THE COAREA FORMULA
FOR METRIC SPACE VALUED MAPS

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Abstract

In this thesis we prove the coarea formula for Lipschitz maps defined on $\mathbb{R}^n$ and taking values in $\mathcal{H}^m$-$\sigma$-finite metric spaces. We do this by first defining a coarea factor for almost every point in the domain. This factor measures the local distortion of the respective measures. The statement we prove is a generalization of H. Federer’s coarea formula in [Fed2]. Furthermore, we generalize this result to Lipschitz maps defined on $\mathcal{H}^n$-rectifiable spaces.

Preliminary to the proof of the coarea formula we investigate the metric differential defined by B. Kirchheim in [Kir]. We state and prove an analogue for Stepanov’s theorem applicable for maps which are defined on $\mathbb{R}^n$ and take values in arbitrary metric spaces. With the help of this theorem we can generalize the coarea formula even further.
Zusammenfassung


Bevor wir die Koflächenformel beweisen, untersuchen wir das metrische Differential, welches von B. Kirchheim in [Kir] definiert wurde. Wir formulieren und beweisen ein Analogon für Stepanovs Theorem, welches auf Abbildungen anwendbar ist, die auf $\mathbb{R}^n$ definiert sind und Werte in einem beliebigen metrischen Raum annehmen. Mithilfe dieses Theorems können wir die Koflächenformel noch weiter verallgemeinern.
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für Timon
“Don’t Panic!”

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Chapter 1

Introduction

In the year 1959 H. Federer proves in [Fed1] the coarea formula for Lipschitz maps between $C^1$-manifolds $X$ and $Y$, namely

$$ \int_A J_m f(x) \, d\mathcal{H}^n(x) = \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A) \, d\mathcal{H}^m(y). $$

Here $n$ and $m$ are the dimensions of $X$ and $Y$ respectively, $A$ is an $\mathcal{H}^m$-measurable subset of $X$ and $J_m f(x)$ is a distortion factor he defines at every $x$ where $f$ is differentiable.

Later, in his book about geometric measure theory [Fed2], he elaborately studies the geometric significance of the quantity

$$ \int_A J_m f(x) \, d\mathcal{L}^n(x) $$

for subsets $A \subset \mathbb{R}^n$ for different values of $m$. He generalizes the coarea formula to Lipschitz maps defined on $\mathcal{H}^m$-rectifiable spaces with $\mathcal{H}^m$-rectifiable image spaces [Fed2, 3.2.22]. This coarea formula is an important tool in geometric measure theory. In 1978 M. Ohtsuka enlarged the set of admissible maps to Lipschitz maps defined on $A \subset \mathbb{R}^n$ taking values in $\mathbb{R}^N$ for some $N \in \mathbb{N}$ with $\mathcal{H}^m$-$\sigma$-finite images $f(A)$ [Oht].

In 1994 B. Kirchheim studies rectifiable sets in metric spaces [Kir] and defines the notion of metric differentiability for Lipschitz maps defined on $\mathbb{R}^n$ with arbitrary metric spaces as images. Wherever it exists the metric differential is a seminorm $mdf_x$ on $\mathbb{R}^n$. It is defined as the seminorm $\tau$
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satisfying

$$\lim_{|v|+|w| \to 0} \frac{d(f(x+v), f(x+w)) - \tau(v-w)}{|v| + |w|} = 0.$$  

When \( f \) takes values in a Euclidean space and \( f \) is differentiable in a point \( x \), \( \operatorname{md} f_x \) exists and equals \( |Df_x(\cdot)| \). Kirchheim shows that this metric differential behaves very nicely and he proves an analogue to Rademacher’s theorem, stating that for any Lipschitz map \( \mathbb{R}^n \to X \) to a metric space \( X \), the metric differential is defined almost everywhere with respect to the Lebesgue measure. Furthermore, he defines the area factor \( J(\operatorname{md} f_x) \) for the metric differential if it is a norm, and he proves the following area formula.

$$\int_A J(\operatorname{md} f_x) \, d\mathcal{L}^n(x) = \int_Y \#\{f^{-1}(y) \cap A\} \, d\mathcal{H}^m(y).$$  

Regarding the results of H. Federer and B. Kirchheim, it is natural to ask the following question: Given is a map \( f : \mathbb{R}^n \to Y \) where \( Y \) is a metric space. Can we define a distortion factor in a point \( x \in \mathbb{R}^n \), analogous to the distortion factor \( J_m f(x) \) by H. Federer? And given this distortion factor, for which maps \( f : \mathbb{R}^n \to Y \) can we state a coarea formula? This is the main question we discuss in this thesis.

Our main result is the following theorem, which answers the above question for specific cases.

**Theorem 1.1** (coarea formula). Suppose \( f : \mathbb{R}^n \to Y \) is a Lipschitz map where \( Y \) is an \( \mathcal{H}^m-\sigma \)-finite metric space where \( n \geq m \geq 1 \). If \( A \subset \mathbb{R}^n \) is \( \mathcal{L}^n \)-measurable, then

$$\int_A C_m(\operatorname{md} f_x) \, d\mathcal{L}^n(x) = \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A) \, d\mathcal{H}^m(y)$$  

and, given an \( \mathcal{L}^n \)-integrable function \( g : \mathbb{R}^n \to \mathbb{R} \),

$$\int_A g(x) C_m(\operatorname{md} f_x) \, d\mathcal{L}^n(x) = \int_Y \int_{f^{-1}(y) \cap A} g(x) \, d\mathcal{H}^{n-m} \, d\mathcal{H}^m(y).$$  

The appearing coarea factor \( C_m(\operatorname{md} f_x) \) is a generalization of the area factor for norms by B. Kirchheim. For a seminorm \( \tau \) on \( \mathbb{R}^n \) with rank less than or equal to \( m \), \( C_m(\tau) \) is defined so that the coarea formula holds for every linear map \( f : \mathbb{R}^n \to V \) into an \( m \)-dimensional normed space \( V \) with \( \operatorname{md} f = \tau \). For seminorms with rank higher than \( m \) we do not define the
coarea factor. We will show that in the setting of Theorem 1.1, where the target space is $\mathcal{H}^m$-$\sigma$-finite, $md f_x$ has rank higher than $m$ only on a set of measure zero. Therefore the left hand side of (1.1) is defined. In Section 3.5 we give an explicit definition of the coarea factor.

In Chapter 3 we study the metric differential and provide a proof for the metric version of Rademacher's theorem (Proposition 3.2). It assures that the term $C_m(md f_x)$ is defined almost everywhere and it is therefore crucial for the coarea formula to make sense.

Stepanov’s theorem is a generalization of Rademacher’s theorem. It states that any map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at almost every point in the set

$$\left\{ x \in \mathbb{R}^n : \limsup_{y \to x} \frac{d(f(y), f(x))}{|y - x|} < \infty \right\}.$$  

We prove that Stepanov’s theorem remains true if we consider maps from $\mathbb{R}^n$ to metric spaces and metric differentiability (Proposition 3.24). With this theorem we can extend the coarea formula from Lipschitz maps to a much larger class of maps (Corollary 4.15).

In Chapter 4 we prove the main theorem (Theorem 1.1). We do this by first choosing a $\lambda > 1$. Dependent on this $\lambda$, we construct adequate partitions of the set $A$, using the results obtained in Chapter 3. This allows us to prove the coarea formula on each set of the partition up to a factor of some powers of $\lambda$. From this we can infer the coarea formula on $A$ up to this factor. Since we do this for every $\lambda > 1$, we can conclude the exact formula.

From the calculations performed to prove the coarea formula, we also conclude Theorem 4.16 about rectifiable level sets. It states that in the chosen setting, for almost every $y \in Y$ the preimage $f^{-1}(y)$ is an $\mathcal{H}^{n-m}$-rectifiable set.

