Dynamic R&D networks
the efficiency and evolution of interfirm collaboration networks

Author(s):
König, Michael David

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Dynamic R&D Networks
The Efficiency and Evolution of Interfirm Collaboration Networks.

A dissertation submitted to the
ETH ZURICH

for the degree of
DOCTOR OF SCIENCES

presented by
MICHAEL DAVID KÖNIG
Diplom-Ingenieur, Technische Universität Wien
born 19. March 1980
citizen of Austria

accepted on the recommendation of
Prof. Dr. Dr. Frank Schweitzer, examiner
Prof. Dr. Francis Bloch, co-examiner

2009
To Chigme
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A Some Results from Matrix Theory

B Graph Perturbation Theory
Abstract  A nation’s competitive advantage hinges on the capacity of its economy to innovate and the efficiency of its efforts in research and development (R&D) (Porter, 1990). High wage countries such as Switzerland are particularly affected by the pressure to develop new products and technologies both fast and efficiently in order to sustain their standard of living. Moreover, as we observe a growing complexity in the technologies used for production and innovation, firms increasingly discover that their in-house innovative capabilities are insufficient for developing these technologies (Rivkin, 2000). Thus firms gather knowledge from other firms through R&D collaborations (Cowan and Jonard, 2004; Sorenson et al., 2006). However, as long as the underlying mechanisms that govern the innovation process, the formation of R&D collaborations and the efficiency of the resulting economy are not sufficiently understood, policy makers and society as a whole incur the risk of getting locked into inefficient states, leading to a loss of competitiveness in the rapidly changing global economy.

The work contained herein contributes to reducing this risk by increasing our understanding of the efficiency and evolution of R&D networks. For this purpose, a new model to investigate the efficiency and evolution of networks of firms exchanging knowledge in R&D partnerships is developed. We examine the efficiency of a given network structure in terms of maximizing aggregate profits in the economy. We show that the efficient network structure depends on the marginal cost of collaboration and the volatility of the technological environment. When the marginal cost is low, the complete graph is efficient. However, high marginal cost implies that the efficient network is sparser and has a core-periphery structure. Next, we examine the evolution of the network structure under alternative scenarios, assuming that the decision on collaborating partners is decentralized.

First, we assume that the formation of links follows a dynamic process in which firms are stochastically selected to revise their collaboration strategies. In these dynamics, firms decide to form a link if the link did not exist before and the link is beneficial to both of them, and decide to delete a link if the link existed before and deletion is beneficial to at least one of them (Jackson and Watts, 2002). We show the existence of multiple equilibrium structures, most of them being inefficient. This is due to the path dependent character of the partner selection process, the presence of knowledge externalities and the presence of severance costs involved in link deletion. We study the properties of the emerging equilibrium networks and we show that they are coherent with the stylized facts of R&D networks.

Second, we analyze a dynamic link formation process in which alliances are formed between firms with high knowledge levels (technological leaders (Powell et al., 1996a)) while the volatile environment introduces interruptions in the connections between firms leading
to the breakdown of some collaborations. We incorporate the idea of bounded rational decision making (Aumann, 1997; Simon, 1953) by restricting the search for new collaboration partners to the neighbors of the neighbors of a firm (Gulati, 1995; Jackson and Rogers, 2007). The volatile environment causes existing links to decay where those links that firms value less than others are more likely to be affected by the decay. We show analytically that there exist stationary networks and that their topological properties match with those of empirical R&D networks. In particular, we find a strong core-periphery structure (due to the presence of inter-linked stars (Goyal and Joshi, 2003)) observed in empirical studies (Baker et al., 2008). We show that the relative size of the core compared to the periphery critically depends on the linking opportunities of firms and the strength of the link decay. Moreover, there exists a sharp transition in efficiency and network density from highly centralized to decentralized networks.

For all dynamic processes governing the formation of links between firms studied in this thesis a tension between efficiency and stability exists. If firms myopically pursue the maximization of profits in an uncertain technological environment or marginal costs of collaboration are high enough, the economy can get locked into inefficient equilibria. Thus, we show that the risks inherent in innovation driven markets are not only present in the uncertain outcomes of firms’ R&D investments but also in the possibility of inefficient network structures emerging from the R&D partner choices of firms.


Zunächst betrachten wir den Fall, dass die Entstehung von Kollaborationen in dem F&E-Netzwerk einem dynamischen Prozess folgt, in dem Firmen zufällig ausgewählt werden, um ihre Kollaborationsstrategien revidieren zu können. Dies bedeutet, dass Firmen entscheiden, ob sie eine Kollaboration eingehen, wenn sie noch nicht besteht oder eine Kollaboration beenden, wenn sie bereits besteht. Ersteres findet nur dann statt, wenn beide Firmen davon profitieren, während letzteres dann stattfindet, wenn zumindest eine der Firmen davon profitiert (Jackson and Watts, 2002). Wir zeigen, dass unter diesen Annah-


Für alle dynamischen Prozesse, die die Entstehung von Kollaborationen zwischen Firmen bestimmen, besteht eine Inkongruenz zwischen Effizienz und Stabilität. Wenn Firmen in einem volatilen Umfeld kurzzeitig die Maximierung ihres Profits verfolgen oder wenn die marginalen Kosten einer Kollaboration hoch genug sind, dann können ineffiziente Gleichgewichte entstehen. Damit wird klar, dass innovationsgetriebenen Märkten einem doppelten Risiko ausgesetzt sind: Zum Einen betrifft dieses Risiko die unsicheren Ergebnisse von F&E-Investitionen. Zum Anderen betrifft es aber auch die Möglichkeit der Entstehung einer ineffizienten Netzwerkstruktur, die sich aus der Partnerwahl der Firmen unter Unsicherheit mit beschränkter Information ergeben kann.
Summary

R&D collaboration networks offer promising and indispensable opportunities as well as immanent economic risks. This thesis shows that the risk inherent in innovation driven markets exists not only at the micro level but also at the macro level of the economy. On the micro level, the risk associated with uncertain innovative efforts shapes firms’ investment decisions. On the macro level, the global structures emerging out of these individual investment decisions are typically characterized by multiple equilibria and possible inefficiency.

Almost all economic investments bear a risk of how the market will respond to a new product (commercial success). Innovators face additional risks. First, their investment into R&D does not necessarily lead to a new technology. Second, if such a new technology is discovered, it has to be put into practice in a new and better product than the already existing ones. This inherent uncertainty of R&D projects often causes agents to invest “too little” (Gerosky, 1995).

Conventional economic theory suggests that interaction on perfect markets leads to efficient economic outcomes. In contrast, this thesis shows that modern high-technology markets, which are characterized by a strong tendency of firms to invest in joint, risky R&D projects (Hagedoorn, 2002; Hagedoorn et al., 2000), are characterized by multiple equilibria, most of them being inefficient. It is not a priori clear which of the possible equilibria will be selected in the evolution of the market. As a reaction to the problems that are not explained by conventional economic theory, there has been an ongoing discussion on how one can achieve that a particular economic outcome, selected from many possible alternatives, guarantees the most efficient use of resources and their allocation (Arthur, 1990, 1994; Arthur et al., 1987). The fact that indeterminacy in market outcomes can exist was already noticed by Marshall (1890). He analyzed a case of oligopolistic market competition that eventually lead to the monopoly of a single firm. However, it was indeterminate in advance which firm would be the monopolist. Similarly, in the formation of R&D networks, the resulting market outcome is the consequence of a path dependent process and the inherent non-ergodicity of the system (Arthur, 1989). The evolution of innovative markets is driven by stochastic events, such as the arrival of new innovations and, as a consequence, the profitability of certain R&D collaborations. This thesis shows that, if firms myopically pursue the maximization of profits in an uncertain technological environment, the economy can get locked into inefficient market outcomes.

In the following paragraphs the different chapters of this thesis are introduced and we elaborate in more detail how this thesis improves our understanding of the efficiency and evolution of dynamic R&D networks.

In Chapter 1 we give a short introduction into the literature on innovation and R&D
network formation. We discuss empirical findings and theoretical explanations on why firms enter collaborative agreements. In doing so, we review the existing literature on industrial organization. Moreover, we point out that these R&D collaborations are inherently dynamic, which serves as a central motivation for the dynamic models of R&D networks that follow.

**Chapter 2** introduces the most important definitions used throughout the thesis. We will discuss some basic graph theoretic concepts and some common network measurements as well as different measures of network centrality.

Next, we will discuss how we model the flow of knowledge, the creation of innovations and the translation of innovations into firms’ profits in a networked economy. The possibility of recombining different knowledge stocks to introduce innovations in the industry is the rationale for R&D collaborations (see Ahuja, 2000; Kogut and Zander, 1992; Powell et al., 1996b; Weitzman, 1998). In **Chapter 3** we formalize this idea by assuming that the arrival rate of innovations is proportional to the growth rate in the knowledge stock of the firm, and that firm’s knowledge growth is a linear combination of the firm’s own knowledge stocks and the knowledge of its R&D partners. We show that, if the period over which collaborations are evaluated is long enough, the expected number of innovations per period turns out to be proportional to the largest eigenvalue of the adjacency matrix associated with the connected component to which the firm belongs. This has several implications. First, as the largest eigenvalue is identical for all firms in the same component, the formation/deletion of a collaboration by a firm has a strong non-rival external effect on all its direct and indirect neighbors. Second, the magnitude of the change in eigenvalue, resulting from creating/severing a collaboration, varies with the topology of the network and the position of the two firms involved in the collaboration, thus implying a strong path-dependent character of partner’s choice decisions. Finally, it can be shown that the largest eigenvalue is related to the number of *all* walks connecting firms in a given component. This reproduces the observation of “multiconnectivity” emerging from the formation of alliances that Powell et al. (2005) have found in R&D networks in the biotech sector.

In Section 3.7 we show that our model can be related to other models in the network formation literature in which agents face a trade-off between the benefit they get from accessing the network and the cost of forming links with other agents (see e.g. Bala and Goyal, 2000; Carayol and Roux, 2005; Haller and Sarangi, 2005; Jackson and Wolinsky, 1996; Vega-Redondo and Goyal, 2007). To this regard, our model shares many similarities and differences with the “connections” model introduced by Jackson and Wolinsky (1996) and with the linear “two-way flow” model without decay introduced by Bala and Goyal
(2000). However, differently from Jackson and Wolinsky’s model the benefit the agent receives from the network in our model does not only depend on the shortest path existing between the agent and its direct and indirect neighbors but it accounts for all possible walks between them. Next, differently from Bala and Goyal’s linear model, in our model the utility of an agent does not only depend on the number of direct and indirect neighbors that can be reached by the agent with its existing connections, but also on how each neighbor can be reached.

A payoff function that considers all walks in the network, and thus comes closer to our approach, has been introduced in Ballester et al. (2006). Their model is a generalization of two other prominent models on innovation networks. Following Zenou (2006) we first consider the Cournot oligopoly game introduced by Goyal and Moraga-Gonzalez (2001) (see also Bala and Goyal (2000); Goyal and Joshi (2003)). Second, we consider a game introduced by Bramoulle and Kranton (2007) in the context of the provision of public goods in networks in which agent’s efforts (in R&D) have positive externalities. We show that the Nash equilibrium efforts in the game of Ballester et al. (2006) are equivalent to the asymptotic shares of knowledge in our model, provided that the local payoff complementarities are strong enough. However, the aforementioned models in this paragraph do not explain how the network comes to exist and treat it as exogenously given. We improve on them by making the network endogenous, emerging from the interactions of firms.

The increasing importance of R&D partnerships has spurred research, both theoretical and empirical, on the consequences of a given structure of the R&D network for technology innovation and diffusion (see among many others Ahuja, 2000; Cowan and Jonard, 2004, 2006; Letterie et al., 2008). To this regard, an important and still unsettled debate concerns the relation between the position of a firm in the network and its performance, and, in particular, whether a densely interconnected network is more conducive to knowledge diffusion and innovation than a network with structural holes (i.e. displaying the presence of hubs indirectly connecting many firms which have no direct link across them). Indeed, clusters of densely and directly connected firms might be seen as fostering collaboration efforts among participants by generating trust and punishment of opportunistic behaviors, and a common language and problem solving heuristics (see e.g. Ahuja, 2000; Coleman, 1988; Cowan and Jonard, 2006; Walker et al., 1997). Conversely, by creating a structural hole in the network, firms may have access to different sources of knowledge spillovers while economizing on the costs of direct collaborations (cf. Burt, 1992; Gargiulo, Martin and Benassi, Mario, 2000; Rowley et al., 2000a). The next chapter will add to this debate by determining the optimal level of structural holes versus strong ties in the core-periphery structure of the efficient network.
In Chapter 4 we analyze the efficiency of possible R&D network structures. We show that the network structure maximizing industry welfare (measured as the sum of firms’ profits) is a function of the marginal cost of collaborations. In particular, the efficient graph always belongs to a specific class of graphs (the class of nested split graphs (Aouchiche et al., 2006) or interlinked stars (Goyal and Joshi, 2003)). Furthermore, when the marginal cost is low the efficient graph coincides with the complete graph, i.e. the one maximizing the number of direct ties. As the marginal costs increase, it is efficient for the industry to organize into networks with a core-periphery structure. More precisely, at high levels of collaboration costs, efficient networks display the presence of hubs, indirectly linking cliques of firms to otherwise disconnected nodes in the network. Relatedly, we show that if collaboration costs and the size of the industry are large enough, the efficient structure for the industry is characterized by significant inequality in profits across firms. In particular, firms having fewer (more) direct connections are also the ones displaying higher (lower) profits. This stems from the fact that all firms in a connected component share the same revenues, depending on their knowledge growth rate, but their costs are different. In addition, profits inequality increases both in the number of firms and in the marginal cost of collaboration.

There is a large body of literature that has investigated the salient features of empirically observed R&D networks (see e.g. Ahuja, 2000; Fleming et al., 2007; Hagedoorn et al., 2006; Hanaki et al., 2007; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). These studies have identified four main structural properties of innovation networks that are invariant across different industries: (i) Networks are sparse, that is, from all possible connections between firms, only a small subset is realized. (ii) Networks are highly clustered, i.e. the neighbors of a firm are likely to be connected among each other. (iii) The distribution of links over the firms tends to be highly heterogeneous with only few firms being connected to many others. (iv) R&D networks are characterized by a core-periphery structure in which a core set of firms has links to a large number of firms that are comparably poorly connected (Baker et al., 2008; Goyal, 2007). Following this wave of empirical research, theoretical models have explored the emergence of R&D networks in a framework with firms being allowed to form any arbitrary pattern of bilateral R&D agreements (see e.g. Goyal and Joshi, 2003; Goyal and Moraga-Gonzalez, 2001). However, these models lead to network structures that are too simple to account for the stylized facts listed above.

We improve on the aforementioned literature by proposing alternative dynamic processes governing the formation of R&D collaborations that are able to reproduce all the stylized facts of empirical networks in the following chapters. Moreover, we compare aggregate profits of the emerging R&D networks with that of the efficient network.
In Chapter 5 we investigate equilibrium selection under a two-sided link creation and deletion mechanism (see Vega-Redondo, 2007, p. 212). Firms are stochastically selected to revise their collaboration strategies. They form a link if it did not exist before and if it is beneficial to both of them. An existing link is deleted if it is beneficial to at least one of the firms selected. We show that multiple equilibrium structures for the same level of collaboration costs do arise in our model. In particular, for the same level of collaboration cost, both the spanning star (i.e. the star encompassing all nodes in the network), as well as the graph composed by disconnected cliques of the same size are possible equilibrium networks. The existence of multiple equilibria implies that efficiency is not necessarily met by equilibrium structures in our model. In addition, we identify the conditions on industry size and collaboration costs under which the efficient network never belongs to the set of possible equilibria. Finally, we show that under this dynamics, equilibrium structures matching the stylized facts of empirical R&D networks are selected.

Next, in Chapter 6 we introduce a dynamic model of local best response link formation that departs from the model in the previous chapter. Firms have only local information when forming links which implies that they search for a potential partner among their neighbors’ neighbors. The creation of a link depends on the level of knowledge that the potential partner is endowed with (Pammolli and Riccaboni, 2002; Riccaboni and Pammolli, 2002). Moreover, we do not assume that each collaboration involves a time-invariant marginal cost. Instead, we assume that firms are living in a volatile environment making some of their links unprofitable. This means that links are exposed to a stochastic decay as uncertain innovations can change the profitability of some R&D collaborations in a rapidly changing technological environment (see also Ehrhardt et al., 2008; Marsili et al., 2004; Vega-Redondo, 2006).

We show that the emerging stationary networks in this network formation process reproduce the empirically observed properties of R&D networks. In particular, by introducing capacity constraints and allowing firms to form links globally, we give an illustration of how the distinction between assortative social networks and dissortative technological networks can be explained, as it has been suggested by Newman (2002, 2003b). We show that R&D networks can exhibit characteristics of both types of networks depending on the number of links a firm can maintain. Moreover, we find that there is a sharp transition in the network density from highly centralized to decentralized networks (see also Guimerà et al., 2002). A similar transition can be observed in the efficiency of stationary networks. We find that high link decay rates and low linking opportunities can generate inefficient network structures. Our analysis reveals that inefficiency can arise from multiple hubs of highly connected firms competing for centrality in the network while it would be efficient to have a network that consists of a core of densely connected firms and a single
hub connecting the firms in the periphery to the core. The competition for becoming a central player in the R&D network can thus be the source of an inefficient operation of the economy.

Finally, Chapter 7 concludes this thesis and proposes some possible extensions for future research.

We have redirected some of the theorems and results from the literature to which we have referred at several occasions in the thesis to the Appendix. Chapter A contains some basic results on matrix theory while Chapter B introduces an important theorem from graph perturbation theory.
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It would have been very difficult to accomplish this work without the support of my
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Michael David König
Zürich, October 2008
Chapter 1

Introduction

In this chapter we will first discuss the characteristics of innovations, knowledge creation and R&D and the problems that can arise when innovations should be supplied by individual firms operating in a competitive market. We then proceed by considering R&D partnerships where firms jointly invest into innovative activities. We discuss empirical findings and theoretical explanations of why firms enter collaborative agreements. We conclude by pointing out that these collaborations are inherently dynamic, thereby giving a central motivation for the dynamic models of R&D networks following in the next chapters.

1.1 The Importance of Research and Development

Economists widely agree on technological change and innovation being the main components of economic growth (Aghion and Howitt, 1998; Tirole, 1988). In the absence of ongoing technological improvement, economic growth can hardly be maintained (Barro and Sala-i Martin, 2004). The close link between R&D and economic performance has also become generally accepted among policy makers. Following this insight, in recent years of economic growth, OECD countries have fostered investments in science, technology and innovation (OECD, 2006, 2008). This can be witnessed in Figure (1.1) which gives an example of R&D investment of OECD countries (and non-member states).

In the following we will discuss the theoretical framework required to describe and understand the preconditions for and effectiveness of R&D when it is undertaken by individual, profit maximizing firms.
<table>
<thead>
<tr>
<th>OECD Member Country</th>
<th>million current PPP $</th>
<th>Non-Member Economy</th>
<th>million current PPP $</th>
</tr>
</thead>
<tbody>
<tr>
<td>Austria (2007)</td>
<td>7 865.3</td>
<td>China (2006)</td>
<td>86 758.2</td>
</tr>
<tr>
<td>Belgium (2006)</td>
<td>6 472.4</td>
<td>Israel (2006)</td>
<td>7 985.1</td>
</tr>
<tr>
<td>Canada (2007)</td>
<td>23 838.9</td>
<td>Romania (2006)</td>
<td>1 066.8</td>
</tr>
<tr>
<td>Finland (2007)</td>
<td>6 283.3</td>
<td>Slovenia (2006)</td>
<td>784.1</td>
</tr>
<tr>
<td>Germany (2006)</td>
<td>66 688.6</td>
<td>Taiwan (2006)</td>
<td>16 552.9</td>
</tr>
<tr>
<td>Greece (2006)</td>
<td>1 734.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hungary (2006)</td>
<td>1 831.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Iceland (2005)</td>
<td>293.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ireland (2007)</td>
<td>2 490.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Italy (2005)</td>
<td>17 827.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Japan (2006)</td>
<td>138 782.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>South Korea (2006)</td>
<td>35 885.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Luxembourg (2006)</td>
<td>542.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mexico (2005)</td>
<td>5 919.0</td>
<td></td>
<td></td>
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<tr>
<td>Netherlands (2006)</td>
<td>9 959.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>New Zealand (2005)</td>
<td>1 189.3</td>
<td></td>
<td></td>
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<tr>
<td>Norway (2006)</td>
<td>3 686.2</td>
<td></td>
<td></td>
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<tr>
<td>Poland (2006)</td>
<td>3 110.0</td>
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<tr>
<td>Portugal (2006)</td>
<td>1 839.5</td>
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<tr>
<td>Slovak Republic (2006)</td>
<td>467.1</td>
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<tr>
<td>Spain (2006)</td>
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<tr>
<td>Sweden (2006)</td>
<td>11 815.3</td>
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<td>Switzerland (2004)</td>
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<tr>
<td>Turkey (2006)</td>
<td>4 883.7</td>
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<td>United Kingdom (2006)</td>
<td>35 590.8</td>
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<tr>
<td>United States (2006)</td>
<td>343 747.5</td>
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<tr>
<td>EU-27 (2006)</td>
<td>242 815.6</td>
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<tr>
<td><strong>Total OECD (2006)</strong></td>
<td>817 768.9</td>
<td></td>
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</tr>
</tbody>
</table>

Table 1.1: Gross domestic spending on R&D in OECD countries and non-member states in 2007. All values are given in million dollars of current purchasing power parity (PPP) (source: OECD (2008)).
1.2 Innovation and Knowledge Production

In general terms an innovation is associated with a new technology or organizational idea (Gersbach and Winkler, 2007). Schumpeter (1942) further distinguishes between an invention and an innovation. An invention is the creation of a new good, service or organizational idea. An innovation is the act by which these new ideas are brought to the market. With respect to this Scotchmer (2004) differentiates between innovations for which the means to bring them to the market are available and those for which they are not. Zaltman et al. (1973) describe three dimensions of innovations: (i) programmed innovations that are scheduled in advance (e.g. the extension of a production line), while non-programmed innovations are not. The latter can be further divided into slack innovations, where the innovation is stimulated by the availability of means, and distress innovations, that act as a response to a pressing problem. Moreover, (ii) one can distinguish between ultimate innovations which are ends in themselves and instrumental innovations, that are tools for ultimate innovations. Finally, (iii) routine innovations encompass the refinement or transfer of existing innovations and radical innovations exhibit real novelty.

Not only the characterization of innovations can include a multitude of dimensions but also the process by which these are made. The process of making an innovation can be explained by three different approaches (Gersbach and Winkler, 2007): (i) The transcendental approach assumes that innovation is mainly driven by highly gifted individuals. (ii) The mechanistic approach assumes that new ideas are mainly the product of the socio-economic environment and not individual workers. And (iii) the series of accidents approach assumes that only the interaction between the researcher and the physical and social environment makes an innovation possible. Through this interaction innovations can be made by chance and trigger multiple subsequent discoveries. However, empirical evidence suggests that none of these approaches is able to explain the process of innovation in isolation, instead, all of these approaches play a role in the production of real life innovations.

Knowledge is a peculiar good when it comes to issues of producing it. Not only are the results of many ventures aimed at producing an innovation uncertain. Private firms may not even have an incentive to undertake such R&D activities if they cannot cover the costs they have incurred for their research activities, because the innovation may diffuse immediately on the market before the inventor can make any profits with it. This problem of approbiability will be discussed in the next Section.
1.3 Market Structure and Innovation

The problem of appropriability can diminish or even interrupt the efficient operation of a market and thus be the source of market failures. Particularly technology driven markets can exhibit serious market failures (Arora et al., 2004; Gerosky, 1995) which make it difficult for innovators to realize a reasonable return from trading the results of their R&D activities. This difficulty stems from the public good character of knowledge, which makes it different from products or services. Knowledge is non-rival, meaning its use by one agent does not diminish its usability by another agent, and sometimes (when knowledge spillovers cannot be avoided, see Section 1.4) non-excludable, meaning that the creator of new knowledge cannot prevent non-payers from using it. The problems associated with trading of knowledge can prevent agents from innovating at all.

There are three generic reasons for failures of markets for technology (Arora et al., 2004; Arrow, 1962; Gerosky, 1995): (i) economies of scale/scope\(^1\), (ii) uncertainty and (iii) externalities.

(i) R&D projects often require huge initial investments and they can exhibit economies of scale since the cost for useful technological information per unit of output declines with the level of output (Wilson, 1975). Besides, Nelson (1959) has shown that economies of scope can apply to innovative agents. The broader an agents’ “technological base”, the more likely it is that any outcome of its R&D activities will be useful for her. The result is that markets for knowledge exchange are often dominated by monopolies.

(ii) Almost all economic investments bear a risk of how the market will respond to the new product (commercial success). Innovators face additional risks. First, their investment into R&D does not necessarily lead to a new technology. Second, if such a new technology is discovered, it has to be put into practice for a new and better product than the already existing ones. This inherent uncertainty of R&D projects often causes agents to invest “too little”.

(iii) Externalities are important when the action of one agent influences the profits of another without compensation between them on the market. Public goods are a typical example of creating externalities. Knowledge is a public good and the returns that innovators can realize from it are often far below their investment in R&D\(^2\).

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\(^1\)Economies of scale express decreasing average costs of production with increasing output. Economies of scope express cost reductions due to the production of various products and services.

\(^2\)Consider an innovator that has developed a new product. A potential buyer can evaluate the value of
The public good character of knowledge can seriously diminish an agent’s incentive to do R&D.

In order to overcome the above mentioned problems associated with the returns on investment into R&D, appropriate incentive mechanisms have to be created that encourage firms to invest into R&D. In general, Von Hippel and Von Krogh (2003) suggest three basic models of encouraging firms to invest into R&D:

(i) The private investment model assumes that innovation is undertaken by private agents investing resources to create an innovation. Society then provides them with limited rights to exclusively use the results of their innovation through patents or other intellectual property rights arrangements, creating a temporary monopoly (Scotchmer, 2004).

(ii) The so called collective action model (Allen, 1983) assumes that agents are creating knowledge as a public good. Knowledge is made public and unconditionally supplied to a public pool accessible to everybody. The problem is that potential beneficiaries could free-ride by waiting until others provide the public good. One solution to this problem is to provide contributors (in this case innovators) with some form of subsidy. Scientific research is such an example where reputation based rewards are granted to scientists for their innovative performance.

(iii) In the private-collective innovation model participants use their private resources for creating new knowledge and then make it publicly available. This is typically observed in open source projects. There are several incentives (Lerner and Tirole, 2002; Von Hippel and Von Krogh, 2003) for agents to participate in open-source projects. These range from signaling the competence of a contributor, the desire of building a community to the expectation of reciprocity from the community members for their efforts.

The discussion so far has assumed that firms can only develop an innovation independently, via “in-house” R&D. However, there exist a large body of empirical studies (e.g. Hagedoorn et al., 2000; Roijakkers and Hagedoorn, 2006) that witness an increasing number of cases in which firms jointly invest in R&D activities. This can alleviate some of the aforementioned issues with R&D investment through cost and risk sharing of firms. When a product only if he has seen the product before. But when the product is disclosed to the potential buyer then he may be able to develop the product by himself. The innovator then will not be able to sell his product. Knowing this, the problem of disclosure can prevent the innovator from making the innovation at all.
talking about joint R&D activities and R&D partnerships, respectively, it is important to understand how such partnerships come to exist and what incentives firms have to enter such a collaborative arrangement. In the next section we discuss attempts in the neoclassical literature to identify the conditions under which co-operative arrangements between rival firms can emerge and we will then narrow this discussion on joint R&D activities. Subsequently, we will point out what is missing in the existing literature, in particular, when modelling firms operating in R&D intensive markets.

1.4 Collaborations between Competitors

Colombo (1998a) suggests that the existing literature (mainly the neoclassical literature on industrial organization) has tried to explain the emergence of interfirm co-operation as the attempt of profit-maximizing firms to internalize externalities. An externality is an external effect of one agent on another that is not compensated through the market. If an agent benefits from the externality, she is not charged for the positive effect. If she is disadvantaged, she is not compensated for the negative effect. The agent causing the effect does not include the consequences of her actions in her decision. Externalities can be the reason for inefficiencies and market failures and arise in multiple contexts. Colombo (1998a) differentiates between (i) horizontal competitive externalities, (ii) technological externalities, (iii) vertical externalities, and (iv) network externalities.

(i) If firms compete on the market, their profits are affected by the output that other firms produce or the prices they set. In this way, firms are exposed to horizontal competitive externalities. However, instead of firms competing against each other they may be able to collude and form a cartel, thus internalizing the competitive externality. In a cartel firms can jointly agree to set prices and quantities that maximize the sum of their profits. Since these firms are acting as a single monopolist consumers’ surplus is negatively affected. However, collusive agreements tend to be unstable since there exist incentives for the colluding firms to deviate from the agreed price or quantity.

(ii) In an oligopolistic industry, a firm’s R&D activities may have a positive or negative effect on the cost or the demand of other firms. In this way, firms are exposed to positive or negative technological externalities. Negative externalities arise from technological competition. In the presence of intellectual property rights a firm can either increase its production by a cost reducing innovation or it can make an innovation substituting the products of its rivals. On the other hand, positive externalities
1.4. Collaborations between Competitors

arise from spillovers. When one agent benefits from knowledge created elsewhere we speak of knowledge spillovers. Knowledge spillovers define (Foray, 2004, p.95) “any original, valuable knowledge generated somewhere that becomes accessible to external agents ... other than the originator”. Knowledge spillovers are associated with the problem of appropriability and they can lead to the failure of markets for knowledge and technological progress. However, spillovers can also have a positive effect on social welfare if they enhance market competition. If the problem of appropriability is mitigated e.g. through the existence of intellectual property rights, then (Foray, 2004, p.91) “competition not only creates incentives to produce new knowledge but it also forces the other agents to increase their own performance through imitation, adoption and absorption of the new knowledge created elsewhere”. On the other hand, if the effectiveness of intellectual property rights is weak and rival firms can employ an innovation without purchasing the right to do so at negligible imitation costs, then a firm’s incentive to invest into R&D independently may be too low. In this case, interfirm collaborations can offer a way out. If positive technological externalities dominate negative technological externalities, it can become profitable and welfare increasing that firms form cartels with joint R&D investments (see Kamien et al., 1992).

(iii) Negative vertical externalities can arise if an upstream monopolist supplies a competitive downstream market. Vertical mergers are welfare increasing if the firms on the competitive downstream market jointly own the upstream supplier. This happens for example in the oil industry where oil companies jointly own pipelines or production fields (Colombo, 1998a).

(iv) Positive network externalities are present in markets where the value of a good depends on the number of units sold. In such markets (Mas-Colell et al., 1995; Tirole, 1988), “a good is more valuable to a user the more users adopt the same good or compatible goods”. Network externalities can cause “vendor lock-in” (e.g. the Microsoft operating system Windows). A vendor lock-in makes a customer dependent on a vendor for products or services. The customer is then unable to use another vendor without substantial switching costs. Vendor lock-in can create barriers to market entry and thus can result in an inefficient monopoly (Arthur, 1989). Co-operative arrangements may overcome lock-in effects by assuring the inter-operability of network goods and make e.g. a platform independent software application available (Katz and Shapiro, 1985, 1994).

The above attempts to explain co-operative agreements between firms in the context of R&D have two main disadvantages (Colombo, 1998a). First, there are significant tech-
nological differences among firms, which are not merely the result of an asymmetric distribution of relevant information but also of differences in firms’ specific resources and experiences. We will further elaborate on this issue in Section 1.5. Second, interfirm collaborations are viewed from a static perspective in which feedbacks related to the dynamic impact of co-operation on the firm’s specific capabilities and its physical and social environment are not considered. We discuss the importance of this dynamic nature of interfirm R&D collaborations in Section 1.10.

1.5 The Resource-Based View of the Firm

Differently to the discussion in the preceding section, where we concentrated on the product-market side of the firm, we now focus on the resource side of the firm. Most products use several resources as inputs and most resources can be used in multiple products. Following Penrose (1959) the firm then may be regarded as a repository of different assets or resources.

The resource-based view of the firm emphasizes that the competitive advantage of a firm mainly rests on its specific resources (Wernerfelt, 1984). This competitive advantage requires that resources are heterogeneous and not easily imitable among firms (Barney, 1991).

In technology driven sectors and high technology industries these valuable resources are determined by the firms’ technological knowledge. The knowledge base of a firm derives from learning and adaptation to its physical and social environment and the behavior of the firm then may feed back to the characteristics of the environment.

Even when firms are exposed to similar environmental conditions there can be considerable differences among firms in their technological knowledge. Simon (1959, 1978) noted that agents may perceive an information set that may not match the actual information set in the environment. This is particularly relevant in technological environments that are characterized by uncertainty where there may exist multiple possible courses of action. In this way, the same information set may lead to different sets of knowledge of the agents operating in the same environment.

Industry evolution and the discovery of new technologies may further enhance technological differences among firms. The knowledge of a firm evolves over time and this evolution is a path dependent process. Firms are continuously adapting to their changing environment and alternating market conditions and they are driven by their previous experiences of success and failures. In this way the cognitive differences and the differentiation of
knowledge over firms in the industry are further enhanced. It is precisely this heterogeneity in the knowledge embodied in different firms and its employees that can give one firm a competitive advantage over another.

Richardson (1972) points out that firms tend to specialize in activities for which their capabilities offer a comparative advantage and continues to suggest that firms may engage in cooperative agreements to co-ordinate their complementary but dissimilar activities. This means that, if knowledge can be transferred, fruitfully combined, and the costs of this transaction are moderate enough (we will discuss the costs of transferring knowledge in Section 1.8) firms have an incentive to form collaborations. Similarly, Colombo (p.11 1998a) states that “firms engage in collaborative ventures in order to combine their own specific assets with others which are possessed by other firms and cannot be reproduced automatically.”

Evidence for the growing importance of R&D collaborations also comes from an increasing number of empirical studies (e.g. Fleming et al., 2007; Hagedoorn et al., 2006; Letterie et al., 2008; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). For example Figure (1.2) shows the number of R&D collaborations in the biotech sector between 1960 and 1998. It can be seen that after a small growth in the 1960s and 1970s, interfirm partnerships flourish during the 1980s. Moreover, the share of high-tech industries increasingly dominates the
Figure 1.2: The share of high-, medium- and low-tech industries in the newly established R&D partnerships between 1960 and 1998 (source: Hagedoorn (2002)).

number of R&D collaboration activities over medium- and low-tech industries.

Powell and Grodal (2006) emphasize the role of knowledge transfer in the innovation process and the generation of new ideas and proposes two types of information exchange. First, one explanation for the exchange of information is the complementarity of assets in the division of innovative labor. Powell and Grodal (2006) give an example from the biotech sector: small university spin-off firms may develop a new drug, but lack the infrastructure to manage and fund costly clinical trials. However, a research hospital may provide the small firm with the necessary facilities. In this way, both parties can develop a product they would not have been able to develop individually.

Second, another form of information sharing occurs when firms exchange existing knowledge in a network and recombine it in novel ways in order to generate new knowledge. This new knowledge can be translated into a profitable innovation and positively affect firm growth. This phenomenon has been coined “recombinant growth” by Weitzman (1998).

The recombination of existing knowledge is gaining importance in the current economy (Foray, 2004) and further accelerates the heterogeneity and dispersion of knowledge among firms. More than ever, today’s economies and, in particular, R&D intensive markets witness the phenomenon that technologies are becoming increasingly complex and inter-
dependent. This increasing complexity of technologies can make a firm’s “in-house” innovative effort insufficient to compete in an R&D intensive economy. Thus, firms have to become more specialized on specific domains of a technology and they tend to rely on knowledge transfers from other firms, which are specialized in different domains, in order to combine complementary domains of knowledge for production.

We will give a further classification of different types of R&D collaborations and information exchange in Section 1.7 where we distinguish between formal and informal collaborations. Within this classification the division of innovative labor can be regarded mainly as a formal arrangement while the exchange of information is occurring to a large extent through informal relationships between firms.

Having acknowledged the existence of differentiated technological capabilities and knowledge bases among firms and the potential benefits that can arise in combining them, we have not yet answered the question of how to allocate and exchange these assets among economic agents and firms. However, we will approach this question in the next section and discuss different institutional frameworks that can regulate this exchange. Inter-organizational networks represent just one form of co-ordinating economic exchange among agents and we will discuss under what conditions networks can be the preferred arrangement for exchanging knowledge between firms.

1.6 Resource Allocation: markets, firms and networks

Ebers (1997) suggests three economic institutions governing the allocation of resources among agents: markets, firms and networks.

(i) Markets are an institutional framework that allows agents to exchange goods, services or information in a decentralized and anonymous way. Markets institutionalize competition among actors for exchange opportunities (Hodgson, 1988). This competition includes informational resources, and bargaining over the conditions of exchange are the coordination mechanisms among agents. Interaction takes place through prices and the identity of the exchange partners hardly matters. Moreover, information flows are limited to information on prices, quantities and qualities. Compared to firms and networks the exchange of information is relatively small.

(ii) Firms are a collective of agents that pool their complementary resources and establish a corporate actor, collective decision-making rules as well as rules that govern the allocation of income and losses from their joint activities and resources (Ebers, 1997; Vanberg, 1982, 1992). The authority that controls the agents’ activities and
prescribes the rules of behavior is accepted with the signing of the employment contract. Members of a firm develop rather close social relationships through their repeated interactions and the relatively small size of the teams they are working in. In this way they develop mutual expectations about conduct and obligations (Blau, 1964).

(iii) Inter-organizational networks institutionalize partner-specific exchange relationships of finite duration, often terminated by the accomplishment of a certain, pre-agreed goal, or unspecified duration between a limited number of selected agents (p.21 Ebers, 1997). Differently to firms, agents do not establish a corporate actor but retain the control over their resources and negotiate their joint resource allocation. Additionally, the exchange of resources entails a flow of information that is much broader than in pure market-based interactions.

Table (1.2) gives a summary of the above forms of economic institutions governing the allocation of resources among agents.

Through networks, firms can gain access to valuable resources and assets that are complementary to their own and that enable them to accomplish tasks and develop capabilities they would not have been able to achieve individually (Powell and Grodal, 2006; Powell, 1990; Powell and Smith-Doerr, 1994). In the following we further elaborate on the potential benefits that networks can provide over other institutional forms like markets or firms for the control of exchanging these resources.

Networked firms may gain competitive advantage over firms that organize their exchange relations in a market-based form because they gain an improved organizational control and level of co-ordination over their resources due to their higher degree and extended scope of information sharing, reciprocal obligations and recurrent joint decision-making (Ebers, 1997).

Networking linkages are preferred to full integration within a firm when there are barriers to the integration of new capabilities and innovations (Ebers, 1997). The imitation of desired strategic capabilities can become difficult when firms have long-term strategic commitments. Networking is a means of circumventing these barriers and allows for a more rapid repositioning. Finally, network forms of collaboration and development do not involve the irreversible commitment associated with the internal development of innovations (Porter and Fuller, 1986).
### 1.6. Resource Allocation: markets, firms and networks

<table>
<thead>
<tr>
<th>Characteristics</th>
<th><strong>Market</strong></th>
<th><strong>Firm</strong></th>
<th><strong>Network</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution of property rights over resources</td>
<td>unilateral decision management and decision control with residual risk bearing by transaction partners</td>
<td>separation of decision management, decision control, and residual risk bearing among transaction partners</td>
<td>unilateral decision control and residual risk bearing combined with periodical joint decision making by transaction partners</td>
</tr>
<tr>
<td>Resource flows among agents</td>
<td>infrequent, discrete acts of exchange of resources</td>
<td>resource pooling of co-specialized resources</td>
<td>repeated partner specific exchange of resources</td>
</tr>
<tr>
<td>Relational expectations among agents</td>
<td>narrow, confine to terms of contract</td>
<td>wider, including contractually unspecified reciprocal obligations and mutual expectations</td>
<td>wider, including contractually unspecified reciprocal obligations and mutual expectations</td>
</tr>
<tr>
<td>Duration of exchange relationship</td>
<td>short-term economic exchange relation with a finite duration</td>
<td>longer-term social relationship, unspecified duration</td>
<td>longer-term social relationship, finite duration (based on goal accomplishment) or unspecified duration</td>
</tr>
<tr>
<td>Information flows among agents</td>
<td>confined to terms of exchange (price, quantity, quality, delivery)</td>
<td>higher degree of information sharing with regard to a wider spectrum of information</td>
<td>higher degree of information sharing with regard to a wider spectrum of information</td>
</tr>
<tr>
<td>Co-ordination mechanism</td>
<td>bargaining and competition</td>
<td>authority and identification</td>
<td>negotiation and concurrence</td>
</tr>
</tbody>
</table>

Table 1.2: Characterization of three economic institutions, markets, firms and networks, governing the allocation of resources among agents (source: Ebers (1997)).
In the next section we will further differentiate the organizational structure encountered in collaborative agreements in R&D networks. Then, in Section 1.8, we discuss the potential costs and their sources in research partnerships.

### 1.7 Types of R&D Collaborations

Hagedoorn et al. (2000) suggest characterization of the organizational structure of research partnerships based on distinguishing between formal and informal agreements and, within the first, between equity based research corporations and research joint ventures. Figure (1.3) illustrates these categories.

![Figure 1.3: The taxonomy of research partnerships distinguished by their organizational structure (source: Hagedoorn et al. (2000)).](image)

Formal research collaborations can be further partitioned into equity joint ventures (RJVs) and non-equity, contractual forms of R&D partnerships.

RJVs are created by at least two firms that combine their resources through the joint ownership of a separate firm. They allow for the spreading of risk, sharing of fixed costs, economies of scale, gaining access to new markets, creating entry barriers for competitors and the sharing of efforts (Hagedoorn et al., 2000). However, half of all equity-based RJVs stay behind their expected benefits or are abandoned entirely (Kogut, 1988a,b).

Non-equity, contractual forms of R&D partnerships cover all agreements in which two or
more firms join their R&D efforts without creating a separate corporation. Contractual arrangements are project-based and have a limited time-horizon. Such a contractual arrangement can be terminated with only a small loss compared to the loss that would be incurred by dissolving a RJV. These properties may be viewed by firms as an advantage over RJVs where an entirely new organizational entity needs to be established.

Recent empirical studies have identified a general trend in the increase and dominance of non-equity based contractual forms of research partnerships over RJVs in the total of all partnerships. Figure (1.4) shows the share of RJVs in the newly established R&D partnerships between 1960 and 1998 in a recent study by Hagedoorn (2002). Clearly, companies seem to increasingly prefer contractual arrangements over joint ventures.

![Figure 1.4: The share of research joint ventures in the newly established R&D partnerships between 1960 and 1998 (source: Hagedoorn (2002)).](image)

### 1.8 Costs of Transmitting Knowledge

The problem of appropiability mainly arises when agents are able to imitate and use new knowledge generated elsewhere at negligible cost. However, the costs of transmitting knowledge, either voluntarily or involuntarily, can vary considerably, and they depend on the nature of technology, the legal and institutional framework within which the new
technology is created and the internal capabilities of the firm to absorb the technology (Powell and Grodal, 2006).

Foray (2004) points at the relatively easy transferability of explicit knowledge in contrast to tacit knowledge. Explicit knowledge is highly codified, for example in blueprints, recipes or manuals. In contrast, tacit knowledge is largely idiosyncratic and non-decomposable. Tacit knowledge requires considerably higher costs for transferring it and significantly more trial-and-error learning for the recipient to absorb it.

Gerosky (p.118 1995) points out that the transfer of technologies may be a declining cost activity in which firms with large R&D investments adopt new technologies faster than less research intensive competitors. The effect of investing into R&D may be twofold: first, firms invest into R&D in order to develop a new technology and second, in the course of their research activities they can develop the ability to assimilate and exploit other existing technologies (increasing their “absorptive capacities” (Cohen and Levinthal, 1989)).

