrow{\eta} \mathcal{T}\mathcal{L}_S(\mathbb{1})
\]

Moreover, the following hold true:

(i) \( \mathcal{L}_S = \mathcal{L}_S(\mathbb{1}) \otimes \mathcal{T} \) and \( \mathcal{T}_S = R_S(\mathbb{1}) \otimes \mathcal{T} \).

(ii) \( \varepsilon : \mathcal{L}_S(\mathbb{1}) \cong \mathcal{L}_S(\mathbb{1}) \otimes \mathcal{L}_S(\mathbb{1}) \) and \( \eta : R_S(\mathbb{1}) \cong R_S(\mathbb{1}) \otimes R_S(\mathbb{1}) \).
(iii) \(L_S\) and \(T_S\) are \(\alpha\)-compactly generated \(\otimes\)-triangulated, with tensor unit \(L_S(1)\) and \(R_S(1)\), respectively (but neither is a \(\otimes\)-\(\Delta\)-subcategory of \(T\), since they have different units).

(iv) Their \(\alpha\)-compact objects are \((L_S)_c = L_S \cap T = J_S\) and \((T_S)_c = (R_S(1)) \subseteq T_S\). (Note that the compact objects of \(L_S\) are also compact in \(T\), but those of \(T_S\) need not be.)

(v) The functors

\[
L_S \otimes - : T \to L_S \quad \text{and} \quad R_S(1) \otimes - : T \to T_S
\]

are \(R_T\)-linear \(\otimes\)-\(\Delta\)-triangle functors commuting with \(\alpha\)-small coproducts.

They take generating sets to generating sets.

(vi) To apply \(T(1, -)\) on \(\mathbb{I} \to_R R_S(1)\) induces the localization \(R_T \to S^{-1} R_T\).

(vii) \(A \in T_S \iff s_A \) is invertible for every \(s \in S\).

(viii) If \(A \in T_c\), then \(\eta : B \to R_S(1) \otimes B\) induces an isomorphism

\[
S^{-1} T(A, B) \cong T(A, R_S(1) \otimes B)
\]

for every \(B \in T\).

Proof. Everything has been proved already, or is otherwise clear. To wit, point (i) follows from \(L_S = \text{Im}(L_S)\) and \(T_S = \text{Im}(R_S)\) together with \(L_S = L_S(1) \otimes -\) and \(R_S = R_S(1) \otimes -\). Point (iv) and (v) are immediate consequences of Neeman localization and of \(L_S, T_S\) being \(\alpha\)-localizing \(\otimes\)-ideals. Point (iii) holds by the same reasons, together with (ii) which shows that \(L_S(1)\) and \(R_S(1)\) act as tensor units. Point (vii) is Lemma 2.3.11 and Lemma 2.3.17 (together with \(q_t\) is fully faithful). Point (viii) is Corollary 2.3.15.

2.3.24. Notation. We shall only apply the theorem to the case when \(S = R_T \setminus p\) for some prime ideal \(p \in \text{Spec } R_T\). It will be convenient to use the following lighter notation:

\[
p_A := L_S(A) \quad , \quad A_p := R_S(A) \quad , \quad T_p := L_S^\perp \quad , \quad q_p := (-)_p : T \to T_p.
\]

We prefer to write \(T_p\) (rather than, say, \(R_p\)) because we will think of this category as the localization of \(T\) at \(p\), and in order not to forget the asymmetry of the situation (cf. parts (iv) and (vi)-(viii) of the theorem).

2.3.25. Remark. The authors of [BIK07] prove very similar results (and much more) for good old genuine compactly generated categories, without need for a tensor structure. Instead of the central ring \(R_T\), they posit a (graded) noetherian ring acting on \(T\) via endomorphisms of \(\text{id}_T\), compatibly with the translation. In the case of a compactly generated tensor triangulated category (with our same hypothesis \((\otimes \square)\) and assuming that all compact objects are strongly dualizable – cf. next Section), they also prove Theorem 2.3.22 for the graded central ring \(R_T\), but only when it is noetherian; see [BIK07, §8]). Wishing to use their theorem, I came across the apparently insurmountable problem that in the \(\alpha\)-relative case Brown representability for the dual does not hold (cf. Remark 1.6.28), and there is no everywhere-defined internal Hom (cf. Remark 2.4.12), both of which \textit{loc. cit.} makes crucial use. This led to the above formulation and proof.

2.4. Strongly dualizable objects

Besides compactness, we will also make use of another ‘smallness’ notion, namely that of a strongly dualizable object. To begin with, let \(\mathcal{M}\) be any \textbf{closed tensor category}, that is a tensor category (= symmetric monoidal category) \(\mathcal{M} = (\mathcal{M}, \otimes, 1)\) together with an \textbf{internal Hom functor}, that is a functor

\[
\text{Hom} : \mathcal{M}^{\text{op}} \times \mathcal{M} \to \mathcal{M}
\]
equipped with isomorphisms

\[(2.4.1) \quad \mathcal{M}(A \otimes B, C) \cong \mathcal{M}(A, \text{Hom}(B, C))\]

natural in \(A, B, C \in \mathcal{M}\). It follows that each \(\text{Hom}(B, -)\) is right adjoint to \(- \otimes B : \mathcal{M} \to \mathcal{M}\). Reassuringly, this is also sufficient. For let \(\mathcal{M}\) be a tensor category such that, for every pair \(B, C \in \mathcal{M}\) there is an object \(\text{Hom}(B, C)\) and an isomorphism \(\phi_{B,C} : \mathcal{M}(- \otimes B, C) \cong \mathcal{M}(-, \text{Hom}(B, C))\). Given \(b : B' \to B\) and \(c : C \to C'\), the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}(- \otimes B, C) & \xrightarrow{\phi_{B,C}} & \mathcal{M}(-, \text{Hom}(B, C)) \\
\mathcal{M}(- \otimes b, c) & & \downarrow{\alpha(b,c)} \\
\mathcal{M}(- \otimes B', C') & \xrightarrow{\phi_{B',C'}} & \mathcal{M}(-, \text{Hom}(B', C'))
\end{array}
\]

defines a natural transformation \(\alpha(b, c)\) which, by Yoneda, is induced by a unique morphism \(\text{Hom}(b, c) : \text{Hom}(B, C) \to \text{Hom}(B', C')\). By the functoriality of \(\mathcal{M}(- \otimes ?, ?)\), this data defines a functor \(\text{Hom} : \mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}\), and the collection of the \(\phi_{B,C}\) provides an isomorphism of functors \(\mathcal{M}^{op} \times \mathcal{M} \to \mathcal{M}\) as in \((2.4.1)\).

2.4.2. Remark. In a closed tensor category, the tensor products always commute with coproducts, because each \(- \otimes A\) has the right adjoint \(\text{Hom}(A, -)\).

2.4.3. Example. Let \(\mathcal{T}\) be a genuine compactly generated tensor triangulated category, where the tensor product commutes with coproducts (as we always assume). Then any two \(B, C \in \mathcal{T}\) yield a decent cohomological functor \(\mathcal{T}(- \otimes B, C)\), which by Brown representability is represented by an object \(\text{Hom}(B, C)\) of \(\mathcal{T}\). As explained above, a choice of such objects and isomorphisms \(\mathcal{T}(- \otimes B, C) \cong \mathcal{T}(-, \text{Hom}(B, C))\) for every pair \((B, C)\) assemble into an internal Hom functor which makes \(\mathcal{T}\) a closed monoidal category.

2.4.4. Remark. In our axiomatisation of a tensor triangulated category, we assume that the tensor product is a triangle functor in each variable. It is natural to ask oneself if this implies, in the Example 2.4.3, that \(\text{Hom} : \mathcal{T}^{op} \times \mathcal{T} \to \mathcal{T}\) is also a triangle functor in both variables (here \(\mathcal{T}^{op}\) is endowed with the negative triangulation, with translation \(\mathcal{T}^{-1}\) and where \((h^{op}, g^{op}, f^{op})\) is distinguished iff \((-f, -g, -h)\) is distinguished in \(\mathcal{T}\)). Indeed, this is true in many important examples and is assumed in [HPS97] as part of their axioms for a stable homotopy category. Now the functor \(\text{Hom}(B, -)\), being right adjoint to the triangle functor \(- \otimes B\), is also canonically a triangle functor by Prop. 1.1.27. Concerning \(\text{Hom}(-, C)\), the natural isomorphisms \(TA \otimes B \cong T(A \otimes B) \cong A \otimes TB\) provide an isomorphism \(\text{Hom}(-, C) \cong T\text{Hom}(B, C)\); but it is not clear to me that, with this isomorphism, \(\text{Hom}(\_, C)\) should automatically preserve distinguished triangles. What is certainly true however, and easily checked, is that \(\text{Hom}(\_, C)\) sends exact triangles to exact triangles, and this is good enough for us. (A triangle is exact if it yields long exact sequences upon application of functors of the form \(\mathcal{T}(X, -)\) and \(\mathcal{T}(-, Y)\).)

Now, in a general closed tensor category \(\mathcal{M}\), by juggling via \((2.4.1)\) the symmetry \(\gamma : A \otimes B \cong B \otimes A\), as well as the unit \(\eta : A \to B \otimes \text{Hom}(A, B)\) and counit \(\varepsilon : \text{Hom}(B, C) \otimes B \to C\) of the adjunction, it is possible to obtain various natural morphisms, notably:

\[
\begin{align*}
\circ : \text{Hom}(B, C) \otimes \text{Hom}(A, B) & \to \text{Hom}(A, C) \\
1 : A & \to \text{Hom}(A, A) \\
\circ : \text{Hom}(A, B) \otimes \text{Hom}(A', B') & \to \text{Hom}(A \otimes A', B \otimes B')
\end{align*}
\]
This “internal” composition $\circ$ is associative, and the identity $1$ behaves as the identity should. The tensor $\otimes$ is compatible with composition and identity in the obvious sense. This interesting structure means that $\mathcal{M}$ is enriched over itself, and that the tensor product is an enriched functor (see [HPS97, Prop. A2.3]).

By specializing $\otimes$ to the case $B = A' = \mathbb{1}$ we get a canonical natural morphism $\text{Hom}(A, \mathbb{1}) \otimes B \to \text{Hom}(A, B)$.

2.4.5. Definition. The Spanier-Whitehead dual of an object $A \in \mathcal{M}$ is $A^\vee := \text{Hom}(A, \mathbb{1})$. An object $A \in \mathcal{M}$ is strongly dualizable if the canonical natural morphism $A^\vee \otimes B \to \text{Hom}(A, B)$ is an isomorphism for all $B$.

2.4.6. Corollary. If $A$ is strongly dualizable, then there are natural isomorphisms

$$\mathcal{M}(B \otimes A, C) \cong \mathcal{M}(B, A^\vee \otimes C)$$

for all $B, C \in \mathcal{M}$.

**Proof.** Simply apply $\mathcal{M}(B, -)$ to $A^\vee \otimes C \cong \text{Hom}(A, C)$, then use the adjunction ($- \otimes A, \text{Hom}(A, -)$). \hfill \square

Introduced in [DP80], strongly dualizable objects were studied systematically in [LMS86, III] under the name of “finite objects”. The theoretical importance of this notion is made clear by their results:

2.4.7. Proposition. (i) The subcategory of strongly dualizable objects contains $\mathbb{1}$, and is closed under taking tensor products and internal Homs. In particular, if $A$ is strongly dualizable so is its dual $A^\vee$.

(ii) If $A$ is strongly dualizable, the natural map $A \to A^\vee$ (the adjoint of $A \otimes A^\vee \xrightarrow{\eta} \text{Hom}(A, \mathbb{1}) \otimes A \xrightarrow{\epsilon} \mathbb{1}$) is an isomorphism.

(iii) If $A, B$ are strongly dualizable, the natural map $\otimes : A^\vee \otimes B^\vee \to (A \otimes B)^\vee$ is an isomorphism.

If $\mathcal{M}_sd$ denotes the full subcategory of strongly dualizable objects, it follows that the restriction $(-)^\vee : \mathcal{M}_sd \to \mathcal{M}_sd$ of Spanier-Whitehead duality is a $\otimes$-equivalence of closed $\otimes$-categories with $(-)^{\vee\vee} \cong \text{id}_{\mathcal{M}_sd}$.

**Proof.** [LMS86, III §1], see also [HPS97, App. A]. \hfill \square

2.4.8. Lemma. Let $A \in \mathcal{M}_sd$. Then $A$ is a retract of $A \otimes A^\vee \otimes A$.

**Proof.** The proof of [LMS86, Prop. 1.3 (1)], that is Proposition 2.4.7 (ii) above, implies that $\eta \otimes A^\vee \otimes A \xrightarrow{\eta} A \otimes A^\vee \otimes A^\vee$ is a split mono. \hfill \square

2.4.9. Proposition. Let $\mathcal{T}$ be a closed tensor triangulated category. Then the strongly dualizable objects of $\mathcal{T}$ form a thick subcategory that contains $\mathbb{1}$ and is closed under taking tensor products and internal Homs. If $\mathbb{1}$ is $(\alpha)$-compact, all strongly dualizable objects are $(\alpha)$-compact.

**Proof.** (Cf. [HPS97, Thm. A.2.5].) The additivity and exactness (as in Remark 2.4.4) of $\text{Hom}(-, \mathbb{1})$ imply that the full subcategory of strongly dualizable objects is thick. The other closure properties are those of Proposition 2.4.7 (i). For the second statement, assume $\mathbb{1} \in \mathcal{T}$, and let $A \in \mathcal{T}_{sd}$; then

$$T(A, \coprod_i B_i) \overset{\text{Cor. 2.4.6}}{=} T(\mathbb{1}, A^\vee \otimes \coprod_i B_i) \overset{\text{Rem. 2.4.2}}{=} T(\mathbb{1}, \coprod_i A^\vee \otimes B_i) \overset{\text{Cor. 2.4.6}}{=} \prod_i T(\mathbb{1}, A^\vee \otimes B_i)$$

(and $|T(A, X)| = |T(\mathbb{1}, A^\vee \otimes X)| < \alpha$) shows that $A$ is $(\alpha)$-compact. \hfill \square
2.4.10. **Corollary.** Let $T$ be a monogenic compactly generated tensor triangulated category and equip it with a closed structure as in Example 2.4.3 (recall that we assume that $\mathbb{1}$ is compact and $\otimes$ commutes with coproducts). Then the strongly dualizable objects of $T$ coincide with its compact objects, i.e., with $\langle \mathbb{1} \rangle$.

**Proof.** It remains to see that compact objects are strongly dualizable. But this follows from the fact that the tensor unit is always strongly dualizable, and that strongly dualizable objects form a thick subcategory of $T$. $\square$

2.4.11. **Definition.** A $\otimes$-category $\mathcal{M}$ is rigid if there is a functor $D: \mathcal{M}^{op} \to \mathcal{M}$ with natural isomorphisms $\mathcal{M}(A \otimes B, C) \cong \mathcal{M}(A, D(B) \otimes C)$.

In a rigid category the bifunctor $D(?) \otimes ?$ provides an internal Hom, and all objects are strongly dualizable. Conversely, we have seen (Prop. 2.4.7) that the subcategory of strongly dualizable objects in a closed monoidal category is rigid by Spanier-Whitehead duality $D(-) := \text{Hom}(-, \mathbb{1})$. For instance, the thick subcategory of strongly dualizable objects in a closed $\otimes$-$\Delta$-category $T$ is a rigid subcategory of compact objects (Example 2.4.3). In particular, if $T$ is monogenic, the thick subcategory of compact objects $T_c = T_{sd}$ is rigid (Cor. 2.4.10).

2.4.1. **The $\alpha$-relative case.** This is all very nice and well but, alas!, the triangulated categories of $C^*$-algebras that we shall encounter in Chapters 4 and 5 have only countable coproducts, and give rise to $\aleph_1$-compactly generated categories, not to genuine ones. This leads to trouble:

2.4.12. **Remark.** If $T \not\simeq 0$ is $\alpha$-compactly generated for some cardinal $\alpha$ such that there is a $\beta < \alpha$ with $2^\beta \geq \alpha$ (e.g. $\alpha$ a successor cardinal), then we claim that the internal Hom is not defined on the whole category. For instance, the object $\text{Hom}(\bigvee \beta \mathbb{1}, \mathbb{1})$ doesn’t exist in $T$. If it existed, the natural isomorphisms

$$\prod \beta T(X, \mathbb{1}) \cong T(\prod \beta X, \mathbb{1})$$

$$\cong T(X \otimes \prod \beta \mathbb{1}, \mathbb{1})$$

$$\cong T(X, \text{Hom}(\prod \beta \mathbb{1}, \mathbb{1}))$$

would imply that it represents the product $\prod \beta \mathbb{1}$, which by Remark 1.6.7 cannot exist in $T$.

This remark applies in particular when $\alpha = \aleph_1$, as in our categories of $C^*$-algebras (simply set $\beta := \aleph_0$), whose internal Hom functor therefore cannot be everywhere-defined. Nonetheless, we may salvage something from this wreck: even in the $\alpha$-relative case, if $T$ is such that

$$T_c \otimes T_c \subseteq T_c$$

(which holds for instance if $T$ is monogenic, see Lemma 2.1.31), then the internal Hom is defined at least on $T_{op}^{op} \times T$. Indeed, if (2.4.13) holds and $A$ is $\alpha$-compact, the functor $T(- \otimes A, B)$ is immediately seen to be $\alpha$-decent on any generating set of $\alpha$-compact objects, and therefore it is representable. Now of course we may still define strongly dualizable objects as those $A$ for which the canonical comparison map $A^\vee \otimes B \to \text{Hom}(A, B)$ exists and is an isomorphism. Thanks to our axiom ($\otimes \Delta$), the argument in the proof of Proposition 2.4.9 goes through unchanged.
showing that strongly dualizable objects are all $\alpha$-compact anyway, and they form a thick subcategory of $\mathcal{T}$.

For ease of reference, we restate this:

2.4.14. Proposition. Let $\mathcal{T}$ be an $\alpha$-compactly generated $\otimes\Delta$-category such that $\mathcal{T}_c \otimes \mathcal{T}_c \subseteq \mathcal{T}_c$ (e.g., $\mathcal{T}$ is monogenic). Then the internal $\text{Hom}(B,C)$ exists for $B$ $\alpha$-compact and $C$ arbitrary. In particular, the subcategory $\mathcal{T}_{sd}$ of strongly dualizable objects is a rigid and thick $\otimes\Delta$-category of $\mathcal{T}_c$. If $\mathcal{T}$ is monogenic, then $\mathcal{T}_{sd} = \mathcal{T}_c$. $\square$

2.5. Classification in the noetherian case

Let $\mathcal{T}$ be an $\alpha$-compactly generated tensor triangulated category. In this section we prove a criterion for a support datum on $\mathcal{T}$ of a certain nice form to be classifying for the $\otimes\Delta$-subcategory $\mathcal{T}_{sd}$ of strongly dualizable objects; this would then provide a ‘computation’ of Balmer’s spectrum $\text{Spc}(\mathcal{T}_{sd})$ by Theorem 2.2.15. Our proof is abstracted from the classification of thick $\otimes$-ideals in the stable module category $\text{stmod}(kG)$ (see [BCR97, Thm. 3.4], cf. Example 2.5.6 below). Note that we shall make crucial and multiple use of a noetherian hypothesis.

2.5.1. Notation. Let $R$ be a graded commutative ring. As before, we write $\text{Spec}_h(R)$ for the Zariski spectrum of its homogeneous prime ideals. If $M$ is a graded $R$-module, its localization at a $p \in \text{Spec}_h(R)$ is defined by $M_p := S_p^{-1}M$, as in (2.3.7). The $R$-support of $M$ is the subset $\text{Supp}_R(M) := \{p \mid M_p \neq 0\} \subseteq \text{Spec}_h(R)$ of the homogeneous spectrum. By Proposition 2.3.8, $\text{Supp}_R(M) = \emptyset \Leftrightarrow M \sim 0$.

We shall have a closer look at $\text{Supp}_R$ in the next subsection.

Recall that, by Proposition 2.1.12, each graded Hom set $\mathcal{T}^\ast(A,B)$ of $\mathcal{T}$ is canonically a graded module over the graded central ring $R^\ast_T(1,1)$. Here is our criterion:

2.5.2. Theorem. Let $\mathcal{T} = (\mathcal{T}, \otimes, 1)$ be an $\alpha$-compactly generated tensor triangulated category (satisfying $\otimes\Delta+$ and $\otimes\biguplus$) such that either

(a) $\alpha$ is the cardinality of a proper class, or
(b) $\mathcal{T}_c \otimes \mathcal{T}_c \subseteq \mathcal{T}_c$.

Let $(X,\sigma)$ be a continuous generalized support datum (Def. 2.2.9) on $\mathcal{T}$ satisfying the following two hypotheses:

(A) There exists a noetherian graded commutative ring $R$, a homomorphism $\phi : R \to R^\ast_T$ preserving the grading and a set $\mathcal{S}$ of strongly dualizable objects containing the tensor unit, such that $X$ is a spectral subspace of $\text{Spec}^h(R)$, and such that the restriction of $\sigma$ on strongly dualizable objects has the following form:

\[
\sigma(A) = X \cap \bigcup_{S \in \mathcal{S}} \text{Supp}_R(\mathcal{T}^\ast(S,A)) \quad \forall A \in \mathcal{T}_{sd}
\]

(here we use $\phi$ to view $\mathcal{T}^\ast(S,A)$ as a graded $R$-module.)

(B) $\text{Ker}(\sigma) = (\text{Ker}(\sigma) \cap \mathcal{T}_{sd})_{\text{loc}}$.

If this is the case, the support datum $(X,\sigma)$ induced on $\mathcal{T}_{sd}/\text{Ker}(\sigma)$ is classifying, so that we have a canonical homeomorphism $\text{Spc}(\mathcal{T}_{sd}/\text{Ker}(\sigma)) \cong X$.

I have stretched the hypotheses as far as they went, but, concerning this work, I am interested in applying the following special case:
2.5. Corollary. Let $T$ be a monogenic $\otimes$-$\Delta$-compactly generated $\otimes$-$\Delta$-category, and let $(X,\sigma)$ be a continuous generalized support datum on $T$ satisfying hypotheses (A) of the theorem, as well as

$$(B') \ (X,\sigma) \text{ detects objects: } \sigma(A) = \emptyset \Rightarrow A \cong 0, \text{ for all } A \in T.$$ 

Then $(X,\sigma|_{T^c})$ is classifying and $\text{Spec}(T^c) \cong X$.

Proof. We know from Lemma 2.1.31 that in the monogenic case hypothesis (b) is satisfied. Moreover, $T_{sd} = T_e$ by Proposition 2.4.14. Obviously $(B')$ implies (B), since it states that $\text{Ker}(\sigma) \cong 0$ and therefore $\text{Ker}(\sigma) \cong 0 \cong (\text{Ker}(\sigma) \cap T_e)_{loc}$. Thus the theorem applies, and moreover $T_{sd}/\text{Ker}(\sigma) \cong T_e$. \hfill $\square$

2.5.5. Remark. The space $\text{Spec}^b(R)$ is spectral, so that a subspace $X \subseteq \text{Spec}^b(R)$ is spectral if and only if it is a patch (or proconstructible), that is if $X$ is an intersection of finite unions of sets $X_i$, each of which is quasi-compact open or the complement of a quasi-compact open in $\text{Spec}^b(R)$ (see [Hoc69, §2]). In the theorem, $R$ is assumed to be noetherian, which implies that $\text{Spec}^b(R)$ is a noetherian topological space: every open subset being quasi-compact. In this case, a subspace $X \subseteq \text{Spec}^b(R)$ is again spectral if and only if it is an intersection of finite unions of open or closed subsets.

2.5.6. Example. Consider $T := \text{StMod}(kG)$, the stable module category of a finite group $G$ and a field $k$ (see e.g. [Car96]); this is a (genuine) compactly generated tensor triangulated category with compact objects $\text{StMod}(kG)_c = \text{StMod}(kG)_{sd} = \text{stmod}(kG)$, the stable category of finite dimensional $kG$-modules. Set $R$ to be the cohomology ring of $G$

$$R := H^*(G,k) \cong \bigoplus_{i \geq 0} \text{End}^i_{\text{StMod}(kG)}(1),$$

and put $S := T_{sd}$ and $X := \text{Proj}(R) = \text{Spec}^b(R) \setminus \{R^>0\}$. The graded ring $R = H^*(G,k)$ is noetherian by the Evens-Venkov Theorem ([Eve91]), and the formula (2.5.3) is known to provide a continuous support datum satisfying $(B')$ ([BCR97, Prop. 2.2]) — in the notation of the next section, we would write $(\text{Proj}(R),\text{supp}_t)$ for this support. Thus Theorem 2.5.2 shows that $(\text{Proj}(R),\text{supp}_t)$ is classifying on $\text{stmod}(kG)$. In particular, there is a canonical homeomorphism $\text{Spec}(\text{stmod}(kG)) \cong \text{Proj}(H^*(G,k))$.

2.5.1. Preliminaries on supports. For this subsection, $T$ is any $\otimes$-$\Delta$-triangulated category. We posit a graded commutative ring $R$ and a grading preserving homomorphism $R \to R^*_T$ which we use to see $T^*$ as a graded $R$-linear category (in the sense of Lemma 2.3.6). In particular, we consider the graded Hom sets $T^*(A,B)$ as graded $R$-modules. Using the module-theoretic support $\text{Supp}_R$ as a building block, we define over $\text{Spec}^b(R)$ various supports for the objects of $T$.

2.5.7. Definition. Let $A \in T$. The total $R$-support of $A$ is the subset

$$\text{Supp}_R(A) := \text{Supp}_R(\text{End}^*_T(A)) = \{p \mid \text{End}^*_T(A)_p \neq 0\} \subseteq \text{Spec}^b(R).$$

For an object $X \in T$, we will also make use of the short-hand

$$\text{Supp}_X(A) := \text{Supp}_R(T^*(X,A)) \subseteq \text{Spec}^b(R)$$

as well as

$$\text{Supp}_S(A) := \bigcup_{X \in S} \text{Supp}_R(T^*(X,A)) \subseteq \text{Spec}^b(R)$$

for a class of objects $S \subseteq T$. 


2.5.8. REMARKS. 
(a) Notice that $\text{Supp}_t = \text{Supp}_T$. Indeed, consider for each $p \in \text{Spec}^h(R)$ the localized graded category $S^{-1}_p T^*(A, B)$ (Lemma 2.3.6). By Yoneda for this category we see that
$$\text{End}_T^r(A)_p = 0 \iff T^*(X, A)_p = 0 \ \forall X \in T$$
for any homogeneous prime $p$, showing that $\text{Supp}_t(A) = \text{Supp}_T(A)$.
(b) Let $E$ be a unital graded $R$-algebra, such as $\text{End}_T^r(A)$. The annihilator of $E$ in $R$ is the homogeneous ideal
$$\text{Ann}_R(E) := \{r \in R^{\text{homog}} | re = 0 \ \forall e \in E\}$$
$$= \{r \in R^{\text{homog}} | r \cdot 1_E = 0\}.$$ 
For a homogeneous prime $p$ of $R$ we have
$$p \in \text{Supp}_R(E) \iff E_p \neq 0$$
$$\iff \text{homogeneous } r \in (R \setminus p) \cap R^{\text{even}} \text{ with } r \cdot 1_E = 0$$
$$\iff \text{homogeneous } r \in R \setminus p \text{ with } r \cdot 1_E = 0$$
$$\iff p \in V(\text{Ann}_R(E)).$$
In particular, the total support $\text{Supp}_t(A)$ of any object $A$ is a closed subset of $\text{Spec}^h(R)$. (Note that $V(\text{Ann}_R(M)) \subseteq \text{Supp}_R(M)$ fails for a general $R$-module $M$; one should assume that $M$ is finite over $R$.)
(c) The total support satisfies the axioms (SD1)-(SD4) of a support datum (Definition 2.2.7), see the next lemma, and it yields closed subsets of $\text{Spec}^h(R)$ by point (b). Thus, the only property of a support datum that $\text{Supp}_t$ may fail to enjoy is the compatibility with the tensor product:
$$\text{SD5): } \text{Supp}_t(A \otimes B) = \text{Supp}_t(A) \cap \text{Supp}_t(B) \text{ for all } A, B \in T.$$ 
(d) The total support detects objects:
$$X \cong 0 \iff \text{End}_T^r(X) \cong 0$$
$$\iff \text{End}_T^r(X)_p \cong 0 \ \forall p \in \text{Spec}(R)$$
$$\iff \text{Supp}_t(X) = \emptyset.$$ 
The second equivalence is Proposition 2.3.8.

2.5.9. LEMMA. The total $R$-support $\text{Supp}_t$ satisfies (SD1)-(SD4).

PROOF. Axiom (SD1) is clear, and (SD2) holds because the translation $T$ is an equivalence. (SD3): Let $A, B \in T$. Then $1_{A \otimes B} = 1_A \otimes 1_B$, so that $r \cdot 1_{A \otimes B} = 0$ if both $r \cdot 1_A = 0$ and $r \cdot 1_B = 0$, for every $r \in R$. Hence $\text{Supp}_t(A \otimes B) = \text{Supp}_t(A) \cup \text{Supp}_t(B)$ by Remark 2.5.8 (b). Let $A \rightarrow B \rightarrow C \rightarrow TA$ be a distinguished triangle in $T$. To prove (SD4), it suffices by Remark 2.5.8 (b) to show that $\text{Ann}_R(\text{End}^r(A)) \cdot \text{Ann}_R(\text{End}^r(B)) \subseteq \text{Ann}_R(\text{End}^r(C))$. The distinguished triangle induces an exact sequence
$$T^*(C, B) \rightarrow T^*(C, C) \rightarrow T^*+1(C, A)$$
of graded $R$-modules. Let $f, g \in R$ be such that $f 1_A = 0$ and $g 1_B = 0$. Then the image of $f 1_C$ in $T^*+1(C, A)$ is zero, so it must come from some element of $T^*(C, B)$, which is annihilated by $g$. Thus $gf \cdot 1_C = 0$. \hfill $\square$

2.5.10. LEMMA. Let $T$ be $\alpha$-compactly generated, and let $\mathcal{G} \subseteq T$ be a generating set of $\alpha$-compact objects. Then
$$\text{Supp}_t(A) = \bigcup_{X \in \mathcal{G}} \text{Supp}_X(A)$$
for every $\alpha$-compact object $A \in T_c$. 

Proof. Let \( p \subseteq R \) be a homogenous prime and let \( A \in T_c \). Then
\[
\text{End}_T(A)_p = 0 \iff T^*(X, A)_p = 0 \quad \forall X \in T
\]
\[
\iff T^*(X, A)_p = 0 \quad \forall X \in G.
\]
The first equivalence is just Remark 2.5.8 (a). For the second one, we need only prove \( \Rightarrow \). Notice that the functor \( T(-, A)_p \) is a cohomological functor on \( T \), since so is \( T(-, A) \), and localization at \( p \) is exact. (But note also that localization doesn’t commute with infinite products in general, so \( T(-, A)_p \) need not be decent.) It follows that the class of objects \( X \) on which \( T^*(X, A)_p \) vanishes is a thick subcategory. Thus if it vanishes on \( G \), it vanishes on \( \langle G \rangle = T_c \) (Prop. 1.7.8), and therefore on \( A \). □

2.5.2. The proof of Theorem 2.5.2. For simplicity, we shall drop the cardinal \( \alpha \) from the notation.³

First of all, note that each of the hypotheses (a) and (b) implies that the subcategory \( T_{sd} \) of strongly dualizable objects is a rigid and thick \( \otimes \Delta \)-subcategory of \( T_c \); if (a) holds, this holds by Ex. 2.4.3 and Prop. 2.4.9; if only (b) is true instead, by Prop. 2.4.14. Therefore if \( \sigma|_{T_{sd}} \) is a support datum, the quotient \( T_{sd}/\text{Ker}(\sigma) \) is also a \( \otimes \Delta \)-category, so that it makes sense to consider its spectrum \( \text{Spc}(T_{sd}/\text{Ker}(\sigma)) \).

Moreover, the very same references show that in both cases \( T_{sd} \subseteq T_c \), and the right adjoint \( \text{Hom}(B, -) \) of \( - \otimes B \) exists at least if \( B \in T_c \). Thus each \( B \in T_{sd} \) has a dual \( B^\vee = \text{Hom}(B, 1) \), and \( \text{Hom}(B, -) = B^\vee \otimes (-) \) (by Cor. 2.4.6).

Second, since \( T_{sd} \subseteq T_c \), the subcategory \( \langle T_{sd} \rangle_{\text{loc}} \subseteq T \) is a compactly generated localizing subcategory having \( T_{sd} \) as compact objects. Thus we may change ‘ambient category’ without affecting neither \( T_{sd} \) nor \( \text{Ker}(\sigma) \cap T_{sd} \): therefore, by substituting \( T \) with \( \langle T_{sd} \rangle_{\text{loc}} \), we may as well assume from now on that \( T_{sd} = T_c \).

Third, since by hypothesis \( (X, \sigma) \) is a continuous generalized support datum on \( T \), the subcategory \( \text{Ker}(\sigma) \) is a localizing (and thus thick) \( \otimes \)-ideal, so that the quotient \( T / \text{Ker}(\sigma) \) is a \( \otimes \Delta \)-category with coproducts and the quotient functor \( q : T \to T / \text{Ker}(\sigma) \) preserves coproducts. The support \( (X, \sigma) \) induces via \( \pi(qA) := \sigma(A) \) a continuous generalized support datum \( (X, \overline{\sigma}) \) on the quotient, and this support detects objects, i.e., it satisfies (B∗) (Lemma 2.2.10). But we have assumed more: hypothesis (B) means that \( q : T \to T / \text{Ker}(\sigma) \) is a Neeman localization. Indeed, (B) says that \( \text{Ker}(\sigma) \) is generated as a localizing subcategory by a set of compact objects of \( T_c \), namely (a skeleton of) \( \text{Ker}(\sigma) \cap T_c \). Hence \( \text{Ker}(\sigma) \) is compactly generated by this set (in particular \( \text{Ker}(\sigma)_c = \text{Ker}(\sigma) \cap T_c \)), the quotient \( T / \text{Ker}(\sigma) \) is again \( \alpha \)-compactly generated (Corollary 1.7.30), and moreover the functor \( F \) induced on the quotients

\[
\begin{array}{ccc}
\text{Ker}(\sigma)_c & \to & T_c \\
\downarrow & & \downarrow \\
T / \text{Ker}(\sigma) & \to & T_c / \text{Ker}(\sigma)_c \\
\uparrow F & & \uparrow F
\end{array}
\]

is fully faithful and it identifies the quotient on compact objects \( T_c / \text{Ker}(\sigma)_c \) with a cofinal subcategory of \( (T / \text{Ker}(\sigma))_c \) (Theorem 1.7.21). Note that \( T_c / \text{Ker}(\sigma)_c \) is the category whose spectrum we want to compute.

2.5.11. Remark. At this point I’m very tempted to use the following

Proposition. ([Bal05, Prop. 3.13]). Let \( K \subseteq L \) be a cofinal \( \otimes \Delta \)-subcategory of the \( \otimes \Delta \)-category \( L \). Then the map \( \text{Spc}(L) \to \text{Spc}(K) \), \( P \to P \cap K \), induced by the inclusion is a homeomorphism.

³It was about time.
Hence for our purposes it seems as if we could substitute $T_c/\mathrm{Ker}(\sigma)_c$ with $(T/\mathrm{Ker}(\sigma))_c$, which would allow us to change the ambient category once more, working directly in the quotient $T = T/\mathrm{Ker}(\sigma)$ with a support datum $\overline{\sigma}$ which detects objects. This would simplify the rest of the proof, but in order to make such a reduction we should check that the support $(X, \overline{\sigma})$ satisfies again hypothesis $(A)$, with $T/\mathrm{Ker}(\sigma)$ instead of $T$. I don’t know how to do this, so we shall stay in $T$. This difficulty suggests that, in a better version of Theorem 2.5.2, one should replace hypothesis $(A)$ with something more natural.

Another reduction that we do make is the following. Note that the map $\text{Spec}(\phi) : \text{Spec}^b(R_\overline{T}) \to \text{Spec}^b(R)$ factors through the subset $\text{Spec}^b(R/\mathrm{Ker}\phi) \subseteq \text{Spec}^b(R)$. Thus, when computing $\sigma$ on compact objects according to the formula (2.5.3) in hypothesis $(A)$, we may take $\phi(R)$ instead of $R$. Hence we may assume that $R$ is a graded subring of $R_\overline{T}$.

To sum up, the rest of the proof may and shall be done under the following

2.5.12. Hypothesis. Assume that $T_{sd} = T_c$ and that $\phi : R \to R_\overline{T}$ is the inclusion of a graded subring.

In the notation of Subsection 2.5.1, hypothesis $(A)$ says that $\sigma(A) = X \cap \text{Supp}_S(A)$ for every $A \in T_c$. As it turns out, $\sigma|_{T_c}$ doesn’t depend on the choice of $S$, but only of $R \subseteq R_\overline{T}$ and $X \subseteq \text{Spec}^b(R)$:

2.5.13. Lemma. For every $A \in T_c$, we have $\sigma(A) = X \cap \text{Supp}_c(A)$.

Proof. Lemma 2.5.10 implies that

$$X \cap \text{Supp}_c(A) \subseteq \bigcup_{C \in T_{c}} \text{Supp}_c(A)$$

$$X \cap \bigcup_{C \in T_{c}} \text{Supp}_c(C^\vee \otimes A)$$

$$\subseteq \bigcup_{C \in T_{c}} X \cap \text{Supp}_S(C^\vee \otimes A)$$

$$= \bigcup_{C \in T_{c}} \sigma(C^\vee \otimes A)$$

$$= \bigcup_{C \in T_{c}} \sigma(C^\vee) \cap \sigma(A) \subseteq \sigma(A)$$

for every $A \in T_{sd}$; the crucial step here is the second equality, which uses the property $T(U \otimes C, V) \cong T(U, C^\vee \otimes V)$ of the strongly dualizable object $C$. The converse inclusion follows immediately from the formula in hypothesis $(A)$ and Lemma 2.5.10. \hfill \Box

2.5.14. Corollary. $(X, \sigma|_{T_c})$ is a support datum on $T_c$, and therefore $(X, \overline{\sigma})$ is a support datum on $T_c/\mathrm{Ker}(\sigma)_c$.