In Chapter 5 we generalize the main theorem to Lipschitz maps defined on $\mathcal{H}^n$-rectifiable spaces. The proof mainly follows the one described in Chapter 4. But the distortion by the chart maps of the corresponding rectifiable sets complicates the calculation. We prove the theorem about rectifiable level sets also for Lipschitz maps with an $\mathcal{H}^n$-rectifiable domain (Theorem 5.14).

In the following chapter we provide two examples. The first one illustrates the necessity of the $\mathcal{H}^m$-$\sigma$-finiteness condition and shows, that we cannot relax it to the weaker condition of $m$-Hausdorff-dimensionality. The other one deals with a purely unrectifiable target space.

The last chapter is devoted to the coarea inequality. Besides being necessary to prove the coarea formula, it is also of independent interest.
proved the coarea inequality for Lipschitz maps between metric spaces, fulfilling some additional conditions [Fed2]. There he states that they might be superfluous. Shortly after, R. O. Davies writes in [Dav] the following: “H. Federer tells me that this work answers a question raised in Geometric measure theory, (...) consequently the supplementary conditions (...) are superfluous.”

We provide a proof of the coarea inequality without the extra conditions. It mainly follows Federer’s proof and makes use of Davies’ result. With the use of the coarea formula we investigate the equality cases of the coarea inequality where $f$ is defined on an $\mathcal{H}^n$-rectifiable space and has an $\mathcal{H}^m$-$\sigma$-finite image.

During the development of this thesis, the coarea formula was proven by M. Karmanova for the case where the metric differential of $f$ has rank $m$ for every point $x \in A$ [Kar]. All our results were produced independently of her paper.
Chapter 2

Preliminaries

2.1 Notation

Given a metric space \((X, d)\), we denote the distance between a point \(x \in X\) and a subset \(A \subset X\) by \(d(x, A) := \inf_{a \in A} d(x, a)\). For a point \(x \in X\), \(B_r(x)\) and \(U_r(x)\) denote the closed and the open ball, respectively, with center \(x\) and radius \(r\). Analogously we define \(B_r(A) := \{y \in X : d(y, A) \leq r\}\) and \(U_r(A) := \{y \in X : d(y, A) < r\}\).

For any norm \(\sigma\) on the Euclidean space \(\mathbb{R}^n\), we denote by \(\mathbb{R}^n_\sigma\) the metric space \((\mathbb{R}^n, \sigma)\). Closed balls in \(\mathbb{R}^n_\sigma\) are denoted by \(B_{\sigma,r}(x)\). For a seminorm \(\sigma\) on \(\mathbb{R}^n\) we denote by the kernel of \(\sigma\) the linear subspace
\[
\ker \sigma := \{x \in \mathbb{R}^n : \sigma(x) = 0\}.
\]
The rank of \(\sigma\) is defined by \(\text{rank} \sigma := n - \dim \ker \sigma\).

Suppose \(L : \mathbb{R}^n \to \mathbb{R}^m\) is a linear map and \(\sigma\) is a seminorm on \(\mathbb{R}^m\). Then we denote by \(\sigma L\) the seminorm on \(\mathbb{R}^n\) defined by \(\sigma L(v) = \sigma(Lv)\) for all \(v \in \mathbb{R}^n\).

Given a \(\lambda \geq 0\), a map \(f\) between metric spaces \((X, d_X)\) and \((Y, d_Y)\) is called a \(\lambda\)-Lipschitz map if
\[
d_Y(f(x), f(x')) \leq \lambda d_X(x, x')
\]
for all \(x, x' \in X\). A map which is \(\lambda\)-Lipschitz for some \(\lambda \geq 0\) is called a Lipschitz map. The smallest constant \(\lambda\) satisfying that \(f\) is \(\lambda\)-Lipschitz is called the Lipschitz constant of \(f\) and denoted by \(\text{Lip} f\). Given a \(\lambda \geq 1\), a map
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$f : (X, d_X) \to (Y, d_Y)$ is called $\lambda$-bi-Lipschitz if

$$\frac{1}{\lambda} d_X(x, x') \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x')$$

for all $x, x' \in X$. Bi-Lipschitz maps and the bi-Lipschitz constant of a map $f$, denoted by $\text{BLip}_f$, are defined analogously.

2.2 Hausdorff measure

For $n \in \mathbb{N}$ we denote by

$$\alpha_n := \mathcal{L}^n(B_1(0)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

the Lebesgue measure of the unit ball in $\mathbb{R}^n$, and set $\alpha_0 = 1$. Suppose $X$ is a metric space. For $0 < \delta \leq \infty$ and $A \subset X$ we define the outer measures

$$\mathcal{H}^n_\delta(A) = \inf \sum_{i=1}^\infty \alpha_n\left(\frac{1}{2}\diam C_i\right)^n$$

on $X$ where the infimum is taken over all coverings $\{C_i\}_{i \in \mathbb{N}}$ of $A$ with $\diam C_i \leq \delta$ for all $i$. Then

$$\mathcal{H}^n(A) := \lim_{\delta \searrow 0} \mathcal{H}^n_\delta(A) = \sup_{\delta > 0} \mathcal{H}^n_\delta(A)$$

is the $n$-dimensional Hausdorff measure of the set $A$. Whereas the outer measures $\mathcal{H}^n_\delta$ in general have very few measurable sets, $\mathcal{H}^n$ is a Borel regular measure on every metric space $X$. The Hausdorff measure can be defined for any real $n \in [0, \infty)$. For every metric space $X$ we can define its Hausdorff dimension as

$$\dim_{\mathcal{H}}(X) = \sup\{n \in [0, \infty) : \mathcal{H}^n(X) > 0\} = \inf\{n \in [0, \infty) : \mathcal{H}^n(X) = 0\} = \inf\{n \in [0, \infty) : \mathcal{H}^n(X) < \infty\}.$$ 

In this work, however, we only consider Hausdorff measures with respect to integral $n$.

On the metric space $\mathbb{R}^n$ endowed with the Euclidean norm, $\mathcal{H}^n$ equals the $n$-dimensional Lebesgue measure $\mathcal{L}^n$. In this work we will always use the term
Hausdorff measure to refer to this measure. For every norm $\sigma$ on $\mathbb{R}^n$ we denote by $\mathcal{H}_\sigma^n$ the $n$-dimensional Hausdorff measure on $\mathbb{R}^n_{\sigma}$. The Hausdorff measure on $\mathbb{R}^n$ with respect to the Euclidean metric we denote by $\mathcal{H}_e^n$ or, if there is no reason of ambiguity, simply by $\mathcal{H}^n$.

The Hausdorff measure has some natural properties. We will use them later without reference.

Lemma 2.1.

i) Suppose $X$ and $Y$ are metric spaces, $A \subset X$ and $f : X \to Y$ a $\lambda$-Lipschitz map. Then we have $\mathcal{H}^n(f(A)) \leq \lambda^n \mathcal{H}^n(A)$.

ii) Suppose $\sigma$ is a norm on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ and $L$ is an automorphism on $\mathbb{R}^n$. Then $\mathcal{H}^n_{\sigma L}(A) = \mathcal{H}^n_{\sigma}(LA)$.

iii) Suppose $\sigma$ is a norm on $\mathbb{R}^n$, then $\mathcal{H}^n_{\sigma}(B_{\sigma,1}(0)) = \alpha_n$.

The first two statements follow directly from the construction of the Hausdorff measure. A proof of the third statement can be found in [Kir, Lemma 6].