The costs associated with the establishment, maintenance and management of collaborations may be called internal costs of inter-organizational networking (Ebers and Grandor, 1997). On the other hand, there exit also costs that affect those firms that are not taking part in the networking arrangement directly. We have addressed this issue in Section 1.4 in the discussion of internalizing externalities. The possible negative consequences for other firms (and consumers) can often arise from collusion between collaborating firms. One can call this negative network externalities. One example are the entry barriers for potential competitors that are created through coalitions of engineering firms (Soda and Usai, 1999). However, networks may also have positive effects on third parties or the efficiency of the whole economy. Such positive network externalities can stem from economies of scale due to a pooling of resources between collaborating firms, stable economic environmental conditions and the development of trust through reciprocal exchange relationships, more informed investment decisions trough joint strategizing and, finally, technological spillovers (Ebers and Grandor, 1997).

Foray (2004) suggests that the knowledge-based economy is developing towards a state in which the costs for acquiring, reproducing and transmitting knowledge are constantly decreasing, spatial and geographical limitations on knowledge exchange are becoming less important and attitudes change towards more open behavior of sharing knowledge instead of hiding it from others. In this state, knowledge externalities will play an increasingly important role.

Even though improvements in information communication technologies will make it easier to transfer and reproduce knowledge, the costs for transferring it can often be considerable, in particular, if this transfer involves tacit knowledge and the pooling of highly
skilled workers. Firms then become fairly selective about potential collaboration partners. Indeed, if firms are not collaborating with all other firms in the economy but only with a particular subset, this will strongly affect the overall interfirm collaboration network. The network attains a particular structure that is shaped by the individual partner choices of firms. In the next section we review theoretical and empirical contributions studying these network topologies.

1.9 R&D Network Topology

A large body of literature has investigated the salient characteristics of empirical R&D networks (see e.g. Ahuja, 2000; Fleming et al., 2007; Hagedoorn et al., 2006; Hanaki et al., 2007; Powell et al., 2005; Roijakkers and Hagedoorn, 2006). These studies have identified four main structural properties of innovation networks that are invariant across the different industries examined: (i) Networks are sparse, that is, from all possible connections between firms, only a small subset is realized. (ii) Networks are characterized by high clustering. This means that the collaborating partners of a firm are likely to be connected among each other. (iii) The distribution of links over the firms tends to be highly heterogeneous with only few firms being connected to many others. (iv) The highly connected firms are forming the core of the R&D network while the firms in the periphery are connected to this core with only a few links. Thus, empirical R&D networks are characterized by a core-periphery structure (Baker et al., 2008; Goyal, 2007).

In Figure (1.5) we show the network topology of patent co-authorships from Fleming et al. (2007). Next, Figure (1.6) shows networks of firms who are linked through university collaborations. Moreover, Figure (1.7) shows an example of a core-periphery structure of an R&D network. In all the underlying studies, authors find the aforementioned characteristics, namely sparse, highly clustered and degree heterogeneous networks with a core-periphery structure.

The increasing importance of R&D partnerships has spurred, both theoretical and empirical, research on the consequences of a given structure of the R&D network for technological innovation and diffusion (see among many others Ahuja, 2000; Cowan and Jonard, 2004, 2006; Letterie et al., 2008). As we have mentioned above, empirical R&D networks are characterized by a strong core-periphery structure. In this structure, a densely connected subgroup of firms is holding links to many sparsely connected firms in the periphery. Coleman (1988) pointed out that dense subgroups of closely linked firms can be a source of

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3In Section 6.4.1 we will give a more detailed overview of the empirically observed characteristics of R&D networks.
Figure 1.5: Networks of Silicon Valley and Boston 128 inventors (as evidenced by joint patent applications) in 1986 – 1990. The figure shows the largest connected component only. Boxed areas indicate examples of highly clustered inventors (source: Fleming et al. (2007), see also Fleming and Frenken (2007)).
1.9. R&D Network Topology

(a) Network of firms collaborating with the university of Berkeley.

(b) Network of firms collaborating with the university of Stanford.

Figure 1.6: Networks of firms where a link indicates that they are advised by the same member of the universities of Berkeley and Stanford. The network of advised firms related to Stanford is bigger, containing 104 firms, than the one of Berkeley, containing 62 firms (source: Assimakopoulos and Kenney (2005), see also Kenney and Richard (2004)).
Figure 1.7: A sample R&D collaboration network in the pharma-biotech industry. The network shows a core-periphery structure. The 12 most linked firms form a densely connected core with a large number of poorly connected firms in the periphery. (source: Baker et al. (2008))

social capital. One can ask if this structure is optimal or if the productivity of the economy can be increased in alternative network structures. However, this is an important and still unsettled debate on the relation between the position of a firm in the network and its performance. In particular, whether a densely interconnected network (high social capital), is more efficient to knowledge diffusion and innovation than a network with structural holes (i.e. displaying the presence of hubs indirectly connecting many firms which have no direct link across them). Indeed, clusters of densely and directly connected firms might be seen as fostering collaboration efforts among participants by generating trust and punishment of opportunistic behaviors, and a common language and problem solving heuristics (see e.g. Ahuja, 2000; Coleman, 1988; Cowan and Jonard, 2006; Walker et al., 1997). On the other hand, by creating a structural hole in the network, firms may have access to different sources of knowledge spillovers while economizing on the costs of direct collaborations at
1.10. The Importance of Network Dynamics

The debate between social capital and structural holes can be incorporated in the concept of clustering. The proponents of the structural holes argument would argue that highly clustered networks diminish the performance of the network while the proponents of the social capital position would claim the opposite. One possibility to combine these opposing views is that they apply to different times in the evolution of the R&D collaboration network (Cowan and Jonard, 2006; Rowley et al., 2000b; Walker et al., 1997) (an important issue that we will further discuss in the next section). In an early phase of the network evolution, that is characteristic for a young industry, new technologies are being explored and there are many possible research directions to take. The most valuable links are those that connect different parts of the network. In this case, it would be most profitable for firms to create structural holes and work as information intermediaries. They can act as information brokers and take advantage of informational synergies (Ebers, 1997). In a more mature industry and a later phase of the network evolution technologies that have proven to be fruitful for further investigation are being exploited. Here, densely connected clusters with a high social capital in which firms can promote the development of a common technology are more valuable.

As the discussion of structural holes and social capital reveals, some important characteristics of R&D collaboration networks can only be understood by taking into account the dynamic nature of interactions between firms. In the next section we will try to give an account of this view and highlight the importance of network dynamics.

1.10 The Importance of Network Dynamics

Many studies on interfirm networks examine networks from a static point of view, taking the network structure as given, rather than viewing the network as the outcome of the individual behavior of firms interacting with each other and adapting to their social and economic environmental conditions. This static view has been criticized by a number of authors. For example, Ebers and Grandor (1997) argue that “...inter-organizational networking should be considered as inherently dynamic”. In the same line Colombo (1998b) states that “actually, there may be important feedbacks between the collaborative ventures firms have entered into and the nature and evolution of firms’ capabilities which can be assessed only in a dynamic framework”. And Powell and Grodal (p.78 2006) state that “concerns with how the parties in a relationship adapt to changing circumstances, or attend to the incentives to adjust the relationship to make improvements remain largely
unexamined”.

Due to the interaction of networking partners, firms’ information bases are prone to change over the course of the relationship and, over time, networked partners gain additional information on their partners’ capabilities, incentives and behavior (Ebers and Grandor, 1997). As a consequence, their changing perceptions, expectations and mutual trust will influence the way in which they shape their future relationship. For instance, Ring and Ven (1994) show that trust is the result of a dynamic process in cooperative inter-organizational relationships.

The substantial transaction costs that can arise from the coordination and exchange of specific assets and capabilities has been intensively investigated by transaction cost theory (Williamson, 1979, 1983). However, the evolution of a firm’s assets and capabilities is not taken into account adequately by transaction cost theory (Colombo, 1998b). The transaction costs associated with R&D arrangements are changing over time. First, a firm’s switching costs to replace a mediocre partner decrease with their own expertise and the size of their knowledge stock. Second, the ability to evaluate the contribution that a partner brings into the collaboration and the means to monitor her behavior increase with the duration of the collaboration. This reduces problems associated with moral hazard and adverse selection. Third, learning costs decrease with the increasing abilities to enhance existing skills and to create new ones in the course of the collaboration.

Some empirical studies have shown that R&D collaborations are exposed to a high risk of failure (Veugelers, 1998). For example Kogut (1988a) found that 20% of all alliances fail after a period of five to six years. However, Ebers and Grandor (1997) point out that the inherent dynamics and instability of R&D collaborations make it misleading to take the longevity of a collaboration as a measure of its performance. Many studies assume that, as long as a collaborations between firms exists, it is mutually rewarding and firms thus have an incentive to uphold the collaboration infinitely. However, the termination of collaborations may not be evidence for the failure of the collaboration but the consequence of the fact that a collaboration has outlived its usefulness. If the conditions under which the collaboration was profitable initially have changed and the cost outweigh the benefits, it can become reasonable for firms to close the relationship. However, some collaborations seem to outlive their usefulness. Amongst other things one reason for this excess inertia can be that, when alliances are difficult and expensive to form, firms may be reluctant to disband them (Inkpen and Ross, 2001). Moreover, there may also be costs associated with the closing of a collaboration directly (Powell and Grodal, 2006), for example by breaking up a contract. Such severance costs can considerably change the link formation dynamics among firms, an issue that we will further explore in Chapter 5.
In this section we have seen that R&D networks should be conceived as the outcome of an evolution of firms’ capabilities, perceptions and interactions which in turn are influenced by the network these firms are embedded in. Following this approach, we propose a dynamic framework for the incentives between horizontally related firms in an R&D intensive economy. The firms enter collaborative agreements and the benefits arise from sharing knowledge about a cost reducing technology or from exchanging knowledge used for making an innovation, thus creating a network of R&D collaborating firms. The incentives of firms to form collaborations shape the network and the network feeds back into the firms’ incentives and behavior. This mutual inter-dependency is an essential feature of the models we will introduce in the following sections. In these models, R&D networks emerge from the dynamic interactions between firms which are in turn shaped by the network evolution.
Chapter 2

Characterization of Networks

2.1 Basic Graph Theory

Consider an industry populated by \( n \) firms. The network \( G \) is the pair \((N, L)\) consisting of the set of firms \( N = \{1, ..., n\} \), representing the population of firms, and a set of links \( L \), representing R&D collaborations among them. We consider undirected networks since an R&D collaboration between firms is a mutual agreement. A link \( ij \in L \), represents the existence of an R&D collaboration between firms \( i \) and \( j \) in \( N \). The number of firms is \( n = |N| \) and the number of collaborations is \( m = |L| \).

The neighborhood of a firm \( i \in N \) is the set \( N_i = \{j \in N : ij \in L\} \). The degree \( d_i \) of a firm \( i \in N \) gives the number of links incident to firm \( i \). Clearly, \( d_i = |N_i| \). The maximum degree is \( \Delta(G) \) and the minimum degree is \( \delta(G) \). Let \( N_i^{(2)} = \bigcup_{j \in N_i} N_j \setminus \{i \cup N_i\} \) denote the second-order neighbors of firm \( i \). Similarly, the \( k \)-th order neighborhood of firm \( i \) is defined recursively from \( N_i^{(0)} = i \), \( N_i^{(1)} = N_i \) and

\[
N_i^{(k)} = \bigcup_{j \in N_i^{(k-1)}} N_j \setminus \left( \bigcup_{l=0}^{k-1} N_i^{(l)} \right) \quad (2.1)
\]

A walk \( W_k \) of length \( k \) (containing \( k \) links) connecting firm \( i_1 \) and \( i_{k+1} \) is a sequence of firms \((i_1, i_2, ..., i_{k+1})\) such that \( i_1 i_2, i_2 i_3, ..., i_k i_{k+1} \in L \). Note that \( N_i^{(k)} \) from above denotes the set of firms which can be reached from firm \( i \) along a walk of length \( k \) and no less than \( k \). A walk is closed if the first and last firm in the sequence are the same, and open if they are different. A trail is a walk with no repeated link. A path is a walk in which no firm is visited twice. \( P_n \) denotes the path of length \( n \) containing \( n \) distinct firms. The

\footnote{We will use the terms graph and network interchangeably.}
geodesic distance is the length of the shortest path between firms \(i\) and \(j\) in \(G\). If the endpoints of a trail are the same (a closed trail) then we refer to it as a circuit. A circuit with no repeated firm except the last is called a cycle. In particular, \(C_n\) denotes the cycle containing \(n\) firms. Note that a cycle is also a circuit but a circuit is not necessarily a cycle.

A subgraph, \(G'\), of \(G\) is the graph of subsets of the firms, \(N(G') \subseteq N(G)\), and links, \(L(G') \subseteq L(G)\). A graph \(G\) is connected, if there is a path connecting every pair of firms. Otherwise \(G\) is disconnected. The components of a graph \(G\) are the maximally connected subgraphs. A component is said to be minimally connected if the removal of a link in the component splits the component into two components.

In a complete graph \(K_n\) every firm is adjacent to every other firm. The graph in which no pair of firms is adjacent is the empty graph \(\overline{K}_n\). A clique \(K_{n'}\), \(n' \leq n\), is a complete subgraph of the network \(G\). An independent set \(\overline{K}_{n'}\) is a clique in which all \(n'\) firms are not pairwise adjacent. A graph is \(k\)-regular if every firm has the same number of links \(d_i = k\) for all \(i \in N\). The complete graph \(K_n\) is \((n-1)\)-regular. The cycle \(C_n\) is 2-regular.

In a bipartite graph there exists a partition of the firms in two disjoint sets \(V_1\) and \(V_2\) such that each link connects a firm in \(V_1\) to a firm in \(V_2\). \(V_1\) and \(V_2\) are independent sets with cardinalities \(n_1\) and \(n_2\), respectively. In a complete bipartite graph \(K_{n_1,n_2}\) each firm in \(V_1\) is connected to each other firm in \(V_2\). The star \(K_{1,n-1}\) is a complete bipartite graph in which \(n_1 = 1\) and \(n_2 = n - 1\). Examples of simple graphs are shown in Figure (2.1).

![Figure 2.1: A cycle \(C_5\) (left), a path \(P_5\) (middle) and the star \(K_{1,4}\) (right). All graphs are undirected and contain 5 nodes.](image)

Let \(A\) be the symmetric \(n \times n\) adjacency matrix of the network \(G\). The element \(a_{ij} \in \{0, 1\}\) indicates if there exists a link between firms \(i\) and \(j\) such that \(a_{ij} = 1\) if \(ij \in L\) and \(a_{ij} = 0\) if \(ij \notin L\). An example of a simple undirected graph containing four firms and its associated adjacency matrix \(A\) is given in Figure (2.2).

The \(k\)-th power of the adjacency matrix is related to walks of length \(k\) in the graph.
2.1. Basic Graph Theory

\[ A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \]

Figure 2.2: On the right hand side an undirected graph consisting of 4 firms and 5 links is shown. To the left, the corresponding adjacency matrix \( A \) is shown. In the first row in the adjacency matrix \( A \) we have the entries, \( a_{11} = 0, a_{12} = 1, a_{13} = 1, a_{14} = 0 \). The entry \( a_{12} = 1 \) indicates that there exists a link between firms 1 and 2. Since collaborations are assumed to be bilateral agreements, there exists a link from firms 2 to 1 as well. Consequently, in the second row we find the entry \( a_{21} = 1 \). This holds for all links and thus \( A \) is symmetric, \( A = A^T \).

In particular, \( (A^k)_{ij} \) gives the number of walks of length \( k \) from firm \( i \) to firm \( j \). We can consider a simple example of a cycle \( C_4 \), as illustrated in Figure (2.3). Let \( U \) be a matrix consisting of one-entries only, that is, \( u_{ij} = 1 \) for all \( i, j \in \{1, ..., 4\} \). For the \( k \)-th power of the adjacency matrix \( A \) of \( C_4 \) we have that \( A^k = 2^{k-1}A \) if \( k \) is odd and \( A^k = 2^{k-1}(U - A) \) if \( k \) is even. This can be shown by induction. Consider \( k \) even and assume that \( A^{k-1} = 2^{k-2}A \). Then \( A^k = A^{k-1}A = 2^{k-2}A^2 = 2^{k-1}(U - A) \), using the fact that \( A^2 = 2(U - A) \). Similarly, the case for \( k \) odd can be computed. Thus, the number of walks of length \( k \) in \( C_4 \) from firms \( i \) to \( j \) is given by \( 2^{k-1} \) if \( k \) is even and firms \( i, j \) are not neighbors (\( a_{ij} = 0 \)) or \( k \) odd and \( i, j \) are neighbors (\( a_{ij} = 1 \)). Otherwise, there does not exist a walk of length \( k \) between \( i \) and \( j \) in \( C_4 \).

The eigenvalues of the adjacency matrix \( A \) are the numbers \( \lambda_1, \lambda_2, ..., \lambda_n \) such that \( Ax_i = \lambda_i x_i \) has a nonzero solution vector \( x_i \), which is an eigenvector associated with \( \lambda_i \) for \( i = 1, ..., n \). Since the adjacency matrix \( A \) of an undirected graph \( G \) is real and symmetric, the eigenvalues of \( A \) are real, \( \lambda_i \in \mathbb{R} \) for all \( i = 1, ..., n \). Moreover, if \( x_i \) and \( x_j \) are eigenvectors for different eigenvalues, \( \lambda_i \neq \lambda_j \), then \( x_i \) and \( x_j \) are orthogonal, i.e. \( x_i^T x_j = 0 \) if \( i \neq j \). In particular, \( \mathbb{R}^n \) has an orthonormal basis consisting of eigenvectors of \( A \). Further, there exist matrices \( S \) and \( D \) such that \( S^T S = SS^T = I \) and \( SAS^T = D \), where \( D \) is the diagonal matrix of eigenvalues of \( A \) and \( I \) is the identity matrix. The Perron-Frobenius eigenvalue \( \lambda_{PF} \) is the largest real eigenvalue of \( A \), see Chapter A in the Appendix), i.e. all eigenvalues \( \lambda_i \) of \( A \) satisfy \( |\lambda_i| \leq \lambda_{PF} \) for \( i = 1, ..., n \) and there exists an associated nonnegative eigenvector \( v \geq 0 \) such that \( Av = \lambda_{PF} v \). For a connected
Chapter 2. Characterization of Networks

A cycle $C_4$ with its associated adjacency matrix $A$. The walks of length one are simply given by the adjacency matrix $A$. One can see in $A^2$ that for each pair of firms there always exists a walk of length 2 if the firms are not neighbors in $C_4$. For example, there are two walks (1, 2, 3) and (1, 4, 3) from firms 1 to 3 in $C_4$. Similarly, there exist 4 walks of length 3 between firms that are neighbors. For example, between firms 1 and 2 there are (1, 4, 3, 2), (1, 4, 1, 2), (1, 2, 1, 2) and (1, 2, 3, 2). $A^2$ and $A^3$ count the number of walks of length 2 and 3 in $C_4$, respectively. We can write $A^2 = 2(U - A)$ and $A^3 = 4A$.

Figure 2.3: A cycle $C_4$ with its associated adjacency matrix $A$. The walks of length one are simply given by the adjacency matrix $A$. One can see in $A^2$ that for each pair of firms there always exists a walk of length 2 if the firms are not neighbors in $C_4$. For example, there are two walks (1, 2, 3) and (1, 4, 3) from firms 1 to 3 in $C_4$. Similarly, there exist 4 walks of length 3 between firms that are neighbors. For example, between firms 1 and 2 there are (1, 4, 3, 2), (1, 4, 1, 2), (1, 2, 1, 2) and (1, 2, 3, 2). $A^2$ and $A^3$ count the number of walks of length 2 and 3 in $C_4$, respectively. We can write $A^2 = 2(U - A)$ and $A^3 = 4A$.

For large $n$, $G(n, m)$ can be described by $G(n, p)$ and $p = m/n$ since the actual number of links generated in $G(n, p)$ is almost always close to the expected value $m$ (West, 2001).
2.2 Network Statistics

This section covers only those network measurements that we will use throughout the thesis. For a more extensive survey see Costa et al. (2007) and also Newman (2003a) as well as Wasserman and Faust (1994). Note that the following definitions assume undirected graphs.

**Degree Distribution**  The degree distribution $p(d)$ gives the proportion of firms in the network $G$ with a given degree $d$. For a random graph $G(n, p)$ we have (Bollobas, 1985)

$$p(d) = \binom{n-1}{d} p^d (1-p)^{n-1-d},$$  \hspace{1cm} (2.2)

with the limit

$$\lim_{n \to \infty} p(d) = \frac{\bar{d}^{d-\bar{d}}}{d!},$$  \hspace{1cm} (2.3)

where $\bar{d} = p(n-1)$ is the average degree in $G(n,p)$. The degree distribution is thus a Poisson distribution for large $n$.

**Clustering**  The clustering coefficient $\mathcal{C}(u)$ for firm $u$ is the proportion of links between the firms within its neighborhood $N_u$ divided by the number of links that could possibly exist between them (Watts and Strogatz, 1998). We have

$$\mathcal{C}(u) = \frac{2|\{(vw : v, w \in N_u \land vw \in L\}|}{d_u(d_u-1)}.$$  \hspace{1cm} (2.4)

The total clustering coefficient is the average of the clustering coefficients over all firms, $\mathcal{C} = \frac{1}{n} \sum_{u \in N} \mathcal{C}(u)$.

A high clustering coefficient $\mathcal{C}$ means that two neighbors of a randomly selected firm are likely to be neighbors of each other. It also indicates a high redundancy of the network. For a complete graph $K_n$ it is trivially $\mathcal{C} = 1$. If $\bar{d} = p(n-1)$ denotes the average degree in a random graph $G(n,p)$, we get $\mathcal{C} = \frac{\bar{d}}{n-1} = p$. For a cycle $C_n$ we have for large $n$ $\mathcal{C} \sim \frac{3}{4}$ (Barrat and Weigt, 2000).

**Assortativity and Nearest Neighbor Connectivity**  Newman (2002) proposed an aggregate measure of firm degree correlations, which is called assortativity. It is defined as

$$\gamma = \frac{\sum_{jk} jk(e_{jk} - q_j q_k)}{\sigma_q^2},$$  \hspace{1cm} (2.5)
where $q_k$ denotes the remaining degree. $q_k$ is the value of the degree distribution of a firm at the end of a randomly chosen link. The distribution of $q_k$ is given by

$$q_k = \frac{(k + 1)p_k + 1}{\sum_j j p_j}. \quad (2.6)$$

$e_{jk}$ is the joint probability distribution of the remaining degrees of the two firms at either end of a randomly chosen link. For undirected networks $e_{ij} = e_{ji}$. Moreover, $\sum_j e_{jk} = 1$, and $\sum_j e_{jk} = q_k$. Finally, the variance is given by $\sigma^2_q = \sum_k k^2 q_k - (\sum_k k q_k)^2$. If a network exhibits positive degree correlations then it is said to be assortative while if it exhibits negative degree correlations it is dissortative. In an assortative network firms with high degree tend to be connected to firms of high degree and firms with low degree tend to be connected to firms with low degree.

Another measure of degree correlation has been introduced by Pastor-Satorras et al. (2001) called average nearest neighbor connectivity. More precisely, the average nearest neighbor connectivity $d_{nn}(d)$ is the average degree of the neighbors of a firm with degree $d$. It is defined by

$$d_{nn}(d) = \sum_{d'} d' P(d'|d), \quad (2.7)$$

where $P(d'|d)$ denotes the probability that a firm with degree $d$ has a neighbor with degree $d'$.

**Characteristic Path Length** The characteristic path length is defined as the number of links in the shortest path between two firms, averaged over all pairs of firms (Watts and Strogatz, 1998). This can be written as

$$L = \frac{1}{n(n-1)} \sum_{u \neq v} d(u, v), \quad (2.8)$$

where $d(u, v)$ is the length of the geodesic (shortest path) between firms $u$ and $v$.

For a $k$-regular graph the average path length is $L = \frac{n}{2k}$. For a complete graph $K_n$ it is $L = 1$. For a cycle $C_n$ we have $L = \frac{n+1}{4}$ if $n$ is odd and $L = \frac{n^2}{4(n-1)}$ if $n$ is even. For a random graph $G(n, p)$ we obtain $L = \frac{ln n}{\bar{d} d}$ where $\bar{d} = p(n-1)$ is the average degree in $G(n, p)$.

If there is no path between two firms in the network, we assume that the length of the shortest path between them is infinity. But this implies that the characteristic path length is not well defined for disconnected networks. However, taking the inverse of the shortest
2.3. Centrality and Centralization

In the following paragraphs we will introduce different measures of \textit{centrality} which incorporate different aspects of a firm’s position in the network (Borgatti, 2005a; Borgatti and Everett, 2006; Freeman, 1978). Degree centrality counts the number of links incident to a firm. Closeness centrality measures how many steps it takes to reach any other firm in the network. Betweenness centrality measures how many paths between any pair of firms pass through a particular firm. Eigenvector centrality measures the importance of a firm as being proportional to the importance of its neighbors. Finally, Bonacich centrality measures the number of weighted walks starting at a particular firm. The different measures of centrality capture different aspects of the position of a firm in a network and therefore the choice of the right measure depends on the particular application under consideration.

While centrality refers to the central position of an individual firm in a network, we can also consider how variable the individual centralities in the network are. \textit{Centralization} measures the extent to which centrality values are concentrated in a few firms in the network.

For analyzing different degrees of centralization we use the centralization index introduced by Freeman (1978) (see also Wasserman and Faust (1994)). Consider a particular centrality measure \( C(u) \) for any firm \( u \in N \) in the network \( G \). Then the centralization \( C \) of \( G \) is given by

\[
C = \frac{\sum_{u \in N} (C(u^*) - C(u))}{\max_{G'} \sum_{v \in N'} (C(v^*) - C(v))},
\]

where \( u^* \) and \( v^* \) are the firms with the highest values of centrality in the network and and the maximum in the denominator is computed over all networks \( G' \) with the same number of firms, \( |N| = |N'| \). Freeman (1978) has shown that for degree, closeness, betweenness and eigenvector centrality this maximum is obtained for the star \( K_{1,n-1} \). For the eigenvector and Bonacich centralization we will take the star as the maximally centralized network as well, assuming that it maximizes the sum of differences in eigenvector and Bonacich

\[
E = \frac{1}{n(n-1)} \sum_{u \neq v} \frac{1}{d(u, v)}.
\]

The network efficiency must not be confused with the efficiency in terms of total profits discussed in Chapter 4. The first is related to shortest paths in the network while the latter measures social welfare: a network is efficient if it maximizes aggregate profits.
centralities. Thus, $K_{1,n-1}$ has a centralization of 1 while, for example, any regular network has centralization 0.

**Degree Centrality** If we consider the degree of a firm as a measure of centrality then its centrality depends on the size of the network (with maximum centrality given by $n-1$). In order to overcome this bias one can consider the normalized degree centrality that divides the degree by $n-1$, yielding a measure in $[0,1]$. That is for a firm $u \in N$ the degree centrality is given by

$$C_d(u) = \frac{d_u}{n-1}$$  \hspace{1cm} (2.11)

We have that $d_u = \sum_{j=1}^{n} a_{uj} = \sum_{j=1}^{n} a_{ju}$ (since $A$ is symmetric). There are several applications of degree centrality, for example the popularity in friendship networks, the diffusion of information and the spread of infections (Wasserman and Faust, 1994).

The maximum sum of degree centrality differences is attained by the star $K_{1,n-1}$. The degree of the central firm $u^*$ in $K_{1,n-1}$ is $n-1$ and the degrees of the peripheral firms $u \neq u^*$ are 1. We get $\sum_{u \in N} (C_d(u^*) - C_d(u)) = (n-1)(n-2) = n^2 - 3n + 2$. Thus, the degree centralization is given by (Freeman, 1978)

$$C_d = \frac{\sum_{u \in N} (C_d(u^*) - C_d(u))}{n^2 - 3n + 2}. \hspace{1cm} (2.12)$$

**Closeness Centrality** Next, we consider a measure of centrality that takes into account by how many links (on the shortest path) a firm is separated from all other firms in the network. The closeness centrality of firm $i \in N$ is defined as (Beauchamp, 1965; Sabidussi, 1966)

$$C_c(u) = \frac{n-1}{\sum_{v \neq u} d(u,v)}, \hspace{1cm} (2.13)$$

where $d(u,v)$ measures the shortest path between firms $u$ and $v$. If a firm has high closeness centrality it can quickly interact with other firms and gather information from them since it has short communication paths to them.

In a star $K_{1,n-1}$ the sum of shortest paths from the central firm $u^*$ to all other firms is given by $n-1$ and the sum of shortest paths for all other firms $u \neq u^*$ is $(n-2)2 + 1 = 2n - 3$. Thus, $C_c(u^*) = 1$, $C_c(u) = \frac{n-1}{2n-3}$ and $\sum_{u \in N} (C_c(u^*) - C_c(u)) = (n-1) \left( 1 - \frac{n-1}{2n-3} \right) = \frac{n^2 - 3n + 2}{2n-3}$. It follows that the closeness centralization for the network $G$ is given by (Freeman, 1978)

$$C_c = \frac{\sum_{u \in N} (C_c(u^*) - C_c(u))}{(n^2 - 3n + 2)/(2n-3)} \hspace{1cm} (2.14)$$
2.3. Centrality and Centralization

**Betweenness Centrality** Firms which are not directly connected might depend on some other firm that lies on a path connecting them. If a firm lies on many such paths, connecting different components in a network, it has a high *betweenness centrality*. Betweenness centrality is defined as (Brandes, 2001; Freeman, 1977)

$$C_b(u) = \sum_{u\neq v\neq w} \frac{g(v, u, w)}{g(v, w)}, \quad (2.15)$$

where $g(v, w)$ denotes the number of shortest paths from firm $v$ to firm $w$ and $g(v, u, w)$ counts the number of paths from firm $v$ to firm $w$ that pass through a particular firm $u$. Normalized betweenness divides simple betweenness by its maximum value.

Freeman (1978) shows that *betweenness centralization* is given by

$$C_b = \frac{\sum_{u\in N} (C_b(u^*) - C_b(u))}{n^3 - 4n^2 + 5n - 2} \quad (2.16)$$

**Eigenvector Centrality** *Eigenvector centrality* measures the importance of a firm based on the importance of its neighbors. Even if a firm is connected to a few others (thus having a low degree centrality) its neighbors may be important, and therewith increase the firm’s importance, giving it a high eigenvector centrality. Let’s assume that the importance of a firm $i \in N$ is measured by $v_i$. Then the eigenvector centrality of firm $i$ is proportional to the sum of the eigenvector centralities of all firms which are connected to $i$ (Newman, 2007).

$$v_i = \frac{1}{\lambda} \sum_{j \in N_i} v_j = \frac{1}{\lambda} \sum_{j=1}^{n} a_{ij} v_j, \quad (2.17)$$

where $N_i$ is the set of firms that are connected to firm $i$, $n$ is the total number of firms and $\lambda$ is a constant. In matrix-vector notation we can write $A v = \lambda v$, which is the eigenvector equation. If the proportionality factor $\lambda$ is given by the largest eigenvalue $\lambda_{PF}$ of the adjacency matrix $A$ then all the elements in the eigenvector must be positive (Horn and Johnson, 1990) and we get a proper measure of centrality.

Next, we consider the star $K_{1,n-1}$. The eigenvector associated with the largest real eigenvalue $\lambda_{PF} = \sqrt{n-1}$ of $K_{1,n-1}$ is given by $v = \frac{1}{\sqrt{2(n-1)}} (1, ..., 1, \sqrt{n-1}, 1, ..., 1)^T$. The highest eigenvector centrality in the star $K_{1,n-1}$ is attained by the central firm $u^*$ and given by $\frac{1}{\sqrt{2}}$. All the remaining peripheral firms $u \neq u^*$ in $K_{1,n-1}$ have eigenvector centrality $\frac{1}{\sqrt{2(n-1)}}$. Thus we have that $\sum_{u \in N} (C_v(u^*) - C_v(u)) = \frac{n-1}{2} (\sqrt{n-1} - 1)$. Assuming that the maximum eigenvector centralization is attained by the star $K_{1,n-1}$, we obtain for
the eigenvector centralization
\[ C_v = \frac{1}{\sqrt{\frac{n-1}{2}} \sqrt{n-1-1}} \sum_{u \in N} (v_u^* - v_u). \] (2.18)

**Bonacich Centrality** In the following we define a related network centrality measure following Bonacich (1987). We consider the matrix
\[ B(G, \alpha) = \sum_{k=0}^{\infty} \alpha^k A^k. \] (2.19)

The Bonacich centrality vector is given by
\[ b(G, \alpha) = B(G, \alpha) \cdot u, \] (2.20)

where \( u = (1, \ldots, 1)^T \). One can show that, if and only if \( \alpha < 1/\lambda_{PF} \), then \( B(G, \alpha) = (I - \alpha A)^{-1} \) exists and is non-negative (Debreu and Herstein, 1953). Therefore, we can write the Bonacich centrality vector as
\[ b(G, \alpha) = \sum_{k=0}^{\infty} \alpha^k A^k \cdot u = (I - \alpha A)^{-1} \cdot u. \] (2.21)

For the components of the Bonacich vector we get
\[ b_i(G, \alpha) = \sum_{k=0}^{\infty} \alpha^k (A^k \cdot u)_i = \sum_{k=0}^{\infty} \alpha^k \sum_{j=1}^{n} (A^k)_{ij}. \] (2.22)

\( A^k \) is the \( k \)-th power of \( A \), with elements \( (A^k)_{ij} \). \( (A^k)_{ij} \) gives the number of path of length \( k \) in \( G \) between firms \( i \) and \( j \). Accordingly, \( \sum_{j=1}^{n} (A^k)_{ij} \) is the sum of all paths of length \( k \) in \( A \) starting from \( i \). It follows that \( b_i(G, \alpha) \) is the sum of all paths in \( G \) starting from \( i \), where the paths of length \( k \) are weighted by their geometrically decaying factor \( \alpha^k \).

We now compute the Bonacich centrality for the star \( K_{1,n-1} \). We have that
\[ (I - \alpha A)^{-1} = \frac{1}{1 - (n-1)\alpha^2} \begin{pmatrix} 1 & \alpha & \cdots & \cdots & \alpha \\ \alpha & 1 - (n-2)\alpha^2 & \alpha^2 & \cdots & \alpha^2 \\ \vdots & \alpha^2 & \ddots & \cdots & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ \alpha & \alpha^2 & \cdots & \alpha^2 & 1 - (n-2)\alpha^2 \end{pmatrix}. \] (2.23)
2.3. Centrality and Centralization

For the expression \((I - \alpha A)^{-1} \cdot u\) from Equation (2.21) we get

\[
b(K_{1,n-1}, \alpha) = \frac{1}{1 - (n-1)\alpha^2} (1 + (n-1)\alpha, 1 + \alpha, \ldots, 1 + \alpha)^T \tag{2.24}
\]

As in the case for the previous centralization measures we assume that the maximum eigenvector centralization is attained by the star \(K_{1,n-1}\) and we obtain for the Bonacich centralization

\[
C_B = \frac{\sum_{u \in N} (b_u(G, \alpha) - b_u(G, \alpha))}{\alpha(n-1)(n-2)/(1 - (n-1)\alpha^2)}. \tag{2.25}
\]

There exists an important relation between Bonacich and eigenvector centralities (see Bonacich, 1987). Let \(\{v = x_1, \ldots, x_n\}\) denote the set of orthonormal eigenvectors and \(\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\) are the associated eigenvalues of the adjacency matrix \(A\). We can write \(A = \sum_{i=1}^n \lambda_i x_i x_i^T\) and similarly for the powers of \(A\) we get \(A^k = \sum_{i=1}^n \lambda_i^k x_i x_i^T\). Assuming that \(\alpha < 1/\lambda_{PF}\), we can write

\[
b(G, \alpha) = (I - \alpha A)^{-1} u = (\sum_{k=0}^\infty \alpha^k A^k) u = (\sum_{k=0}^\infty \alpha^k \sum_{i=1}^n \lambda_i^k x_i x_i^T) u = (\sum_{i=1}^n (\sum_{k=0}^\infty \alpha^k \lambda_i^k) x_i x_i^T) u = \sum_{i=1}^n \frac{1}{1-\alpha \lambda_i} x_i x_i^T u. \tag{2.26}
\]

As \(\alpha\) approaches \(1/\lambda_{PF}\) from below all terms in the last equation from above become negligible compared to the first one and we obtain

\[
\lim_{\alpha \to 1/\lambda_{PF}} (1 - \alpha \lambda_{PF})b(G, \alpha) = (v^T u) v = v \sum_{j=1}^n v_j. \tag{2.27}
\]

We find that the Bonacich centrality \(b(G, \alpha)\) is proportional to the eigenvector centrality \(v\) as \(\alpha\) approaches \(1/\lambda_{PF}\) from below. Thus, if there are strong weights on the paths then Bonacich and eigenvector centrality orderings coincide. As an example consider the star \(K_{1,4}\) with \(n = 5\) firms. We have that \(\lambda_{PF} = 2\) and \(v = \frac{1}{2\sqrt{2}}(2, 1, 1, 1, 1)^T\) with \(v^T v = 1\).

Using the expression in Equation (2.24) we get \(b_1(K_{1,4}, \alpha) = \frac{1+4\alpha}{1-4\alpha^2}\) and \(b_{i>1}(K_{1,4}, \alpha) = \frac{1+\alpha}{1-4\alpha^2}\). With the limits \(\lim_{\alpha \to 1/2} (1-2\alpha) b_1(K_{1,4}, \alpha) = \frac{3}{2}\) and \(\lim_{\alpha \to 1/2} (1-2\alpha) b_{i>1}(K_{1,4}, \alpha) = \frac{3}{4}\), we find that

\[
\lim_{\alpha \to 1/2} (1 - 2\alpha)b(K_{1,4}, \alpha) = \frac{2}{3\sqrt{2}}v. \tag{2.28}
\]

Moreover, similarly to the eigenvector centrality, we can write

\[
\lim_{\alpha \to 1/\lambda_{PF}} \frac{b_i(G, \alpha)}{\sum_{j \in N_i} b_j(G, \alpha)} = \frac{1}{\lambda_{PF}}. \tag{2.29}
\]

The above equation can be easily verified for the case of the star \(K_{1,n-1}\) using Equation (2.24).
Chapter 3

Innovation and Profit from R&D Collaborations

There are various approaches in the literature to explain when and why firms form collaborations. Nevertheless, the rationale of firms to engage in collaborations can be traced back to two main motives (see also Ebers and Grandor, 1997).

On one hand, firms attempt to increase their revenues through inter-organizational networking. Through networking, firms can gain access to complementary resources and assets and enhance their competitiveness by jointly developing a new product or service innovations which increase their revenues (Ebers and Grandor, 1997).

On the other hand, inter-organizational collaborations can be motivated by contingent cost reductions. Cost reductions can be the result of economies of scale and scope for joint research activities (Contractor and Lorange, 1988). Goyal and Moraga-Gonzalez (2001) has studied R&D networks assuming that firms form collaborations in order to share knowledge about a cost reducing technology (see also Gersbach and Schmuzler, 2003b; Kamien et al., 1992). We will further discuss this model in Section 3.7.3 in this chapter. Moreover, firms may economize on the costs of organizing, managing and coordinating their research activities (Hennart, 1991; Thorrelli, 1986). Finally, firms can also reduce their risk when investing in uncertain innovations in collaboration with other firms (Contractor and Lorange, 1988).

In this chapter we follow the first approach and propose a model in which firms can jointly invest into R&D activities and therewith create profitable innovations that increase their revenues. We will briefly consider the effect of independent R&D activities and we discuss the distribution of knowledge and profits following from our assumptions.
3.1 A Schumpeterian Approach

In this section we introduce a model in which firms exploit R&D collaborations to introduce innovations to the industry. Decisions over R&D partners are made at discrete times $t = T, 2T, 3T, ...$ where the length of a period is given by $T > 0$. Innovations are introduced during each period $(t, t + T]$. The rewards from each innovation are assumed to be appropriable so that an innovation returns a value equal to the constant $V > 0$. We follow a Schumpeterian approach similar to the theoretical literature on innovation and endogenous technical change (see e.g. Aghion and Howitt, 1998; Reinganum, 1983, 1985; Winter, 1984), in which we assume that the introduction of innovations by a firm $i \in N$ is governed by a non-homogeneous Poisson process with arrival rate equal to $h_i(\tau)$, where $\tau \geq 0$ indicates the time variable within a period. Thus, the probability that an innovation is introduced by firm $i$ in the interval $d\tau$, is equal to $h_i(\tau)d\tau$.

Denote by $\tau_0$ the arrival time, within the interval $(t, t + T]$, of the first innovation. The probability that an innovation is introduced by firm $i$ in the interval $(0, \tau_0]$ is exponentially distributed,

$$\text{Prob}(\tau < \tau_0) = 1 - e^{-\int_0^{\tau_0} h_i(\tau)d\tau}. \quad (3.1)$$

Thus, the higher is the arrival rate $h_i(\tau)$ the more likely it is that firm $i$ introduces an innovation before time $\tau_0$.

Moreover we assume that, for any firm $i$, the arrival rate of innovations is proportional to the growth rate $\rho_i(\tau)$ of knowledge

$$h_i(\tau) = b\rho_i(\tau), \quad b > 0. \quad (3.2)$$

In other words, the higher the growth rate of new ideas, the more likely it is that the firm will be able to innovate. Similar to Carayol and Roux (2003a,b) expected revenues of firm $i$ in a period $(t, t + T]$ are given by the value $V$ of each innovation times the expected number of innovations in the period. Note that in Equation (3.2) the innovation process starts anew at the beginning of every period $(t, t + T]$, taking as initial condition the stock of knowledge at the end of the previous period $(t - 1, t - 1 + T]$. In addition, let us set $\tau \in (0, T]$. From Equation (3.2) the expected number of innovations in a period $(t, t + T]$ can be written as

$$\int_0^T h_i(\tau)d\tau = b\int_0^T \rho_i(\tau)d\tau. \quad (3.3)$$

In turn, the growth rate of knowledge is affected by the network of collaborations as follows. In each period $(t, t + T]$, new knowledge is generated by recombining the existing knowledge

\footnote{Note that both, the innovation arrival rate $h_i(\tau)$ and the growth rate of knowledge $\rho_i(\tau)$ are flow variables and are measured per unit of time.}
3.1. A Schumpeterian Approach

stocks of firms in the economy through the existing network of R&D collaborations (see Kogut and Zander, 1992; Weitzman, 1998). More precisely, let us denote by \( x_i(\tau) \) the stock of knowledge of firm \( i \) at time \( \tau \in (t, t + T] \). Then, new knowledge within firm \( i \) is generated according to

\[
\dot{x}_i(\tau) = \sum_{j=1}^{n} a_{ij}(t)x_j(\tau), \tag{3.4}
\]

where \( a_{ij}(t) \) are the elements of the adjacency matrix \( A(G(t)) \) (defined in Chapter 2) corresponding to the network of R&D collaborations\(^2\). In vector-matrix notation, Equation (3.4) reads \( \dot{x}(\tau) = A(G(t))x(\tau) \). Note also that in Equation (3.4) for non-negative initial values of \( x(0) \geq 0 \), we have that \( \dot{x}(\tau) \geq 0 \) as well as \( x(\tau) \geq 0 \), that is, all firms have non-negative, growing knowledge stocks.

The growth rate of knowledge of firm \( i \), \( \rho_i(\tau) = \dot{x}_i(\tau)/x_i(\tau) \), in Equation (3.4) is directly affected by the growth rate of knowledge of its neighbors, whose growth rate is affected by the growth rate of their neighbors, and so on. Therefore, the topology of the whole network of R&D collaborations (including all direct and indirect paths along which knowledge can flow between the firms), influences the innovation process within the firm.

Collaborations also imply a cost for the firms. Within a period \((t, t + T]\) each collaboration involves a fixed cost\(^3\) per unit of time equal to \( \bar{c} \). Moreover, we assume that firms are risk-neutral. Finally, if we denote by \( G_i(t) \) the connected component to which firm \( i \) belongs in the period, then expected profits for the firm at the beginning of the period can be written as

\[
\tilde{\pi}_i(G_i(t), c, t) = bV \int_{0}^{T} \rho_i(\tau)d\tau - \bar{c}Td_i(t), \tag{3.5}
\]

where \( d_i(t) \) is the degree of the firm at time \( t \) and during the period.

The timing of events in each period \((t, t + T]\) runs as follows: at the beginning \( t \) of the period, the network of R&D collaborations is determined (only one link is added or removed at time \( t \), based on expected profits), and remains fixed throughout the period \((t, t + T]\). During the period \((t, t + T]\), firms recombine their knowledge stocks through the network while they also bear the costs of their collaborations. As a result, innovations are introduced and the rents accrue to the firms.