Proof. By Remark 2.5.8, $\text{Supp}_c$ is a support datum on $T$ as soon as it satisfies (SD5): in particular each set $\text{Supp}_c(A) \subseteq \text{Spec}^b R$ is closed. It follows that $X \cap \text{Supp}_c(A)$ is closed in $X$ for each $A \in T_c$. By hypothesis, $(X, \sigma)$ is a generalized support datum on $T$, so in particular it satisfies (SD5). Now on $T_c$ we have $\sigma(A) = X \cap \text{Supp}_c(A)$ by Lemma 2.5.10, so that, uniting the best of both, $(X, \sigma|_{T_c})$ is a support datum. The statement about the quotient is in Lemma 2.2.10. \hfill \Box

2.5.15. Corollary. The equality $\sigma(A) = \sigma(A^\vee)$ holds for $A \in T_c$. 

Proof. Note that $\text{Supp}_t(A) = \text{Supp}_t(A^\vee)$ always holds for a strongly dualizable $A$, since
\[ \text{End}^*_T(A) \cong \text{End}^*_{T^r}(A^\vee) = \text{End}^*_T(A^\vee) \]
(indeed, $(-)^\vee$ is a duality functor on $T_{sd}$). By Lemma 2.5.13, $\sigma(A) = X \cap \text{Supp}_t(A)$ and the claim follows. \hfill $\square$

2.5.16. Remark. Another consequence of Lemma 2.5.13 is that, if the choice for $X$ is the whole of $\text{Spec}^h R$, then $\sigma(A) = \text{Supp}_t(A)$ on compact objects. But $\text{Supp}_t$ detects objects (Remark 2.5.8 (d)), and therefore $T_c \cap \text{Ker}(\sigma) \simeq 0$. In this case, hypothesis (B) becomes equivalent to (B'): $\text{Ker}(\sigma) \simeq 0$. The rest of the proof would simplify accordingly.

2.5.17. Lemma. Let $Z \subseteq X$ be a closed subset. Then there is an object $A_Z \in T_c$ with $\sigma(A_Z) = Z$.

Proof. The closed subset has the form $Z = X \cap V(I)$ for some homogeneous ideal $I \subseteq R$. Since $R$ is noetherian, $I$ is generated by finitely many homogeneous elements, say $I = (r_1, \ldots, r_n)$. Let $C_i$ be the cone of $r_i : \mathbb{I} \to T^m \mathbb{I}$. It is strongly dualizable and compact, and

Claim 1: $\text{Supp}_{S}(C_i) = V((r_i))$.

Indeed, let $S \in \mathcal{S}$. By applying $T^*(S, -)_p$ to the triangle $\mathbb{I} \to T^m \mathbb{I} \to C_i \to T\mathbb{I}$, we obtain an exact sequence
\[ T^*(S, \mathbb{I})_p \xrightarrow{r_i} T^{*+m}(S, \mathbb{I})_p \xrightarrow{T^*(S, C_i)_p} T^{*+1}(S, \mathbb{I})_p \]
of graded $R$-modules. Note that the first morphism is multiplication by $r_i$ (indeed, recall from Prop. 2.1.12 that in a tensor triangulated category, the three actions of $R^*_T$ on the graded Hom sets given by tensoring with $\mathbb{I}$ on the right or left, or by composition, all coincide – up to a sign). It is invertible whenever $r_i \in R^*_p$, and the latter is equivalent to $r_i \not\in p$. Hence $r_i \not\in p$ implies that $T^*(S, C_i)_p = 0$ for all $S \in \mathcal{S}$, that is $p \not\in \text{Supp}_{S}(C_i)$. On the other hand, if we assume that $p \not\in \text{Supp}_{S}(C_i)$ then in particular $T^*(\mathbb{I}, C_i)_p = 0$. Setting $S := \mathbb{I}$ in the above exact sequence, we see that $r_i : R_p \to R_p$ is invertible, i.e. $r_i \not\in p$. Thus $p \in \text{Supp}_{S}(C_i)$ iff $r_i \in p$, as claimed.

Now it suffices to set $A_Z := C_1 \otimes \cdots \otimes C_n$ (which is again a strongly dualizable and compact object), because
\[
\begin{align*}
\sigma(A_Z) & \overset{(SD5)}{=} \sigma(C_1) \cap \cdots \cap \sigma(C_n) \\
& \overset{\text{Hyp. (A)}}{=} X \cap \text{Supp}_{S}(C_1) \cap \cdots \cap \text{Supp}_{S}(C_n) \\
& \overset{\text{Claim 1}}{=} X \cap V((r_1)) \cap \cdots \cap V((r_n)) \\
& = X \cap V(I) = Z,
\end{align*}
\]

as desired. \hfill $\square$

For the rest of the proof, let’s introduce some short-hand notation. For a subset $Z \subseteq X$, write
\[
\begin{align*}
C_Z & := \{ A \mid \sigma(A) \subseteq Z \} \subseteq T_c \\
T_Z & := (C_Z)_{\text{loc}} \subseteq T.
\end{align*}
\]

Notice that, since $\sigma$ is a generalized support datum on $T_c$, $C_Z$ is a thick $\otimes$-ideal of $T_c$ (Lemma 2.2.10) and therefore $T_Z$ is a localizing $\otimes$-ideal of $T$ (Lemma 2.1.25). By Proposition 1.7.15, the pair $(T_Z, T_Z^\perp)$ is a complementary pair of localizing subcategories of $T$, with gluing triangle
\[ L_{C_Z}(A) \longrightarrow A \longrightarrow R_{C_Z}(A) \longrightarrow TL_{C_Z}(A) \]
for all $A \in T$ (cf. Notation 1.7.17).
2.5.18. **Lemma.** If $A \in \mathcal{T}_Z$, then $\sigma(A) \subseteq Z$.

**Proof.** By Brown representability, there is a distinguished triangle $B \to B \to A \to TB$ where $B = \coprod_i B_i$ is a coproduct of objects in $\mathcal{C}_Z$, which are supported on $Z$ by definition. Consequently

$$\sigma(A) \subseteq \sigma(\coprod_i B_i) = \bigcup_i \sigma(B_i) \subseteq Z$$

because $\sigma$ is compatible with triangles (SD3) and is continuous. \hfill \Box

Now reconsider the situation: we have a commutative diagram

$$\begin{array}{ccc}
\text{Ker}(\sigma) & \longrightarrow & \mathcal{T} \\
\cup & \cup & \cup \\
\text{Ker}(\sigma)_c & \longrightarrow & \mathcal{T}_c/\text{Ker}(\sigma)_c
\end{array}$$

where the rows are short exact sequences of $\otimes$-$\Delta$-categories, and the subcategory $\mathcal{T}_c/\text{Ker}(\sigma)_c \subseteq (\mathcal{T}/\text{Ker}(\sigma))_c$ is cofinal. Note that $\text{Ker}(\sigma)_c = \mathcal{C}_0$ in our notation above. Indeed, by our hypothesis (B), $\text{Ker}(\sigma) = \mathcal{T}_0$ since it is generated by its compact objects.

What we need to do, is relate supports of objects with thick $\otimes$-ideal in the quotient.

2.5.19. **Lemma.** Let $q : \mathcal{T}_c \to \mathcal{T}_c/\text{Ker}(\sigma)_c$ denote the quotient functor as above, and let $A \in \mathcal{T}_c$. Then we have an equality $q(C_{\sigma(A)}) = q(\sigma(A))$ of subcategories of the quotient.

With a slight abuse of language, write

$$\mathcal{C}_{\sigma(q(A))} := \{ qB \mid \sigma(qB) \subseteq \sigma(qA) \} \subseteq \mathcal{T}_c/\text{Ker}(\sigma)_c,$$

for any object $qA$ in the quotient, and note that it is also a thick $\otimes$-ideal (because $\sigma$ is again a support datum). The previous lemma is our stepping stone to the following key result:

2.5.20. **Corollary.** Let $A \in \mathcal{T}_c$. Then $\mathcal{C}_{\sigma(q(A))} = q(\mathcal{A})$ in $\mathcal{T}_c/\text{Ker}(\sigma)_c$.

**Proof.** Note that there is an equality $q(C_{\sigma(A)}) = \mathcal{C}_{\sigma(q(A))}$, as one sees immediately from the definitions of the two categories. Also, we see that $q(\sigma(A)) \subseteq (\sigma(\mathcal{A}))_\otimes$, because, by definition, a triangle in the quotient $\mathcal{T}_c/\text{Ker}(\sigma)_c$ is distinguished iff it is isomorphic to one coming from $\mathcal{T}_c$, and because $q$ is a $\otimes$-functor which is bijective on objects. Since $qA \in q(\sigma(A))$, we have the other inclusion, and thus equality, as soon as we can show that $q(\sigma(A))$ is thick (it is clearly a $\otimes$-ideal). Therefore

$$\mathcal{C}_{\sigma(q(A))} = q(C_{\sigma(A)}) \overset{\text{Lemma 2.5.19}}{=} q(\sigma(A)) \subseteq q(\sigma(A))$$

and we have equality, because we know that $\mathcal{C}_{\sigma(q(A))}$ is thick in $\mathcal{T}_c/\text{Ker}(\sigma)_c$. \hfill \Box

**Proof of Lemma 2.5.19.** Let $Z := \sigma(A)$. Now we’ll be working upstairs, in $\mathcal{T}$. Since $\mathcal{C}_Z$ is a thick $\otimes$-ideal of $\mathcal{T}_c$, we have $\mathcal{A} \subseteq \mathcal{C}_Z$, so that $L_{\mathcal{A}} \alpha \otimes L_{\mathcal{C}_Z} \cong L_{\mathcal{A}}$ by Proposition 1.7.19. Thus for any $B \in \mathcal{T}$, the gluing triangle for $\mathcal{A} \otimes$ and the object $L_{\mathcal{C}_Z}(B)$ has the form

$$\begin{array}{c}
L_{\mathcal{A}}(B) \longrightarrow L_{\mathcal{C}_Z}(B) \longrightarrow R_{\mathcal{A}}(B) \longrightarrow TL_{\mathcal{A}}(B).
\end{array}$$

Recall from Proposition 1.7.19 (ii) (or rather its proof) that

$$\mathcal{A} \otimes = \mathcal{T}_c \cap \text{Im}(L_{\mathcal{A}}) \quad \text{and} \quad \mathcal{C}_Z = \mathcal{T}_c \cap \text{Im}(L_{\mathcal{C}_Z}),$$
where as usual Im denotes the essential image of the respective functors $T \to T$.

Let $D := R_{i(A)\otimes}Lc_z(B)$ denote the third object in the natural triangle (2.5.21). We are going to prove that $\sigma(D) \subseteq \emptyset$, i.e., that $qC \cong 0$ in $T/\text{Ker}(\sigma)$. Together with (2.5.21) and (2.5.22), this would then imply the required equality $q(\langle A \rangle) = q(C_Z)$, because $B \in T$ was an arbitrary object. Hence we are left with proving

Claim: $\sigma(D) = \emptyset$.

Indeed, since the first two objects in (2.5.21) belong to $T_Z$, so does $D$. Therefore $\sigma(D) \subseteq Z$, by Lemma 2.5.18. Let $C$ be any compact object of $T$. Then

$$0 \cong T(C \otimes A, D) \cong T(C, A^\vee \otimes D)$$

because $C \otimes A \in \langle A \rangle$, and $D = R_{i(A)\otimes} \cdots \in \langle A \rangle$ (for the first isomorphism), and because $A \in T_c = T_{nd}$ (for the second one). But then $A^\vee \otimes D \cong 0$, because compact objects generate $T$. Using this, we conclude that

$$\sigma(D) = \sigma(A) \cap \sigma(D)$$

$$\cong_{\text{Cor. 2.5.15}} \sigma(A) \cap \sigma(D)$$

$$\cong_{(SD5)} \sigma(A^\vee) \cap \sigma(D)$$

$$\cong_{(SD1)} \sigma(A^\vee \otimes D) \cong_{\emptyset} 0,$$

as we had claimed. \hfill \Box

2.5.23. Remark. Note that the second-to-last line of the proof is the only place where we have used the hypothesis that $\sigma$ satisfies (SD5) for all objects of $T$, not only for the compact ones. Indeed, we only need the equation $\sigma(A \otimes D) = \sigma(A) \cap \sigma(D)$ for $A$ compact and $D$ arbitrary.

Finally, we have at hand all the ingredients for proving that $(X, \sigma)$ is classifying for the category $T_c/\text{Ker}(\sigma)$. To avoid clutter, from now on we will simply write $\sigma$ also for $\sigma$ (everything now will take place in the quotient anyway, so there'll be no confusion).

Note that, since $R$ is graded noetherian, its homogeneous spectrum $\text{Spec}^h(R)$ is topologically noetherian, i.e., all its open subsets are quasi-compact. The same clearly holds for the subspace $X$. Hence every union of closed subsets of $X$ is a Thomason subset (Remark 2.2.16 (c)). Also, since $T_c = T_{nd}$ is rigid, every thick $\otimes$-ideal is radical (Remark 2.2.16 (b)). We have seen already that $(X, \sigma)$ is a support datum on $T_c/\text{Ker}(\sigma)$ (Cor. 2.5.14), and the space $X$ is spectral by definition. Therefore, in order to see that $(X, \sigma)$ is classifying (Def. 2.2.14), it remains to prove:

2.5.24. Lemma. The assignments

$$Y \mapsto C_Y = \{A \in T_c/\text{Ker}(\sigma) \mid \sigma(A) \in Y\}$$

$$C \mapsto \sigma(C) := \bigcup_{A \in C} \sigma(A),$$

sending a union $Y$ of closed subsets of $X$ to the thick $\otimes$-ideal $C_Y$, respectively sending a thick $\otimes$-ideal $C$ of $T_c/\text{Ker}(\sigma)_c$ to the union of closed subsets $\sigma(C)$, are inverse to each other.

Proof. Given a thick $\otimes$-ideal $C \subseteq T_c/\text{Ker}(\sigma)$, clearly $C \subseteq C_{\sigma(C)}$. In order to prove the other inclusion, consider an $A \in C_{\sigma(C)}$, i.e., $\sigma(A) \subseteq \bigcup_{B \in C} \sigma(B)$. Since the set

$$\sigma(A) \cong_{2.5.13} \text{Supp}_R(A) \cong_{\text{Rem. 2.5.8}} X \cap V(\text{Ann}(\text{End}^*_R(A)))$$

is a closed subset of the noetherian space $X$, it consists of a finite union $Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets $Z_i$. By irreducibility, each $Z_i$ is contained in a $\sigma(B_i)$,
for some compact $B_i \in \mathcal{C}$. Therefore
\[
\sigma(A) \subseteq \sigma(B_1) \cup \cdots \cup \sigma(B_n) = \sigma(B_1 \oplus \cdots \oplus B_n)
\]
from which we deduce
\[
A \in \mathcal{C}_{\sigma(B_1 \oplus \cdots \oplus B_n)} = \langle B_1 \oplus \cdots \oplus B_n \rangle \subseteq \mathcal{C}
\]
thanks to Corollary 2.5.20. This proves that $\mathcal{C} = \mathcal{C}_{\sigma(\mathcal{C})}$.

Conversely, let $Y = \bigcup_i Z_i$ be a union of closed subsets of $X$. Clearly $\sigma(\mathcal{C}_Y) \subseteq Y$
by definition (indeed for any subset $Y \subseteq X$). By Lemma 2.5.17, there are compact objects $A_i$ with $\sigma(A_i) = Z_i$. But then $A_i \in \mathcal{C}_{Z_i} \subseteq \mathcal{C}_Y$, and thus $Y = \bigcup_i Z_i \subseteq \sigma(\mathcal{C}_Y)$.
So we have proved that $\sigma(\mathcal{C}_Y) = Y$, concluding the verification that the functions
$Y \mapsto \mathcal{C}_Y$ and $\mathcal{C} \mapsto \sigma(\mathcal{C})$ are the inverse of each other. \hfill \square

This ends the proof of Theorem 2.5.2.
Relative homological algebra in triangulated categories

In order to do some rudimentary homological algebra, one needs very little: a pointed category, some notion of ‘exact sequence’, and a related notion of ‘projective object’ ([EM65]). If the category carries a triangulated structure, a remarkably rich theory ensues. This was initiated by Christensen [Chr98] and Beligiannis [Bel00b]. We follow Meyer and Nest [MN07] [Mey08] [Mey], who built on their work and also applied it to the categories we are interested in.

3.1. Homological functors and ideals

Let $T$ be a triangulated category.

3.1.1. Definition. In what follows, a stable abelian category $A = (A,T)$ will simply mean an abelian category $A$ equipped with an automorphism $T$. A stable exact functor $F := (F,\varphi) : A \to B$ between stable abelian categories is a pair $(F,\varphi)$, where $F : A \to B$ is an exact functor, and $\varphi : FT \cong TF$ is a specified isomorphism. Similarly, a stable homological functor on $T$ is a pair $(F,\varphi)$ consisting of a homological functor $F : T \to A$ to a stable abelian category $A = (A,T)$, together with an isomorphism $\varphi : FT \cong TF$; it follows that $F$ sends every distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ to an exact sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{\varphi F(h)} TF(A)$$

in $A$. A stable cohomological functor on a triangulated category $T$ is just a stable homological functor $F : T^{op} \to A$.

Note that every stable homological functor is also a homological functor (Def. 1.1.17), and that every homological functor $H : T \to A$ has an associated stable homological functor $H_* := \{H_i\}_i : T \to A^Z$ into the category of $Z$-graded objects in $A$, where translation is simply the shift of degree $T(A_i) := A_{i+1}$.

3.1.2. Example. The universal homological functor, i.e., the Yoneda embedding $h : T \to \text{Coh}(T)$ into the Freyd category, is a stable homological functor. The translation of $\text{Coh}(T)$ is induced by that of $T$ in the obvious way by setting $T(hA) := h(TA)$ on representable objects, so that we may choose $\varphi := \text{id} : hT = Th$. Indeed, as we already noted (Remark 1.5.11), this is the universal stable homological functor out of $T$.

3.1.3. Remark. An n-periodic homological functor, i.e., one for which there is an isomorphism $H_* \cong H_{*+n}$, gives rise to a stable functor into $A^{Z/n}$, the stable category of $Z/n$-graded objects. If the translation functor of $T$ itself is n-periodic ($T \cong T^n$), every homological functor on $T$ is n-periodic. This is the case in the various Kasparov categories, with $n = 2$, because of Bott periodicity.
3.1.4. Definition. A **homological ideal** in a triangulated category $T$ is an ideal $I$ of morphisms of $T$ — that is: $I \subseteq \text{Mor}(T)$ defines an additive subcategory of $T$ with the same objects and such that $\text{Mor}(T) \cap I \subseteq I$ — which is the kernel on morphisms of some stable homological functor $F$:

$$I = I_F := \ker(F) := \{ f \in \text{Mor}(T) \mid F(f) = 0 \}.$$ 

Thus, in particular, a homological ideal is stable: $I^{\pm 1}(T) \subseteq I$. Note that different functors can give rise to the same homological ideal, but the resulting homological algebra in $T$ only depends on the ideal.

3.1.5. Example. Let $A$ be an abelian category, and let $\text{Ho}(A)$ be the homotopy category of complexes, where morphisms are chain homotopy classes of morphisms of complexes in $A$. We will denote complexes by

$$A := (A_n, d_n) := (\cdots \rightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots).$$

Taking the homology of complexes defines a stable homological functor $H_\ast : \text{Ho}(A) \rightarrow A^\mathbb{Z}$, $A \mapsto \{H_n(A_n)\}_{n}$, to the stable abelian category of $\mathbb{Z}$-graded objects of $A$. The associated homological ideal is

$$I_{H_\ast} = \{ f \in \text{Mor}(\text{Ho}(A)) \mid H_\ast(f) = 0 \}.$$ 

Homological algebra in the triangulated category $\text{Ho}(A)$ relative to $I_{H_\ast}$ recovers (parts of) the usual homological algebra for abelian categories.

In the following, we will consider a fixed triangulated category $T$ together with a fixed homological ideal $I$. It is with respect to the choice of $I$ that the homological algebra in $T$ will be relative to. Heuristically, one should think of morphisms in $I$ as being ‘almost zero’, and of distinguished triangles

$$(3.1.6) \quad A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA$$

whose connecting map $h$ belongs to $I$ as ‘short exact sequences’ relative to $I$. Here is some convenient terminology capturing this intuition.

3.1.7. Definition. An object $A \in T$ is **$I$-contractible** if $1_A \in I(A, A)$. Let (3.1.6) be a distinguished triangle of $T$. If $h \in I$, we will say that the triangle is $I$-**exact**, and also that $f$ is $I$-**monic**, that $g$ is $I$-**epic**, and that $h$ is $I$-**phantom**. If $g$ and $h$ are both in $I$, then $f$ is an $I$-**equivalence**. Note that $f$ is an $I$-equivalence if and only if $C = \text{cone}(f)$ is $I$-contractible. If $F$ is a stable homological functor with $I = \ker(F)$, we will also write $F$-monic, $F$-exact, etc., instead of $I_F$-monic, $I_F$-exact, etc.

3.1.8. Remark. Rather that basing the theory on a homological ideal (whose definition involves the choice of a stable homological functor, and whose formal properties relevant for homological algebra are not clear) one could follow Christensen [Chr98] in axiomatising the pair $(I, P)$ consisting of an ideal $I$ together with the class of $I$-projective objects (see below). If there are ‘enough $I$-projective objects’, the two frameworks are easily seen to be equivalent. Conceptually though, the best way to capture homological algebra in a triangulated category is perhaps the equivalent approach of Beligiannis [Bel00b], who gives axioms for the class of $I$-exact triangles which are highly reminiscent of the axioms of admissible exact sequences in an exact category. This should be appealing to algebraic $K$- theorists.1

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1 Had I seen the light earlier...
3.2. Projective objects and resolutions

3.2.1. Definition. A homological functor $H : T \to A$ is said to be $I$-exact if $I \subseteq \ker(H)$. An object $A \in T$ is $I$-projective if the functor $T(A, -) : T \to \text{Ab}$ is $I$-exact. Dually, $A$ is $I$-injective if $T(-, A) : T \to \text{Ab}^{\text{op}}$ is $I$-exact.

3.2.2. Remark. We will denote by $\text{Proj}(T, I)$, resp. $\text{inj}(T, I)$, the full subcategory of $I$-projective, resp. $I$-injective, objects of $T$. Clearly, $\text{Proj}(T, I)$ is closed under taking direct summands and arbitrary coproducts, and dually, $\text{inj}(T, I)$ is closed under taking direct summands and arbitrary products.

3.2.3. Remark. A homological functor $H$ is $I$-exact iff its associated stable functor $H_* : T \to \text{Ab}^I$ is so; this is because $I$ is stable (under translation). We see that the $I$-exactness of a homological functor $H$ is equivalent to each of the following:

(i) $H$ sends $I$-epimorphisms to epimorphisms of $A$.
(ii) $H$ sends $I$-monomorphisms to monomorphisms of $A$.
(iii) $H$ turns each $I$-exact triangle $A \to B \to C \to TA$ to a short exact sequence $0 \to H(A) \to H(B) \to H(C) \to 0$.
(iv) $H$ maps $I$-exact chain complexes of $T$ (see next definition) to exact complexes in $A$.

The equivalence of (i) and (ii) should warn the reader from drawing too much intuition from the context of abelian categories: there is no room for ‘right’ or ‘left’ (I-)exact functors here (cf. also Lemma 3.4.6).

3.2.4. Remark. If $P$ is an $I$-projective object and $A \to B \to C \to TA$ is an $I$-exact triangle, we obtain a short exact sequence $0 \to T(P, A) \to T(P, B) \to T(P, C) \to 0$. Thus $I$-projective objects enjoy the usual lifting property of projectives, only with respect to $I$-epis, not to categorical epis. Conversely, an object satisfying the lifting property with respect to $I$-epis is $I$-projective.

Since $I$-injective objects are nothing but the $I$-projectives of $T^{\text{op}}$, we might restrict attention to projectives in what follows, trusting in the reader’s ability to dualize the results for themselves.

3.2.5. Definition. For an additive category $C$, we will denote by $\text{Ch}(C)$ the category of chain complexes in $C$.

(i) A complex $A_* = (A_\bullet, d_\bullet) \in \text{Ch}(T)$ is said to be $I$-decomposable, if there exist $I$-exact triangles in $T$

$$B_{n+1} \overset{g_n}{\longrightarrow} A_n \overset{f_n}{\longrightarrow} B_n \overset{b_n \in I}{\longrightarrow} TB_{n+1} \quad (n \in \mathbb{Z})$$

such that $d_n = g_{n-1} \circ f_n$ for all $n$.

(ii) Given a complex $A_* \in \text{Ch}(T)$, choose a distinguished triangle

$$A_n \overset{d_n}{\longrightarrow} A_{n-1} \overset{a_n}{\longrightarrow} X_n \overset{b_n}{\longrightarrow} TA_n$$

on each differential $d_n$. We say that $A_*$ is $I$-exact at $n$ (or $I$-acyclic at $n$) if the composite $Ta_{n+1} \circ b_n : X_n \to TX_{n+1}$ belongs to $I$.

Clearly the definition is independent from the choices. As usual, a complex is $I$-exact or $I$-acyclic if it is $I$-exact at $n$ for every $n \in \mathbb{Z}$. 

\[ \cdots \longrightarrow A_{n+1} \overset{d_n}{\longrightarrow} A_n \overset{d_n}{\longrightarrow} A_{n-1} \longrightarrow \cdots \]

\[ X_{n+1} \overset{a_n}{\longrightarrow} A_n \overset{b_n}{\longrightarrow} X_n \overset{Tb_{n+1}}{\longrightarrow} X_{n+1} \]
(iii) Let $A \in T$. An $\mathcal{I}$-projective resolution of $A$ is a complex $\cdots \to P_n \to \cdots \to P_1 \to P_0$ of $\mathcal{I}$-projectives together with an "augmentation" $P_0 \to A$, such that

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0,$$

is $\mathcal{I}$-exact. (In particular, the augmentation $P_0 \to A$ is $\mathcal{I}$-epi.)

(iv) We say that $T$ has enough $\mathcal{I}$-projective objects if for every object $A$ of $T$ there is a $\mathcal{I}$-projective $P$ and a $\mathcal{I}$-epi $P \to A$.

3.2.6. Lemma. Let $F : T \to A$ be a stable homological functor into a stable abelian category, such that $\mathcal{I} = \mathcal{I}_F$. Then a chain complex $A_\bullet \in \text{Ch}(T)$ is $\mathcal{I}$-exact at $n$ if and only if the sequence

$$(3.2.7) \quad F(A_{n+1}) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1})$$

is exact in $A$. \hfill $\square$

Proof. ([MN07, Lemma 3.9].) With the notation of Definition 3.2.5(ii), the complex $A_\bullet$ is exact at $n$ iff $T(a_{n+1}b_n) \in \mathcal{I}$ iff $a_{n+1}T^{-1}b_n \in \mathcal{I}$ iff $F(a_{n+1}) \circ F(T^{-1}b_n) = 0$ iff $\text{Im}(F(a_{n+1})) \subseteq \text{Ker}(F(a_{n+1}))$. But from the exactness of the sequences

$$FT^{-1}X_n \xrightarrow{FT^{-1}b_n} FA_n \xrightarrow{Fd_n} FA_{n-1}$$

$$FA_{n+1} \xrightarrow{Fd_{n+1}} FA_n \xrightarrow{F(a_{n+1})} FX_{n+1}$$

we deduce that $\text{Im}(FT^{-1}b_n) = \text{Ker}(Fd_n)$ and $\text{Im}(Fd_{n+1}) = \text{Ker}(F(a_{n+1}))$, so that the latter condition is equivalent to $\text{Ker}(Fd_n) \subseteq \text{Im}(Fd_{n+1})$. Since $d_{n+1}d_n = 0$, this is in turn equivalent to the exactness of $(3.2.7)$. \hfill $\square$

3.2.8. Corollary. $\mathcal{I}$-decomposable complexes are $\mathcal{I}$-exact. \hfill $\square$

3.2.9. Remark. It is easy to find examples of $\mathcal{I}$-exact complexes which are not $\mathcal{I}$-decomposable, already with $\mathcal{I} = \text{Mor}(T)$ or $\mathcal{I} = \emptyset$ ([MN07, Example 3.11, Example 3.19]). Therefore in relative homological algebra, perhaps surprisingly, exactness and decomposability of short exact sequences are different notions, contrary to what happens in abelian category.

3.2.10. Definition. Denote by $\text{Ac}(T, \mathcal{I})$ the strictly full subcategory of $\text{Ho}(T)$ of $\mathcal{I}$-acyclic complexes.

3.2.11. Corollary. $\text{Ac}(T, \mathcal{I})$ is a thick triangulated subcategory of $\text{Ho}(T)$. \hfill $\square$

Proof. Choose a stable homological functor $F : T \to A$ with $\mathcal{I} = \mathcal{I}_F$. In particular, $F$ is an additive functor between additive categories, so it induces a triangle functor $F := \text{Ho}(F) : \text{Ho}(T) \to \text{Ho}(A)$. By Lemma 3.2.6 a complex $A_\bullet$ is $\mathcal{I}$-acyclic if $FA_\bullet$ is acyclic in $\text{Ho}(A)$, iff $FA_\bullet \cong 0$ in the derived category of $A$; this last equivalence, because the category $\text{Ac}(A) \subseteq \text{Ho}(A)$ of acyclic complexes in an abelian category is thick, and $\text{D}(A) = \text{Ho}(A)/\text{Ac}(A)$ by definition (see e.g. [Wei94]). Hence $\text{Ac}(T, \mathcal{I}) = \text{Ker}(\text{Ho}(T) \to \text{Ho}(A) \to \text{D}(A))$ is thick. \hfill $\square$

3.2.12. Definition. The Verdier quotient

$$\text{D}(T, \mathcal{I}) := \text{Ho}(T)/\text{Ac}(T, \mathcal{I})$$

is called the derived category of $T$ relative to $\mathcal{I}$.

Note that an $\mathcal{I}$-epi $P \to A$ is the same as a complex $P \to A \to 0$ which is $\mathcal{I}$-exact at $A$. If $P$ is $\mathcal{I}$-projective, we may call such a $P \to A$ a one step $\mathcal{I}$-projective resolution of $A$. By successively choosing such one-step resolutions, one proves:
3.2.13. **Lemma.** If $T$ has enough $I$-projective objects, then every object has an $I$-projective resolution (and vice-versa). Moreover, for each $A \in T$ the projective resolution can be chosen so that the augmented complex $P_* \to A$ is $I$-decomposable. □

The next nice properties of projective objects can also be proved as the analogous statements for abelian categories.

3.2.14. **Lemma.** (Comparison Lemma). Let $f : A \to B$ be a morphism in $T$, let $P_* \to A$ be a (non necessarily $I$-exact) complex of projectives such that $P_n = 0$ for $n < 0$, and equipped with an augmentation $P_0 \to A$, and let $E_* \to B$ be any (non necessarily projective) resolution of $B$. Then there exists a lift of $f$ to a chain map $P_* \to E_*$, unique up to chain homotopy.

If $f \in I(A, B)$, then the lift $P_* \to E_*$ is homotopic to zero.

**Proof.** Exactly as for abelian categories. Use here that the augmented complex $(P_0 \to A)$ is $I$-acyclic and that the functor $T(P_n, -)$ is $I$-exact to obtain, by Lemma 3.2.6, a long exact sequence

$$
\cdots \to T(P_n, E_1) \to \cdots \to T(P_n, E_0) \to T(P_n, B) \to 0
$$

for every $n > 0$. For the last claim, it is enough to note that $T(P, f) = 0$ whenever $f \in I$ and $P$ is $I$-projective, so that all the liftings can be chosen to be the zero map. □

3.2.15. **Proposition.** ([Mn07, Prop. 3.26].)

(i) Given two $I$-projective resolutions $P_* \to A$ and $P'_* \to A'$, every morphism $A \to A'$ can be lifted to a chain map $P_* \to P'_*$, and this lifting is unique up to chain homotopy. Thus the choice of a projective resolution for each object $A \in T$ induces a functor

$$
P : T \to \text{Ho}_+(\text{Proj}(T, I)) \subseteq \text{Ho}(T)
$$

to the homotopy category of (right bounded) complexes (of $I$-projectives), and any two classes of choices yield canonically isomorphic functors.

(ii) Given an $I$-exact triangle $E = (A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} TA)$, there exists a canonical map $\delta_{E, P}$ such that the following triangle is distinguished in $\text{Ho}(T)$:

$$
P(A) \xrightarrow{P(f)} P(B) \xrightarrow{P(g)} P(C) \xrightarrow{\delta_{E, P}} P(A)[1].
$$

**Sketch of proof.** Part (i) is an immediate consequence of Lemma 3.2.14. For part (ii), the idea is to consider the diagram in $\text{Ch}(T)$

$$
\begin{array}{ccc}
PA_\bullet & \xrightarrow{f} & PB_\bullet \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}
\xrightarrow{\ell} \begin{array}{ccc}
C_\bullet & \rightarrow & PA_\bullet[1] \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & TA
\end{array}
$$

obtained by lifting $f$ to projective resolutions according to (i). Here $C_\bullet$ is the mapping cone construction for the chain map $f$, so that the first row is a standard distinguished triangle in $\text{Ho}(T)$. The chain map $\ell : C_\bullet \to C$ is $(C_0)_0 = (PA_\bullet)[1] \oplus (PB_\bullet)_0 = B \oplus C$ in degree zero and zero elsewhere. Since $C_\bullet$ has $I$-projective entries by construction, one is left with checking that, if $E$ is $I$-exact, the augmented complex $(C_\bullet, \ell)$ is $I$-exact, and therefore it is an $I$-projective resolution of $C$. □
3.2.16. Remark. It is perhaps a good time to recall that the triangulation of \( \text{Ho}(\mathcal{T}) \) does not depend in any way on the given one in \( \mathcal{T} \) (indeed, it comes for free with any additive category). Also, the shift of complexes \((-)[1]\) is ‘orthogonal’ to the given translation \( T \) of \( \mathcal{T} \). In particular, the canonical fully faithful functor \( T \to \text{Ho}(\mathcal{T}) \) sending an object \( A \) to the complex with \( A \) concentrated in degree zero, does not commute with the translation and is not a triangle functor.

3.2.17. Proposition. Assume one of the following two hypotheses:

(i) \( \mathcal{I} = \ker(F) \), where \( F : \mathcal{T} \to \mathcal{A} \) is a stable homological functor to a stable abelian category \( \mathcal{A} \). Assume that \( F \) has a partial left adjoint \( G : \mathcal{P} \to \mathcal{T} \), defined on a subcategory \( \mathcal{P} \subseteq \text{Proj}(\mathcal{A}) \) such that for every \( A \in \mathcal{T} \) there exists an epimorphism \( P \to F(A) \) with \( P \in \mathcal{P} \).

(ii) \( \mathcal{I} = \ker(F) \), where \( F : \mathcal{T} \to \mathcal{S} \) is a triangle functor to a triangulated category \( \mathcal{S} \). Assume that \( F \) has a left adjoint \( G : \mathcal{S} \to \mathcal{T} \).

Then \( (\mathcal{T}, \mathcal{I}) \) has enough projective objects, and \( \text{Proj}(\mathcal{A}, \mathcal{T}) \) consists of all summands of objects in the image of \( G \).

Proof. ([MN07, Prop. 3.37.]) The proof is the same in both cases, so let \( \mathcal{C} \) be either \( \mathcal{P} \subseteq \text{Proj}(\mathcal{A}) \) or \( \mathcal{S} \). If \( f \in \mathcal{I} \), then \( T(GA, f) \cong C(A, FA) = 0 \) because \( \mathcal{I} \subseteq \ker(F) \). Thus \( \text{Im}(G) \subseteq \text{Proj}(\mathcal{T}, \mathcal{I}) \). Now if \( A \in \mathcal{T} \), by hypothesis there is an epimorphism \( p : P \to F(A) \) with \( P \in \mathcal{C} \) (we can choose the identity \( 1_{FA} \) for \( \mathcal{T}, \mathcal{I} \)). So its adjoint \( p' : G(P) \to A \) is a one-step \( \mathcal{I} \)-projective resolution, provided it is \( \mathcal{I} \)-epi, or equivalently, that \( F(p') \) is an epimorphism. Consider the commutative square on the left.

\[
\begin{array}{ccc}
G(P) & \longrightarrow & G(P) \\
\downarrow & & \downarrow \\
G(P) & \longrightarrow & A
\end{array} \quad \begin{array}{ccc}
P & \longrightarrow & FG(P) \\
\downarrow & & \downarrow \\
P & \longrightarrow & F(A)
\end{array}
\]

Via the natural isomorphism of the partial adjunction, it corresponds to the commutative square on the right. In particular, \( F(p') \) is an epimorphism since \( p \) is one by hypothesis. This proves that there are enough \( \mathcal{I} \)-projective objects in \( \mathcal{T} \). If moreover \( A \in \text{Proj}(\mathcal{T}, \mathcal{I}) \), then \( p' : G(P) \to A \) is a split epi (by the lifting property of \( \mathcal{I} \)-projectives for \( \mathcal{I} \)-epimorphisms). This shows that every \( \mathcal{I} \)-projective is a summand of one of the form \( G(P) \), \( P \in \mathcal{C} \). \( \square \)

3.3. Operations on homological ideals

The intersection of any family \( \{\mathcal{I}_i\}_{i \in I} \) of ideals of \( \mathcal{T} \) is again an ideal. If the \( \mathcal{I}_i \) are homological, then so is their intersection, at least in the case where each has enough projectives:

3.3.1. Proposition. Let \( \mathcal{I}_i \ (i \in I) \) be a set of homological ideals in \( \mathcal{T} \), each with enough projective objects, and suppose that \( \mathcal{T} \) has coproducts of cardinality \( |I| \). Then \( \mathcal{I} := \bigcap \mathcal{I}_i \) is also homological with enough projectives, and the \( \mathcal{I} \)-projectives are exactly the summands of objects of the form \( \bigsqcup P_i \), with \( P_i \in \mathcal{I}_i \).