In Section 4.5, as well as in the proof of the coarea inequality in Section 7.1, the problem arises that we need a functional similar to an integral with respect to the outer measures $\mathcal{H}_\delta^m$ on a metric space $Y$. Since in general these outer measures possess very few measurable sets, we cannot expect the appearing functions to be measurable. In both sections 4.5 and 7.1 the upper integral $\int^*$ is very hard to calculate and hence not suitable either. We will instead use the weighted integral defined as follows:

Definition 2.2 (weighted integral). Let $X$ be an $\mathcal{H}^m$-$\sigma$-finite metric space and $f : X \to [0, \infty]$. For $\delta \in (0, \infty]$ we define the weighted integral of $f$ with respect to $\mathcal{H}_\delta^m$ by

$$\int_X^* f \, d\mathcal{H}_\delta^m = \inf \sum_{C \in \mathcal{C}} u(C) \mathcal{H}_\delta^m(C)$$

where the infimum is taken over all countable families $\mathcal{C}$ of subsets of $X$ and $[0, \infty]$-valued functions $u$ on $\mathcal{C}$ such that

$$\sum_{C \in \mathcal{C}} u(C) \chi_C \geq f.$$  \hspace{1cm} (2.1)

This definition is equivalent to the one Federer uses in the proof of the coarea inequality, although we state the definition in a slightly different way.
Obviously, the definition remains unchanged if we demand (2.1) to hold only \( H^m_\delta \)-almost everywhere.

An elaborate study of this integral and the measure underlying it can be found in [How], where a measure developed from the weighted integral is referred to as \textit{weighted Hausdorff measure}.

We will prove the following proposition and theorem in Chapter 7. Since we use them both in the proof of the coarea formula, we already state them at this point. Federer proves them under the additional condition that \( Y \) is proper, or that the set \( \{ y \in Y : f(y) > 0 \} \) in Proposition 2.3 or the set \( \{ y \in Y : \mathcal{H}^{n-m}(f^{-1}(y) \cap A) > 0 \} \) in Theorem 2.4 is \( \mathcal{H}^m \)-\( \sigma \)-finite [Fed2, 2.10.24 – 2.10.25]. This would satisfy our requirements, since we consider \( \mathcal{H}^m \)-\( \sigma \)-finite target spaces. However, the coarea inequality is of independent interest. Therefore we provide the proofs without the extra conditions.

**Proposition 2.3.** Suppose \( Y \) is metric space and \( f : Y \to [0, \infty] \). Then

\[
\lim_{\delta \searrow 0} \int_{Y}^* f \ d\mathcal{H}^m_\delta = \int_{Y}^* f \ d\mathcal{H}^m.
\]

**Theorem 2.4** (Coarea inequality). If \( f : X \to Y \) is a Lipschitz map between metric spaces and \( A \subset X, 0 \leq m \leq n \), then

\[
\int_{Y}^* \mathcal{H}^{n-m}(f^{-1}(y) \cap A) \ d\mathcal{H}^m(y) \leq \text{(Lip } f\text{)}^m \frac{\alpha_{n-m} \alpha_m}{\alpha_n} \mathcal{H}^n(A).
\]

**Lemma and Definition 2.5.** Let \( A \subset \mathbb{R}^n \) be an \( \mathcal{H}^n \)-measurable subset of \( \mathbb{R}^n \). Then \( \mathcal{H}^n \)-almost every point \( a \in A \) fulfills

\[
\lim_{r \searrow 0} \frac{\mathcal{H}^n(B_r(a) \cap A)}{\mathcal{H}^n(B_r(a))} = 1.
\]

We call such a point density point of \( A \).

The statement follows directly from the Lebesgue differentiation theorem.

**Lemma 2.6.** Let \( (X, d) \) be a metric space and let \( n \in \mathbb{N} \). Suppose there exists \( \delta_0 \in (0, \infty) \) such that

\[
\mathcal{H}^n_{\delta_0}(X) = 0.
\]

Then

\[
\mathcal{H}^n_{\delta}(X) = 0
\]

for all \( \delta \in (0, \infty] \), which implies \( \mathcal{H}^n(X) = 0 \) as well.
Proof. By construction it suffices showing
\[ H^n_\infty(X) = 0 \Rightarrow H^n(X) = 0 \]
to prove the statement. Assume \( H^n_\infty(X) = 0 \). Then there exists a sequence of covers \( \{C_i\}_{i \in \mathbb{N}} \) of \( X \), such that
\[ \lim_{i \to \infty} \sum_{C \in C_i} \text{diam}(C)^n = 0. \]
Let \( d_i = \frac{1}{2} \sup_{C \in C_i} \text{diam}(C) \) for all \( i \in \mathbb{N} \). We have \( d_i^n \leq \sum_{C \in C_i} \text{diam}(C)^n \) and therefore \( d_i \to 0 \) as \( i \to \infty \). Let \( \delta > 0 \). For \( i \) large enough, \( 2d_i \leq \delta \), and therefore \( C_i \) is a \( \delta \)-cover of \( X \). Thus \( H^n_\delta(X) = 0 \). This holds for every \( \delta > 0 \), and therefore \( H^n(X) = 0 \).

2.3 Covering theorems

In this work we will use some well-known covering theorems.

**Theorem 2.7** (Besicovitch’s covering theorem). There is a constant \( N = N(n) \) depending only on \( n \in \mathbb{N} \) with the following property: Suppose \( A \) is a bounded subset of \( \mathbb{R}^n \) and \( C \) is a family of closed balls in \( \mathbb{R}^n \) such that each point of \( A \) is the center of some ball belonging to \( C \). Then there exist disjoint families \( D_1, \ldots, D_N \subset C \) such that
\[ A \subset \bigcup_{i=1}^N \bigcup_{D \in D_i} D. \]

Cf. [Mat, 2.7], [Fed2, 2.8.14 and 2.8.15].

**Theorem 2.8** (Lusin’s theorem). Suppose \( X \) is a metric space and \( \nu \) a Borel regular measure on \( X \). Suppose \( f : X \to Y \) is a measurable map with values in a separable metric space \( Y \), \( A \) is a \( \nu \)-measurable, \( \nu \)-\( \sigma \)-finite subset of \( X \) and \( \varepsilon > 0 \). Then \( A \) contains a closed set \( C \) such that \( \nu(A \setminus C) < \varepsilon \) and \( f|_C \) is continuous.

Cf. [Fed2, 2.3.5].

**Theorem 2.9** (Egorov’s theorem). Suppose \( X \) is a metric space and \( \nu \) a measure on \( X \). Suppose \( f_1, f_2, f_3, \ldots \) and \( g \) are \( \nu \)-measurable maps with values in a separable metric space \( Y \). Suppose \( A \subset X \) with \( \nu(A) < \infty \). If
\[ f_n(x) \to g(x) \text{ as } n \to \infty \text{ for almost all } x \in A \]
and \( \varepsilon > 0 \), then there exists a \( \nu \)-measurable set \( B \subset A \) with \( \nu(A \setminus B) < \varepsilon \) such that

\[
f_n(x) \to g(x) \text{ uniformly for } x \in B \text{ as } n \to \infty.
\]

Cf. [Fed2, 2.3.7].
Chapter 3

Metric differential

In their papers [Kir] and [AK], Ambrosio and Kirchheim established the following definition.

**Definition 3.1** (metric differential). Let $Y$ be a metric space. We say that a map $f : \mathbb{R}^n \to Y$ is metrically differentiable at $x \in \mathbb{R}^n$ if there exists a seminorm $\sigma$ on $\mathbb{R}^n$ such that

$$
\lim_{|v| + |w| \to 0} \frac{d(f(x + v), f(x + w)) - \sigma(v - w)}{|v| + |w|} = 0.
$$

Then we call $\sigma$ the metric differential at $x$ and denote it by $\text{md}_f x$.

3.1 Rademacher’s theorem

Rademacher’s Theorem states that any Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere. The following proposition is the corresponding statement for Lipschitz maps $f : \mathbb{R}^n \to Y$, where $Y$ is a metric space. This theorem was proved by Kirchheim in [Kir, Theorem 2]. Since it is essential for this work and the constructions proving it are very helpful for the understanding of the behaviour of the metric differential, we provide a proof of the theorem.