\(^2\)In Equation (3.4) we are assuming that the process of creation of ideas at the firm level is cumulative, in that larger knowledge stocks (of the firm and of its collaborators) lead to higher knowledge growth. This property of knowledge dynamics has often been emphasized in innovation studies (e.g. Dosi, 1993).

\(^3\)In Chapter 6 we will depart from this assumption (setting \( c = 0 \)) and introduce a model in which the cost of a collaboration is (i) stochastically changing and (ii) depending on the particular partner with whom the collaboration is maintained.
Chapter 3. Innovation and Profit from R&D Collaborations

The expression for expected profits in Equation (3.5) can be directly related to the structure of the network of collaborations. For this purpose, the next proposition establishes a relation between, on one hand, the asymptotic growth rate of ideas, the asymptotic relative stock of knowledge and the rate of convergence, and, on the other hand, the eigenvalues and eigenvectors of the adjacency matrix $A(G_i(t))$ of the connected component of firm $i$.

**Proposition 1** Consider the eigenvalues $\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ associated with the adjacency matrix $A(G_i(t))$ of the connected component $G_i(t)$ of firm $i \in N(G_i(t))$. Then the following results hold:

(i) The asymptotic knowledge growth rate of a firm $i$ is constant and equal to the largest real eigenvalue (Perron-Frobenius eigenvalue) of the adjacency matrix $A(G_i(t))$

$$\lim_{\tau \to \infty} \rho_i(\tau) = \lambda_{PF}(G_i(t)).$$

The rate of convergence is $O \left( e^{-|\lambda_{PF}(G_i(t)) - \lambda_2(G_i(t))|} \right)$ as $\tau \to \infty$.

(ii) The asymptotic value of a firm $i$’s relative knowledge stock equals the element $v_i$ of the eigenvector $v$ associated with the eigenvalue $\lambda_{PF}(G_i(t))$

$$\lim_{\tau \to \infty} \frac{x_i(\tau)}{\sum_{j=1}^{n} x_j(\tau)} = v_i.$$  

**Proof 1** (i) The general solution (Braun, 1993; Horn and Johnson, 1990; Khalil, 2002) of the system of linear ordinary differential equations in Equation (3.4) is

$$x(t) = e^{At}x(0),$$

where $x(0)$ is the initial distribution of knowledge and $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is the matrix exponential. The real, symmetric matrix $A(G)$ is diagonalizable (Haemers, 2006) and thus, the matrix exponential can be written as

$$e^{At} = S \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix} S^{-1}.$$  

$\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ are the real eigenvalues of $A$ and $S$ is a non-singular matrix whose columns are the eigenvectors of $A$. From Equation (3.9) we can see that the general solution of Equation (3.4) can be written as (Zwillinger, 1998)

$$x(t) = \sum_{j=1}^{n} c_j v_j e^{\lambda_j t},$$
3.1. A Schumpeterian Approach

where $c_i$ are unknown constants that are determined by the initial values $x(0) = \sum_{j=1}^{n} c_j v_j$, $\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the real eigenvalues of $A$ and $v_1, \ldots, v_n$ the corresponding eigenvectors. In Equation (3.10) only those eigenvalues and corresponding eigenvectors of the adjacency matrix of the connected component $G_i$ of firm $i$ appear. All other eigenvalues have vanishing eigenvector components and do not contribute to the trajectory. This is intuitively clear since firms in disconnected components have decoupled equations of the form (3.4) and their trajectories can be computed independently. We get

$$
\lambda_{PF} - \frac{\dot{x}_i(t)}{x_i(t)} = \frac{\lambda_{PF} \dot{x}_i(t) - \dot{x}_i(t)}{x_i(t)} = \sum_{j=1}^{n} c_j v_j e^{\lambda_j t} (\lambda_{PF} - \lambda_j)
= \sum_{j=2}^{n} c_j v_j e^{\lambda_j t} (\lambda_{PF} - \lambda_j)
= \frac{\sum_{j=1}^{n} c_j v_j e^{\lambda_j t}}{\sum_{j=1}^{n} c_j v_j e^{\lambda_j t}}.
$$

(3.11)

In the numerator of Equation (3.11) we obtain a sum of exponentials with one exponential term less than in the denominator, namely the one with the largest real eigenvalue. We have that the sum of exponentials converges to the exponential with the largest real eigenvalue. Consider for example $ae^{\lambda_1 t} + be^{\lambda_2 t} = ae^{\lambda_1 t} (1 + \frac{b}{a} e^{(\lambda_2 - \lambda_1) t}) \sim ae^{\lambda_1 t}$ for large $t$. Thus we get

$$
\lambda_{PF} - \lim_{t \to \infty} \frac{\dot{x}_i(t)}{x_i(t)} = \lim_{t \to \infty} \frac{c_2 v_2 e^{\lambda_2 t} (\lambda_{PF} - \lambda_2)}{c_1 v_1 e^{\lambda_{PF} t}} \lim_{t \to \infty} e^{-(\lambda_{PF} - \lambda_2) t} = 0.
$$

(3.12)

Next, we compute a lower bound for the difference $\lambda_{PF} - \lambda_2$ and thus the order of convergence. Consider the real eigenvalues $\lambda_{PF} = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of the adjacency matrix $A$. We have that $\sum_{j=1}^{n} \lambda_j^2 = \text{tr}(A^2) = 2m$ (Bollobas, 1998). Thus, we get

$$
\lambda_2^2 = 2m - \lambda_{PF}^2 - \sum_{j=3}^{n} \lambda_j^2
\leq 2m - \lambda_{PF}^2
\leq 2m - \left(\frac{2m}{n}\right)^2
= \frac{2m(n^2 - 2m)}{n^2}.
$$

(3.13)

Here we have used the fact that $\lambda_{PF} \geq \frac{2m}{n}$ (Bollobas, 1998). Therefore we get

$$
\lambda_{PF} - \lambda_2 \geq \frac{2m - \sqrt{2m(n^2 - 2m)}}{n},
$$

(3.14)

which is positive and a monotonically increasing function for $n^2/4 < m \leq n(n-1)/2$. We thus have a fast convergence for dense networks. We note however, that in many cases the largest real eigenvalue is well separated from the second largest real eigenvalue. For example, in a random graph $G(n, p)$, the largest real eigenvalue grows with $n$ (keeping $p$ constant) while the upper bound on the second largest real eigenvalue grows with $\sqrt{n}$ (Restrepo et al., 2007).
(ii) We show that the knowledge shares converge to the eigenvector $v$ associated with the largest real eigenvalue $\lambda_{PF}$ of $A$. The relative values of knowledge (shares) are given by

$$y_i = \frac{x_i}{\sum_{j=1}^{n} x_j}; \quad \sum_{j=1}^{n} y_j = 1.$$  

(3.15)

Rewriting Equation (3.4) by means of Equation (3.15) gives us the dynamics of the shares of knowledge

$$\dot{y}_i = \sum_{j=1}^{n} a_{ji}y_j - y_i \sum_{k,j=1}^{n} a_{jk}y_j.$$  

(3.16)

Equation (3.16) preserves the normalization of $y$. If we consider a normalized eigenvector $y^{(\lambda)}$, $\sum_{j=1}^{n} y_j^{(\lambda)} = 1$, associated with an eigenvalue $\lambda$ of $A$ we have

$$\sum_{j=1}^{n} a_{ij}y_j^{(\lambda)} = \sum_{j=1}^{n} a_{ji}y_j^{(\lambda)} = \lambda y_i^{(\lambda)}.$$  

(3.17)

Inserting $y^{(\lambda)}$ into Equation (3.16) yields

$$\dot{y}_i = \sum_{j=1}^{n} a_{ji}y_j^{(\lambda)} - y_i^{(\lambda)} \sum_{k,j=1}^{n} a_{jk}y_j^{(\lambda)}$$

$$= \lambda y_i^{(\lambda)} - y_i^{(\lambda)} \sum_{k,j=1}^{n} a_{jk}y_j^{(\lambda)}$$

$$= \lambda y_i^{(\lambda)} - \lambda y_i^{(\lambda)} = 0.$$  

(3.18)

Thus, $y^{(\lambda)}$ is a stationary solution of Equation (3.16). It can be shown that an eigenvector $v$ associated with the largest real eigenvalue $\lambda_{PF}$ of $A$ is the stable fixed point of Equation (3.16)$^4$ (see also Jain and Krishna, 1998, 2002; Krishna, 2003).

For a more detailed discussion of the asymptotic distribution of knowledge shares we refer to Section (3.5).

This concludes the proof. □

Item (i) in Proposition (1) states that the knowledge dynamics defined in Equation (3.4) converge, for a given R&D network, to a steady state characterized by a constant growth rate of ideas. In addition, the constant growth rate depends on the topology of the connected component that the firm belongs to (through the largest eigenvalue $\lambda_{PF}(G_i(t))$). This implies that, in the steady state, the arrival rate of an innovation is constant and

$^4$If the largest real eigenvalue has multiplicity more than one then the stable fixed point can be written as a linear combination of the associated eigenvector and generalized eigenvectors (Braun, 1993).
equal to $b \lambda_{PF}$. Moreover, item (ii) implies that the topology of the connected component $G_i(t)$ determines the distribution of relative values of the knowledge stocks between firms in the same component. Finally, the rate of convergence to the steady state is determined by the eigenvalues of $A(G_i)^5$.

An important assumption in our model is that the growth of knowledge is much faster than the formation of R&D collaborations. This is equivalent to saying that $\tau$ is measured in time units much smaller than those used to measure $t$. In other words, $t = k \tau$, with $k$ large$^6$. Under this assumption, the expected number of innovations per unit of time can be approximated (taking the limit $k \to \infty$) with the largest real eigenvalue of a firm’s connected component.

**Corollary 1** The expected number of innovations of firm $i \in N$ per unit of time in a period $(t, t + T]$ tends to a limit proportional to the largest real eigenvalue of firm $i$’s connected component $G_i$.

$$\lim_{k \to \infty} \frac{b}{kT} \int_0^{kT} \rho_i(\tau) d\tau = b \lambda_{PF}(G_i(t)). \quad (3.19)$$

**Proof 2** The proof follows directly from an application of the following lemma.

**Lemma 1** Consider a continuous function $f : [0, \infty) \to \mathbb{R}$ that converges to a finite value $\lambda$, i.e. $\lim_{t \to \infty} f(t) = \lambda < \infty$. Then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) dt = \lambda. \quad (3.20)$$

**Proof 3** Denote $F(T) = \frac{1}{T} \int_0^T f(t) dt$. We can write

$$F(T) = \frac{1}{T} \int_0^{\tau'} f(t) dt + \frac{1}{T} \int_{\tau'}^T f(t) dt. \quad (3.21)$$

$^5$In general, the convergence in a connected component to its largest real eigenvalue is always guaranteed. In addition, more dense networks are characterized by a faster convergence (see the proof of Proposition (1) in the appendix). However, the convergence can be slow for sparse networks and particular network topologies. For a recent application and discussion of the convergence properties of the social network matrices see Jackson and Golub (2007).

$^6$t and $\tau$ stand for the numerical values in different units of the slow and fast time scales with $t = k \tau$. For example, if $\tau$ is measured in hours and $t$ is measured in years, this would give $k \approx 10^7$ and the two time scales would differ in seven orders of magnitude.
The first integral in the above expression is finite since any continuous function on the compact set \([0, \tau']\) has a maximum denoted by \(c\). Since \(f(t)\) converges to \(\lambda\), for any \(\epsilon'\), we can find a \(\tau'(\epsilon')\) such that for all \(t \geq \tau'\) we have \(|f(t) - \lambda| < \epsilon'\). Thus we get

\[
\begin{align*}
|F(T) - \lambda| &= \left| \frac{1}{T} \left( \int_0^{\tau'} f(t) dt + \int_{\tau'}^T f(t) dt - \lambda T \right) \right| \\
&\leq \frac{1}{T} \left( |c|\tau' + \left| \int_{\tau'}^T f(t) dt - \lambda T \right| \right) \\
&\leq \frac{1}{T} \left( |c|\tau' + \int_{\tau'}^T |f(t)| dt - \lambda dt + (T - \tau' - T)\lambda \right) \\
&\leq \frac{1}{T} (|c|\tau' + \epsilon'(T - \tau') - \tau'\lambda) \\
&= \frac{|c|\tau'}{T} + \frac{T-\tau'\epsilon'}{T} \\
&\leq \frac{|c|\tau'}{T} + \epsilon'.
\end{align*}
\]

We define

\[
\epsilon = \frac{|c|\tau'}{T} + \epsilon' \quad \tau = \frac{|c|\tau'}{c - \epsilon'}. \tag{3.23}
\]

Since \(\frac{\partial}{\partial T} = -\frac{|c|\tau'}{T^2} < 0\) we have that \(|F(T) - \lambda| < \epsilon\) for \(T > \tau\). For any \(\epsilon > 0\) we can find an \(\epsilon' < \epsilon\) (e.g. \(\epsilon' = \epsilon/2\)) and the corresponding \(\tau'(\epsilon')\) from which we compute \(\tau(\epsilon)\) such that \(|F(T) - \lambda| < \epsilon\) for all \(T > \tau(\epsilon)\). This means that \(\lim_{T \to \infty} F(T) = \lambda\). \(\Box\)

Thus, the corollary follows. \(\Box\)

Expected profits of the firm at beginning of the period \((t, t + T]\) can now be written as

\[
\tilde{\pi}_i(G_i(t), c, t) = b\lambda_{PF}(G_i(t))VT - \tilde{c}d_i(t)T. \tag{3.24}
\]

Applying an affine transformation to the above equation, we finally obtain expected profits per unit of time in the period between \(t\) and \(t + T\),

\[
\pi_i(G_i(t), c, t) = \lambda_{PF}(G_i(t)) - cd_i(t), \tag{3.25}
\]

where \(c = \frac{\tilde{c}}{b}\) is the marginal cost of link formation (rescaled by the factor \(1/bV\)).\(^7\) Since the largest eigenvalue \(\lambda_{PF}(G_i(t))\) in Equation (3.25) is the same for all firms in a connected component, the expected revenues from R&D collaborations will be the same for all the members of \(G_i\). Nonetheless, profits from R&D collaborations vary, in general, across firms, since each firm may have a different number of collaborations. The following lemma\(^8\) characterizes the relation between the largest eigenvalue of a connected component and the creation or removal of R&D collaborations.

\(^7\)The introduction of linear and homogeneous in-house R&D activities in Equation (3.4) for the dynamics of knowledge stocks would not alter the functional form of profits (up to a constant). See also the discussion on basic research in Section 3.2.

\(^8\)A proof of the foregoing lemma can be found in Cvetkovic et al. (1995).
3.2. Introducing Basic Research

**Lemma 2** Denote \( G' = (N', L') \) the graph obtained from the graph \( G = (N, L) \) by the addition or removal of a link. Then

(i) \( \lambda_{PF}(G') \geq \lambda_{PF}(G) \) if \( ij \notin L \) and \( \lambda_{PF}(G') \leq \lambda_{PF}(G) \) if \( ij \in L \).

(ii) \( \lambda_{PF}(G') \leq \lambda_{PF}(K_n) = n - 1 \).

(iii) \( |\lambda_{PF}(G') - \lambda_{PF}(G)| \leq 1 \).

Thus, the largest real eigenvalue in a component is a non decreasing function of the number of links. In addition, it is a bounded function, since its value can never be higher than the one associated with the complete graph \( K_n \). Finally, the change in the eigenvalue is itself a bounded function, since its value must be less than one. The preceding observations deliver two central properties of the model.

First, since the probability of innovation is the same for all firms in a given connected component and it is affected by each link, the creation (deletion) of a collaboration by one firm has a positive (negative) non-rival external effect on all its direct and indirect neighbors in the component. As we will discuss in the model introduced in Section 5.1, this property also implies that the network can evolve into equilibria that are socially inefficient.

Second, the marginal revenue from R&D collaborations is always a positive (albeit bounded) function of the number of links. This means that the creation (deletion) of a new R&D collaboration increases (decreases) the probability of an innovation and thus the expected revenue. Moreover, the revenue itself is a bounded function of the number of links. The last property does not imply that the revenue is also a concave function of the number of links\(^9\). However, we will show in the model discussed in Section 5.1 that this implies that in the presence of marginal costs \( c > 0 \), as the network grows in the number of links, the highest marginal revenue that can actually be obtained from the creation of a new link or from the removal of an existing one can become very small.

### 3.2 Introducing Basic Research

Hitherto, we have assumed that firms can solely create innovations through joint R&D activities. However, there may be situations in which a government allocates resources for

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\(^9\)Incidentally, note that \( \lambda_{PF} \) is not even determined as a function of \( m \), because, for a given \( m \), there are many different ways to arrange the links among the firms, resulting in different values of \( \lambda_{PF} \).
basic research from which all firms in the sector can benefit (as a public good) (see also Gersbach et al., 2008; Salter and Martin, 2001).

In a more general setting than the one we have considered in the previous section we can assume that the growth of knowledge depends on two factors: (i) a constant increase in knowledge due to basic R&D and (ii) R&D collaborations in which an increase in the knowledge stock of a firm may be assumed to be proportional to the knowledge levels of its R&D partners. Let us assume that the increase in a firm’s stock of knowledge due to basic research is proportional to its current stock of knowledge. This implies that a firm can benefit more from basic research if it is technologically advanced and has accumulated some knowledge already (Cohen and Levinthal, 1989). Let \(\gamma \in \mathbb{R}_+\) and consider the following knowledge growth dynamics

\[
\dot{x}_i = \sum_{j=1}^n a_{ij}x_j + \gamma x_i.
\]  

(3.26)

With the asymptotic limit

\[
\lim_{t \to \infty} \rho_i(t) = \lim_{t \to \infty} \frac{\dot{x}_i}{x_i} = \lambda_{PF}(G_i) + \gamma. 
\]  

(3.27)

Profits \(\tilde{\pi}\) are then given by

\[
\tilde{\pi}_i = (b\lambda_{PF}(G_i) + b\gamma - \tilde{c}d_i)V.
\]  

(3.28)

Applying an affine transformation \(\pi_i = \alpha\tilde{\pi} + \beta, \alpha > 0\) with \(\alpha = 1/bV, \beta = -\gamma\) and \(c = \tilde{c}/b\) yields

\[
\pi_i = \lambda_{PF}(G_i) - cd_i.
\]  

(3.29)

Thus, including a constant increase in knowledge due to basic research does not alter the profit function of firms (up to an affine transformation). However, it is straightforward to show that it increases aggregate profits of the economy.

### 3.3 The Ex Ante and Ex Post Number of Innovations

In this section we discuss two alternative but equivalent ways to evaluate the benefit of a collaboration on the basis of expected unitary profits from Equation (3.25). On one hand, firms can form expectations on future innovations and profits they will make. If these expected profits are positive, they start a new collaboration, otherwise not. In this case firms are forward looking and evaluate the number of innovations they will make \textit{ex ante}. On the other hand, firms can form collaborations by trial and error. After a certain
time they evaluate the collaboration on the basis of the innovations and profits it has generated. Here, firms are backward looking and they count the number of innovations they have made \textit{ex post}. In this section we will show that both behavioral assumptions, forward and backward looking behavior, use the profit function in Equation (3.25) as the basis on which collaborations are evaluated, \textit{ex ante} or \textit{ex post}, if the evaluation period is long enough.

The actually observed number of innovations in a certain period approaches the expected number of innovations if the period is long enough. In order to see this, we make the following observations. We denote the number of innovations that firm \(i\) makes between time \(t\) and \(t+T\) by \(N_i(t, t+T) = N_i(T+t) - N_i(t)\). The expected number of innovations in the period is \(h_i(t, t+T) = \int_{t}^{t+T} h_i(\tau) d\tau\). We have

\[
\text{Prob}(N_i(T+t) - N_i(t) = k) = \text{Prob}(N_i(t, t+T) = k) = e^{-h_i(t,t+T)} \frac{h_i(t, t+T)^k}{k!}.
\] (3.30)

Now we can derive an asymptotic relation between the observed number of innovations \(N_i(t, t+T)\) (in a particular observation or sample of the Poisson process) and the expected number of innovations \(h_i(t, t+T)\) in the interval \((t, t+T]\).

\textbf{Lemma 3} Let \(N_i(t, t+T)\) be a non-homogeneous Poisson process as defined above with expected value \(h_i(t, t+T)\). Then

\[
\text{Prob}\left(\lim_{T \to \infty} \frac{N_i(t, t+T)}{h_i(t, t+T)} = 1\right) = 1
\] (3.31)

\textbf{Proof 4} We consider the expression for the number of innovations in the time interval \((t, t+T]\) given by \(\int_{t}^{t+T} h_i(t') dt' = \int_{0}^{T} h_i(s') ds'\) where we have substituted \(t'\) with the shifted time \(s' = t' - t\). Now we can consider the non-homogeneous Poisson process \(N(0, T) = N(T)\), denoting the number of events in the time interval \((0, T]\). Moreover, by introducing the modified time \(s = h(0, T) = \int_{0}^{T} h_i(\tau) d\tau\) we obtain a homogeneous Poisson process \(Y(s) = N(t)\) with expected value \(E(Y(s)) = 1\) (Daley and Vere-Jones, 2003). Using the law of large numbers for the homogeneous Poisson process (Durrett, 2004) we get

\[
\text{Prob}\left(\lim_{s \to \infty} \frac{Y(s)}{s} = 1\right) = \text{Prob}\left(\lim_{T \to \infty} \frac{N(T)}{h(0, T)} = 1\right) = 1
\] (3.32)

where \(s \to \infty\) implies \(T \to \infty\). □

Lemma (3) implies that the observed number of innovations per unit of time \(N_i(t, t+T)/T\) equals the expected number of innovations per unit of time \(h_i(t, t+T)/T\) if \(T\) is large.
enough. Note that, if a firm has increased its innovation arrival rate from one period to the next \( h_i(t, t + T) > h_i(t - 1, t - 1 + T) \) then \( \text{Prob}(N_i(t, t + T) > N_i(t - 1, t - 1 + T)) = 1 \) if \( T \) is large enough. This means that with probability one an increase in the innovation arrival rate increases the observed number of innovations in a period long enough.

Lemma (3) allows us to make two different assumptions on the behavior of firms that yield identical link formation decisions. Firms can (i) ex ante form expectations on their future profits or (ii) firms can ex post evaluate the consequences of their linking activities from their past profits.

(i) On one hand, we can assume that firms are forming ex ante expectations on the profits they will earn by creating an additional collaboration. If the time horizon of the firms over which they form these expectations is long enough, then expected profits are determined by Equation (3.25).

(ii) On the other hand, firms may experiment with different collaborations and evaluate them after a certain trial time. In this case firms are not assumed to form any expectations ex ante but they are evaluating the consequences of their actions ex post. From Lemma (3) we know that the average number of innovations generated in a particular realization of the Poisson innovation process converges to the expected value. Thus, we can take expected profits in Equation (3.25) as a sufficient approximation to the actually observed profits a firm is earning ex post, if the time over which the collaboration was maintained is long enough.

In the following chapters we will restrain to the first interpretation in which firms are forming ex ante expectations on their future profits (forward looking behavior). This allows us to use stability concepts from the literature on network formation. However, this rational expectations approach requires full knowledge of firms on their connected component. Nevertheless, we have shown in this section that an alternative interpretation can be made assuming that firms are evaluating the consequences of their actions ex post (backward-looking behavior) in which case firms would be facing the same profit function from Equation (3.25).

### 3.4 Marginal Revenues and the Relative Stock of Knowledge

In the profit function from Equation (3.25) the increase in revenues due to the formation of a new link depends on the increase in the largest real eigenvalue (see Lemma (2)).
There exists a close relationship between the increase in the largest real eigenvalue $\Delta \lambda_{PF}$ (proportional to a firm’s revenues) due to the formation of the link $ij$ and the eigenvector components $v_i$ and $v_j$ that determine the asymptotic shares of knowledge of firms $i$ and $j$ (see Proposition (1)). Let $G + ij$ denote the network obtained from the network $G$ by adding the link $ij$ between firms $i$ and $j$ in the same component $G$. From Theorem (2) in Appendix B we see that, in a first-order approximation\(^{10}\), the increase in profits due to the establishment of a new collaboration $ij$ is $\Delta \pi_i = \Delta \pi_j = \Delta \lambda_{PF}(G + ij) - c \approx 2v_iv_j - c$. Here $v_i = \lim_{t \to \infty} x_i / \sum_{k=1}^{n} x_k$ and $v_j = \lim_{t \to \infty} x_j / \sum_{k=1}^{n} x_k$ are the asymptotic shares of knowledge of the firms. Thus, in a first-order approximation, a link between two firms creates higher profits the higher their asymptotic shares of knowledge are.

The relationship between the change in the largest real eigenvalue and the eigenvector components allows us to give alternative interpretations of the link formation processes we will study in the following chapters. In Chapter 5 a link formation process in which a link $ij$ is formed if $\Delta \lambda_{PF}(G + ij) > c$ for both firms $i$ and $j$ is introduced. From Theorem (2) we conclude that, in a first-order approximation, this is equivalent to a link formation process in which a link between two firms $i$ and $j$ is created if the product of their levels of output and R&D effort, respectively, is larger than the marginal cost $c$ associated with the creation of the link. In Chapter 6 we will study a link formation process where a link $ij$ is formed between firms $i$ and $j$ with the highest eigenvector components $v_i$ and $v_j$. Together with Theorem (2) this implies that, in a first-order approximation, firms are (strict) preferentially forming a new link to another firm that yields the highest increase in their profits $\pi_i$ and $\pi_j$. We further discuss the the asymptotic distribution of knowledge in the next Section 3.5 and compare it to the distribution of profits in Section 3.6.

### 3.5 The Distribution of Knowledge

In this section we analyze in more detail the distribution of the asymptotic shares of knowledge. In Proposition (1) we have seen already that the knowledge shares converge to the eigenvector $\mathbf{v}$ associated with the largest real eigenvalue $\lambda_{PF}$ of the adjacency matrix $\mathbf{A}$. This follows from the fact, that an eigenvector associated with the largest real eigenvalue of $\mathbf{A}$ is the stable fixed point of the dynamics of the knowledge shares $\mathbf{x} / \sum_{j=1}^{n} x_j$ from Equation (3.16). The general result is stated in the next proposition.

\(^{10}\)For large networks and in particular for networks with skewed degree distributions, the creation of an additional link has a small effect on the spectral properties of the network (Restrepo et al., 2006). In this case, the approximations of the perturbation analysis in Appendix B hold. For more detail see Theorem (2). However, for an exact relationship between the largest real eigenvalue and the associated eigenvector when a link is added to the network, we refer to Rowlinson (1990).
Proposition 2 (Krishna (2003)) Consider the the simplex \( S = \{ y = (y_1, ..., y_n)^T \in \mathbb{R}_+^n : 0 \leq y_i \leq 1 \text{ for all } i = 1, ..., n, \sum_{i=1}^n y_i = 1, \} \) and the following system of ordinary differential equations

\[
\dot{y}_i = \sum_{j=1}^n a_{ij} y_j - y_i \sum_{k,j=1}^n a_{jk} y_j, \quad i = 1, ..., n, \tag{3.33}
\]

where \( y \in S \) and \( a_{ij} \) is the \( ij \)-th element of the adjacency matrix \( A \) of the graph \( G \). Then we have the following:

(i) Every eigenvector \( v \) of \( A \) in the simplex \( S \) is a fixed point of Equation (3.33) and vice versa.

(ii) Starting from any initial condition in the simplex \( S \), the trajectory of Equation (3.33) converges to some fixed point in \( S \).

(iii) For generic initial conditions in \( S \), the fixed point of Equation (3.33) is an eigenvector \( v \) associated with the largest real eigenvalue \( \lambda_{PF} \) of \( A \).

(iv) If \( A \) has a unique (up to constant multiples) eigenvector \( v \), it is the unique stable attractor of Equation (3.33).

In particular, if the adjacency matrix \( A \) is primitive, the eigenvector \( v \) is unique and there is a unique stable attractor. Interpreting these results in our model, one can state the following proposition.

Proposition 3 If the adjacency matrix \( A \) of the graph \( G \) of collaborations between firms is primitive, there is a unique asymptotic distribution of relative values of knowledge \( x_1 / \sum_{j=1}^n x_j \) given by the eigenvector \( v \) associated with the largest real eigenvalue \( \lambda_{PF} \) of \( A \).

If the assumption of primitivity of the matrix fails, in particular, if the matrix is non-negative but not irreducible, then there are, in general, several eigenvectors associated with the largest real eigenvalue \( \lambda_{PF} \) of \( A \) and thus several possible equilibria for the relative values of knowledge, depending on the initial conditions.

It is useful to look at an alternative but equivalent way to characterize a primitive graph. A graph \( G \) is primitive if and only if it is connected and the greatest common divisor of the set of length of all cycles in \( G \) is one (Xu, 2003). This means, for instance, that the connected graph consisting of two connected firms is not primitive as the only cycle has length two (since the link is undirected a walk can go forward and backward along the
3.5. The Distribution of Knowledge

link). Similarly, a path or a tree is not primitive neither, since all cycles have even length. However, if we add one link in order to form a triangle, the graph becomes primitive. The same is true, if we add links in order to form any cycle of odd length. We can state the following result.

**Proposition 4** If the graph $G$ is connected, the presence of one cycle of odd length is a sufficient condition for the primitivity of $G$ and hence for the uniqueness of the relative knowledge distribution $x/\sum_{j=1}^{n} x_j$ given by the eigenvector $v$ associated with the largest real eigenvalue $\lambda_{PF}$ of $A$.

We now discuss the relation between walks in the graph and growth rate of knowledge. In our model, a walk in the graph corresponds to a sequence of firms contributing to their individual knowledge to neighbors in the walk in order to generate a sequence of knowledge recombined. As mentioned in Section 2.1, each component of the power $k$ of the adjacency matrix, $(A^k)_{ij}$, gives the number of walks of length $k$ from firms $i$ to $j$. Considering the vector $u = (1, \ldots, 1)^T$, we have that $n_k := u^T A^k u$ is the number of all walks of length $k$ among all firms in $G$. Since the adjacency matrix is symmetric we have that $u = \sum_i a_i w_i$ where $w_i$ is the eigenvector of $A$ associated with the eigenvalue $\lambda_i$. It follows that $n_k = \sum_i |a_i|^2 \lambda_i^k$. For large $k$, we have approximately $n_k \sim \lambda_{PF}^k$ (Chung and Lu, 2007), and we get

$$\frac{n_k - n_{k-1}}{n_{k-1}} \sim \lambda_{PF} - 1. \quad (3.34)$$

Thus, the largest real eigenvalue $\lambda_{PF}$ of the graph measures the growth rate of the number of walks of length $k$ when the length increases by one, as well as the growth factor of the number of knowledge recombinations in the network of collaborations$^{11}$. As we have seen in Proposition (1), $\lambda_{PF}$ also coincides with the asymptotic growth rate of knowledge in time. Therefore, the faster the number of walks in the graph (and thus the possible knowledge recombinations) grows with the length of these walks, the faster grow the asymptotic knowledge shares of the firms in time. One should not confuse these two growth rates, one in time and the other one with respect to walk length (which does not vary in time, as we are analyzing a static network).

A similar interpretation can be derived from the Rayleigh-Ritz theorem (Horn and Johnson, 1990) which states that

$$\lambda_{PF} = \max_{x \neq 0} \frac{x^T A x}{x^T x}, \quad (3.35)$$

$^{11}$We point out that knowledge does not decay along a path connecting firms. This means that firms have full and non-attenuated access to the knowledge in their connected component. For a different approach we refer to Jackson and Wolinsky (1996) and the discussion in Section 3.7.1.
where the maximum is obtained for the eigenvector $v$ associated with $\lambda_{PF}$. Here, $x$ can be any vector in $\mathbb{R}^n$. $x_i x_j$ can be interpreted as the result of the recombination of the knowledge of firms $i$ and $j$ if they are connected. Accordingly, one can interpret the right-hand side of Equation (3.35) as the maximum number of total knowledge recombinations, $\mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_{i,j} x_i a_{ij} x_j$, normalized to the sum of squares of knowledge, $\mathbf{x}^T \mathbf{x} = \sum_{i=1}^n x_i^2$, generated in the economy. Some other results relate $\lambda_{PF}$ to the number of links or the degree of the firms in the graph. For instance, the Perron-Frobenius theorem (see Chapter A in the Appendix) states that $\lambda_{PF}$ is bounded from below and above by the minimum and maximum degree respectively ($d_i = \sum_j a_{ij}$ is the degree of firm $i$). This means, that the higher (minimum or maximum) is the degree of the firms in the graph, the higher is $\lambda_{PF}$ and thus the asymptotic knowledge growth rate. We denote the maximum degree in $G$ by $\Delta$. Then, we can give a better lower bound for the largest real eigenvalue $\sqrt{\Delta} \leq \lambda_{PF}(G) \leq \Delta$. We refer to Cvetkovic et al. (1995); Cvetkovic and Rowlinson (1990) for other inequalities involving $\lambda_{PF}$.

There is also a result about the inequality of the growth rate of knowledge across firms. For a primitive matrix $\mathbf{A}$ one can show (Boyd, 2006) that the eigenvector associated with the eigenvalue $\lambda_{PF}$ is the solution to the following optimization problem

$$\max_{\mathbf{x} > 0} \min_{1 \leq i \leq n} \frac{\sum_{j=1}^n a_{ij} x_j}{x_i} \quad (3.36)$$

where $\sum_{j=1}^n a_{ij} x_j = (\mathbf{A} \mathbf{x})_i = \dot{x}_i$. The eigenvector associated with the largest real eigenvalue $\lambda_{PF}$ of $\mathbf{A}$ is the vector that maximizes the minimum growth factor over all firms $i$ and also minimizes the maximum growth factor.

**Proposition 5** If the graph $G$ is primitive, the unique stable distribution of relative knowledge values $\mathbf{x}/\sum_{j=1}^n x_j$ to which the dynamics in Equation (3.4) converges, is also the distribution that minimizes the difference between maximum and minimum growth rates across firms.

From the results above we can conclude that the profit function of firm $i$ in Equation (3.25) increases with the number of walks in the connected component to which firm $i$ belongs. On the other hand, profits decrease with the degree of the firm. Therefore it is best for a firm to be able to reach other firms through many walks while not having too many links.
3.6 Disparity in the Distribution of Profits and Knowledge

Firms with the largest knowledge stock are only the most profitable firms if the marginal cost for forming links vanishes. However, if positive costs for forming links are taken into account, then significant differences in the profitability of a firm and its share of the total knowledge produced in the industry can exist.

Figure (3.1) shows an example in which relative profits of central firms (with a higher degree) are smaller than those of peripheral firms (with a smaller degree) compared to the maximum profits in the network. This stems from the fact that all firms in one connected component have the same expected knowledge growth rate (given by the largest real eigenvalue) and accordingly their expected returns from their research activities are identical. However, central firms have a higher degree and therefore their costs of collaboration are higher. This means that firms with a higher degree earn lower profits than firms with a smaller degree. This is a main difference to the profit function introduced in Goyal and Moraga-Gonzalez (2001) (see Section 3.7.3). There, the central firms in the network earn higher profits than the peripheral firms. However, our results would match with the ones from Goyal and Moraga-Gonzalez (2001) if we compare asymptotic knowledge shares instead of profits. From the knowledge growth function in Equation (3.4), it follows that central firms have a higher share of the total knowledge produced in the industry than peripheral firms. In order to see this, we consider the star \( K_{1,n-1} \) as an example of a network with a typical core-periphery structure. Following Proposition (1) the asymptotic relative stock of knowledge is determined by the eigenvector associated with the largest real eigenvalue \( \lambda_{PF} \) of \( A \), that is, \( \lim_{t \to \infty} x_i / \sum_{j=1}^{n} x_j v_i \), where \( v_i \) is the \( i \)-th component of the eigenvector \( v \) associated with the eigenvalue \( \lambda_{PF} \). In a star \( K_{1,n-1} \), the central firm has an eigenvector component \( 1/\sqrt{2} \) while all other firms have an eigenvector component \( 1/\sqrt{2(n-1)} \). In this example, we see that the central firm indeed has the highest eigenvector component and therewith the highest relative stock of knowledge.

As we have discussed in this section, there may be significant differences between firms’ profits and knowledge stocks. In Chapter 4 we will proceed by analyzing the differences in profits in the efficient network (that is maximizing aggregate profits). Moreover, in Chapters 5 and 6 we will study two particular network formation processes which give rise to networks that can exhibit large inequalities in profits and the shares of knowledge of firms in the industry. However, before we continue our analysis of the distribution of profits and knowledge, we compare our model with other prominent models on R&D networks in the literature.
Chapter 3. Innovation and Profit from R&D Collaborations

Figure 3.1: The distribution of profits and the distribution of knowledge for a star $K_{1,9}$. In (a) the brightness of the node color indicates profits and in (b) the relative stock of knowledge. Firms with highest (lowest) profits have the lowest (highest) shares of knowledge in the industry.

3.7 Relation to the Literature

In the following we will discuss several prominent models on R&D collaboration networks from the literature. We will discuss the similarities and differences to the model we have introduced in the previous sections and we will work out how it can extend and improve on the existing literature.\textsuperscript{12}

3.7.1 The Connections Model

For instance, the utility function proposed in the “connections” model of Jackson and Wolinsky (1996) is given by

$$ u_i = \sum_{j=1}^{n} \delta^{d(i,j)} - cd_i, $$

where $0 < \delta < 1$ and $d(i,j)$ is the length of the shortest path from node $i$ to node $j$, $i, j \in N = \{1, ..., n\}$ in the network $G$.

The difference between the profit function in (3.25) and the utility function in (3.37) becomes apparent in the benefit term. While Equation (3.37) considers the shortest path between firm $i$ and $j$ only, our model takes all possible walks from firm $i$ to the other firms in the connected component into account.\textsuperscript{13} Recall that, in our model, a walk represents

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\textsuperscript{12}Here we we only consider models in which an R&D collaboration is not viewed as an exclusive arrangement. Being part of a collaboration does not prevent a firm from being part of another. Therefore, we do not discuss coalition network formation models such as the one introduced by Bloch (1995).

\textsuperscript{13}It has been argued that in several settings paths that are not the shortest may have a significant impact on the information that is transmitted from one agent to another (see e.g. Stephenson and Zelen,
a sequence of knowledge recombinations among the firms along that walk. Not all recombination of knowledge translate into a successful innovation. However, the more walks there are in the component, the higher is the number of possible knowledge recombinations available. It turns out that the likelihood for a successful innovation is increased. Indeed, the largest eigenvalue $\lambda_{PF}(G_i)$ of the adjacency matrix of a connected component $G_i$, is related to the number of possible walks in that component (more precisely, the growth rate in the number of walks of length $k$ tends to $\lambda_{PF}(G_i)$; this property has been discussed in Section 3.5). Thus, the larger is $\lambda_{PF}(G_i)$, the larger is the number of possible knowledge recombinations via direct and indirect R&D collaborations. From Equation (3.25) we can conclude that profits of firm $i$ grow with the number of walks in the connected component to which firm $i$ belongs. On the other hand, profits decrease with the degree $d_i$ of the firm. Therefore, it is best for a firm to be able to reach other firms through many walks while not having too many links to pay for. This observation is not shared with the connections model. It becomes apparent if one considers the following simple example. The revenues of the hub in a star $K_{1,n-1}$ and a node in a complete graph $K_n$ in Equation (3.37) are identical, because the shortest paths to all the other nodes are one link long in both cases. This is not the case in our model where these two graphs generate very different revenues. A node in the complete graph $K_n$ can reach the other nodes through many different paths and this generates a much higher revenue than the one of the hub in a star $K_{1,n-1}$.

### 3.7.2 The Two-Way Flow Model

In their linear “two-way flow” model, Bala and Goyal (2000) introduce a utility function of the form

$$u_i = |G_i| - cd_i,$$  \hspace{1cm} (3.38)

where $|G_i|$, $G_i \subseteq G$, is the size of the connected component of firm $i \in N = \{1, ..., n\}$ in the network $G$. This means that the utility of firm $i$ grows with the number of all firms in the network which can be reached by firm $i$ across at least one path. The number of links and the number of paths between $i$ and the other firms do not matter because the benefit flow across the network is assumed to be independent of its topology. In contrast, in our model, the topological properties of the component that the firm belongs to are critical for its profits. Consider the following simple examples. According to Equation (3.38), revenues for a firm in the complete graph $K_n$, in the clique $K_{1,n-1}$ and in the cycle $C_n$, are identical. However, in our model the revenues a firm earns from being part of a

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1989; Wasserman and Faust, 1994). Moreover, empirical studies on R&D networks (cf. Powell et al., 2005; Riccaboni and Pammolli, 2002) bring support to the claim that firms are forming R&D collaborations in a way that increases the number of walks in the network.
clique $K_n$ are higher than in a star $K_{1,n-1}$, which in turn are higher than in a cycle $C_n$ for the same number of firms. This ranking can be understood if one considers the possible walks in these graphs. The number of walks is highest in the complete graph $K_n$ while it is smallest in the cycle $C_n$ (that contains only one walk). While all these graphs encompass the same number of firms in the connected component, they differ significantly in the way the links are arranged among the firms.

3.7.3 Network Games with Local Payoff Complementarities

A payoff function that considers all walks in the network and thus comes closer to our model has been introduced in Ballester et al. (2006). The authors study a general network game with local payoff complementarities and globally-uniform payoff substitutabilities. Player $i \in N =\{1,...,n\}$ selects an effort $x_i \in \mathbb{R}_+$ and receives the following linear-quadratic payoff

$$u_i(x) = \alpha x_i - \frac{\beta}{2} x_i^2 - \gamma \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} a_{ij} x_i x_j.$$  \hspace{1cm} (3.39)

Now we can turn to the equilibrium analysis of the game in which agents set their payoffs simultaneously on a (static) network $G$ and receive a payoff given by the above Equation (3.39). An interior Nash equilibrium in pure strategies $x^* \in \mathbb{R}_n^+$ of the simultaneous move game derives from the first order conditions $\frac{\partial u_i(x^*)}{\partial x_i} = 0$ for all $i = 1, ..., n$. Ballester et al. (2006) show that if $\lambda/\beta < 1/\lambda_{PF}$ this condition is also sufficient and one obtains a unique Nash equilibrium solution.

**Proposition 6 (Ballester et al. (2006))** Denote $\lambda^* = \lambda/\beta$ and let $b_i(G, \lambda^*)$ be the Bonacich centrality of parameter $\lambda^*$ introduced in Section 2.3. For $\lambda^* < \lambda_{PF}$ the unique Nash equilibrium solution of the simultaneous $n$-player move game with payoffs given by Equation (3.39) and strategy space $\mathbb{R}_n^+$ is given by

$$x_i^* = \frac{\alpha}{\beta + \gamma \sum_{j=1}^{n} b_j(G, \lambda^*) b_i(G, \lambda^*)}$$ \hspace{1cm} (3.40)

for all $i = 1, ..., n$.

In the following we will give two examples in the context of R&D networks in which the Nash equilibrium actions of the agents are given by their network centrality, as it is the case in Equation (3.40).