Proof. This is [Chr98, Prop. 3.1], because a homological ideal \( \mathcal{I} \) with enough projective objects \( \mathcal{P} := \text{Proj}(\mathcal{T}, \mathcal{I}) \) is the same as a projective class \( (\mathcal{I}, \mathcal{P}) \) in the sense of Christensen. \( \square \)
3.4. Relative derived functors

Given two ideals in $T$, one can also define their **product** $I_1 \circ I_2$ as follows:

$$I_1 \circ I_2(A, B) := \bigcup_{X \in \text{obj}(T)} \{ f_1 \circ f_2 \mid f_1 \in I_1(X, B), f_2 \in I_2(A, X) \}.$$ 

This operation also preserves homological ideals with enough projectives, and Christensen provides a description of its projective objects.

3.3.2. **Proposition.** Let $I_1$ and $I_2$ be two homological ideals of $T$, each with enough projectives. Then $I_1 \circ I_2$ is also homological with enough projectives, and the latter are precisely the summands of objects $A \in T$, such that there exists a distinguished triangle $P_2 \to A \to P_1 \to TP_2$ with $P_1 \in \text{Proj}(T, I_1)$ and $P_2 \in \text{Proj}(T, I_2)$.

**Proof.** [Chr98, Prop. 3.3].

3.4. Relative derived functors

Ideally, we should now define and study the *total* left and right derived functors $L_n F$ and $R_n F : \text{D}(T, I) \to \text{D}(A)$ of, say, an additive functor $F : T \to A$ to an abelian or exact category $A$. This would take too many pages and will not be needed later, so we limit ourselves to constructing the “satellites” $L_n F$ and $R_n F : T \to A$ by hand, as in [MN07] and [Bel00b, p. 287]. These functors will provide the input of the ABC spectral sequence of the next section.

3.4.1. **Definition.** Let $F : T \to A$ be an additive functor to an abelian category $A$. If $(T, I)$ has enough projectives, for every $n \geq 0$ define the $n$-th **derived functor** $L_n^I F$ **relative to** $I$ to be the composition

$$L_n^I F : T \xrightarrow{P} \text{Ho}(T) \xrightarrow{F} \text{Ho}(A) \xrightarrow{H_n} A,$$

where $P$ is a functorial $I$-projective resolution as in Proposition 3.2.15 (i). Dually, if $(T, I)$ has enough injectives define the $n$-th **right derived functor** $R_n^I F$ **relative to** $I$ as

$$R_n^I F : T \xrightarrow{I} \text{Ho}(T) \xrightarrow{F} \text{Ho}(A) \xrightarrow{H_n} A,$$

where $I$ is a functorial injective resolution.

3.4.2. **Remarks.** (a) By the Comparison Lemma (Lemma 3.2.14), by using any projective (resp. injective) resolution at hand one would compute the same derived functors, up to unique isomorphism.

(b) Consider an additive functor $G : T^{\text{op}} \to A$, and assume that $T$ has enough $I$-projectives. Then $T^{\text{op}}$ has enough injectives and we may use them to compute the right derived functors $R_n^I G : T^{\text{op}} \to A$:

$$R_n^I G : T^{\text{op}} \xrightarrow{P} \text{Ho}(T^{\text{op}}) \xrightarrow{G} \text{Ho}(A) \xrightarrow{H_n} A.$$

The reason we write upper indices in this case will become apparent in Corollary 3.4.5 below.

3.4.4. **Example.** Assume that $(T, I)$ has enough projectives. Then the $I$-**relative Ext functors** are defined to be the derived functors $\text{Ext}_n^I_{T, I}(-, B) := R_n^I T(\_, B) : T^{\text{op}} \to \text{Ab}$. They may be computed by $I$-projective resolutions as in the above remark. The augmentation $P_0 \to A$ provides a natural transformation $T(\_, B) \Rightarrow \text{Ext}_n^I_{T, I}(-, B)$ which in general is neither injective nor surjective.
3.4.5. Corollary. Assume that \((T,I)\) has enough projectives. As above, let \(F : T \rightarrow A\) and \(G : T^{\text{op}} \rightarrow A\) be additive functors to an abelian category. Then each \(I\)-exact triangle \(A \rightarrow B \rightarrow C \rightarrow TA\) in \(T\) induces long exact sequences
\[
\cdots \rightarrow \mathbb{L}_{n+1}^IF(C) \rightarrow \mathbb{L}_{n}^IF(A) \rightarrow \mathbb{L}_{n}^IF(B) \rightarrow \mathbb{L}_{n}^IF(C) \rightarrow \mathbb{L}_{n-1}^IF(A) \rightarrow \cdots
\]
\[
\cdots \leftarrow \mathbb{R}_{n+1}^IF(C) \leftarrow \mathbb{R}_{n}^IF(A) \leftarrow \mathbb{R}_{n}^IF(B) \leftarrow \mathbb{R}_{n}^IF(C) \leftarrow \mathbb{R}_{n-1}^IF(A) \leftarrow \cdots
\]
in \(A\).

Proof. The given \(I\)-exact triangle produces an exact triangle \(PA \rightarrow PB \rightarrow PC \rightarrow PA[1]\) in \(\text{Ho}(T)\) by Proposition 3.2.15 (ii). The triangle functors \(F = \text{Ho}(F), G = \text{Ho}(G)\) send it to distinguished triangles in \(\text{Ho}(A)\), whose long exact sequences in homology are the above ones by definition.

At this point one could proceed to list the nice properties of derived functors. For instance, each \(\mathbb{L}_{n}^IF\) descends to a functor \(\mathbb{L}_{n}^IF : T/I \rightarrow A\), since \(I \subseteq \ker(\mathbb{L}_{n}^IF)\). Of particular interest are the relative Ext groups \(\text{Ext}_{T,I}^n(A,B)\). We refer to [MN07, §3.5] for more, and content ourselves by quoting the following result.

3.4.6. Lemma ([MN07, Lemma 3.29]). Assume that \((T,I)\) has enough projectives. For a homological functor \(H : T \rightarrow A\), the following assertions are equivalent.

(i) \(H\) is \(I\)-exact (i.e., \(I \subseteq \ker(H)\)).

(ii) \(\mathbb{L}_{n}H(A) \cong H(A)\) and \(\mathbb{L}_{n}H(A) \cong 0\) for all \(n > 0\) and \(A \in T\).

(iii) \(\text{Ho}_0H(A) \cong H(A)\) for all \(A \in T\).

The analogous statement holds for the right derived functors of a cohomological functor \(H : T^{\text{op}} \rightarrow A\).

\[\square\]

3.5. The universal \(I\)-exact functor

Given any triangulated category \(T\), we have seen in Subsection 1.5.1 that the Yoneda embedding into the category of coherent functors, \(h : T \rightarrow \text{Coh}(T)\), provides the universal (stable) homological functor out of \(T\). Given a homological ideal \(I\), this can be modified to yield the universal \(I\)-exact (stable) homological functor out of \(T\). The ‘absolute case’ is recovered by choosing the ideal \(I = 0\).

Recall that a homological functor \(H : T \rightarrow A\) is called \(I\)-exact if \(I \subseteq \ker(H)\).

3.5.1. Theorem. For every triangulated category \(T\) and every homological ideal \(I\), there exists a universal \(I\)-exact stable homological functor \(h_I : T \rightarrow \text{Coh}(T,I)\): for every \(I\)-exact (stable) homological functor \(H : T \rightarrow A\) to some stable abelian category, there is a (stable) exact functor \(\tilde{H}\), unique up to unique iso, such that \(\tilde{H} \circ h_I = H\).

Idea of proof. The category \(\text{Coh}(T,I)\) can be obtained by localizing the Freyd category \(\text{Coh}(T)\) at a suitable Serre subcategory; this is the approach of [Bel00b, §3], where \(\text{Coh}(T,I)\) is called the Steenrod category. The Serre subcategory to be killed is that of all images in \(\text{Coh}(T)\) of morphisms \(h(f)\) with \(f \in I\). Alternatively, one can construct the functor as the composite \(T \rightarrow D(T,I) \rightarrow \mathcal{N},\) where \(\mathcal{N}\) is the heart of the canonical t-structure of the relative derived category \(D(T,I)\). See [MN07, §3.2.1] for more details.

3.5.2. Remark. By construction we have \(I = \ker(h_I)\). Thus this theorem, and the next corollary, are particularly interesting when relative homological algebra is treated axiomatically — as in Beligiannis’ or Christensen’s settings — where one doesn’t assume on the outset a stable homological functor \(F\) with \(I = \ker(F)\) and the availability of an an abelian category where to test \(I\)-exactness.
3.5.3. Corollary. Let \( f : A_\bullet \to B_\bullet \) be a chain map in \( \text{Ch}(\mathcal{T}) \). Then the following are equivalent:

1. \( f \) is invertible in \( \text{D}(\mathcal{T}, \mathcal{I}) \).
2. \( \text{cone}(f) \in \text{Ch}(\mathcal{T}) \) is \( \mathcal{I} \)-acyclic.
3. \( h_\mathcal{T}(\text{cone}(f)) \in \text{Ch}\text{Coh}(\mathcal{T}, \mathcal{I}) \) is acyclic.
4. \( H_n h_\mathcal{T}(f) \) is an isomorphism in \( \text{Coh}(\mathcal{T}, \mathcal{I}) \) for every \( n \in \mathbb{Z} \).
5. \( h_\mathcal{T}(f) \) is invertible in \( \text{D}(\text{Coh}(\mathcal{T}, \mathcal{I})) \) (it’s a quasi-isomorphism).

In particular, the universal \( \mathcal{I} \)-exact functor descends to the derived categories:

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{h_\mathcal{T}} & \text{Coh}(\mathcal{T}, \mathcal{I}) \\
\downarrow & & \downarrow \\
\text{D}(\mathcal{T}, \mathcal{I}) & \xrightarrow{h_\mathcal{T}} & \text{D}(\text{Coh}(\mathcal{T}, \mathcal{I})).
\end{array}
\]

We record en passant the compatibility of these constructions with coproducts:

3.5.4. Lemma. Assume that \( \mathcal{T} \) has all \( \alpha \)-small coproducts, and that the homological ideal \( \mathcal{I} \) is closed under taking \( \alpha \)-small coproducts of morphisms. Then

1. \( \text{Coh}(\mathcal{T}, \mathcal{I}) \) has \( \alpha \)-coproducts and \( h_\mathcal{T} : \mathcal{T} \to \text{Coh}(\mathcal{T}, \mathcal{I}) \) commutes with them.
2. \( \text{D}(\mathcal{T}, \mathcal{I}) \) has \( \alpha \)-coproducts and the canonical functor \( i : \mathcal{T} \to \text{D}(\mathcal{T}, \mathcal{I}) \) preserves them.

Proof. Point (i) was already proved for the Freyd embedding (Lemma 1.5.5(ii)). Now it is enough to check that the Serre subcategory \( \mathcal{S} = \mathcal{S}(\mathcal{I}) \) of \( \text{Coh}(\mathcal{T}) \) used to construct \( \text{Coh}(\mathcal{T}, \mathcal{I}) = \text{Coh}(\mathcal{T}) / \mathcal{S} \) is closed under taking coproducts of objects. This is clear, since \( \mathcal{S} \) is the precisely the full subcategory of \( \text{Coh}(\mathcal{T}) \) of objects of the form \( \text{Im}(h_{\mathcal{C}_1} \to h_{\mathcal{C}_0}) \), for some \( i \in \mathcal{I} \) (cf. [Bel00b, § 3.2]).

(ii): In \( \text{Ch}(\mathcal{T}) \), coproducts can be constructed objectwise in \( \mathcal{T} \); in particular, \( \mathcal{T} \to \text{Ch}(\mathcal{T}) \) preserves them. The ideal \( \mathcal{I} \) is closed under taking coproducts of maps iff the subcategory \( \text{Ac}(\mathcal{T}, \mathcal{I}) \) is closed under taking coproducts of complexes (recall for this that a coproduct of distinguished triangles is always automatically distinguished), iff \( \text{D}(\mathcal{T}, \mathcal{I}) \) has coproducts and the localisation functor \( \text{Ch}(\mathcal{T}) \to \text{D}(\mathcal{T}, \mathcal{I}) \) preserves them.

This universal \( \mathcal{I} \)-exact functor can be used to bring down to Earth the relative derived functors of the previous sections, by identifying them with suitable derived functors in the more familiar context of abelian categories. But for this to be useful, one should have some criterion for recognizing the Steenrod category \( \text{Coh}(\mathcal{T}, \mathcal{I}) \) as something already known. The next result does precisely that.

3.5.5. Theorem. Let \( F : \mathcal{T} \to \mathcal{A} \) be a stable \( \mathcal{I} \)-exact functor with \( \mathcal{I} = \ker F \), and suppose that \( \mathcal{T} \) is idempotent complete. Then the following are equivalent:

1. \( \mathcal{T} \) has enough \( \mathcal{I} \)-projective objects and \( F \) is the universal \( \mathcal{I} \)-exact stable homological functor.
2. The following holds:
   (a) \( \mathcal{A} \) has enough projective objects,
   (b) \( F \) has a partial left adjoint \( F^\dagger \) defined on \( \text{Proj}(\mathcal{A}) \).
   (c) \( F \circ F^\dagger \cong \text{id}_{\text{Proj}(\mathcal{A})} \).

Actually, if \( \mathcal{T} \) has coproducts, in (b) it suffices for \( F^\dagger \) to be defined on a subcategory \( \mathcal{P} \subseteq \text{Proj}(\mathcal{A}) \) which generates \( \text{Proj}(\mathcal{A}) \) by taking coproducts and summands (or quotients) in \( \mathcal{A} \), because then \( F^\dagger \) can be extended to all projectives, retaining property (c).
Proof. See [MN07, Theorem 3.39, Remark 3.41]. We sketch the proof of \(\text{‘(ii)⇒(i)’}\), because it yields interesting observations. First of all:

3.5.6. Lemma. If the conditions in (i) of the theorem are satisfied, the functors \(F\) and \(F^\dagger\) restrict to equivalences \(F : \Proj(T,I) \simeq \Proj(A) : F^\dagger\), quasi-inverse to each other.

Proof. It is easily checked that the hypotheses of Proposition 3.2.17 (i) are satisfied, so \(T\) has enough \(I\)-projectives, and each \(I\)-projective is a summand of some \(F^\dagger(A), A \in A\). By (c), we only have to show that \(F^\dagger \circ F \cong \text{id on } \Proj(T,I)\). But (c) already yields an isomorphism \(F^\dagger F |_{\text{Im}(F^\dagger)} \cong \text{id}_{\text{Im}(F^\dagger)}\), which then can be extended to direct summands. \(\square\)

3.5.7. Lemma. If \(F : T \to A\) satisfies the conditions in Thm. 3.5.5 (ii), then it enjoys the following property. Given any (stable) homological functor \(H : T \to B\) to some (stable) abelian category, there exists a (stable) right exact functor \(\hat{H} : A \to B\), unique up to unique isomorphism, such that \(\hat{H} \circ F(P) = H(P)\) for all \(P \in \Proj(T,I)\).

Sketch of proof. For objects of the form \(F(P) \in A\) with \(P \in \Proj(T,I)\) we must set \(\hat{H}(F(P)) := H(P)\). For a general \(A \in T\), we may choose a projective resolution \(Q_\bullet \to A\) in \(A\), because \(A\) has enough projectives by (a). By Lemma 3.5.6 we may assume that \(Q_\bullet = F(P_\bullet)\) for a complex \(\cdots \to P_1 \to P_0\) of \(I\)-projectives. Then we define \(\hat{H}(A) := H_0(H(P_0)) = \text{Coker}(F(P_1 \to P_0)) \in B\). We leave it to the reader to verify that this defines a functor \(\hat{H} : A \to B\) with the required properties. \(\square\)

End of the proof of ‘(ii)⇒(i)’. Now let \(H\) be also \(I\)-exact. Then we may set \(\hat{H}F = H\) on the whole category \(T\). We claim that moreover, in this case, \(\hat{H}\) is exact. This would prove that \(F\) is the universal \(I\)-exact functor. Let \(A \in A\) and choose a projective resolution of \(A\) of the form \(F(P_\bullet) = H_0(P_\bullet)\). Then we compute \(L_n \hat{H}(A) = H_n(\hat{H}F(P_\bullet)) = H_n(HP_\bullet) = 0\) for all \(n \neq 0\), by Lemma 3.2.6 and the \(I\)-exactness of \(F\) and \(H\). Hence \(\hat{H}\) is exact. \(\square\)

Here is an immediate consequence of Lemma 3.5.6.

3.5.8. Corollary. Assume that \(T\) has enough \(I\)-projectives, and let \(h_I : T \to \text{Coh}(T,I)\) be the universal \(I\)-exact functor. Then \(h_I\) induces an equivalence \(\text{Ch}(\Proj(T,I)) \simeq \text{Ch}(\Proj(\text{Coh}(T,I)))\). For every \(A \in T\), it induces a bijection between isomorphism classes in \(\text{Ch}(T)\) of \(I\)-projective resolutions of \(A\) and isomorphism classes in \(\text{ChCoh}(T,I)\) of projective resolutions of \(h_I(A)\).

Proof. For the second part, note that by the partial adjunction we have \(T(P,B) \cong T(F^\dagger FP,B) \cong \text{Coh}(h_I(P,h_IB))\) for all \(P \in \Proj(T,I)\) and all \(B \in T\). Therefore, we have also a bijection of the augmentations \(P_0 \to A \to 0\) and \(h_IP_0 \to h_IA \to 0\). \(\square\)

Finally, the more general extension property of the universal \(I\)-exact functor that appears in Lemma 3.5.7 yields the above-mentioned method for computing relative derived functors:

3.5.9. Proposition. Assume that \(T\) has enough \(I\)-projectives, and let \(h_I : T \to \text{Coh}(T,I)\) be the universal \(I\)-exact functor. Then for every (stable) homological functor \(H : T \to B\) to an abelian category, there exists a (stable) right exact functor \(\hat{H} : \text{Coh}(T,I) \to B\), unique up to unique isomorphism, such that \(H|_{\Proj(T,I)} = \hat{H} \circ h_I|_{\Proj(T,I)}\). There are canonical isomorphisms

\[ L^n_I H \cong L^n \hat{H} \circ h_I \]
of functors $T \to B$ for all $n \in \mathbb{Z}$.

Assume moreover that $T$ has all $\alpha$-small coproducts and $I$ is closed under $\alpha$-small coproducts. Then if $H$ preserves $\alpha$-small coproducts so does $\tilde{H}$. □

In concrete situations it may be possible to give more explicit formulas for the functors $\tilde{H}$, cf. [MN07, §5] and Section 5.3 below.

3.6. The ABC spectral sequence

In his sequel [Mey08] to [MN07], Ralf Meyer studies the generalized Adams spectral sequence in this context. He calls it the ABC spectral sequence, short for Adams [Ada74], Brinkmann [Bri68] and Christensen [Chr98], and so we shall do too (cf. also [Tho03, §4.4.5]).

3.6.1. Hypothesis. In this section, $T$ will be a triangulated category with $\alpha$-small coproducts (where as usual $\alpha$ is some uncountable infinite regular cardinal; see Section 1.7), and $I$ will be a homological ideal in $T$ with enough projectives, which is closed under taking $\alpha$-small coproducts of morphisms.

The spectral sequence is constructed as follow ([Mey08, §3.1-2]). Fix an object $A \in T$. A phantom tower over $A$ is a diagram in $T$ of the form

\[
A = N_0 \xrightarrow{\iota_0} N_1 \xrightarrow{\iota_1} N_2 \xrightarrow{\iota_2} \cdots \]

where the triangles “$\triangle$” commute (that is, $\delta_{n+1} = \epsilon_n \pi_{n+1}$) and the triangles “$\nabla$” are rotated $I$-exact triangles (that is, each triangle $(3.6.2)$ $P_n \xrightarrow{\pi_n} N_n \xrightarrow{\iota_{n+1}^n} N_{n+1} \xrightarrow{\epsilon_n} TP_n$ is distinguished and $\iota_{n+1}^n \in I$). Moreover, the objects $P_n$ are required to be $I$-projective. It follows that the complex $(T \cdot P, T \cdot \delta)$ displayed on the second row is an $(I$-decomposable) $I$-projective resolution of $A$. Since $(T, I)$ has enough projectives, one can construct a phantom tower recursively by choosing a one step resolution $\pi_n : P_n \to N_n$ and a distinguished triangle containing it, as in (3.6.2) ([Mey08, Lemma 3.3]).

Now let $F : T \to \text{Ab}$ be a homological functor you are interested in. By applying $F$ to a phantom tower for $A$ and “rolling up”, one produces an exact couple, and thus a spectral sequence, in a well-known way. More precisely, start by extending the phantom tower toward the left by setting $P_n := 0$, $N_n := A$ and $\iota_{n+1}^n := 1_A$ for all $n < 0$. Then define bigraded abelian groups

\[
D := \bigoplus_{p,q \in \mathbb{Z}} D_{p,q}, \quad D_{p,q} := F_{p+q+1}(N_{p+1})
\]

and homogeneous homomorphisms

\[
E := \bigoplus_{p,q \in \mathbb{Z}} E_{p,q}, \quad E_{p,q} := F_{p+q}(P_p)
\]

and homogeneous homomorphisms

\[
D \xrightarrow{i} D \xleftarrow{i} E
\]
with components

\[ i_{pq} := F_{p+q+1}(\epsilon_{p+1}) : D_{p,q} \rightarrow D_{p+1,q-1} \]

\[ j_{pq} := F_{p+q+1}(\epsilon_p) : D_{p,q} \rightarrow E_{p,q} \]

\[ k_{pq} := F_{p+q}(\alpha_p) : E_{p,q} \rightarrow D_{p-1,q} \]

Thus \( \text{deg}(i) = (1,-1) \), \( \text{deg}(j) = (0,0) \) and \( \text{deg}(k) = (-1,0) \). Since the triangles (3.6.2) are distinguished and \( F \) is homological, the sequences

\[ \cdots \rightarrow D_{p,q} \xrightarrow{j_{p,q}} D_{p+1,q-1} \xrightarrow{j_{p+1,q-1}} E_{p+1,q-1} \xrightarrow{k_{p+1,q-1}} D_{p,q-1} \rightarrow \cdots \]

are exact for all \( p,q \in \mathbb{Z} \) and therefore \( (D,E,i,j,k) \) is an exact couple. Following the general method, one can define for \( r \geq 1 \) its \( r \)-th derived exact couple \( (D^r, E^r, i^{(r)}, j^{(r)}, k^{(r)}) \), with the first one being the original one. The resulting homologically indexed spectral sequence has \( r+1 \)-th page \( E^{r+1} \) given by the homology of the differential bigraded module \( (E^r, d^{(r)} := j^{(r)}k^{(r)}) \), where \( \text{deg}(d^{(r)}) = (-r,r-1) \).

This spectral sequence is the ABC spectral sequence associated to \( (T,I) \), the homological functor \( F \), the object \( A \in T \) and the chosen phantom tower on \( A \). Given a cohomological functor \( G : T^{op} \rightarrow \text{Ab} \) instead, one can similarly produce a cohomological spectral sequence, but we won’t go into that.

By direct inspection of the definitions, one can easily recognize the second page \( E^2 \) and conclude on its functorial properties:

3.6.4. Theorem. The second page of the ABC spectral sequence identifies canonically with the left derived functors of \( F \) with respect to \( I \):

\[ E^2_{pq} \cong L^2_p F_q(A) = L^2_p F(T^{-q}A). \]

Moreover, from the second page onwards the spectral sequence doesn’t depend on the choice of a phantom tower over \( A \), and it is functorial in \( A \) (it defines a functor from \( T \) to the category of homological spectral sequences of abelian groups).

Proof. [Mey08, Theorem 3.15].

For a homological functor \( F \) and for objects \( A \) which can be constructed from projective ones by taking finitely many cones or summands, the ABC spectral sequence converges towards \( F(A) \). More precisely, recall the definition of the \( n \)-power \( T^n = T^{n-1} \circ T \) of an ideal from Section 3.3). Since \( I \) has enough projectives, \( T^n \) is also a homological ideal with enough projectives; these are obtained recursively from \( T \)-projective objects by taking summands of cones, at most \( n-1 \) times (see Proposition 3.3.2).

3.6.5. Proposition. Consider a homological functor \( F : T \rightarrow \text{Ab} \); let \( m \in \mathbb{N} \).

(i) Let \( A \in \text{Proj}(T, T^{m+1}) \). Then the ABC spectral sequence for \( F \) and \( A \) collapses at \( E^{m+2} \) and converges towards \( F_*(A) \). Its limit page \( E^{m+2} \) is supported in the region \( 0 \leq p \leq m \).

(ii) Let \( A \) be an object having an \( T \)-projective resolution of length \( m \). Then the ABC spectral sequence for \( F \) and \( A \) is supported from the second page onwards on the region \( 0 \leq p \leq m \), so that it collapses at \( E^{m+1} \). If moreover \( A \in (\text{Proj}(T,T))_{\text{loc}} \), than it converges towards \( F_*(A) \).

Proof. [Mey08, Proposition 3.17].

In order to discuss the more general convergence properties of the ABC spectral sequence, we must be able to describe its target, which in general will not be the functor \( F \). The following beautiful theorem is one of the main results of [Mey08],
and is obtained by thoroughly analysing the phantom tower and comparing it with another tower (the “$P$-cellular approximation of $A$”) which also yields the ABC spectral sequence.

3.6.6. **Theorem.** Let $T$ and $I$ be as in Hypothesis 3.6.1. Then the pair

$$(\langle \text{Proj}(T,I) \rangle_{\text{loc}}, \text{Ker}(h_I))$$

with the localizing subcategory generated by $I$-projectives on the left and the subcategory of $I$-contractible objects on the right, is a complementary pair of localizing subcategories of $T$ (Definition 1.4.21).

**Proof.** [Mey08, Theorem 4.6]. □

3.6.7. **Remark.** Note that Theorem 3.6.6 is quite different in spirit from Proposition 1.7.15, as it produces a pair of localizing subcategories out of a homological ideal with enough projectives, without having to know anything at all about the compact objects of $T$.

3.6.8. **Notation.** For short, we will write $L_I := \langle \text{Proj}(T,I) \rangle_{\text{loc}}$ and $R_I := \text{Ker}(h_I)$ for the two localizing subcategories of the theorem. Predictably then, the gluing triangle for this complementary pair will be written $L_I A \longrightarrow A \longrightarrow R_I A \longrightarrow T L_I A$.

3.6.9. **Remark.** (a) Note that, if $I := \ker(F)$ for a (decent) stable homological functor $F : T \rightarrow A$, then $R_I = \text{Ker}(F)$. If moreover $F$ has a partial left adjoint $G$ defined on projectives, which can be used to provide $(T,T)$ with enough projectives as in Proposition 3.2.17, then $L_I = \langle G(\text{Proj}(A)) \rangle_{\text{loc}}$.

(b) With the hypotheses of this section (see Hyp. 3.6.1), $T$ has ($\alpha$-small) coproducts and so does $\text{Coh}(T,I)$; the universal $I$-exact functor $h_I$ commutes with them (Lemma 3.5.4). Moreover, $\text{Coh}(T,I)$ has enough projectives. Let $P$ be a set of projective generators (in the sense that every projective in $\text{Coh}(T,I)$ is isomorphic to a summand of a coproduct of objects in $P$). Then the partial left adjoint $h_I^!$ preserves coproducts and $L_I = \langle h_I^!(P) \rangle_{\text{loc}}$.

3.6.10. **Definition.** Let $F : T \rightarrow C$ be any functor. Then define the **localization of $F$ at $I$**, written $L_I F$, to be the composite functor

$$L_I F : T \xrightarrow{L_I} L_I T \xrightarrow{F} A.$$

It is the localization of $F$ with respect to $R_I \subseteq T$, i.e., the right Kan extension of $F$ along the localization functor $T \rightarrow T/R_I$ (see [Mac98]). The canonical natural transformation $L_I F \Rightarrow F$ is obtained simply by applying $F$ to $L_I A \rightarrow A$. Thus if $A \in L_I$, it yields a canonical identification $L_I F(A) \cong F(A)$ for all $F$.

It turns out that the functor $L_I F$ is the appropriate target of the ABC spectral sequence for $F$, at least when $F$ is nice.

3.6.11. **Theorem.** Let $(T,I)$ be as in Hypothesis 3.6.1, let $F : T \rightarrow \text{Ab}$ be a decent homological functor and let $A$ be an object of $T$. Then the ABC spectral sequence for $F$ and $A$ converges (strongly) towards $L_I F(T^{\rightarrow} A)$. Therefore, if $A \in L_I$, it converges towards $F_*(A)$.

**Proof.** [Mey08, Theorem 4.15]. □
3.6.12. Remark. (a) By applying to the phantom tower of an object a cohomological functor $G : T^{\text{op}} \to \text{Ab}$, one obtains a cohomologically indexed ABC spectral sequence, containing the right derived functors $\mathbb{R}^n G$ on the second page. In this case, the discussion of convergence is complicated by the non-exactness of filtering limits of abelian groups, in a familiar way. The correct general target here is provided by the localized functor $\mathbb{R}^LG := G \circ L_T : T^{\text{op}} \to \text{Ab}$.

(b) It is apparent from [Mey08] that the construction of the ABC spectral sequence works equally well for a homological functor $F : T \to \mathcal{A}$ whose target is a general Grothendieck abelian category, such as a category of modules. Also, the proofs of the above convergence results go through in this general case, since they use nothing more than the exactness of filtering colimits.
CHAPTER 4

Equivariant Kasparov theory

We begin by defining the complex representation ring of a compact group $G$, and by studying its Zariski spectrum in case $G$ is a compact Lie group (Section 4.1). Then, in Section 4.2, we work our way towards a conceptual discussion of the $G$-equivariant Kasparov category $KK^G$, where $G$ is a second countable locally compact group; its objects are separable $C^*$-algebras with a continuous $G$-action, and its Hom sets are the $G$-equivariant bivariant $K$-theory groups $KK^G(A,B)$. The category $KK^G$ comes equipped with a tensor triangulated structure, where $\otimes$ is given by the minimal tensor product of $C^*$-algebras with the diagonal $G$-action, and where the tensor unit $1_1$ is simply the algebra of complex numbers equipped with the trivial action. If $G$ is compact, the homological functor $KK^G(1_1,-)$ identifies with the equivariant topological $K$-theory of $G$-$C^*$-algebras and, by specializing to commutative algebras, it recovers the equivariant $K$-cohomology of locally compact $G$-spaces; in particular, the central ring $R_{KK^G} = \text{End}_{KK^G}(1)$ can be identified with $R(G)$. In the last section we recall Ralf Meyer’s and Ryszard Nest’s elegant use of the triangulated category $KK^G$ in reformulating the Baum-Connes conjecture for $G$ with coefficients in $A \in KK^G$.

Our only original contribution to this chapter is the following observation (Theorem 4.3.7): If the Balmer spectrum $\text{Spc}(KK^G)$ is covered by the spectra $\text{Spc}(KK^H)$, where $H$ runs through all the compact subgroups of $G$, then a strong version of the Baum-Connes conjecture holds for $G$.

Ample references are provided throughout.

4.1. The representation ring of a compact Lie group

We present here Segal’s homonymous classic [Seg68a]. Let $G$ be a compact group. All subgroups considered are closed.

4.1.1. Definition. Consider the set $\text{Rep}(G)$ of isomorphism classes of finite dimensional continuous linear representations $\alpha : G \rightarrow GL(V)$ (for short, $G$-modules). The direct sum and the tensor product of $G$-modules make $\text{Rep}(G)$ into a commutative semiring, having as multiplicative identity the class of the trivial one dimensional $G$-module. Define the complex representation ring, or character ring, of $G$ to be the Grothendieck group completion $R(G) := K_0(\text{Rep}(G))$ consisting of formal differences $[(V, \alpha)] - [(V', \alpha')]$. As a ring, $R(G)$ can be identified with a subring of $C(G, \mathbb{C})$, the ring of continuous complex valued functions on $G$. More precisely, given a $G$-module $M = (V, \alpha)$, its character is the continuous function $\chi_M : g \mapsto \text{Tr}(\alpha_g)$ given by the trace. This induces a map $R(G) \rightarrow C(G, \mathbb{C})$, which can be shown to be an injective ring homomorphism.

4.1.2. Remarks. (a) Every finite dimensional representation is isomorphic to a unitary representation $G \rightarrow U(n)$. 

97
(b) Every unitary representation of a compact group $G$ on a complex Hilbert space is a direct sum of irreducible representations, and all irreducible representations are finite dimensional (e.g. [Fol95, Thm 5.2]). It follows that, as an abelian group, $R(G)$ is free on the set $\hat{G}$ of equivalence classes of irreducible $G$-modules.

(c) When $G$ is abelian, then every irreducible representation is one-dimensional and $\hat{G}$ is an abelian group under $\otimes$, identifiable with the Pontrjagin dual $\text{Hom}(G, U(1))$. In this case $R(G)$ is the group ring $\mathbb{Z}[\hat{G}]$.

(d) The representation ring is contravariantly functorial for continuous group homomorphisms $f : H \to G$. We shall write $\text{res}^H_G : R(G) \to R(H)$ for the ring homomorphism induced by the inclusion of a closed subgroup $H \leq G$.

4.1.3. Hypothesis. For the rest of the section, $G$ is a compact Lie group (e.g. a finite group); then its closed subgroup will also be compact Lie groups. This is used for Theorem 4.1.5, as well as in the explicit construction of an induction homomorphism $R(H) \to R(G)$ for closed subgroups $H \leq G$, which we shall not mention but is used by Segal in the proof of Theorem 4.1.7.

4.1.4. Examples. (a) If $G = \{1\}$, then $R(G) \cong \mathbb{Z}$.

(b) If $G = \mathbb{Z}/n$, then $R(G) \cong \mathbb{Z}[X]/(X^n - 1)$.

(c) If $G = \mathbb{T}$ ($= \mathbb{R}/\mathbb{Z} = U(1)$), then $R(G) \cong \mathbb{Z}[X, X^{-1}]$.

(d) For a diagram $H_1 \to G \leftarrow H_2$ of abelian groups one has

$$R(H_1) \otimes_{R(G)} R(H_2) \cong R(H_1 \times_G H_1)$$

([Seg68a, remark after Lemma 3.4]).

(e) A group $G$ is (topologically) cyclic if $G = \{g^n \mid n \in \mathbb{Z}\}$ for some element $g \in G$. Thus a cyclic group $G$ is a product of a torus $\mathbb{T}^n$ with a finite cyclic group $\mathbb{Z}/m$. Combining (a)-(d) we see that in this case

$$R(G) \cong \mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}, Y]/(Y^n - 1).$$

(f) If $G = U(n)$, let $T \leq G$ be a maximal torus. Then $\text{res} : R(G) \to R(T) \cong \mathbb{Z}[X_i, X_i^{-1} : i = 1 \ldots n]$ is injective. Moreover, we may identify $R(G)$ with the subring $R(G) = \mathbb{Z}[S_1, \ldots, S_n, S_n^{-1}]$, where $S_k$ is the $k$-th symmetric function in the variables $X_1, \ldots, X_n$ ([Seg68a, Prop. 3.1]).

By embedding the compact Lie group $G$ in a unitary group $U(n)$, one may use the last example to prove the following fundamental result of Atiyah:

4.1.5. Theorem. [Seg68a, Prop. 3.2, Cor. 3.3]. Assume Hypothesis 4.1.3. If $H \leq G$ is a closed subgroup, then $R(H)$ is a finitely generated $R(G)$-module.

Thus every $R(G)$ is finite over some $R(U(n))$. From Example 4.1.4 (f) we get:

4.1.6. Corollary. $R(G)$ is a finitely generated ring. In particular it is noetherian.

Here are the first results about $\text{Spec} R(G)$, leading to Segal’s concept of ‘support’ of a prime ideal.

4.1.7. Theorem. Assume Hypothesis 4.1.3. Then

(a) For every prime ideal $\mathfrak{p} \in \text{Spec} R(G)$ there exists a minimal (with respect to inclusion) closed subgroup $S \leq G$ such that there exists a prime $\mathfrak{q} \in \text{Spec} R(S)$ lying above $\mathfrak{p}$, i.e., with $(\text{res}_S^G)^{-1}(\mathfrak{q}) = \mathfrak{p}$.

(b) The minimal subgroup $S$ as in (a) is uniquely determined up to conjugacy in $G$. Call this conjugacy class, or any of its representatives, the support of $\mathfrak{p}$, in symbols $\text{supp}(\mathfrak{p})$.

(c) The support $S = \text{supp}(\mathfrak{p})$ of $\mathfrak{p}$ is a (topologically) cyclic group, as in Example 4.1.4 (e).
4.1.8. Definition. A subgroup $S \leq G$ is a **Cartan subgroup** if it is cyclic and of finite index in its normalizer $N_G(S)$. The finite group $W_S := N(S)/S$ is called the **Weyl group** of $S$.

4.1.9. Proposition. [Seg68a, Prop. 1.5]. Let $G^0$ be the connected component of the identity of $G$. Then the projection $G \to G/G^0$ to the finite quotient induces a surjection
\[ \{ \text{Cartan subgroups of } G \} \to \{ \text{cyclic subgroups of } G/G^0 \} \]
which induces a bijection on conjugacy classes.

The Cartan subgroups of $G$ yield a description of $\text{Spec } R(G)$, as follows. Fix a $p \in \text{Spec } \mathbb{Z}$, and let $\kappa(p)$ denote the prime field of characteristic $p$ (so for instance $\kappa(0) = \mathbb{Q}$). For every $H \leq G$, we write $R_p(H) := R(H) \otimes_{\mathbb{Z}} \kappa(p)$. Let $S$ be a cyclic subgroup of $G$ with generator $g \in S$ (i.e. $S = \langle g \rangle$). Let $p_S \leq R(S) = \mathbb{Z}[\hat{S}]$ be the prime ideal of those characters vanishing on $g$, and let $\bar{R}_p(S) := R_p(S)/p_S$ (this doesn’t depend on the choice of generator $g$). Then the Weyl group $W_S = N_G(S)/S$ acts on $S$ by conjugation, which induces an action on $\bar{R}_p(S)$. So we may consider the ring of fixed points $(\bar{R}_p(S))^{W_S}$. Now the inclusions $S \hookrightarrow G$ induce a homomorphism
\[ f_p : R_p(G) \to \prod_S (\bar{R}_p(S))^{W_S}, \]
where we let $S$ run through the conjugacy classes of $p$-regular Cartan subgroups of $G$; here $p$-regular means that $p \nmid [S : S^0]$, where $S^0$ is the component of the identity in $S$. This homomorphism describes the fiber of $\text{Spec } R(G)$ over $p$.