**Theorem 3.2.** Suppose $f : \mathbb{R}^n \to Y$ is a Lipschitz map and $B$ is the set where $\text{md}_f x$ exists. Then $B$ is a Borel set and $\mathcal{H}^n(\mathbb{R}^n \setminus B) = 0$. 
As in the proof of Rademacher’s theorem we first prove it for $n = 1$, i.e. for Lipschitz curves in $Y$. Lebesgue’s Theorem states that any Lipschitz function $\mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere. From this one can conclude the one-dimensional version of the statement, Lemma 3.4.

**Definition 3.3** (metric derivative). Suppose $Y$ is a metric space and $(a, b) \subset \mathbb{R}^n$ and let $\gamma : (a, b) \to Y$ be a Lipschitz map. Then we define the metric derivative at a point $x \in (a, b)$ as the limit

$$
\lim_{t \to 0} \frac{d(\gamma(x + t), \gamma(x))}{|t|}
$$

whenever it exists and, in this case, denote it by $|\dot{\gamma}|(x)$.

**Lemma 3.4.** For each Lipschitz curve $\gamma : (a, b) \to Y$ the metric derivative exists at $\mathcal{H}^1$-almost every point in $(a, b)$.

The proof of this lemma can be found in [AT, 4.1.6].

**Corollary 3.5.** Given a metric space $Y$, a Lipschitz map $f : \mathbb{R}^n \to Y$ and a nonzero vector $v \in \mathbb{R}^n$, the limit

$$
\lim_{t \to 0} \frac{d(f(x + tv), f(x))}{|t|}
$$

exists at $\mathcal{H}^n$-almost every point $x \in \mathbb{R}^n$.

**Proof.** We apply Lemma 3.4 on the curves $\gamma_x(t) : \mathbb{R} \to Y, t \to x + tv$. Then the statement follows directly from Fubini’s theorem.

For the proof of Proposition 3.2 we take a detour over an equivalent definition of differentiability.

**Definition 3.6.** Suppose $f : \mathbb{R}^n \to Y$ is a Lipschitz map. Then we say $f$ is $*$-differentiable at $x$ if for all $u, v \in \mathbb{R}^n$

i) $$
\lim_{t \to 0} \frac{d(f(x + tu), f(x + tv))}{|t|} = \gamma_x(u, v)
$$
for some $\gamma_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

ii) $\gamma_x(u, v) = \gamma_x(u - v, 0)$. 
Lemma 3.7. A Lipschitz map \( f : \mathbb{R}^n \to Y \) is \( \ast \)-differentiable at \( x \in \mathbb{R} \) if and only if it is metrically differentiable at \( x \).

Proof. Let \( x \in \mathbb{R}^n \) be such that \( f \) is \( \ast \)-differentiable at \( x \). Then \( \sigma_x(u) = \gamma_x(u,0) \) defines a seminorm on \( \mathbb{R}^n \). Indeed, we have
\[
\sigma_x(\lambda u) = \lim_{t \to 0} \frac{d(f(x + t\lambda u), f(x))}{|t|} = |\lambda| \lim_{t \to 0} \frac{d(f(x + tu), f(x))}{|t|} = |\lambda| \sigma_x(u)
\]
and
\[
\sigma_x(u + v) = \lim_{t \to 0} \frac{d(f(x + tu + tv), f(x))}{|t|} \\
\leq \lim_{t \to 0} \frac{d(f(x + tu + tv), f(x + tu))}{|t|} + \lim_{t \to 0} \frac{d(f(x + tu), f(x))}{|t|} \\
= \gamma_x(u + v, u) + \gamma_x(u, 0) = \sigma_x(v) + \sigma_x(u).
\]

We denote now \( \frac{d(f(x+tu), f(x+tv))}{|t|} \) by \( Q_{x,t}(u, v) \). As shown above we know that for fixed \( u, v \)
\[
\lim_{t \to 0} Q_{x,t}(u, v) = \sigma_x(u - v).
\]

It remains to show that
\[
\lim_{t \to 0} |Q_{x,t}(u, v) - \sigma_x(u - v)| = 0
\]
uniformly on a neighborhood of \( 0 \in \mathbb{R}^n \times \mathbb{R}^n \).

Suppose \( D = B_1(0) \times B_1(0) \in \mathbb{R}^n \times \mathbb{R}^n \). We endow \( \mathbb{R}^n \times \mathbb{R}^n \) with the metric \( \tilde{d} = |\cdot| + |\cdot| \) and we denote the Lipschitz constant of \( f \) by \( L \). From
\[
|Q_{x,t}(u, v) - Q_{x,t}(u', v')| \\
= \frac{1}{|t|} |d(f(x + tu), f(x + tv)) - d(f(x + tu'), f(x + tv'))| \\
\leq \frac{1}{|t|} \left| d(f(x + tu), f(x + tu')) + d(f(x + tu'), f(x + tv)) \\
+ d(f(x + tv'), f(x + tv)) - d(f(x + tu'), f(x + tv')) \right| \\
= \frac{1}{|t|} \left| d(f(x + tu), f(x + tu')) + d(f(x + tv), f(x + tv')) \right| \\
\leq L(|u - u'| + |v - v'|) \\
= L \tilde{d}((u, v), (u', v'))
\]
it follows that
\[ |\gamma_x(u, v) - \gamma_x(u', v')| \leq L \tilde{d}((u, v)(u', v')). \]

Hence \( \gamma_x \) is \( L \)-Lipschitz.

Suppose \( \varepsilon > 0 \) and let \( S_\varepsilon \subset D \) be a finite \( \varepsilon \)-net. For \( (u, v) \in D \) choose \( (u, v) \varepsilon \in S_\varepsilon \) with \( \tilde{d}((u, v), (u, v)\varepsilon) \leq \varepsilon \). Let \( t_\varepsilon > 0 \) be such that \( |Q_{x,t}(u, v) - \gamma_x(u, v)| \leq \varepsilon \) for all \( 0 < t \leq t_\varepsilon \) and \( (u, v) \in S_\varepsilon \). Thus for every \( (u, v) \in D \) and every \( 0 < t \leq t_\varepsilon \) we get
\[
|Q_{x,t}(u, v) - \gamma_x(u, v)| \leq |Q_{x,t}((u, v)\varepsilon) - \gamma_x((u, v)\varepsilon)| + 2L \varepsilon \\
\leq \varepsilon + 2L \varepsilon = (2L + 1)\varepsilon.
\]

It follows for \( (u, v) \in \mathbb{R}^n \times \mathbb{R}^n \) with \( |u| + |v| = t \leq t_\varepsilon \)
\[
\left| \frac{d(f(x + u), f(x + v)) - \sigma_x(u - v)}{|u| + |v|} \right| \\
= \left| \frac{d(f(x + t\frac{u}{t}), f(x + t\frac{v}{t}))}{t} - \sigma_x(u - v) \right| \\
\leq (2L + 1)\varepsilon.
\]

Hence \( f \) is metrically differentiable in \( x \). It is clear that metric differentiability implies \( \ast \)-differentiability.

**Proof of Theorem 3.2.** By Lemma 3.7, the set where \( f \) is metrically differentiable coincides with the set where \( f \) is \( \ast \)-differentiable. We will show that the latter is a Borel set.

Wherever \( \gamma_x \) exists, it is positively 1-homogeneous in \( (u, v) \). Therefore we can assume \( u, v \in B_1(0) \) what we do from now on. For every \( x \in \mathbb{R}^n \) and \( t \neq 0 \), \( Q_{x,t}(u, v) \) is Lipschitz in \( u \) and \( v \) and, wherever it exists, \( \gamma_x(u, v) \) is Lipschitz in \( u \) and \( v \) as well. Therefore it suffices that the conditions for \( \ast \)-differentiability of \( f \) in a point \( x \) are satisfied for \( u, v \) belonging to a dense set in \( B_1(0) \) for \( f \) to be \( \ast \)-differentiable in \( x \). For this purpose we choose a dense, countable subset \( S \subset B_1(0) \).