(i) An application of the payoff function introduced in Equation (3.39) are R&D collaboration networks. Following Bala and Goyal (2000); Goyal and Joshi (2003); Goyal
and Moraga-Gonzalez (2001); Zenou (2006) we consider a Cournot oligopoly game in which a set $N = \{1, ..., n\}$ of firms is competing on a homogeneous product market. The total output of the firms is given by $x = \sum_{i=1}^{n} x_i$. Firms face an inverse linear demand of the form $p(x) = \Phi - \sum_{i=1}^{n} x_i$. Firm $i \in N$ then sets her quantity $x_i$ in order to maximize its profit $\pi_i$ given by

$$\pi_i(x, G) = \left( \Phi - \sum_{j=1}^{n} x_j \right) x_i - c_i x_i.$$  \hfill (3.41)

Firms can reduce their costs for production by establishing an R&D collaboration with another firm. The amount of this cost reduction depends on the effort the firms invest into the collaboration (see also Gersbach and Schmutzler, 2003a,b; Kamien et al., 1992, for similar models where research effort reduces production costs). Given the collaboration network $G$, every firm sets an R&D effort level unilaterally\footnote{See Kamien et al. (1992) for a similar model of competitive RJVs in which firms unilaterally choose their R&D effort levels.}. We assume that firms can only jointly develop a cost reducing technology. Given the effort levels $e_i$, marginal cost $c_i$ of firm $i$ is given by

$$c_i = c_0 - \sum_{j \in N_i} e_j = c_0 - \sum_{j=1}^{n} a_{ij} e_j,$$  \hfill (3.42)

and $a_{ij} = 1$ if firms $i$ and $j$ set up a collaboration (0 otherwise) and $a_{ii} = 0$. $A$ is the adjacency matrix with elements $a_{ij} \in \{0, 1\}$. Firms’ profits are then given by

$$\pi_i(x, G) = \left( \Phi - c_0 \right) x_i - \sum_{j=1}^{n} x_i x_j + \sum_{j=1}^{n} x_i e_j.$$  \hfill (3.43)

If we further assume that the R&D collaboration effort $e_j$ of firm $j$ is proportional to firm $j$’s output $x_j$, i.e. $e_j = \lambda x_j$, then we can write Equation (3.41) as follows\footnote{Compared to the original model by Goyal and Moraga-Gonzalez (2001) we have set $\gamma = \beta = 0$. $\gamma = 0$ implies that there are no (direct) costs for R&D collaborations and $\beta = 0$ implies that there are no spillovers from non-collaborating firms. Note however, that the introduction of direct costs for the effort invested into collaborations would still give a payoff function of the form in Equation (3.39).}.

$$\pi_i(x, G) = \alpha x_i - \sum_{j=1}^{n} x_i x_j + \lambda \sum_{j=1}^{n} a_{ij} x_i x_j,$$  \hfill (3.44)

where we have denoted $\alpha = \Phi - c_0$. Firms set their quantities $x$ and make profits given by Equation (3.44). The profit function in Equation (3.44) is a special case of the utility function in Equation (3.39) if we set $\beta = 1$ and $\gamma = 0$. Therefore, equilibrium profits are given by Equation (3.40) with the corresponding parameter values.
Another example of a game in which an agent’s actions have positive externalities and actions are strategic substitutes has been introduced by Bramoulle and Kranton (2007) in the context of the provision of public goods in networks. The authors consider strategic experimentation in R&D in which innovations resulting from R&D activities, are shared with the neighbors in the network. Firm \( i \in N = \{1, \ldots, n\} \) invests an amount \( x_i \in \mathbb{R}_+ \) in its R&D activities. \( x = (x_1, \ldots, x_n)^T \) is the effort profile of all firms. Let \( \bar{x} = \sum_{j \in N} x_j = \sum_{j=1}^n a_{ij} x_j \) be the total contribution of the neighbors of firm \( i \). Further we assume that the effort of the firms are not perfect substitutes. The level of substitutability between the efforts of the neighbors of a firm and its own effort is measured by the parameter \( \rho \in [0, 1) \). \( \rho = 1 \) would correspond to perfect substitutes. Each firm receives benefits from its own and its neighbors’ efforts according to a strictly concave benefit function \( b(\cdot) \), where \( b(0) = 0 \) and \( b' > 0, b'' < 0 \). The individual marginal cost of effort is constant and equal to \( c \). The profits of firm \( i \) from the effort profile \( x \) and network \( G \) may then be written as

\[
\pi_i(x) = b \left( x_i + \rho \sum_{j=1}^n a_{ij} x_j \right) - cx_i. \tag{3.45}
\]

Let \( \bar{A} \) be the matrix obtained from \( A \) with elements \( \bar{a}_{ij} = 1 - a_{ij} \) and \( \bar{a}_{ii} = 0 \) for all \( i, j = 1, \ldots, n \). The network \( \bar{G} \) associated with \( \bar{A} \) is the complement of \( G \). Two firms are adjacent in \( \bar{G} \) if and only if they are not adjacent in \( G \). Bramoulle et al. (2008); Calvó-Armengol (2007) show that if \( 1 > \rho \lambda_{PF} \), the equilibrium efforts are interior and uniquely defined by

\[
x_i^* = \frac{1 + \rho b(G, \rho)}{b_i(G, \rho)}, \quad \text{for all } i = 1, \ldots, n. \tag{3.46}
\]

In the following we work out the similarities between the revenues in our model, given in Equation (3.25), and the Nash equilibrium output, given by Equation (3.40). In order to simplify our analysis, we choose parameter values \( \alpha = \beta = 1 \) and \( \gamma = 0 \) such that equilibrium efforts and payoffs are given by \( b_i(G, \lambda^*) \) and \( \frac{1}{2} b_i(G, \lambda^*)^2 \), respectively, for all \( i = 1, \ldots, n \) (similar to Corbo et al., 2006).

Bonacich (p.161 1991) has shown that the eigenvector associated with the largest real eigenvalue of the adjacency matrix is the limit of the Bonacich centrality defined in Equation (2.20) as \( \lambda \) approaches \( 1/\lambda_{PF} \) from below (see also Chapter 2, Section 2.3). This finding can be carried over to the equilibrium outputs and payoffs. There exists an asymptotic result as the network externalities grow, that is, as \( \lambda \) approaches \( 1/\lambda_{PF} \) from below, stated in the following corollary.
3.7. Relation to the Literature

Corollary 2 Let $\lambda_{PF}$ be the largest real eigenvalue of $A(G)$ and $v$ the associated eigenvector, that is $(Av)_i = \lambda_{PF}v_i$, for all $i = 1, ..., n$. Further assume that $\sum_{j=1}^n v_j = 1$. Setting $\alpha = \beta = 1$ and $\gamma = 0$ in the payoff function from Equation (3.39), implies that in the Nash equilibrium we have

$$\lim_{\lambda^* \uparrow 1/\lambda_{PF}} (1 - \lambda^* \lambda_{PF})x^* = \lim_{\lambda^* \uparrow 1/\lambda_{PF}} (1 - \lambda^* \lambda_{PF})b(G, \lambda^*) = v. \quad (3.47)$$

From Corollary (2) we thus find that Nash equilibrium outputs are proportional to the asymptotic shares of knowledge from Proposition (1) (see also Section 3.4). In Chapter 6 we will introduce a link formation process in which firms with high asymptotic knowledge shares are forming collaborations with each other. In the light of the previous discussion we find that this link formation process is equivalent to a setup in which links are formed between firms with the highest Nash equilibrium outputs based on the payoff function in Equation (3.40) with appropriate parameter values. In both cases, the establishment of a new collaboration is more likely between firms with high productivity, either in knowledge or output. This result is supported by a number of empirical studies on interfirm collaboration networks (Gulati and Gargiulo, 1999; Ijiri and Simon, 1977; Riccaboni and Pammolli, 2002).

All the models discussed in Section 3.7.3 assume that the network is exogenously given. However, following the discussion in Section 1.10, R&D collaboration networks must be considered as the endogenous outcome of the individual link formation decisions of firms over time. Accordingly, we improve on the models discussed in this section by introducing different network formation processes in Chapters 5 and 6.

However, before analyzing the network evolution, we first study the efficient networks that maximize aggregate profits as a measure for social welfare as well as aggregate output in production and knowledge creation in the next chapter. We will further analyze the distribution of profits in the efficient network and the conditions under which this distribution is skewed, creating inequality in individual profits.
Chapter 4

Efficient Networks

In the model presented in the previous section, firms face a trade-off between increasing the probability to innovate by forming R&D collaborations and the cost of sharing knowledge with other firms in the industry. In this section we investigate how this trade-off can be managed in order to yield the best outcome for the industry. First, we show that there is an interval for the marginal cost of link formation, \( c \in [0, 1] \), in which the network that maximizes social welfare, that is the efficient graph, is a connected graph. We will show in Section 5.1 that this interval is the one of main interest since for values above this interval, \( c > 1 \), firms do not have the incentive to form any additional collaborations. In particular, this implies that an empty network is stable.

We then investigate the topology of the efficient graph, and show that it belongs to a well defined class of connected graphs, the “nested split graphs” characterized by a core-periphery structure. In particular, for \( c \) small enough, the efficient graph is the complete graph. On the other hand, for higher values of \( c \) and a larger number of firms, the efficient graph is sparser and shows a strong degree heterogeneity. In addition, we show that it is characterized by significant inequality in profits.

4.1 Efficient Networks

Following Jackson and Wolinsky (1996), we define industry welfare as the sum of firms’ individual profits

\[
\Pi(G, c) = \sum_{i=1}^{n} \pi_i(G_i) = \sum_{i=1}^{n} (\lambda_{PF}(G_i) - cd_i) = \sum_{i=1}^{n} \lambda_{PF}(G_i) - 2mc. \tag{4.1}
\]
We are interested in the solution of the following social planner’s problem. Let \( G(n) \) denote the set of all possible graphs with \( n \) firms. For a given value of cost \( c \), the social planner’s solution is given by

\[
G^* = \arg \max_{G \in \mathcal{G}(n)} \Pi(G, c).
\]  

(4.2)

A graph \( G^* \) solving the maximization problem (4.2), will be denoted as “efficient”.

In order to solve this problem we begin by identifying an interval for the marginal cost \( c \) in which industry welfare is increased by connecting two disconnected components of the network. The following lemma can be stated.

**Lemma 4** Consider a graph \( G \) consisting of two disconnected components \( G_1 \) and \( G_2 \), with \( n_1 \), \( n_2 \) firms, \( m_1 \), \( m_2 \) links, eigenvalues \( \lambda_{PF}(G_1) \), \( \lambda_{PF}(G_2) \) and total profits \( \Pi(G_1) = n_1\lambda_{PF}(G_1) - 2m_1c \), \( \Pi(G_2) = n_2\lambda_{PF}(G_2) - 2m_2c \). We further assume that \( c \in [0,1] \). Then there exists a connected graph \( G' \) with \( n = n_1 + n_2 \) firms that has higher total profits than \( G \), that is \( \Pi(G') \geq \Pi(G) = \Pi(G_1) + \Pi(G_2) \).

**Proof 5** Since \( G_1 \) and \( G_2 \) are connected, we have that \( m_1 \geq n_1 - 1 \) and \( m_2 \geq n_2 - 1 \) (West, 2001). We now consider different cases for the number of links in the components.

1. \( m_1 \geq n_1 \) and \( m_2 \geq n_2 \): Assume that the largest eigenvalue of \( G_1 \) is \( \lambda_{PF}(G_1) \geq \lambda_{PF}(G_2) \). Let \( G' \) be the graph obtained as follows: for each firm in \( G_2 \) we rewire one incident link to a firm in \( G_1 \). In this way, all firms in \( G_2 \) are connected to \( G_1 \). The number of rewired links is \( n_2 \) (and there are at least that many links since \( m_2 \geq n_2 \) by assumption). There exists a relationship between the largest real eigenvalue of a graph and those of its subgraphs (Cvetkovic et al., 1995): if \( H \) is a subgraph of \( G \), \( H \subseteq G \), then \( \lambda_{PF}(H) \leq \lambda_{PF}(G) \). Therefore, \( \lambda_{PF}(G') \geq \lambda_{PF}(G_1) \geq \lambda_{PF}(G_2) \). Total profits of \( G' \) are

\[
\Pi(G') = (n_1 + n_2)\lambda_{PF}(G') - 2(m_1 + m_2)c \\
\geq n_1\lambda_{PF}(G_1) + n_2\lambda_{PF}(G_2) - 2(m_1 + m_2)c \\
= \Pi(G_1) + \Pi(G_2).
\]  

(4.3)

2. \( m_1 \geq n_1 \) and \( m_2 = n_2 - 1 \): If \( m_2 = n_2 - 1 \) then the largest real eigenvalue of \( G_2 \) is at most the one of the star \( K_{1,n_2-1} \) with \( \lambda_{PF}(G_2) \leq \sqrt{n_2-1} \) (Hong, 1993).

We construct the graph \( G \) by connecting all firms of \( K_{1,n_2-1} \) to a single firm in \( G_1 \) and including the remaining isolated firm by adding one more link. The graph \( G \) has an eigenvalue \( \lambda_{PF}(G) \geq \lambda_{PF}(K_{1,n_1+n_2-1}) = \sqrt{n_1+n_2-1} \). Otherwise, the links in \( G \) are redistributed to form a star \( K_{1,n_1+n_2-1} \) and the remaining links are attached at
random. Since $\lambda_{PF}$ is an increasing function of the number of links in the graph the inequality follows. We obtain

$$
\Pi(G) = (n_1 + n_2)\lambda_{PF}(G) - 2(m + 1)c
$$

$$
\Pi(G_1) + \Pi(G_2) = n_1\lambda_{PF}(G_1) + n_2\sqrt{n_2 - 1} - 2\left(m_1 + (n_2 - 1)c\right)
$$

(4.4)

Thus, we get

$$
\Pi(G) - (\Pi(G_1) + \Pi(G_2)) = \Pi(G) - (\Pi(G_1) + \Pi(K_{1,n_2-1}))
$$

$$= n_1\left(\lambda_{PF}(G) - \lambda_{PF}(G_1) \geq 0\right)
$$

$$+ n_2(\lambda_{PF}(G) - \sqrt{n_2 - 1}) - 2c
$$

$$\geq n_2(\lambda_{PF}(G) - \sqrt{n_2 - 1}) - 2c.
$$

(4.5)

With $\lambda_{PF}(G) \geq \sqrt{n_1 + n_2 - 1} \geq \sqrt{n_2 + 1}$ if $n_1 \geq 2$ (by assumption). If the last inequality above is larger than 0, we have that

$$
\sqrt{n_2 + 1} - \sqrt{n_2 - 1} \geq \frac{2c}{n_2}.
$$

(4.6)

If $0 \leq c \leq 1$, this inequality is true if $n_2 \geq 2$ (by assumption).

(iii) $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$: If $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$, then both components are stars, $K_{1,n_1-1}$ and $K_{1,n_2-1}$ with eigenvalues $\sqrt{n_1 - 1}$ and $\sqrt{n_2 - 1}$. Construct the graph $G$ by connecting $n_2 - 1$ firms from $K_{1,n_2-1}$ to the central firm in $K_{1,n_1-1}$. Then attach a link to the remaining isolated firm to obtain a star $G = K_{1,n_1+n_2-1}$. Then

$$
\Pi(G) = \Pi(K_{1,n_1+n_2-1}) = (n_1 + n_2)\sqrt{n_1 + n_2 - 1} - 2c(n_1 + n_2 - 1)
$$

(4.7)

For the difference we get

$$
\Pi(G) - (\Pi(K_{1,n_2-1}) + \Pi(K_{1,n_2-1})) = n_1(\sqrt{n_1 + n_2 - 1} - \sqrt{n_1 - 1})
$$

$$+ n_2(\sqrt{n_1 + n_2 - 1} - \sqrt{n_2 - 1}) - 2c
$$

$$\geq (n_1 + n_2)(\sqrt{n_1 + n_2 - 1} - \sqrt{n_1 - 1}).
$$

(4.8)

Without loss of generality, we have assumed that $n_1 \geq n_2$. The expression above is larger or equal than 0 if

$$
\frac{(n_1 + n_2)(\sqrt{n_1 + n_2 - 1} - \sqrt{n_1 - 1})}{\sqrt{m_1 + 1} - \sqrt{m_1 - 1}} \geq 2 \geq 2c,
$$

(4.9)

with $n_2 \geq 2$. We get

$$
\sqrt{n_1 + 1} - \sqrt{n_1 - 1} \geq \frac{2}{n_1 + 2},
$$

(4.10)

and the last inequality holds for $n_1 \geq 2$. 
(iv) $n_1 \geq 2$ and $n_2 = 1$: We have one isolated firm and a connected graph $G_1$. Total profits are $\Pi(G) = n_1 \lambda_{PF}(G_1) - 2m_1c$. Denote the graph $G'$ obtained by connecting the isolated firm to $G_1$. Then

$$
\Pi(G') = (n_1 + 1)\lambda_{PF}(G') - 2(m_1 + 1)c \\
\geq \Pi(G) + (\lambda_{PF}(G') - 2c),
$$

(4.11)

We now consider three more cases:

(1) If $n_1 \geq 4$, then $m_1 \geq n_1 - 1$ (since $G_1$ is connected by assumption). We can construct a star $K_{1,n_1-1}$ plus additional links from $G_1$ and connect the isolated firms to it. Denote the resulting graph $G'$. Then, $\lambda_{PF}(G') \geq \lambda_{PF}(K_{1,n_1}) = \sqrt{n_1} \geq 2$. Thus, $\Pi(G') - \Pi(G) \geq 0$ if $\lambda_{PF}(G') \geq 2 \geq 2c$ for $c \in [0, 1]$.

(2) If $n_1 = 3$, then $G_1$ is either a path $P_3$ of length 3 or a cycle $C_3$ containing 3 firms. We connect the isolated firm to $G_1$. In the case of $G_1 = P_3$ we get

$$
\Pi(G') - \Pi(G) = \frac{4\sqrt{3} - 6c - (3\sqrt{2} - 4c)}{\Pi(G')} - \frac{3\sqrt{2} - 4c}{\Pi(G)} \\
= 2.69 - 2c > 0,
$$

(4.12)

where the last inequality follows from $c \in [0, 1]$. In the case of $G_1 = C_3$ we obtain

$$
\Pi(G') - \Pi(G) = \frac{42.17 - 8c - (32 - 6c)}{\Pi(G')} - \frac{32 - 6c}{\Pi(G)} = 2.68 - 2c > 0
$$

(4.13)

again, using $c \in [0, 1]$.

(3) For $n_1 = 2$ we connect the isolated firm to $G_1 = P_2$ and again denote the resulting connected graph $G'$. We have that

$$
\Pi(G') - \Pi(G) = \frac{3\sqrt{2} - 4c - (21 - 2c)}{\Pi(G')} - \frac{21 - 2c}{\Pi(G)} = 2.24 - 2c > 0
$$

(4.14)

with $c \in [0, 1]$.

(v) $n_1 = 1$ and $n_2 = 1$: We have two isolated firms with total profits $\Pi(G) = 0$. If we connect the firms via a link we have $\Pi(G') = 2(1 - c)$. Since $0 \leq c \leq 1$ total profits in the connected graph $G'$ are higher.

The above cases consider all possible cases of disconnected graphs and show that total profits $\Pi$ can be increased by connecting them. □
Thus, for $c \in [0, 1]$, connecting two previously disconnected components of the graph yields total profits larger than the respective total profits of the disconnected components. From this it follows immediately that the efficient network is connected.

**Proposition 7** Let $\mathcal{H}(n, m)$ denote the set of connected graphs having $n$ firms and $m$ links. If $c \in [0, 1]$ then $G^* \in \mathcal{H}(n, m)$.

**Proof 6** For a contradiction assume that the efficient graph $G$ is disconnected (and all connected graphs have smaller total profits than $G$). Since $G$ is disconnected then it has at least two components. With Proposition (4) each pair of components can be connected, resulting in a graph with higher total profits. Ultimately all components of $G$ can be connected, yielding a connected graph $G'$ with at least the total profits of $G$. This is a contradiction to the assumption that the efficient graph is disconnected. □

This means that, in order to guarantee an efficient knowledge production in the economy, each firm must have (direct or indirect) access to the knowledge of all other firms in the industry. Since the efficient graph is connected, Equation (4.1) for total profits simplifies to

$$\Pi(G, c) = n\lambda_{PF}(G) - 2mc.$$  \hspace{1cm} (4.15)

This implies that, for any given values of $n$ and $m$, the efficient graph is also the one with maximal $\lambda_{PF}$. In other words, for $c \in [0, 1]$, the efficient graph $G^*$ belongs to the set of connected graphs that maximize $\lambda_{PF}(G)$, denoted by $\mathcal{H}^*(n, m)$. As a result, the efficient graph belongs to a special class of graphs characterized by well defined topological properties\(^1\). In order to fully describe these properties we first need to introduce some basic definitions.

Brualdi and Solheid (1986) show that the graphs in the set $\mathcal{H}^*(n, m)$ have a stepwise adjacency matrix $A$, defined as follows.

**Definition 1** (Brualdi and Solheid (1986)) In a stepwise matrix $A$, the elements $a_{ij}$ satisfy the following condition. If $i < j$ and $a_{ij} = 1$, then $a_{hk} = 1$ whenever $h < k \leq j$ and $h \leq i$.

The above definition says that if the adjacency matrix has an element equal to one, $a_{ij} = 1$, then the element above in the matrix is also one, $a_{i,j-1} = 1$, and the element to the left

---

\(^1\)The efficient networks are similar to those obtained in the model of Ballester et al. (2006). More precisely, within that model Corbo et al. (2006) show that the networks that maximize welfare are given by the graphs with maximal eigenvalue.
in the matrix is one, \( a_{i-1,j} = 1 \). Consequently, all preceding elements to the left and above are one. In this way, the one-elements are separated from the zero-elements in the adjacency matrix along a line which has the form of a step function. This fact has brought about the name stepwise matrix. An example of a stepwise matrix is shown in Figure (4.1, right). The graphs associated with a stepwise adjacency matrix are called \textit{nested split graphs} (Aouchiche et al., 2006).

Nested split graphs have a nested neighborhood structure: the set of neighbors of each firm is contained in the set of neighbors of the next higher degree firm. However, before providing the formal definition of these graphs, we provide an intuition of its structure with the help of the representation in Figure (4.1, left). In particular, we consider a nested split graph that is also connected, since this will be our focus later on. First, the firms in a nested split graph can be partitioned in subsets of firms with different properties. In Figure (4.1, left) each circle represents a subset of firms (and not an actual firm of the network). Furthermore, we denote the partition of the graph as \( \mathcal{P} = U \cup V \), where \( U \) and \( V \) consist of subsets, \( U = \{ U_1, U_2, ..., U_k \} \) and \( V = \{ V_1, V_2, ..., V_k \} \) respectively. Recall the notation from Section 2.1 in which \( K_n \) denotes the complete graph with \( n \) firms and \( \bar{K}_n \) the empty graph with \( n \) isolated firms. Then, for example, in Figure (4.1, left) the sets are \( U_1 = K_2, U_2 = K_2 \cup K_1 \) and \( U_3 = K_2 \cup K_1 \cup K_1 \) and \( V_1 = \bar{K}_2, V_2 = \bar{K}_2 \) and \( V_3 = \bar{K}_2 \) respectively. Of course, the subgraph \( K_2 \) is simply a complete graph since it contains only two firms, and even more so \( K_1 \) is a subgraph consisting of a single firm only.

The subsets \( U_i \) and \( V_i \) differ in the fact that in \( U_i \) all firms are connected to each other while in \( V_i \) there exist no links between the firms. Moreover, there exist also links between firms belonging to different subsets. Indeed, the neighborhood of the firms in each set \( V_i \) is precisely the set \( U_i \). In Figure (4.1, left) a line between two subsets indicates that there exists a link between each firm in one subgraph to each firm in the other subgraph. For example the firms in \( V_1 \) at the top right of the figure are all connected to the firms in \( U_1 \) at the top left. Similarly, the firms in \( V_2 \) are connected to the firms in \( U_2 \) and the firms in \( V_3 \) are connected to the firms in \( U_3 \). Additionally, the set \( U \) as well as any union of the subsets in \( U \) form a complete subgraph or clique. Similarly, any union of the sets in \( V \) form an independent set. Notice also, that all the firms in one set have the same degree. Next to the sets in Figure (4.1, left) the degree of the firms in a subset is indicated. The degree of a firm in a set can be easily derived from the adjacency matrix shown in Figure (4.1, right) by counting the number of ones in a row corresponding to a particular firm in a set. For example the set \( K_2 \), top left in the figure, corresponds to two firms whose links are indicated in the first two rows of the adjacency matrix.

With the preceding discussion in mind, we can now give a more formal definition of a
Definition 2 (Cvetkovic et al. (2007)) In a nested split graph, the set of nodes have a partition $\mathcal{P} = U \cup V$ with the following properties.

(i) $U$ induces a clique, and $V$ induces an independent set. This also holds for any union of subsets in $U$ and $V$.

(ii) $U$ has subsets $U_1, ..., U_k$ such that $U_1 \supset ... \supset U_k$ and the neighborhood of each node in $V_i$ is $U_i$, for any $i = 1, ..., k$.

A distinctive feature of nested split graphs is that they are characterized by a core-periphery structure (see also Goyal, 2007). The firms in the set $U$ form a clique or the core of the network. These densely linked firms are connected to the firms in the independent set $V$ which in turn are not connected among themselves, the periphery of the network. Such core-periphery structures have been observed in a recent study on firm collaborations in the pharmaceutical and biotech sector by Baker et al. (2008).

If a nested split graph is connected we call it a connected nested split graph. As we mentioned above, the representation and the adjacency matrix in Figure (4.1) show a connected nested split graph. From the stepwise property of the adjacency matrix it follows that a connected nested split graph contains at least one spanning star, that is, there is at least one firm that is connected to all other firms. This property can also be seen in Figure (4.1), where the first row of the adjacency matrix is entirely filled with ones, which indicates the presence of a spanning star. Nested split graphs are also known as “threshold networks”. These networks are defined as follows\(^2\).

Definition 3 (Mahadev and Peled (1995)) For a threshold network $G = (V, L)$ the following are equivalent:

(i) $G$ is a threshold network.

(ii) $G$ is a nested split graph from Definition (2).

(iii) $G$ does not have an alternating 4-cycle. This is a configuration of nodes $a, b, c, d$ such that $ab, cd \in L$ and $ac, bd \notin L$. This induces a path $P_4$, a cycle $C_4$ or a the union of two two-cliques $K_2 \cup K_2$.

\(^2\)In the following, $\lceil x \rceil$, where $x$ is a real valued number $x \in \mathbb{R}$, denotes the smallest integer larger or equal than $x$ (the ceiling of $x$). Similarly, $\lfloor x \rfloor$ the largest integer smaller or equal than $x$ (the floor of $x$).
(iv) $G$ can be constructed from a one-node graph by repeatedly adding an isolated node or a dominating node that is connected to all other nodes. For example a star $K_{1,n-1}$ is constructed by adding $n-1$ isolated nodes and one dominating node connecting to all already existing nodes in the network.

(v) The vicinal preorder (reflexive, transitive) of $G$ is total (no pairs of nodes are incomparable).

(vi) Consider $m$ distinct degrees $d_{(1)} < d_{(2)} < \ldots < d_{(m)}$ in $G$. Let $D_i$ be the $i$-th set of nodes with identical degree, that is $u, v \in D_i$ if and only if $d_u = d_v = d_{(i)}$. For each $v \in D_i$,

$$N_v = \bigcup_{j=1}^{k} D_{m+1-j}, \quad k = 1, \ldots, \lfloor \frac{m}{2} \rfloor$$
$$N_v \cup v = \bigcup_{j=1}^{k} D_{m+1-j}, \quad k = \lfloor \frac{m}{2} \rfloor + 1, \ldots, m,$$

in other words, for $x \in D_i$ and $y \in D_j$, $x$ is adjacent to $y$ if and only if $i + j > m$.

(vii) Consider the degree sequence $k_1 < k_2 < \ldots < k_m$. Then

$$k_{l+1} = k_l + |D_{m-l}|, \quad l = 0, \ldots, m, \quad k \neq \lfloor \frac{m}{2} \rfloor.$$

(viii) There exists a non-negative real vector $w \in \mathbb{R}^m$ (weights) and a real number $t \in \mathbb{R}$ (threshold) such that for distinct nodes $u$ and $v$,

$$w_u + w_v > t \quad \text{if and only if } uv \in L.$$
Threshold networks have already been used in several models trying to reproduce the stylized facts of real-world networks (Boguña and Pastor-Satorras, 2003; Caldarelli et al., 2002; Servedio et al., 2004). Moreover, Garlaschelli et al. (2005) have applied a threshold network model to describe large market investments.

We point out that nested split graphs have also appeared in the recent literature on economic networks. Nested split graphs are so called “inter-linked stars” introduced in Goyal and Joshi (2003).

Definition 4 (Goyal and Joshi (2003)) Let \( h(G) = \{ h_1(G), h_2(G), ..., h_m(G) \} \) be a partition of the nodes in \( G \) such that \( N_i = N_j \) if and only if \( i, j \in h_l(G) \) for some \( l \in \{1, 2, ..., m\} \). The groups are numbered in ascending order of links. An inter-linked star is a partition with two features:

(i) \( N_i = n - 1 \) for all nodes \( i \in h_m(G) \), and

(ii) \( N_j = h_m \) for all nodes \( j \in h_1(G) \).

However, the above definition does not specify the nested neighborhood structure that characterizes nested split graphs.

Subsequently, Goyal et al. (2006) identified inter-linked stars in the network of scientific collaborations among economists. The wider applicability of nested split graphs suggests that a network formation process that generates these graphs as it is stated in Definition (9) may be of general relevance for understanding economic and social networks.

In Proposition (7) we have shown that \( G^* \) is connected and we know that \( G^* \) has a stepwise adjacency matrix. From the above discussion we can further conclude that \( G^* \) is a connected nested split graph and it contains at least one spanning star as a subgraph.

The determination of the exact topology of \( G^* \), for given \( n \) and \( c \), is simple for \( c \in [0, 1/2] \) (see Proposition 8). In contrast, for \( c > 1/2 \) this problem requires the determination of the graph with the largest eigenvalue among all graphs in \( \mathcal{H}^*(n, m) \) (for a fixed \( n \) and arbitrary \( n - 1 \leq m \leq n(n - 1)/2 \)). This is still an unresolved research problem in Spectral Graph Theory (Aouchiche et al., 2006). However, it turns out that the value of total profits associated with the efficient graph \( G^* \) can be approximated by the total profits associated with a special type of a connected nested split graph. Following Bell (1991) we denote this graph by \( F_{n,d} \).

Definition 5 \( F_{n,d} \) is the graph obtained from the complete graph \( K_d \) with \( d \) nodes and a subset of \( n - d \) disconnected nodes, by adding \( n - d \) links connecting one node in \( K_d \) to each of the \( n - d \) disconnected nodes.
Notice that the complete graph and the spanning star are particular cases of connected nested split graphs (and of the graph $F_{n,d}$): the star is $K_{1,n} = F_{n,1}$ and the complete graph is $K_n = F_{n,n}$. Figure (4.2) shows the graph $F_{10,7}$ together with its adjacency matrix and Figure (4.6) shows several examples of this type of graph for $n = 10$. Moreover, note that the number of links in $F_{n,d}$ is given by $m = \binom{d}{2} + (n - d)$.

Figure 4.2: The connected nested split graph $F_{10,7}$ (left) and its associated adjacency matrix (right).

As discussed in more detail in the proof of the next proposition, the maximum relative discrepancy of total profits between $F_{n,d}$ and the efficient graph $G^*$ is relatively small and vanishes for large $n$. For $n = 100$ we get an error below 2%, while for $n = 200$ the error is below 1% as can be seen in Figure (4.7). The higher the number $n$ of firms, the closer total profits of $F_{n,d}$ get to total profits of $G^*$. Thus, in order to determine the efficient network $G^*$, if $n$ is small one can search through all connected nested split graphs and identify the one with highest total profits, while for large $n$, one can use $F_{n,d}$ as a good approximation.

Bringing the above results together, we can state the following proposition, which characterizes the topology of the efficient graph $G^*$ with $n$ firms in the industry as a function of the marginal cost of collaboration $c \in [0,1]$.

**Proposition 8** Let $G^*$ be the efficient graph for a given number $n$ of firms and $F_{n,d}$ be the graph introduced in Definition (5).

(i) If $c \in [0,1]$ then $G^*$ is a connected nested split graph.

(ii) Denote the relative error in total profits between the efficient graph and the graph $F_{n,d}$ as $\epsilon = (\Pi(G^*) - \Pi(F_{n,d}))/\Pi(F_{n,d})$. If $c \in [0,1]$, then the relative error is bounded from above as follows

$$\epsilon \leq \frac{2c(2c-1)n - 5c^2}{n^2 + 2c(1-2c)n + 9c^2}.$$  \hfill (4.19)
4.1. Efficient Networks

and vanishes for large $n$, i.e. $\lim_{n \to \infty} \epsilon = 0$.

(iii) If $c \in [0, 0.5]$ then $G^*$ is the complete graph $K_n$.

(iv) If $c > n$ then $G^*$ is the empty graph $\bar{K}_n$.

Figure 4.3: Illustration of the range of efficient graphs as a function of the cost of collaboration. For costs $0 \leq c \leq 0.5$ the efficient graph is the complete graph $K_n$. In the region $0.5 < c \leq 1$ the connected graphs with stepwise matrices are efficient (see Definition (1)) or equivalently the connected nested split graphs. Note that for $n$ large, $F_{n,d}$ attains total profits of $G^*$ with a vanishing relative error in aggregate profits and thus can be seen as an approximation for $G^*$.

Proof 7

(i) From Lemma (4) we know that the efficient graph is connected. Moreover, (Brualdi and Solheid, 1986) have shown that among the connected graphs, the graphs with maximal eigenvalue have a stepwise adjacency matrix. We have mentioned already that these graphs are referred to connected nested split graphs (Aouchiche et al., 2006).

(ii) We have introduced the graph $F_{n,d}$ in Section 4.1. In order to prove the claim, we derive a lower bound for the total profits of $F_{n,d}$, as well as an upper bound for the total profit of the efficient graph $G^*$. We then show that, if one chooses $d$ appropriately, the relative difference between the two bounds vanishes for large $n$. Let us start with the lower bound. Recall that $F_{n,d}$ is the graph obtained from a complete graph $K_d$ of $d$ firms and $n - d$ isolated firms by connecting each isolated firm to one and the same firm of $K_d$ via one link. The number of links $m$ in this graph is
determined by the size $d$ of the clique, $m(d) = \binom{n}{2} + (n - d)$. Since $F_{n,d}$ contains $K_d$ as a subgraph, the largest real eigenvalue of $F_{n,d}$ is larger or equal to the one of $K_d$, which is $\lambda_{PF}(K_d) = d - 1$. Therefore, total profits of the graph $F_{n,d}$ are bounded from below as follows:

$$\Pi(F_{n,d}) = n\lambda_{PF}(F_{n,d}) - 2m(d)c \geq n(d - 1) - 2m(d)c. \quad (4.20)$$

Since the inequality above is valid for any integer $d$, such that $1 \leq d \leq n$, we are interested in the value of $d$ that maximizes the right hand side of Equation (4.20), that is

$$d = \arg\max_{1 \leq k \leq n} \{n(k - 1) - 2m(k)c\}, \quad (4.21)$$

where $m(k) = \binom{n}{2} + (n - k)$ and $k \in \mathbb{N}_+$. By computing the first and second derivative of the objective function $n(k - 1) - 2m(k)c$ with respect to $k$, one finds that its maximum occurs for $k = \frac{n+3c}{2c}$. For simplicity, one can take $d$ as the closest integer to this value.$^3$ Notice that, as a consequence, $d$ converges to $\frac{n}{2c}$ for large $n$.

Replacing $d = \frac{n+3c}{2c}$ in Equation (4.20), we obtain a lower bound, which is independent of $d$, and given by

$$\Pi(F_{n,d}, c) \geq \frac{n^2 + n(2c - 8c^2) + 9c^2}{4c}. \quad (4.22)$$

We now derive an upper bound for total profits of the efficient network $G^*$. The largest real eigenvalue of a connected graph is at most $\sqrt{2m - n + 1}$ (Hong, 1993). From this, it follows immediately that total profits of $G^*$ are bounded by

$$\Pi(G^*, c) \leq n\sqrt{2m - n + 1} - 2mc. \quad (4.23)$$

We have shown already that for cost $c \leq 1/2$ the efficient graph is complete. Therefore, we are interested in values of cost $c > 1/2$. Assuming that $c > 0.5$, the number $m$ of links that maximizes the right hand side of Equation (4.23) is $m = \frac{n^2 + 4nc^2 - 4c^2}{8c^2}$. Replacing such a value of $m$, we obtain an upper bound that is independent of the number $m$ of links,

$$\Pi(G^*, c) \leq \frac{n^2 - 4nc^2 + 4c^2}{4c}. \quad (4.24)$$

At this point, combining Equation (4.22) and (4.24), we obtain that the relative difference $\epsilon$ in the total profits of the graph $F_{n,d}$ and the graph $G^*$ is bounded from above by

$$\epsilon = \frac{\Pi(G^*, c) - \Pi(F_{n,d}, c)}{\Pi(F_{n,d}, c)} \leq \frac{2c(2c - 1)n - 5c^2}{n^2 + 2c(1 - 2c)n + 9c^2}. \quad (4.25)$$

$^3$The results on the relative error that we obtain later in this proof are still valid under this assumption.
The expression on the right hand side of the above inequality converges to zero for large \( n \), and therefore the relative difference in total profits vanishes.

(iii) Since for the complete graph \( K_n \), \( \lambda_{PF} = n - 1 \) and \( m = \frac{n(n-1)}{2} \), its total profits are given by

\[
\Pi(K_n) = n(n-1) - 2\frac{n(n-1)}{2}c = n(n-1)(1-c).
\]

(4.26)

On the other hand, the largest real eigenvalue \( \lambda_{PF} \) of a graph \( G \) with \( m \) links is bounded from above so that \( \lambda_{PF} \leq \frac{1}{2}(\sqrt{8m+1} - 1) \) (Stanley, 1987)\(^4\). For total profits we then have

\[
\Pi = \sum_{i=1}^{n} \lambda_{PF}(G_i) - 2mc \\
\leq n \max_{1 \leq i \leq n} \lambda_{PF}(G_i) - 2mc \\
\leq \frac{n}{2}(\sqrt{8m+1} - 1) - 2cm \\
= b(n,m,c),
\]

with \( n \leq m \leq \binom{n}{2} \). For fixed cost \( c \) and number of firms \( n \), the number of links maximizing Equation (4.27) is given by \( m^* = \frac{n^2-c^2}{8c^2} \) if \( \frac{n^2-c^2}{8c^2} < \binom{n}{2} \) and \( m^* = \frac{n(n-1)}{2} \) if \( \frac{n^2-c^2}{8c^2} > \binom{n}{2} \). The graph with the latter number of links is the complete graph \( K_n \).

Inserting \( m^* \) into Equation (4.27) yields

\[
b(n,m^*,c) = \begin{cases} 
\frac{n}{2}(\sqrt{\frac{n^2-c^2}{c^2}} + 1) - \frac{n^2-c^2}{4c}, & \text{if } c > \frac{n}{2n-1}, \\
(n(n-1)(1-c) = \Pi(K_n), & \text{if } c < \frac{n}{2n-1}.
\end{cases}
\]

(4.28)

The bound for \( c \leq \frac{n}{2n-1} \sim \frac{1}{2} \) in the limit of large \( n \) coincides with total profits of the complete graph \( K_n \). Therefore \( K_n \) is the efficient graph in that region of cost.

(iv) If \( c = n \) then the number of links maximizing Equation (4.27) is given by \( m^* = 0 \) and the efficient graph is the empty graph \( \bar{K}_n \).

Finally, in Figure (4.5) we show total profits and the number of links of \( F_{n,d} \) for \( n = 10 \). Note that in this case, \( F_{n,d} \) is not an approximation but the exact efficient network. □

Figure (4.3) gives a graphical representation of the results on network efficiency in Proposition (8). For the particular case of \( n = 10 \) the connected graph with maximal eigenvalue is known (Aouchiche et al., 2006) and so is the efficient network \( G^* \). In this case, the efficient graph is \( F_{n,d} \) itself (without any approximation). In Figure (4.6) the efficient graphs for values of cost \( c \in [0,1] \) and this system size \( n = 10 \) are shown. Moreover, Figure (4.5) in

\(^4\)Notice that a similar result can be obtained using an alternative bound for connected graphs, \( \lambda_{PF} \leq \sqrt{2m - n + 1} \) due to (Hong, 1993).
Figure 4.4: Upper bound \( b(n, m^*, c) \) of Equation (4.28) for \( n = 100 \) and varying costs \( c \). For \( c \leq \frac{n}{2n-1} \) the upper bound corresponds to the complete graph \( K_n \).

Figure 4.5: Total profits \( \Pi \) (left) and the number of links \( m \) (right) in the efficient networks \( F_{n,d} \), with \( n = 10 \). The plot on the right shows that, with this value of \( n \), the efficient network coincides with the complete graph (\( m = n(n-1)/2 = 45 \)) for \( c \in [0, 0.6] \), while it differs significantly from the complete graph for higher value of \( c \). Note that in Proposition (8), item (iii), \( c \leq 0.5 \) is a sufficient condition for the efficient graph to be complete, but not a necessary one.

The appendix shows the corresponding total profits and number of links. We observe that, with increasing marginal cost, the efficient network becomes more sparse and the degree heterogeneity increases. For any value of cost larger than 0.6 the efficient network consists of a densely connected cluster (clique) and one firm that acts as a hub (star) connecting the remaining firms to the cluster.

If we consider \( F_{n,d} \) as the efficient network, we can make the following observation. From the topological structure of \( F_{n,d} \) it follows that, when marginal cost of link formation is
4.1. Efficient Networks

Figure 4.6: Efficient graphs for values of cost $c = 0.55, 0.65, 0.75, 0.85$ and $n = 10$. The density of the efficient graph is decreasing and the degree heterogeneity is increasing with increasing cost.

high, it is efficient to concentrate knowledge creation in a small and dense cluster with one firm acting as a hub connecting all the peripheral firms to the cluster. As the marginal cost of link formation decreases, knowledge recombination becomes cheaper and it is efficient that a larger fraction of firms participates in the densely connected cluster. Finally, in the region of small marginal cost, $0 \leq c \leq 0.5$, it is efficient for all firms to take part in a densely connected cluster, therewith establishing as many collaborations as possible. In this case, the fully connected graph is the one that maximizes industry profits.

An important final remark concerns the relation to the efficient graphs found in models akin to ours. Similar to both Jackson and Watts (2002) and Bala and Goyal (2000), we find that the efficient graph is always connected and that it includes, depending on the cost $c$, the star and the complete graph. However, differently from the model of Bala and Goyal, the efficient graph is in general not minimally connected (removing one link does not necessarily make the graph disconnected). Moreover, differently from the model of Jackson and Watts, in our model the set of efficient graphs is not limited to the star and the complete graph, but it includes a whole class of graphs that can be seen as intermediate graphs between these two extreme cases.

$^5$Note that for $c = 0$ the problem in (4.2) can be reduced to the problem of maximizing total knowledge growth in the steady state for a given number of firms, in which case the complete graph is the solution.
4.2 Efficiency and Profit Distribution

Previous work on R&D networks (see Cowan and Jonard, 2004) has emphasized the emergence of a trade-off between the efficiency (in terms of knowledge diffusion) and inequality (in terms of firms’ knowledge levels). A similar trade-off between efficiency and profits emerges also in this model if the marginal cost of link formation and the number of firms operating in the industry are high enough. We measure this inequality through profit variance.

In Proposition (7) we have shown that for $c \in [0,1]$ the efficient graph is connected, $G^* \in \mathcal{H}(n,k)$. Thus, the returns from collaborations in an efficient graph are identical for all firms (since they have the same largest real eigenvalue) but the cost is different and proportional to the degree of the firm. More formally, let us define by $\sigma_d^2$ the variance of profits associated with the graph $G$. It follows for a graph $G \in \mathcal{H}(n,m)$ that

$$\sigma_d^2(G) = c^2 \sigma_d^2(G), \quad (4.29)$$

where $\sigma_d^2$ is the degree variance. Since the degree is by definition homogeneous in a complete graph, from Proposition (8) it follows that for $c \leq 0.5$, profit inequality is zero and no tension between efficiency and equality arises.

For higher values of costs, we can take $F_{n,d}$ as a sufficient approximation to the efficient network $G^*$, and from the properties of $F_{n,d}$ we can conclude that the efficient network is characterized by considerable degree heterogeneity and profit inequality. More precisely, the following proposition can be stated.