4.1.10. Proposition. [Seg68a, Prop. 3.5]. Let $G$ be a compact Lie group, and let $f_p$ be the ring homomorphism constructed above. Then the kernel of $f_p$ is the nilradical of $R_p(G)$, and for every $x$ in the codomain, there is a $k > 0$ with $x^k \in \text{Im}(f_p)$. It follows that the map on spectra
\[ \text{Spec}(f_p) : \bigoplus_S (\text{Spec } \bar{R}_p(S))/W_S \to \text{Spec } R_p(G) \]
is a homeomorphism.

4.1.11. Remark. If $G$ is a finite group, its $p$-regular Cartan subgroups are precisely all its cyclic subgroups whose order is not divisible by $p$.

Using the proposition and its proof one can give a more precise account of the prime ideals of $\text{Spec } R(G)$:

4.1.12. Proposition. [Seg68a, Prop. 3.7]. Let $G$ be a compact Lie group.

(i) A prime $p \in \text{Spec } R(G)$ is maximal iff $\text{supp}(p)$ is finite and $\text{char } \kappa(p) \neq 0$.

(ii) The minimal primes of $\text{Spec } R(G)$ are in bijection with the conjugacy classes of $p$-regular Cartan subgroups of $G$; or, alternatively, with the conjugacy classes of cyclic subgroups of $G/G^0$.

(iii) For a closed subgroup $H \leq G$ the following are equivalent:
\begin{itemize}
  \item[(a)] $p$ comes from $R(H)$.
  \item[(b)] $p$ contains the kernel of restriction $\text{res}^H_G : R(G) \to R(H)$.
  \item[(c)] $R(H)p \neq 0$, where $R(H)$ is localized as an $R(G)$-module.
  \item[(d)] $\text{supp}(p)$ is conjugate to a subgroup of $H$.
\end{itemize}

(iv) If $p \subseteq p'$, then $\text{supp}(p')$ is conjugate to a subgroup of $\text{supp}(p)$. 

Proof. See [Seg68a, §3].
4.1.1. Finite cyclic groups. We want to expand Example 4.1.4 (b), in order to record the following fact. Let $S$ be a finite cyclic group, and let $q \in \text{Spec } R(S)$ be a prime ideal of its representation ring such that $\text{supp}(q) = S$. That is, $q$ does not come from any proper subgroup of $S$.

4.1.13. Proposition. $R(S)_q$ is a discrete valuation ring, or a field. In any case, it is a hereditary ring (it has global dimension at most one).

Sketch of proof. See [Phi85, Prop. 6.2.2, Lemma 6.4.2] for details. Write $n := |S|$. There is an isomorphism $R(S) = \mathbb{Z}[\bar{S}] \cong \mathbb{Z}[X]/(X^n - 1)$, where $X$ corresponds to a homomorphism $\chi : S = \langle \sigma | \sigma^n = 1 \rangle \to \mathbb{C}^\times$ sending the generator $\sigma$ to a primitive complex $n$th root of unity $\zeta$. Note that $X^n - 1 = \prod_n \phi_d(X)$, where $\phi_d$ is the $d$th cyclotomic polynomial. Phillips shows that, with the hypothesis $\text{supp}(q) = S$, the prime $q$ contains (the element of $R(S)$ corresponding to) $\phi_n$, but it doesn’t contain any other $\phi_d$, for $d \neq n$. He uses this to show that $R(S)_q \cong (\mathbb{Z}[X]/(\phi_n))_q \cong \mathbb{Z}[\zeta]_q$, where $q'$ is a prime ideal of $\mathbb{Z}[\zeta] \subset \mathbb{C}$. Now this ring has the right properties, because $\mathbb{Z}[\zeta]$ is a Dedekind domain.

4.1.14. Remark. Note that $R(S)$ itself is no Dedekind domain – indeed, it is not even a domain.

4.1.15. Remark. It follows from all the above that, if $G$ is a finite group and $p \in \text{Spec } R(G)$, there exists a cyclic subgroup $S \leq G$ (choose one in the conjugacy class of $\text{supp}(p)$) and there is a prime $q \in \text{Spec } R(S)$ such that $(\text{res}_G^S)^{-1}q = p$ and such that every $R(S)_q$-module has a projective resolution of length one. This is perhaps the single most crucial ‘natural fact’ that will allow us to define the generalized support datum $\sigma$ of Section 5.4.

4.2. $G$-$C^*$-algebras and Kasparov theory

I am going to commit a didactical crime, for which I apologize in advance. In particular, I will describe Gennadi Kasparov’s “equivariant bivariant K-theory” KK$^G$ for (ungraded, complex, separable) $G$-$C^*$-algebras from a purely algebraic point of view, as a tensor triangulated category with certain properties, hardly giving any motivation for its study. The reader can rest assured that there is plenty of the latter, as amply supplied in the referenced works (e.g. [Bla98, §24] [MN06, §1]).

4.2.1. From $R(G)$ to $KK^G$. In [Seg68b], Segal defined and studied an equivariant version of (complex) topological $K$-theory for locally compact spaces. More precisely, for every compact group $G$ he constructed a cohomology theory $K^*_G$ on pairs $(X,Y)$ of compact $G$-spaces (i.e., $X$ is a compact space with a continuous $G$-action and $Y$ is a $G$-invariant closed subspace of $X$). Among the nice properties of equivariant $K$-theory are:

- When $G$ is trivial, it is the usual topological $K$-theory of Atiyah and Hirzebruch [AH59].
- $K^0_G(X) := K^0_G(X, \emptyset)$ classifies isomorphism classes of $G$-equivariant complex vector bundles over the compact space $X$. (It is the group completion of the semiring thereof.)
- $K^1_G(pt) = 0$ and $K^0_G(pt)$ identifies with the complex representation ring $R(G)$, i.e., it classifies finite $G$-modules. Moreover, each $K^*_G(X,Y)$ carries a natural $R(G)$-action.

\[\text{Compact and locally compact spaces are always assumed to be Hausdorff.}\]
Bott periodicity holds: there is a natural isomorphism
\[ K^*_C(S^2X, S^2Y) \cong K^*_C(X, Y), \]
where \( S \) denotes the suspension \( SX := (X \times [0, 1])/(X \times \{0, 1\}) \).

4.2.1. REMARK. (a) Because of excision \( K^*_C(X, Y) \cong K^*_C(X/Y, pt) \), one may equivalently consider \( K^*_C \) as being defined on the category of compact \( G \)-spaces \((X, x_0)\) with a chosen \( G \)-fixed base-point, and base-point preserving equivariant maps. The suspension \( S \) becomes then the reduced suspension \( \Sigma(X, x_0) = (S^1, \text{pt}) \wedge (X, x_0) = ([0, 1] \times X)/(\{(0, 1) \times X\} \cup ([0, 1] \times x_0)) \).

(b) Note that the Alexandroff one-point compactification \( U \mapsto (U^+, \infty) \), which adds one \( G \)-invariant “point at infinity” \( \infty \), as one says, to a locally compact \( G \)-space \( U \), and is functorial on proper equivariant maps, may be used for extending equivariant \( K \)-theory to general (pairs of) locally compact \( G \)-spaces. On the other hand, the notation \( K^*_C(X - Y) = K^*_C(X, Y) \) is also used. For instance \( K^*_C(\mathbb{R}) = K^*_C(S^1, \text{pt}) \), where the circle \( S^1 \) is to be considered with the trivial \( G \)-action.

Consider the contravariant functor
\[(4.2.2) \quad (X, x_0) \mapsto C(X, x_0),\]
which assigns to a pointed \( G \)-space \((X, x_0)\) the algebra of complex-valued continuous functions vanishing at the base-point. It is well-known that \( A := C(X, x_0) \) carries the structure of a commutative \( C^* \)-algebra, and that the \( G \)-action on \( X \) defines via \( g \cdot f(x) := f(g^{-1} \cdot x) \) a continuous \( G \)-action on \( A \) by \( G \)-equivariant \( * \)-homomorphisms \((C^* \text{-algebras are equipped with the } C^* \text{-norm topology; since } * \text{-homomorphisms are automatically continuous, the continuity of the action reduces to checking that } G \mapsto A, g \mapsto g \cdot a, \text{is continuous for each } a \in A). \) Call any \( C^* \)-algebra equipped with a continuous \( G \)-action a \( C^* \)-\( G \)-\textit{algebra}.\footnote{A standard reference is \cite{Ped79}, where \( C^* \)-\textit{algebras are called ‘}C*-dynamical systems’.}

4.2.3. THEOREM. (Equivariant Gelfand duality). The functor \((4.2.2)\) is a contravariant equivalence between the category of pointed compact \( G \)-spaces and pointed maps, and the category of (non necessarily unital) commutative \( G \)-\( C^* \)-\textit{algebras and (non unital) \( G \)-equivariant *-homomorphisms.}

\textbf{Sketch of proof.} Assume first that \( G = \{1\} \). The classical Gelfand duality theorem (see any textbook on \( C^* \)-algebras) says that the contravariant functor \( X \mapsto C(X) = \{f : X \to \mathbb{C} \} \) from compact spaces to unital commutative \( C^* \)-algebras is an equivalence, with quasi-inverse given by assigning to an algebra \( A \) its spectrum of closed maximal ideals. It follows formally that pointed spaces (spaces under the point) are contravariantly equivalent to augmented algebras (algebras over \( \mathbb{C} \)). But the latter category is equivalent to that of (possibly non unital) commutative \( C^* \)-algebras, by sending an augmented unital algebra \((A, A \twoheadrightarrow \mathbb{C}) \) to \( \text{Ker}(\alpha) \), and conversely by sending a general commutative algebra \( B \) to its minimal unitalization \( B^+ \). Now, given a pointed space \((X, x_0)\), if we apply \( C \) to \([x_0] \mapsto X \) we get a surjection \( C(X) \twoheadrightarrow \mathbb{C} \) whose kernel naturally identifies with \( C(X, x_0) \). Thus we see that the equivalence is given by \((4.2.2)\). The equivariant version is obtained by noticing that continuous actions on a compact space \( X \) are in bijection with continuous actions on \( C(X) \), and that all the above trafficking restricts to equivariant maps and equivariant *-homomorphisms.

Hence, roughly speaking, one may regard \( G \)-spaces as commutative \( C^* \)-\textit{algebras.} Now, by identifying \( G \)-vector bundles on a space \( X \) with the right kind of \( C(X) \)-modules \( (\text{à la Swan-Serre}) \), equivariant \( K \)-theory can be nicely extended\footnote{See \cite{Phi85, \S 2} for a detailed treatment of this theory.} on
general, non necessarily commutative, $G$-$C^*$-algebras, where it becomes an essential and fashionable ingredient of harmonic analysis and non-commutative geometry. This is remarkable, because, for instance, in the case $G = \{1\}$ not even ordinary Eilenberg-Steenrod cohomology (with compact supports) can be extended from commutative to general $C^*$-algebras — at least, if one wants the extended theory to satisfy some natural axioms ([Bla98, Ex. 22.4.2]). As it turns out, equivariant $K$-theory $K^G$ of $G$-$C^*$-algebras (note the switch of indices: now it is a covariant functor) plays a central rôle similar to that of stable homotopy theory for spaces. More precisely, both are the homology theory defined by mapping out of the ten-
functor) plays a central rôle similar to that of stable homotopy theory for spaces.

4.2.2. The equivariant Kasparov category $KK^G$. Let $G$ be a locally compact group. Using [Mey07], we give a rapid definition of the equivariant Kasparov category $KK^G$. We refer to loc. cit. for more details and for references.

Some working experience suggests that ‘nice’ homology theories for $G$-$C^*$-algebras should satisfy at least the following reasonable properties.

4.2.4. Definition. Let $F : G$-$C^*$-alg $\to \mathcal{A}$ be a functor from $G$-$C^*$-algebras to some additive category.

(i) $F$ is homotopy invariant if $F([A[0,1] \xrightarrow{ev_0} A])$ is an isomorphism for every $A$ (see the Appendix, esp. Section A.3, for homotopy of $C^*$-algebras and the definition of $ev_0$).

(ii) $F$ is split-exact if for every split extension $J \hookrightarrow A \xrightarrow{q} A/J$ (where $J$ is an invariant closed two-sided ideal, and the quotient map $q$ has a splitting $s : A/J \to A$ in $G$-$C^*$-alg) we have an isomorphism $F(J) \oplus F(A/J) \cong F(A)$ given by $F(j)$ and $F(s)$.

(iii) $F$ is (equivariant $C^*$-) stable if $F([\mathbb{K}(H_1) \xrightarrow{j} \mathbb{K}(H_1 \oplus H_2)])$ is an isomorphism, whenever $H_1, H_2$ are non-zero $G$-Hilbert spaces, $\mathbb{K}(H)$ denotes the algebra of compact operators on $H$ endowed with the $G$-action $(g \cdot T)(\xi) := gT(g^{-1} \xi)$, and $i$ is the $*$-homomorphism induced by $H_1 \subset H_1 \oplus H_2$.

4.2.5. Definition. If it exists, we denote by $KK^G$ the additive category which is the target of the universal functor $F_{univ} : G$-$C^*$-alg $\to KK^G$ with properties (i)-(iii) above, through which any other functor $F : G$-$C^*$-alg $\to \mathcal{A}$ satisfying (i)-(iii) must factor uniquely. We call it the equivariant Kasparov category. Write $KK := KK^{(1)}$ if the group $G$ is trivial.

4.2.6. Remark. Actually $F_{univ}$ trivially exists, because each of the three properties can be expressed as requiring that $F$ inverts a certain class of morphisms. Hence we may construct $KK^G$ as the (large) localization of $G$-$C^*$-alg by the union of these classes (cf. Section 1.2). In particular, we may take the objects of $KK^G$ to be $G$-$C^*$-algebras. It is not hard then to see that the localized category is additive. (In the Appendix we formalize somewhat this direct approach, noting that a structure of triangulated category appears quite early in the construction). What is interesting is that $KK^G$ enjoys a wealth of nice properties, such as Bott periodicity in both variables, which all theories (functors) factoring through it will automatically enjoy too.

4.2.7. Remark. Kasparov’s original definition of $KK$ and $KK^G$ (see [Kas80] [Kas88]) is quite explicit and very involved; we will not even attempt to discuss it here. In the beginning, the (non-equivariant) theory wasn’t even explicitly recognized as forming a category (the “Kasparov product” combining in one complicated operation the composition $f \circ g$ of morphisms – i.e., of elements of the “bivariant
4.2. G-C*-algebras and Kasparov theory

K-theory groups" $\text{KK}(A, B)$, with the tensor bifunctor $f \otimes g$. It was some time before Nigel Higson [Hig87], based on work of Joachim Cuntz [Cun87], discovered the above simple universal property for $\text{KK}$. This was generalized to $\text{KK}^G$ by Thomsen [Ths98] and Meyer [Mey00]. Triangulated categories explicitly entered the domain only with [Tho03].

In order for the underlying analytic constructions to work nicely, one needs:

4.2.8. Hypothesis. From now on, we assume that all C*-algebras are separable (as topological spaces: they have a dense countable subset), i.e., we substitute $G$-C*-alg with its full subcategory $G$-C*sep of separable algebras. Accordingly, in Def. 4.2.4 (iii) we only consider separable G-Hilbert spaces. We also require $G$ to be second countable locally compact (i.e., the topology of $G$ has a countable basis).

4.2.9. Theorem. The category $\text{KK}^G$ carries the structure of a tensor triangulated category, as follows:

- The tensor product is given by the minimal (spacial) tensor product $A \otimes B$ of C*-algebras equipped with the diagonal action. The tensor unit is $\mathbb{I} = \varepsilon^G C$, the complex numbers equipped with the trivial $G$-action. This tensor structure is the unique extension of the minimal tensor product on $G$-C*sep making the canonical functor $F_{\text{uni}}$, a monoidal functor.
- The translation $T = \Sigma^{-1}$ is the inverse of the suspension functor $\Sigma : \text{KK}^G \to \text{KK}^G$, defined by $\Sigma A := C(S^1, \text{pt}) \otimes A$ (where $S^1$ has the trivial action; more often, one sees the notation $\Sigma = C_0(\mathbb{R}) \otimes -$).
- The triangulation consists of the cone triangles (cf. Section A.5).

This structure satisfies the axiom $(\otimes \Delta +)$ of Section 2.1. Moreover:

- $\text{KK}^G$ has all countable coproducts, which are given by the the C*-direct sum of algebras (with coordinatewise action). The tensor product preserves coproducts.
- (Bott periodicity). There is a natural isomorphism $\Sigma \circ \Sigma \cong \text{id}_{\text{KK}^G}$.
- For compact $G$, we have a natural isomorphism $\text{KK}^G_{\text{c}}(\mathbb{I}, A) \cong K^G_{\text{c}}(A)$ recovering equivariant K-theory of (separable) G-C*-algebras, and therefore Segal’s equivariant K-theory of G-spaces $K^G_{\text{c}}(X, x_0) = K^G_{\text{c}}(C(X, x_0))$. In particular, the central ring $R_{\text{KK}^G}(\mathbb{I}, \mathbb{I})$ is $R(G)$ in even degrees and 0 in odd degrees.

Proof. It is proved in [MN06, §2 and App. A] that the given translation and triangles satisfy the axioms of a triangulated category, and that the minimal tensor product extends to $\text{KK}^G$ and is triangulated in each variable. We reprove this in our Appendix A, making sure that $(\otimes \Delta +)$ is satisfied.

The rest needs Kasparov’s construction of $\text{KK}^G$. Countable C*-direct sums become coproducts in $\text{KK}^G$ by [Kas88, Thm. 2.9]: they are preserved by $\otimes$ because the minimal tensor product of C*-algebra commutes with C*-direct sums already in G-C*alg. Agreement with equivariant K-theory is more or less immediate from Kasparov’s definition of the groups $\text{KK}^G(A, B)$ ([Kas88, Rem. 1 p. 160]). Finally, see e.g. [Bla98, §19.2] to find an isomorphism $\mathbb{C} \cong C_0(\mathbb{R})^{\otimes 2}$ in $\text{KK}^G$ and therefore in $\text{KK}^G$, implying Bott periodicity.

4.2.10. Convention. Let $G$ be compact. Because of Bott periodicity, and because $\text{KK}^G(T \mathbb{I}, \mathbb{I}) = 0$, we won’t need to distinguish the commutative ring $R(G)$ from the periodic graded ring $R_{\text{KK}^G}$. Similarly, we may consider the graded $R_{\text{KK}^G}$-modules $\text{KK}^G_{\text{c}}(A, B)$ as being $\mathbb{Z}/2$-graded $R(G)$-modules. In particular for $G$ compact, we will view equivariant K-theory $K^G_{\text{c}} = \text{KK}^G(T^* \mathbb{I}, -)$ as a stable homological functor from $\text{KK}^G$ to the stable abelian category $R(G)\text{-Mod}^{\mathbb{Z}/2}$ of
$\mathbb{Z}/2$-graded $R(G)$-modules. The latter is abelian with enough projectives (given by the usual $\mathbb{Z}/2$-graded projective $R(G)$-modules), and it has a right exact tensor structure given by the usual $\mathbb{Z}/2$-graded tensor product of $R(G)$-modules:

$$(M \otimes N)_0 := (M_0 \otimes N_0) \oplus (M_1 \otimes N_1) \quad (M \otimes N)_1 := (M_0 \otimes N_1) \oplus (M_1 \otimes N_0)$$

where $(-)_0, (-)_1$ denote the even resp. odd graded part of a module. Derived functors, such as Tor, will be computed in this abelian category.

### 4.2.3. The rich functoriality of $KK^G$.

One may ‘vary $G$’ instead of $C^*$-algebras, in various interesting ways. Once again, we refer to [MN06, §3.2] and [Mey07] for details and for proofs. (As in loc. cit., we use the notational convention that indices $F^G$ indicate a functor $KK^G \rightarrow KK^G$, and an empty index refers to the trivial group $G = \{1\}$.) Although we shall make scarce use of these functors, we give the complete picture because of its nice symmetry, suggestive of deeper structure. Note that all functors considered below can be easily shown to be triangle functors which commute with coproducts.

First of all, if $H$ is a closed subgroup of $G$ we may simply restrict the action. This yields a restriction tensor triangle functor

$$\text{Res}^H_G : KK^G \rightarrow KK^H,$$

which specializes to $\text{res}^H_G : R(G) = KK^G(\mathbb{1}, \mathbb{1}) \rightarrow KK^H(\mathbb{1}, \mathbb{1}) = R(H)$ if $G$ is compact (see Rem. 4.1.2 (d)). Another obvious tensor triangle functor is given by the trivialization

$$\tau^G : KK \rightarrow KK^G$$

which assigns to every separable $C^*$-algebra the trivial $G$-action. We clearly have $\text{Res}_G^H \circ \tau^G = \tau^H$ for all $H \leq G$. For a closed subgroup $H \leq G$, there is also an induction functor

$$\text{Ind}_H^G : KK^H \rightarrow KK^G.$$

(For an $H$-$C^*$-algebra $A$ one defines $\text{Ind}_H^G(A)$ to be the $C^*$-algebra of bounded continuous functions $f : G \rightarrow A$ such that $f(x) = h \cdot f(xh)$ ($x \in G, h \in H$) and such that the induced map $f : xH \mapsto \|f(x)\|_A$ vanishes at infinity, with $G$ acting on the left by translation.) Interestingly, the same induction functor is left or right adjoint to restriction, according to what kind of groups one considers:

#### 4.2.11. Proposition. Let $H \leq G$ be a closed subgroup.

(i) There is a ‘Frobenius formula’ $\text{Ind}_H^G(A \otimes \text{Res}_G^H B) \cong \text{Ind}_H^G(A) \otimes B$.

(ii) If $G/H$ is discrete, then $KK^H(A, \text{Res}_G^H B) \cong KK^G(\text{Ind}_H^G A, B)$.

(iii) If $G/H$ is compact, then $KK^H(\text{Res}_G^H A, B) \cong KK^G(A, \text{Ind}_H^G B)$.

(All isomorphisms are natural in $A$ and $B$). $\square$

A more refined construction is the reduced descent, or reduced cross product functor

$$\text{Des}_G = G \ltimes (-) : KK^G \rightarrow KK.$$

(For the definition at the level of $G$-$C^*$-algebras, see [Bla06, II.10] and [Ped79, §7.7]; the construction of the reduced cross product $G \ltimes A$ specializes to the more famous reduced group $C^*$-algebra $C^*_r(G) = G \ltimes (\tau^G \mathbb{C})$.) The relation of descent to trivialization is strikingly similar to that between induction and restriction:

#### 4.2.12. Proposition.

(i) $G \ltimes (A \otimes \tau^G B) \cong (G \ltimes A) \otimes B$.

(ii) If $G$ is discrete, then $KK^G(A, \tau^G B) \cong KK(G \ltimes A, B)$.

(iii) If $G$ is compact, then $KK^G(\tau^G A, B) \cong KK(A, G \ltimes B)$. $\square$

#### 4.2.13. Remark. Choosing $A = \mathbb{C}$ in Proposition 4.2.12 (iii) yields the identification $K^G_0(A) \cong K_*(G \ltimes A)$ known as the Green-Julg Theorem. It reduces the computation of equivariant $K$-theory for compact groups to ordinary $K$-theory.
To complete the picture, compare the formula $\text{Res}_G^H \circ \tau^G = \tau^H$ with:

4.2.14. **Theorem (Green’s Imprimitivity).** For every closed subgroup $H \leq G$ there is a natural isomorphism $G \rtimes (\text{Ind}_H^G A) \cong H \rtimes A$. □

4.2.4. **The tensor triangulated category $T^G$.** Let $G$ be a compact group.

4.2.15. **Lemma.** The tensor unit $\mathbb{1} = \tau^G \mathbb{C}$ is an $\aleph_1$-compactly generated object of $KK^G$ (Def. 1.7.1).

**Proof.** Let $A \in KK^G$. Since $G$ is compact, by the Green-Julg theorem ([Bla98, Thm. 11.7.1], cf. Rem. 4.2.13) there is an isomorphism $KK^G_G(\mathbb{1}, A) = K^G_G(A) \cong K_*(G \rtimes A)$, where $G \rtimes A \in KK$ is the reduced cross product $C^*$-algebra. Since $A$ is separable, so is $G \rtimes A$. It is well-known that the (ordinary) $K$-theory groups of a separable algebra are countable. Since $G \rtimes (-)$ and $K_*$ commute with $C^*$-direct sums, this shows that $\mathbb{1}$ is an $\aleph_1$-compact object of $KK^G$. □

Therefore we may consider equivariant $K$-theory as a stable homological functor

$$h := K_*^G : KK^G \to R(G)\text{-Mod}_{\mathbb{Z}/2}^G$$

with target the stable abelian category of $\mathbb{Z}/2$-graded countably generated $R(G)$-modules.

We consider in $KK^G$ the homological ideal (Def. 3.1.4) $I_G^G = \ker(K_*^G)$.

4.2.16. **Proposition.** The functor $h$ is the universal $K_*^G$-exact functor (Sec. 3.5).

**Proof.** See [MN07, Thm. 5.5]. It is quite easy to construct for $h$ a partial left-adjoint and right-inverse $h^!$ defined on projective modules, by sending

$$R(G) \mapsto h^!(R(G)) := \tau^G \mathbb{C} \quad \text{and} \quad R(G)[1] \mapsto T(\tau^G \mathbb{C})$$

and by extending this functor to free graded modules and their direct summands. Then one uses the criterion of Theorem 3.5.5. (cf. the proof of Proposition 5.3.1).

It follows from the proposition and the results of Section 3.5 that $h$ restricts to an equivalence

$$\text{Proj}(KK^G, I_G^G) \simeq \text{Proj}(R(G)\text{-Mod}_{\mathbb{Z}/2}^G)$$

between $K_*^G$-projective objects in $KK^G$ and projective countable graded $R(G)$-modules. Since the $K_*^G$-projective objects of $KK^G$ are precisely the direct summands of coproducts of copies of $\tau^G \mathbb{C}$ and its translations, by Theorem 3.6.6 the pair

$$\langle (\tau^G \mathbb{C})_{\text{loc}}, \ker(K_*^G) \rangle$$

is a complementary pair of localizing subcategories of $KK^G$ (Def. 1.4.22). In particular, by considering the gluing triangle for this pair we immediately see:

4.2.17. **Proposition.** Every separable $G$-$C^*$-algebra $A \in KK^G$ has a natural universal approximation $\varepsilon_A : LA \to A$ with $LA \in (\tau^G \mathbb{C})_{\text{loc}}$ which is an isomorphism in $K$-theory: $K_*^G(\varepsilon_A) : K_*^G(LA) \cong K_*^G(A)$. □

This motivates the next

4.2.18. **Definition.** Denote by

$$T^G := (\tau^G \mathbb{C})_{\text{loc}} \subset KK^G$$

the $\aleph_1$-localizing subcategory generated by the tensor unit in $KK^G$. In other words, $T^G$ is the smallest subcategory of $KK^G$ containing $\mathbb{1} = \tau^G \mathbb{C}$ which is closed under the formation of countable coproducts and mapping cones.
4. EQUIVARIANT KASPAROV THEORY

4.2.19. **Proposition.** $T^G$ is a monogenic $\aleph_1$-compactly generated tensor triangulated category (Def. 2.1.30).

**Proof.** By Lemma 4.2.15, $T^G$ is $\aleph_1$-compactly generated. The tensor structure of $\text{KK}^G$ restricts to $(\mathbb{I}) = T_c$ (Prop. 2.1.28) and therefore to $T^G$, as it’s easily seen. We know that $\otimes$ commutes with countable coproducts (see Thm. 4.2.9), so $T^G$ satisfies the required axiom $(\otimes \bigvee)$. □

4.3. The Baum-Connes Conjecture and the spectrum

We briefly describe some results of [MN06], to which we add a geometric twist providing motivation for studying the Balmer spectrum of $\text{KK}^G$ and of related categories.

4.3.1. **Definition.** Consider the two full subcategories of $\text{KK}^G$

$$\text{CI}^G := \bigcup_{H \leq G \text{ compact}} \text{Im}(\text{Ind}^G_H),$$

(for ‘compactly induced’), and

$$\text{CC}^G := \bigcap_{H \leq G \text{ compact}} \text{Ker}(\text{Res}^G_H)$$

(for ‘compactly contractible’). Clearly, they are ($\aleph_1$-) localizing subcategories. They are also $\otimes$-ideals: $\text{CC}^G$ because each $\text{Res}^G_H$ is a $\otimes$-functor and $\text{CI}^G$ because of the Frobenius formula (Prop. 4.2.11 (i)).

4.3.2. **Theorem ([MN06, Thm. 4.7]).** The pair $(\text{CI}^G, \text{CC}^G)$ of subcategories of $\text{KK}^G$ is a complementary pair (Def. 1.4.21) of localizing $\otimes$-ideals. □

The gluing triangle for this pair at the tensor unit $\mathbb{I}^G = r^G C$ is called the **Dirac triangle for $G$**. Write

$$P^G \xrightarrow{D} \mathbb{I}^G \longrightarrow N^G \longrightarrow T P^G$$

for it. The approximation morphism $D : P^G \rightarrow \mathbb{I}^G$ is the **Dirac morphism**. Since both subcategories are $\otimes$-ideals, the gluing triangle $P^G(A) \xrightarrow{D(A)} A \rightarrow N^G(A) \rightarrow T P^G(A)$ for this pair at any other object $A \in \text{KK}^G$ is obtained by tensoring the Dirac triangle with $A$ (Lemma 2.1.22).

4.3.3. **Definition.** Let $A \in \text{KK}^G$. One says that the **Baum-Connes conjecture for $G$ with coefficients** $A$ holds if the homomorphism

$$K_\ast(G \ltimes D(A)) : K_\ast(G \ltimes P^G(A)) \longrightarrow K_\ast(G \ltimes A)$$

is an isomorphism.

The main result of [MN06] is a proof that the homomorphism (4.3.4) is naturally isomorphic to the so-called “assembly” or “index” map for the group $G$ with coefficients $A$, showing that the above formulation of the Baum-Connes conjecture is equivalent to the usual formulation with coefficients for locally compact groups ([BCH94]). This triangular formulation is very convenient for proving so-called “permanence properties”, i.e., for deducing the validity of the conjecture for certain groups from its validity for some others. It is also conceptually transparent: it states that, at the object $A$, the functor $K_\ast \circ G \ltimes (-) : \text{KK}^G \rightarrow \text{Ab}^{G/2}$ coincides with its localization $\mathbb{L}(K_\ast G \ltimes (-))$ with respect to the triangulated subcategory $\text{CC}^G \subseteq \text{KK}^G$. It is only natural then to ask if the analog of the Baum-Connes conjecture holds for other functors on $G$-$\mathcal{C}^*$-algebras. Obviously, the analog of the
conjecture holds for all functors if the approximation \( P_G(A) \to A \) is itself an isomorphism in \( KK^G \); and the conjecture is true for all coefficients \( A \in KK^G \) if it holds with trivial coefficients \( A = 1_1 \) since \( (P_G(A) \to A) \cong A \otimes (P_G \to 1_1) \).

In [HK01], Higson and Kasparov proved that the conjecture holds for the class of second countable locally compact groups with the Haagerup approximation property (= \( a\)-\( T \)-menable groups). These are groups admitting a proper and isometric action on a Hilbert space, in a suitable sense. They form a rather large class containing all amenable groups. The proof’s method involves the construction of a left inverse \( \eta : 1_1 \to P_G \) (called a dual Dirac morphism) to the Dirac morphism \( D : P_G \to 1_1 \).

4.3.5. Lemma. The following are equivalent:

(i) There exists an \( \eta \in KK^G(1_1, P_G) \) with \( \eta \circ D = 1_{P_G} \).
(ii) \( KK^G(N_G, P_G) = 0 \).
(iii) \( CC^G \perp CI^G \).
(iv) There is a triangle equivalence \( KK^G \simeq CI^G \times CC^G \) (where the product category is equipped with the evident coordinatewise translation and triangulation).

Assume that (i)-(iv) hold, and consider the idempotent (called the “\( \gamma \)-element”) \( \gamma := D \circ \eta : 1_1 \to 1_1 \).

Then \( \gamma \downarrow A : A \to A \) has the property that \( \gamma \downarrow A = 1 \) iff \( A \in CI^G \) and \( \gamma \downarrow A = 0 \) iff \( A \in CC^G \).

Proof. This is the easy part of [MN06, Thm. 8.3] (or: Exercise). \( \square \)

4.3.6. Corollary. If it exists, the dual Dirac morphism \( \eta \) is unique. Moreover: It exists and \( \gamma = 1 : 1_1 \to 1_1 \) iff \( CI^G \simeq 0 \) iff \( CI^G \simeq KK^G \).

What Higson and Kasparov actually show in [HK01] is that, if \( G \) has the Haagerup approximation property, the dual Dirac morphism exists and \( \gamma = 1_{1_1} \). Hence for such groups \( KK^G \simeq CI^G \), and the analog of the Baum-Connes conjecture holds for all functors defined on \( KK^G \) and for all coefficients ([MN06, Thm. 8.5]).

All of this was already in [MN06]. We contribute the following very easy but quite intriguing observation, which may serve as a motivation for studying the spectrum (Def. 2.2.1) of tensor triangulated categories arising in this context.

4.3.7. Theorem. Assume that

\[
(4.3.8) \quad \text{Spc}(KK^G) = \bigcup_{H \leq G \text{ compact}} \text{Spc}(\text{Res}_G^H) \left( \text{Spc}(KK^H) \right),
\]

that is, assume that the spectrum of \( KK^G \) is covered by the spectra of \( KK^H \) as \( H \) runs through the compact subgroups of \( G \). Then the Dirac morphism \( D : P_G \to 1_1 \) is an isomorphism, so that the inclusion \( CI^G \hookrightarrow KK^G \) is an equivalence and the analog of the Baum-Connes conjecture holds for all functors and all coefficients.

Proof. By Proposition 2.2.17 (a), an object \( A \in KK^G \) is \( \otimes \)-nilpotent if and only if it belongs in each prime \( \otimes \)-ideal \( P \in \text{Spc}(KK^G) \). Thus if the covering hypothesis (4.3.8) holds, we have

\begin{align*}
A \text{ is } \otimes \text{-nilpotent} & \iff A \in P \quad \forall P \in \text{Spc}(KK^G) \iff A \in (\text{Res}_G^H)^{-1} Q \quad \forall Q \in \text{Spc}(KK^H), \forall H \iff \text{Res}_G^H(A) \in Q \quad \forall Q \in \text{Spc}(KK^H), \forall H
\end{align*}
where $H$ varies among all compact subgroups of $G$. Now consider $A := N^G = N^G(\mathbb{1}^G)$. The object $N^G$ satisfies the latter condition, because by construction $N^G \in CC^G = \bigcap_H \text{Ker}(\text{Res}_H^G)$. Thus $N^G$ is a $\otimes$-nilpotent object. But $N^G$ is also $\otimes$-idempotent: $N^G \cong N^G \otimes N^G$, because the endofunctor $A \mapsto A$ is idempotent by the general properties of gluing triangles, and because it is isomorphic to $A \mapsto N^G \otimes A$ (Lemma 2.1.22). Therefore $N^G \cong 0$, which is equivalent to the Dirac morphism for $G$ being an isomorphism (Cor. 1.1.16), which is equivalent to $CC^G \cong 0$ and $\text{Cl}^G \hookrightarrow KK^G$ being an equivalence.

4.3.9. Remark. Before getting too enthusiastic, we should point out that the problem of computing $\text{Spc}(|KK^G|)$ in terms of $G$ may well turn out to be hopeless, since the structural properties of $KK^G$ are rather mysterious – especially if one does not know yet whether the Baum-Connes conjecture holds for $G$. In the next and last chapter I take a first step in this direction, by tackling the problem of computing $\text{Spc}(T_c^G)$ for $G$ compact, which looks easier but still interesting (cf. Prop. 4.2.17).
Towards a computation of the spectrum of

$$\mathcal{T}_c^G = \langle \tau^G C \rangle$$

Let $G$ be a compact Lie group, and let $\text{KK}^G$ be the equivariant Kasparov category of separable $G$-$C^*$-algebras (Subsection 4.2.4). Let $\mathcal{T}^G$ be the localizing subcategory of $\text{KK}^G$ generated by the tensor unit $\mathbb{1} = \tau^G C$. It is a monogenic ($\aleph_1$-) compactly generated tensor triangulated category (Def. 1.7.1, see Subsec. 4.2.4). Its category of compact objects $\mathcal{T}^G_c$ is $\langle \mathbb{1} \rangle$, the thick subcategory of $\text{KK}^G$ generated by $\mathbb{1}$. Recall also that we have Bott periodicity $T^2 \cong \text{id}$, and that the central ring $\text{R}_{\mathcal{T}^G} = \text{End}_{\mathcal{T}^G}(\mathbb{1})$ is just $R(G)$, the complex representation ring of $G$. In particular it is noetherian (see Subsec. 4.1).

We conjecture that the Balmer spectrum $\text{Sp}(\mathcal{T}^G_c)$ (Def. 2.2.1) is homeomorphic to the Zariski spectrum $\text{Spec}(R(G))$. We are going to prove this for the trivial group and provide evidence for the general case.

5.0.10. Notation. If $R$ is a commutative ring, consider it as a $\mathbb{Z}/2$-graded ring concentrated in degree zero, and write $R\text{-}\text{Mod}^{\mathbb{Z}/2}$ for the category of $\mathbb{Z}/2$-graded $R$-modules and grading preserving $R$-linear maps. It is a stable abelian category (Def. 3.1.1) with enough projectives, which are simply the degreewise projective $R$-modules. It has a derived category $D(R\text{-}\text{Mod}^{\mathbb{Z}/2})$. Endow $R\text{-}\text{Mod}^{\mathbb{Z}/2}$ with the usual tensor product $\otimes_R$ of graded $R$-modules, and write $\otimes^L_R$ for its left derived functor, which we only need to be defined on the derived category of right-bounded complexes $D_+(R\text{-}\text{Mod}^{\mathbb{Z}/2})$ (cf. Convention 4.2.10).