We define \( \hat{B} \subset \mathbb{R}^n \) to be the set where the first condition of \( \ast \)-differentiability is fulfilled. And for all \( u, v \in S \), we define \( \hat{B}_{u,v} \) to be the set of points \( x \in \mathbb{R}^n \) where \( \lim_{t \to 0} Q_{x,t}(u, v) \) exists. For fixed \( u, v \in S \) and \( t \neq 0 \), \( x \mapsto Q_{x,t}(u, v) \) is continuous. Hence \( \hat{B}_{u,v} \) is a Borel set. It follows that \( \hat{B} = \bigcap_{u,v \in S} \hat{B}_{u,v} \) is a Borel set as well.

Let now \( B \subset \hat{B} \) be the set of points that fulfill the second condition of \( \ast \)-differentiability as well, i.e. \( B \) consists of all points \( x \in \mathbb{R}^n \) in which \( f \) is
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We know that on $\tilde{B}$, for every $u, v \in \mathbb{R}^n$, the function $\gamma_x(u, v)$ is Borel measurable. Hence

$$B_{u,v} = \{ x \in \tilde{B} : \gamma_x(u-v,0) - \gamma_x(u,v) = 0 \}$$

are Borel sets and $B = \bigcap_{u,v \in S} B_{u,v}$ as well.

To prove that $\mathcal{H}^n(\mathbb{R}^n \setminus B) = 0$, we first fix $u$ and $v$ in $S$. Consider the set of $x \in \mathbb{R}^n$ where $\lim_{t \to 0} Q_{x,t}(u - v, 0)$ exists. By Corollary 3.5 we know that this set is almost all of $\mathbb{R}^n$. By Lusin’s theorem we can cover almost all of it by a sequence of sets $\{L_i\}_{i \geq 1}$ such that $\sigma_x(u - v)$ is continuous on $L_i$ for each $i \geq 1$. By Egorov’s theorem we can find a sequence of subsets $M_{ik}$ covering almost all of $L_i$ such that on each $M_{ik}$, $Q_{x,t}(u - v, 0)$ converges uniformly to $\sigma_x(u - v)$ for $t \to 0$.

Consider now a density point $x_0 \in M_{ik}$. Choose $\varepsilon > 0$. Let $\delta > 0$ be such that the following conditions hold:

- for every $x \in M_{ik}$ with $|x - x_0| < \delta$,
  $$|\sigma_x(u - v) - \sigma_{x_0}(u - v)| < \varepsilon,$$

- for every $x \in M_{ik}$ and every $0 < t < \delta$,
  $$\left| \frac{d(f(x + tv(u - v)), f(x)) - \sigma_x(u - v)}{t} \right| < \varepsilon,$$

- for every $0 < t < \delta$, $B_t(x_0) \cap M_{ik}$ is $t\varepsilon$-dense in $B_t(x_0)$.

For every $0 < t < \delta$ we choose a $v_t \in \mathbb{R}^n$ such that $|v_t| < \delta$, $x_0 + v_t \in M_{ik}$ and $|v_t - tv| \leq t\varepsilon$. Then for every $0 < t < \delta$ we have

$$|d(f(x_0 + tv), f(x_0 + tv)) - d(f(x_0 + v_t + tv(u - v)), f(x_0 + v_t))|$$

$$\leq |d(f(x_0 + tv), f(x_0 + tv + v_t - tv)) + d(f(x_0 + tv + v_t - tv), f(x_0 + v_t))$$

$$+ d(f(x_0 + v_t), f(x_0 + tv)) - d(f(x_0 + v_t + tv(u - v)), f(x_0 + v_t))|$$

$$= |d(f(x_0 + tv), f(x_0 + tv + v_t - tv)) + d(f(x_0 + v_t), f(x_0 + tv))|$$

$$\leq 2Lt\varepsilon.$$

and

$$\left| \frac{d(f(x_0 + v_t + tv(u - v)), f(x_0 + v_t))}{t} - \sigma_{x_0+v_t}(u - v) \right| < \varepsilon.$$
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Combined with the first condition on \( \delta \), this gives us for every \( 0 < t < \delta \)

\[
\left| \sigma_{x_0}(u - v) - \frac{d(f(x_0 + tu), f(x_0 + tv))}{t} \right| < (2L + 2)\varepsilon.
\]

Hence \( \lim_{t \to 0} \frac{d(f(x_0 + tu), f(x_0 + tv))}{t} \) exists and equals \( \sigma_{x_0}(u - v) \).

Thus \( x_0 \) lies in \( \tilde{B}_{u,v} \). Since almost every point in \( \mathbb{R}^n \) is a density point of some \( M_{ik} \), we get \( \mathcal{H}^n(\mathbb{R}^n \setminus \tilde{B}_{u,v}) = 0 \). This holds for all \( u, v \in S \), and therefore \( \mathcal{H}^n(\mathbb{R}^n \setminus \tilde{B}) = 0 \).

For fixed \( u, v \in S \), almost every point of \( \tilde{B} \) is a density point of some \( M_{ik} \), thus \( \mathcal{H}^n(\tilde{B} \setminus B_{u,v}) = 0 \). This holds again for all \( u, v \in S \), and therefore \( \mathcal{H}(\tilde{B} \setminus B) = 0 \). It follows \( \mathcal{H}^n(\mathbb{R}^n \setminus B) = 0 \). \( \square \)

3.2 Partitions

In this section the goal is to prove that there are measurable partitions of the set \( B \), such that on each subset we have a lot of information about the metric differential. There are two important conclusions. In Lemma 3.13 we consider the set where \( \text{md}_{f_x} \) is a norm, and we construct a partition of it such that we can control the shape of the respective norms. In Lemma 3.17, we consider a set where \( \text{md}_{f_x} \) has constant rank and construct a partition of it such that we can control the direction of the kernel of the respective seminorms.

From now on, we denote by \( C(S^{n-1}) \) the space of continuous functions on \( S^{n-1} \) endowed with the supremum norm.

**Lemma 3.8.** Suppose \( f : \mathbb{R}^n \to Y \) is a Lipschitz map and let \( B \subset \mathbb{R}^n \) be the Borel set where \( \text{md}_{f_x} \) is defined. Then the map

\[
\sigma : B \to C(S^{n-1})
\]

\[
x \mapsto \text{md}_{f_x}(\cdot)
\]

is Borel measurable.

**Proof.** Suppose \( g \in C(S^{n-1}) \) and \( r > 0 \). We will show that the preimage of the closed ball \( B_r(g) \subset C(S^{n-1}) \) under \( \sigma \) is Borel measurable.

Let \( D = \{v_i\}_{i \geq 1} \) be a dense subset of \( S^{n-1} \). We define the sets

\[
B_{r,i}(g) = \{ h \in C(S^{n-1}) : |g(v_i) - h(v_i)| \leq r \}
\]
with the property
\[ B_r(g) = \bigcap_{i \geq 1} B_{r,i}(g). \]

By
\[ \sigma^{-1}B_r(g) = \bigcap_{i \geq 1} \sigma^{-1}B_{r,i}(g) \]
\[ = \bigcap_{i \geq 1} \left\{ x \in B : \lim_{t \to 0} \frac{d(f(x), f(x + tv_i))}{t} \in [g(v_i) - r, g(v_i) + r] \right\} \]
\[ = \bigcap_{i \geq 1} \bigcup_{j \geq 1} \bigcap_{k \geq j} \left\{ x \in B : \left| \frac{d(f(x), f(x + \frac{1}{k}v_i))}{k} - h(v_i) \right| \leq r \right\} \]
we see that it \( \sigma^{-1}(B_r(g)) \) a Borel set. And since \( C(S^{n-1}) \) is separable, the closed balls generate the Borel \( \sigma \)-algebra. Therefore \( \sigma \) is Borel measurable.