**Proposition 9** Let $F_{n,d}$ be the graph defined in (5) and $\bar{d} = 2m/n$ the average degree. Then

(i) The degree variance is growing quadratically with the number of firms, i.e.

$$\sigma_d^2(F_{n,d}) = O(n^2). \quad (4.30)$$

(ii) Let $c > 0.5$. For large $n$ the coefficient of variation of degree, $c_v(F_{n,d}) = \sigma_d(F_{n,d})/\bar{d}$, tends to a constant depending on the cost,

$$\lim_{n \to \infty} c_v(F_{n,d}) = \sqrt{2c - 1}. \quad (4.31)$$

(iii) Consider the random graph $G(n,m)$ with $n$ firms and $m$ links. For large $n$, the degree variance of the graph $F_{n,d}$ is larger by a factor $n$ than the variance of a random graph with equal number of firms and links

$$\sigma_d^2(F_{n,d})/\sigma_d^2(G(n,m)) = O(n). \quad (4.32)$$
4.2. Efficiency and Profit Distribution

Proof 8

(i) With \( \sum_i d_i = 2m \) we can write the degree variance as follows

\[
\sigma^2_d = \frac{1}{n} \sum_{i=1}^{n} \left( d_i - \frac{2m}{n} \right)^2 = \frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left( \frac{2m}{n} \right)^2.
\]  

(4.33)

Using then the fact that the graph \( F_{n,d} \) contains one firm with degree \( n - 1 \) (the hub), \( d - 1 \) firms with degree \( d - 1 \) (those in the clique) and \( n - d \) firms with degree 1, we get

\[
\sigma^2_d(F_{n,d}) = \frac{1}{n} \left( (n - 1)^2 + (d - 1)^3 + (n - d) \right) - \left( \frac{2m}{n} \right)^2.
\]  

(4.34)

We now replace in the equation above the value of \( d \) that maximizes total profits for the graph \( F_{n,d} \), \( d = \frac{n + 3c}{2c} \), as found from Equation (4.21), as well as the corresponding value of \( m \), given by \( m(d) = \frac{n^2 + 8c^2n - 9c^2}{8c} \). As a result, one obtains the degree variance and this expression is of quadratic order in \( n \), \( \sigma^2_d = O(n^2) \).

(ii) The coefficient of variation of the degree is defined as \( c_v = \sigma_d / \bar{d} \). Recalling that the average degree is \( \bar{d} = 2m/n \) and replacing, as above, the value of \( d \) that maximizes total profits, \( d = \frac{n + 3c}{2c} \), and the corresponding value of \( m \), one obtains an expression in \( n \) and \( c \). The limit of large \( n \) for this expression is well defined and equal to

\[
\lim_{n \to \infty} c_v = \sqrt{2c - 1}.
\]  

(4.35)

(iii) Gutman and Paule (2002) have shown that the degree variance of a random graph \( G(n,m) \) with \( n \) firms and \( m \) links is given by

\[
\sigma^2_d(G(n,m)) = \frac{2m(n^2 - n - 2m)}{n^3 + n^2}.
\]  

(4.36)

Replacing \( m \) as in (ii), the expression above turns out to be of order \( O(n) \), and consistently, the ratio of (4.34) and (4.36) is of order \( O(n) \).

This concludes the proof. \( \square \)

The results of this proposition are illustrated in Figure (4.7). The coefficient of variation of degree, \( c_v \), increases with cost (Figure (4.7), top-left). It also increases with the number of firms up to the finite limit of \( \sqrt{2c - 1} \) for large \( n \). Equation (4.29) implies that the inequality in profits increases with cost as well. Moreover, the degree variance of \( F_{n,d} \) is many times larger than the degree variance of a random graph \( G(n,m) \) with the same density (Figure (4.7), bottom-left). It follows that, for higher values of marginal cost \( 0.5 < c \leq 1 \), the industry displays an inequality in profits significantly larger than the one that could be observed if collaborations were formed at random.
Figure 4.7: Properties of the graph $F_{n,d}$ as a function of the cost of collaboration. Upper bound $\bar{\epsilon}$ on the relative error $\epsilon$ in the approximation of the efficient graph $G^*$ (top, left); degree variance $\sigma^2_d(F_{n,d})$ (bottom, left); degree coefficient of variation $c_v(F_{n,d})$ (top, right); ratio of degree variance of $F_{n,d}$ and degree variance of a random graph $G(n, m)$ of the same size and density, $\sigma^2_d(F_{n,d})/\sigma^2_d(G(n, m))$ (bottom, right) for $n = 50$, $n = 100$, $n = 200$ and cost $c \in [0.5, 1]$. 
Chapter 5

Myopic Pairwise Link Formation

In this chapter, we propose a model in which firms innovate by recombining their knowledge with that of other firms in the industry, via a network of costly R&D collaborations. Building on the results we have obtained in the previous chapters, we study the emergence of pairwise stable structures by employing the notion of “improving path” (cf. Jackson and Watts, 2002), and assuming that link deletion is subject to severance costs. We show that the existence of multiple stable structures. In addition, we study the relation between network stability and efficiency. Finally, we investigate equilibrium selection under a two-sided myopic link dynamics and we show that the model is able to generate stable structures that match the properties of empirically observed R&D networks.

5.1 Network Formation

The analysis contained in the previous chapter assumes that the structure of the network is fixed. In this way, it is possible to study which network topologies maximize industry welfare. In this section we depart from this static network perspective, and we investigate how the structure of the network evolves whenever firms are allowed to endogenously choose the partners with whom they want to collaborate.

Jackson and Watts (2002) we consider a network formation process in which the creation of a new link requires the bilateral agreement of the two parties involved. However, the deletion of a link requires the unilateral decision of one of the two firms only. Consistently, as network equilibrium criterion, we adopt the definition of pairwise stability, as in Jackson and Watts (2002). Based on this definition of stability, we derive the conditions on the value of cost for which structures like the empty graph, the complete graph or the star are stable. Among the possible stable graphs, we find also a disconnected graph consisting
of multiple cliques of the same size\textsuperscript{1}. A first important finding here is the co-existence of multiple equilibrium networks for the same value of cost.

However, these relatively simple structures are not the only stable networks emerging in our model. Since it is increasingly difficult to derive general proofs of stability for more complex structures, we follow the argument in Vega-Redondo (cf. 2007, p. 208) and we perform a dynamic study of network stability. We model explicitly the evolution process in which, at the beginning of each period, a pair of firms decides whether to form or delete a link, based on the expected profits this action brings about. This investigation, performed through computer simulation, shows that there exist a multitude of complex structures which are pairwise stable. Remarkably, these networks display topological properties that are consistent with the stylized facts of R&D networks, in a region of the parameters of the model.

5.2 Improving Paths and Equilibrium Networks

We consider a process of network evolution in which firms form or delete one link at a time based on the marginal profits they expect from that action. In other words, new links are created whenever the increase in the probability of innovation, i.e. the marginal revenue of a new collaboration, is greater than the marginal cost of a collaboration, with the gain being strict for at least one of the firms in the selected pair. Likewise, link deletion occurs whenever the saving in marginal cost from removing a collaboration are enough to compensate for the decrease in marginal revenue. However, given its unilateral nature, we assume that removing a collaboration involves severance costs\textsuperscript{2} so that the savings in marginal costs from removing a collaboration is reduced by a factor $\alpha$.

Following Jackson and Watts (2002), we call improving path, a sequence of networks $\{G(t)\}_{t\in\mathbb{N}}$ such that (i) any two consecutive networks, $G(t)$ and $G(t+1)$, differ by one link only, (ii) if the link is added, both firms benefit from the new link, at least one of them strictly, and (iii) if a link is deleted, at least one of the two firms strictly benefits from the deletion.

Improving paths emanating from any initial network must either lead to an equilibrium network structure, in which no pair of firms has an incentive to form a link, and no single firm has an incentive to remove a link, or to a cycle, in which a finite number of networks

\textsuperscript{1}This equilibrium is similar to the equilibrium networks found in Bloch (1995).

\textsuperscript{2}These severance costs can be associated with the legal procedures needed to unilaterally bring a contract to an end, or it can have a different nature, e.g. they can be associated with the loss of reputation for managers breaking long-lasting collaborations.
5.2. Improving Paths and Equilibrium Networks

is repeatedly visited (see Lemma (1) in Jackson and Watts, 2002). In this section we investigate the existence of both equilibrium networks and cycles.

Let \( G \) denote the current graph \( G(t) \) at time \( t \). Further, denote by \( G + ij \) the graph obtained from \( G \) by adding the link \( ij \). Similarly, let \( G − ij \) denote the graph obtained by removing the edge \( ij \). Denote by \( \lambda_i(G) \) the largest eigenvalue \( \lambda_{PF}(G_i) \) of the connected component \( G_i \) to which the firm \( i \) belongs. Note that, although link deletion implies that the degree of \( i \) is reduced by one (and so is the cost for firm \( i \)), the firm saves only a fraction of the cost due to the presence of the severance costs \( v(c) = (1 − \alpha)c \). Thus, the change in profits of firm \( i \) induced by the removal of a link are given by

\[
\pi_i(G − ij) − \pi_i(G) = \lambda_i(G − ij) − (d_i − 1)c − v(c) − (\lambda_i(G) − d_ic)
\]

\[
= c − v(c) − (\lambda_i(G) − \lambda_i(G − ij))
\]

\[
= \alpha c − (\lambda_i(G) − \lambda_i(G − ij))
\]

where \( \alpha \in [0, 1] \). Obviously, the firm will only remove a link if this action increases her profits. With the above notation we can now give the definition of a pairwise stable network.

**Definition 6** The graph \( G = (N, L) \) is pairwise stable if

(i) \( \forall ij \in L, \pi_i(G) \geq \pi_i(G − ij) \) and \( \pi_j(G) \geq \pi_j(G − ij) \) or, equivalently, \( \forall ij \in L, \lambda_i(G) − \lambda_i(G − ij) \geq \alpha c \) and \( \lambda_j(G) − \lambda_j(G − ij) \geq \alpha c \)

(ii) \( \forall ij \notin L, \) if \( \pi_i(G + ij) > \pi_i(G) \) then \( \pi_j(G + ij) < \pi_j(G) \), and, if \( \pi_j(G + ij) > \pi_j(G) \)

then \( \pi_i(G + ij) < \pi_i(G) \) or, equivalently, \( \forall ij \notin L, \) if \( \lambda_i(G + ij) − \lambda_i(G) > c \) then

\( \lambda_j(G + ij) − \lambda_j(G) < c \), and, if \( \lambda_j(G + ij) − \lambda_j(G) > c \) then \( \lambda_i(G + ij) − \lambda_i(G) < c \)

Before moving to the analysis of the existence of stable graphs, we give an explanation about why in our model the network might stop evolving along an improving path and finally reach an equilibrium. Let us consider an improving path along which the number \( m \) of links is increasing from \( m_1 = 0 \), corresponding to the empty graph, to at most \( m_2 = n(n − 1)/2 \), corresponding to the complete graph \( K_n \). Figure (5.1) shows some instances of improving paths and their corresponding densities (in terms of the number of links) for different values of the cost \( c \), but with severance cost \( \alpha = 0 \). Note that a vanishing value of \( \alpha \) implies that no links are removed since then the severance cost exceeds any potential gains that could be realized by saving the cost for that link.

For comparison, the figure shows also the straight line with slope \( \frac{2}{n} \), equal to the average increase of \( \lambda_{PF} \) going from the empty graph to the complete graph. In contrast, along any
improving path the trajectory of $\lambda_{PF}(m)$ starts off above such straight line. This is stated in the following Lemma and has an important implication.

**Lemma 5** Along any improving path in which the number $m$ of links is increasing, $\lambda_{PF}(m)$ increases with $m$ faster than $\frac{2}{n}$, in a set of integers $I = \{0, 1, 2, ..., m_0\}$ with $m_0 < n(n - 1)/2$.

**Proof 9** The assumptions on the improving path require that we add one link at a time. Starting from an empty network, the first link added yields a pair (i.e. a path\(^3\) $P_2$ of length 2), with an eigenvalue of $\lambda_{PF} = 1 > \frac{2}{n}$ for $n > 2$. If a second link is added to one of the firms of the pair by attaching another firm to them, a path of three firms is formed, with associated largest real eigenvalue $\lambda_{PF} = 2 \cos(\pi/4) = 1.41$ (see Table (5.4)). $1.41 > \frac{4}{n}$ for $n \geq 3$. Therefore, we can always find an integer $m_3 \geq 1$, such that $\lambda_{PF}(m) > \frac{4}{n}$ for $m \in [0, m_3] \cap N_+$, with $m_3 \leq n(n - 1)/2$. $\square$

Since, in addition, the sum of the increments of $\lambda_{PF}(m)$ has to be constant, this means that any improving path has to cross the straight line at some point before reaching the complete graph. In other words, for some value of $m$ the marginal revenue becomes smaller than the marginal cost (for any value of cost and $n$ large enough), implying that the evolution stops. This is stated more precisely in the following proposition.

**Proposition 10** Along an improving path in which the number $m$ of links increases, marginal profits become negative for some value of $m^* \leq n(n - 1)/2$, for any value of cost $c$ and $n$ large enough.

**Proof 10** Along an improving path the number $m$ of links can vary only by one or zero in absolute value. Here, we restrict ourselves to the improving paths in which the number of links increases from $m_1 = 0$ to at most $m_2 = n(n - 1)/2$. From Proposition (2) we know that the largest real eigenvalue is bounded, $\lambda_{PF}(m) \leq n - 1$. Taking the average between the extreme values $\lambda_{PF}(m_1) = 0$ and $\lambda_{PF}(m_2) = n - 1$ we get an average increase of $\lambda_{PF}(m)$ per additional link of $\frac{2}{n}$. This fact is depicted by the straight line in Figure (5.1) which has a slope of $\frac{2}{n}$ and intersects the origin. Moreover, Figure (5.1) shows an example of an improving path that reaches 50% of the density of a complete graph before it arrives at a stable network. Let us now define $y_m = \lambda_{PF}(m) - \frac{2}{n} m$. $y_m$ is just the difference between the largest real eigenvalue of the improving path and the straight line in Figure (5.1). Since

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\(^3\)The term path refers to a particular type of graph, as we have discussed in Section 2.1. On the other hand, in Section 5.2 we have introduced the term improving path which refers to a sequence in the space of graphs.
we have that \( y_{m_1} = 0 \) and \( y_{m_2} = 0 \), it obviously holds that the sum of the increments \( \Delta y_m = y_m - y_{m-1} \) in \( I = [m_1, m_2] \cap \mathbb{N}_+ \), have to be zero, that is
\[
\sum_{m=m_1+1}^{m_2} \Delta y_m = 0. \tag{5.2}
\]

However, Lemma (5) ensures that along the improving path, \( y_m \) starts off positive. In a case that \( I = I_1 \cup I_2 \), with \( I_1 = [m_1, m_3] \cap \mathbb{N}_+ \) and \( I_2 = [m_3, m_2] \cap \mathbb{N}_+ \). Equation (5.2) on the increments of \( y \) implies that
\[
\sum_{m \in I_1} \Delta y_m = - \sum_{m \in I_2} \Delta y_m = -b. \tag{5.3}
\]

Denoting by \( \langle \Delta y \rangle_{I_2} \) the average increment in the set \( I_2 \), we have
\[
\sum_{m \in I_2} \Delta y_m = \langle \Delta y \rangle_{I_2}(m_2 - m_3). \tag{5.4}
\]

There must be some increments that are smaller or equal to the average increment. Hence, there must exists a value \( m^* \) such that
\[
\Delta y_{m^*} \leq - \frac{b}{m_2 - m_3} < 0, \tag{5.5}
\]
or, equivalently, there exists an \( m^* \) such that
\[
\Delta \lambda_{PF}(m^*) < \frac{2}{n}. \tag{5.6}
\]

For any given cost \( c \), we can find an \( n \) large enough and an \( m^* \) such that \( \Delta \lambda_{PF}(m^*) < c \). This means that the marginal revenue is smaller than the cost, for some value of \( m^* \). This concludes the proof. \( \square \)

In Section 5.3 we will show with Proposition (19) that for large enough networks and costs smaller than \( 1/2 \), no improving path will ever reach the complete graph.

In light of the foregoing results we now proceed to investigate the stability of specific network structures. From a straightforward application of the properties of the marginal revenues from collaboration (cf. item (iii) in Lemma (2)) it follows that, on one hand, when marginal costs are zero, \( c = 0 \), links will always be created and no existing link will be deleted. Then the unique equilibrium is the complete graph \( K_n \).

**Proposition 11** If costs are zero, \( c = 0 \), then the complete graph \( K_n \) is the unique stable network.
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Figure 5.1: Largest real eigenvalue $\lambda_{PF}$ of a network of $n = 20$ firms as a function of the number $m$ of links, along three specific improving paths for costs $c = 0.0$, $c = 0.05$, $c = 0.15$ and severance cost parameter $\alpha = 0$. The number of links $m$ and the largest real eigenvalue $\lambda_{PF}$ are normalized to their maximum values (attained by the complete graph). The improving path for cost $c = 0.15$ reaches 22% of the density of a complete graph before it arrives at a stable network while the improving path for cost $c = 0.05$ reaches already 50% of the maximum density. Obviously the improving path for $c = 0$ reaches the complete graph.

Proof 11 If costs are zero, $c = 0$, then the change in eigenvalue equals the change in profits. Since (in a connected graph $G$) each link created strictly increases $\lambda_{PF}$ (Horn and Johnson, 1990) and accordingly profits, the complete graph $K_n$ is reached eventually. □

On the other hand, when the difference between marginal costs $c$ and severance costs $v(c)$ is larger than one, it is profitable to remove any link and the only equilibrium is the empty graph $\bar{K}_n$.

Proposition 12 For cost $c' = \alpha c > 1$ the empty graph $\bar{K}_n$ is the unique stable network.

Proof 12 There exists the following bound on the change in eigenvalue by the removal and creation of a link (Cvetkovic et al., 1995): If the graphs $G, G'$ differ in one link only then $|\lambda_{PF}(G') - \lambda_{PF}(G)| \leq 1$. A link is created if $\Delta \lambda_{PF} > c$. Thus, no link is created if $c = 1$. On the other hand, a link is removed, if $\Delta \lambda_{PF} < c'$. And thus, all links are removed if $c' > 1$ and we obtain an empty graph $\bar{K}_n$. □

Besides the foregoing extreme situations, the determination of stable networks becomes quite involved. This is because, in general, the marginal revenue from a collaboration depends on the topology of the graph. In addition, for a given topology, it varies with
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Figure 5.2: Upper and lower bounds for the size of the stable cliques (left) and the number of different sizes of stable cliques (right) for $\alpha = 1$ and $\alpha = 0.1$.

the position of the firm which is chosen to create or delete a link. Starting from an initial graph $G_0$ this property implies that different network trajectories can be explored, according to the particular pair of firms that is allowed to revise its collaboration strategy at the beginning of each period. Thus, improving paths have a strong path dependent character in this model and multiple equilibrium networks might be possible for the same level of marginal costs. In what follows we show that, on one hand, multiple pairwise stable networks exist for the same value of marginal cost $c \in (0,1)$ and severance costs $v(c)$. On the other hand, we identify a region of costs in the same interval in which stable networks do not exist and a sequence of networks is repeatedly visited\(^4\). In the following proposition, we show that a set of disconnected cliques of the same size can be a stable network, if their size falls within a certain interval that depends on the marginal cost of collaboration $c$ and on the severance cost parameter $\alpha$.

**Proposition 13** Consider costs $c, c' = \alpha c$ and $\alpha \in [0,1]$. If the network $G$ consists of a set of $k$ equally sized, disconnected cliques $K_n^1, K_n^2, ..., K_n^k$ ($G$ having $kn$ firms in total) then $G$ is stable if\(^5\)

$$\left\lfloor \frac{1 + c(1 - c)}{c} \right\rfloor \leq n \leq \left\lceil \frac{2 - c'(1 - c')}{c'} \right\rceil.$$  \hspace{1cm} (5.7)

**Proof 13** The structure of the proof is as follows. We want to show that the graph $G$ consisting of $k$ cliques of the same size is stable, that is, no link is removed or created.

\(^4\)This is a cycle in the space of network trajectories, to not confuse with the specific graph called cycle.

\(^5\)In the following, $\lfloor x \rfloor$, where $x$ is a real valued number $x \in \mathbb{R}$, denotes the smallest integer larger or equal than $x$ (the ceiling of $x$). Similarly, $\lceil x \rceil$ the largest integer smaller or equal than $x$ (the floor of $x$).
For the removal, we can focus on links between firms in the same clique, since these are the only links in $G$. Thus, in Proposition (14) we show that, for any pair of firms in the same clique, the link is not removed as long as the size $n$ of the clique is smaller than a given bound $b_r$. In particular, we will show that, $b_r = \lceil \frac{2-c'(1-c')}{c} \rceil$.

For the creation of links, we can focus on links between firms in different cliques, since these are the only new links that can be added to the graph. Thus, in Proposition (15) we show that for any pair of firms belonging to different cliques, a link between them is not created as long as the size $n$ of the clique is larger than another bound $b_c$, and we will show that $b_c = \lceil 1+c(1-c) \rceil$.

It turns out that the bound for the removal, $b_r$, is larger than the bound for the creation, $b_c$, for any value of $c \in [0,1]$, as it is shown in Figure (5.2). However, since the size $n$ of the clique has to be an integer, the interval $[b_c, b_r]$ needs to contain at least one integer. This can be done constructively. We explored the interval $c \in [0,1]$ with cost increments of $10^{-3}$ and we counted the number of integer values that fall within $[b_c, b_r]$. As it is shown in Figure (5.6), for $c < 0.35$, there is always at least one integer in between the two bounds, while for $c < 0.2$, there are always several integers falling in between the two bounds.

This is a remarkable finding as it implies that for the values of cost given above, there exists a multiplicity of equilibria. Indeed, for a given value of cost, the stable graphs are all the configurations with cliques of the same size $n$, where $n$ varies among the integers included in the interval $[b_c, b_r]$.

Propositions (13) and (14), used for this proof, are given below. $\square$

**Proposition 14** Consider a clique $K_n$ and denote by $K_n - ij$ the graph obtained from $K_n$ by removing a link $ij$. Then $\lambda_{PF}(K_n) - \lambda_{PF}(K_n - ij) > c'$ if $n \leq \lceil \frac{2-c'(1-c')}{c} \rceil$.

**Proof 14** Denote the matrix obtained from the adjacency matrix $A$ of $K_n - ij$, and subtracting the variable $\lambda$ on the diagonal of $A$ by $M = A - \lambda I$. $M$ is a block matrix of the form

$$M = \begin{pmatrix} K & B^T \\ B & D \end{pmatrix},$$

(5.8)
with submatrices

\[
K = \begin{pmatrix}
-\lambda & 1 & \cdots & \cdots & 1 \\
1 & -\lambda & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \\
1 & \cdots & 1 & -\lambda \\
\end{pmatrix}_{(n-2)\times(n-2)}, \\
B = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}_{2\times n}, \\
D = \begin{pmatrix}
-\lambda & 0 \\
0 & -\lambda \\
\end{pmatrix}_{2\times 2}.
\]  

(5.9)

(5.10)

(5.11)

Since \( M \) is a block-matrix (Horn and Johnson, 1990) we can write

\[
\det(M) = \det(K) \det(P).
\]  

(5.12)

We have the following lemma.

**Lemma 6**

\[
\det \begin{pmatrix}
a & 1 & \cdots & \cdots & 1 \\
1 & a & \ddots & & \\
\vdots & \ddots & \ddots & \ddots & \\
1 & \cdots & 1 & a \\
\end{pmatrix}_{n\times n} = ((n-1) + a) (a-1)^{n-1}.
\]  

(5.13)

**Proof 15** The above determinant can be written as \( \det(U - (1-a)I) \), where \( U \) is a matrix consisting of all ones, \( u_{ij} = 1 \) \( i, j = 1, \ldots, n \) and \( I \) is the identity matrix. Hence, the eigenvalues of the above matrix are minus \( 1 - a \) the eigenvalues of \( U \). \( U \) has eigenvalues \( n \) and \( 0 \) with multiplicities \( 1 \) and \( n-1 \) respectively (Horn and Johnson, 1990). Therefore, we can write for the determinant \( (n-(1-a)) (0-(1-a))^{n-1} = ((n-1) + a) (a-1)^{n-1} \).

\( \square \)

Thus, we get for the determinant of \( K \)

\[
\det K = -((n-1) - \lambda) (1 + \lambda)^{n-1}.
\]  

(5.14)

\(^{6}\)The numbers at the bottom right of the matrix indicate the dimension of the matrix.
The Schur complement is $P = D - BK^{-1}B^T$. Multiplying the inverse of $K$ with $B$ from the left and $B^T$ from the right we obtain

$$BK^{-1}B^T = ||K^{-1}||_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$  \hspace{1cm} (5.15)

where $||K^{-1}||_1$ is the sum of all elements in the matrix $K^{-1}$ (the $l_1$ norm of the matrix $K^{-1}$ (Horn and Johnson, 1990)). By computing $K^{-1}K = I$ one can verify that

$$K^{-1} = \begin{pmatrix} \frac{n-4-\lambda}{(\lambda-(n-3))(1+\lambda)} & \frac{1}{(\lambda-(n-3))(1+\lambda)} & \cdots \\ \frac{1}{(\lambda-(n-3))(1+\lambda)} & \cdots & \ddots \\ \vdots & & & \ddots \\ \end{pmatrix}. \hspace{1cm} (5.16)$$

And, by summation over the elements in $K^{-1}$, we obtain $||K^{-1}||_1 = \frac{n-2}{(n-3)-\lambda}$. Consequently, the determinant of the Schur complement $P$ is given by

$$\det(P) = (1 + \lambda)^{n-3} \lambda \left( \lambda^2 - (n-3)\lambda - 2(n-2) \right).$$  \hspace{1cm} (5.17)

The largest real eigenvalue of $K_n - ij$ is given by the root of

$$\lambda^2 - (n-3)\lambda - 2(n-2) = 0.$$  \hspace{1cm} (5.18)

Thus we get

$$\lambda_{PF} = \frac{1}{2} \left( n - 3 + \sqrt{n^2 + 2n - 7} \right).$$  \hspace{1cm} (5.19)

For the change in eigenvalue $\Delta \lambda_{PF} = \lambda_{PF}(K_n) - \lambda_{PF}(K_n - ij)$ we obtain

$$\Delta \lambda_{PF} = \frac{1}{2} \left( n + 1 - \sqrt{n^2 + 2n - 7} \right),$$  \hspace{1cm} (5.20)

since $\lambda_{PF}(K_n) = n - 1$. This is a decreasing function in $n$. Then for $n \in \mathbb{N}$, $\Delta \lambda_{PF} > c'$ if

$$n \leq \left[ \frac{2 - c'(1-c')}{c'} \right].$$  \hspace{1cm} (5.21)

For $c' = 2 - \sqrt{2} = 0.586$ we have $n \leq 3$ and for $c' = 1$ we obtain $n \leq 2$. \square

**Proposition 15** Denote the graph consisting of two disconnected cliques by $G$ and the graph obtained from $G$ by connecting the two cliques in $G$ via a link by $G'$. Then for $n \geq \left[ \frac{1+e(1-e)}{e^2} \right]$ we have $\lambda_{PF}(G') - \lambda_{PF}(G) < c$. 
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Denote the adjacency matrix of the graph obtained by connecting two complete subgraphs $K_n$ and $K_n$ via a link, see Figure (5.3), by $A$. And denote the matrix obtained by subtracting the variable $\lambda$ on the diagonal of $A$ by $M = A - \lambda I$. The eigenvalues of $A$ are given by the roots of the determinant of $M$. $M$ has the form of a block matrix with the submatrices $K$ and $B$. We have

$$M = \begin{pmatrix} K & B \\ B^T & K \end{pmatrix}, \quad (5.22)$$

$$K = \begin{pmatrix} -\lambda & 1 & \cdots & \cdots & 1 \\ 1 & -\lambda & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ 1 & \cdots & 1 & -\lambda \end{pmatrix}_{n \times n}, \quad (5.23)$$

Due to the symmetry of the graph we can consider a matrix of the following form, where we have put the one on the diagonal indicating the link between the cliques,

$$B = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}_{n \times n}, \quad (5.24)$$

with the Schur complement

$$P = K - B^T K^{-1} B. \quad (5.25)$$

For the determinant of $M$ we have $\det M = \det(K) \det(P)$. The determinant of $K$ is given by

$$\det(K) = (1 + \lambda)^{n-3}(\lambda - n + 3). \quad (5.26)$$
The inverse of $K$ is already given in (5.16). W.l.o.g. the Schur complement $P$ is given by

$$P = \begin{pmatrix} -\lambda & 1 & \cdots & \cdots & 1 \\ 1 & -\lambda & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\lambda & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & -\lambda + \frac{\lambda-(n-2)}{(\lambda-(n-1))(1+\lambda)} \end{pmatrix}. \quad (5.27)$$

In the next step we make use of the following lemma.

**Lemma 7**

$$\det \begin{pmatrix} b & 1 & \cdots & 1 \\ 1 & a & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ a & 1 & \cdots & 1 \\ 1 & \cdots & 1 & a \end{pmatrix}_{n \times n} = (1 - n + (n-2)b + ab)(a-1)^{n-2}. \quad (5.28)$$

**Proof 17** We give a proof by induction. For $n = 2$ we get

$$\det \begin{pmatrix} b & 1 \\ 1 & a \end{pmatrix}_{2 \times 2} = ab - 1 = (b(2-2) + ab - (2-1))(a-1)^0. \quad (5.29)$$

For $n = 3$ we get

$$\det \begin{pmatrix} b & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}_{3 \times 3} = a^2b - 2a + 2 - b = (b + ab - 2)(a-1). \quad (5.30)$$

For the induction step we apply a Laplace expansion of the determinant in (5.28) into Minors.

$$b \det \begin{pmatrix} a & 1 & \cdots & \cdots & 1 \\ 1 & a & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & a \end{pmatrix}_{(n-1) \times (n-1)} \quad - (n-1) \det \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & a & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & a \end{pmatrix}_{(n-1) \times (n-1)} \quad (5.31)$$

Note that here the matrix $K$ has dimension $n \times n$.
For the first determinant we can use Lemma (6) and for the second the induction hypothesis in order to obtain
\[ b((n - 2) + a)(a - 1)^{n-2} - (n - 1)(a - 1)^{n-2}. \] (5.32)
\[ \square \]

Now we can compute the determinant of the Schur complement \( P \)
\[ \det P = - (1 - n + (n - 2)q - \lambda q) (1 + \lambda)^{n-2} \]
\[ q := \frac{-(n-2)+\lambda}{-(n-1)+\lambda(1+\lambda)} - \lambda. \] (5.33)
\( \lambda_{PF} \) is given by the largest root of \( \det P = 0 \). We obtain \( \lambda_{PF} = \frac{1}{2}(n - 1 + \sqrt{n^2 - 2n + 5}) \).

The change in the largest real eigenvalue is
\[ \Delta \lambda_{PF} = \frac{1}{2}(n - 1 + \sqrt{n^2 - 2n + 5}) - (n - 1) \]
\[ = \frac{1}{2}(1 - n + \sqrt{(n - 2)n + 5}), \] (5.34)

since \( \lambda_{PF}(K_n) = n - 1 \). Thus, \( \Delta \lambda_{PF} < c \) if
\[ n \geq \left\lceil \frac{1 + c(1 - c)}{c} \right\rceil. \] (5.35)

For costs \( c = 0.5 \) we get \( n \geq 2 \) and for \( c = 1 \) we get \( n \geq 1 \). \( \square \)

From Proposition (13) it follows immediately that for a given value of cost \( c \) there exist multiple integer values \( n \) (the size of the clique) that fit into the interval spanned by the upper and lower bounds in Equation (13). This is discussed in more details in the proof of Proposition (13) (see appendix) and implies that multiple equilibrium networks exist for a given value of marginal cost \( c \) and severance cost \( v(c) \).

Moreover, note that the homogeneous size of the cliques is only a sufficient condition for stability but it is not necessary. Indeed, the equilibrium networks obtained with computer simulations show clearly that there exist also equilibria with disconnected cliques of different sizes (see e.g. Figure (5.13), bottom-right). The requirement of having cliques of the same size appears in Proposition (13) only to allow for an analytical treatment.

Equally sized disconnected cliques are not the only possible stable networks structures in the interval \( c \in (0, 1) \) and \( \alpha \in [0, 1] \). The next proposition shows that the spanning star, i.e. the star encompassing all firms, can be pairwise stable as well, if the size of the star (and therewith the number of firms in the industry) falls within a certain region that depends on the cost \( c \) and on the severance cost parameter \( \alpha \).
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Proof 18

Figure 5.4: Upper and lower bounds for the sizes of the stars (left) and the number of stars of different sizes (right) for \( \alpha = 1 \) and \( \alpha = 0.1 \). One can see that for \( \alpha = 1 \) and cost larger than 0.4 a stable spanning star does not exist.

![Graph showing upper and lower bounds for star sizes and number of stars](image)

Figure 5.5: A star \( K_{1,8} \) and (i) the creation of a link or (ii) the removal of a link.

**Proposition 16** Consider costs \( c, c' = \alpha c, \alpha \in [0,1] \). The network \( G \) consisting of a spanning star \( K_{1,n-1} \) with \( \lceil \frac{2}{c} \rceil \leq n \leq \lfloor \frac{1+c'^2(6+c'^2)}{4c'^2} \rfloor \) is stable.

In order to proof the stability of the spanning star \( K_{1,n-1} \), that connects all firms in the network, we have to consider two cases: (i) the creation of a link and (ii) the removal of a link (see Figure (5.5)).

(i) We consider the creation of a link between the firms in the star. The normalized eigenvector associated with the largest real eigenvalue \( \lambda_{PF} \) is given by

\[
\frac{1}{\sqrt{2(n-1)}}(1, \ldots, 1, \sqrt{n-1}, 1, \ldots, 1)^T.
\]

Maas (1987) found an upper bound for the largest real eigenvalue \( \lambda_{PF} \) and the cor-
responding eigenvector \( \mathbf{x} \) of an undirected graph \( G \) if a link \( ij \) is added

\[
\lambda_{PF}(G + ij) - \lambda_{PF}(G) < 1 + \delta - \frac{\delta(1 + \delta)(2 + \delta)}{(x_i + x_j)^2 + \delta(2 + \delta + 2x_i x_j)},
\]

(5.37)

where \( \delta \) denotes the minimum degree in the graph \( G \). Applying Equation (5.37) to the star \( K_{1,n-1} \) gives \( \Delta \lambda_{PF} = \lambda_{PF}(K_{1,n-1} + ij) - \lambda_{PF}(K_{1,n-1}) < \frac{2}{n} \). The link \( ij \) is not created if \( \Delta \lambda_{PF} < c \) or equivalently

\[
2 \frac{n}{c} > 2.
\]

This is a decreasing function in \( c \).

(ii) The change in eigenvalue by removing a link from \( K_{1,n-1} \) is given by \( \Delta \lambda_{PF} = \lambda_{PF}(K_{1,n-2}) - \lambda_{PF}(K_{1,n-1}) = \sqrt{n - 1} - \sqrt{n - 2} \). A link is not removed from the star if \( \Delta \lambda_{PF} > c' \) or equivalently

\[
2 < n < \frac{1 + c'^2(6 + c'^2)}{4c'^2}.
\]

(5.39)

Putting the bounds obtained in (i) and (ii) together we get the desired proposition. The number of different sizes of stars that are stable are shown in Figure (18). □

The foregoing results have two important implications in relation to the literature. First, the stable graphs are not necessarily connected. Second, in general they are not minimally connected. Indeed, the multiple clique equilibrium is a disconnected graph in which each component is complete and thus not minimally connected. This is an important feature that for instance distinguishes our model from the “connections” model in Jackson and Watts (2002) and from the linear “two-way flow” model Bala and Goyal (2000). In both such models, the equilibrium networks are always connected, while in the latter they are also minimally connected. Furthermore, both models find that the spanning star is stable for intermediate values of the cost of collaboration. However, differently from both models, in our model the spanning star is never the unique stable network. Indeed, the next proposition combines together the results of the previous two propositions, the conditions under which the link formation dynamics defined in (7) may lead to two different pairwise stable network topologies for the same level of marginal cost \( c \) and severance cost parameter \( \alpha \), namely (i) the set of disconnected equally sized cliques or (ii) the spanning star.

---

8Equation (5.37) is an upper bound and the number of stable stars derived from it may actually be higher.
Proposition 17 Consider costs \( c, c' = \alpha c, \alpha \in [0, 1] \) and the network \( G \) with \( n \) firms such that \( \left\lceil \frac{2}{c} \right\rceil \leq n \leq \left\lfloor \frac{1+c^2(6+c^2)}{c} \right\rfloor \). If there exists an integer \( k \leq n \), \( \text{mod} (n, k) = 0 \) such that \( \left\lceil \frac{1+c(1-c')}{c} \right\rceil \leq k \leq \left\lfloor \frac{2-c(1-c')}{c'} \right\rfloor \) then \( G \) can be stable for at least two cases.

(i) \( G \) consists of disconnected cliques \( K^1_k, \ldots, K^d_k \), \( n = kd \) or

(ii) \( G \) consists of a spanning star \( K_{1,n-1} \).

There are at least two stable networks for the same level of marginal cost \( c \) (degenerate cost region).

The multiplicity of equilibria stated in the above proposition is illustrated in Figure (5.6). The plot shows the number of different values of size of cliques when the configuration of multiple cliques and the spanning star are both stable. One can see that for smaller values of \( \alpha \) the number of stable networks increases. Furthermore, Figure (5.7) shows two examples of possible equilibrium networks obtained with \( n = 20, c = 0.3 \) and \( \alpha = 0.1 \).

Not all values of marginal cost \( c \) and severance cost parameter \( \alpha \) lead to pairwise stable networks. Consistently with the concept of improving path (cf. Jackson and Watts, 2002, Lemma (1)) the next proposition shows that in the interval \((0.586, 0.618)\), there exists a cycle of repeatedly visited networks.

Proposition 18 For values of cost \( 2 - \sqrt{2} = 0.586 < c < \frac{1}{2} (\sqrt{5} - 1) = 0.618 \) and \( \alpha \in [0.707, 1] \), the improving path leads to a cycle of networks. In such cycle, a sequence of paths \((P_2, \{P_2, P_2\}, P_3, P_3, P_2)\) is repeatedly visited.
5.2. Improving Paths and Equilibrium Networks

Figure 5.7: An example of two possible (different) equilibrium networks for cost $c = 0.3$ and $\alpha = 0.1$ with $n = 20$ firms. A set of disconnected cliques of the same size (left) and a spanning star (right)

Proof 19 A link between two disconnected firms is created if the largest real eigenvalue of the connected component of the firms after the link is created increases more than the cost, i.e. $\Delta \lambda_{PF} > c$. Similarly an existing link is removed if the largest real eigenvalue of the connected component of the firms after the link is removed does not decrease more than the cost, i.e. $|\Delta \lambda_{PF}| < c' = \alpha c$, with $c, c', \alpha \in [0, 1]$. We therefore have to consider the change in eigenvalue by the creation or removal of a link and compare it to the cost.

The proof of Proposition (18) is composed of two steps. (i) We show that in every period $t$ in the network formation process $\Gamma(G) = G(0), G(1), ...$ the network $G(t)$, $t \geq 1$, consists only of graphs from the set $S = \{\emptyset, P_2, P_3, P_4\}$, where $\emptyset$ denotes the set of isolated firms. (ii) We show that there exists a cycle, i.e. a sequence of repeatedly visited graphs, $C = (P_2, \{P_2, P_2\}, P_3, P_4)$, in which each graph is an improvement over the previous graph in the sequence $C$ (Jackson and Watts, 2002). Since all the graphs in the set $S$ can be found in the cycle $C$, starting from any of the graphs in $S$, the network formation process will proceed to the next graph in the cycle $C$. Therefore, for the given values of $\alpha$ and cost $c, c'$ respectively, we can infer that there does not exist a pairwise stable equilibrium network.

1. We give a proof by induction on the periods $t \geq 1$ of the network formation process $\Gamma(G)$. The induction basis is period $t = 1$. The network $G(1)$ is obtained from the empty network $G(0) = K_n$ (initial network) by the formation of a link and thus contains only a $P_2$ and isolated firms, both graphs are contained in the set $S$. Now we assume that the network at time $t > 1$ consists only of graphs in the set $S$ (induction hypothesis). The induction step consists in showing from $G(t)$ to $G(t + 1)$, no other
graphs than the ones in the set $S$ will be created. This will conclude this part of the proof. In order to prove the induction step, we observe that in the network formation process $\Gamma(G)$, at time $t$, a pair of firms, say $i$ and $j$, is selected at random. Either $i$ and $j$ are already connected in $G(t)$ or they are not. In any case, they both belong by assumption to one of the graphs in $S$. All the possible cases can be grouped as follows.

(a) Both firms are isolated. We show that the empty graph evolves into a $P_2$. The creation of a link between two isolated firms results in $\Delta \lambda_{PF} = 1$. Since by assumption $c < 1$, the link is indeed created.

(b) At least one of the firms, say $i$, is part of a $P_2$. In this case, we show that the only possible evolution step is from two $P_2$ to one $P_4$.
   
   i. Link creation: Figure (5.8) shows all possible distinct graphs that can be obtained depending on which graph belongs the second firm, $j$, and in which position. Each of these possible graphs is named with a number in the following way. For instance, when $j$ is in another $P_2$, the possible positions in that $P_2$ result both in one same graph labelled as 1. When $j$ is in a $P_3$, there are two possible distinct resulting graphs, labelled as 2.1 and 2.2. Similarly, we label the graphs resulting in the remaining case that $j$ is in a $P_4$. Table (5.1) report the increase of the largest eigenvalue of the graph when the link is created in all the possible cases. For instance, consider the graph 2.1 resulting from a $P_2$ and a $P_3$ with the creation of a link. Before the creation of the link, firm $i$ is in a $P_2$ which has $\lambda_{PF} = 1$ and firm $j$ is in a $P_3$ graph which has $\lambda_{PF} = \sqrt{2}$. Since the link formation rule requires that both firms will benefit after the creation of the link, we have to consider the worst case for the initial graph, which means the highest of the two values, i.e. $\lambda_{PF} = \sqrt{2} = 1.414$. For the resulting graph 2.1 we have $\lambda'_{PF} = \sqrt{3} = 1.732$, and therefore an increase of eigenvalue $\Delta \lambda_{PF} = 0.318$ which is smaller than the cost $c = 0.586$. It follows that this link will not be created. After analyzing all the other cases, we can see that only the case 1 results in an increase in the largest real eigenvalue $\Delta \lambda_{PF} = 0.618$ that is higher than the lower bound of the cost $c > 2 - \sqrt{2} = 0.586$. This implies that the only possible evolution step at this point is the formation of one $P_4$ starting from two $P_2$. Notice that $P_4$ is in the set $S$.

ii. Link deletion: A link is deleted if this beneficial to at least one of the two firms concurrent to the link, or, equivalently, if $|\Delta \lambda_{PF}| < c' = c\alpha$. In the case we are considering, by assumption at least one of the firms is in a
5.2. Improving Paths and Equilibrium Networks

Figure 5.8: All possible graphs for link creation when at least one of the selected firms is part of a $P_2$. We have labeled all the possible cases or links respectively with numbers shown next to the dashed links.

Table 5.1: Change in eigenvalue for link creation when at least one of the selected firms is part of a $P_2$. The numbers in the first row in the table refer to the possible links indicated by the same numbers in Figure (5.8). The maximum increase in the largest real eigenvalue is given by the creation of a link between the two pairs, indicated by 1 in Figure (5.8).

$c$) At least one of the firms is part of a $P_3$. In this case, we show that if $\alpha \in [0.707, 1]$ then the only possible evolution step is from one $P_3$ to one $P_2$ and one isolated firm.

i. Link creation: Figure (5.9) shows all possible graphs that can be obtained by adding a link when at least one of the selected firms is part of a $P_3$. Table (5.2) shows the increase in eigenvalue for all these possible graphs. From the Table we can see that in none of the cases the increase in eigenvalue is
higher than the lower bound of the cost. Thus, no link is created.

Figure 5.9: All possible cases for link creation when at least one of the selected firms is part of a $P_3$. We have labeled all the possible cases or links respectively with numbers shown next to the dashed links.