5.1. A trivial example

Consider the case of the trivial group $G = \{1\}$. Then $\text{KK}^G$ is the original non-equivariant Kasparov category $\text{KK}$. Here $\text{KK}(T^*C, -) = K_*(-)$ is the ordinary topological $K$-theory of separable complex $C^*$-algebras, and the central ring $\text{R}_{\text{KK}}$ is $R(\{1\}) \cong \mathbb{Z}$ (Example 4.1.4 (a)). The localizing subcategory generated by the tensor unit, that we shall denote $\text{Boot} := \langle C \rangle_{\text{loc}} \subseteq \text{KK}$, contains exactly the $C^*$-algebras of the so-called Bootstrap class. In other words, it can be shown — essentially by translating ‘localizing triangulated subcategory of $\text{KK}$’ back to classical terms, and by using standard properties of nuclear $C^*$-algebras — that $\text{Boot}$ is precisely the smallest class of separable nuclear $C^*$-algebras with the following properties:

1. $C, C_0(\mathbb{R}) \in \text{Boot}$.
2. It is closed under countable $C^*$-direct limits, in particular countable $C^*$-direct sums.
3. If $0 \to J \to A \to B \to 0$ is an extension of $C^*$-algebras and two out the three algebras are in $\text{Boot}$, so is the third.
4. $\text{Boot}$ is closed under $\text{KK}$-equivalence, i.e., isomorphism in $\text{KK}$ (this of course includes stabilization by compact operators and taking matrix algebras).
and an isomorphism \( A \) of \( \text{Proj} \) equivalence length one, say \( 0 \).

5. THE SPECTRUM OF \( \mathcal{T}_c^G = (\tau_c \mathcal{C}) \)

(See [MN06, §2.5] and [Bla98, §22.3.4] for explanations). Note the conceptual clarity – and concrete opaqueness – gained by recognizing \( \text{Boot} \) as a localizing subcategory of the triangulated category \( \mathcal{KK} \). We are going to show that the Balmer spectrum of its subcategory of compact objects, \( \text{Boot}_c = \langle \mathcal{C} \rangle \), is simply \( \text{Spec}(\mathbb{Z}) \).

5.1.1. The direct approach. We recall some classical results.

5.1.1. Theorem. (Universal Coefficient Theorem, UCT). Let \( A \) and \( B \) be separable \( \mathbb{C} \)-algebras, with \( A \in \text{Boot} \). Then there is a short exact sequence of \( \mathbb{Z}/2 \)-graded abelian groups

\[
\begin{eqnarray*}
\text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B)) & \longrightarrow & \mathbb{K}K_*(A,B) \longrightarrow \text{Hom}(K_*(A), K_*(B)),
\end{eqnarray*}
\]

where the map marked by \(+1\) has degree one, and the second map is the \( K \)-theory functor \( K_* = \mathbb{K}K_*(-,-) \). The sequence is natural and splits unnaturally.

Proof. See [RS87], or [Bla98, §23]. (The result can also be obtained as a special case of the ABC spectral sequence, see Section 3.6, for the homological ideal \( I = \text{Ker}(K_*) \) in \( \mathcal{KK} \)).

In fact, \( A \) satisfies the UCT (for all \( B \)) if and only if it lies in the Bootstrap category ([MN06, §6.2]). As an application, we can characterize the compact objects of the Bootstrap category. But first recall:

5.1.3. Lemma. Let \( M \) be any \( \mathbb{Z}/2 \)-graded countable abelian group. Then there is an \( A \in \text{Boot} \) with \( K_*(A) \cong M \).

Proof. This is very easy. Choose a (degree-wise) free resolution of \( M \) of length one, say \( 0 \to P_1 \to P_0 \to M \to 0 \). Then realize \( P_1 \to P_0 \) as a morphism \( f : Q_1 \to Q_0 \) in \( \text{Proj}(\mathbb{K}K, I_{K_*}) \) (recall from Subsection 4.2.4 that \( K_* \) induces an equivalence \( \text{Proj}(\mathbb{K}K, I_{K_*}) \cong \text{Proj}(\text{Ab}^{\mathbb{Z}/2}_* \mathbb{C})) \)). Since the \( P_i \) are coproducts of copies of \( \mathbb{C} \) or \( T\mathbb{C} \), they belong in \( \text{Boot} \). Finally, apply \( K_* \) to a distinguished triangle \( Q_1 \to Q_0 \to \text{cone}(f) \to TQ_1 \) containing \( f \) to get the 2-periodic exact sequence

\[
\begin{eqnarray*}
P_1 & \longrightarrow & P_0 \longrightarrow K_*(\text{cone}(f)) \longrightarrow P_1[1] \longrightarrow P_0[1].
\end{eqnarray*}
\]

The last map is injective and therefore \( K_*(\text{cone}(f)) \cong M \).

5.1.4. Corollary. Consider objects \( A,B \in \text{Boot} \) such that \( K_*A \cong K_*B \) as graded abelian groups. Then there exists an isomorphism \( A \cong B \) in \( \text{Boot} \).

Proof. Because of the surjectivity (and additivity) of the second homomorphism in the UCT, we may lift the isomorphism \( K_*A \cong K_*B \) to a morphism \( f : A \to B \) in \( \text{Boot} \). Since \( \langle \mathbb{C}, T\mathbb{C} \rangle \) generates \( \text{Boot} \), the condition \( \text{cone}(f) \cong 0 \) is equivalent to \( K_*(\text{cone}(f)) \cong 0 \). But \( K_*(f) \) is an isomorphism, and thus \( f : A \cong B \).

5.1.5. Corollary. Let \( A \in \text{Boot} \). Assume that there is an isomorphism \( K_*(A) \cong M_1 \oplus M_2 \) of \( \mathbb{Z}/2 \)-graded abelian groups. Then there are \( A_1, A_2 \in \text{Boot} \) and an isomorphism \( A_1 \oplus A_2 \cong A \) in \( \text{Boot} \).

Proof. Use Lemma 5.1.3 to get \( A_1, A_2 \in \text{Boot} \) with \( K_*A_1 \cong M_1 \) and \( K_*A_2 \cong M_2 \). Now use Corollary 5.1.4.

5.1.6. Lemma. An object \( A \in \text{Boot} \) is \((\mathfrak{N}_-)-\)compact if and only if \( K_*(A) \) is a finitely generated abelian group.

Proof. Let \( \prod_i B_i \) be a countable coproduct in \( \mathcal{KK} \). Since \( A \) belongs to the Bootstrap category, we can use the Universal Coefficient Theorem to obtain the
following commutative diagram with exact rows. (Note that the canonical vertical maps are automatically injective.]

\[
\begin{array}{ccc}
\Ext(K_\ast(A), \bigoplus_i K_\ast(B_i)) & \longrightarrow & \KK_\ast(A, \bigsqcup_i B_i) \\
\downarrow & & \downarrow \\
\bigoplus_i \Ext(K_\ast(A), K_\ast(B_i)) & \longrightarrow & \bigoplus_i \KK_\ast(A, B_i) \\
\end{array}
\]

\[
\begin{array}{ccc}
\Ext(K_\ast(A), \bigoplus_i K_\ast(B_i)) & \longrightarrow & \Hom(K_\ast(A), \bigoplus_i K_\ast(B_i)) \\
\downarrow & & \downarrow \\
\bigoplus_i \Ext(K_\ast(A), K_\ast(B_i)) & \longrightarrow & \bigoplus_i \Hom(K_\ast(A), K_\ast(B_i)) \\
\end{array}
\]

Now if \( K_\ast(A) \) is finitely generated, every homomorphism \( K_\ast(A) \to \bigoplus_i K_\ast(B_i) \) must factor through some finite subsum of \( \bigoplus_i K_\ast(B_i) \). This is the same as saying that the third vertical map is an isomorphism. It also implies that the first one is an isomorphism, because in order to compute the \( \Ext \) groups we can choose a (length-one) projective resolution of \( K_\ast(A) \) consisting of \( (\mathbb{Z}/2\text{-graded}) \) finitely generated free groups. We conclude by the 5-lemma that the middle vertical map is also an isomorphism, i.e., \( A \in \KK_\ast \) and thus \( A \in \Boot_c \).

Conversely, assume that \( A \in \Boot_c \). Then the middle vertical map is an iso (for all coproducts \( \bigsqcup_i B_i \) in \( \Boot \)). Let \( f \in \Hom(K_\ast A, \bigoplus_i K_\ast(B_i)) \). It can be realized as a morphism \( A \to \bigsqcup_i B_i \), which by hypothesis must factor through some finite subsum, say: \( A \xrightarrow{f'} B' \to \bigsqcup_i B_i \). The image of \( f' \) in \( \bigoplus \Hom(K_\ast A, K_\ast B_i) \) maps to \( f \).

Hence the third vertical map is also surjective and thus an isomorphism. Since by Lemma 5.1.3 every countable coproduct of countable \( \mathbb{Z}/2\text{-graded} \) abelian groups can be realised as \( \bigoplus_j K_\ast(B_j) \) for some \( B_j \in \Boot \), it follows that \( K_\ast(A) \) is finitely generated. (Indeed, if that were not the case we could choose a set \( \{m_n\}_{n \geq 0} \) of generators for \( M := K_\ast(A) \) such that \( M_j \not\subseteq M_{j+1} \), where \( M_j \) denotes the subgroup generated by \( \{m_0, \ldots, m_j\} \). Let \( \pi_j : M \to M/M_j \) be the canonical projection to the quotient. Then \( \pi := (\pi_j) : M \to \prod_j M/M_j \) restricts to a homomorphism \( M \to \bigoplus_j M/M_j \), because \( \pi_j(m_k) \neq 0 \) only for \( 0 \leq j < k \) and thus \( \pi(x) \) is almost everywhere zero for all elements \( x \in M \). Clearly, \( \pi \) does not factor through any finite partial sum.)

A classical companion of the UCT is:

5.1.8. Theorem. (Künneth Theorem for Tensor Product, KTP). Let \( A \) and \( B \) be separable \( C^* \)-algebras with \( A \in \Boot \). Then there is a natural short exact sequence of \( \mathbb{Z}/2\text{-graded} \) abelian groups

\[
\begin{array}{ccc}
K_\ast(A) \otimes K_\ast(B) & \xrightarrow{\alpha} & K_\ast(A \otimes B) \\
\downarrow & & \downarrow \\
\Tor_1^\mathbb{Z}(K_\ast(A), K_\ast(B)) & \longrightarrow & K_\ast(A \otimes B)
\end{array}
\]

which splits unnaturally. The homomorphism \( \alpha \) is the one induced by the monoidal structure \( \otimes : \KK(C, A) \otimes \KK(C, B) \to \KK(C, A \otimes B) \).

Proof. See [RS87], or [Bla98, §23].

\[
\begin{array}{ccc}
K_\ast(A \otimes B)_{(p)} & \cong 0 \\
\downarrow & & \downarrow \\
K_\ast(A)_{(p)} \otimes K_\ast(B)_{(p)} & \cong 0 or K_\ast(B)_{(p)} \cong 0.
\end{array}
\]

Proof. Since \( \Tor_1^\mathbb{Z}(K_\ast(A), K_\ast(B))_{(p)} = \Tor_1^\mathbb{Z}(K_\ast(A)_{(p)}, K_\ast(B)_{(p)}) \) and \( (K_\ast(A) \otimes K_\ast(B))_{(p)} = K_\ast(A)_{(p)} \otimes K_\ast(B)_{(p)} \) ([Wei94, Cor. 3.2.10]), the implication ‘\( \Longleftarrow \)’ follows by localizing the exact sequence (5.1.9) at \( p \). Conversely, if \( K_\ast(A \otimes B) = 0 \), then (5.1.9) (localized at \( p \)) implies that \( K_\ast(A)_{(p)} \otimes_{\mathbb{Z}(p)} K_\ast(B)_{(p)} = 0 \). Since \( A, B \in \Boot_c \), the \( \mathbb{Z}(p) \)-modules \( K_\ast(A)_{(p)} \) and \( K_\ast(B)_{(p)} \) are finitely generated (Lemma 5.1.6), and we conclude with Nakayama’s Lemma that either \( K_\ast(A)_{(p)} = 0 \) or \( K_\ast(B)_{(p)} = 0 \) ([AM69, Exercise 3 p.31]).

Now we have everything we need to prove:
5.1.11. Theorem. The canonical morphism (Prop. 2.2.25)

\[ (5.1.12) \quad \rho : \text{Spec}(\text{Boot}_c) \to \text{Spec}(\text{R}_{\text{Boot}_c}) = \text{Spec}(\mathbb{Z}) \]

sending a thick prime \( \otimes \)-ideal \( P \) to \( \rho(P) = \{ f \in \text{KK}(\mathbb{C}, \mathbb{C}) \mid \text{cone}(f) \notin P \} \), is an isomorphism of locally ringed spaces (cf. Subsec. 2.2.1).

Proof. By Corollary 5.1.4, the isomorphism type of an object \( A \in \text{Boot} \) is determined by its graded \( K \)-theory group \( K_*(A) \). Since we're dealing only with compact objects, we need only consider finitely generated abelian groups.

Therefore every full subcategory of \( \text{Boot}_c \), such as a thick \( \otimes \)-prime \( P \in \text{Spec}(\text{Boot}_c) \), is determined by the range of the isomorphism types of the finitely generated \( \mathbb{Z}/2 \)-graded \( K \)-theory groups of its objects. Recall also that if \( K_*(A) \cong M_1 \oplus M_2 \) (as graded groups), then there is a corresponding splitting \( A \cong B_1 \oplus B_2 \) with \( K_*(B_i) \cong M_i, \ i = 1, 2 \) (Cor. 5.1.5). These facts, as well as the classification of finitely generated abelian groups, will be used without mention. For each prime \( p \in \text{Spec}(\mathbb{Z}) \), consider the following full subcategory of \( \text{Boot}_c \):

\[ P(p) := \text{Ker}(K_*(-)(p)) = \{ A \in \text{Boot}_c \mid K_*(A)(p) \cong 0 \}. \]

Thus if \( p = 0 \), \( T(p) \) consists of those compact \( C^* \)-algebras with torsion \( K \)-theory, and if \( p \) is a prime number, it contains those compact \( C^* \)-algebras \( A \) whose \( K \)-theory \( K_*(A) \) is torsion, but doesn't contain any \( p \)-primary torsion. Since \( K_*(-)(p) = \text{Boot}_c(-)(p) \) is a homological functor, each \( P(p) \) is a thick subcategory. By Corollary 5.1.10, each \( P(p) \) is actually a thick prime \( \otimes \)-ideal, and so belongs to \( \text{Spec}(\text{Boot}_c) \).

Note also that we have strict inclusions \( P(p) \subseteq \overline{P(0)} \) for \( p \neq 0 \), because by Lemma 5.1.3 there exist \( C^* \)-algebras with \( K \)-theory of any prescribed torsion.

Claim: \( \rho(P(p)) = (p) \in \text{Spec}(\mathbb{Z}). \)

Proof. More precisely, let \( n \mapsto f_n \) be the canonical isomorphism \( \mathbb{Z} \cong R(\{1\}) = \text{RKK} = \text{R}_{\text{Boot}_c} \) sending \( 1 \) to \( 1_C : C \to C \). In \( K \)-theory, a distinguished triangle containing \( f_n \) yields an exact sequence

\[ 0 \to K_1(\text{cone}(f_n)) \to \mathbb{Z} \to K_0(\text{cone}(f_n)) \to 0, \]

showing that \( K_0(\text{cone}(f_n)) \cong \mathbb{Z}/n \) and \( K_1(\text{cone}(f_n)) \cong 0 \) if \( n \neq 0 \), as well as \( \text{cone}(f_0) \cong C \oplus TC \) (Cor. 5.1.4). For the following we will use the shorthand

\[ C_n := \text{cone}(f_n). \]

Now we see that

\[ \rho(P(p)) = \{ f_n \in \text{R}_{\text{Boot}_c} \mid C_n \notin P(p) \} = \{ f_n \in \text{R}_{\text{Boot}_c} \mid n \notin p\mathbb{Z} \} = p \cdot \text{R}_{\text{Boot}_c}, \]

proving the claim.

We are going to show that every \( P \in \text{Spec}(\text{Boot}_c) \) belongs to \( \{ P(p) \mid (p) \in \text{Spec}(\mathbb{Z}) \} \). Then clearly \( \text{Spec}(\text{Boot}_c) \cong \text{Spec}(\mathbb{Z}) \) as topological spaces, by definition of the respective topologies.

Fix a \( P \in \text{Spec}(\text{Boot}_c) \). Then \( P \subseteq P(0) \): if there is an object \( A \in P \setminus P(0) \), then \( K_*(A) \) has a non-torsion element and therefore a summand isomorphic to \( \mathbb{Z} \).

Thus \( C \in P \) by thickness and therefore \( P = \text{Boot}_c \). Hence \( P(0) \) is the generic point of the spectrum: \( \{ P(0) \} = \text{Spec}(\text{Boot}_c) \). Assume now that \( P \subseteq P(0) \).

Claim 1: \( P \subseteq P(p) \) for some prime number \( p \neq 0 \).

In fact, if \( P \nsubseteq P(p) \), then there exists \( A \in P \) such that \( K_*(A) \) consists of \( p \)-primary torsion. Using the Claim 2 below, we deduce that \( P \) contains all algebras with torsion \( K \)-theory, i.e. \( P = P(0) \), in contradiction with our assumption.
Claim 2: If a thick \( \otimes \)-ideal \( \mathcal{J} \subseteq \text{Boot}_c \) contains an \( A \) with \( K_\ast(A) \cong \mathbb{Z}/p^\ell \) (concentrated in degree 0 or 1) for some \( \ell \geq 1 \), then it contains also \( C_p \). It follows that \( \mathcal{J} \) contains \( \langle C_p \rangle = \langle C_p \rangle_\otimes \), i.e., the objects with (finitely generated) \( p \)-primary torsion \( K \)-theory.

In order to prove Claim 2, notice that \( \langle f_p \rangle^\ell = f_{p^\ell} \), because \( \mathbb{Z} \cong \text{R}_{\text{Boot}_c} \) is a ring isomorphism. Now Lemma 2.2.20 implies that \( \langle C_p \rangle_\otimes = \langle A \rangle_\otimes \), proving Claim 2, and therefore also Claim 1.

Hence we are left with the case of a \( \mathcal{P} \in \text{Spec}(\text{Boot}_c) \) with \( \mathcal{P} \subseteq \mathcal{P}(p) \) for some prime \( p \). We must show the absurdity of such a thing. By hypothesis there is some \( A \in \mathcal{P}(p) \setminus \mathcal{P} \), and we can assume by Claim 2 that \( A = C_q \) for some prime \( q \neq p \). Since \( \mathcal{P} \subseteq \mathcal{P}(p) \) we also have \( C_p \notin \mathcal{P} \). But for every two distinguished primes \( p \neq q \) we have \( C_p \otimes C_q \cong 0 \notin \mathcal{P} \) by the Künneth Theorem, and since \( \mathcal{P} \) is a tensor prime, this implies that \( C_p \in \mathcal{P} \) or \( C_q \in \mathcal{P} \), in contradiction with the above. Thus the case \( \mathcal{P} \subseteq \mathcal{P}(p) \) doesn’t occur, and we have a complete picture of the spectrum of \( \text{Boot}_c \).

To prove that \( \text{Spec}(\text{Boot}_c) \cong \text{Spec}(\mathbb{Z}) \) is an isomorphism of ringed spaces, it is enough to look at the stalks. Now \( \text{Boot}_c / \mathcal{P}(p) = W(p)^{-1}\text{Boot}_c \), where \( W(p) \) stands for the multiplicative system of morphisms \( f \in \text{Mor}(\text{Boot}_c) \) with \( \text{cone}(f) \in \mathcal{P}(p) \). Hence the isomorphism \( \text{End}_{\text{Boot}_c}(\mathcal{C}) \cong \mathbb{Z} \) induces \( \text{End}_{\mathcal{P}(p)}(\mathcal{C}) \cong \mathbb{Z}_{(p)} \), as wished.

5.1.2. The overkill approach. I want to sketch another, far too complicated proof that \( \text{Spec}(\text{Boot}_c) \cong \text{Spec}(\mathbb{Z}) \), making use of the abstract criterion of Section 2.5. The purpose of this gimmick is to serve as an introduction to the equivariant situation, as well as to provide evidence for Conjecture 5.5.1. We refer to the next few sections for more details.

For each \( (p) \in \text{Spec}(\mathbb{Z}) \), consider the following composition of functors:

\[
F_{(p)} : \text{Boot} \xrightarrow{K_\ast(-)_{(p)}} \mathbb{Z}/p^\ell \text{-Mod}_{\mathbb{Z}/2^\ell} \xrightarrow{(-) \otimes \kappa(p)} D(\mathbb{Z}/p^\ell \text{-Mod}_{\mathbb{Z}/2^\ell}) \overset{(\ast)}{\xrightarrow{\kappa(p)-\mathbb{V}\mathbb{S}_{2^\ell}}} \mathbb{V}^{\mathbb{Z}/2^\ell} \times \mathbb{Z}
\]

Here \( \kappa(p) \) is the prime field of characteristic \( p \) (i.e., the residue field of \( \text{Spec}(\mathbb{Z}) \) at \( (p) \)); the equivalence \( (\ast) \) identifies a complex of \( (\mathbb{Z}/2^\ell \text{-graded}) \kappa(p) \text{-vector spaces} \) with the \( \mathbb{Z}/p^\ell \)-graded vector space of its homology. Since \( K_\ast(-)_{(p)} \) is homological and preserves coproducts, and since the dotted composition reflects isomorphisms, it follows that the object-kernel \( \text{Ker}(F_{(p)}) \subseteq \text{Boot} \) is a localizing subcategory. It is even a prime \( \otimes \)-ideal, because of the following

5.1.13. Lemma. \( F_{(p)}(A \otimes B) \cong 0 \iff F_{(p)}(A) \cong 0 \) or \( F_{(p)}(B) \cong 0 \).

Proof. Let \( A, B \in \text{Boot} \). By the Künneth Theorem 5.1.8, the group \( K_\ast(A \otimes B)_{(p)} \) vanishes if and only if both \( K_\ast(A)_{(p)} \otimes K_\ast(B)_{(p)} \) and \( \text{Tor}_1(K_\ast(A)_{(p)}, K_\ast(B)_{(p)}) \) vanish. The latter groups are the homology groups of \( K_\ast(A)_{(p)} \otimes^L_{\mathbb{Z}(p)} K_\ast(B)_{(p)} \), so the latter condition is equivalent to the vanishing of \( K_\ast(A)_{(p)} \otimes^L_{\mathbb{Z}(p)} K_\ast(B)_{(p)} \) in the derived category. Now the functor \( (-) \otimes^L_{\mathbb{Z}(p)} \kappa(p) \) is easily seen to commute with the respective derived tensor products \( \otimes^L_{\mathbb{Z}(p)} \) and \( \otimes^L_{\kappa(p)} \), and the equivalence \( (\ast) \) identifies the latter with the usual tensor product of graded vector spaces. Now we conclude with the observation that, in the tensor category \( (\kappa(p)-\mathbb{V}\mathbb{S})_{\mathbb{Z}/2^\ell} \times \mathbb{Z} \), we may have \( M \otimes N \cong 0 \) only if one of the factors is already zero. □
Thus we have a bunch of functors \( \{ F(p) \} \) indexed by the points of Spec(\( \mathbb{Z} \)), each of them with a localizing prime \( \otimes \)-ideal of Boot for object-kernel. It is easy to see that this implies:

5.1.14. Corollary. Let \( \sigma(A) := \{(p) \in \text{Spec}(\mathbb{Z}) \mid F(p) \neq 0\} \) for every \( A \in \text{Boot} \). Then the pair \((\text{Spec}(\mathbb{Z}), \sigma)\) is a continuous generalized support datum on Boot.

We want to apply the criterion of Corollary 2.5.4 to Boot and \( \sigma = \) with the choices \( R := R^*_{\text{Boot}} \) (concentrated in \( \mathbb{Z} \)-degree zero), \( X := \text{Spec}(R) = \text{Spec}(\mathbb{Z}) \) and \( S := \{ | | \} \) —, which would imply \( \text{Spec}(\text{Boot}) \cong \text{Spec}(\mathbb{Z}) \). But first we have to prove the following two conditions:

(A) For all \( A \in \text{Boot}, \sigma(A) = \text{Supp}_Z(K_\ast A) \overset{\text{Def.}}{=} \{(p) \mid K_\ast(A)(p) \neq 0\} \).

(B') For all \( A \in \text{Boot}, \sigma(A) = \emptyset \Rightarrow A \cong 0 \).

Proof of (A). This holds by Nakayama’s Lemma, as in the proof of Corollary 5.1.10.

Proof of (B’). We have \( A \cong 0 \) iff \( K_\ast(A) \cong 0 \). Now, it is a general fact that, for a commutative noetherian ring \( R \), an \( R \)-module \( M \) is isomorphic to zero if and only if \( M \otimes \kappa(p) \cong 0 \) for every \( p \in \text{Spec}(R) \) (see [Nee92b, Lemma 2.12]). Concluding: \( A \cong 0 \) iff \( F(p)(A) = K_\ast(A) \otimes \kappa(p) \cong 0 \) for all \( p \), iff \( \sigma(A) = \emptyset \).

5.2. Localization of equivariant \( K \)-theory

Let \( G \) be a compact group, and let \( p \in \text{Spec}(R(G)) \). We wish to apply the abstract results of Section 2.3 with \( T = T^G \) and \( S = R(G) \setminus p \). Thus we consider the thick \( \otimes \)-ideal

\[ J_p := \langle \text{cone}(s) \mid s \in R(G) \setminus p \rangle \otimes \subseteq T^G \]

of compact objects and the localizing \( \otimes \)-ideal

\[ L_p := \langle J_p \rangle := \langle J_S \rangle_{\text{loc}} \subseteq T^G \]

that it generates. Now Theorem 2.3.22 specializes to

5.2.1. Theorem. The pair

\[ (L_p, T^G_p := L^\perp_p) \]

is a complementary pair (Def. 1.4.21) of localizing \( \otimes \)-ideals in \( T \). The gluing triangle for an object \( A \in T \) is obtained by tensoring \( A \) with the gluing triangle for the \( \otimes \)-unit

\[ p \mathbb{1} \xrightarrow{\varepsilon} \mathbb{1} \xrightarrow{\eta} p \mathbb{1} \to T(p \mathbb{1}). \]

Moreover, the following hold true:

(i) \( L_p = p \mathbb{1} \otimes T^G \) and \( T^G_p = \mathbb{1} \otimes T^G \).

(ii) \( \varepsilon : p \mathbb{1} \cong \mathbb{1} \otimes p \mathbb{1} \) and \( \eta : \mathbb{1} \otimes \cong \mathbb{1} \otimes p \mathbb{1} \).

(iii) \( L_p \) and \( T_p \) are monogenic \( \mathcal{N}_1 \)-compactly generated \( \otimes \)-triangulated, with tensor unit \( p \mathbb{1} \) and \( \mathbb{1} \mathbb{1} \), respectively (note however that neither is a \( \otimes \)-\( \Delta \)-subcategory of \( T^G \), since they have different units).

(iv) Their compact objects are \( (L_p)_c = J_p \) and \( (T^G_p)_c = (p \otimes T^G_c) \subseteq T_p^G \).

(Note that the compact objects of \( L_p \) are also compact in \( T^G \), but those of \( T_p^G \) need not be.)

(v) The functors

\[ p \mathbb{1} \otimes (-) : T^G \to L_p \quad \text{and} \quad \mathbb{1} \otimes (-) : T^G \to T_p^G \]

are \( R(G) \)-linear \( \otimes \)-\( \Delta \)-triangle functors commuting with coproducts.

(vi) \( A \in T_p^G \Leftrightarrow s|_A^\ast \) is invertible for every \( s \in R(G) \setminus p \).
3. A G-equivariant Künneth spectral sequence

Let G be compact and p ∈ Spec R(G). Consider the p-local subcategory
\[ T^G_p := \mathbb{I}_p \otimes T^G = \langle \mathbb{I}_p \rangle_{\text{loc}} \subseteq T^G \]
of the previous section. Denote by
\[ h := K^G_p = K^G_p(-) : T^G_p \rightarrow R(G)_p \text{-Mod}^{Z/2}_{\infty} \]
the restriction of equivariant K-theory \( K^G_p \) to \( T^G_p \), which is canonically isomorphic to \( K^G(-)_p \) (Thm. 5.2.1 (vii)). The target category of \( h \) is the stable abelian category of \( \mathbb{Z}/2 \)-graded countably generated (whence the notation “\( \infty \)”) \( R(G)_p \)-modules. We consider on \( T^G_p \) the restriction \( \mathcal{I} \) of the homological ideal \( \ker(K^G_p) \) of \( T^G \), i.e., \( \mathcal{I} = \ker(h) \).

5.3.1. Proposition. The functor \( h \) is the universal \( \mathcal{I} \)-exact (stable homological) functor on \( T^G_p \), and \( T^G_p \) has enough \( \mathcal{I} \)-projective objects.

Proof. We use Meyer and Nest’s criterion Theorem 3.5.5. Since \( T^G_p \) is idempotent complete (having countable coproducts, see Prop. 1.1.37), and since \( h \) is \( \mathcal{I} \)-exact stable homological and \( R(G)_p \text{-Mod}^{Z/2}_{\infty} \) has enough projectives, it remains to construct for \( h \) a partial left adjoint
\[ h^! : \text{Proj}(R(G)_p \text{-Mod}^{Z/2}_{\infty}) \rightarrow T^G_p \]
defined on projective objects, such that
\[ (5.3.2) \quad h \circ h^!(P) \cong P \]
for every \( P \). Set \( h^!(R(G)_p) := \mathbb{I}_p \). Then indeed
\[ hh^!(R(G)_p) = K^G_p(\mathbb{I}_p) \cong R(G)_p. \]
Now we may easily extend \( h^! \) to an additive functor defined on free (and then projective) graded modules. Then (5.3.2) is checked by an easy computation, using that \( h \) commutes with coproducts and the translation. (Copy the beginning of the proof of [MN07, Thm. 5.5] for more details.) \( \square \)

5.3.3. Proposition. Let \( F : T^G_p \rightarrow \text{Ab} \) be a homological functor, and assume that it preserves coproducts. Then for all \( n \in \mathbb{Z} \) there is a canonical isomorphism
\[ (5.3.4) \quad L^n_F_* \cong \text{Tor}_n^{R(G)_p}(F_*(\mathbb{I}_p), h(-)) \]
of functors \( T^G_p \rightarrow \text{Ab}^{Z/2} \). (On the left hand side we have the left derived functors of \( F_* \) with respect to \( \mathcal{I} \), Def. 3.4.1; on the right hand side, the left derived functors of the tensor product of graded modules, i.e., the homology of \( \otimes_{R(G)_p}^L \): the \( R(G)_p \)-action on \( F_*(\mathbb{I}_p) \) is that induced by the functoriality of \( F_* \).)

Proof. (Note by inspecting the definitions that \( L^*_F((F_*)_n) = (L^*_F F)_n \).) By universality of \( h \) and Proposition 3.5.9, there exists a coproduct-preserving stable (= grading preserving) right exact functor
\[ \tilde{F}_* : R(G)_p \text{-Mod}^{Z/2}_{\infty} \rightarrow \text{Ab}^{Z/2} \]
such that $\tilde{F}_* \circ h(P) = F_*(P)$ for $I$-projective objects $P$; moreover, there are canonical isomorphisms

\[(5.3.5)\quad L^n_p F_* \cong (L_n \tilde{F}_*) \circ h\]

for all $n \in \mathbb{Z}$. Therefore we are left with computing $\tilde{F}_*$ and its left derived functors, for which we follow the second part of the proof of [MN07, Thm. 5.5]. We claim:

5.3.6. Lemma. There is a natural isomorphism

\[(5.3.7)\quad \tilde{F}_*(M) \cong F_*(\mathbb{I}_p) \otimes_{R(G)_p} M\]

of graded abelian groups, for $M \in R(G)_p\text{-Mod}_{\mathbb{Z}/2}$.

To prove the lemma, notice first that (5.3.7) holds for the free module $M = R(G)_p$, because there are canonical isomorphisms of graded $R(G)_p$-modules

$$\tilde{F}_*(R(G)_p) = F_*(\mathbb{I}_p) \cong F_*(\mathbb{I}_p) \otimes_{R(G)_p} R(G)_p.$$  

We may extend this to all $\mathbb{Z}/2$-graded free modules in the obvious way. Since both $\tilde{F}_*$ and $F_*(\mathbb{I}_p) \otimes (-)$ are right exact functors, we may compute them for $-$ and extend the natural isomorphism onto $-$ a general graded module $M$ by use of a free presentation $P \to P' \to M \to 0$.

The proposition follows then from the lemma: by taking left derived functors of (5.3.7) we get $L_n \tilde{F}_* \cong \text{Tor}^R_{n+p}(F_*(\mathbb{I}_p), -)$, and by combining this with (5.3.5) we find the wanted isomorphism (5.3.4).

5.3.8. Remark. Let $F : T^G_p \to \text{Ab}$ be an additive functor. Since $T^G_p$ is an $R(G)_p$-linear category, $F$ lifts to a functor $T^G_p \to R(G)_p\text{-Mod}_{\mathbb{Z}/2}$ (simply by $r \cdot a := F(r \cdot_A a)$ for $a : \mathbb{Z} \to F(A)$). This is for instance how we regarded $F_*(\mathbb{I}_p)$ as a graded $R(G)_p$-module in Proposition 5.3.3. It is clear from the proof that the isomorphism (5.3.4) is actually an isomorphism of graded $R(G)_p$-modules.

Now we may specialize Meyer's ABC spectral sequence, discussed in Section 3.6, to the homological pair $(T^G_p, I)$.

Let $F : T^G_p \to \text{Ab}$ be a homological functor which commutes with coproducts. Since $h$ also commutes with coproducts, the ideal $I$ is closed under coproducts, and thus by Theorem 3.6.11, for every $A \in T^G_p$ there exists a strongly convergent spectral sequence

\[(5.3.9)\quad E^2_{pq} = L_p^I F_n(A) \xrightarrow{n \to \mathbb{Z}/2} L^I_p F_n(A).\]

The abutment group $L^I_p F_n(A)$ is the ‘localization at $\text{Ker}(h)$’ of the functor $F_n = F \circ T^{-n} : T^G_p \to \text{Ab}$. But here $\text{Ker}(h) \simeq 0$, which implies that the localization of $F_n$ (or indeed, of any functor) is the functor itself: $L^I_p F_n(A) = F_n(A)$ if $A \in T^G_p$.

Now choose $F$ to be the functor $K^G_G(B \otimes -)_p : T^G_p \to \text{Ab}$, for some fixed $B \in T^G$. According to Proposition 5.3.3, its left derived functors are

\[
L^I_p F_{*+q}(A) \cong \text{Tor}^R_{n+p}(K^G_G(B \otimes \mathbb{I}_p), h_*(A)) \\
\cong \text{Tor}^R_{n+p}(K^G_G(B)_p, K^G_G(A)_p) \\
= \text{Tor}^R_{n+p}(K^G_G(A)_p, K^G_G(B)_p)
\]

for all $A \in T^G_p$. (Note that the natural $R(G)_p$-action on $F_*(-)$ is indeed the one coming from its functoriality, as required by the proposition.) Since $K^G_G(D \otimes \mathbb{I}_p) = K^G_G(D)_p$ for all $D \in T^G$, by precomposing both $F$ and its derived functors with $(-) \otimes \mathbb{I}_p : T^G \to T^G_p$, we may consider general $A \in T^G_p$. Putting it all together, we have just proved:
5.3.10. **Theorem.** (Künneth Spectral Sequence). Let $A, B \in T^G$ and let $p \in \text{Spec} \, R(G)$. Then there is a strongly convergent spectral sequence

$$
E_{pq}^2 = \text{Tor}_p^{R(G)}(A_p, B_q) \xrightarrow{n=q} \text{Tor}_n^{R(G)}(A \otimes B)_p
$$

of $\mathbb{Z}/2$-graded $R(G)_p$-modules. \qed

5.3.11. **Theorem.** (Phillips-Künneth Formula). Specializing further, I am now able to prove a result of Phillips ([Phi85, Theorem 6.4.6]). The reason I bothered to reprove his theorem is that I wasn’t able to clearly relate the hypotheses made in loc. cit. on the $C^*$-algebras with those I need here.

5.3.12. **Theorem.** (Phillips-Künneth Formula). Let $S$ be a finite cyclic group, and let $q \in \text{Spec} \, R(S)$ such that the Segal support of $q$ is $S$ itself (see Thm. 4.1.7). Then there is a natural short exact sequence

$$
K^S_q(A) \otimes_{R(S)_q} K^S_q(B) \rightarrow K^S_q(A \otimes B) \xrightarrow{+1} \text{Tor}_1^{R(S)_q}(K^S_q(A)_q, K^S_q(B)_q)
$$

for arbitrary objects $A, B \in T^S$ (the $+1$ indicates a map of $\mathbb{Z}/2$-degree 1).

**Proof.** Recall that with our hypotheses, Proposition 4.1.13 says that the local ring $R(S)_q$ is as good as they make them: it is hereditary. This means that every object $A \in T^S_q$ has a $K^S_q$-projective resolution of length one. It follows by Theorem 3.6.5 (ii) that the ABC spectral sequence for $A$ and any homological functor $F$ is concentrated in the 0th and 1st columns, and moreover it collapses at the second page. This is true in particular for the Künneth spectral sequence of Theorem 5.3.11. It is now a general fact that the spectral sequence decomposes into short exact sequences of the required form. \qed

**Second proof.** For those who don’t like spectral sequences, it is possible to use instead [MN07, Thm. 4.4], which (among other things) says the following. Let $(\mathcal{T}, \mathcal{I})$ be a triangulated category with a homological ideal, and let $A \in \mathcal{T}$ have an $\mathcal{I}$-projective resolution of length one. Assume that

$$(*) \quad T(A, X) = 0 \quad \text{for all $\mathcal{I}$-contractible $X \in \mathcal{T}$ (that is, with $1_X \in \mathcal{I}$).}
$$

Then for any homological functor $H : \mathcal{T} \rightarrow \mathcal{A}$ there is a short exact sequence

$$
0 \rightarrow \mathbb{L}_0^\mathcal{T} H_*(A) \rightarrow H_*(A) \rightarrow \mathbb{L}_1^\mathcal{T} H_{*-1}(A) \rightarrow 0
$$

in $\mathcal{A}$.