We define \( S \subset C(S^{n-1}) \) to be the set consisting of all functions which are restrictions to \( S^{n-1} \) of a seminorm. Furthermore, we define for every \( 0 \leq k \leq n \) the set \( S^{(k)} \) consisting of all functions which are restrictions to \( S^{n-1} \) of a seminorm of rank \( k \). We say a function in \( S^{(k)} \) has rank \( k \) as well.

**Lemma 3.9.** \( S \) is closed in \( C(S^{n-1}) \).

**Proof.** A function \( f \in C(S^{n-1}) \) belongs to the set \( S \) if and only if \( f \) it is symmetric, nonnegative and the following form of the triangle inequality holds:

For every quadruple \( (s, t, v, w) \) with \( s, t \in \mathbb{R} \) and \( v, w \in S^{n-1} \subset \mathbb{R}^n \) with \( |sv + tw| = 1 \) the inequality
\[ f(sv + tw) \leq s \cdot f(v) + t \cdot f(w) \]
holds.

These properties are preserved by taking limits in the Banach space \( C(S^{n-1}) \).

**Lemma 3.10.** Suppose \( 0 \leq m \leq n \). Then the set \( \bigcup_{k=0}^{m} S^{(k)} \) is closed in \( C(S^{n-1}) \).

**Proof.** For \( m = n \) we already proved the statement. So we assume \( m < n \). Suppose \( f_i \) is a sequence in \( \bigcup_{k=0}^{m} S^{(k)} \) converging to \( f \in S \). By passing to a subsequence we can assume all \( f_i \) to have the same rank \( r \leq m \). Let \( q \) be the rank of \( f \). We will show \( q \leq r \) what implies the statement.
The set \( \{ v \in S^{n-1} : f(v) = 0 \} \) is an isometric copy of \( S^{n-1-q} \subset S^{n-1} \) which we denote by \( K \). Let now \( \varepsilon > 0 \) be sufficiently small such that there exists no isometric copy of \( S^{n-q} \) in \( U_{\varepsilon}(S^{n-1-q}) \subset S^{n-1} \). Consider the set \( C = S^{n-1} \setminus U_{\varepsilon}(K) \subset S \). This is a compact set on which \( f \) is positive and we can find a \( \delta > 0 \) such that \( f > \delta \) on \( C \). We can choose \( i \) large enough such that

\[
|f - f_i|_\infty < \delta/2.
\]

It follows for every \( v \in S' \)

\[
f_i(v) \geq f(v) - |f(v) - f_i(v)| \geq f(v) - \delta/2 > 0.
\]

Therefore \( K_i = \{ v \in S^{n-1} : f_i(v) = 0 \} \subset U_{\varepsilon}(K) \). But \( K_i \) is an isometric copy of \( S^{n-1-r} \subset S^{n-1} \). Together with the assumption for \( \varepsilon \) we conclude that \( n - 1 - r \leq n - 1 - q \) and therefore \( q \leq r \). \( \square \)

Together with the Borel measurability of \( \text{md} f \), this lemma directly implies the following corollary.

**Corollary 3.11.** For every \( 0 \leq m \leq n \), let \( B^{(m)} \subset B \) be the set of points \( x \in \mathbb{R}^n \) where \( \text{md} f_x \) exists and has rank \( m \). Then \( B^{(m)} \) is a Borel set for every \( 0 \leq m \leq n \).

**Definition 3.12 (\( \lambda \)-close).** Given \( \lambda \geq 1 \), two norms \( \tau \) and \( \rho \) are said to be \( \lambda \)-close if

\[
\lambda^{-1}\tau(v) \leq \rho(v) \leq \lambda\tau(v)
\]

for every \( v \in \mathbb{R}^n \).

The following lemma is essential in the proof of the coarea formula. It gives us a Borel partition of \( B^{(n)} \) such that on each subset we have a lot of information about \( f \) and \( \text{md} f_x \).

**Lemma 3.13.** Suppose \( f : \mathbb{R}^n \to Y \) is a Lipschitz map and \( \lambda > 1 \). Then there exists a Borel partition \( \{ B_i \}_{i \in \mathbb{N}} \) of \( B^{(n)} \) and a sequence of norms \( \sigma_i \) on \( \mathbb{R}^n \) such that \( \text{md} f_x \) is \( \lambda \)-close to \( \sigma_i \) and

\[
\lambda^{-1}\sigma_i(x - x') \leq d(f(x), f(x')) \leq \lambda\sigma_i(x - x'),
\]

for all \( x, x' \in B_i \).

Cf. [Kir, Lemma 4].

**Proof.** We choose a sequence of functions \( f_i \in S^{(n)} \subset C(S^{n-1}) \) which is dense \( S^{(n)} \). They induce a sequence of norms \( \{ \sigma_i \}_{i \in \mathbb{N}} \) on \( \mathbb{R}^n \). This sequence has the
property that for every norm $\sigma$ on $\mathbb{R}$ and every $\mu > 0$ there is an $i \in \mathbb{N}$ such that $\sigma$ is $\lambda$-close to $\sigma_i$.

Given $\lambda > 1$, choose $\delta > 0$ such that $\lambda^{-1} + \delta < 1 < \lambda - \delta$. For $i, k \in \mathbb{N}$, denote by $B_{i,k}$ the Borel set of all $x \in B(n)$ with

i) $(\lambda^{-1} + \delta)\sigma_i(v) \leq md f_x(v) \leq (\lambda - \delta)\sigma_i(v)$ for all $v \in \mathbb{R}$,

ii) $|d(f(x + v), f(x)) - md f_x(v)| \leq \delta \sigma_i(v)$ for all $|v| < 1/k$.

To see that the $B_{i,k}$ cover $B$, let $x \in B$ and choose $i \geq 1$ such that i) holds. Let $c_i > 0$ be such that $|v| \leq c_i \sigma_i(v)$ for all $v \in \mathbb{R}^n$, and pick $k \geq 1$ such that ii) holds with $\delta \sigma_i(v)$ replaced by $(\delta/c_i)|v|$. Then $x \in B_{i,k}$. Let now $C \subset B_{i,k}$ be a set with diam $C \leq 1/k$. Then

$$
\begin{align*}
\text{d}(f(x + v), f(x)) &\leq \text{md} f_x(v) + \delta \sigma_i(v) \leq \lambda \sigma_i(v), \\
\text{d}(f(x + v), f(x)) &\geq \text{md} f_x(v) - \delta \sigma_i(v) \geq \lambda^{-1} \sigma_i(v),
\end{align*}
$$

whenever $x, x + v \in C$. By subdividing and relabeling the sets $B_{i,k}$ appropriately we obtain the desired partition.

We denote by $G(k, n)$ the Grassmannian of $k$-dimensional subspaces in $\mathbb{R}^n$. For a subspace $W \subset \mathbb{R}^n$, $p_W$ denotes the orthogonal projection $\mathbb{R}^n \to W$. On $G(k, n)$ we define the metric

$$
d_G(W, W') = \|p_W - p_{W'}\|
$$

where $\| \cdot \|$ denotes the operator norm. Endowed with this metric, $G(k, n)$ is a compact metric space (cf. [Mat, 3.9]).

For the proceeding results, we need the following estimates.