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Table 5.2: Change in eigenvalue for link creation when at least one of the selected firms is part of a $P_3$. The numbers in the first row in the table refer to the possible links indicated by the same numbers in Figure (5.9). The maximum increase in the largest real eigenvalue is given by the creation of a triangle, indicated by 5 in Figure (5.9). However, does not exceed the minimum value of cost $c \geq 0.586$ and so the corresponding firms do not form this link.

ii. Link deletion: The removal of a link from $P_3$ results in a change in eigenvalue of $\Delta \lambda_{PF} = \sqrt{2}-1 = 0.414$. The lower bound for the cost is $c > 2-\sqrt{2}$. We have that $|\Delta \lambda_{PF}| \leq c' = \alpha c$ if $\alpha \geq \frac{\sqrt{2}-1}{2-\sqrt{2}} = 0.707$. Therefore, if we restrict the values of $\alpha$ to the interval $[0.707, 1]$ then the link is removed and we obtain a single connected pair $P_2$ and one isolated firm. Both are contained in the set of graphs $S = \{\emptyset, P_2, P_3, P_4\}$.
(d) At least one of the firms is part of a $P_4$. In this case, we show that if $\alpha \in [0.707, 1]$ then the only possible evolution step is from one $P_4$ to one $P_3$ and an isolated firm.

i. Link creation: Figure (5.10) shows the possible graphs that can be obtained by adding a link when at least one of the selected firms is part of a $P_4$. Table (5.2) shows the corresponding increase in eigenvalue. From the Table we can see that in none of the cases the increase in eigenvalue is higher than the lower bound of the cost. Thus, no link is created.

![Figure 5.10](image)

Figure 5.10: All possible cases for link creation when at least one of the selected firms is part of a $P_4$.

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</table>

Table 5.3: Change in eigenvalue for link creation when at least one of the selected firms is part of a $P_4$.

ii. Link deletion: The change in eigenvalue is either (A) $\Delta \lambda_{PF} = 0.204$ if the first or last link in $P_4$ is removed or (B) it is $\Delta \lambda_{PF} = 0.618$ if the second link in the middle of $P_4$ is removed.
In case (A) the change in eigenvalue is given by $\Delta \lambda_{PF} = \sqrt{2} - \frac{1}{2}(1+\sqrt{5})$. If $c' = \alpha c \geq \sqrt{2} - \frac{1}{2}(1+\sqrt{5})$, then $|\Delta \lambda_{PF}| \leq c' = \alpha c$ and the link is removed. This means that, for $c > 2 - \sqrt{2}$ we must have that $\alpha \geq \frac{|\sqrt{2} - \frac{1}{2}(1+\sqrt{5})|}{2 - \sqrt{2}} \approx 0.348$, which is certainly true since we have assumed that $\alpha \geq \frac{\sqrt{2} - 1}{2 - \sqrt{2}} = 0.707$. Thus, the link is removed under the above made assumptions on cost and $\alpha$ and we obtain a path of length three, $P_3$, which is in the set of graphs $\{\emptyset, P_2, P_3, P_4\}$.

In case (B) we have that $\Delta \lambda_{PF} = 1 - \frac{1}{2}(1 + \sqrt{5})$. This link is removed for $|\Delta \lambda_{PF}| \leq c' = \alpha c$ implying that $\alpha \geq \frac{|1 - \frac{1}{2}(1+\sqrt{5})|}{2 - \sqrt{2}} \approx 1.055$. Since we have assumed that $\alpha \in [0,1]$ this cannot be true. Therefore, the link is not removed.

Notice that in all the cases the graphs created belong to the set $S$, as we wanted to prove.

2. From the preceding analysis we can infer two facts: First, the individual profits of the firms involved in the creation or removal of a link always increase along the closed sequence of graphs $C = (P_2, \{P_2, P_2\}, P_4, P_3, P_2)$, as it is illustrated in Figure (5.11). Therefore, this is an improving path (Jackson and Watts, 2002) which is cyclical and never reaches an equilibrium. Notice that, the firms responsible for the creation or deletion of the links along the sequence are different and individual profits of a given firm are not increasing at every step. Along the improving path, the individual profits of the firms involved in the link creation or removal increase, while the profits of the others may decrease. This highlights the effects of the externalities inherent in our model on the individual profits of the firms.

Second, since at every step of the sequence there is only one possible network evolution step and since all the non-empty graphs of the set $S$ are also in the cycle $C$, we can conclude that $C$ is the only improving path in the given range of parameters.

The graphs which are repeatedly visited are illustrated in Figure (5.11). The fact that for some values of the parameters of the model there exists no stable network has also been found by Jackson and Watts (2002) and by Haller et al. (2007); Haller and Sarangi (2005).
5.3 Stability vs. Efficiency

We have shown that for the same level of marginal cost there exist multiple equilibrium structures associated with different values of total profits. This indicates that stable networks can, in general, be inefficient. In particular, we have shown that in the marginal cost interval \( c \in [0, 1] \), graphs that are not connected can be stable (cf. Propositions (13) and (17)), while in that cost region the efficient graph is always connected (cf. Proposition (7)). The multiplicity of equilibria and the consequent possible inefficiency of the network evolution process stems from externalities inherent in the process of knowledge recombination, described in Chapter 3. Indeed, when a firm decides to create or delete a link it takes into account its private marginal revenue from collaboration (given by the change in the largest eigenvalue of its connected component), but neglects social marginal revenues inherent to that decision. The latter is equal to the sum of changes in the largest eigenvalue of all firms belonging to the same connected component. Thus, it may well be that creating a link is not profitable for the individual firm although it would be profitable from the industry point of view.

Furthermore, the efficient network may not even belong to the set of equilibria, as it is shown in the next proposition.

**Proposition 19** Consider a network of size \( n \geq 2 \). For cost \( c < \frac{1}{2} \) the equilibrium network is not efficient.

**Proof 20** The change in the largest real eigenvalue, \( \Delta \lambda_{PF} \) of a graph \( G \) with \( m \) links and
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Figure 5.12: Maximal value of cost $c$ for which the complete graph $K_n$ can be obtained as an equilibrium network.

$n$ firms, by adding one link to the graph is bounded by

$$\Delta \lambda_{PF} \leq \frac{1}{2}(-1 + \sqrt{1 + 8(m + 1)}) - \frac{2m}{n}. \quad (5.40)$$

The above inequality can be obtained as follows. The average degree of the graph is $\bar{d} = \frac{2m}{n}$. A lower bound on the largest real eigenvalue is given by $\lambda_{PF} \geq \bar{d}$ (Cvetkovic and Rowlinson, 1990). An upper bound on the largest real eigenvalue is given by $\lambda_{PF} \leq \frac{1}{2}(-1 + \sqrt{1 + 8m})$ (Stanley, 1987). Combining the two bounds yields the inequality in (5.40).

We apply the bound of Equation (5.40) on the change in the largest real eigenvalue, $\Delta \lambda_{PF}$, by adding a link to the graph $G$ with $m$ links. Solving the equation $\Delta \lambda_{PF} = c$ for $m$ yields the maximal number $m^*$ of links that can be added to a graph of $n$ firms when the cost is $c$, $m^*(n, c) = \frac{n}{4}(-1 - 2c + n + \sqrt{n^2 + 9 - 2n(1 + 2c)})$. Notice that $m^*(n, c)$ decreases with increasing cost $c$. Imposing now this expression to be equal to one link less than the number of links in a complete graph $K_n$ with $n$ firms, $\binom{n}{2} - 1 = \frac{n(n-1)}{2} - 1$, we get $c^* = \frac{2}{n}$. Thus, if costs exceed this value then the increase in eigenvalue corresponding to the creation of the link that would make the graph complete, is smaller than the cost. Notice that $c^*$ decreases with $n$ and tends to 0 for large $n$, as plotted in Figure (5.12), and therefore for any given $c$ there is an $n$ large enough such that the complete graph cannot be reached. \hfill \Box

This result can be explained in the following way. Proposition (8) states that, when the marginal cost of link formation is less or equal to 1/2, the complete graph is the efficient graph. However, if the number $n$ of firms in the industry is large enough, the individual marginal revenue of a collaboration is bounded from above by a value decreasing with $n$ (see the proof of the Proposition (19) in the appendix). In particular, for $n \geq \frac{2}{c}$ the upper
bound is always smaller than the marginal cost \( c \). Therefore the complete graph is not stable\(^9\).

An exhaustive discussion of efficiency and stability would require the determination of individual and total profits of firms under all possible network configurations. Both require the computation of the largest eigenvalue. Unfortunately, there is no general closed form solution available for any graph. However, one can provide general results for some special classes of graphs. Based on these findings Table (5.4) summarizes the results on efficiency and stability discussed so far and compares them with results for other well known classes of graphs in the literature.

Three graphs in the table deserve a special attention. The first is the empty graph, which is never stable nor efficient in the interval \([0, 1)\). The second one is the complete graph, which is efficient in \([0, 0.5]\), but is never stable for \( c > 0 \) (see Proposition (10) and Proposition (19)). The third graph is the star, which can be stable but is never efficient\(^{10}\) in \([0, 1]\). In other words, both the star and the complete graph are never stable and efficient at the same time. This is a first important difference with respect to the literature, e.g. the models in Jackson and Watts (2002) and Bala and Goyal (2000), where, at least in an interval of the parameters considered, the star (or, respectively, the complete graph) can be efficient and stable. In our model the tension between efficiency and stability is more pronounced. We were not able to find any efficient graph which is also stable, except from the trivial case of \( c = 0 \) in which, due the absence of collaboration cost, the complete graph is both stable and efficient.

Moreover, it is interesting to review the properties of the other graphs listed in the table and their mutual relations. A \( k \)-regular graph, i.e. a graph in which all firms have the same degree, yields a revenue proportional to the degree of the firms, regardless of the size of the graph. This means that when the degree is small the performance in terms of aggregate profits of this graph is rather poor. However, the complete graph is a particular case of regular graph in which all firms have degree \( n - 1 \). In this case, the regular graph can be efficient.

The set of cliques of the same size, is stable for particular values of their size \( d \), depending on the level of costs. It can also be efficient, in the particular case of one set containing all firms, i.e. the complete graph. In this case however, it is never stable, as noted above.

\(^9\)Another (degenerate) region of the parameter space in which the network dynamics leads to inefficient equilibrium outcomes is the one in which marginal cost is in the open interval \((2 - \sqrt{2}, \frac{1}{2} (\sqrt{5} - 1)) = (0.586, 0.618)\). In that case (cf. Proposition (18)), for any number of firms in the industry the dynamics gets stuck into a cycle of networks, none of which is efficient.

\(^{10}\)One can show that for \( c < \frac{n}{n-1+\sqrt{n-1}} \sim 1 \) for \( n \to \infty \), \( K_n \) has a higher performance than \( K_{1,n-1} \). E.g. for \( n = 100 \) we get \( c < 0.918 \).
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The set of identical cliques is also a particular case of k-regular graph, because the firms in each clique have the same degree. In a path, the degree of the firms is 2, except from the two firms at the beginning and at the end of the path. In this sense, the graph is similar to a 2-regular graph. Indeed its eigenvalue is a little smaller than the one of 2-regular graph. When the network evolves starting from an empty graph, the first connected graph that is formed is indeed a path of length 2, possibly followed by a path of length 3 (see the proof of Lemma (5) in the appendix). In such transition, the largest eigenvalue of the component jumps from 0 to 1 and then to $2\cos\left(\frac{\pi}{4}\right) > 1$. Instead, when the graph is almost complete, the addition of a new link yields a negligible increase in the eigenvalue. Notice that the path of length 3 is also a star with one hub and two peripheral firms.

A cycle is a closed path and it is in particular a 2-regular graph. In a cycle there is only one walk, which yields a revenue independent of the number of participating firms. In particular, because of this in our model the path is never an efficient graph. As we already noticed in Section 3.7, this is a consequence of the payoff function which differs in this respect from the one used in other models in the literature (e.g. Bala and Goyal, 2000).

We also list in the table the bipartite graph because of its relation to the notion of structural holes (cf. Burt, 1992). In a bipartite graph, firms can be grouped in two separate classes so that links connect only firms of one class to firms of the other class. Consider for example a network consisting of few hubs, disconnected among them, and of many peripheral firms, connected only to one or more hubs. The hubs fill the structural holes among the the peripheral firms. This network is also a bipartite graph, since the hubs and the peripheral firms form two separate classes of firms. Notice that the star is a particular case of a bipartite graph. In our model, the bipartite graph is not efficient nor stable.

Finally, concerning the largest eigenvalue of $F_{n,d}$ an exact solution is given by the largest root of the cubic polynomial $x^3 - (d-2)x^2 - (n-1)x + (d-2)(n-d)$ (Bell, 1991). From this exact solution, one can show that for a fixed value of $d$, $\lim_{n \to \infty} \Delta \lambda_{PF} = 0$ and thus it is always profitable to remove a link if $n$ is large (however large the severance cost or small the marginal cost may be).

5.4 Topological Properties of Stable Networks

The empirical research on R&D partnerships has investigated in depth the topological patterns of networks of knowledge exchange. From this literature (see e.g. Ahuja, 2000; Fleming et al., 2007; Hanaki et al., 2007; Powell et al., 2005), three features emerge as
### Table 5.4: Summary of the largest real eigenvalue, total profits, efficiency and stability for different types of networks.

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<tbody>
<tr>
<td>empty graph</td>
<td>$\lambda_{PF} = 0$</td>
<td>$\Pi = 0$</td>
<td>$c &gt; n$</td>
<td>$c &gt; 1$</td>
</tr>
<tr>
<td>$G = K_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>complete graph</td>
<td>$\lambda_{PF} = n - 1$</td>
<td>$\Pi = (1 - c)n(n - 1)$</td>
<td>$c \leq \frac{1}{2}$</td>
<td>$c = 0$</td>
</tr>
<tr>
<td>$G = K_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>k-regular graph</td>
<td>$\lambda_{PF} = k - 1$</td>
<td>$\Pi = n(k - 1)(1 - c)$</td>
<td>if $k = n$</td>
<td>see cliques</td>
</tr>
<tr>
<td>$G = K_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>path</td>
<td>$\lambda_{PF} = 2\cos\left(\frac{\pi}{n+1}\right)$</td>
<td>$\Pi = 2\cos\left(\frac{\pi}{n+1}\right) - (n - 1)c$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$G = P_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>star</td>
<td>$\lambda_{PF} = \sqrt{n - 1}$</td>
<td>$\Pi = n\sqrt{n - 1} - 2(n - 1)c$</td>
<td>not in $0 &lt; c &lt; 1$</td>
<td>$\left[\frac{2}{c}\right] \leq n \leq \left[\frac{1 + c^2(6 + c'^2)}{4c'^2}\right]$</td>
</tr>
<tr>
<td>$G = K_{1,n-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cycle</td>
<td>$\lambda_{PF} = 2$</td>
<td>$\Pi = 2n(1 - c)$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$G = C_n$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>bipartite graph</td>
<td>$\lambda_{PF} = \sqrt{n_1n_2}$</td>
<td>$\Pi = (n_1 + n_2)\sqrt{n_1n_2} - n_1n_2c$</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$G = K_{n_1,n_2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G = F_{n,d}$</td>
<td>$\lambda_{PF} \geq d - 1$</td>
<td>$\Pi = \lambda_{PF}(F_{n,d}) - 2c\left(\binom{n}{2} + (n - d)\right)$</td>
<td>with good approx. $^a$</td>
<td>no</td>
</tr>
<tr>
<td>$G = {K_1^d, ..., K_l^d}$</td>
<td>$\lambda_{PF} = d - 1$</td>
<td>$\Pi = n(d - 1)(1 - c)$</td>
<td>if $l = 1, d = n$, see $K_n$</td>
<td>$\left[\frac{1 + c(1-c)}{c}\right] \leq d \leq \left[\frac{2-c(1-c)}{c}\right]$</td>
</tr>
</tbody>
</table>

$^a\forall c$, and for large $n$, total profits of this graph tends to the one of efficient graph, $\lim_{n \to \infty} \epsilon = 0$

$^b$We have $l$ cliques of identical size $d$. 
robust stylized facts: (i) R&D networks are sparse, that is the number of actual links is much less than the number of possible links. (ii) Networks are highly clustered. This means that the collaboration partners of a firm are likely to be connected among each other. (iii) The distribution of links over the firms is characterized by high dispersion, with few firms being connected to many others.

The analytical study of equilibrium networks in Section 5.1 has pointed to the existence of equilibrium networks that match some of the stylized facts mentioned above. Indeed, equally sized cliques are characterized by a high clustering, while the spanning star shows high degree heterogeneity. All these networks belong to the set of possible equilibria structures in our model.

In this section we define an explicit process of network evolution that is a particular case of an improving path and we analyze by means of computer simulations the structural properties of stable networks in our model. In this way we explore the existence of more complex stable network structures, beyond those described in the previous section. Furthermore, we investigate whether our model is also able to generate pairwise stable structures that feature, at the same time, all the stylized facts of R&D networks.

There are several possible processes which would be consistent with the definition of an improving path. In this work, we investigate a stochastic process in which all pairs of firms have the same probability to be selected to revise their R&D collaboration strategy (cf. Vega-Redondo, 2007, p. 212).

**Definition 7 (Myopic Pairwise Dynamics)** Let \( G = (N, L) \) denote the current graph \( G(t) \) at time \( t \). We define the network formation process \( \Gamma(G) \) as follows. At the beginning of each period (at times \( t = 0, T, 2T, ... \)) a single pair of firms, \( i \) and \( j \), is uniformly selected at random from the set \( N \) of firms.

(i) If the link \( ij \) does not currently exist, \( ij \not\in L \), then it is created whenever neither firm is harmed by the creation and at least one of them strictly gains, i.e.

\[
\pi_i(G + ij, c) \geq \pi_i(G, c) \wedge \pi_j(G + ij, c) \geq \pi_j(G, c) \wedge \\
\pi_i(G + ij, c) > \pi_i(G, c) \vee \pi_j(G + ij, c) > \pi_j(G, c).
\]

or equivalently

\[
\lambda_i(G + ij) - \lambda_i(G) \geq c \wedge \lambda_j(G + ij) - \lambda_j(G) \geq c \wedge \\
\lambda_i(G + ij) - \lambda_i(G) > c \vee \lambda_j(G + ij) - \lambda_j(G) > c.
\]

(ii) If the link \( ij \) is currently in place, \( ij \in L \), then it is removed whenever at least one of the firms strictly gains from the change, with link deletion involving the severance
cost $v(c) = (1 - \alpha)c$, and $\alpha \in [0, 1]$. More formally

$$\pi_i(G - ij, c, v) > \pi_i(G, c, v) \lor \pi_j(G - ij, c, v) > \pi_j(G, c, v),$$

(5.43)

or equivalently:

$$\lambda_i(G) - \lambda_i(G - ij) < \alpha c \lor \lambda_j(G) - \lambda_j(G - ij) < \alpha c$$

(5.44)

Note that, in the evolution of the network defined above, the only element of stochasticity is the sequence of the pairs of firms chosen to create or delete links.

We study stable network structures arising from this process in computational experiments

11 conducted in a large region of the model’s parameter space. More precisely, we carried out multiple (50 repetitions for each parameter choice) computer simulations of the network dynamics defined in (7) with a fixed number $n$ of firms in the industry

12 ($n = 50$), starting each from an empty network $\bar{K}_n$. For each simulation we selected a value for the marginal cost $c$ in the interval $[0, 1]$ and a value for the severance cost parameter $\alpha$ in the interval

13 $[0, 0.5]$. As the number of chosen values were respectively 12 for the marginal cost and 5 for the severance cost parameter, the total number of computer simulations summed up to 3000. The results of the aforementioned Monte-Carlo experiments are shown in the Figures from (5.13) to (5.15).

The plots in Figure (5.13) show typical equilibrium networks obtained in simulations for marginal cost of link formation equal to 0.15 and different values of the severance cost parameter $\alpha$. Recall that severance cost are equal to $v = (1 - \alpha)c$, and thus are inversely related to the parameter $\alpha$. As the plots reveal, in this region of the parameter space the dynamics in our model is able to generate equilibrium structures displaying the complex features that characterize R&D networks observed in reality (see e.g. Fleming et al., 2007). In particular, for very high severance costs the equilibrium network contains a giant component with a high degree heterogeneity. On the other hand, as the severance cost associated with link deletion fall down (increasing values of $\alpha$), we observe a significant increase in the cliquishness of the network, and a reduction of degree heterogeneity.

11When simulating the network evolution discussed in Section 5.1 the largest real eigenvalue of the network has to be computed many times. Since the largest real eigenvalue of a graph can be computed in polynomial time (Hong, 1993) our model is well suited for numerical investigations.

12Choosing a different, possibly higher, number of firms would have not altered the results, as only the size (the number of firms) of the connected components, and not the total size of the system matters for the dynamics.

13Preliminary simulation studies with values of $\alpha$ greater than 0.5 for the severance cost did not reveal the presence of any striking difference in the results.
Figure 5.13: Equilibrium networks for $n = 50$, $c = 0.15$, (a) $\alpha = 0.0$, (b) $\alpha = 0.1$, and (c) $\alpha = 1.0$ starting from an empty network. Relative profits (compared to the firm with highest profits in the network) are indicated with different shades, meaning that nodes representing firms with higher relative profits are shown in a lighter shade. The network plots use the Fruchterman-Reingold algorithm (Fruchterman and Reingold, 1991).
5.4. Topological Properties of Stable Networks

Figure 5.14: Average degree \( \langle \bar{d} \rangle \) and degree variance \( \langle \sigma_d^2 \rangle \) in the equilibrium network for \( n = 50, c \in [0, 1] \), starting from an empty network (averaged over 50 simulations).

The insights coming from the foregoing qualitative study are confirmed by a more quantitative analysis of the topological properties of equilibrium graphs. The plots in Figure (5.14) display respectively the mean and the variance of the network degree distribution as functions of the marginal cost \( c \) and severance cost parameter \( \alpha \). The mean degree is inversely related to the sparseness of the graph, while degree variance captures the degree heterogeneity. As the plots in the figure make clear, higher cost of R&D collaboration lead to graphs that are more sparse. On the other hand, degree heterogeneity reaches a peak for values of marginal cost close to 0.1, and then falls down as collaboration costs increase. In addition, degree heterogeneity increases with severance costs (decreasing \( \alpha \)).

The presence of clusters of highly interconnected firms is a key feature of empirically observed R&D networks (cf. stylized fact number (iii)). As the plots in Figure (5.15) show, this feature is also a characteristic for the equilibrium networks generated by the model. In particular, the average clustering coefficient (Figure (5.15), top-left) is close to one in a wide region of the explored parameter space \( (c \in (0, 0.5), \alpha > 0) \). Moreover, it is a decreasing function of severance costs \( v = (1 - \alpha) c \). Finally, note that clustering becomes zero for values of costs greater or equal to 0.7. Further information on the topological features of R&D clusters can be gathered by looking at the average number of connected components, at their average size and the concentration of their size (Figure (5.15), top-right, bottom-left and bottom-right respectively). As the plots in the figure indicate, the number of connected components is an increasing function of collaboration costs while the average size and its concentration variables are negatively related to collaboration costs. In addition, as the costs of link severance increase the number of components increases, while component size and its concentration decrease.

Joining together the foregoing results we can conclude that sparse equilibrium networks
organized in clusters of highly interconnected firms are a distinctive feature of the network dynamics in our model. Moreover, low values of the R&D collaboration costs and high values of the costs of link severance lead to equilibrium structures characterized by a small number of large components, with a highly dispersed degree distribution. As collaboration costs increase and as link severance costs decrease, we observe that equilibrium networks tend to be more and more organized in size homogeneous cliques having only few connections among them.

5.5 Transient Network Dynamics

In this section we briefly study the transition of the initial network to the equilibrium network for vanishing severance costs \( v(c) = 0 \). For this purpose, in Figure (5.16) we show snap-shots of a sample trajectory of the network formation process for \( \alpha = 1.0, c = 0.175 \).
with $n = 100$ firms at times $t = 100$, $t = 200$, $t = 1000$ and the equilibrium network, reached at approximately $t = 15 \cdot 10^4$. Besides, different network statistics are shown in Figure (5.17). We find that the network evolves from a graph with low clustering to high clustering. The average degree and correspondingly the network density is increasing over time while the degree variance decreases over time and is almost zero in the equilibrium network. This fact is also indicated by the degree distributions at two points in time, one at an early stage at $t = 427$ and the other at $t = 1710$ in a later stage of the network evolution. The first distribution shows a broader spectrum of distinct degrees while the latter distribution is concentrated around the average degree. A homogeneous degree distribution is then characteristic for the equilibrium network where the firms in the size-homogeneous cliques have (approximately) identical degrees.

We have discussed in Section 1.9 that the network topology can significantly change from displaying low clustering in an early stage of the evolution to a network with high clustering in a later stage of the evolution. Previous authors have explained this transition by a change in the technological regime of the industry (Cowan and Jonard, 2006; Rowley et al., 2000b; Walker et al., 1997). In an early stage new technologies are explored and firms tend to connect to distant part in the network. These networks are characterized by structural holes and a low value of clustering. As technologies mature, firms prefer to form densely connected clusters with a high social capital. Accordingly, these networks exhibit high clustering.

In this section we may provide an alternative explanation to the transition of networks with low clustering to networks with high clustering over time. In our model, the technological regime does not change. Instead, marginal profits from forming an R&D collaborations change with the structure of the network. Recall that a link is formed if the increase in the number of possible knowledge recombinations, as measured by the increase in the largest real eigenvalue, generates marginal revenues that are above the marginal cost of a collaboration. In an early stage of the network evolution, when the network is sparse, marginal revenues are rarely below marginal cost and so many randomly created links are beneficial for the two firms involved. The network formation thus resembles the growth of a random graph in an early stage of the evolution. Note that the clustering coefficient of the random graph is vanishes. However, with increasing density of the network, marginal revenues can become less than marginal cost for many links that are formed at random. In this case, firms become more selective about their collaborations and they create only those links that correspond to the highest increase in the largest real eigenvalue $\lambda_{PF}$. It is known that, for a given number of nodes and links, the graph with the highest value of $\lambda_{PF}$ consists of a clique, while some nodes may be attached to the clique (Cvetkovic and Rowlinson, 1990). This indicates that high values of $\lambda_{PF}$ are associated with densely
connected clusters. Since firms form only those links that give rise to the highest increase in $\lambda_{PF}$, the clustering of the network increases in a later stage of the network evolution. Therefore, we observe a transition from low clustering to high clustering as the network evolves over time.

Finally, we point out the similarity between the equilibrium networks obtained for low values of $\alpha = 0.1$ in Figure (5.13) and the networks at an early state of the network evolution in Figure (5.16). In both cases these networks are characterized by (i) sparseness, (ii) significantly higher values of clustering than one would observe in a random graph of comparable density and (iii) heterogeneous degree distributions with the presence of hubs. We have noted already that these topological characteristics can also be found in empirical networks. The model discussed in this section is able to generate networks that match the empirically observed properties of R&D networks either in the equilibrium for high values of the severance cost or for low severance costs in an earlier stage of the network evolution.
Figure 5.16: Transient networks for $\alpha = 1.0$, $c = 0.175$ and $n = 100$ at times (a) $t = 100$, (b) $t = 200$, (c) $t = 1000$ and (d) the equilibrium network for $t = 15 \cdot 10^4$ starting from an empty network. Relative profits (compared to the firm with highest profits in the network) are indicated with different shades, meaning that nodes with higher relative profits are shown in a lighter shade.
Figure 5.17: A sample network evolution for $\alpha = 1.0$, $c = 1.75$ and $n = 100$. The average degree (top, left) increases while the degree variance (top, right) decreases over time and almost vanishes for the equilibrium network. The degree distribution (bottom, left) is skewed in the early phases (after one fourth of the time to the equilibrium) of the network evolution and becomes peaked around a single value in the equilibrium. The clustering coefficient (bottom, right) increases over time and attains its maximum value in the equilibrium.
Chapter 6

Local Best Response Link Formation

In this chapter we introduce a dynamic model of link formation that departs from the model in the previous chapter in mainly two ways. First, firms are assumed to have certain preferences for forming a collaboration with their potential partners. This is different from the assumptions we made underlying the link creation mechanism that we studied in the previous section. There, firms meet at random and form a collaboration if it increases their profits. This happens even when there would exist a more profitable collaboration. These myopic link formation dynamics have originally been proposed by Jackson and Wolinsky (1996). In this chapter we depart from the assumption that firms meet at random and that they have no preferences over their potential collaborations, while we keep the assumption that forming a collaboration between two firms involves mutual consent. Second, we do not assume that each collaboration involves a time-invariant marginal cost. Instead, firms live in a volatile environment which eventually renders some of their collaborations unprofitable. This means that collaborations are exposed to a stochastic link decay. We will discuss each of these extensions in the following paragraphs.

There are several empirical studies that try to identify the mechanisms driving the formation of inter-firm collaborations. Orsenigo et al. (2001); Riccaboni and Pammolli (2002) have investigated network and firm growth in life sciences and information and communication technologies. The authors find that new collaborations among existing firms in the network are driven by a “preferential attachment” mechanism (Barabasi and Albert, 1999). This means that firms with many collaborations are more likely to receive a new collaboration. This preferential attachment mechanism leads to a “rich-get-richer” effect.

Subsequently, Gay and Dousset (2005) have argued that networks in biotechnology are driven by preferential attachment to firms holding key technologies instead of companies having a high degree. Not the number of collaborations but the “fitness” in terms of knowledge and technology makes a firm a preferred partner for a collaboration. Instead of
a “rich-get-richer” process the authors call this mechanism of attachment a “fit-get-richer” process (see also Ravasz and Barabási, 2003).

Similarly, Powell et al. (2005) have studied interorganizational collaborations in the life sciences. The authors compare different mechanisms under which collaborations are formed. Similarly to the previous empirical studies, they find evidence that firms in the biotech sector preferentially connect to other firms with high degree. But they also find evidence for other mechanisms that influence the partner choice. Firms choose new partners based on their similarity to previous partners and firms try to meet the choices of their existing partners if they form a new collaboration. Finally, they find strong empirical evidence of firms connecting to each other in order to create multiple independent pathways between them.

We incorporate the aforementioned observations into our model as follows. We assume that a firm is a technological leader if it has a higher level of knowledge than other firms. In the line of a “fit-get-richer” mechanism suggested by Riccaboni and Pammolli (2002), firms then preferentially connect to the firm with the highest knowledge level. We have shown in Proposition (1) that this firm is also the firm with the highest eigenvector centrality. Thus, we study a link formation process based on centrality, following the empirical literature\(^1\) that has revealed that firms with high centrality tend to form alliances with each other (Gulati and Gargiulo, 1999; Kogut et al., 1992; Powell et al., 1996a; Walker, 2005). Moreover, from the properties of the eigenvector centrality (see Section 2.3) it follows that the link to the firm with the highest knowledge level maximizes the number of independent paths in the connected component of the firm. This is in line with the findings of Powell et al. (2005). Therefore, both empirical observations are reproduced in this model.

Goyal (2007) points out that firms that are close in the current collaboration network are more likely to form a new collaboration (see also the empirical study by Gulati and Gargiulo (1999)). Following this observation, we assume that firms search for new partners

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\(^1\)We assume that link formation is based on centrality. As such, our link formation process is relevant to all situations in which the partner choices of agents are influenced by the centrality of their position in the network and that of their potential partners. There are numerous situations in which centrality plays an important role. It is a fundamental measure of the importance of actors in social networks dating back to early works like Bavelas (1948) (see Wasserman and Faust (1994) for an introduction and survey). Among many other examples centrality has been used to investigate influence in exchange networks (Cook et al., 1983), importance of actors in peer networks (Ballester et al., 2006; Durlauf, 2004), creativity of workers (Perry-Smith and Shalley, 2003), their performance (Mehra et al., 2001), power in organizations (Brass, 1984), the flow of information (Borgatti, 2005b; Stephenson and Zelen, 1989), the formation and performance of (Boje and Whetten, 1981; Powell et al., 1996a; Uzzi, 1997) as well as the success of open-source projects (Grewal et al., 2006).
among their neighbors’ neighbors. This local search mechanism can be interpreted in three different ways. First, firms perceive the partners of their partners as more similar to themselves. Second, firms try to match the partner choices of their neighbors. Both mechanisms have been found to play a role in inter-firm collaboration networks in the biotech industry as the study of Powell et al. (2005) has shown. Third, we assume that firms have only limited information on which other firms operate in the same technological domain and what their knowledge capacities are. However, firms may have information on the partners of their partners and they know the most technologically advanced firms among these. In this way, we introduce a bounded rational link formation process which requires only local information of the network the firm is embedded in.\(^2\)

We note here that the centrality based link formation process we have described above (and which will be introduced more formally in Section 6.1) can be applied in the same manner to other prominent models in the literature on R&D networks, in particular to those we have discussed briefly in Chapter 3.7 where equilibrium efforts and payoffs of the agents in the network are determined by their eigenvector centrality.

There is a second difference of the modelling approach followed in this chapter compared to the model we have analyzed in the previous chapter. It concerns the costs of link formation. Here we assume that there are no constant marginal costs \(c\) associated with establishing a collaboration. Since we neglect marginal costs \(c\), the collaboration with the firm having the lowest knowledge level yields lowest profits.\(^3\)

Links can also be removed in the course of the network evolution. We assume that firms live in a volatile environment that renders some of their collaborations unprofitable due to stochastic shifts in external conditions. We assume that firms value links to firms with higher knowledge levels more than to firms with lower knowledge levels. The less valued links are then more likely to be interrupted by stochastic events than the more valued ones. We incorporate this idea by exposing the least preferred collaborations, that are, the links to those firms with the lowest level of knowledge, to a stochastic link decay.

In general, there are several situations in which links can decay. For example, a link may decay when a connection is interrupted due to errors, when the conditions under which the link was initially formed are no longer effective or when the opportunity cost for maintaining the link in terms of time or other resources may have increased and thereby making the collaboration unprofitable. Moreover, volatility expresses the fact that there

\(^2\)A similar approach to local link formation in which agents form links to their neighbors’ neighbors has been developed by Jackson and Rogers (2007). However, in their model links are formed at random whereas in our model links are formed as a mutual best response of both firms involved.

\(^3\)This property follows directly from the monotonicity of the eigenvalue (see Lemma (2)) that determines marginal revenues and profits of the firm when marginal costs are neglected.
are constraints on the number of links that a firm can maintain.

Most models in the literature on network formation have paid little attention to the environment firms are embedded in or they have assumed that environmental conditions remain sufficiently static such that one can simply neglect them. In this model we do not consider a stationary economic environment, but we assume that firms are embedded in a volatile environment that requires them to continually adapt to changing conditions (see also Vega-Redondo, 2007, for a general discussion). From this point of view, the network evolution can be regarded as the interplay between two competing forces: Search for central partners with high stocks of knowledge and stochastic link decay in a volatile environment.

Similar approaches to network evolution have been adopted by Ehrhardt et al. (2008); Marsili et al. (2004); Vega-Redondo (2006). However, differently to these models, we assume that link decay is not independently and identically distributed over all links but depends on the knowledge and centrality of the firm incident to the link. Moreover, the authors study quasi-random networks in which no correlation between the agents and their neighbors exist. This is not a realistic assumption as social and economic networks tend to be correlated. For example Newman (2002, 2003b) finds strong degree correlations, Goyal et al. (2006) find clustering degree correlations and Gay and Dousset (2005) observe betweenness centrality degree correlations. In this model we find strong correlations between various network statistics and we will discuss this issue further in Section 6.4.1.

This chapter is organized as follows. In Section 6.1 we introduce the network formation process and discuss its basic properties. A general discussion of the networks encountered during the evolution in terms of their topology and centralization is given in Section 2.2 and Section 2.3, respectively. Section 6.4 shows that stationary networks exist and can be computed analytically. After deriving the stationary networks we analyze their properties, in terms of topology and centralization (Sections 6.4.1 and 6.4.2). In Section 6.5 we study their efficiency from the point of view of maximizing aggregate profits. We investigate the efficiency of different stationary networks as a function of the volatility of the environment. We proceed in Section 6.6 by including capacity constraints in the number of links that a firm can maintain and combine it with a global search mechanism for new collaboration partners.

6.1 Network Formation

In the following we introduce a network formation process that incorporates the idea that firms with high centrality are more likely to connect to each other (Gulati and Gargiulo,
and that the presence of common neighbors enhances the likelihood of firms to form a new link between them (Gulati, 1995).

Let time be measured at countable dates \( t \in \{1, 2, 3, \ldots\} \). Consider a network \( G(t) \) with firms \( N = \{1, \ldots, n\} \). Let \( N_i = \{k \in N : ik \in G(t)\} \) be the set of neighbors of firm \( i \in N \). Let \( N_i^{(2)} = \bigcup_{j \in N_i} N_j \setminus \{i \cup N_i\} \) denote the second-order neighbors of firm \( i \). We assume that firms have a preference to form links with other firms with the highest knowledge levels among the neighbors of their neighbors. At every time \( t \), a firm \( i \), selected uniformly at random from the set \( N \), enjoys an updating opportunity of her current links with probability \( p_i \in (0, 1) \). If a firm receives such an opportunity, it initiates a link to the firm \( j \) with the highest knowledge in her second-order neighborhood \( N_i^{(2)} \). Firm \( j \) is said to be the local best response of firm \( i \) given the network \( G(t) \). Firm \( j \) accepts the link if \( i \) also has the highest knowledge in her second-order neighborhood\(^4\). That is, firm \( i \) is also a local best response of firm \( j \). Note, that the connectivity relation is symmetric such that \( j \) is a second-order neighbor of \( i \) if \( i \) is a second order neighbor of \( j \). Moreover, as we will show in Section 6.2, in this network formation process, firm \( i \) is always a local best response of firm \( j \) if firm \( j \) is a local best response of firm \( i \).

In the following we give a formal definition of the local best responses of a firm given the prevailing network \( G(t) \).

**Definition 8** Consider a network \( G(t) = (N, L(t)) \) with firms \( N = \{1, \ldots, n\} \) and a set of links \( L(t) \). Let \( G(t) + ij \) be the graph obtained from \( G(t) \) by the addition of the link \( ij \notin L(t) \) between firms \( i \in N \) and \( j \in N \). Further, let \( v = (v_1, \ldots, v_n) \) denote the profile of asymptotic knowledge shares of the firms in \( G(t) \) given by Equation (3.7) in Proposition (1). Then firm \( j \) is a local best response of firm \( i \) if \( v_j \geq v_k \) for all \( j, k \in N_i^{(2)} \). Firm \( j \) may not be unique. The set of firm \( i \)'s local best responses is denoted by \( BR_i(G(t)) \). If firm \( i \) does not have any second-order neighbors, \( N_i^{(2)} = \emptyset \), then firm \( j \) is a local best response of firm \( i \) if \( v_j \geq v_k \) for all \( j, k \in N \setminus \{i \cup N_i\} \).

Note that the local best response strategies in the network games studied by Bala and Goyal (2000); Haller et al. (2007); Haller and Sarangi (2005) allow an agent to remove or create an arbitrary number of links. In contrast we restrict link formation (the strategy space) of a firm to one additional link only. We can view the link formation process as an evolutionary process of continuous and asynchronous adaptation of firms to the current network. The restriction in the number of links that a firm can form at a time simply means

\(^4\)Here (and also in the link formation process described in the previous chapter) we follow the “Consent Game” introduced by Myerson (1991, p. 448) in the sense that a link \( ij \) is formed only if both \( i \) and \( j \) want to form the link.
that the establishment of a new collaboration takes time and, after a new collaboration has been established, another firm is already adjusting its links to the network.

Moreover, here we can omit the removal of links. For vanishing marginal costs $c = 0$ the profit function given by Equation (3.25) is monotonically increasing with the number of links, that is, we always have that $\pi_i(G(t) + ij) > \pi_i(G(t))$ and $\pi_j(G(t) + ij) > \pi_j(G(t))$. On the other hand, the removal of a link would always decrease a firm’s profits. Firms would therefore always prefer the creation of a new link over the removal of an existing one.\(^5\) Therefore, link removal is strictly dominated by link creation. However, as we will see later, links can be removed as the result of a stochastic link decay.

As we have discussed in the previous section, the network of firms is exposed to a volatile environment which induces links to decay. With probability $q_i \in (0, 1)$, we assume that a link of a randomly selected firm $i$ decays. However, not all links decay with the same probability. We assume that the link to the firm $j$ with the lowest knowledge among the neighbors $N_i$ of firm $i$ is the least important collaboration to firm $i$. We assume that the link to the least important partner of firm $i$ decays before any other link is affected by the decay.

We assume that during the time interval from $t$ to $t + 1$ a firm $i$ is selected and either link creation (with probability $p_i$) or link decay (with probability $q_i$) occur. Note that taking into account the possibility of a firm remaining quiescent only modifies the time-scale of the process discussed, thus yielding identical results to the model proposed. This implies that, without any loss of generality, it is possible to assume $p_i + q_i = 1$. Accordingly, we will use one parameter $p_i$ and $q_i = 1 - p_i$ to denote the rate at which links are formed or decay, respectively.

**Definition 9** We define the network formation process $\Gamma_2(G)$ as a sequence of networks $G(0), G(1), G(2), \ldots$ in which at every step $t = 0, 1, 2, \ldots$, a firm $i$ is uniformly selected at random, $i \sim U\{1, \ldots, n\}$. Let $G(t) + ij$ denote the network obtained from the current network $G(t)$ by adding the link between firm $i$ and firm $j$ and $G(t) - ij$ the network obtained from the removal of the link between firm $i$ and firm $j$, respectively. Moreover, let $v = (v_1, \ldots, v_n)$ denote the profile of asymptotic knowledge shares of the firms in $G(t)$ given by Equation (3.7) in Proposition (1). Then one of the following two events occurs:

(i) With probability $p_i \in (0, 1)$, firm $i$ initiates a link to a local best response firm $j \in BR_i(G(t))$. The link $ij$ is created if $i \in BR_j(G(t))$ is also a local best response of $j$, given the current network $G(t)$. If $BR_i(G(t)) = \emptyset$ or $BR_j(G(t)) = \emptyset$, nothing happens.

\(^5\)Without link decay, firms would add links until reaching the complete graph $K_n$. 

(ii) With probability $q_i = 1 - p_i \in (0, 1)$ the link $ij \in G(t)$ decays, such that is $v_j(G(t)) \leq v_k(G(t))$ for all $j, k \in N_i$. If firm $i$ does not have any links, nothing happens.

The network formation process $\Gamma_1(G)$ introduced in Definition (7) portrays the random encounters of firms and the establishment of a collaboration if it is mutually beneficial. In contrast, the network formation process $\Gamma_2(G)$ in Definition (9) assumes an asynchronous updating of the linking strategies of the firms. Moreover, we note that without link decay, the sequence of local best responses converges to the complete network $K_n$.

If a link is created as a mutual local best response between firms, it maximizes their profits under certain conditions. In particular, we have the following condition on the increase in profits.

**Proposition 20** Consider the profit function in Equation (3.25) with vanishing marginal cost $c = 0$. If $\lambda_{PF}(G) \gg \lambda_2(G)$ and $j \in BR_i(G)$ then $\pi_i(G + ij) \geq \pi_i(G + ik)$ for all $k \in N_i^{(2)}$.

**Proof 21** For vanishing marginal cost $c = 0$ the increase in eigenvalue is equivalent to the increase in profits. If $\lambda_{PF}(G) \gg \lambda_2(G)$ we can apply the first-order approximation on the increase in eigenvalue by the creation of a link from Theorem (2) stated in the Appendix B. Under this approximation, the increase in eigenvalue depends only on the eigenvector components of the connecting firms. Therefore, the link to the firm with the highest eigenvector component maximizes both, the increase in eigenvalue and the increase in profits. $\square$

Networks with skewed degree distribution are known to have a large spectral gap $\lambda_{PF} \gg \lambda_2$ (Dorogovtsev et al., 2003; Farkas et al., 2001; Goh et al., 2001; Mihail and Papadimitriou, 2002). We will show that stationary networks from the above link formation process $\Gamma_2(G)$ (9) exhibit skewed degree distributions for values of $\alpha$ below $1/2$. In this case, the condition $\lambda_{PF} \gg \lambda_2$ is fulfilled. If firm $i$ creates a link to a firm with a high asymptotic knowledge share and eigenvector centrality, respectively, then this link is also maximizing the profits of firm $i$.

### 6.2 Network Formation and Nested Split Graphs

In this section we will show that an essential property of the link formation process $\Gamma_2(G)$ introduced in Definition (9) is that it produces a well defined class of graphs denoted
as nested split graphs. We have introduced nested split graphs already in Definition (2) in Section 4.1. They include many common networks such as the star or the complete network. Moreover, as the name already indicates, they have a nested neighborhood structure. This means that the set of neighbors of each firm is contained in the set of neighbors of each higher degree firm. Nested split graphs have particular topological properties and an associated adjacency matrix with a well defined structure, namely that of a stepwise matrix. In Section 4.1 we have already introduced these types of graphs and in Section 6.3 we will give a definition and further analyze their topological structure.