Set $T := T^S_q$, $\mathcal{I} := \mathcal{I}_{K^S_q}$ and $H := K^S(- \otimes B) = K^S(- \otimes B)_q$, for a fixed $B \in T^S_q$. By Proposition 5.3.1, (homotopy classes of) $\mathcal{I}$-projective resolutions of $A$ are in bijection with (homotopy classes of) $R(S)_q$-projective resolutions of $h(A) = K^S_q(A)$. Since the ring $R(S)_q$ is hereditary, we deduce that every $A \in T^S_q$ has $\mathcal{I}$-projective length at most one. Hypothesis $(*)$ is also satisfied, since by definition an object $X$ is $\mathcal{I}$-contractible if $h_*(X) = \mathcal{T}^S_q(T^{-n} \mathbb{1}_q, X) \cong 0$, which in turn is equivalent to $X \cong 0$, since $\mathbb{1}_q$ generates $T^S_q$. Thus we have the short exact sequence (5.3.12) of $\mathcal{I}$-derived functors. Now use Proposition 5.3.3 to identify $\mathbb{L}_1^\mathcal{T} H_*$ with the functors appearing in the statement of the theorem. \qed

5.3.13. **Corollary.** Let $A, B \in T^S_q = (\mathbb{1}_q)_{loc} \subseteq T^S_q$. Then $K^S_q(A \otimes B)_q = 0$ if and only if $K^S_q(A)_q \otimes_{R(S)_q} K^S_q(B)_q$ vanishes in $D(R(S)_q \text{-Mod}^{Z/2})$. 

The first equivalence is immediate from the Phillips-Künneth Formula; the second
extension of scalars $f$ and let $\pi$ because, by definition, the Tor functors are computed as the homology of the
derived tensor product of graded modules.

5.3.2. The ‘residue field object’ of a prime ideal. We keep $(S, q)$ as
above: $S$ is a finite cyclic group and $q \in \text{Spec } R(S)$ has $\text{supp}(q) = S$. Let $\kappa(q) := R(S)_q/q R(S)_q$ denote the residue field of $R(S)$ at the prime ideal $q$.

5.3.14. Lemma. There is an object $\kappa_q \in \mathbb{I}_q \otimes T^S$ such that $K^S_q(\kappa_q) \cong \kappa(q)$ (concentrated in degree zero).

Proof. By Proposition 4.1.13, the local ring $R(S)_q$ is either a field or a discrete valuation ring; in the first case, $\kappa_q := \mathbb{I}_q$ does the job, because $K^S_q(\mathbb{I}_q) = R(S)_q \cong \kappa(q)$ is already the residue field. In the second case, the maximal ideal $m := q R(S)_q$

is generated by a single non-zero divisor, say $\pi$, which under the identification $R(S)_q = \text{End}_{T^S}(\mathbb{I}_q)$ corresponds to a morphism $\chi : \mathbb{I}_q \rightarrow \mathbb{I}_q$. Complete $\chi$ to a
distinguished triangle: $\mathbb{I}_q \rightarrow \mathbb{I}_q \rightarrow \text{cone}(\chi) \rightarrow T\mathbb{I}_q$. By applying $h = K^S_q(-)$ to it we obtain a periodic exact sequence

$$R(S)_q \xrightarrow{K^S_q(\chi) = \pi} R(S)_q \xrightarrow{} K^S_0(\text{cone}(\chi)) \xrightarrow{} 0 \xrightarrow{} 0$$

Hence $K^S_q(\text{cone}(\chi)) = R(S)_q/m = \kappa(q)$; moreover, we see that $K^S_q(\text{cone}(\chi)) = 0$, because $\pi$ is a non-zero divisor. Therefore we may set $\kappa_q := \text{cone}(\chi)$. □

5.3.15. Corollary. Let $A \in T^S$. Then $K^S_q(\kappa_q \otimes A) = 0$ if and only if the
derived tensor product $\kappa(q) \otimes_{R(S)_q} K^S_q(A)_q$ vanishes in $D(R(S)_q\text{-Mod}^{\mathbb{S}/2})$.

Proof. Since $\kappa_q \cong \mathbb{I}_q \otimes \kappa_q$, we may substitute $A$ with $\mathbb{I}_q \otimes A$ and $K^S_q(\kappa_q \otimes A)$ with $K^S_q(\kappa_q \otimes A)_q$, and then we apply Corollary 5.3.13 with $B := \kappa_q$. □

We recall a couple of standard facts of homological algebra:

5.3.16. Lemma. Let $f : R \rightarrow S$ be a homomorphism of (graded) unital rings, and let $L f^* = S \otimes_R (-) : D_+(R) \rightarrow D_+(S)$ be the left derived functor of the extension of scalars $f^* : M \rightarrow S \otimes_R M$ for (graded) left $R$-modules. Then there is a

quasi-isomorphism

$$L f^*(M_\bullet) \otimes_S L f^*(N_\bullet) \simeq L f^*(M_\bullet \otimes_R N_\bullet)$$

in $D_+(S)$, natural in $M_\bullet, N_\bullet \in D_+(R)$ (Indeed, $L f^*$ is a tensor triangle functor.)

Proof. See e.g. [Wei94, Lemma 10.6.7]. To prove it, substitute $M_\bullet$ and $N_\bullet$ with quasi-isomorphic complexes of $R$-flat modules, so that $\otimes_R^1 = \otimes_R$ on them, and then consider the canonical isomorphism $(S \otimes_R M) \otimes_S (S \otimes_R N) \cong S \otimes_R (M \otimes_R N)$ extended to complexes. Clearly, the same statement holds for a homomorphism $f : R \rightarrow S$ of graded rings, and using the tensor product of graded modules. □

5.3.17. Lemma. Let $k$ be a (graded) field. Then there is an equivalence of the
derived category $D(k)$ with the category $k\text{-VS}^Z$ of $\mathbb{Z}$-graded (graded) $k$-vector spaces, identifying $\otimes^1_k$ with the tensor product of graded vector spaces.
Proof. Since $k$-modules are free and thus flat, $\otimes^h_k$ reduces to the tensor product of complexes anyway. Thus the fully faithful functor $k\text{-}VS^\mathbb{Z} \hookrightarrow D(k)$ sending a $\mathbb{Z}$-graded vector space to the corresponding complex with zero differentials is a tensor functor. It is also essentially surjective, because every complex $V_\bullet = (V_n, d_n)$ is quasi-isomorphic to the graded vector space of its homology (there are inclusions $H_n(V_\bullet) \hookrightarrow \text{Ker}(d_n) \subseteq V_n$, since $H_n$ is a quotient of $\text{Ker}(d_n)$ by definition and everything splits; these assemble to a quasi-isomorphism $(H_\bullet, 0) \to V_\bullet$).

5.3.18. Proposition. Let $(S, q)$ be as above. Then

$$K^S_q(\kappa_q \otimes A \otimes B) \neq 0 \iff K^S_q(\kappa_q \otimes A) \neq 0 \text{ and } K^S_q(\kappa_q \otimes B) \neq 0$$

for all $A, B \in T^S$.

Proof. Once again, note that $\kappa_q \cong \kappa_q \otimes \mathbb{I}_q$, since it belongs to $T^S_q = \mathbb{I}_q \otimes T^S$. Therefore we have isomorphisms

$$\kappa_q \otimes A \otimes B \cong \kappa_q \otimes (\mathbb{I}_q \otimes A) \otimes (\mathbb{I}_q \otimes B)$$

by the idempotency of $\mathbb{I}_q \otimes (-)$, so we may as well assume that $A, B \in T^S_q$ and thus $K^S_q = K^S_q(-)$. Now look:

$$K^S_q(\kappa_q \otimes A \otimes B) \neq 0 \iff \kappa(q) \otimes^h K^S_q(A \otimes B) \neq 0$$

$$\iff \kappa(q) \otimes^h (K^S_qA \otimes^h K^S_qB) \neq 0$$

$$\iff (\kappa(q) \otimes^h K^S_qA) \otimes^h_L (\kappa(q) \otimes^h K^S_qB) \neq 0$$

$$\iff \kappa(q) \otimes^h K^S_qX \neq 0 \quad (X = A, B)$$

$$\iff K^S_q(\kappa_q \otimes X) \neq 0 \quad (X = A, B).$$

The first, second and last equivalences follow from Corollary 5.3.15. The third is Lemma 5.3.16 (for $\mathbb{Z}/2$-graded modules and the quotient homomorphism $R(G)_q \to \kappa(q)$ for $f$). The fourth is Lemma 5.3.17 together with the fact that, in the category $\kappa(q)\text{-}VS^\mathbb{Z}/2 \times \mathbb{Z}$ of $\mathbb{Z}$-graded $\mathbb{Z}/2$-graded vector spaces, a product $V_\bullet \otimes W_\bullet$ vanishes only if one of the factor vanishes (just consider bases).

5.4. The generalized support datum $\sigma$

Now let $G$ be a finite group, and consider, as usual, the monogenic $\mathbb{R}_1$-compactly generated $\otimes\Delta$-category $T^G = (\tau^G G)_\text{loc} \subseteq KK^G$. We are going to define a continuous generalized support datum on $T^G$, and we will give reasons for conjecturing that its restriction to $T^G_c = (\tau^G C)_\text{loc}$, the subcategory of compact objects, is a classifying support datum. This would show that $\text{Spc}(T^G_c) \cong \text{Spec}(R(G))$.

Recall from Section 4.1 that for every prime ideal $p \in \text{Spec } R(G)$ there exists a finite cyclic subgroup $S \leq G$, unique up to conjugacy in $G$ (the Segal support $\text{supp}(p)$), such that: There exists a prime ideal $q \in \text{Spec } R(S)$ with $(\text{res}_G^S)^{-1}q = p$, and moreover $S$ is minimal with respect to inclusion among the subgroups with this property. It follows that $q$ also cannot come from proper subgroups of $S$, i.e.: $\text{supp}(q) = S$.

From now on, we fix for each $p \in \text{Spec } R(G)$ an $S \leq G$ and a $q \in \text{Spec } R(S)$ as above. Thus each pair $(S, q)$ satisfies the hypotheses ($S$ is finite cyclic and $\text{supp}(q) = S$) for the Phillips-Künneth formula and the existence of the residue field object $\kappa_q \in T^S$ (see Section 5.3). We will consider the collection of functors

$$F_p : T^G \xrightarrow{\text{Res}_G^S} T^S \xrightarrow{\kappa_q \otimes (-)} (\mathbb{I}_q)_{\text{loc}} \xrightarrow{K^S_q(-)q} R(S)\text{-}\text{Mod}^{\mathbb{Z}/2}$$
indexed by \( p \in \text{Spec}(R(G)) \). Recall that the restriction functor \( \text{Res}^G_S \) is a tensor triangle functor, which specializes at the \( \otimes \)-unit to the restriction homomorphism \( \text{res}^G_S : R(G) \to R(S) \). (Note that the functor \( \text{Res}^G_S : KK^G \to KK^S \) obviously restricts to \( T^G = (\tau^G\mathbb{C})_{\text{loc}} \to T^S = (\tau^S\mathbb{C})_{\text{loc}} \) since \( \text{Res}^G_S(\tau^G\mathbb{C}) = \tau^S\mathbb{C} \).)

5.4.1. **Lemma.** Let \( T \) be a \( \otimes\Delta \)-category, and let \( \{ F_x : T \to A_x \}_{x \in X} \) be a family of stable homological functors on \( T \) such that \( F_x(\varnothing) \neq 0 \), indexed on some topological space \( X \). Then \( (X, \sigma(A) := \{ x \mid F_x(A) \neq 0 \}) \) defines a pre-support datum on \( T \) (Def. 2.2.9). If each \( F_x \) preserves coproducts, then \( (X, \sigma) \) is also continuous.

**Proof.** (Note that this lemma is a sort of converse of Lemma 2.2.10.) We have to check that \( (X, \sigma) \) satisfies Axioms (SD1)-(SD4) of Def. 2.2.7. Since each \( F_x \) is stable homological, each \( \text{Ker}(F_x) \) is a thick triangulated subcategory, implying that \( (X, \sigma) \) satisfies (SD2), (SD3) and (SD4) (to see this, work with the complements \( X \setminus \sigma(A) \)). Obviously \( \sigma(\varnothing) = \emptyset \), and the hypothesis \( F_x(\varnothing) \neq 0 \) guarantees that \( \sigma(\varnothing) = X \); hence (SD1) holds too. If each \( F_x \) sends coproducts to coproducts, each \( \text{Ker}(F_x) \) is a localizing subcategory, and therefore we compute \( \sigma(\bigsqcup A_i) = \bigcup \sigma(A_i) \), showing that \( (X, \sigma) \) is continuous. \( \Box \)

5.4.2. **Definition.** For an \( A \in T^G \) we consider the subset
\[
\sigma(A) := \{ p \mid F_p(A) \neq 0 \} \subseteq \text{Spec}(R(G)).
\]

5.4.3. **Theorem.** The pair \( (\text{Spec}(R(G)), \sigma) \) is a continuous generalized support datum on \( T^G \) (Def. 2.2.9).

**Proof.** Being a composition of two triangle functors, followed by a stable homological one, each \( F_p \) is a stable homological functor. Moreover it preserves coproducts, since each factor does, and \( F_p(\varnothing) = \kappa(q) \neq 0 \) (Lemma 5.3.14). Thus by Lemma 5.4.1 we know that \( (\text{Spec}(R(G)), \sigma) \) is a continuous generalized support datum as soon as we have shown the compatibility with the tensor product:

**Claim:** (SD5) holds, that is: \( \sigma(A \otimes B) = \sigma(A) \cap \sigma(B) \) for all \( A, B \in T^G \).

But now we just have to apply the definitions and what we've done so far:
\[
p \in \sigma(A \otimes B) \iff F_p(A \otimes B) \neq 0 \\
\iff K^G_S(\kappa_q \otimes \text{Res}(A) \otimes \text{Res}(B)) \neq 0 \\
\iff K^G_S(\kappa_q \otimes \text{Res}(A)) \neq 0 \land K^S_x(\kappa_q \otimes \text{Res}(B)) \neq 0 \\
\iff F_p(A) \neq 0 \land F_p(B) \neq 0 \\
\iff p \in \sigma(A) \cap \sigma(B).
\]

The third, crucial equivalence holds by Proposition 5.3.18. Hence the claim is true, and with it the proposition. \( \Box \)

5.4.4. **Remark.** Note that \( \ker(F_p) \) does not depend on the \( S \) we chose in the conjugacy class of \( \text{supp}(p) \), since, if \( S \) and \( S' \) are conjugated in \( G \), then one easily finds an isomorphism \( KK^G \cong KK^{S'} \) identifying \( \text{Res}^G_S \) with \( \text{Res}^{S'}_S \). Nor it depends on the chosen \( q \in \text{Spec}(R(S)) \) lying over \( p \), because (for fixed \( S \)) these are all permuted by the Weyl group \( W_S \) through isomorphisms \( R(S) \to R(S) \) (Prop. 4.1.10); thus if \( (\text{res}^G_S)^{-1}q = p = (\text{res}^{S'}_S)^{-1}q' \), one readily constructs an isomorphism \( \kappa_p \cong \kappa_{p'} \) of residue field objects.
We have continuous maps
\[ \text{Spec}(T^G) \rightarrow \text{Spec}(T^G_{c}) \xrightarrow{\rho} \text{Spec}(R(G)). \]
The first one is simply restriction, and it sends $\text{Ker}(F_p)$ to the thick prime $\otimes$-ideal $\mathcal{P}_p := \text{Ker}(F_p) \cap T^G_{c} \subseteq \text{Spec}(T^G_{c})$. The second one is the natural comparison of Subsection 2.2.2, and we claim that is sends $\mathcal{P}_p$ to $p$.

5.4.5. Lemma. If $A \in T^G_{c}$, the $R(G)$-module $K^G_{c}(A)$ is finitely generated.

Proof. We prove this by induction on the length of $A \in \mathbb{I} = T^G_{c}$. The assertion is certainly true for the generator $\mathbb{I} = \tau^G_{c}$. Let $A \in \mathbb{I}$ have length greater or equal to one. Then there exists a triangle $A_1 \rightarrow A_0 \rightarrow B \rightarrow TA_1$ such that $A_0$ and $A_1$ have strictly shorter length, and such that $A$ is a direct summand of $B$. Hence by induction hypothesis $K^G_{c}(A_i)$ is finitely generated ($i = 0, 1$). Since $R(G)$ is noetherian, we deduce from the exact sequence
\[ \cdots \rightarrow K^G_{c}(A_0) \rightarrow K^G_{c}(B) \rightarrow K^G_{c-1}(A_1) \rightarrow \cdots \]
that $K^G_{c}(B)$, and its direct summand $K^G_{c}(A)$, are also finitely generated.

5.4.6. Lemma. For every compact object $A \in T^G_{c}$, the following are equivalent:

(i) $p \not\in \sigma(A)$, that is, $F_p(A) = K^S_{\mathbb{F}}(\kappa_\mathbb{F} \otimes R^S_{c} A) \cong 0$ in $R(S)_q \text{-Mod}^{G/2}$.

(ii) $K^S_{\mathbb{F}}(\text{Res}^S_{c} A)_q \cong 0$ in $R(S)_q \text{-Mod}^{G/2}$.

Proof. We know by Corollary 5.3.15 that condition (i) is equivalent to the vanishing of $X_\bullet := \kappa(\mathbb{F}) \oplus_{R(S)_q} K^S_{\mathbb{F}}(\text{Res} A)_p$ in $D(\kappa(\mathbb{F}) \text{-VS}^{G/2})$. Let us show that this is equivalent to (ii). Indeed, since $\text{Res}^S_{c} (\tau^G_{c} \mathbb{C}) = \tau^S \mathbb{C}$, the functor $\text{Res}^S_{c}$ restricts to $(\tau^G_{c} \mathbb{C}) \rightarrow (\tau^S \mathbb{C})$, i.e., it sends compact objects to compact objects. Hence if $A$ is compact in $T^G$, by Lemma 5.4.5 (applied to $S$) the $R(S)_q$-module $M := K^S_{\mathbb{F}}(\text{Res} A)_q$ is finitely generated. Since $R(S)_q$ is a noetherian ring of global dimension one, we find a length-one resolution of $M$ by finitely generated projectives, say $P_\bullet = (\cdots \rightarrow P_1 \xrightarrow{d} P_0 \rightarrow \cdots)$. Moreover, since $R(S)_q$ is local and the $P_i$ finitely generated, we may replace the complex $P_\bullet$ by the minimal, that is, such that $d(P_i) \subseteq mP_i$ where $m := qR(S)_q$ denotes the maximal ideal (see [Rob80]). Now $X_\bullet = k(\mathbb{F}) \otimes_{M} k(\mathbb{F}) = (P_0/mP_0 \rightarrow P_1/mP_1)$; so $X_\bullet \cong 0$ if $P_i/mP_i = 0$ ($i = 0, 1$). By Nakayama (or because projectives over a local ring are free), the latter condition is equivalent to $P_i = 0$ ($i = 0, 1$), i.e., to $M \cong 0$.

5.4.7. Corollary. The restriction of $(\text{Spec } R(G), \sigma)$ to $T^G_{c}$ is a support datum.

Proof. We still need to show that $\sigma(A)$ is a closed subset of $\text{Spec } R(G)$ when $A$ is compact. By Lemma 5.4.6 we have
\[ \sigma(A) = \{ p \mid K^S_{\mathbb{F}}(\text{Res}^S_{c} A)_q \neq 0 \} \subseteq \text{Spec } R(G), \]
where the pair $(S, \mathbb{F})$ has been chosen for each $p$. But $\sigma$ would not change if we allow $(S, \mathbb{F})$ to vary among all the possible choices; hence
\[ \sigma(A) = \bigcup_S \text{Spec}(\text{Res}^S_{c})^{-1}(\text{Supp}_{R(S)} K^S_{\mathbb{F}}(\text{Res}^S_{c} A)) \]
where $S$ varies among the cyclic subgroups of $G$. Since $\text{Res}^S_{c} A \in T^S_{c}$, the $R(S)$-module $K^S_{\mathbb{F}}(\text{Res}^S_{c} A)$ is finitely generated. Therefore its support is closed in $\text{Spec } R(S)$, and we conclude that the finite union $\sigma(A)$ is a closed subset of $\text{Spec } R(G)$.

5.4.8. Corollary. The natural map $\rho : \text{Spec}(T^G_{c}) \rightarrow \text{Spec}(R(G))$ maps $\mathcal{P}_p$ to $p$. In particular, it is surjective.
5. THE SPECTRUM OF $\mathcal{T}_{G}^{c} = \langle r^{c} G \rangle$

Proof. In order to show that $\rho(\mathcal{P}_{p}) = p$, recall the definition of $\rho$:

$$\rho(\mathcal{P}) \overset{\text{Def.}}{=} \{ f \in R(G) \mid \text{cone}(f) \not\in \mathcal{P} \}$$

for every $\mathcal{P} \in \text{Spec}(\mathcal{T}_{G}^{c})$. Thus, for an $f \in R(G)$ we have equivalences

$$f \not\in \rho(\mathcal{P}_{p}) \iff \text{cone}(f) \in \mathcal{P}_{p} \iff F_{p}(\text{cone}(f)) = 0$$

(we used that $\text{Res}_{G}^{S}$ is a triangle functor specializing to $\text{res}_{G}^{S}$), as well as

$$f \not\in \mathcal{P}_{p} \iff f \in (R(G)_{p})^{\times}.$$  

Consider the following commutative diagram:

$$
\begin{array}{ccc}
R(G) & \overset{\text{res}_{G}^{S}}{\longrightarrow} & R(S) \\
\downarrow{\ell}_{p} & & \downarrow{\ell}_{q} \\
R(G)_{p} & \longrightarrow & R(S)_{q}
\end{array}
$$

where the vertical maps are the localization homomorphism of rings at the indicated prime. Since $p = \text{res}^{-1}(q)$, the lower horizontal map is a local homomorphism of local rings, and we deduce that $\ell_{p}(f)$ is invertible if and only if $\ell_{q}(\text{res} f)$ is invertible. This proves that $\rho(\mathcal{P}_{p}) = p$. \qed

5.4.9. Remark. Corollary 5.4.8 only shows that $\rho$ is surjective for a finite group $G$. Quite recently, Paul Balmer has found an elegant simple proof showing that, if $\mathcal{T}$ is any $\otimes$-$\Delta$-category whose graded central ring $R^{G}_{*}T$ is noetherian, then both $\rho : \text{Spec} \mathcal{T} \rightarrow \text{Spec} R^{G}_{*}T$ and its graded version $\text{Spc} \mathcal{T} \rightarrow \text{Spec}^{h} R^{G}_{*}T$ are surjective. Hence the value of our corollary lies not so much in relation to the surjectivity of $\rho$, as in providing evidence that our $\sigma$ might well be the classifying support; see the next and last section.

5.5. Final remarks and conjectures

We strongly suspect that $\text{Spc}(\mathcal{T}_{G}^{c}) \cong \text{Spec}(R(G))$ for every finite group $G$. More precisely:

5.5.1. Conjecture. The continuous generalized support datum

$$(\text{Spec} R(G), \sigma)$$

on $\mathcal{T}_{G}^{c}$ (Def. 5.4.2) satisfies the hypotheses (A) and (B’) of Corollary 2.5.4. Thus its restriction to $\mathcal{T}_{G}^{c}$ is classifying (Def. 2.2.14), so that by Theorem 2.2.15 there is a canonical homeomorphism $\text{Spc}(\mathcal{T}_{G}^{c}) \cong \text{Spec}(R(G))$.

The two conditions translate here as:

(A) For all $A \in \mathcal{T}_{G}^{c}$, $\sigma(A) = \text{Supp}_{R(G)}(K^{G}_{*}A) \overset{\text{Def.}}{=} \{ p \mid K^{G}_{*}(A)_{p} \neq 0 \}$.

(B’) For all $A \in \mathcal{T}_{G}^{c}$, $\sigma(A) = \emptyset \Rightarrow A = 0$.

We bring the following evidence for Conjecture 5.5.1.

• The conjecture holds for the trivial group (see Subsection 5.1.2).
• The generalized support $\sigma$ is a ‘genuine’ support datum on $T^G_c$ (Cor. 5.4.7), as required by the conjecture. In particular, by the universal property of Balmer’s spectrum there is a continuous map $f : \text{Spec}(R(G)) \to \text{Spc}(T^G_c)$ defined by $p \mapsto f(p) = \{ A \mid p \not\in \sigma(A) \} = \text{Ker}(F_p) \cap T^G_c \overset{\text{Def.}}{=} \mathcal{P}_p$. This $f$ is the candidate homeomorphism of the conjecture.

• Encouragingly, the comparison map $\rho : \text{Spc}(T^G_c) \to \text{Spec}(R(G))$ sends the thick $\otimes$-prime $\mathcal{P}_p$ back to $p$ (Cor. 5.4.8), showing that $f$ is topologically split injective.

5.5.2. Remark. If one restricts attention to commutative $G$-$C^*$-algebras, then Conjecture 5.5.1 looks even more plausible. Specifically, it follows from Segal’s localization theorem for equivariant $K$-theory of $G$-spaces [Seg68b, Prop. 4.1] that condition (A) holds at least for unital commutative $G$-$C^*$-algebras ($\equiv$ compact $G$-spaces); condition (B') should be provable along the same lines. It remains to extend this to non-unital commutative algebras, and to understand the relation between general $G$-$C^*$-algebras in $T^G$ and commutative ones.

5.5.3. Remark. I also suspect that $\text{Spc}(T^G_c) \cong \text{Spec}(R(G))$ when $G$ is a general compact Lie group. The only significant difference from the finite case is that now Segal’s support $\text{supp}(p)$ of a prime ideal $p \in \text{Spec}(R(G))$ is in general not a finite cyclic group any more, but is instead the product of a finite cyclic group with a torus. The local rings of the representation ring $R(\text{supp}(p))$ may not be hereditary anymore — but they shouldn’t be too wild either, see Example 4.1.4 (e). Thus one might be able to construct residue field objects $\kappa_p$ in this case too, so that the same definition of $\sigma : \text{obj}(T^G) \to 2^{\text{Spec}(R(G))}$ applies for all compact Lie groups. More work will be needed to prove the crucial compatibility (SD5) with the tensor product.

Thank you, dear reader, for indulging me so far.
APPENDIX A

The Spanier-Whitehead category

The grandmother of all triangulated categories is the stable homotopy category of Edwin Spanier and J.H.C. Whitehead, whose objects are (formal desuspensions of) finite CW-complexes with base-point, and the morphisms are given by stabilizing pointed homotopy classes of pointed maps with respect to the suspension functor \( X \mapsto S^1 \wedge X \). The resulting situation inspired Puppe’s axioms for distinguished triangles (i.e., homotopy cofiber sequences) \([\text{Pup62}]\). Proving the axioms (including Verdier’s Octahedron Axiom) involves finding certain homotopies. Subsequently, the ‘same’ proof – the same homotopies, more or less – has been used in various other contexts, especially in the theory of operator algebras where a little ‘naive’ homotopy was all that was required, so that the modern algebraic techniques of model categories and simplicial sets have not yet taken root.

In this appendix we give the classical proof, in a somewhat formalized version.

A.1. Preliminaries on spaces

As in the main text, tensor category will be synonymous with symmetric monoidal category with unit object; a tensor functor is a (always strong) monoidal functor ([Mac98, VII,XI]).

In the following we will consider a fixed category, denoted \( \text{Cpt} \), of compact spaces. This should be a full subcategory of the category of compact Hausdorff spaces, containing all finite CW-complexes and having push-outs (at least of injective maps along any other) and finite products. We write \( \text{Cpt}_2 \) for the category of pairs \((X,Y)\) of spaces \( X,Y \in \text{Cpt} \) together with an inclusion \( Y \hookrightarrow X \). As usual, a map of pairs \( f : (X,Y) \rightarrow (X',Y') \) is just a map \( f : X \rightarrow X' \) such that \( f(Y) \subseteq Y' \).

Without further notice, we will identify \( \text{Cpt} \) as a full subcategory of \( \text{Cpt}_2 \) via \( X \mapsto (X,\emptyset) \). We remark that there is also a faithful functor \( \text{Cpt} \to \text{Cpt}_+ \), \( X \mapsto X_+ := (X \sqcup \text{pt}, \text{pt}) \) which adjoins a disjoint base-point. This functor sends products to smash products: \((X \times X')_+ \cong X_+ \wedge X'_+ \) (recall the definition: \((X,x_0) \wedge (X',x'_0) := (X \times X'/(x_0 \times X' \cup (X \times x'_0)))\), pt) for pointed spaces \((X,x_0),(X',x'_0) \in \text{Cpt}_+ \).

One can define a smash product \( \wedge : \text{Cpt}_2 \times \text{Cpt}_2 \rightarrow \text{Cpt}_2 \) on pairs by

\[
(X,Y) \wedge (X',Y') := (X \times X', (X \times Y') \cup (Y \times X')).
\]

It is easily checked that this product (together with obvious structural isomorphisms) makes \( \text{Cpt}_2 \) into a tensor category with unit \( \text{pt} = (\text{pt},\emptyset) \). Moreover, the ‘quotient functor’

\[
Q : (X,Y) \mapsto (X/Y, \{Y\}) , \quad \text{Cpt}_2 \rightarrow \text{Cpt},
\]

is a tensor functor from pairs to pointed spaces. We have \( Q(X,\emptyset) = X_+ \), because by definition the quotient \( X/Y \) is the push-out of \( \text{pt} \sqcup Y \subseteq X \) (in particular, \( Q \) sends the smash unit \( \text{pt} \) to the smash unit \( \text{pt}_+ \)). To sum up, we have the commutative
of tensor categories, where of course \( \text{Cpt} = (\text{Cpt}, \cdot, \text{pt}) \). The inclusion \( J : \text{Cpt}_a \hookrightarrow \text{Cpt}_2 \) that views a pointed space \((X, x_0)\) as a pair is not a tensor functor, because it sends \( \text{pt}_+ \) to \( \text{pt}_+ \neq \text{pt} \). Nonetheless, \( QJ = \text{id}_{\text{Cpt}_a} \), and there is a natural transformation \( \eta : \text{id}_{\text{Cpt}_2} \Rightarrow JQ \) with components \((X, Y) \to (X/Y, \text{pt})\), which restrict on \( \text{Cpt} \) to the inclusions \( X \hookrightarrow X_+ \). We will be studying certain functors \( \text{Cpt}_2^{\text{op}} \to A \) which by hypothesis invert \( \eta \). Thus at all effects we will identify \( \text{Cpt}_2 \) with \( \text{Cpt}_a \), i.e., each pair \((X, Y)\) with \((X/Y, \text{pt})\) and, taking some care with maps, also with the (noncompact!) space \( X - Y \).

Let \( I := [0, 1] \). If \( Y \subseteq X \) then \( I \times Y \subseteq I \times X \) and we define the cylinder over \((X, Y)\) to be the pair \((I \times X, I \times Y)\). A homotopy between \( f_0 \) and \( f_1 : (X, Y) \to (X', Y') \) is a map \( H : (I \times X, I \times Y) \to (X', Y') \) with \( H \circ i_0 = f_0 \) and \( H \circ i_1 = f_1 \), where \( i_0 \) and \( i_1 \) are the obvious inclusions of pairs \((X, Y) \subseteq (I \times X, I \times Y)\) at \( t = 0 \) resp. \( t = 1 \). Homotopy is an equivalence relation on the \( \text{Hom} \) sets \( \text{Hom}_{\text{Cpt}_2}((X, Y), (X', Y')) \) compatible with composition. If we restrict this equivalence relation to \( \text{Cpt} \) we find the usual (free) homotopy of spaces. Notice also that the quotient functor takes a homotopy \( H : (I \times X, I \times Y) \to (X', Y') \) from \( f_0 \) to \( f_1 \) to a base-point preserving, or pointed, homotopy
\[
Q(H) : I_+ \land (X/Y, \text{pt}) = (I \times X/I \times Y, \text{pt}) \to (X'/Y', \text{pt})
\]
from \( Q(f_0) \) to \( Q(f_1) \). In fact, when restricted to the subcategory \( \text{Cpt}_a \), the homotopy relation of \( \text{Cpt}_2 \) coincides with the base-point preserving homotopy relation. Thus \( Q \) and \( J \) descend to the homotopy categories.

### A.2. Admissible categories of algebras

In this section we want to consider a ‘category of algebras’ \( A \), by which we mean the following.

**A.2.1. Definition.** A category of algebras is a category \( A \) with a zero object, denoted \( 0 \) (i.e., an object which is initial and final), where pull-backs of epimorphisms along arbitrary morphisms exist (and therefore also finite products \( A \times B \) and kernels of epimorphisms exist). In \( A \) we will consider extensions (or short exact sequences), that is diagrams of the form \( A \rightarrowtail B \twoheadrightarrow C \) such that \( p \) is an epimorphism and \( i \) is the kernel of \( p \). We require of \( A \) two more intrinsic properties:

- The pull-back of an epimorphism is an epimorphism.
- (Three lemma). Given a commuting diagram in \( A \)
\[
\begin{array}{ccc}
A & \rightarrow & B & \rightarrow & C \\
\downarrow f & & \downarrow g & & \downarrow h \\
A' & \rightarrow & B' & \rightarrow & C'
\end{array}
\]
where the rows are extensions, if \( f \) and \( h \) are isomorphisms so is \( g \).

**A.2.2. Remark.** Clearly, the 3-lemma holds for example if there exists a functor \( U : A \to B \) to an abelian category \( B \) reflecting isomorphisms and sending each extension in \( A \) to an exact sequence of \( B \). Thus \( A \) could be an abelian category itself, or an exact category. Note however that the \( A \) that we would like to consider are not additive categories.
A.2. Example. Consider the category $A = (\text{Cpt}_*)^{\text{op}}$. The epimorphisms in $\text{Cpt}_*^{\text{op}}$ are the monomorphisms of $\text{Cpt}_*$, and these are the injective maps $f : (X, x_0) \to (Y, y_0)$. Their kernels in $\text{Cpt}_*^{\text{op}}$ correspond to the quotients $(X, x_0)/(Y, y_0)$. The push-outs of injective maps are injective, and the 3-lemma holds (indeed, this is true in the category of all topological spaces).

We are going to apply the 3-lemma in the following (equivalent) form:

A.2.4. Lemma. Let $A$ be a category of algebras as in Def. A.2.1, and consider in it a morphism of exact sequences:

$$
\begin{array}{cccc}
A & \to & B & \to & C \\
\downarrow f & & \downarrow g & & \downarrow \ell \\
A' & \to & B' & \to & C'.
\end{array}
$$

Then the right hand square is a pull-back if and only if $f$ is an isomorphism.

Proof. If the right square is a pull-back, define an $A' \to B$ with components $i'$ and $0 : A' \to C$. It must factor through the kernel $i$ via an $f' : A' \to A$. We compute $i' f = (0, i') f = (0, g i) = i$. Since $i$ is a kernel it is a monomorphism, from which we deduce that $f = 1_A$. We also have $i' f f' = g i f' = i'$ which implies $f f' = 1_{A'}$, this time because $i'$ is a kernel and a mono. Hence $f$ is invertible. Now assume instead that $f$ is invertible, and consider the following commutative diagram:

$$
\begin{array}{cccc}
A & \to & B & \to & C \\
\downarrow f & & \downarrow g & & \downarrow k \\
\ker(p) & \to & P & \to & C \\
\downarrow m & & \downarrow p & & \downarrow \ell \\
A' & \to & B' & \to & C'.
\end{array}
$$

Here the bottom-right square is a pull-back, and $k$ (resp. $\ell$ and $m$) are induced by the universal property of pull-backs (resp. kernels). Notice that $p : P \to C$ is an epi, since we assumed that in $A$ the pull-back of an epi is again epi. By the first half of this lemma, $m$ is an isomorphism, and since $f$ is an isomorphism by hypothesis, so is $\ell$. We conclude with the 3-lemma that $k$ is an isomorphism. □

We also want our category of algebras $A$ to come equipped with a sort of ‘exact action’, or representation, of the compact spaces, as made precise in the following definition.

A.2.5. Definition. An admissible category of algebras is a pair $(A, C)$, where $A$ is a category of algebras (Def. A.2.1: with 0, where pull-backs of epis exist and are epis, and where the 3-lemma holds), and $C$ is a functor $(\text{Cpt}_2)^{\text{op}} \times A \to A$, written

$$(X, Y), A \mapsto C(X, Y; A)$$

or sometimes $A^{X \to Y}$ or $A(X - Y)$, subject to the following axioms.

• (Action). The functor $C$ makes $A$ into a module over the symmetric monoidal category $\text{Cpt}_2 = (\text{Cpt}_2, \wedge, \text{pt})$, in the sense that there are given isomorphisms

$$C((X, Y), (X', Y'; A)) \cong C((X, Y) \wedge (X', Y'; A)) \quad \text{and} \quad C(\text{pt}; A) \cong A$$

natural in $(X, Y), (X', Y') \in \text{Cpt}_*$ and $A \in A$. 

• \textbf{(Exactness).} The action is exact in the following sense. For every pair \((X,Y) \in \Cpt_2\), the functor \(C(X,Y;?) : \mathcal{A} \to \mathcal{A}\) sends extensions to extensions.

• \textbf{(Additivity).} For every pair \((X,Y)\), the functor \(C(X,Y;?) : \mathcal{A} \to \mathcal{A}\) preserves finite products (in particular it sends 0 to 0).

The next and last two axioms specify the behaviour of the contravariant functors \(C(\cdot; A)\) for a fixed ‘algebra’ \(A \in \mathcal{A}\).