**Lemma 3.14.** Suppose $0 < k < n$ and $W, W' \in G(k, n)$ such that $W^\perp$ and $W'$ are transversal, and define

$$AH(W, W') := \inf\{\varepsilon > 0 : (W \cap S^{n-1}) \subset U_\varepsilon(W \cap S^{n-1})\}.$$

Then

i) $\text{BLip}(p_W|_{W'}) \leq \frac{1}{\sqrt{1 - d_G(W, W')}}$,

ii) $AH(W, W') \leq 2 d_G(W, W').$
Proof. Let \( v \) be in \( W' \) with \( |v| = 1 \) such that \( |p_W(v)| \) is maximal. Then i) follows by

\[
\text{BLip}_P W |_{W'} = \sup_{w \in W', |w| = 1} \frac{1}{|P_W(w)|} = \frac{1}{|P_W(v)|} = \frac{1}{\sqrt{1 - |v - P_W(v)|^2}} = \frac{1}{\sqrt{1 - |(P'_W - P_W)(v)|^2}} \leq \frac{1}{\sqrt{1 - d_G(W, W')^2}}.
\]

We have

\[
AH(W, W') = \sup\{d(w, W' \cap S^{n-1}) : w \in W \cap S^{n-1}\}.
\]

To prove ii), let \( w_0 \in W \cap S^{n-1} \) be a point where the maximum is achieved. Then we have

\[
AH(W, W') = d(w_0, W' \cap S^{n-1}) \leq 2 d(w_0, W') = 2 (p_W - p_W')(w_0) \leq 2 d_G(W, W').
\]

\[\square\]

Lemma 3.15. Suppose \( W, W' \subset \mathbb{R}^n \) are two \( k \)-dimensional subspaces where \( 0 < k < n \) such that \( W' \perp W \) are transversal. Then \( \text{BLip}_P W |_{W'} = \text{BLip}_P W' |_{W' \perp} \).

Proof. Assume \( W \neq W' \). Let \( v \in W' \) with \( |v| = 1 \) maximising \( d(v, W) \). Define \( \tilde{v} = \frac{v - P_W(v)}{|v - P_W(v)|} \), a unit vector in \( W' \perp \). A simple argument of symmetry shows that \( d(\tilde{v}, v^\perp) = d(v, W) \). Since \( W' \perp \subset v^\perp \), we get \( d(\tilde{v}, W' \perp) \geq d(v, W) \). By Pythagoras’ theorem we see that

\[
\text{BLip}_P W |_{W'} \leq \text{BLip}_P W' |_{W' \perp}
\]

holds. And since the argument works identically in the other direction, equality holds. \[\square\]
Lemma 3.16. Suppose \( f : \mathbb{R}^n \to Y \) is a Lipschitz map. Then the map
\[
\ker : S^{(m)} \to G(n - m, n)
\]
is continuous.

Proof. Suppose \( g \in S^{(m)} \) and \( \varepsilon > 0 \). Consider the compact set
\[
C = S^{n-1} \setminus U_\varepsilon(S^{n-1} \cap \ker g).
\]
The function \( g \) is strictly positive on \( C \) thus we can find a \( \delta_\varepsilon > 0 \) with \( g > \delta_\varepsilon \) on \( C \) as well and we conclude
\[
(S^{n-1} \cap \ker h) \subset U_\varepsilon(S^{n-1} \cap \ker g)
\]
and therefore \( AH(\ker h, \ker g) \leq \varepsilon \). Given \( \eta > 0 \), by Lemma 3.14 i) we can set \( \varepsilon = \frac{1}{2} \eta \), and then \( |g - h|_\infty < \delta_\varepsilon \) implies \( d_h(\ker g, \ker h) < \eta \).

Lemma 3.17. Suppose \( f : \mathbb{R}^n \to Y \) is a Lipschitz map, \( \lambda > 1 \) and \( m \in \{1, \ldots, n - 1\} \). Then there exists a finite Borel partition \( \{B_i\}_{1 \leq i \leq N} \) of \( B^{(m)} \) and a sequence of \((n - m)\)-dimensional subspaces \( K_i \) of \( \mathbb{R}^n \) such that for every \( x \in B_i \), the orthogonal projections
\[
\ker md f_x \to K_i, \quad K_i \to \ker md f_x,
\]
\[
(\ker md f_x)^\perp \to K_i^\perp, \quad K_i^\perp \to (\ker md f_x)^\perp
\]
are all \( \lambda \)-bi-Lipschitz.

Proof. Choose a \( \delta > 0 \) small enough such that \( \frac{1}{\sqrt{1 - \delta^2}} \leq \lambda \). Since the metric space \((G(n - m, n), d_G)\) is compact, we find a finite Borel partition \( \{G_i\}_{1 \leq i \leq N} \) of \( G(n - m, n) \) by sets with diameter less than \( \delta \). By Lemma 3.14 ii), any two subspaces \( W, W' \in G(n - m, n) \) fulfill
\[
\text{BLip}(\rho_W|W') \leq \lambda.
\]
By Lemmata 3.8 and 3.16 the map
\[
\ker md f : B^{(m)} \to G(n - m, n), \quad x \mapsto \ker md f_x
\]
is Borel measurable. Hence the subsets \((\ker md f)^{-1}(G_i) \in B^{(m)}\) are Borel sets and fulfill the desired condition of the first two projections. By Lemma 3.15 the condition is fulfilled for the latter two projections as well.
3.3 Arbitrary domains

The notion of the metric differential of a map \( f \) defined as above makes sense if \( f \) is defined on \( \mathbb{R}^n \), or more generally, on an open subset of \( \mathbb{R}^n \). However, we can generalize this notion to maps defined on arbitrary subsets of \( \mathbb{R}^n \) without losing the existence and uniqueness almost everywhere.

**Definition 3.18** (metric differential). Let \( X \) be a metric space and suppose \( A \subset \mathbb{R}^n \) is an arbitrary subset. Let \( f : A \to X \) be a Lipschitz map. We say \( f \) is metrically differentiable at \( x \in A \) if there exists a unique seminorm \( \sigma \) satisfying

\[
\lim_{|v|+|w| \to 0} \frac{d(f(x+v), f(x+w)) - \sigma(v-w)}{|v| + |w|} = 0.
\]  

(3.1)

In this case we call \( \sigma \) the metric differential at \( x \) and denote it by \( \text{md} f_x \).

Remark that for open sets, the two definitions coincide. In this definition we do, unlike in the definition on \( \mathbb{R}^n \), demand the seminorm \( \sigma \) to be unique. Without this property, there could exist several seminorms fulfilling (3.1) at a point, as we see in the following example.

**Example 3.1.** Let \( A \subset \mathbb{R}^2 \) be the set \( \{(t,0) : t \in \mathbb{R}\} \) and let \( f : A \to \mathbb{R}^2 \) be the identity map. Then every (semi)norm \( \sigma \) satisfying \( \sigma((1,0)) = 1 \) fulfills (3.1).

**Lemma 3.19.** Suppose \( A \subset \mathbb{R}^n \) and let \( f : A \to X \) be a Lipschitz map to a metric space \( X \). For every \( x \in A \) that is a density point of \( A \) the following holds: If there exists a seminorm \( \sigma \) on \( \mathbb{R}^n \) satisfying (3.1) then \( \sigma \) is unique with this property.

**Proof.** Let \( x \in A \) be a density point of \( A \) and assume \( \sigma \) and \( \tau \) are two seminorms satisfying (3.1). Let \( c \in \mathbb{R} \) such that \( \sigma(v) \leq c|v| \) and \( \tau(v) \leq c|v| \) for all \( v \in \mathbb{R}^n \). Combining the properties of \( \sigma \) and \( \tau \) gives us

\[
\lim_{|v|+|w| \to 0} \frac{\sigma(v-w) - \tau(v-w)}{|v| + |w|} = 0
\]

Together with the density property of \( x \) in \( A \), we know that for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( \sigma(v) - \tau(v) \leq \varepsilon|v| \) for every \( v \in \mathbb{R}^n \) with \( |v| \leq \delta \) and \( x + v \in A \), and such that for every \( v \in \mathbb{R}^n \) with \( |v| \leq \delta \) there exists a \( v' \in \mathbb{R}^n \)
such that $|v'| \leq \delta$, $x + v' \in A$ and $|v - v'| \leq \varepsilon |v|$. Thus we get for every $v \in \mathbb{R}^n$ with $|v| \leq \delta/2$

$$\begin{align*}
\sigma(v) - \tau(v) &\leq \sigma(v') - \tau(v') + 2c \varepsilon |v| \\
&\leq \varepsilon |v'| + 2c \varepsilon |v| \\
&\leq 2\varepsilon |v| + 2c \varepsilon |v| \\
&\leq \varepsilon (2 + 2c) |v|.
\end{align*}$$

We conclude $\sigma(v) = \tau(v)$ for all $v \in \mathbb{R}^n$.