**Proposition 21** Let $\Gamma_2(G)$ denote the the network formation process introduced in Definition (9). Then the network $G(t)$ generated by $\Gamma_2(G)$ is a nested split graph with a stepwise adjacency matrix $A$.

**Proof 22** We give a proof by induction. The induction basis is trivial. Starting from an empty graph $G(0) = \bar{K}_n$, which has a trivial stepwise adjacency matrix, we can omit the removal of a link. Then we select a firm and connect it to another one. This creates a path of length two whose adjacency matrix is stepwise. Thus $G(1)$ has a stepwise matrix. Next we consider the induction step $G(t)$ to $G(t + 1)$.

We consider the creation of a link $ij$. By the induction hypothesis $G(t)$ has a stepwise adjacency matrix. Suppose firm $j$ is a local best response of firm $i$. Let $G' = G + ij$ be the graph obtained from the current graph $G$ by the addition of a link $ij$. $G'$ consists again of a connected component and possibly isolated firms.

For the local best response $j$ of a randomly selected firm $i$ we have that $j \in BR_i(G(t))$ if and only if $x_j \geq x_k$ for all $ik / \in L(t)$. Moreover, since $G(t)$ is a nested split graph we can apply the following lemma which tells us that $d_i > d_j$ implies $v_i > v_j$.

**Lemma 8** Consider a $n \times n$ stepwise matrix $A$, let $v$ be the eigenvector associated with the largest real eigenvalue of $A$ and let $d_i = \sum_{j=1}^{n} a_{ij}$ for $i = 1, \ldots, n$. If and only if $d_i > d_j$ then $v_i > v_j$.

**Proof 23** A graph having a stepwise adjacency matrix is a nested split graph $G$. A nested split graph has a nested neighborhood structure. The neighborhood $N_j$ of a firm $j$ is contained in the neighborhood $N_i$ of the next higher degree firm $i$ with $|N_i| = d_i > |N_j| = d_j$ with $N_j \subset N_i$. For the eigenvector components we can write for any $1 \leq i \leq n$

$$\sum_{k=1}^{n} a_{ik}v_k = \sum_{k \in N_i} v_k = \lambda_{PF}v_i \tag{6.1}$$

\footnote{This situation has been considered by Grassi et al. (2007, p. 247).}
and similarly for \(1 \leq j \leq n\)

\[
\sum_{k=1}^{n} a_{jk} v_k = \sum_{k \in N_j} v_k = \lambda_{PF} v_j \tag{6.2}
\]

Since \(N_j \subset N_i\) and \(d_j = |N_j| < |N_i| = d_i\) we get

\[
\frac{v_i}{v_j} = \frac{\sum_{k \in N_i} v_k}{\sum_{k \in N_j} v_k} < 1 \tag{6.3}
\]

Conversely, in a nested split graph we must either have \(N_i \subset N_j\) or \(N_j \subset N_i\). Assuming that \(v_i > v_j\) we can conclude from the above equation that \(N_j \subset N_i\) and therefore \(|N_i| = d_i > |N_j| = d_j\). If there are \(l\) distinct degrees in \(G\) then the ordering of degrees \(d_1 > d_2 > \ldots > d_l\) is equivalent to the ordering of the eigenvector components \(v_1 > v_2 > \ldots > v_l\). □

(8) we know that firm \(j\) has the highest eigenvector component as well as the highest degree among the firms not already connected to \(i\). Conversely, from the properties of a stepwise matrix we know that \(i\) has the highest eigenvector component and the highest degree among the firms not already connected to \(j\) as well. Moreover, in a nested split graph with a stepwise matrix, all firms are separated from each other through at most two links.\(^7\) Thus, firms \(i\) and \(j\) are both second-order neighbors. From the properties of a stepwise matrix \(A(G(t))\) (see Definition (1)) we therefore know that the matrix \(A(G(t) + ij)\) is also stepwise.

We give an example in Figure (6.1). Let the firms be numbered by the rows respectively columns of the adjacency matrix \(A\). Two possible positions for the creation of a link from firm 4, either to firm 7 or to firm 10 are indicated with asterisks. Since in a stepwise matrix the firm with the higher degree has a higher eigenvector component, firm 7 with degree \(d_7 = 3\) has a higher eigenvector component than firm 10 with degree \(d_{10} = 1\). From all firms that are not connected to firm 4, that is firms 7, 8, 9 and 10, firm 7 has the highest degree and thus the highest eigenvector component. Finally, note that all firms are at most two links separated from each other.

For the removal of a link a similar argument can be applied. The firm with the smallest eigenvector component also has the smallest degree among the neighbors of a firm. From the properties of the stepwise matrix \(A(G(t))\) it then follows that the matrix \(A(G(t) - ij)\) is stepwise.

\(^7\)From this property it follows that the local best response firm is also a global best response. However, this fact is not imposed ex ante but it follows from the special properties of the network that is generated by the link formation process.
Chapter 6. Local Best Response Link Formation

Figure 6.1: Two possible positions for the creation of a link from firm 4, either to firm 7 or to firm 10, are indicated with asterisks. Firm 7 has degree 3 while firm 10 has degree 1. Since firms with higher degree have higher eigenvector components, firm 7 has a higher eigenvector component than firm 10. 4

Thus, in any step \( t \geq 0 \) in the network formation process \( \Gamma_2(G) \), \( G(t) \) is a nested split graph with an associated stepwise adjacency matrix \( A \). □

From Proposition (21) we can directly obtain a result, which is stated in the next proposition, on the connectivity of the network \( G(t) \).

**Corollary 3** The network \( G(t), t = 0, 1, 2, \ldots \) generated by the network formation process \( \Gamma_2(G) \) from Definition (9) consists of one connected component and possibly isolated firms.

**Proof 24** In a nested split graph the firms with maximum degree are connected among each other and to all lower degree firms which are not isolated. Therefore, the nested split graph consists of a single connected component and possibly isolated firms. □

### 6.3 Properties of Nested Split Graphs

In this section we will discuss the topological properties of nested split graphs that arise from the network formation process \( \Gamma_2(G) \) in more detail. In Section 6.3.1 we derive several network statistics for nested split graphs. In the next Section 6.3.2 we then analyze different measures of centrality in a nested split graph. The important result of Section 6.3.2 is that degree, closeness, and eigenvector centrality have the same ordering in a nested split graph. If the ordering is not strict, this also holds for betweenness centrality.
6.3. Properties of Nested Split Graphs

Figure 6.2: Illustration of the degree partition $z_1, ..., z_k$ for $k = 6$ and the associated entries in the stepwise adjacency matrix. The set $z_6$ contains two firms with degree 2 and the set $z_6$ contains two firms with degree 9.

In the following we consider a nested split graph with distinct positive firm degrees $d_{(1)} > d_{(2)} > ... > d_{(k)}$. Let $z_i$, with $1 \leq i \leq k$, denote the set of firms in the graph $G$ with the $i$-th largest degree, that is $z_i = \{ v \in G \mid d_v = d_{(i)} \}$. The sequence $z_1, ..., z_k$ is called the degree partition of $G$ (Mahadev and Peled, 1995). We will refer to the set $z_i$ and its cardinality with the same expression and only distinguish between the two where it is necessary. An illustration showing the the partition $z_0$ and $z_k$ with the corresponding entries in the stepwise adjacency matrix is shown in Figure (6.2). In the notation of the previous section we have that $V_i = z_{k-i+1}$ where $1 \leq i \leq \frac{k}{2}$. Similarly, $U_i = \sum_{j=1}^{i} z_i$ where $1 \leq i \leq \frac{k}{2}$.

6.3.1 Network Statistics

In this section we will derive the expressions for the degree distribution, the clustering coefficient, neighbor connectivity and the characteristic path length in a connected nested split graph. All measures that we will discuss have been defined in Section 2.2.

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8A disconnected nested split graph consist of a single connected component and isolated firms. The results we give in this section can then be applied to the connected component of the graph.
Degree Distribution

The degree distribution $p(d)$ gives the proportion of firms in the network $G$ with a given degree $d$ (see Section 2.2). Note that all nodes in a set $z_i$, $i = 1, ..., k$ have the same degree. An illustration of a connected nested split graph is shown in Figure (6.3). In a connected

\[ d = \sum_{j=1}^{k} z_j - 1 \]

Figure 6.3: Representation of a connected nested split graph with $k = 6$ sets of firms with distinct degrees. The degrees of the firms in a particular set are shown next to the set.

For a nested split graph the degree of a firm $u$ in the set $z_i$ is given by

\[ d_{u \in z_i} = \begin{cases} 
\sum_{j=1}^{k-i+1} z_j - 1, & \text{if } 1 \leq i \leq \frac{k}{2}, \\
\sum_{j=1}^{k-i+1} z_j, & \text{if } \frac{k}{2} + 1 \leq i \leq k.
\end{cases} \]  

(6.4)

Using the above expression for the degrees we can write

\[ p(d) = \begin{cases} 
\frac{z_i}{n}, & \text{if } \exists 1 \leq i \leq \frac{k}{2} \text{ s.t. } d = \sum_{j=1}^{k-i+1} z_j - 1 \\
0, & \text{or if } \exists \frac{k}{2} + 1 \leq i \leq k \text{ s.t. } d = \sum_{j=1}^{k-i+1} z_j,
\end{cases} \]  

(6.5)

Clustering Coefficient

In the following we consider a connected nested split graph without the isolated firms in the set $z_k$. Note that for all firms in the independent sets, $u \in z_i$ with $i \leq \frac{k}{2} + 1$, the clustering coefficient is one, since the neighbors of a firm $u$ are all connected among each other.
Next, we consider the firms \( u \in z_i \) with \( i \leq \frac{k}{2} \) and degree \( d_u = \sum_{j=1}^{k-i+1} z_j \). The neighbors of firm \( u \) in the cliques are all connected among each other with a total of \( \frac{1}{2} \sum_{j=1}^{k} z_j \left( \sum_{j=1}^{k} z_j - 1 \right) \) links. The neighbors of in the independent sets are not connected. The pairs of neighbors for which one firm is in a clique and one firm is in an independent set contribute \( \sum_{j=i}^{k} z_{k-j+1} \left( \sum_{l=1}^{j} z_l - 1 \right) \) links, excluding the firm in the clique itself (see also Figure (6.3)). Putting the above contributions together, we get that the clustering coefficient is given by

\[
\mathcal{C}(u \in z_i) = \begin{cases} 
1, & \text{if } \frac{k}{2} + 1 \leq i \leq k, \\
\frac{1}{d_u(d_u-1)} \left[ \sum_{j=1}^{k} z_j \left( \sum_{j=1}^{k} z_j - 1 \right) + 2 \sum_{j=i}^{k} z_{k-j+1} \left( \sum_{l=1}^{j} z_l - 1 \right) \right], & \text{if } 1 \leq i \leq \frac{k}{2}.
\end{cases}
\] (6.6)

**Assortativity and Neighbor Connectivity**

the nearest neighbor connectivity for nested split graphs let us consider a firm \( u \in z_i \) with \( i = \frac{k}{2} + 1, \ldots, k \) corresponding to the independent sets. From Equation (6.4) we know that the number of neighbors (degree) of firm \( u \) is given by \( \sum_{j=1}^{k-i+1} z_j \). The neighbors of firm \( u \) are the firms in the cliques with degrees given in Equation (6.4). Thus the number of neighbors of the neighbors of \( u \) in the sets \( z_j \) is \( \sum_{j=1}^{k-i+1} z_j - 1 \). Putting the above results together, we obtain the following expression for the nearest neighbor connectivity of firm \( u \).

\[
d_{nn}(u \in z_i) = \frac{1}{z_j} \sum_{j=1}^{k-i+1} z_j \left( \sum_{l=1}^{k-j+1} z_l - 1 \right),
\] (6.7)

for \( i = \frac{k}{2} + 1, \ldots, n \). Next we consider a firm \( u \) in the set \( z_i \) with \( 1 \leq i \leq \frac{k}{2} \) corresponding to the cliques. The number of neighbors of firm \( u \) is given by its degree from Equation (6.4), that is \( \sum_{j=1}^{k-i+1} z_j - 1 \). The number of neighbors of firm \( v \in z_j \) in the cliques is given by \( \sum_{l=1}^{k-i+1} z_l - 1 \). Since firm \( u \) is connected to all firms in the cliques we can sum over all their neighborhoods. The number of neighbors of firm \( w \in z_s \) in the independent sets is given by \( \sum_{t=1}^{k-s+1} z_t \). Firm \( u \) is connected to all firms in the independent sets up to the set \( z_{k-i+1} \). Thus, the nearest neighbor connectivity of firm \( u \) is given by

\[
d_{nn}(u \in z_i) = \frac{1}{\sum_{j=1}^{k-i+1} z_j - 1} \left[ \frac{1}{z_j} \sum_{j=1}^{k-i+1} z_j \left( \sum_{l=1}^{k-j+1} z_l - 1 \right) + \sum_{s=\frac{k}{2}+1}^{k-s+1} z_s \sum_{t=1}^{z_t} \right],
\] (6.8)

for \( i = 1, \ldots, \frac{k}{2} \). When the nearest neighbor connectivity is a monotonic increasing function of the degree \( d \), then the network is assortative, while, if it is monotonic decreasing with \( d \), it is dissortative. Nested split graphs are dissortative, since for \( d_u \in z_i < d_v \in z_j \) it
follows that $d_{nn}(u) > d_{nn}(v)$. The higher is the degree of a firm in the cliques, the more neighbors it has from the independent sets with low degrees.

**Characteristic Path Length**

characteristic path length for nested split graphs. If we consider a firm $u \in z_i$ with $1 \leq i \leq k$, we get the following expression for the shortest paths

$$d(u, v) = \begin{cases} 
1 & \text{for all } v \in \bigcup_{j=1}^{k-i+1} z_j, \\
2 & \text{for all } v \in \bigcup_{j=k-i+2}^{k} z_j.
\end{cases} \quad (6.9)$$

In order to compute the average path length we have to consider all pairs of firms in the graph and compute the length of the shortest path between them, which is given in Equation (6.9).

We first consider all pairs of firms in the cliques. All the firms are adjacent to each other and thus the shortest path between them has length one. Moreover, there are $\frac{1}{2} \sum_{j=1}^{\frac{k}{2}} z_j \left( \sum_{j=1}^{\frac{k}{2}} z_j - 1 \right)$ pairs of firms in the cliques.

Next we consider all pairs of firms in the cliques. From Equation (6.9) we know that all of them are two links away from each other. And there are $\frac{1}{2} \sum_{j=\frac{k}{2}+1}^{k} z_j \left( \sum_{j=\frac{k}{2}+1}^{k} z_j - 1 \right)$ pairs of firms in which both firms stem from an independent set.

Now we consider the pairs of firms in which one firm is in the independent set $z_k$. Then there are $z_k z_1$ pairs of firms with shortest path 1 and $z_k \left( \sum_{j=2}^{k} z_j - 1 \right)$ pairs of firms with shortest path 2. Similarly, we can consider the pairs in which one firm is in the set $z_{k-1}$. Then we have $z_{k-1} (z_1 + z_2)$ pairs of firms with shortest path 1 and $z_{k-1} \left( \sum_{j=3}^{k} z_j - 1 \right)$ pairs of firms with shortest path 2. In the same way, all firms in the independent sets can be considered. Finally, if one firm is in the set $z_{\frac{k}{2}+1}$ then we have $z_{\frac{k}{2}+1} \sum_{j=1}^{\frac{k}{2}} z_j$ pairs of firms with distance 1 and $z_{\frac{k}{2}+1} \left( \sum_{j=\frac{k}{2}+1}^{k} z_j - 1 \right)$ pairs with distance 2.

Therefore, the average path length $L$ defined in Equation (2.8) is given by the following relation:

$$n(n - 1) L = \frac{1}{2} \sum_{j=1}^{\frac{k}{2}} z_j \left( \sum_{j=1}^{\frac{k}{2}} z_j - 1 \right) + \\
2 \frac{1}{2} \sum_{j=\frac{k}{2}+1}^{k} z_j \left( \sum_{j=\frac{k}{2}+1}^{k} z_j - 1 \right) + \\
z_k z_1 + 2 z_k \left( \sum_{j=2}^{k} z_j - 1 \right) + \\
z_{k-1} (z_1 + z_2) + 2 z_{k-1} \left( \sum_{j=3}^{k} z_j - 1 \right) + \\
... + \\
z_{\frac{k}{2}+1} \sum_{j=1}^{\frac{k}{2}} z_j + 2 z_{\frac{k}{2}+1} \left( \sum_{j=\frac{k}{2}+1}^{k} z_j - 1 \right). \quad (6.10)$$
Thus we get

\[ n(n - 1)L = \frac{1}{2} \sum_{j=1}^{\frac{n}{2}} z_j \left( \sum_{j=1}^{\frac{n}{2}} z_j - 1 \right) + \sum_{j=\frac{n}{2}+1}^{k} z_j \left( \sum_{j=\frac{n}{2}+1}^{k} z_j - 1 \right) + \sum_{l=\frac{n}{2}+1}^{k} z_l \left[ \sum_{j=1}^{k-l+1} z_j + 2 \left( \sum_{j=k-l+2}^{k} z_j \right) \right]. \]  

(6.11)

Finally we observe, that a connected nested split graph is characterized by short path lengths, which are at most two.

### 6.3.2 Centrality

In this section we study different measures of centrality (Wasserman and Faust, 1994) and compare them with the eigenvector centrality, if the network is a connected nested split graph. In particular, we derive the expressions for degree, closeness and betweenness centrality. All centrality measures considered in this section have been defined in Section 2.3.

**Degree Centrality**

The degree of a firm \( u \) in the set \( z_i \) is given by Equation (6.4). The degree centrality of a firm \( u \in N \) is given by the proportion of firms that are adjacent to \( u \) (Wasserman and Faust, 1994). Thus, we obtain the normalized degree centrality simply by division of Equation (6.4) with the maximum degree \( n - 1 \). This yields

\[ C_d(u \in z_i) = \begin{cases} 
\frac{1}{n-1} \sum_{j=1}^{k-i+1} z_j - 1, & \text{if } 1 \leq i \leq \frac{k}{2}, \\
\frac{1}{n-1} \sum_{j=1}^{k-i+1} z_j, & \text{if } \frac{k}{2} + 1 \leq i \leq k.
\end{cases} \]  

(6.12)

We observe that the degree centrality (as well as the degree) is decreasing with increasing index \( i \) of the set to which the firm belongs.

**Closeness Centrality**

From the shortest paths we can compute the closeness centrality in a nested split graph.

\[ C_c(u \in z_i) = \begin{cases} 
\frac{\sum_{j=1}^{k-i+1} z_j + 2 \sum_{j=k-i+2}^{n-1} z_j - 1}{\sum_{j=1}^{n-1} z_j + 2 \sum_{j=k-i+2}^{n-1} z_j - 2}, & \text{if } 1 \leq i \leq \frac{k}{2}, \\
\frac{\sum_{j=1}^{k-i+1} z_j + 2 \sum_{j=k-i+2}^{n-1} z_j - 2}{\sum_{j=1}^{n-1} z_j + 2 \sum_{j=k-i+2}^{n-1} z_j - 2}, & \text{if } \frac{k}{2} + 1 \leq i \leq k.
\end{cases} \]  

(6.13)
Note that we have subtracted 1 and 2 respectively since the sums in the denominator include the firm \( u \) itself. Note that \( C_c(u \in x_1) = 1 \). We have that closeness centrality is identical for all firms in the same set. Moreover, closeness centrality is decreasing with increasing order of the firm degrees. This means that the smaller is the degree, the smaller is the closeness centrality of the firm.

**Betweenness Centrality**

We first observe that the betweenness centrality for all firms in the independent sets \( z_i \) with \( \frac{k}{2} + 1 \leq i \leq k \) is zero. For the remaining firms, betweenness centrality in nested split graphs has already been computed by Hagberg et al. (2006). Here we report their results adapted to our notation. The betweenness can be computed from \( C_b(u \in z_{k+1}^{k+1}) = 0 \) and the following recursive relation

\[
C_b(u \in z_{i-1}) = \begin{cases} 
C_b(u \in z_{i}) + \frac{\sum_{j=1}^{i} z_j}{\sum_{j=1}^{i} z_j} & \text{if } 1 \leq i \leq \frac{k}{2} \\
0 & \text{if } \frac{k}{2} + 1 \leq i \leq k.
\end{cases}
\]

From the above equation we observe that betweenness centrality is increasing with degree such that the firms in \( z_1 \) have the highest betweenness centrality, the firms in \( z_2 \) the second highest betweenness centrality and so on. Thus, the ordering of betweenness centralities follows the degree ordering for all firms in the cliques while the firms in the independent sets have vanishing betweenness centrality.

**Eigenvector Centrality**

In the case of nested split graphs we can give the following proposition for the eigenvector associated with the largest real eigenvalue. We point out that the proof follows already from a more general result in Grassi et al. (2007).

**Proposition 22** Consider a nested split graph \( G \) and firms \( i, j \in G \) with degree \( d_i, d_j \). Let \( \mathbf{v} \) denote the eigenvector associated with the largest real eigenvalue of the adjacency matrix of \( G \). If and only if \( d_i > d_j \) then \( v_i > v_j \).

**Proof 25** A graph having a stepwise adjacency matrix is a nested split graph \( G \). A nested split graph has a nested neighborhood structure. The neighborhood \( N_j \) of a firm \( j \) is contained in the neighborhood \( N_i \) of the next higher degree firm \( i \) with \( |N_i| = d_i > |N_j| = d_j \).
with $N_j \subset N_i$. For the eigenvector component $i$ we can write
\[
\sum_{k=1}^{n} a_{ik}v_k = \sum_{k \in N_i} v_k = \lambda_{PF}v_i
\] (6.15)
and similarly for $j$
\[
\sum_{k=1}^{n} a_{jk}v_k = \sum_{k \in N_j} v_k = \lambda_{PF}v_j
\] (6.16)
Since $N_j \subset N_i$ and $d_j = |N_j| < |N_i| = d_i$ we get
\[
\frac{v_i}{v_j} = \frac{\sum_{k \in N_i} v_k}{\sum_{k \in N_j} v_k} < 1
\] (6.17)
Conversely, in a nested split graph we must either have $N_i \subset N_j$ or $N_j \subset N_i$. Assuming that $v_i > v_j$ we can conclude from the above equation that $N_j \subset N_i$ and therefore $|N_i| = d_i > |N_j| = d_j$. It follows that, if there are $l$ distinct degrees in $G$ then the ordering of degrees $d_1 > d_2 > ... > d_l$ is equivalent to the ordering of the eigenvector components $v_1 > v_2 > ... > v_l$. □

Finally, we make the following observation on the ordering of different centrality measures in a nested split graph.

**Proposition 23** Consider a nested split graph $G$ with firms $i \in N$. Let $C_d$, $C_c$, $C_b$, $C_v$ denote the degree, closeness, betweenness and eigenvector centrality in $G$, respectively. Then, for any $l, m \in \{d, c, v\}$, $l \neq m$ and $i, j \in N$ we have that
\[
C_l(i) \geq C_l(j) \Leftrightarrow C_m(i) \geq C_m(j), \quad (6.18)
\]
and
\[
C_l(i) \geq C_l(j) \Rightarrow C_b(i) \geq C_b(j). \quad (6.19)
\]

**Proof 26** The proof follows directly from the expressions of the different values of centrality in a nested split graph. Note however, that all firms in the independent sets have vanishing betweenness centrality and thus we have that if two firms have both vanishing betweenness centrality, this may not imply that they also have the same degree, eigenvector or closeness centrality. □
If and only if a firm $i$ has the $k$-th highest degree centrality then $i$ is the firm with the $k$-th highest closeness or eigenvector centrality. Moreover, if a firm $i$ has the $k$-th highest degree centrality (this also holds for closeness and eigenvector centrality respectively) then it also has the $k$-th highest betweenness centrality. The orderings of degree, closeness and eigenvector centralities coincide and this also applies in a weak sense to betweenness centrality. The above proposition has an important implication.

**Corollary 4** If firms are forming links to the firm with the highest degree centrality, they form a link to the firm having the highest centrality in all other centrality measures considered. The network formation process $\Gamma_2(G)$ defined in (9) thus is independent of any particular centrality measure.

### 6.4 Stationary Networks

In this section we will show that the network formation process $\Gamma_2(G)$ converges to a stationary network $\bar{G}$. In the following Section 6.4.1 we analyze the topological properties of the stationary network. In Section 6.4.2 we study its degree of centralization.

In the following we make a simplifying assumption on the probability $p_i$ with which links are created and the probability $q_i = 1 - p_i$ with which they decay for a randomly selected firm $i \in N$. We assume that these probabilities are homogeneously distributed over firms. This means that we assume $p_i = \alpha = \text{const.}$ and $q_i = 1 - \alpha$ respectively for all $i \in N$. The results obtained in the previous section nonetheless hold for heterogeneous probabilities.

Moreover, let the set $z_i$ contain the firms with degree $i = 0, ..., n-1$ in the network $G$. Note that this is different to the preceding sections where the sets were indexed by the ranking of the degrees while here we index them by the actual values of the degrees.

We first give a definition of a stationary network.\(^9\)

**Definition 10** Let $z(t) = (z_0(t), z_1(t), ..., z_{n-1}(t))$ be the distribution of firms with degree 0,1,...,n−1 in the network $G(t)$ generated by the network formation process $\Gamma_2(G)$ from Definition (9). Consider the expected change $E(z(t+1) - z(t)) = E(\Delta z)$. Then a stationary network $\bar{G}$ is the nested split graph corresponding to an expected distribution $z$ with $E(\Delta z) = 0$.

Next, we make an assumption on the distribution of the number of firms with a certain degree. Let there be $k$ distinct degrees in $\bar{G}$. Then the sets corresponding to the degrees

\(^9\)Note that the a nested split graph is uniquely defined by its degree distribution (up to graph isomorphisms).
larger than $10 \left\lfloor \frac{\Delta}{2} \right\rfloor$ contain only one firm. Later on, we will show that this assumption holds for the stationary network $\bar{G}$ in general.

**Corollary 5** Let $G$ be a nested split graph. If $G$ contains isolated firms then the number $k$ of distinct degrees in $G$ is odd, otherwise, $k$ is even.

**Proof 27** The proof follows from the symmetry of the stepwise adjacency matrix. An illustration is given in Figure (6.4) for a graph with isolated firms. The number $s$ of non-empty sets $z_d$ with firms of degree larger than half of the maximum degree, $\Delta/2 < d \leq n-1$, must be equal to the number of sets with firms of degree smaller than to half of the maximum degree, $0 \leq d < \Delta/2$. The total number of non-empty sets $k$ (or equivalently the number of distinct degrees in $G$) is given by the number of these sets plus the set containing the isolated firms. This means that $k = 2s + 1$. Therefore, $k$ must be odd, if $G$ contains isolated firms and even otherwise. \(\square\)

---

10In the following, $\lceil x \rceil$, $x \in \mathbb{R}$, denotes the smallest integer larger or equal than $x$ (the ceiling of $x$). Similarly, $\lfloor x \rfloor$ is the largest integer smaller or equal than $x$ (the floor of $x$).
Having defined a stationary network in Definition (10) and observed that the number of non-empty sets \( k \) must be odd, we can now derive its explicit form in the following proposition.

**Proposition 24**  The network formation process \( \Gamma_2(G) \) defined in (9) has a stationary network \( \tilde{G} \) defined in (10). Let \( z_d \) denote the set containing the firms with degree \( 0 \leq d \leq n - 1 \) in \( \tilde{G} \) and \( k \) be the number of non-empty sets. Then for the network \( \tilde{G} \) we have that

\[
\begin{align*}
z_0 &= \frac{1 - \alpha}{\alpha} z_1 + \frac{1 - 2\alpha}{\alpha}, \quad (6.20) \\
z_d &= \alpha z_{d+1} + (1 - \alpha) z_{d-1}, \quad 0 \leq d \leq \left\lfloor \frac{k}{2} \right\rfloor, \quad (6.21) \\
z_{\left\lfloor \frac{k}{2} \right\rfloor-1} &= z_{\left\lfloor \frac{k}{2} \right\rfloor}^2 + \frac{1 - 3\alpha}{\alpha} z_{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1 - 2\alpha}{\alpha}, \quad (6.22) \\
z_d &= 1, \quad d \in \{\left\lfloor \frac{k}{2} \right\rfloor + 1 + z_{\left\lfloor \frac{k}{2} \right\rfloor}, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1 + \sum_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor + 1} z_{\left\lfloor \frac{k}{2} \right\rfloor - j\}.
\end{align*}
\]

**Proof 28**  Let there be \( k \) distinct degrees in \( \tilde{G} \). In this proof we assume that if a firm has a degree larger than \( \left\lfloor \frac{k}{2} \right\rfloor \) then there is no other firm with the same degree. However, the results we obtain in this proof are also valid in general. In the following we consider four different regions of the stepwise adjacency matrix and analyze the expected change of the size of a particular set \( z_d \) of firms with degree \( 0 \leq d \leq n - 1 \).

(i) We first consider the set \( z_{\left\lfloor \frac{k}{2} \right\rfloor} \) of firms with degree \( \left\lfloor \frac{k}{2} \right\rfloor \). If we take a look at the stepwise adjacency matrix and the position of the firms in this matrix, we find that they are located exactly at the intersection of the stepfunction with the diagonal of the matrix. An illustration of this is shown in Figure (6.5).

We can compute the expected change in the number of firms with degree \( \left\lfloor \frac{k}{2} \right\rfloor \).

First, we consider the creation of a link. With probability \( \alpha \) a firm in the set with firm degrees \( \left\lfloor \frac{k}{2} \right\rfloor + z_{\left\lfloor \frac{k}{2} \right\rfloor} \) is chosen to create a link. By assumption, this set contains only one firm and so it contributes \( \frac{\alpha}{n} \) in expectation. Next, a firm from the set \( z_{\left\lfloor \frac{k}{2} \right\rfloor} \) can be selected to create a link with probability \( \alpha \). In this way a new set at the intersection with the diagonal is created containing two firms. The expected contribution to the change of the size of the set at the diagonal is \( \alpha (2 - z_{\left\lfloor \frac{k}{2} \right\rfloor}) \frac{1}{n} z_{\left\lfloor \frac{k}{2} \right\rfloor} \). Next, a firm from the set \( z_{\left\lfloor \frac{k}{2} \right\rfloor-1} \) can be selected for the creation of a link. These firms have a degree one less than the firms in the set \( z_{\left\lfloor \frac{k}{2} \right\rfloor} \). Thus, the number of firms in the set \( z_{\left\lfloor \frac{k}{2} \right\rfloor} \) increases by \( \frac{\alpha}{n} z_{\left\lfloor \frac{k}{2} \right\rfloor-1} \).
Next, we consider the removal of a link. With probability $1 - \alpha$ a firm in the set $z_{\lfloor k/2 \rfloor}$ removes a link. Thus, the expected decrease in the size of the set $z_{\lfloor k/2 \rfloor}$ is $-\frac{1-\alpha}{n} z_{\lfloor k/2 \rfloor}$. Moreover, a firm in the set with firm degrees $\lfloor \frac{k}{2} \rfloor + z_{\lfloor k/2 \rfloor}$ and size one removes a link and the expected change is $-\frac{1-\alpha}{n}$.

Putting the above contributions for the creation and removal of a link together we obtain for the expected change in the number of firms with degree $\lfloor k/2 \rfloor$

$$E(\Delta z_{\lfloor k/2 \rfloor}) = \alpha \left( \frac{1}{n} + (2 - z_{\lfloor k/2 \rfloor}) \frac{1}{n} z_{\lfloor k/2 \rfloor} + \frac{1}{n} z_{\lfloor k/2 \rfloor} - 1 \right) - (1 - \alpha) \left( \frac{1}{n} z_{\lfloor k/2 \rfloor} + \frac{1}{n} \right)$$

(6.24)

In the stationary network $\bar{G}$ the expected change in the size of the sets is vanishing. Setting $E(\Delta z_{\lfloor k/2 \rfloor}) = 0$ we obtain for $\bar{G}$

$$z_{\lfloor k/2 \rfloor}^2 + \frac{1 - 3\alpha}{\alpha} z_{\lfloor k/2 \rfloor} + \frac{1 - 2\alpha}{\alpha} = z_{\lfloor k/2 \rfloor} - 1$$

(6.25)

Figure 6.5: Representation of a region in the stepwise matrix. The stepfunction separating the zero entries in the adjacency matrix from the one entries is shown with a thick line. We show the region where the stepfunction intersects the diagonal of the matrix.
(ii) We consider the firms in the classes \( z_{\lfloor \frac{k}{2} \rfloor - i} \) with degrees between 0 and \( \lfloor \frac{k}{2} \rfloor \), that is \( 0 \leq i \leq \lfloor \frac{k}{2} \rfloor \). All the relevant sets of firms with degrees larger than \( \lfloor \frac{k}{2} \rfloor \) have size one. We give an illustration of this situation in Figure (6.6).

Let us investigate the creation of a link. With probability \( \frac{\alpha}{n} \) a link is created from a higher degree firm to a firm in the set \( z_{\lfloor \frac{k}{2} \rfloor - i} \) yielding a contribution to the expected change of the size of that class is \( -\frac{\alpha}{n} \). Similarly, if a link is created from a higher degree firm to a firm in the set \( z_{\lfloor \frac{k}{2} \rfloor - (i+1)} \) then the expected change of the size of that class of \( \frac{\alpha}{n} \). If a firm in \( z_{\lfloor \frac{k}{2} \rfloor - i} \) is selected for creation, then we get an expected decrease of \( -\frac{\alpha}{n} z_{\lfloor \frac{k}{2} \rfloor - i} \).

Now we consider the removal of a link. If a link is removed from a higher degree firm to a firm in the set \( z_{\lfloor \frac{k}{2} \rfloor - i} \), the expected change of the size of that class is \( -\frac{1}{n} - \frac{\alpha}{n} \). If a isolated firm removes a link, the expected change in the number of isolated firms is \( -\frac{\alpha}{n} z_{\lfloor \frac{k}{2} \rfloor - (i-1)} \). Moreover, if a firm in the class \( z_{\lfloor \frac{k}{2} \rfloor - (i-1)} \) is selected for removing a link, we get an expected increase of \( \frac{1}{n} - \frac{1}{n} z_{\lfloor \frac{k}{2} \rfloor - (i+1)} \). If a firm in \( z_{\lfloor \frac{k}{2} \rfloor - i} \) is selected for removing a link, we get an expected change of \( -\frac{1}{n} - \frac{\alpha}{n} z_{\lfloor \frac{k}{2} \rfloor - i} \).

Thus, the expected change in the number of firms with degree \( \lfloor \frac{k}{2} \rfloor - i \) is given by

\[
E(\Delta z_{\lfloor \frac{k}{2} \rfloor - i}) = \alpha \left( \frac{1}{n} - \frac{1}{n} + \frac{1}{n} z_{\lfloor \frac{k}{2} \rfloor - (i+1)} - \frac{1}{n} z_{\lfloor \frac{k}{2} \rfloor - i} \right) + (1 - \alpha) \left( -\frac{1}{n} + \frac{1}{n} + \frac{1}{n} z_{\lfloor \frac{k}{2} \rfloor - (i-1)} - \frac{1}{n} z_{\lfloor \frac{k}{2} \rfloor - i} \right).
\]

(6.26)

For \( E(\Delta z_{\lfloor \frac{k}{2} \rfloor - i}) = 0 \) we obtain

\[
z_{\lfloor \frac{k}{2} \rfloor - i} = \alpha z_{\lfloor \frac{k}{2} \rfloor - (i+1)} + (1 - \alpha) z_{\lfloor \frac{k}{2} \rfloor - (i-1)}.
\]

(6.27)

(iii) In the following we consider the firms with degree zero in the set \( z_0 \). In a representation of the stepwise adjacency matrix in Figure (6.7) this is the set of firms where the stepfunction to the right of the diagonal touches the boundary.

The expected change of the set \( z_0 \) due to the creation of a link has the following contributions. The firm with the highest degree can create a link to an isolated firm and thus decrease the number of isolated firms by one. The expected change from this link is \( -\frac{\alpha}{n} \). If a isolated firm creates a link, the expected change in the number of isolated firms is \( -\frac{\alpha}{n} z_{\lfloor \frac{k}{2} \rfloor - i} \).

On the other hand, the removal of links can also affect the size of the set \( z_0 \). If the highest degree firm removes a link, an additional isolated firm is created yielding an expected increase in the size of \( z_0 \) of \( \frac{1}{n} - \frac{\alpha}{n} \). Next, if a firm with degree one removes a
Figure 6.6: Representation of a region in the stepwise matrix. The stepfunction separating the zero entries in the adjacency matrix from the one entries is shown with a thick line. We show the region to the right of the diagonal in the middle between the intersection of the stepfunction with the diagonal and the boundary of the matrix.

link, the number of isolated firms further increases. This gives an expected change of $z_0$ of $(1 - \alpha) \frac{z_1 n}{n}$.

Thus, the expected change in the number of firms with degree 0 is

$$E(\Delta z_0) = -\alpha \left( \frac{1}{n} + \frac{z_0}{n} \right) + (1 - \alpha) \left( \frac{1}{n} + \frac{z_1}{n} \right). \quad (6.28)$$

Setting $E(\Delta z_{\lfloor \frac{k}{2} \rfloor - i}) = 0$ it follows that

$$z_0 = \frac{1 - \alpha}{\alpha} z_1 + \frac{1 - 2\alpha}{\alpha}. \quad (6.29)$$

(iv) We consider the boundary of the adjacency matrix to the left of the diagonal as illustrated in Figure (6.8). Let the ordered set of degrees be \{d_{(1)}, d_{(2)}, \ldots d_{(\lfloor \frac{k}{2} \rfloor + 1)}\}. Then the expected change in the number of firms with degree larger than $\lfloor \frac{k}{2} \rfloor$ vanishes, $E(\Delta z_{d_{(1)}}) = E(\Delta z_{d_{(2)}}) = \ldots = E(\Delta z_{d_{(\lfloor \frac{k}{2} \rfloor + 1)}} = 0.$
Figure 6.7: Representation of a region in the stepwise matrix. The stepfunction separating the zero entries in the adjacency matrix from the one entries is shown with a thick line. We show the region to the right of the diagonal where the stepfunction touches the boundary of the matrix.

This concludes the proof. □

Since Proposition (24) delivers a complete description of the stationary network $\bar{G}$ we can compute its adjacency matrix $A$ for different values of $\alpha$. This is shown in Figure (6.9). We observe the transition from sparse stationary networks containing few hubs and many firms with small degree to a quite homogeneous network with many firms having high degrees. Moreover, this transition is sharp around $\alpha = 1/2$. In Figure (6.10) we show particular networks arising of the network formation process $\Gamma_2(G)$ for the same values of $\alpha$. Again, we can identify the sharp transition from hub-dominated networks to homogeneous, almost complete ones.

From the above equations we can derive the number $k$ of distinct degrees in $\bar{G}$, that is the number of nonempty sets $z_d \neq \emptyset$ with $0 \leq d \leq n - 1$, assuming that $k$ is odd. This implies that we have isolated firms in the stationary network. However, it is straightforward to obtain the result for $k$ even when $\bar{G}$ is connected.

**Proposition 25** Assume that the number $k$ of distinct degrees in the stationary network
6.4. Stationary Networks

Figure 6.8: Representation of a region in the stepwise matrix. The stepfunction separating the zero entries in the adjacency matrix from the one entries is shown with a thick line. We show the region to the left of the diagonal where the stepfunction touches the boundary of the matrix.

$\bar{G}$ is odd. Then

$$k = 2\frac{\ln \left( 1 - \frac{(2\alpha - 1) n}{3\alpha} \right)}{\ln \left( \frac{1-\alpha}{\alpha} \right)} - 1. \quad (6.30)$$

**Proof 29** In the following we will compute an expression for the number $k$ of distinct degrees in the stationary network $\bar{G}$, assuming that $k$ is odd. From Equations (6.21) and (6.20) we get

$$z_{k-1}^{\frac{k-1}{2}} = 1 - \frac{\alpha}{\alpha} z_{k-1}^{\frac{k-1}{2}} - (i-1) + \frac{1-2\alpha}{\alpha}, \quad (6.31)$$

for all $0 \leq i \leq \frac{k-1}{2}$. Further, combining Equations (6.31) and (6.22) we obtain

$$z_{k-1}^{\frac{k-1}{2}} = 2. \quad (6.32)$$

We note however, that for $\alpha = 1/2$ we find in numerical simulations that this overestimates the sizes of the sets where we find that $z_{k-1}^{\frac{k-1}{2}} = 1$. The solution of the recurrence equation in (6.31) together with $z_{k-1}^{\frac{k-1}{2}} = 2$ gives

$$z_{k-1}^{\frac{k-1}{2}} = 3 \left( \frac{1-\alpha}{\alpha} \right)^i - 1, \quad (6.33)$$
where \(0 \leq i \leq \frac{k-1}{2}\). Moreover, we have that the sum of the sizes of the sets must be equal to the total number of firms in the network. This can be written as

\[
\sum_{d=0}^{n-1} z_d = n = \frac{k-1}{2} + \sum_{d=0}^{\frac{k-1}{2}} z_{\frac{k-1}{2} - d}.
\]  

(6.34)

Using Equation (6.34) together with Equation (6.33) we can write

\[
n = \frac{k-1}{2} + 3 \sum_{d=0}^{\frac{k-1}{2}} \left( \frac{1-\alpha}{\alpha} \right)^d - \frac{k-1}{2}
\]

\[
= 3 \frac{1 - \left( \frac{1-\alpha}{\alpha} \right)}{1 - \left( \frac{1-\alpha}{\alpha} \right)^{\frac{k-1}{2}}} 
\]

(6.35)

and

\[
k + \frac{1}{2} \ln \left( \frac{1-\alpha}{\alpha} \right) = \ln \left( 1 - \frac{(2\alpha - 1)n}{3\alpha} \right).
\]  

(6.36)

Thus, the number \(k\) of distinct degrees in the stationary network \(\bar{G}\) is given by

\[
k = 2 \frac{\ln \left( 1 - \frac{(2\alpha - 1)n}{3\alpha} \right)}{\ln \left( \frac{1-\alpha}{\alpha} \right)} - 1.
\]  

(6.37)

For \(\alpha = 1/2\) we get \(k = \frac{n-3}{3}\). This underestimates the number of non-empty sets that we obtain by numerical simulations where we find that \(k = n\) for \(\alpha = 1/2\). For \(\alpha = 1/4\) and \(n = 39\) we get \(k = 5\) and for \(\alpha = 1/4\) and \(n = 9840\) we get \(k = 15\). These numerical examples show that, for small values of \(\alpha\), the size \(n\) of the system has to be rather large in order to have many non-empty sets in the stationary state. Moreover, if there are only a few non-empty sets we observe that the degree distribution is highly skewed and exhibits fat tails. If the links are distributed in such a way then there are many firms having small degrees and only a few firms having a very large degree. □

In Figure (6.11) the number of links \(m\) and the number \(k\) of distinct degrees in \(\bar{G}\) as a function of \(\alpha\) is shown. We see that there is a sharp transition from sparse to dense networks around \(\alpha = 1/2\) while the number of non-empty classes \(k\) reaches a peak at \(\alpha = 1/2\).

The above numerical examples, also depict the fact that a given system size \(n\) and an arbitrarily chosen value for the parameter \(\alpha\), does not necessarily yield a formal solution of the set of Equation (6.33) and Equation (6.34). In order to show that the stationary networks smoothly depend on the mentioned parameters, we included results of the numerical simulation of the model in the respective figures.
6.4. Stationary Networks

Figure 6.9: Adjacency matrices of stationary networks of \( n = 1000 \) firms for different values of parameter \( \alpha: \alpha = 0.4 \) (top-left plot), \( \alpha = 0.42 \) (top-center plot), \( \alpha = 0.48 \) (top-right plot), \( \alpha = 0.495 \) (bottom-left plot), \( \alpha = 0.5 \) (bottom-center plot), and \( \alpha = 0.52 \) (bottom-right plot). The corresponding matrix obtained by means of numerical simulation is represented in a dotted/white scale: the white area represents the zero entries and the dotted area the one entries in the adjacency matrix. The superimposed blue line represents the theoretical curve separating the one entries from the zero entries in the matrix. The matrix top-left for \( \alpha = 0.4 \) is corresponding to an inter-linked star with a few hubs and a large number of low-degree firms while the matrix bottom-right for \( \alpha = 0.52 \) corresponds to an almost complete network. Thus, there is a sharp transition from sparse to densely connected stationary networks around \( \alpha = 0.5 \). Networks of smaller size for the same values of \( \alpha \) can be seen in Figure (6.10).