• \textbf{(Extension).} For every object \(A \in \mathcal{A}\) and every triple \(Z \subseteq Y \subseteq X\) of compact spaces, the application of \(C\) to the inclusions \((Y,Z) \subseteq (X,Z) \subseteq (X,Y)\) yields an extension

\[
C(X,Y; A) \longrightarrow C(X,Z; A) \longrightarrow C(Y,Z; A).
\]

• \textbf{(Excision).} For any pair \((X,Y)\) and every \(A \in \mathcal{A}\), the canonical projection \(\eta : (X,Y) \to (X/Y, \{Y\})\) induces an isomorphism

\[
C(X/Y, \{Y\}; A) \cong C(X,Y; A).
\]

A.2.6. \textbf{Remarks.} (a) By Excision, to give an action \(C\) as in the definition is equivalent to giving an action restricted to pointed spaces,

\[
\tilde{C} : (\Cpt_2)^{\text{op}} \times \mathcal{A} \to \mathcal{A}.
\]

It suffices then to set \(C(X,Y; A) := \tilde{C}(X/Y, \text{pt}; A)\). The functor \(\tilde{C}\) must satisfy the obvious pointed versions of the axioms (safe for Excision). In particular, the Extension axiom would take the form

• \textbf{(Extension')}\). The functor \(\tilde{C}\) sends an inclusion \((Y,y_0) \subseteq (X,x_0)\) to an extension of algebras \(\tilde{C}(X/Y, \{Y\}) \rightarrow \tilde{C}(X,x_0) \rightarrow \tilde{C}(Y,y_0)\).

Strictly speaking, we could have done everything just with pointed spaces, but in practice it is convenient to have access to pairs, e.g. when cutting and gluing homotopies.

(b) The Extension axiom applied to \(\emptyset \subseteq X \subseteq X\) implies that \(C(X,X; A) \cong \ker(\text{id}) \simeq 0\) for any \(X \in \Cpt_2\) and \(A \in \mathcal{A}\).

(c) The Excision axiom takes also an alternative form, see Corollary A.2.7. As an immediate consequence, we see that \(C\) doesn’t distinguish between a space \(X \in \Cpt\) and its Alexandrov compactification \(X_+ = (X \sqcup \text{pt}, \text{pt})\). Hence we need not worry about the two different ways (namely \(X \mapsto (X,\emptyset)\) and \(X \mapsto X_+\)) of seeing \(\Cpt\) inside of \(\Cpt_2\). Beware though that \(0 = C(\text{pt}, \text{pt}; A) \neq C(\text{pt}; A) = A\).

Note that the pair \((\text{pt}, \text{pt})\) is a zero object in \(\Cpt_2\).

A.2.7. \textbf{Corollary.} Let \((X,Y) \in \Cpt_2\) and let \(U \subseteq Y\) be such that the subspaces \(Y - U \subseteq Y\) and \(X - U \subseteq X\) are objects of \(\Cpt\) (in particular \(U\) is open in \(Y\)). Then the inclusion \((X - U, Y - U) \subseteq (X,Y)\) induces for every algebra \(A\) an isomorphism \(C(X,Y; A) \cong C(X - U, Y - U; A)\).

\textbf{Proof.} The inclusion \((X - U, Y - U) \subseteq (X,Y)\) induces the upper horizontal morphism in the commutative square

\[
\begin{array}{ccc}
C(X,Y; A) & \longrightarrow & C(X - U, Y - U) \\
\downarrow & & \downarrow \\
C(X/Y, \text{pt}; A) & \longrightarrow & C((X - U)/(Y - U), \text{pt}; A).
\end{array}
\]

The lower horizontal morphism is an isomorphism since \(((X - U)/(Y - U), \text{pt}) \rightarrow (X/Y, \text{pt})\) is one, and the two vertical arrows are invertible by Excision. \(\square\)
A.2.8. Proposition (Mayer-Vietoris). Consider a square in $\mathsf{Cpt}_2$ as the one on the right hand side,

\[
\begin{array}{ccc}
X' & \to & Y' \\
\downarrow^f \downarrow^{(*)} & & \downarrow^g \downarrow^g \\
X & \to & Y \\
\end{array}
\]

where the horizontal maps are inclusions of pairs, and such that on the bigger spaces it gives a push-out $(*)$. (This is in particular a push-out square in $\mathsf{Cpt}_2$.) Then for every algebra $A \in \mathcal{A}$, the following square is a pull-back.

\[
\begin{array}{ccc}
C(X',Z';A) & \to & C(Y',Z';A) \\
\downarrow^{f^*} \downarrow^{(*)} & & \downarrow^{g^*} \downarrow^{(*)} \\
C(X,Z;A) & \to & C(Y,Z;A) \\
\end{array}
\]

Proof. By the Extension axiom applied to the map of triples $f : (Z \subset Y \subset X) \to (Z' \subset Y' \subset X')$ we obtain a morphism of extensions

\[
\begin{array}{ccc}
C(X',Y';A) & \to & C(X',Z';A) \to C(Y',Z';A) \\
\downarrow^{h^*} \downarrow^{f^*} \downarrow^{(*)} & & \downarrow^{g^*} \downarrow^{(*)} \\
C(X,Y;A) & \to & C(X,Z;A) \to C(Y,Z;A) \\
\end{array}
\]

The morphism $h^*$ is induced by the restriction of $f$ fitting in the commutative square

\[
\begin{array}{ccc}
(X,Y) & \to & (X',Y') \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
(X/Y,pt) & \to & (X'/Y',pt). \\
\end{array}
\]

The bottom map is an isomorphism because of the push-out $(*)$ (cf. Lemma A.2.4!), and the two vertical ones induce isomorphisms on $C(i_!;A)$ by the Excision axiom. Hence $h^*$ is an isomorphism, and it follows by lemma A.2.4 that $(**)$ is a pull-back, and we win. \hfill \Box

A.2.9. Remark. (a) Note that Mayer-Vietoris implies Excision, in the presence of the other axioms. Indeed, consider the push-out in $\mathsf{Cpt}_2$ with horizontal inclusions

\[
\begin{array}{ccc}
X & \to & Y \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
(X/Y,pt) & \to & (pt,pt) \\
\end{array}
\]

and apply the Extension axiom to obtain the morphism of short exact sequences

\[
\begin{array}{ccc}
C(X,Y;A) & \to & C(X;A) \to C(Y;A) \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
C(X/Y,pt;A) & \to & C(pt,pt;A). \\
\end{array}
\]

The right hand side square is a pull-back by Mayer-Vietoris. Moreover, $C(pt,pt;A) = 0$ by Extension (Remark A.2.6.b). It follows by the uniqueness of kernels that the
appropriate map induces an isomorphism \( C(X, Y; A) \cong C(X/\text{pt}; A) \) (or conclude with Lemma A.2.4 instead).

(b) We also notice that the Extension axiom follows from Mayer-Vietoris together with a weaker form of itself, which says that each pair \((X, X)\) has trivial images \( C(X, X; A) \). To see this, consider a triple \( Z \subset Y \subset X \). We apply Mayer-Vietoris to the push-out of inclusions

\[
\begin{array}{ccc}
(X, Y) & \hookrightarrow & (X, Z) \\
\downarrow & & \downarrow \\
(Y, Y) & \hookrightarrow & (Y, Z)
\end{array}
\]

\[
\begin{array}{ccc}
& C(X, Y; A) & \longrightarrow C(X, Z; A) \\
\downarrow & & \downarrow \\
& C(Y, Y; A) & \longrightarrow C(Y, Z; A).
\end{array}
\]

Since \( C(Y, Y; A) = 0 \) by hypothesis, this means that the upper-right corner is an exact sequence, as wished.

For later use, we also record the following corollary of the Exactness axiom.

A.2.10. Lemma. For every pair \((X, Y) \in \text{Cpt}_2\), the functor \( C(X, Y; ?) : \mathcal{A} \to \mathcal{A} \) preserves those pull-backs where one of the maps \( A \to D \leftarrow B \) is an epi.

Proof. Say that \( A \to D \) is the epi. Since in \( \mathcal{A} \) the pull-back of an epi is an epi, and since kernels exist, there is a diagram

\[
\begin{array}{ccc}
\text{ker}(p') & \rightarrow & P \rightarrow B \\
\downarrow & & \downarrow & & \downarrow \\
\text{ker}(p) & \rightarrow & A \rightarrow D
\end{array}
\]

with exact lines. Note that \( f \) is an isomorphism (A.2.4). If we apply \( C(X, Y; ?) \) to the above diagram, we obtain a similar one where the lines are still exact by Exactness, and where the leftmost map is still an isomorphism. Therefore the appropriate square is a pull-back by Lemma A.2.4. \( \square \)

A.2.11. Definition. An admissible tensor category of algebras is a pair \((\mathcal{A}, \tilde{C})\) where:

- \( \mathcal{A} = (\mathcal{A}, \otimes, \text{I}) \) is a tensor category with a 0 object, where the pull-back of an epi exists and is an epi, and where the 3-lemma holds.
- \( \tilde{C} \) is a tensor functor \( \tilde{C} : (\text{Cpt}_2, \wedge, \text{pt})^{\text{op}} \to (\mathcal{A}, \otimes, \text{I}) \) (or defined on \((\text{Cpt}_*, \wedge, \text{pt}_+)\)^{\text{op}} instead, cf. Remark A.2.6.a, in which case one drops Excision below).

This data is subject to the following axioms.

(Exactness). The functors \( \tilde{C}(X, Y) \otimes ? : \mathcal{A} \to \mathcal{A} \) preserve exact sequences.

(Additivity). The functors \( \tilde{C}(X, Y) \otimes ? \) preserve finite products.

(Extension). The functor \( \tilde{C} \) sends a triple \( Z \subset Y \subset X \) to an extension of algebras \( \tilde{C}(X, Y) \hookrightarrow \tilde{C}(X, Y) \otimes \tilde{C}(Y, Z) \).

(Excision). \( \tilde{C} \) sends the natural map \( (X, Y) \to (X/\text{pt}; A) \) to an isomorphism.
A.2. ADMISSION OF CATEGORIES OF ALGEBRAS

A.2.12. Remark. It is clear that an admissible tensor category of algebras yields an admissible category in the sense of Definition A.2.5, if we set

\[ C : (\text{Cpt}_{2})^{op} \times \mathcal{A} \to \mathcal{A}, \]

\[ ((X, Y), A) \mapsto \tilde{C}(X, Y) \otimes A. \]

In particular, the Action axiom is satisfied since the required natural isomorphisms are supplied by combining the given structural isomorphisms 1 1 ∼ \tilde{C}(pt) and \tilde{C}(X, Y) \otimes \tilde{C}(X', Y') ∼ \tilde{C}((X, Y) \wedge (X', Y')) of \tilde{C} with the left unit and associativity identifications of the tensor product of \mathcal{A}.

In the following we will carelessly denote both \mathcal{C} and \tilde{C} by the letter \mathcal{C}.

A.2.13. Example. Let \((\mathcal{A}, \otimes, \mathbb{I})\) be the category of C*-algebras and ∗-homomorphisms, where \otimes is either the minimal or the maximal tensor product of C*-algebras (they coincide when one of the two algebras is nuclear; for instance if it’s commutative). Then the classical functor

\[ C(X, Y; A) := \{ f : X \to A \mid f \text{ is continuous and } f|_Y \equiv 0 \} \]

defined on pairs of compact Hausdorff spaces yields (all commutative) C*-algebras and it satisfies Additivity, Extension and Excision. It is also well-known that there is a natural isomorphism \( C(X, Y) \otimes A \simeq C(X, Y; A) \), where \( C(X, Y) := C(X, Y; \mathbb{C}) \).

Indeed, the functor \( C(\cdot) : \text{Cpt}_{2}^{op} \to \mathcal{A} \) is a tensor functor from pairs of compact spaces to C*-algebras (Exercise: prove this using the classical Gelfand duality between the category of compact spaces and the category of unital C*-algebras and unital ∗-homomorphisms).

Instead of all C*-algebras one can restrict attention to some interesting full monoidal subcategories, such as for instance: (i) Commutative C*-algebras, (ii) separable C*-algebras, (iii) nuclear C*-algebras, etc. (iv) any combination of the previous ones. Notice that the category of unital C*-algebras and unital morphisms is not allowed, since we need kernels.

It is clear that a tensor category \((\mathcal{A}, \otimes, \mathbb{I})\) of ‘algebras’ containing commutative C*-algebras as a subcategory would also satisfy the axioms. For instance:

A.2.14. Example. Pro-C*-algebras and continuous ∗-homomorphisms ([Phi88]).

A.2.15. Example. Separable C*-algebras and asymptotic ∗-homomorphisms (e.g. [Tho03]).

A.2.16. Example. The category \mathcal{A} of (possibly non-unital) C*-categories and ∗-functors, endowed with the minimal tensor product ([GLR85] [Mit02]). In truth, we need a zero object (i.e., we need \mathcal{A} to be a pointed category), but this can be easily achieved by considering the comma category \( 0 \downarrow \mathcal{A} \).

A.2.17. Example. Equivariant versions of all the previous ones. For instance (cf. Chapter 4), we might take for \mathcal{A} the category of separable G-C*-algebras for a second countable locally compact group G, where \otimes is the minimal tensor product (endowed with the diagonal G-action) and \mathbb{I} = \tau^G \mathbb{C} is the the algebra of complex numbers with the trivial G-action.

The more purely topological examples can be treated without any mention whatsoever of C*-algebras.

A.2.18. Example. Take for \((\mathcal{A}, \otimes, \mathbb{I})\) the opposite of the monoidal category \((\text{Cpt}_{*}, \wedge, \text{pt}_{*})\) itself, and take the monoidal ‘quotient’ functor \( Q : (\text{Cpt}_{2})^{op} \to \text{Cpt}_{*}^{op} \).
for the action \( C \). Then \( \text{Cpt}^{\ast}_{G} \) is a category of algebras (Ex. A.2.3), and the axioms of an admissible tensor category of algebras are readily verified.

A.2.19. Example. The opposite of any suitable tensor category of pointed spaces containing \( \text{Cpt}_{\ast} \) as a tensor subcategory.

A.2.20. Example. For instance, let \( G \) be a group, and let \( \text{Cpt}^{G} \) be the category of pointed compact \( G \)-spaces. If we endow \( (X, x_{0}) \wedge (Y, y_{0}) \) with the diagonal \( G \)-action, we see that \( (\text{Cpt}^{G}, \wedge, \text{pt})^{\text{op}} \) provides an example as in A.2.19. Indeed, there is a fully faithful isomorphism-reflecting tensor functor \( \tau : \text{Cpt}_{\ast} \hookrightarrow \text{Cpt}^{G} \) assigning the trivial \( G \)-action to every space.

### A.3. Homotopy of algebras

Using our action of spaces \( C \) we can now do (na"{i}ve) homotopy in \( \mathcal{A} \). For any algebra \( A \in \mathcal{A} \), we will use the following familiar terminology:

\[
\begin{align*}
A^{[0,1]} &= C([0, 1], A) : \text{the cylinder over } A \\
A^{[0,1]} &= C([0, 1], 0; A) : \text{the cone over } A \\
A^{[0,1]} &= C([0, 1], \{0, 1\}; A) : \text{the suspension of } A.
\end{align*}
\]

We will also write \( \Sigma : \mathcal{A} \to \mathcal{A} \), \( A \mapsto A^{[0,1]} := \Sigma A \), for the suspension functor. Note that the projection \( ([0, 1], \{0, 1\}) \to (S^{1}, 1) \) induces an isomorphism \( \Sigma \cong C(S^{1}, 1, ?) \) by Excision.

The inclusion \( \{t\} \subset [0, 1] \) induces the evaluation map

\[ \text{ev}_{t} : A^{[0,1]} \to C(\text{pt}; A) = A, \]

which is an epi by Extension. As a special case, the inclusion maps \( \{0\} \subset [0, 1] \subset ([0, 1], \{0\}) \) induce the cylinder extension of \( A \):

\[ A^{[0,1]} \to A^{[0,1]} \xrightarrow{\text{ev}_{0}} A. \]

(\text{the choice of 0 over 1 is of course just conventional). Similarly, the maps \( \{1\} \subset ([0, 1], \{0\}) \subset ([0, 1], \{0, 1\}) \) give rise to the cone extension of \( A \):

\[ A^{[0,1]} \to A^{[0,1]} \xrightarrow{\text{ev}_{1}} A. \]

Two parallel morphisms \( f, g : A \to B \) in \( \mathcal{A} \) are said to be homotopic, in symbols \( f \sim g \), if there exists a homotopy from \( f \) to \( g \), that is a morphism \( H : A \to B^{[0,1]} \) such that \( \text{ev}_{0} \circ H = f \) and \( \text{ev}_{1} \circ H = g \).

A.3.1. Lemma. Homotopy defines an equivalence relation on each \( \text{Hom}_{\mathcal{A}}(A, B) \), compatible with composition (i.e., it is a categorical ideal of \( \mathcal{A} \)).

Proof. Reflexivity is clear, and symmetry is obtained by exchanging the rôles of 0 and 1 via the map \( \tau : [0, 1] \to [0, 1], t \mapsto 1 - t \). Transitivity is given, as usual, by amalgamation of intervals, which takes here the following form. Apply \( C(\tau; B) \) to the push-out square of inclusions on the left hand side to obtain by Mayer-Vietoris the pull-back on the right:

\[
\begin{array}{ccc}
A^{[0,1]} & \xrightarrow{\text{id}} & A^{[0,1]} \\
C_{\frac{1}{2}(1+\epsilon)} & \xrightarrow{\epsilon} & C_{\frac{1}{2}(1+\epsilon)} \\
[0, 1] \cup \{0\} & \xrightarrow{0} & [0, 1] \\
B^{[0,1]} & \xrightarrow{\text{ev}_{1}} & B^{[0,1]} \\
C_{\frac{1}{2}(1+\epsilon)} \cup \{0\} & \xrightarrow{\text{ev}_{0}} & C_{\frac{1}{2}(1+\epsilon)} \\
\end{array}
\]

Let \( f, g, h \in \text{Hom}_{\mathcal{A}}(A, B) \). Given homotopies \( H \) and \( L : A \to B^{[0,1]} \) between \( f, g \) and \( g, h \) respectively, we have \( \text{ev}_{0}L = g = \text{ev}_{1}H \) and by the pull-back property
there exists a unique $M$ fitting into the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{M} & B^{[0,1]} \\
\downarrow & & \downarrow \\
B^{[0,1]} & \xrightarrow{L} & B^{[0,1]} \\
\downarrow & & \downarrow \\
B^{[0,1]} & \xrightarrow{ev_0} & B
\end{array}
\]

This $M$ is the required homotopy $f \sim h$.

As for compatibility with composition, consider morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
B & \xrightarrow{g_0} & C \\
\downarrow & & \downarrow \\
C & \xrightarrow{h} & D
\end{array}
\]

and a homotopy $H : B \to C^{[0,1]}$ between $g_0$ and $g_1$. Then $L := h^{[0,1]} \circ H \circ f : A \to D^{[0,1]}$ is a homotopy $h \circ f \sim h \circ g_1 f$. □

A.3.2. Definition. We can therefore define the **homotopy category** $A/\sim$, where morphisms are homotopy classes of morphisms in $A$.

A.3.3. Proposition. (a) The canonical functor $\text{can} : A \to A/\sim$, $f \mapsto [f]$, sends finite products to finite products in the homotopy category, and $0$ to a zero object.

(b) The functor $\text{Cpt}_2^{\text{op}} \times A \to A$ descends to the homotopy categories:

\[
\begin{array}{ccc}
\text{Cpt}_2^{\text{op}} \times A & \xrightarrow{C} & A \\
\downarrow \text{can} & & \downarrow \text{can} \\
\text{Cpt}_2/\sim \times A/\sim & \xrightarrow{C} & A/\sim
\end{array}
\]

Moreover $C(X,Y; ?) : A/\sim \to A/\sim$ still satisfies the Additivity axiom, i.e. it preserves $0$ and finite products, for any fixed $(X,Y) \in \text{Cpt}_2$. In particular, this holds for the suspension functor $\Sigma = C(S^1, 1; ?)$:

\[
\begin{array}{ccc}
A & \xrightarrow{\Sigma} & A \\
\downarrow \text{can} & & \downarrow \text{can} \\
A/\sim & \xrightarrow{\Sigma} & A/\sim
\end{array}
\]

(We still shamelessly denote the induced functors by $C$ and $\Sigma$.)

**Proof.** (a) It is obvious that $0$ is initial and final also in $A/\sim$. Of the universal property of a product in $A/\sim$ we only have to check uniqueness, since existence is clearly inherited from $A$. Let $p_B : B \leftarrow B \times C \to C : p_C$ be a product in $A$, and consider a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \text{can} & & \downarrow \text{can} \\
A/\sim & \xrightarrow{\Sigma} & A/\sim
\end{array}
\]

that commutes in the homotopy category. In other words, there are homotopies $H : p_B h \sim f$ and $L : p_C h \sim g$. By the Additivity axiom we can combine these into a homotopy

\[
A^{(H,L)} : B^{[0,1]} \times C^{[0,1]} \xrightarrow{\sim \text{Add}} (B \times C)^{[0,1]}
\]
between \( h \) and \((f,g): A \to B \times C\), because in \( \mathcal{A} \) we compute:

\[
e v_0(H,L) = (e v_0 H, e v_0 L) = (p c h, p g h) = h,
\]

\[
e v_1(H,L) = (e v_1 H, e v_1 L) = (f,g).
\]

Therefore any two maps \( h \) as above are homotopic.

(b) By the Excision axiom, each functor \( C(\; ; A): (\text{Cpt}_2)^{\text{op}} \to \mathcal{A} \) sends the vertical maps in

\[
(I \times X, I \times Y) \xrightarrow{H} (X', Y')
\]

\[
(I \times X/I \times Y, \text{pt}) = I+ \land (X'/Y', \text{pt}) \xrightarrow{\pi} (X'/Y', \text{pt})
\]
to isomorphisms and thus identifies homotopies \( H \) between maps \((X,Y) \to (X',Y')\) with the base-point preserving homotopies \( H \) of the corresponding maps in \( \text{Cpt}_2 \).

Hence it is enough to show that \( C(\; ; A) \) sends a base-point preserving homotopy in \( \text{Cpt}_2 \) between \( f,g : (X,x_0) \to (X',x'_0) \) to a homotopy in \( \mathcal{A} \) between the induced maps. This is clear, because a map \( H : I+ \land (X,x_0) \to (X',x'_0) \) goes to

\[
C(X', x'_0; A) \to C(I+ \land (X,x_0); A) = C(I+; C(X,x_0; A)) = C(X,x_0; A)[0,1],
\]

by the Excision and Action axioms, and this is what we decided to call a homotopy in \( \mathcal{A} \) between \( f^* \) and \( g^* \).

We still have to show that \( C \) preserves homotopies in the \( \mathcal{A} \)-variable, so let \( H : A \to B[0,1] \) be one. Then if we apply \( C(X,Y;?) \) we obtain a homotopy \( C(X,Y; A) \to C(X,Y; B)^{[0,1]} = C(X,Y; B)^{[0,1]} \) by the Action axiom and by the symmetry of the smash product in \( \text{Cpt}_2 \).

The statement about the induced functors \( C(X,Y;?) \) still satisfying Additivity is obvious, because these functors are calculated in \( \mathcal{A} \) and therefore send product diagrams to product diagrams.

We will also use the following observation:

**A.3.4. Lemma.** Consider a pull-back diagram in \( \mathcal{A} \)

\[
(A.3.5) \quad \begin{array}{ccc}
P & \to & B \\
\downarrow & & \downarrow q \\
A & \to & D
\end{array}
\]

where one of \( p \) or \( q \) is an epi. Let \( f_0 : T \to A \) and \( g_0 : T \to B \) be compatible morphisms inducing \((f_0, g_0) : T \to P\), and similarly let \( f_1, g_1 \) induce \((f_1, g_1) : T \to P\). If there are homotopies \( H : f_0 \sim f_1 \) and \( L : g_0 \sim g_1 \) compatible over \( D \), i.e. if \( p^{[0,1]} H = q^{[0,1]} L \), then also \((f_0, g_0) \sim (f_1, g_1)\).

**Proof.** Combine homotopies in the obvious way, using that \( C([0,1],?) \) sends the pull-back \( P \) to another pull-back by Lemma A.2.10. \( \Box \)

**A.4. Inverting the suspension**

There is a well-known easy way of formally inverting an endofunctor. We give the construction together with some general facts.

**A.4.1. Construction.** Let \( \mathcal{C} \) be any category and let \( F : \mathcal{C} \to \mathcal{C} \) be any functor. Define a new category \( \mathcal{C}_F \) having as objects all pairs \((A,n)\) with \( A \) an object of \( \mathcal{C} \) and \( n \in \mathbb{Z} \), and with \( \text{Hom} \) sets

\[
\text{Hom}_{\mathcal{C}_F}((A,n),(B,m)) := \text{colim}_{k \to \infty} \text{Hom}_{\mathcal{C}}(F^{n+k}A, F^{m+k}B).
\]
Composition and identities are the evident choices. There is a canonical functor $\text{can} : C \to C_F$ given by $A \mapsto (A,0)$. Moreover, $F' : (A,n) \mapsto (A,n+1)$ defines an automorphism of $C_F$ with inverse $F'^{-1}$ given by $(A,n) \mapsto (A,n-1)$. The automorphism $F'$ extends $F$ in the sense that there is an isomorphism of functors $\iota : \text{can} \circ F \Rightarrow F' \circ \text{can}$, given by the identity of $FA \in C$ considered as a morphism $\iota_A : (FA,0) \to (A,1)$.

\begin{equation}
\begin{array}{ccc}
C & \xrightarrow{\text{can}} & C_F \\
F & \cong & F' \\
C & \xrightarrow{\text{can}} & C_F 
\end{array}
\end{equation}

\text{(A.4.2)}

A.4.3. Remark. If $F : C \to C$ is a self-equivalence, the functor $\text{can} : C \to C_F$ is clearly fully-faithful and essentially surjective, hence an equivalence. Thus we see that by substituting the pair $(C,F)$ with $(C_F,F')$ one can always replace a mere self-equivalence with an automorphism.

This construction enjoys a nice 2-universal property, which we now make precise. Consider pairs $(C,F)$ consisting of a category together with an endomorphism. A compatible functor $(c,\varphi) : (C,F) \to (D,G)$ between such pairs consists of a functor $c : C \to D$ together with an isomorphism of functors $\varphi : cF \cong Gc$. We say that $(c,\varphi)$ is strictly compatible if $cF = Gc$ and $\varphi = \text{id}$. We can compose compatible functors in the evident way, and the composition of two strictly compatible ones is strictly compatible. A morphism of compatible functors $\alpha : (c,\varphi) \to (d,\psi)$ is a natural transformation $\alpha : c \Rightarrow d$ such that $G(\alpha)\varphi = \psi\alpha_F$ (cf. the definition of morphism of triangle functors).

We denote by $\text{Hom}((C,F),(D,G))$ the category with objects the compatible functors $(C,F) \to (D,G)$ and with morphisms thereof as arrows. Write

$\text{Hom}_{\text{str}}((C,F),(D,G))$

for the full subcategory of strictly compatible functors.

A.4.4. Theorem. (Universal property, [KV87, §2]). Keep notation as above. Then $\text{can},\iota$ is a compatible functor $(C,F) \to (C_F,F')$, and composition with $(\text{can},\iota)$ induces an isomorphism of categories

$\psi = \text{Hom}_{\text{str}}((C_F,F'),(D,G)) \cong \text{Hom}((C,F),(D,G))$

for every pair $(D,G)$ where $G$ is an automorphism.

\begin{proof}
Straightforward, if tiresome, verification (see [Dell04] if necessary).
\end{proof}

In particular, given a functor $c : C \to D$ with an automorphism $G : D \to D$ and an isomorphism $cF \cong Gc$, then there exists a unique functor $\tilde{c} : C_F \to D$ such that $\tilde{c} \circ \text{can} = c$ and $\tilde{c}F' = G\tilde{c}$.

We record another property.

A.4.5. Lemma. Let $J$ be any finite category (i.e. with finitely many morphisms). Then:

(a) If $C$ has all $J$-shaped limits and $F$ preserves them, then also $C_F$ has all $J$-shaped limits, and both $F'$ and can preserve such limits.

(b) The dual statement holds for $J$-shaped colimits.

\begin{proof}
Ditto.
\end{proof}

A.4.6. Remark. If $T = (T,T',E)$ is a triangulated category, one sees that the triple $(T_F,T',E')$ is a triangulated category too, where we define $E'$ to contain all triangles in $(T_F,T')$ isomorphic to $(-T')^n([f,g,i \circ h])$ for $[f,g,h] \in E$ and $n \in \mathbb{Z}$.
Then (can, i) : C → C_T is obviously a triangle functor, indeed a triangle equivalence ([KV87, §2], cf. Remark 1.1.5). By its 2-universal property, Construction A.4.1 can be used to substitute triangulated categories and triangle functors with triangulated categories whose translation is an automorphism and strictly compatible triangle functors.

Returning now to our admissible category of algebras A, we use the above construction to invert the suspension Σ.

A.4.7. PROPOSITION. Let (A, C) be as an admissible category of algebras (Def. A.2.5), and let A/∼ be its homotopy category. Then the category SW(A, C) := (A/∼)_Σ, obtained by formally inverting the suspension Σ : A/∼ → A/∼ as in Construction A.4.1, is an additive category. Moreover, the extended functor Σ' : SW(A, C) → SW(A, C) is additive.

A.4.8. DEFINITION. We call the category SW(A) := SW(A, C) the Spanier-Whitehead category of the admissible category A, and we persist in calling Σ' the suspension functor. We will happily and sloppily write Σ when meaning Σ' : SW(A) → SW(A) and Σ^n A (n ∈ ℤ) when meaning Σ^n can(A) = (A, n) ∈ SW and so on, generally omitting ‘can’ from the notation.

Proof of A.4.7. The argument is a very classical one, but we’d like to spell it out once more to show how formal it is. Recall that the pointed circle (S^1, 1) = Q([0, 1], {0, 1}) has a cogroup structure in the homotopy category Cpt/∼, where the comultiplication is given by the map δ : (S^1, 1) → (S^1, 1) ∨ (S^1, 1) which winds the circle successively along two copies of itself, and the inverse is τ : (S^1, 1) → (S^1, 1), t → 1 − t (and of course in the pointed category Cpt/∼ the counit ε : (S^1, 1) → (pt, pt) = 0 is forced). By definition Σ is C(S^1, 1 ; ?) : A/∼ → A/∼; since C preserves 0 and sends coproducts to products, we see that the natural transformations

\[ δ^* : C(S^1, 1 ; ?) × C(S^1, 1 ; ?) = C((S^1, 1) ∨ (S^1, 1) ; ?) \rightarrow C(S^1, 1 ; ?) \]

\[ τ^* : C(S^1, 1 ; ?) \rightarrow C(S^1, 1 ; ?) \]

make (Σ, δ^*, τ^*) a group object in (End(A/∼), ×, 0), that is, they equip the sets Hom_{A/∼}(A, ΣB) with a group structure natural in A and B. The next well-known lemma implies that the bifunctor Hom_{A/∼}( · , Σ^n(·)) (n ≥ 2) lands in the category of abelian groups and that Σ preserves this group structure. Hence Hom_{SW}( · , Σ^n(·)) lifts to abelian groups, in other words SW is a Z-category. (Alternatively, one may find a homotopy showing that the higher spheres (S^n, pt) are abelian cogroups.) Finally, in A/∼ we have a 0 object and all finite products, so by Lemma A.4.5 we have them also in SW, and they are preserved by can : A/∼ → SW and Σ : SW → SW. In a Z-category with 0, finite products automatically provide direct sums. Hence SW is additive and Σ : SW → SW is an additive functor.

A.4.9. LEMMA. (cf. [Hel69, Cor. 7.2]). Let C be a category with finite products and 0, and let G : C → C be an endofunctor preserving them. If (G, μ, ε) is a group structure on G, then μG = Gμ : G^2 × G^2 ⇒ G^2, εG = G1 : G^2 ⇒ G^2, and moreover G^2 = G ◦ G equipped with this structure is an abelian group.

Proof. Since μ : G × G ⇒ G is natural and G preserves products, we have a commutative diagram

\[ G(G × G × G) \rightarrow G(G × G) × G(G × G) \rightarrow G(G × G) \]

\[ G(G × G) \rightarrow G^2 × G^2 \rightarrow G^2 \]

\[ G′ \]

\[ G′ \]

\[ G′ \]

\[ G′ \]
For every $A, B \in C$, this means that the two group structures on the set $\text{Hom}_C(A, G^2B)$, one given by $\mu G$ and the other by $G\mu$, are mutually distributive, in the sense that they satisfy the equation

$$ (x *_{\mu G} y) *_{G\mu} (x' *_{\mu G} y') = (x *_{G\mu} x') *_{\mu G} (y *_{G\mu} y'). $$

Moreover, $*_{\mu G}$ and $*_{G\mu}$ share a common two-sided neutral element, namely $A \to 0 \to G^2B$. By a short calculation ([Spa66, Thm. 1.6.8]), this is enough to show that the two group structures coincide and are commutative. □

A.4.10. Remark. We recall that the sum in the Hom sets of an additive category is unique, because it can be recovered from the direct sums. Any other (co)group object would have done the same job.

A.5. The triangulation

Next we endow the additive category with automorphism $(\text{SW}(A, C), \Sigma^{-1})$ with the structure of a triangulated category. Thus the translation functor will be $\Sigma^{-1}$. Nonetheless, we want to keep using $\Sigma$, so we will write triangles in the form $\Sigma X \to X'' \to X' \to X$.

Let $f : A \to B$ be any morphism in $A$. The mapping cone of $f$, written $\text{cone}(f)$, is defined to be the pull-back of the lower right corner $B^{[0,1]} \to B \leftarrow A$: $\Sigma B \xrightarrow{\epsilon(f)} \text{cone}(f) \xrightarrow{\epsilon(f)} A \xrightarrow{f} B$.

The lower line is the cone extension of $B$, and it is easily seen that the kernel of the epi $\epsilon(f)$ is isomorphic to $\Sigma B$. Thus the upper line is also exact, and will be called the cone extension of $f$. The cone triangle of $f$ is the following diagram in $A$ (or its image in $\text{SW}$):

$$ (A.5.1) \quad \Sigma B \xrightarrow{\epsilon(f)} \text{cone}(f) \xrightarrow{\epsilon(f)} A \xrightarrow{f} B. $$

A.5.2. Definition. A distinguished triangle in $\text{SW}$ is a diagram

$$ \Sigma X \to X'' \to X' \to X $$

which is isomorphic in $\text{SW}$ to the image under $(-\Sigma)^n$, for some $n \in \mathbb{Z}$, of some cone triangle as in (A.5.1).

A.5.3. Theorem. Let $A = (A, C)$ be an admissible category of algebras (Def. A.2.5). Then the following are true.

(a) The Spanier-Whitehead category $\text{SW}(A) := \text{SW}(A, C) := (A/\sim)_\Sigma$ together with the inverse suspension $\Sigma^{-1}$ as translation functor and with the triangulation of Def. A.5.2 is a triangulated category. In particular, by choosing $A := \text{Cpt}^{\text{op}}$ and $C := Q$ (Ex. A.2.18), we see that $\text{SW}(\text{Cpt}^{\text{op}})$ is a triangulated category (and so is the opposite category, the original Spanier-Whitehead category).

(b) The functor $C$ extends to a triangle bifunctor of the Spanier-Whitehead categories, which is an action of $\text{SW}(\text{Cpt}^{\text{op}})$ on $\text{SW}(A)$:

$$ \begin{array}{ccc}
\text{Cpt}^{\text{op}} \times A & \xrightarrow{C} & A \\
\downarrow & & \downarrow \\
\text{SW}(\text{Cpt}^{\text{op}}) \times \text{SW}(A) & \xrightarrow{C} & \text{SW}(A).
\end{array} $$
(c) If $\mathcal{A}$ is an admissible tensor category of algebras (Def. A.2.11), then its tensor structure extends to a tensor structure on the suspension category, so that the canonical functor $\mathcal{A} \to \text{SW}(\mathcal{A})$ is a tensor functor.

$$
\begin{array}{ccc}
\mathcal{A} \times \mathcal{A} & \overset{\otimes}{\to} & \mathcal{A} \\
\downarrow & & \downarrow \\
\text{SW}(\mathcal{A}) \times \text{SW}(\mathcal{A}) & \overset{\otimes}{\to} & \text{SW}(\mathcal{A})
\end{array}
$$

If moreover $\otimes$ preserves mapping cones in each variable (that is, if each $A \otimes (\cdot)$ takes the pull-back square defining cone$(f)$ to a pull-back square defining cone$(A \otimes f)$), then it extends canonically to a structure of tensor triangulated category on $\text{SW}(\mathcal{A})$, as in Def. 2.1.1. In terms of $\Sigma = T^{-1}$, this requires that $\gamma_{\Sigma, \Sigma} = -1_{\Sigma \otimes \Sigma}$, where $\gamma$ is the symmetry isomorphism.

The following, often used, compatibility also holds: there are in $\text{SW}(\mathcal{A})$ natural isomorphisms $r_{A,B}: \Sigma(A \otimes B) \to \Sigma(A) \otimes B$ and $\ell_{A,B}: \Sigma(A \otimes B) \to A \otimes \Sigma(B)$ making $\otimes$ a triangle functor in the two variables, and such that the following two triangles commute, and the square anticommutes:

$$
\begin{array}{ccc}
\Sigma(1 \otimes A) & \overset{\lambda_{\Sigma, A}}{\to} & \Sigma A \\
\downarrow & & \downarrow \\
\Sigma(\Sigma A \otimes 1) & \overset{\rho_{\Sigma, A}}{\to} & \Sigma A \otimes \Sigma
\end{array}
\quad
\begin{array}{ccc}
\Sigma(A \otimes 1) & \overset{\Sigma \rho_{\Sigma, A}}{\to} & \Sigma(A \otimes B) \\
\downarrow & & \downarrow \\
\Sigma\Sigma(A \otimes B) & \overset{\Sigma \rho_{\Sigma, A} \rho_{\Sigma, B}}{\to} & \Sigma A \otimes \Sigma B
\end{array}
$$

(here $\lambda : 1 \otimes X \cong X$ and $\rho : X \otimes 1 \cong X$ are the unit isomorphisms of the tensor structure).