6.4.1 Stationary Network Statistics

In this section we show that the network formation process \( \Gamma_2(G) \) leads to stationary networks with properties that are shared with the main stylized facts observed in real social and economic networks. These properties can be summarized as follows (Jackson, 2008; Jackson and Rogers, 2007).

(i) The average number of connections between pairs of firms in inter-firm collaboration networks is small (Goyal, 2007). Baker et al. (2008) report on a collaboration network in the biotech sector where about 80% of the firms were at a distance of two links
Figure 6.10: Sample networks of $n = 50$ firms for different values of parameter $\alpha$: $\alpha = 0.4$ (top-left plot), $\alpha = 0.42$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.495$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). Nodes with brighter colors have a higher eigenvector centrality. The networks for small values of $\alpha$ are characterized by the presence of a hub and a growing cluster attached to the hub with a number of low-degree firms attached to this cluster via the hub. This shows the typical core-periphery structure of a nested split graph. With increasing values of $\alpha$ the density of the network increases until the network becomes almost complete for high $\alpha$. The network plots have been generated using a Fruchterman-Reingold algorithm (Fruchterman and Reingold, 1991).

from the core group containing less that 1% of all firms. On average, the distance between firms was 4.

(ii) Interfirm collaboration networks exhibit high clustering (Cowan and Jonard, 2004; Gay and Dousset, 2005; Powell et al., 2005). This means that the neighbors of a firm are likely to be connected. Networks that are characterized by high clustering and short average path length are referred to “small worlds” (Watts and Strogatz, 1998).

(iii) The distribution of degrees is highly heterogeneous. Baker et al. (2008) find that in a
6.4. Stationary Networks

Figure 6.11: On the left hand side we show the number of links $m$ of the stationary network $\bar{G}$. The number $k$ of distinct degrees from Equation (6.30) found in the stationary network $\bar{G}$ for different values of $\alpha$ are shown. We show the results obtained by recourse of numerical results (symbols) and respecting theoretical predictions (lines) of the model. The figure illustrates the sharp transition in network density at $\alpha = 1/2$.

sample of an R&D collaboration network from the biotech sector, less than 1% of the firms where involved in 32% of all the collaborations. While some authors (Powell et al., 2005) find power law degree distributions, others (Ricaboni and Panmolli, 2002) find exponential degree distributions in social networks.

(iv) Goyal et al. (2006) have found that there exists an inverse relationship between the clustering coefficient of a firm and her degree. The neighbors of a high degree firm are less likely to be connected among each other than the neighbors of a firm with low degree. This means that social networks are characterized by a negative clustering-degree correlation.

(v) In a recent study by Gay and Dousset (2005) the authors show that there exists a correlation between betweenness centrality and the degree of a firm. This correlation between degree and betweenness centrality is characteristic for nested split graphs (see Proposition (23)). Since our network formation process generates nested split graphs we reproduce this empirical finding in our model. Accordingly, Figure (6.12) shows betweenness centrality degree correlation.

(vi) Networks in economic and social contexts often exhibit degree-degree correlations. Newman (2002, 2003b) has shown that many social networks tend to be positively correlated, i.e. the network is said to be assortative, while technological networks like the internet (Pastor-Satorras et al., 2001) display negative correlations, i.e. the network is said to be dissortative. Others, however, find also negative correlations in
social networks such as in the Ham radio network of interactions between amateur radio operators (Jackson and Rogers, 2007) or the affiliation network in a Karate club (Zachary, 1977). Networks in economic contexts may have features of both technological and social relationships (Jackson, 2008). Therefore there exist examples with positive degree correlations, such as in the network between venture capitalists (Mas et al., 2007), as well as negative degree correlations, such as in the world trade web (Serrano and Boguñá, 2003) and the network of banks (De Masi and Gallegati, 2007) exist.

(vii) Goyal (2007) points out that R&D collaboration networks exhibit a core-periphery structure. The author reports on a data set of contracts between pharmaceutical and biotechnology firms (see Baker et al., 2008). The 12 most linked firms were densely connected among each other forming the core of the R&D network while these firms also had a large number of links to over 1300 peripheral firms which had themselves only a few links. Since nested split graphs are characterized by such
a core-periphery structure where the core of firms is comprised of the firms in the cliques and the periphery of the firms in the independent sets (see Figure (4.1)) our stationary networks agree with this empirical observation.

In the following sections we more explicitly show that our model reproduces all the above mentioned properties. More precisely, we show that the stationary networks emerging in our link formation process are characterized by short path lengths, high clustering, exponential degree distributions with power law tail, correlations between different centrality measures, a distinctive core-periphery structure (both characteristic for nested split graphs) and negative degree-clustering correlation. Moreover, we show that, if firms can form any number of links (they face no “capacity constraints”) then stationary networks are dissortative. However, if one takes into account capacity constraints (in the number of links a firm can maintain), and allows for random global attachment between firms, we keep all the above mentioned network statistics, while, at the same time, yielding assortative stationary networks. This example shows that capacity constraints can be the cause of assortativity.

Degree Distribution

Figure (6.13) shows the degree distribution for different values of the linking probability $\alpha$. It follows an exponential decay for lower values of $\alpha$ with a power-law tail for values of $\alpha$ approaching $1/2$. Similar degree distributions have been found in empirical studies of organizational networks, e.g. in a recent study by Guimera et al. (2006) and the empirical analysis of biotech collaboration networks by Riccaboni and Pammolli (2002). For $\alpha = 1/2$ the degree distribution is uniform while for larger values of $\alpha$ most of the firms have a degree close to the maximum degree.

Clustering

The clustering coefficient for stationary networks is shown in Figure (6.14). We find that for almost all values of $\alpha$, clustering in the stationary networks is high. This finding is in agreement with the empirical literature on interfirm networks.

Moreover, Goyal et al. (2006) have shown that there exists a negative correlation between the clustering coefficient of a firm and its degree. We find this property in the stationary networks as well, as shown in Figure (6.15). The figure also shows that for $\alpha$ approaching $1/2$ from below the clustering-degree correlation decreases and vanishes for $\alpha = 1/2$. 
Figure 6.13: Degree distribution for different values of parameter $\alpha$ and a network size $N = 1000$: $\alpha = 0.4$ (top-left plot), $\alpha = 0.42$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.49$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). The solid line represents the results obtained by numerical simulations while the dashed line shows the theoretical prediction. The theoretical prediction fails to capture the tail of the distribution. However, the mass in the tails is very low (caused by rare events) and the largest part of the numerical distribution is correctly described by the theoretical distribution. We find that the degree distribution is exponential for values of $\alpha$ below 0.5 and it is characterized by a power law tail for $\alpha$ approaching 0.5 from below. For $\alpha = 0.5$ the degree distribution is uniform and for larger values of $\alpha$ most of the firms have high degrees while only few have low degrees.

**Assortativity and Neighbor Connectivity**

We now turn to the study of correlations between the degrees of firms. This property is usually measured by the network assortativity (Newman, 2002, 2003b) and the nearest neighbor connectivity (Pastor-Satorras et al., 2001). In our basic model without capacity constraints (see Section 6.6 for an extension including capacity constraints in the number of links a firm can maintain) we observe a negative degree correlation implying that stationary networks are dissortative.

The assortativity and nearest neighbor connectivity for different values of $\alpha$ is shown in Figure (6.17). Clearly, stationary networks are dissortative while the degree of dissortativity decreases with increasing $\alpha$. However, if we recall the core-periphery structure that is
Figure 6.14: We show the clustering coefficient obtained by recourse of numerical simulations (symbols) and respecting theoretical predictions (lines) of the model. Each curve corresponds to a different system size ranging from $n = 100$ and $n = 1000$ to $n = 5000$. For the total clustering coefficient we consider only those firms that are not isolated.

Figure 6.15: The figure shows the clustering as a function of the degree of the firms in the network. We can see that stationary networks exhibit a negative clustering-degree correlation for $\alpha = 0.4$ (top-left plot), $\alpha = 0.42$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.49$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). The black curve corresponds to the results obtained by numerical simulations and the red curve to the prediction of the theoretical model.
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characteristic for a nested split graph, we can see that high degree firms forming the core of the network are connected among each other while it is only the low degree firms in the periphery of the network that are not connected among each other. In this sense firms with high degrees tend to be connected to other firms with high degree. Considering only the subset of firms with high degrees, we would call the network assortative. However, the firms with low degrees, that are only connected to firms with high degrees but are disconnected to firms with low degree, are so numerous in the stationary network (obtained for low values of $\alpha$) that we obtain an overall negative value for the assortativity of the network.

The dissortativity of stationary networks simply reflects the fact that the stationary networks are strongly centralized for values of $\alpha$ below 1/2. By way of illustration how centralization can imply dissortativity, we consider the highly centralized star $K_{1,n-1}$. $K_{1,n-1}$ is completely dissortative with $\gamma = -1$. This follows from the fact that the peripheral firms all have minimum degree one and are only connected to the central firm with maximum degree while the central firm is only connected to the firms with minimum degree. In this sense the dissortativity is simply a measure of centralization in the stationary network. We will further analyze centralization in Section 6.4.2.

Figure 6.16: We show the network assortativity $\gamma$ obtained by recourse of numerical simulations (symbols) and the corresponding theoretical predictions (lines) of the model for networks with $n = 100$, $n = 1000$ and $n = 5000$ firms. The figure shows that stationary networks are dissortative for $\alpha$ less than 0.5.

Characteristics Path Length

We show in Figure (6.18) the characteristic path length $L$ and the network efficiency $E$, defined in Section 2.2. From these figures one can see that the characteristic path length

\[ \alpha \]

\[ N=100 \]

\[ N=1000 \]

\[ N=5000 \]
6.4. Stationary Networks

Figure 6.17: We show the average nearest neighbor connectivity for $\alpha = 0.4$ (top-left plot), $\alpha = 0.42$ (top-center plot), $\alpha = 0.48$ (top-right plot), $\alpha = 0.49$ (bottom-left plot), $\alpha = 0.5$ (bottom-center plot), and $\alpha = 0.52$ (bottom-right plot). The solid lines represent the results of simulations and the dashed lines the corresponding theoretical predictions. From the figure we can infer that stationary networks are dissortative. However, the degree correlation vanishes as $\alpha$ approaches 0.5.

Figure 6.18: On the left hand side the characteristic path length $L$ of the network $\overline{G}$ and on the right the results for the network efficiency $E$ are drawn. We show the measures obtained by recourse of numerical simulations (symbols) and the corresponding theoretical predictions (lines) of the model.
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$L$ never exceeds a distance of two. This means that for all parameter values of $\alpha$, the stationary networks are characterized by short distances between firms. Together with the high clustering shown in this section, stationary networks can therefore be called “small worlds” (Watts and Strogatz, 1998). In terms of short average distance between firms, stationary networks $\tilde{G}$ are efficient for values of $\alpha$ larger than $1/2$, while for values of $\alpha$ smaller than $1/2$ they fail to be efficient. However, the short average distance for larger values of $\alpha$ is attained at the expense of a large number of links with a high network density.

6.4.2 Centrality in Stationary Networks

We have mentioned already the core-periphery structure formed in stationary networks. In these structures, a large part of the link density is centered in the core of the network. In this section we analyze the degree of centralization in stationary networks in more detail. Apart from understanding the stationary network structure it is important to analyze centralization since it indicates the vulnerability and resilience of the network to the failure of individual, central firms. For instance, the most centralized network, the star, is highly vulnerable against failures of the firm in the center. Once the firm in the center is removed, the whole network is split into disconnected firms and total as well as individual profits vanish. As we will show, a sharp transition in the centralization as a function of the link creation probability $\alpha$ exists. In the next Section 6.5 we will also find such a transition in the aggregate profits and welfare in the stationary network.

From Figure (6.19), showing degree, closeness, betweenness and eigenvector centralization, we clearly see that there exists a phase transition at $\alpha = 1/2$ from highly centralized to highly decentralized networks. This means that for low rates of linking opportunities $\alpha$ (and a strong link decay) the stationary network is strongly polarized, composed mainly of a star (or an inter-linked star (Goyal and Joshi, 2003)), while for high rates of linking opportunities $\alpha$ (and a weak link decay) stationary networks are largely homogeneous. We can also see that the transition between these states is sharp.

Our findings are in line with previous works studying the optimal internal communication structure of organizations (e.g. Guimerà et al., 2002). Other works (Calvó-Armengol and Martí, 2007; Dodds et al., 2003; Dupouet and Yildizoglu, 2006; Huberman and Hogg, 1995) have discussed the conditions under which informal organizational networks outperform centralized structures in complex, changing environments. Similar to Ehrhardt et al. (2006) we find sharp transitions between largely homogeneous, decentralized networks to heterogeneous, centralized networks. Stationary networks are strongly centralized for a
6.5 Efficiency and Stationary Networks

We now turn to the investigation of the optimality and efficiency of stationary networks $\bar{G}$. Assuming that marginal costs $c = 0$, we know from total profits in Equation (4.15) in Section 4.1 and the monotonicity of the largest real eigenvalue, stated in Lemma (2), that aggregate profits increase with network density. It follows that the efficient network $G^*$ is the complete graph $K_n$. Comparing stationary networks $\bar{G}$ with the complete graph $K_n$
shows that for values of $\alpha$ below $1/2$ stationary networks are highly inefficient and a sharp transition occurs for increasing values of $\alpha$ around $1/2$. However, we can take the lower density of the stationary networks for values of $\alpha < 1/2$ as an exogenous environmental constraint. In this case we must not compare stationary networks with the complete graph but with (sparser) networks of the same density. In this way we find that networks for lower values of $\alpha$ come close to being efficient, if the network size is small enough. In general, however, we find that for $\alpha$ approaching $1/2$ from below, stationary networks are inefficient. We will discuss these results and the reasons for the possible inefficiency in the following paragraphs.

From Equation (4.15) we know that for marginal cost $c = 0$, the efficient network $G^*$ is given by the graph with the maximal largest real eigenvalue $\lambda_{PF}$. Thus, as a measure of efficiency, we consider the largest real eigenvalue $\lambda_{PF}$ instead of aggregate profits of stationary networks. In Figure (6.20) we compare the largest real eigenvalue of the stationary network $\bar{G}$ relative to the corresponding eigenvalue of the complete graph $K_n$, which is $n - 1$. This ratio measures the relation between the efficient and the stationary network and the possible welfare losses associated with it. One can clearly see from Figure (6.20) that high link decay rates and low linking opportunities (corresponding to low values of $\alpha$) can generate highly inefficient network structures.

However, the assessment of efficiency of stationary networks $\bar{G}$ depends on the restrictions we impose on the efficient network $G^*$, for example, in the maximum number of links. Hence, the associated measurement of efficiency of stationary networks can change significantly. We have seen in Figure (6.11) that, for values of $\alpha$ below $1/2$, stationary networks $\bar{G}$ are sparse and are characterized by a low number of links $m$ and a greater number of isolated firms. In this sense, we can interpret $\alpha$ as a measure of the density of the network. If we treat the low density of the stationary networks for values of $\alpha < 1/2$ as an exogenous environmental constraint, we must compare networks of the same density instead of comparing the eigenvalue of the sparse stationary network $\bar{G}$ to the one of the dense complete graph $K_n$. In this manner, we show in Figure (6.20) the ratio of the largest real eigenvalue of $\bar{G}$ to the graph with the largest real eigenvalue, approximated by $F_{n,d}$ (see see Proposition (8)), with the same density as $\bar{G}$ and number of non-isolated firms. From this figure we can see that also networks for lower values of $\alpha$ come close to being efficient. Only for $\alpha$ approaching $1/2$ from below stationary networks are inefficient. However, this effect becomes stronger, the larger the network is.

The region of inefficiency below $\alpha = 1/2$ can be explained by comparing the structure of the stationary network $\bar{G}$ with the network $F_{n,d}$. $\bar{G}$ consists of a few stars and a large number of isolated firms. However, $F_{n,d}$ consists of a densely connected core of firms and
a single spanning star. The core-periphery structure in the graph $F_{n,d}$ differs from the stationary network $\bar{G}$ in the sense that $F_{n,d}$ contains a larger core and a single hub only. From Figure (6.20) we can therefore conclude that an R&D network with a larger core and a hub connecting the remaining firms to the core generates a higher social welfare than a network with several competing hubs and a large number of poorly connected firms. The competition for becoming a central player in the R&D network can thus be the source of inefficiency.

Figure 6.20: On the left hand side we show the fraction of the largest real eigenvalues of the stationary networks and the complete graph $K_n$ (with an eigenvalue of $n-1$). To the right, the eigenvalue of the stationary network is computed relative to the graph $F_{n,d}$, which is an approximation to the efficient network (see Proposition (8)), for the same number of links. The Figures show results obtained by recourse of numerical results (symbols) and respecting theoretical predictions (lines) of the model for different values of $\alpha$ and number of firms $n = 100$, $n = 1000$ and $n = 5000$.

In the next section we will introduce capacity constraints in the number of links a firm can maintain in the link formation process. We also allow for non-local search for new collaborations. Moreover, we will analyze the efficiency of the networks that arise. Stationary networks in this extended link formation process show the same qualitative properties as before. However, there exists a significant difference in the stationary networks emerging in the extended framework, namely, that these networks can become assortative for certain parameter values.

6.6 Introducing Capacity Constraints and Global Search

In the link formation process introduced in Definition (9), a firm initiates a link to its most preferred partner and the link is accepted only if this potential partner also prefers
the initiating firm over others. An alternative to these dynamics is to allow firms not to accept a link from another firm, if they are involved in too many collaborations already. The underlying assumption is that firms face capacity constraints in the number of links they can maintain. Such constraints can arise from information overload associated with the maintenance of too many collaborations at a time (see also Arenas et al., 2004; Dodds et al., 2003; Fagiolo, 2005; Guimerà et al., 2002; Huberman and Hogg, 1995).

In addition, we extend the range of firms among which a firm is searching for a possible partner. We assume that firms are not only searching for new contacts among their neighbors’ neighbors but also among all firms in the industry. However, in the spirit of the link formation process in the previous sections, firms preferably connect to their neighbors’ neighbors and search for new contacts at random only if this fails. This means that, if capacity constraints prevent a firm from forming a link locally, we assume that the firm tries to link to a firm from the whole population at random. Since we have assumed that the time-invariant marginal costs \(c\) vanish in the profit function given by Equation (3.25) any additional link increases the profits of a firm, including a link that is formed at random. Thus, the random formation of links is compatible with the incentives of firms wanting to increase their profits. In this way we introduce a global search mechanism in the link formation process (see Marsili et al., 2004; Vega-Redondo, 2006, for a similar approach).

In the following we discuss in more detail how capacity constraints enter into the link formation decisions of firms. We assume that capacity constraints arise from the fact that a firm can only interact with one other firm out of her neighborhood at a time. Each neighbor of a firm requests information with probability \(\beta\). Assuming that information requests are independent, the probability that a firm \(i \in N\) with \(d_i\) links does not receive any information requests from her neighbors is given by \((1-\beta)^{d_i}\). If a firm does not receive such an information request, she can accept an additional link otherwise not.

Moreover, we allow for the formation of links between firms that are not connected through a common neighbor. This means that firms search globally for new contacts (see also Vega-Redondo, 2007) if they cannot connect to the firm with highest centrality among their neighbors’ neighbors. When a firm \(i\) is selected, she tries to connect to the firm \(j\) with the highest degree in her neighborhood. However, firm \(j \in N_i\) only accepts the link with probability \((1-\beta)^{d_j}\). If firm \(j\) rejects the link firm \(i\) selects another firm \(k \in N \backslash \{N_i \cup i\}\) at random. This means that firm \(i\) is searching for a partner globally. The link \(ik\) is accepted by firm \(k\) with probability \((1-\beta)^{d_k}\). If this link is rejected then firm \(i\) continues to search for a partner until it finds a firm that accepts the link. Figure (6.21) gives an illustration of this link formation process.
In the link formation process introduced in Definition (9) links were created or removed to firms with the highest or lowest eigenvector centrality. As the ordering of eigenvector and degree centrality coincide in this model (see Proposition (23)), firms could be selected by their degree centrality without changing the network formation process. In this section we introduce a link formation process in which firms are selected on the basis of their degree centrality like in the original model. However, we assume that capacity constraints can inhibit firms from forming an additional link. On the other hand, if capacity constraints become very weak, we recover the original link formation process.

Taking into account the above mentioned capacity constraints in the number of links a firm can form and the possibility to create links outside the second-order neighborhood, we generalize the link formation process $\Gamma^2(G)$ introduced in Section 6.1 as follows.

**Definition 11** We define the network formation process $\Gamma_3(G)$ as a sequence of networks, $G(0), G(1), G(2), \ldots$, in which at every step $t = 0, 1, 2, \ldots$, a firm $i \in N = \{1, 2, \ldots, n\}$ is uniformly selected at random, $i \sim U\{1, \ldots, n\}$. Then one of the following events occurs:

(i) With probability $p_i = \alpha$ firm $i$ receives the opportunity to create an additional link. Let $j$ be the firm in $N_i^{(2)}$ with the highest degree, that is $d_j \geq d_k$ for all $j, k \in N_i^{(2)}$.
   (a) With probability $(1 - \beta)^{d_j}$ the link $ij$ is formed.
   (b) Otherwise firm $i$ connects to a randomly selected firm $k \in N \setminus \{N_i \cup i\}$ with probability $(1 - (1 - \beta)^{d_j})(1 - \beta)^{d_k}$.

(ii) With probability $q_i = 1 - p_i = 1 - \alpha$, the link to the firm $j$ in $N_i$ with the smallest degree $d_j \leq d_k$ for all $j, k \in N_i$, decays. If firm $i$ does not have any links, nothing happens.

We further assume that if a firm is not free to accept an additional link, another firm is selected, until a link is formed. In this way, the values of $\alpha$ in the generalized process $\Gamma_3(G)$ are comparable with the basic process $\Gamma_2(G)$ without capacity constraints where $\alpha$ is a measure of the network density.

We now investigate the properties of stationary networks $\bar{G}$ emerging from the extended network formation process $\Gamma_3(G)$ by means of computer simulations for values of $\alpha \in [0.2, 0.5]$ and $\beta \in [0.01, 1]$. We consider a set of $n = 200$ firms and use a sample of 30 to 40 simulation runs from which we compute the average as an approximation to the stationary network $\bar{G}$.

In Figure (6.22) we show the characteristic path length $L$ and the efficiency $E$ in terms of short connections in the network. The plots indicate that stationary networks in the
Chapter 6. Local Best Response Link Formation

Figure 6.21: Probabilities with which a randomly selected firm $i$ creates a link or one of its links decays, if capacity constraints are taken into account. We have assumed that the firm is neither isolated nor fully connected. The link $ij$ is created locally with probability $(1 - \beta)^{d_j}$. If this fails, the link $ik$ is formed with probability $(1 - (1 - \beta)^{d_j})(1 - \beta)^{d_k}$. If the link $ik$ is not formed either, then firm $i$ continues to search for a new partner (globally) until it finds another firm that accepts the link.

Figure 6.22: On the left hand side the characteristic path length $L$ and on the right the network efficiency $E$ for the stationary network $\tilde{G}$ are shown. These measures for the network’s topology are obtained by recourse of numerical simulations of the extended model with capacity constrains for different values of $\alpha$ and $\beta$ in a system comprised of $n = 200$ firms.

extended model exhibit short path lengths between the firms. However, we find that stationary networks may not simply consist of one connected component and possibly isolated firms but they may consist of multiple components. Moreover, we find a giant component encompassing at least 90% of the firms in all the simulations we studied exists.
6.6. Introducing Capacity Constraints and Global Search

Figure 6.23: On the left hand side we show the clustering coefficient obtained by means of numerical simulations for the extended model with capacity constraints and different values of \( \alpha \) and \( \beta \) in a system with \( n = 200 \) firms. On the right we show the corresponding network assortativity. Each curve corresponds to a different value of \( \alpha \). Only firms that are not isolated are considered.

We now turn to the analysis of the degree correlations in the stationary networks. Here we find a key difference with respect to the results obtained in the previous sections. Figure (6.23) shows clustering and assortativity of stationary networks for different values of \( \alpha \) and \( \beta \). For values of \( \beta \) around 0.1 and \( \alpha \in [0.45, 0.5] \) stationary networks are assortative while displaying a high clustering (albeit lower than in the basic model without capacity constraints). Thus, the introduction of capacity constraints can induce assortative networks.

We can further analyze the degree distribution of stationary networks and we find that it is highly skewed following an exponential function. This means that stationary networks emerging from the link formation process \( \Gamma_3 \) are characterized by a high inequality in the number of collaborations of firms.

The results for different centralization measures show a similar behavior as in Section 6.4.2. There exists a sharp, albeit less pronounced, transition from highly centralized, heterogeneous networks to homogeneous networks at \( \alpha 1/2 \). The similarity in the centralization of stationary networks compared to the results we obtained in the previous sections indicates that stationary networks exhibit a core-periphery structure.

Next we study the efficiency of stationary networks. On the left hand side of Figure (6.24) we show the fraction of the largest real eigenvalue of the stationary network and the complete network. On the right hand side, we show the fraction of the eigenvalues of the stationary network and the graph \( F_{n,d} \), with the same number of links and non-isolated firms as the stationary network. The figure resembles the findings in Section 6.5. For
Figure 6.24: We show the fraction of the largest real eigenvalue of stationary networks and the complete graph, obtained by numerical simulations for the model with capacity constraints for different values of $\alpha$ and $\beta$ in a population of $n = 200$ firms. On the right, we show the fraction of the largest real eigenvalue of stationary networks and the graph $F_{n,d}$.

values of $\alpha < 1/2$ stationary networks are highly inefficient with respect to the complete network while the extent of inefficiency can be drastically reduced if one takes the lower network density associated with a high link decay rate into account. We also find that increasing values of $\beta$ strongly reduce network efficiency.

In this section we have studied different network statistics for different values of $\alpha$ and $\beta$. We find that, by introducing capacity constraints and global search, stationary networks become assortative while they exhibit an exponential degree distribution, high clustering, short average path length and negative clustering-degree correlation. These characteristics can be found in social and economic networks as well. Thus, our model is able to reproduce all characteristics of real world networks, ranging from assortative to dissortative networks.

Our findings have an implication for the distinction between assortative and dissortative networks in the literature. As we have discussed in the preceding sections, our network formation process generates stationary networks that are characterized by negative degree-degree correlation and dissortativity respectively. On the other hand, capacity constraints can introduce assortativity. This effect may shed some light on the origin of the distinction between technological and social networks suggested in Newman (2002, 2003b), where technological networks are characterized by dissortativity and social networks by assortativity. Following our findings, technological networks face capacity constraints to a much lower extent than social networks. Consider for example the internet as a prototype of a technological network and the email network in an organization as a prototype of a social network. The number of hyper-links a website can contain may not be as limited as the number of social contacts (measured e.g. by mutual email exchange) an individual
in an organization can maintain. Thus, the distinction between technological and social networks and the degree of assortativity and degree-degree correlations can be derived from the severity of capacity constraints imposed on the number of links an agent can maintain.\textsuperscript{11}

\textsuperscript{11}Ramasco et al. (2004) have made a similar observation in a collaboration network model where agents were aging and thereby were prevented from participating in an unlimited number of collaborations.
Chapter 7

Conclusion

In this thesis, we have investigated the efficiency and the evolution of networks of knowledge exchange across firms. We show that R&D networks face the risk of getting locked in inefficient network structures. For this purpose we have developed a model in which firms recombine their knowledge stocks with those of other firms in the industry in order to introduce innovations in the market. Since each collaboration is costly, firms face a trade-off between the benefits of new collaborations (in terms of an increase in the expected number of innovations per period) and the cost associated with it. Furthermore, we showed that under mild conditions on the horizon over which the performance of R&D collaborations is evaluated, the benefit the firm receives from the network depends on the growth rate of all walks existing across firms in their connected component. To this end, our model extends other popular models in the network formation literature (cf. the “connections” model in Jackson and Wolinsky, 1996, and the linear “two-way flow” model without decay in Bala and Goyal, 2000).

Within the foregoing framework, we characterized the topology of the efficient graph for any level of marginal cost of collaboration. We showed that, when the marginal cost of maintaining collaborations is low, the efficient network is the complete graph. Thus, when collaboration costs are low, a network of densely connected firms maximizes total profits in the industry. On the other hand, as the marginal cost of collaboration increases it is better for the industry network to display the presence of structural holes. In particular, for intermediate costs of collaboration the efficient graph belongs to the class of nested split graphs, characterized by the presence of a hub linking a clique to a set of disconnected firms. Furthermore, we showed that nested split graphs are characterized by significant cross-firm profit inequality, increasing both in collaboration costs and size of the industry. Finally, we showed that for very large costs of collaboration the empty graph is efficient.

In the following we introduced two different link formation dynamics which represent
alternative ways of how firms choose partners and how the costs of collaboration are taken into account.

In the first link formation dynamics, we studied the existence of equilibrium graphs generated from a two-sided myopic pairwise dynamics (cf. Vega-Redondo, 2007, p. 212), and the relation between equilibrium and efficiency. For this purpose, we employed the notion of “improving path” (cf. Jackson and Watts, 2002), and we assumed that the deletion of existing connections involves a severance cost. In line with the concept of improving path, we identified regions of collaboration and severance costs in which there exist pairwise stable networks and others, where no equilibrium is reached. As far as pairwise stable networks were concerned, we showed that different network structures are stable for the same level of costs. In particular, we identified regions of collaboration and severance costs in which (i) the spanning star (i.e. the star encompassing all firms in the network), and (ii) the class of size-homogeneous disconnected cliques are stable. In turn, the source of multiplicity of equilibria lies in (i) the strong path dependency involved in partner selection, (ii) in the presence of external effects affecting marginal revenue of collaborations for firms belonging to the same connected component and (iii) the inertia arising from the presence of a severance cost associated with link deletion. The presence of multiple stable structures for the same level of collaboration costs implies that, in general, efficient structures are not attained in our model. Furthermore, we identified a region of the size of the industry and of costs in which the efficient graph is never attained.

We then investigated the topological characteristics of pairwise stable graphs in this link formation dynamics, to see whether they are able to replicate the stylized facts on empirically observed R&D networks. To this end, we studied the properties of equilibria via computer simulations. The results of our simulations show the existence of a region of low marginal costs of collaboration and high costs of link deletion in which the aforementioned dynamics are able to select pairwise stable structures matching the stylized facts of R&D networks.

In the second link formation dynamics, we have introduced a local best response network formation process. In this process, link creation and removal is based on the position of the firms in the network measured by their asymptotic share of knowledge in the industry. We show that this share is given by their eigenvector centrality. We do not assume that firms have full knowledge of the underlying network but firms are boundedly rational when taking their linking decisions. This means that links are formed to the neighbors of the neighbors of a firm. Differently to the link formation dynamics in the previous chapter, where we have assumed that the marginal cost of a collaboration is time-invariant, we incorporated the idea that a firm’s connections are exposed to a volatile environment. This
means that links can stochastically decay as uncertain innovations change the profitability of some R&D collaborations in a rapidly changing technological environment (see also Ehrhardt et al., 2008; Marsili et al., 2004; Vega-Redondo, 2006).

We have shown that this type of link formation dynamics is independent of any particular measure of centrality that we have considered. Moreover, it leads to stationary networks in which firms are continually adapting their links to the changing environment. The emerging stationary networks exhibit empirically observed properties of R&D networks. In particular, by introducing capacity constraints and allowing firms to form links globally, we gave an illustration of how the distinction between assortative social networks and dissortative technological networks can be explained, giving an explanation for the distinction introduced by (Newman, 2002, 2003b). Moreover, we found a sharp transition in the network density from highly centralized to decentralized networks. A similar transition can be observed in the efficiency of stationary networks. In particular, we find that inefficiency can arise from multiple highly connected firms competing for centrality while it would be efficient to have a network characterized by a core of densely connected firms and a single hub connecting the firms in the periphery to the core.

We have shown that for all dynamic processes governing the formation of R&D collaborations between firms investigated in this thesis a tension between efficiency and stability exists. Therefore, this thesis shows that the risks inherent in R&D intensive markets are not only present in the uncertain outcomes of firms’ R&D investments but also in the possibility of inefficient network structures emerging from the R&D partner choices of firms.

We have tried to keep the models introduced in the thesis simple to obtain a deeper (analytical) understanding of their properties while basing them on reasonable assumptions on the behavior of firms and their interactions. However, it would be desirable to extend the present work along several dimensions.

First, the model could be extended to account for industry demand, for example like in Goyal and Moraga-Gonzalez (2001). In this way, one could then study how the efficiency and dynamics of network structure may change when firms operate in markets that are interdependent. Second, one could investigate whether the foregoing results about the properties of stable networks are robust to different link updating algorithms. For example, one could study the effect on the network dynamics of introducing firms pursuing different strategies, for instance of the kind explored in Bala and Goyal (2000). Likewise, one could depart from the strong assumptions made on firms’ knowledge of the network and about their ability to forecast the stream of innovations out of a given network of collaborations. Instead one might study the efficiency and emergence of network structures under more
simple rules of firm behavior, for example of the kind suggested in the empirical work by Powell et al. (2005). Third, we have introduced a volatile environment in the local best response dynamics that brings about the stochastic decay of links. There we have assumed that it is always the link to the firm with the lowest share of knowledge (the least important link) that is prone to decay. However, one could also consider some other firm characteristics that have an effect on the stability of their collaborations. For example, firms with many links already should be more experienced with managing a collaboration and thus may have a lower decay rate or a higher one, as they have less vested interest in one particular collaboration. Moreover, the exogenous decay of links could be made endogenous. Fourth, we assumed that the knowledge bases of firms in the industry were sufficiently homogeneous to be transferred across firms. However, the process of knowledge transfer across firms is likely to be shaped by the degree of tacitness, as well as by the existing technological complementarities across sectors and firms’ knowledge bases (Dosi, 1993; Orsenigo et al., 2001). A further analysis of R&D network dynamics and efficiency should therefore embed all the foregoing ingredients related to industry technology, and try to investigate how they may affect the revenues and costs of the process of knowledge recombination. Finally, even though we have tried to reproduce empirically observed patterns of R&D networks, it would be desirable to fit the parameters used in our models to existing data more accurately. This would allow us to get a more comprehensive picture of the goodness of fit of our models to real R&D collaboration networks.
Appendix A

Some Results from Matrix Theory

The main result of this chapter is the Perron-Frobenius theorem for real and non-negative $n \times n$ matrices. In particular, we apply it to the real and non-negative symmetric adjacency matrix $A$. However, before we provide the theorem we introduce some basic characteristics of (adjacency) matrices and the corresponding properties of the underlying graphs.

As noted already in Chapter 2, Section 2.1 a walk in the graph is an alternating sequence of nodes and links. The $k$-th power of the adjacency matrix is related to walks of length $k$ in the graph. In particular, $(A^k)_{ij}$ gives the number of walks of length $k$ from node $i$ to node $j$ (Godsil and Royle, 2001). A connected component of a graph is a maximal subgraph in which there exists a walk from every node to every other node. The graph is connected when the only connected component is the graph itself. If the adjacency matrix can be decomposed in blocks, each block corresponds to a connected component.

An $n \times n$ matrix $A$ is said to be a reducible matrix if and only if for some permutation matrix $P$, the matrix $P^TAP$ is block upper triangular. If a square matrix is not reducible, it is said to be an irreducible matrix. The adjacency matrix of a connected graph is always irreducible (Horn and Johnson, 1990) and in particular it cannot be decomposed in multiple blocks. Irreducible matrices can be primitive or cyclic (imprimitive) (Seneta, 2006). This distinction is important because some result about the convergence of the knowledge values holds only for graphs with a primitive adjacency matrix.

For a primitive, non-negative matrix $A$ it is $A^k > 0$ for some positive integer $k \leq (n-1)n^n$ (Horn and Johnson, 1990). This means that, $A$ is primitive if, for some $k$, there is a walk of length $k$ from every node to every other node. Notice that this definition is a much more restrictive than the one of an irreducible (or connected) graph in which it is required that there exits a walk from every node to every other node, but not necessarily of the same length. A graph is said to be primitive if its associated adjacency matrix is primitive.
In the following, we repeat here the Perron-Frobenius theorem in a formulation convenient to our context (Seneta, 2006).

**Theorem 1 (The Perron-Frobenius Theorem)** Let $A$ be a non-negative matrix. Then (i) the Perron-Frobenius eigenvalue $\lambda_{PF}$ is an eigenvalue of $A$ such that all other eigenvalues are smaller or equal in absolute value; (ii) $\lambda_{PF}$ is associated to one or more non-negative eigenvectors and, (iii) $\lambda_{PF}$ is bounded from below and above as follows: $\min_i \sum_j a_{ij} \leq \lambda_{PF} \leq \max_i \sum_j a_{ij}$.

If, in addition, $A$ is an irreducible matrix, then (iv) $\lambda_{PF}$ has multiplicity one and (v) the associated eigenvector is positive.

If, in addition, $A$ is a primitive matrix, then (vi) $\lambda_{PF}$ is strictly greater in absolute value than all other eigenvalues.

Notice that, going from non-negative to irreducible matrices the eigenspace of $\lambda_{PF}$ reduces from several non-negative eigenvectors to only one positive eigenvector.
Appendix B

Graph Perturbation Theory

The following theorem can be found in Cvetkovic et al. (1995). It relates the change in the largest real eigenvalue \( \lambda_{PF} \) to the eigenvector \( v \) associated with \( \lambda_{PF} \) due to the creation (or removal) of a link in the graph \( G \).

**Theorem 2 (Cvetkovic et al. (1995))** Suppose the network \( G \) has nodes 1,...,\( n \) and adjacency matrix \( A \) with eigenvalues \( \lambda_{PF} > \lambda_2 > ... > \lambda_m \) and spectral decomposition \( A = \lambda_{PF}P_1 + \lambda_2P_2 + ... + \lambda_mP_m \). If \( \{e_1,...,e_n\} \) is a standard basis of \( \mathbb{R}^n \) then define the angles of \( G \) as \( \alpha_{ij} = ||P_ie_j|| \). In particular, if \( x \) is the eigenvector associated with the largest real eigenvalue \( \lambda_{PF} \) of \( G \) then \( x_i = \alpha_{1,i} \) for \( i = 1,...,n \). Let \( u,v \) be non-adjacent nodes of the connected graph \( G \). If \( \lambda_{PF} - \lambda_2 > 2 \left( 1 + (x_u^2 + x_v^2)^{1/4} \right) \), in particular if \( \lambda_{PF} - \lambda_2 > 4 \), then

\[
\lambda_{PF}(G + uv) = \lambda_{PF}(G) + \sum_{r=1}^{\infty} c_r
\]

\[
= \lambda_{PF}(G) + 2x_u x_v + \sum_{i=2}^{m} \frac{x_u^2 \alpha_{iu}^2 + 2x_u x_v \alpha_{iu} + x_v^2 \alpha_{iv}^2}{\lambda_{PF} - \lambda_i} + \sum_{r=3}^{\infty} c_r \quad \text{(B.1)}
\]

where \( c_r \) are recursively defined functions of \( \lambda_{PF}(G), \alpha_{iu}, \alpha_{iv} \) and \( (P_i)_{uv} \) for \( i = 1,...,m \). In particular, if \( \lambda_{PF} \gg \lambda_2 \) then the increase in the largest real eigenvalue due to the addition of an edge \( uv \) to \( G \) is given by the first-order approximation

\[
\lambda_{PF}(G + uv) = \lambda_{PF}(G) + 2x_u x_v \quad \text{(B.2)}
\]

In what follows we compute a lower bound for the difference \( \lambda_{PF} - \lambda_2 \). Consider the real eigenvalues \( \lambda_{PF} = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \) of the adjacency matrix \( A \). We have that
\[ \sum_{j=1}^{n} \lambda_j^2 = \text{tr}(A^2) = 2m \] (Bollobas, 1998). Thus, we get

\[ \lambda_2^2 = 2m - \lambda_{PF}^2 - \sum_{j=3}^{n} \lambda_j^2 \]
\[ \leq 2m - \lambda_{PF}^2 \]
\[ \leq 2m - \left( \frac{2m}{n} \right)^2 \]
\[ = \frac{2m(n^2 - 2m)}{n^2}. \]  

(B.3)

Here we use the fact that \( \lambda_{PF} \geq \frac{2m}{n} \) (Bollobas, 1998). Therefore we get

\[ \lambda_{PF} - \lambda_2 \geq \frac{2m - \sqrt{2m(n^2 - 2m)}}{n}, \]  

(B.4)

which is positive and a monotonic increasing function for \( n^2 / 4 < m \leq n(n - 1)/2 \). We thus have a large spectral gap for dense networks. We note however, that in many cases the largest real eigenvalue is well separated from the second largest real eigenvalue. For example, in a random graph \( G(n, p) \), the largest real eigenvalue grows with \( n \) (keeping \( p \) constant) while the upper bound on the second largest real eigenvalue grows as \( \sqrt{n} \) (Restrepo et al., 2007). Finally, Farkas et al. (2001); Goh et al. (2001); Mihail and Papadimitriou (2002) have shown that the assumption \( \lambda_{PF} \gg \lambda_2 \) is valid for scale-free networks and graphs with highly skewed degree distributions.
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6.22 On the left hand side the characteristic path length $L$ and on the right the
the network efficiency $E$ for the stationary network $\bar{G}$ are shown. These
measures for the network’s topology are obtained by recourse of numerical
simulations of the extended model with capacity constrains for different
values of $\alpha$ and $\beta$ in a system comprised of $n = 200$ firms.

6.23 On the left hand side we show the clustering coefficient obtained by means
of numerical simulations for the extended model with capacity constrains
and different values of $\alpha$ and $\beta$ in a system with $n = 200$ firms. On the right
we show the corresponding network assortativity. Each curve corresponds
to a different value of $\alpha$. Only firms that are not isolated are considered.

6.24 We show the fraction of the largest real eigenvalue of stationary networks
and the complete graph, obtained by numerical simulations for the model
with capacity constraints for different values of $\alpha$ and $\beta$ in a population
of $n = 200$ firms. On the right, we show the fraction of the largest real
eigenvalue of stationary networks and the graph $F_{n,d}$.
Bibliography


Curriculum Vitae

Personal Information

Name Michael David König
Nationality Austrian
Date of Birth 19-th March, 1980
Address Hochstrasse 20, CH-8044 Zurich, Switzerland
Phone +41 44 632 84 18 / +41 77 432 0 623
E-mail mkoenig@ethz.ch

Education

1998 Matura (Austrian GCE A-levels equivalent)
    Bundesgymnasium Feldkirch, Austria
1999 Military Service (compulsory in Austria)
2000 – 2005 Study of Theoretical, Numerical and Mathematical Physics
    Graduate Engineer (Dipl. Ing. of Technical Physics), with honors
    Technical University Vienna, Austria
2005 Master Thesis
    Max-Planck-Institute for Plasmaphysics, Munich, Germany
    Title: Tokamak Edge Gradient Stiffness under Extreme Particles
    Flux
2005 – Present Ph.D. Student
    Chair of Systems Design, ETH Zurich, Switzerland
    Title: Dynamic R&D Networks. The Efficiency and Evolution of
    Interfirm Collaboration Networks.
2008 – Present Study of Economics
    University of Zurich, Switzerland
Publications

See the following page.

Selected Grants

Federal Government Grant
EURATOM Mobility Support
Wilhelm-Else-Heraeus Scholarship

Selected Employments

2004 Research Project: Transport and Dynamics of Burning Plasma
Austrian Research Center Seibersdorf
2004 Algorithms for the Analysis of Partial Discharge Impulses
BAUR Measurement Engineering

Languages

German native speaker
English spoken: excellent, written: excellent
French spoken: good, written: good
Latin written: excellent
List of Publications

The following publications contain parts of the material presented in this thesis.


2. On Algebraic Graph Theory and the Dynamics of Innovation Networks, Michael D. König, Stefano Battiston, Mauro Napoletano and Frank Schweitzer, Networks and Heterogeneous Media, Vol. 3, Num. 2, June 2008


4. The Efficiency and Evolution of R&D Networks, Michael D. König, Stefano Battiston, Mauro Napoletano and Frank Schweitzer, Center of Economic Research at ETH Zurich, Working Paper No. 08/95, 2008