**Proof of Theorem A.5.3, part (a).** We already know that $\text{SW}$ is an additive category and that $\Sigma$ is an additive functor (A.4.7). Axiom (TR1) is trivially satisfied: the collection of distinguished triangles is replete by its very definition, every morphism fits in a distinguished triangle (because every morphism in $\text{SW}$ is an iterated (de)suspension of the canonical image of some $f$ in $\mathcal{A}$), and for every object $(A, n) \in \text{SW}$ the following triangle is distinguished:

$$
\begin{array}{ccc}
\Sigma(A, n) & \to & 0 \\
\downarrow & & \downarrow \\
(A, n) & \overset{\text{id}}{\to} & (A, n)
\end{array}
$$

(because cone$(1_A) = A^{[0,1]} \sim 0$), where the last homotopy is induced by a based homotopy $(\text{pt}, \text{pt}) \sim ([0,1], 0)$).

We now check the other axioms of a triangulated category in a series of lemmas. The following is an immediate corollary of Lemma A.2.10.

**A.5.5. Lemma.** The suspension $\Sigma : \mathcal{A} \to \mathcal{A}$ sends a cone triangle (in $\mathcal{A}$) to another cone triangle, up to signs. In particular, it respects cones: $\Sigma\text{cone}(f) = \text{cone}(\Sigma f)$. More precisely, $\Sigma$ sends the pull-back defining cone$(f)$ to that which defines cone$(\Sigma f)$.

**A.5.6. Lemma (Rotation axiom).** A triangle $(u'', u', u) : \Sigma X \to X'' \to X' \to X$ in $\text{SW}$ is distinguished if and only if the “rotated triangle” $(-\Sigma u, u'', u') : \Sigma X' \to \Sigma X \to X'' \to X'$ is.
A.5. THE TRIANGULATION

Proof. By the definition of distinguished triangle and by Lemma A.5.5, it suffices to consider the following situation in \(A\).

\[
\begin{array}{c}
A^{[0,1]} \xrightarrow{f^{[0,1]}_\tau} B^{[0,1]} \xrightarrow{\epsilon} \text{cone}(f) \xrightarrow{\epsilon} A \xrightarrow{f} B \\
\end{array}
\]

The first line is the cone triangle of \(f\), prolonged to the left. Here \(\tau : A^{[0,1]} \rightarrow A^{[0,1]}\) denotes the ‘inverting’ morphism induced by \(t \mapsto 1 - t\), so that the canonical image of \(f^{[0,1]}_\tau \circ \tau\) is \(\Sigma \text{can}(f)\) in \(\text{SW}\). The second line is the cone triangle of \(\epsilon : \text{cone}(f) \rightarrow A\), and \(\theta = (0, \iota)\) is given by the pull-back defining \(\text{cone}(\epsilon)\). Note also that \(\iota(\epsilon) = (\omega, 0)\) by definition, where \(\omega : A^{[0,1]} \rightarrow A^{[0,1]}\) denotes here the natural morphism induced by \(((0, 1), 0) \subset ([0, 1], [0, 1])\). Thus the square \(\mathfrak{A}\) commutes by definition. It is clear that the ‘only if’ implication in the lemma follows from applying the ‘if’ implication twice. So if we can prove that square \(\mathfrak{B}\) homotopy commutes and that \(\theta\) is a homotopy equivalence, we are done, since the candidate triangle in the first row would than be isomorphic in \(\text{SW}\) to a cone triangle and would therefore be distinguished. This is proved in the next two lemmas.

A.5.7. Lemma. Square \(\mathfrak{A}\) commutes up to homotopy.

Proof. By the definition of cones we have the two contiguous pull-back squares

\[
\begin{array}{c}
\xymatrix{ \text{cone}(\epsilon) \ar[r]^{\iota(\epsilon)} & \text{cone}(f) \ar[r]^j & B^{[0,1]} \\
A^{[0,1]} \ar[u]_{\iota} \ar[r]^{\iota(\epsilon)} & A \ar[u]_{\iota(\epsilon)} \ar[r]^j & B \ar[u]_{\iota(\epsilon)} }
\end{array}
\]

Therefore the outer square is also a pull-back, and we will write \(\text{cone}(\epsilon) = A^{[0,1]} \times_B B^{[0,1]}\). In these coordinates we have

\[
\iota(\epsilon) = (\omega, 0) \quad \text{and} \quad \theta f^{[0,1]} = (0, \iota) = (0, f^{[0,1]}).\]

We want to construct a homotopy \(H : (\omega, 0) \sim (0, f^{[0,1]}), \) i.e. a morphism \(H : A^{[0,1]} \rightarrow \text{cone}(\epsilon)^{[0,1]}\) with

\[
\text{ev}_0 H = (\omega, 0) \quad \text{and} \quad \text{ev}_1 H = (0, f^{[0,1]}).
\]

Because of Lemma A.2.10 we can further identify

\[
\text{cone}(\epsilon)^{[0,1]} = A^{[0,1]} \times_B [0, 1]^{[0,1]}.
\]

by applying the cone functor \((?)^{[0,1]}\) to the outer pull-back in (A.5.8). We will define our homotopy as \(H := (h_1, h_2)\) for appropriate morphisms into the two factors. Their definition requires some surgery (or rather patchwork), as follows.

By the Action axiom, for every algebra \(D\) we can identify \((D[0, 1])^{[0,1]}\) with \(D^{[0,1]_2 \times 0} \times [0, 1])\). We will use the notation \(t \in [0, 1]\) and \(r \in [0, 1]\) respectively for the first and second coordinates in the product \([0, 1]_2\) (\(r\) will eventually be the ‘homotopy’ coordinate). Now subdivide \([0, 1]_2^2\) as follows:

\[
\begin{align*}
Z & := \{(t, r) \mid 0 \leq t \leq r\} \subset [0, 1]_2 \\
W & := \{(t, r) \mid r \leq t \leq 1\} \subset [0, 1]_2.
\end{align*}
\]

There is a push-out of pairs

\[
\begin{array}{c}
\xymatrix{ ([0, 1]_2^2, 0 \times [0, 1]) \ar[r]_{\iota} & (Z, 0 \times [0, 1]) \\
(W, 0 \times 0) \ar[u] \ar[r] & (W \cap Z, 0 \times 0) \ar[u] }
\end{array}
\]
and therefore by Mayer-Vietoris a pull-back of algebras
(A.5.10)\[ \begin{array}{c}
A[0,1]^{[0,1]} \xrightarrow{} A^Z-0\times[0,1] \\
\downarrow \hspace{1cm} \downarrow \\
A^W-0\times0 \xrightarrow{} A^W\cap Z-0\times0.
\end{array} \]

Define \( h_1 := (0, \alpha_1) : A[0,1] \to A[0,1]^{[0,1]} \) via pull-back (A.5.10) with components \( 0 : A[0,1] \to A^Z-0\times[0,1] \) and
\[ \alpha_1 : A[0,1]^{[0,1]} \xrightarrow{(t,r) \mapsto t-r} A^W-0\times0. \]
Similarly, the subdivision of \([0,1]^2\) into the subsets
\( Z' := \{(t,r) \mid 0 \leq t \leq 1 - r\} \)
\( W' := \{(t,r) \mid 1 - r \leq t \leq 1\} \)
yields the pull-back square
(A.5.11)\[ \begin{array}{c}
B[0,1]^{[0,1]} \xrightarrow{} B^Z-0\times[0,1] \\
\downarrow \hspace{1cm} \downarrow \\
B^W-0\times1 \xrightarrow{} B^W\cap Z'-0\times1.
\end{array} \]
We define \( h_2 := (0, \alpha_2) \) via pull-back (A.5.11) with components 0 and
\[ \alpha_2 : A[0,1]^{[0,1]} \xrightarrow{(t,r) \mapsto t-r} B^W-0\times1 \xrightarrow{f^W-0\times1} B^W-0\times1, \]
It is then straightforward to check that \( H = (h_1, h_2) \) is well-defined. Moreover
\[ \text{ev}_t^r H = (\text{ev}_t^r h_1, \text{ev}_t^r h_2) = (\omega, 0), \]
\[ \text{ev}_t^r H = (\text{ev}_t^r h_1, \text{ev}_t^r h_2) = (0, \text{ev}_t^r \alpha_2) = (0, f(t,0,1, \tau)). \]
Thus, square \( \mathfrak{A} \) homotopy commutes.

A.5.12. REMARK. In the case of \( C^*\)-algebras (Ex. A.2.13), in order to define the homotopy \( H \) we could have just given an explicit formula such as
\[ A[0,1]^{[0,1]} \xrightarrow{H} \text{cone}(\epsilon)^{[0,1]} = \left(A[0,1] \times_B B[0,1]\right)^{[0,1]} \]
\[ a_t \mapsto \left( \begin{array}{c}
0 \\
a_{t-r}
\end{array} \right) \begin{cases} 0 & 0 \leq t \leq r \\ 1-r & r \leq t \leq 1 \end{cases} \times \begin{cases} 0 & 0 \leq t \leq 1 - r \\ 1-r & 1-r \leq t \leq 1 \end{cases} \] \( f_{a_{2t-2t}} \).

Here something like \( a_t \in A^X \) denotes a general element, i.e. a map \( X \to A \), while \( a_{u(t)} \) simply denotes the composite \( a \circ u : X \to X \to B \). The image \( H(a_t) \) is here written as a function of \( r \in [0,1] \) with values in the pull-back cone(\( \epsilon \)).

In a general category \( \mathcal{A} \) as in Definition A.2.5 we cannot work with elements and functions, but the reasoning used to find the homotopies is exactly the same. In fact, the homotopy given in the proof is the translation in categorical terms of the above explicit formula.

A.5.13. LEMMA. The morphism \( \theta = (0, \iota) : B[0,1] \to \text{cone}(\epsilon) \) is a homotopy equivalence.

A.5.14. REMARK. In the case of \( C^*\)-algebras, in order to prove Lemma A.5.13 we would define a homotopy inverse of \( \theta \) by
\[ \text{cone}(\epsilon) = A[0,1] \times_B B[0,1] \xrightarrow{\theta'} B[0,1], \]
\[ (a_t, b_t) \mapsto \begin{cases}
b_{2t} & 0 \leq t \leq \frac{1}{2} \\ f_{a_{2t-2t}} & \frac{1}{2} \leq t \leq 1 \end{cases}. \]
Given in \( \mathcal{A} \), there is a morphism making the two squares on its side commute.

By Mayer-Vietoris there is a pull-back square (A.5.16) \( \theta \). This is straightforward and is left to the reader. Thus \( \theta \) is a homotopy equivalence, and this ends the proof of Lemma A.5.6.

**Lemma (Morphism axiom).** Given in \( \mathbf{SW}(\mathcal{A}) \) the solid arrow diagram

\[
\begin{array}{c}
\Sigma X \\
\Sigma u
\end{array}
\quad \begin{array}{c}
X'' \\
\epsilon
\end{array}
\quad \begin{array}{c}
X' \\
u
\end{array}
\quad \begin{array}{c}
X
\end{array}
\quad \begin{array}{c}
\Sigma Y \\
\Sigma v
\end{array}
\quad \begin{array}{c}
Y'' \\
\epsilon'
\end{array}
\quad \begin{array}{c}
Y' \\
v'
\end{array}
\quad \begin{array}{c}
Y
\end{array}
\]

with distinguished lines and commutative square \( \epsilon \), there always exists a dotted morphism making the two squares on its side commute.

**Proof.** (Lazy readers may want to skip to Remark A.5.23 right now.) Again by Lemma A.5.5, to prove the Morphism axiom it is enough to prove the following. Given in \( \mathcal{A} \) the solid-arrow diagram

\[
\begin{array}{c}
B^{[0,1]} \\
g^{[0,1]}
\end{array}
\quad \begin{array}{c}
\conef \quad \text{cone}(f) \\
1
\end{array}
\quad \begin{array}{c}
B^{[0,1]} \\
B'
\end{array}
\quad \begin{array}{c}
\conef' \\
2
\end{array}
\quad \begin{array}{c}
\conef' \\
\text{cone}(f')
\end{array}
\quad \begin{array}{c}
A' \\
h
\end{array}
\quad \begin{array}{c}
A \\
f
\end{array}
\quad \begin{array}{c}
\conef \\
\text{cone}(f)
\end{array}
\quad \begin{array}{c}
B \\
g
\end{array}
\]

where the lines are cone triangles, and where the rightmost square homotopy commutes, then there is a \( \gamma \) which makes the squares \( \begin{array}{c} 1 \\ 2 \end{array} \) homotopy commute. (Note that we are working in \( \mathcal{A} \).)

Choose a homotopy \( H : gf \sim f'h : A \to B^{[0,1]} \). We construct \( \gamma \) as follows. Let \( \alpha \) be the composition

\[\alpha : \begin{array}{c}
\conef \\
\text{cone}(f)
\end{array} \to \begin{array}{c}
A \\
f
\end{array} \to \begin{array}{c}
B^{[0,1]} \\
h
\end{array} \to \begin{array}{c}
B' \\
g
\end{array} \]

We evaluate for this map:

(A.5.16) \[\text{ev}_1 \alpha = \text{ev}_1 H \epsilon = f'h \epsilon,\]

(A.5.17) \[\text{ev}_2 \alpha = \text{ev}_0 H \epsilon = g f \epsilon.\]

By Mayer-Vietoris there is a pull-back square

(A.5.18) \[
\begin{array}{c}
B^{[0,1]} \\
\conef'
\end{array}
\quad \begin{array}{c}
B'^{[2,1]} \\
\text{ev}_1'
\end{array}
\quad \begin{array}{c}
B' \\
\text{ev}_2'
\end{array}
\quad \begin{array}{c}
B^{[0,1]} \\
\conef'
\end{array}
\]

Then

\[
\begin{array}{c}
B^{[0,1]} \\
b_t
\end{array}
\quad \begin{array}{c}
\{ b_t(2-r) \\
0 \leq t \leq \frac{1}{2}(r+1) \\
\frac{1}{2}(r+1) \leq t \leq 1 \}
\end{array}
\quad \begin{array}{c}
(\mathcal{A}^{[0,1]})^{[0,1]} \\
\{ b_t(r-2) \\
0 \leq t \leq 1 - r \\
1 - r \leq t \leq 1 \}
\end{array}
\quad \begin{array}{c}
B^{[0,1]} \\
\{ b_t r \\
0 \leq t \leq \frac{r+1}{2} \\
\frac{r+1}{2} \leq t \leq 1 \}
\end{array}
\]

defines a homotopy \( \theta' \sim 1_{B^{[0,1]}} \) and

\[
\begin{array}{c}
\text{cone}(\epsilon) \\
\text{cone}(\epsilon)^{[0,1]}
\end{array}
\quad \begin{array}{c}
(a_t, b_t) \\
\{ 0 \\
0 \leq t \leq 1 - r \\
1 - r \leq t \leq 1 \}
\end{array}
\quad \begin{array}{c}
(\mathcal{A}^{[0,1]})^{[0,1]}
\end{array}
\quad \begin{array}{c}
(a_t, b_t) \\
\{ 0 \\
0 \leq t \leq 1 - r \\
1 - r \leq t \leq 1 \}
\end{array}
\quad \begin{array}{c}
\text{cone}(\epsilon)^{[0,1]}
\end{array}
\quad \begin{array}{c}
(\mathcal{A}^{[0,1]})^{[0,1]}
\end{array}
\quad \begin{array}{c}
(\mathcal{A}^{[0,1]})^{[0,1]}
\end{array}
\]

a homotopy \( \theta' \sim 1_{\text{cone}(\epsilon)} \) (here again \( r \) is the 'homotopy variable' \( \in [0,1] \)).
Consider the morphism
\[ \alpha' : \text{cone}(f) \to B^{[0,1]} \xrightarrow{\gamma} B^{[0,1]} \to B^{[0,1]} \xrightarrow{\epsilon} B^{[0,1]}, \]
where \( \tilde{f} \) is the pull-back of \( f : A \to B \) along \( \epsilon_1 : B^{[0,1]} \to B \). One sees that \( \epsilon_1 \alpha' = \epsilon_1 g^{[0,1]} \tilde{f} = g f \epsilon \) (A.5.17) \( \epsilon_1 \alpha \), so the pull-back (A.5.18) induces a morphism \( \beta : \text{cone}(f) \to B^{[0,1]} \). Because of (A.5.16), we have
\[ (A.5.19) \quad \epsilon_1 \beta = f' \epsilon : \text{cone}(f) \to B'. \]
Finally, thanks to this last equality we can set \( \gamma := (\beta, h \epsilon) \) by the pull-back defining \( \text{cone}(f') \):

Since \( h \epsilon = \epsilon' \gamma \), we see that \( [2] \) actually commutes in \( A \).

We will now show that \( [1] \) commutes up to homotopy, i.e. that \( \gamma \epsilon \sim \epsilon' g^{[0,1]} \).
Notice first that
\[ (A.5.20) \quad \gamma \epsilon = (\beta, h \epsilon) \epsilon = (\beta \epsilon, 0) \quad \text{(since } \epsilon l = 0) \]
\[ (A.5.21) \quad \text{and } \epsilon' g^{[0,1]} = (\omega, 0) \circ g^{[0,1]} = (\omega g^{[0,1]}, 0), \]
where \( \omega : B^{[0,1]} \to B^{[0,1]} \) is induced by \( \{[0,1], 0\} \subset ([0,1], \{0,1\}) \) and where \( \epsilon' = (\omega, 0) \) holds by definition. Therefore, by (A.5.20), (A.5.21) and Lemma A.3.4 it suffices to find a homotopy \( L : \beta t \sim \omega g^{[0,1]} \) between the first components \( B^{[0,1]} \to B^{[0,1]} \).

As before, we use the Action axiom to identify \( B'[0,1][0,1] \) with \( B'[0,1][0,1] \). We will use the notation \( t \in [0,1] \) and \( r \in [0,1] \) respectively for the two coordinates.
Subdivide \( [0,1]^2 \) as follows:
\[ Z := \{(t,r) \mid 0 \leq t \leq \frac{1}{2} (r + 1)\} \subset [0,1]^2 \]
\[ W := \{(t,r) \mid \frac{1}{2} (r + 1) \leq t \leq 1\} \subset [0,1]^2. \]
There is a push-out of pairs
\[ (0,1)[0,1] (0 \times [0,1]) \]
and therefore by Mayer-Vietoris a pull-back of algebras
\[ (A.5.22) \quad \xrightarrow{\zeta} \quad B'[0,1][0,1] \to B'W(0 \times [0,1]) \]
\[ \xrightarrow{\zeta} \quad B'Z \to B'Z \to W. \]

Let \( c : B[0,1] \to B[0,1][0,1] \) be induced by the projection \( [0,1] \to pt \), and let \( \phi : B'[0,1][0,1] \to B'W(0 \times [0,1]) \) be the morphism induced by the obvious homeomorphism of pairs \( (0,1)^2, 0 \times [0,1] \simeq (W, 0 \times [0,1]) \). Then we can define \( \delta \) to be the
composition
\[ \delta : B[0,1] \xrightarrow{\epsilon} B[0,1][[0,1]] \xrightarrow{g[[0,1]]} B'[0,1][[0,1]] \xrightarrow{\delta} B''[[W+0\times[0,1]]]. \]

Since clearly \( \zeta \delta = 0 \), the pull-back (A.5.22) defines a morphism \( L := (\delta,0) : B[0,1] \to B'[0,1][[0,1]] \). Now it only remains to show that \( \epsilon \nu_0L = \beta \epsilon \) and \( \epsilon \nu_1L = \omega g[[0,1]] \), and we are done (where these \( \epsilon \nu_i \) are taken with respect to \( r \)). But these identities are clear from the definitions.

A.5.23. Remark. (cf. Remark A.5.12.) Once again, in the case of \( C^\ast \)-algebras, in order to define \( \gamma : \text{cone}(f) \to \text{cone}(f') \) we could have just written:
\[ \gamma : \text{cone}(f) = B[[0,1]]_B A \longrightarrow (B[[0,1]]_B A') = \text{cone}(f') \]
\[ (b_t,a) \longmapsto \left\{ \begin{array}{l} \tilde{b} \circ b_{2t} \quad 0 \leq t \leq \frac{1}{2} \\ H_{2t-1} \quad \frac{1}{2} \leq t \leq 1 \end{array} \right\}, \phi(h) \]

A similar explicit formula defines a homotopy \( L' : \gamma_t \sim \gamma_t'[[0,1]] \):
\[ L' : B[0,1] \longrightarrow (B[[0,1]]_B A')[[0,1]] = \text{cone}(f')[[0,1]] \]
\[ b_t \longmapsto \left\{ \begin{array}{l} \tilde{g} \circ b_{2t/(r+1)} \quad 0 \leq t \leq (r+1)/2 \\ (r+1)/2 \leq t \leq 1 \end{array} \right\}, 0 \).

The image \( L'(b_t) \) is here a function of \( r \in [0,1] \) with values in the pull-back \( \text{cone}(f') \).

A.5.24. Lemma (Octahedron axiom). Given \( f : A \to B, g : B \to C \) and \( h = gf : A \to B \) in \( SW(A) \), and given distinguished triangles \( [f',f],[g',g] \) and \( [h'',h',h] \), then there exist morphisms \( \alpha : \text{cone}(gf) \to \text{cone}(g) \) and \( \beta : \text{cone}(f) \to \text{cone}(gf) \) such that:
(i) \([f'g',\beta,\alpha] \) is distinguished,
(ii) \( \alpha h'' = \Sigma g' \) and \( h' \beta = f' \),
(iii) \( fh' = g' \alpha \) and \( h'' \Sigma (g) = \beta f'' \).

Proof. We give the proof in terms of elements and functions. By the usual arguments, we can assume that \( f, g \) are morphisms in \( A \) and that \( h = fg \) commutes in \( A \), and the general case can be reduced to the following commutative diagram of algebras, where the lines and columns are mapping cone triangles and the maps \( k \) and \( \ell \) are induced by the functoriality of pull-backs.

\[
\begin{array}{c}
\Sigma \text{cone}(g) \xrightarrow{\Sigma \epsilon(g)} \Sigma B \\
\downarrow \epsilon(g) \downarrow \downarrow \\
\text{cone}(k) \xrightarrow{\epsilon(k)} \text{cone}(f) \\
\downarrow \epsilon(k) \downarrow \\
\Sigma C \xrightarrow{\epsilon(gf)} \text{cone}(gf) \xrightarrow{\epsilon(af)} A \xrightarrow{gf} C \\
\downarrow \downarrow \\
\Sigma C \xrightarrow{\epsilon(g)} \text{cone}(g) \xrightarrow{\epsilon(g)} B \xrightarrow{g} C
\end{array}
\]

We claim that \( \ell \) is a homotopy equivalence. Namely, writing
\[ \text{cone}(k) = (C[[0,1]]_C B)[[0,1]]_B A, \]
we simply have $\ell : (c_{t,s}, b_s, a) \mapsto (b_s, a)$, so that the morphism
\[
\ell' : \text{cone}(f) \to \text{cone}(k)
\]
\[
(b_s, a) \mapsto (g(b_{st}), b_s, a)
\]
defines a section of $\ell$. Then the morphism
\[
H : \text{cone}(k) \to \text{cone}(k)^{(0,1]}
\]
\[
(c_{t,s}, b_s, a) \mapsto (c_{r+(1-r)t, rst+(1-r)s}, b_s, a),
\]
(with $r \in [0, 1]$) is a homotopy $1_{\text{cone}(k)} \sim \ell\ell'$. Thus $\ell'$ is the inclusion of a homotopy retract, and in particular $\ell, \ell'$ are homotopy inverse to each other. Of the commutativity conditions in (ii) and (iii), only $h''r\Sigma(g) = \beta f''$ is not displayed in the above diagram. Here it translates as the equation $\epsilon(gf)\Sigma(y) = \epsilon(k)\ell'\epsilon(f)$ (in $\mathcal{A}$), which is easily seen to hold. □

**Proof of Theorem A.5.3, Part (c).** Let $(\mathcal{M}, \otimes, \mathbb{I})$ be any tensor category, and let $s$ be any object of $\mathcal{M}$ (we denote the objects with lower case letters for practical reasons). One can always formally invert $s$ — by which one means to invert the endofunctor $s\otimes? : \mathcal{M} \to \mathcal{M}$, as in Construction A.4.1. We'll see now a necessary and sufficient condition on $s$ for the monoidal structure to extend to $\mathcal{M}_s := \mathcal{M}_{(s\otimes)}$.

Write $\alpha : x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$, $\rho : x \otimes \mathbb{I} \cong x$, $\lambda : \mathbb{I} \otimes x \cong x$ and $\gamma : x \otimes y \cong y \otimes x$ for the given structural isomorphisms. Note that viewing the symmetry $\gamma$ as a transposition induces a canonical action of the symmetric group $S_n$ on each $n$-fold tensor product $x^\otimes$. (More generally: Mac Lane's coherence theorem for symmetric monoidal categories says that, given a list $(x_1, \ldots, x_n)$ of objects and permutations $\zeta, \xi \in S_n$, there is a unique morphism $\xi_j\zeta_i : \otimes_i x_i \to \otimes_j x_j$ constructed out of the structure maps of $\otimes$ — if one takes care to distinguish different occurrences of the same object; see [Mac98, XI §1] for details, or the pleasant [JS93]).

**A.5.25. Proposition.** Let $\mathcal{M}$ and $s$ be as above. Then it is possible to complement the formula $(a, n) \circ' (b, m) := (a \circ b, n + m)$ (in the evident way) to a monoidal structure $\circ'$ which lifts $\circ$ along the canonical functor
\[
\begin{array}{c}
\mathcal{M} \times \mathcal{M} \xrightarrow{\circ} \mathcal{M} \\
can \downarrow \quad \downarrow \text{can} \\
\mathcal{M}_s \times \mathcal{M}_s \xrightarrow{\circ'} \mathcal{M}_s \\
\end{array}
\]
if and only if the cyclic permutation of the factors $\sigma : s^\otimes \to s^\otimes$ becomes equal in $\mathcal{M}_s$ to the identity of $s^\otimes$.

**Proof.** See [Rio02, §5] for a proof and explanations. New structure maps
\[
\alpha' : (a, n) \circ' (b, m) \circ' (c, \ell) = (a \circ (b \circ c), n + m + \ell)
\]
\[
\to ((a \circ b) \circ c, n + m + \ell) = ((a, n) \circ' (b, m)) \circ' (c, \ell)
\]
\[
\lambda' : (\mathbb{I}, 0) \circ' (a, n) = (\mathbb{I} \circ a, n) \to (a, n)
\]
\[
\rho' : (a, n) \circ' (\mathbb{I}, 0) = (a \circ \mathbb{I}, n) \to (a, n)
\]
\[
\gamma' : (a, n) \circ' (b, m) = (a \circ b, n + m) \to (b \circ a, n + m) = (b, m) \circ' (a, m)
\]
are immediately produced from those of $\mathcal{M}$. (Clearly, if $\mathcal{M}$ is strict so is $\mathcal{M}_s$). Notice that an “evident way” to tensor two morphisms $\varphi : (a, n) \to (a', n')$ and $\psi : (b, m) \to (b', m')$ is the following. Recall that $\varphi$ is represented by some $f$:

\[\footnote{apparently attributed to Voevodsky} \]
s^{i+n} \odot A \to s^{i+n'} \odot A' \in \mathcal{M}$, and similarly $\psi = [g : s^{i+m} \odot B \to s^{i+m'} \odot B']$ (for some $i > -n, -n'$ and $j > -m, -m'$). One then defines $\varphi \odot' \psi : (a \odot b, n + m) \to (a' \odot b', n' + m')$ to be the class of the dotted arrow

$$s^{i+n} \odot a \odot s^{i+m} \odot b \overset{1 \otimes 1}{\longrightarrow} s^{i+j+n+m} \odot a \odot b$$

making the diagram commute in $\mathcal{M}$. By the proof in loc. cit., any other definition of $\varphi \odot' \psi$ using $\varphi, \psi$ and the structural isomorphisms (so that it yields a bifunctor agreeing with $\odot$) would yield the same morphism in $\mathcal{M}_s$.

Now consider an admissible tensor category of algebras $\mathcal{A}$. If we picture $(S^1, 1)^{\otimes 3} \in \text{Cpt}_s$ as the Alexandrov compactification $\mathbb{R}^3 \cup \infty$, we can recognize $\sigma$ as the map given by the cyclic permutation of the three cartesian coordinates in $\mathbb{R}^3$, and which leaves $\infty$ fixed. Rotation around the axis $\{x = y = z\}$ gives a homotopy $\text{id} \sim \sigma$ that leaves the axis and the basepoint $\infty$ fixed. Thus the cyclic permutation $\sigma : (S^1, 1)^{\otimes 3} \to (S^1, 1)^{\otimes 3}$ is homotopic (rel base-point) to the identity, and so its image $\sigma : C(S^1, 1)^{\otimes 3} \to C(S^1, 1)^{\otimes 3}$ equals the identity id, and we can apply the above proposition to extend the monoidal structure $\odot$ to $\text{SW}$.

Note that for every algebra $A$, the functor $A \otimes ? : A \to A$ commutes with the suspension functor $C(S^1, 1) \otimes ?$, simply via the symmetry $\gamma$ of $\otimes$. By hypothesis $\otimes : \text{SW} \times \text{SW} \to \text{SW}$ preserves cones (i.e., $D \otimes f$ sends the cone pull-back of $f$ to the cone pull-back of $D \otimes f$). This would already show that, on $\text{SW}$, the tensor functor is triangulated at each variable. But we want more.

Define the natural isomorphisms $\ell$ and $r$ as follows. Here $\alpha$ and $\gamma$ are, respectively, the associativity and symmetry isomorphisms of the extended $\otimes$ (we also write $\cdot$ instead of $\otimes$ for readability):

$$r_{A,B} := \alpha_{\Sigma, A, B} : \Sigma \cdot (A \cdot B) \longrightarrow (\Sigma \cdot A) \cdot B$$

$$\ell_{A,B} := \alpha_{\text{per}, A, B} : \Sigma \cdot (A \cdot B) \longrightarrow \Sigma \cdot (\Sigma \cdot B)$$

Notice that, by the coherence theorem, any other combination of the structural isomorphisms $\alpha$ and $\gamma$ would have defined the same two morphisms $r$ and $\ell$.

The commutativity of the two triangles in (A.5.4) is assured by the coherence theorem for $\alpha, \rho, \lambda, \gamma$ (the right triangle already by the coherence theorem for $\alpha, \rho, \lambda$ [Mac98, VII.2]).

Look at the following diagram, where we keep track of the two different occurrences of $\Sigma = C(S^1, 1)$ by denoting them $\Sigma_1$ and $\Sigma_2$ respectively.

$$\Sigma_1 \cdot (\Sigma_2 \cdot (A \cdot B)) \longrightarrow \Sigma_2 \cdot (\Sigma_1 \cdot (A \cdot B)) \longrightarrow (\Sigma_2 \cdot (A \cdot B))$$

$$\Sigma_1 \cdot (\Sigma_2 \cdot (A \cdot B)) \longrightarrow (\Sigma_2 \cdot (A \cdot B)) \longrightarrow (\Sigma_2 \cdot A) \cdot (\Sigma_2 \cdot B).$$

This diagram commutes by the coherence theorem, and is the same as the square in (A.5.4) except that here we need an extra $\gamma_{\Sigma, \Sigma}$ to get from $\Sigma_1 \Sigma_2 AB$ to $\Sigma_2 \Sigma_1 AB$. Thus, if we show that $\gamma : \Sigma^{\otimes 2} \to \Sigma^{\otimes 2}$ is equal to $-1$ in $\text{SW}$ (which we must prove
anyway for part (c) of the theorem), we have shown that the square in (A.5.4) anticommutes. This can be seen in Cpt*, by checking that the coordinate exchange $(S^1,1)^{op} 	o (S^1,1)^{op}$ which gives the symmetry $\gamma$ is (homotopic to) the inverse $\tau$ of the abelian cogroup object $(S^1,1)^{op}$, which yields $-1$. □

Proof of Theorem A.5.3. Part (b). We already know that $C$ descends to the homotopy categories (Prop. A.3.3). Now we extend it to the suspension categories in the only way that makes sense:

$$C'((X,Y,n),(A,m)) := (C(X,Y;A), n + m).$$

In order to prove that this is well defined, we use the 2-universal property of Theorticategories in the only way that makes sense: the homotopy categories (Prop. A.3.3). Now we extend it to the suspension category of the abelian cogroup object $(S^1,1)$ setting, this corresponds to inverting the canonical comparison $c$.

We have just seen that, given a pointed category $A$ with a suitable right action of compact spaces, one may produce a triangulated category

$$(A, \Sigma \text{Suspension}^{-1}, \text{cone triangles})$$

in the most ancestral topological way, by smashing the few necessary homotopies right onto the objects of $A$. This is easy and ‘cheap’, in the sense that it avoids Quillen model structures and simplicial methods. The main drawback of this construction is that $SW(A)$ doesn’t have infinite coproducts, so that only ‘few’ cohomology theories are representable in it.

Also, one may wish to find another triangulated category that better reflects the properties of the (co)homology theories that one wants to consider on $A$. But the latter is easy, and we already know how to do it. Namely, $SW(A)$ is a (tensor) triangulated category, so it can be localized at a suitable class of morphisms and it will still yield a (tensor) triangulated category (Lemma 2.1.21). For instance, when studying some kind of rings or algebras, the most useful theories are those which satisfy an excision theorem with respect to some class $\mathcal{E}$ of extensions: given an $E = (J \to A \to \text{cone}(p)) \in \mathcal{E}$, the long exact sequences produced by such theories will relate information on $A$ to information on the ideal $J$ and the quotient $Q$. In our setting, this corresponds to inverting the canonical comparison $u_E : J \to \text{cone}(p)$.
for all extensions in $E$. (The morphism $u_E$ is simply $u_E := (j, 0) : J \to A \times Q Q^{[0,1]} = \text{cone}(p)$.) In this way, every homological functor $H : \text{SW}(A)/\langle \text{cone}(u_E) | E \in E \rangle \to Ab$ gives rise to long exact sequences

$$\cdots \to H_{n+1}(Q) \to H_n(J) \to H_n(A) \to H_n(Q) \to H_{n-1}(J) \to \cdots$$

for extensions in the given class $E$.

As an illustration we consider $G$-$C^*$-algebras.

**A.6.1. Equivariant KK-theory.** Let $G$ be a second countable locally compact group, and consider the tensor category $A := G$-$C^*$ of separable $G$-$C^*$-algebras as in Example A.2.17. Write $\text{SW}^G$ for the Spanier-Whitehead category $\text{SW}(A)$. Define the following classes of morphisms in it:

$$\mathcal{E}_{\text{split}} := \{ u_E | E \text{ is a split extension in } A \}$$

$$\mathcal{E}_{\text{stable}} := \{ \text{ morphisms } \mathbb{K}(H_1) \to \mathbb{K}(H_1 \oplus H_2) \text{ as in Def. 4.2.4(iii) } \}$$

Consider the thick $\otimes$-ideal of $\text{SW}^G$ generated by their cones:

$$\mathcal{J} := \langle \text{cone}(f) | f \in \mathcal{E}_{\text{split}} \cup \mathcal{E}_{\text{stable}} \rangle$$

**A.6.1. Proposition.** The tensor triangulated category $\text{SW}^G/\mathcal{J}$ is equivalent to the equivariant Kasparov category $\text{KK}^G$ (Def. 4.2.5).

**Proof.** By definition, $F_{\text{univ}}$ factors through the homotopy category. We know (by Bott periodicity) that $\Sigma$ is a self-equivalence of $\text{KK}^G$, so the canonical functor $\text{KK}^G \to (\text{KK}^G)_\Sigma$ is an equivalence. Write $p$ for the functor $\text{SW}^G \to (\text{KK}^G)_\Sigma$ induced by the Spanier-Whitehead construction.

$$\begin{array}{ccc}
G-C^*_{\text{sep}} & \to & G-C^*_{\text{sep}/\mathcal{J}} \\
\downarrow F_{\text{univ}} & & \downarrow q \\
\text{KK}^G & \to & (\text{KK}^G)_\Sigma \\
\uparrow p & & \uparrow \sim \\
\text{SW}^G/\mathcal{J} & \to & (\text{KK}^G)_\Sigma
\end{array}$$

Let $E = (J \xrightarrow{j} A \xrightarrow{f} Q)$ be an extension in $G-C^*_{\text{sep}}$ and consider the commutative diagram

$$\begin{array}{ccc}
J & \to & A \\
\uparrow u_E & & \uparrow f \\
\Sigma Q & \to & \text{cone}(f) \\
\uparrow & & \uparrow \\
J & \to & A \\
\downarrow u_E & & \downarrow f \\
\Sigma Q & \to & A \\
\downarrow & & \downarrow \\
\Sigma Q & \to & Q.
\end{array}$$

The second row is a distinguished triangle in $\text{SW}^G$, so a right inverse of $f : A \to Q$ yields an isomorphism $\text{cone}(f) \oplus Q \cong A$ in $\text{SW}^G$ (Cor. 1.1.31); since $q$ inverts $u_E$, we have $J \oplus Q \cong A$ in $\text{SW}^G/\mathcal{J}$. Thus $q$ is split exact, and it is also $C^*$-stable by definition. Therefore it factors through $p$. We still don’t know that this is an isomorphism $(\text{KK}^G)_\Sigma \cong \text{SW}^G/\mathcal{J}$, because $q$ might still invert a possibly bigger class of morphisms than $p$. But we know from [MN06, App. A] that $\Sigma^{-1}$ and cone triangles make $(\text{KK}^G)_\Sigma$ into a triangulated category, and the minimal tensor product extends to a tensor product which preserves distinguished triangles in each variable. Thus $p$ is clearly a tensor triangle functor, showing that $\text{Ker}(p) = \text{Ker}(q)$, showing that they are both universal for inverting the same class of morphisms. □

**A.6.2. Remark.** If instead of $\mathcal{E}_{\text{split}}$ we take the class $\{ u_E \}$ for all extensions, then the category $E^G = \text{SW}^G/\mathcal{J}$ is called **equivariant E-theory**. By construction, it satisfies excision for all extensions. The ‘distance’ from $KK$-theory to $E$-theory is not so great as one may think: they are known to coincide for nuclear $C^*$-algebras. Also, it can be shown, most usefully, that $\text{KK}^G$ automatically inverts $u_E$ for all extensions $J \xrightarrow{j} A \xrightarrow{f} Q$ which are split in a much weaker sense, i.e.,
those where \( f \) has a \( G \)-equivariant ‘completely positive contractive’ linear section ([CS86] [MN06, §2.3]).
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BIBLIOGRAPHY 151


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