To prove the next proposition we need the Kuratowski embedding. This embedding is a standard construction in metric geometry. It describes an isometric embedding of a metric space $X$ into $\bar{X} = l^\infty(X)$, the space of bounded functions on $X$. One big advantage of this embedding is that we can extend any Lipschitz maps $f : A \rightarrow l^\infty(X)$ defined on a subset $A$ of a metric space $Z$ to a map $\bar{f} : Z \rightarrow l^\infty(X)$ with the same Lipschitz constant. The construction works as follows: Fix a basepoint $x_0 \in X$ and define

$$X \longrightarrow l^\infty(X)$$

$$x \longmapsto s^x := d(x, \cdot) - d(x_0, \cdot).$$

It can easily be checked that this yields an isometric embedding.

Suppose now $Z$ is a metric space, $A \subset Z$ and $f : A \rightarrow l^\infty(X)$ is a $\lambda$-Lipschitz map. We then can view $f$ as $(f_x)_{x \in X}$ where $f_x : A \rightarrow \mathbb{R}$ for every $x \in X$. We can extend every such function $f_x$ with

$$\tilde{f}_x(z) := \inf\{f_x(a) + \lambda d(a, x) : a \in A\}.$$ 

The resulting map $\tilde{f} : Z \rightarrow l^\infty(X)$ turns out to be $\lambda$-Lipschitz as well.

With the help of the Kuratowski embedding we can generalize the statements from the last section to arbitrary subsets of $\mathbb{R}^n$. First we formulate the more general version of Proposition 3.2:

**Proposition 3.20.** Suppose $A \subset \mathbb{R}^n$ is $\mathcal{H}^p$-measurable and $f : A \rightarrow X$ is a Lipschitz map to a metric space $X$. Then for almost every $x \in A$, $\text{md} f_x$ exists.

**Proof.** Let $\bar{X} = l^\infty(X)$. As seen above we can embed $X$ in $\bar{X}$ via the Kuratowski embedding and we can also extend $f$ to a Lipschitz map $\bar{f} : \mathbb{R}^n \rightarrow \bar{X}$ with $\bar{f}|_A = f$. Since $\bar{f}$ is defined on the whole $\mathbb{R}^n$, we can apply Theorem...
3.2. Thus the metric differential (in the original sense) \( \text{md} \bar{f}_x \) exists at almost every point \( x \in \mathbb{R}^n \) and therefore at almost every point \( x \in A \). Let now \( x \in A \) be a density point of \( A \) such that \( \text{md} \bar{f}_x \) exists. Since \( \bar{f} \) and \( f \) coincide on \( A \), we know that (3.1) holds for \( \sigma = \text{md} \bar{f}_x \). Lemma 3.19 tells us that \( \text{md} \bar{f}_x \) is the only seminorm satisfying (3.1). Almost every \( x \in A \) is a density point of \( A \) where \( \text{md} \bar{f}_x \) exists, and therefore, by setting \( \text{md} f_x = \text{md} \bar{f}_x \) the statement holds.

In the last section, Corollary 3.11 and Lemmata 3.13 and 3.17 provided Borel partitions of the Borel set, where the metric differential existed. We can generalize these three statements to the case where \( f \) is defined on an \( \mathcal{H}^n \)-measurable subset of \( \mathbb{R}^n \).

**Lemma 3.21** (Generalization of Corollary 3.11). Suppose \( A \subset \mathbb{R}^n \) is an \( \mathcal{H}^n \)-measurable set, \( f : A \to Y \) is Lipschitz and define \( \bar{A} := \{ x \in A : \text{md} f_x \text{ exists} \} \). For every \( 0 \leq m \leq n \), let \( A^{(m)} \) be the set of points where \( \text{md} f_x \) exists and has rank \( m \). Then \( \{ A^{(m)} \}_{0 \leq m \leq n} \) is an \( \mathcal{H}^n \)-measurable partition of \( \bar{A} \).

**Proof.** Let \( \bar{f} : \mathbb{R}^n \to \bar{Y} \) be an extension of \( f : A \to Y \subset \bar{Y} = l^\infty(Y) \). For every point \( x \in \bar{A} \), \( \text{md} \bar{f}_x \) exists as well and equals \( \text{md} f_x \). Thus \( \bar{A} \in B \). Corollary 3.11 gives us a Borel set \( B^{(m)} \subset B \) for every \( 0 \leq m \leq n \). Then \( A^{(m)} = B^{(m)} \cap \bar{A} \) is \( \mathcal{H}^n \)-measurable.

Lemmata 3.13 and 3.17 are generalized analogously, hence we omit the proofs here.

**Lemma 3.22.** Suppose \( A \subset \mathbb{R}^n \), \( f : A \to Y \) is a Lipschitz map and let \( \lambda > 1 \) be. Then there exists a partition \( \{ A_i \}_{i \geq 1} \) of \( A^{(n)} \) and a sequence of norms \( \sigma_i \) on \( \mathbb{R}^n \) such that \( \text{md} f_x \) is \( \lambda \)-close to \( \sigma_i \) and

\[
\lambda^{-1} \sigma_i(x - x') \leq d(f(x), f(x')) \leq \lambda \sigma_i(x - x'),
\]

for all \( x, x' \in B_i \).

**Lemma 3.23.** Suppose \( A \subset \mathbb{R}^n \) is \( \mathcal{H}^n \)-measurable, \( f : A \to Y \) is a Lipschitz map, \( \lambda > 1 \) and \( m \in \{ 1, \ldots, n - 1 \} \). Then there exists a finite \( \mathcal{H}^n \)-measurable partition \( \{ A_i \}_{1 \leq i \leq N} \) of \( A^{(m)} \) and a sequence of \( (n - m) \)-dimensional subspaces \( K_i \) of \( \mathbb{R}^n \) such that for every \( x \in A_i \), the orthogonal projections

\[
\ker \text{md} f_x \to K_i, \quad K_i \to \ker \text{md} f_x,
\]

\[
(\ker \text{md} f_x)^\perp \to K_i^\perp, \quad K_i^\perp \to (\ker \text{md} f_x)^\perp
\]

are all \( \lambda \)-bi-Lipschitz.
3.4 Stepanov’s theorem

Stepanov’s theorem is a generalization of Rademacher’s theorem, cf. [Fed2, 3.1.9]. It states that a map \( f : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at almost every point \( x \) where \( \limsup_{y \to x} |f(y) - f(x)|/|x - y| < \infty \) holds. The following theorem is the corresponding statement for Lipschitz maps \( f : \mathbb{R}^n \to Y \).

**Proposition 3.24.** Suppose \( Y \) is a metric space and \( f : \mathbb{R}^n \to Y \). Then

i) \( f \) is metrically differentiable at \( \mathcal{H}^n \)-almost all points in the set

\[
L(f) := \left\{ x : \limsup_{y \to x} \frac{d(f(y), f(x))}{|x - y|} < \infty \right\},
\]

ii) there exists a countable partition \( \{C_i\}_{i \in \mathbb{N}} \) of \( L(f) \) such that for every \( i \in \mathbb{N} \), \( f|_{C_i} \) is Lipschitz. If \( f \) is measurable, then the sets \( C_i \) can be chosen to be \( \mathcal{H}^n \)-measurable.

**Proof.** We start by proving ii). For every \( j \in \mathbb{N} \) we define the set

\[
C_j := \left\{ x : d(f(x), f(y)) \leq j|x - y| \quad \forall y \in \mathcal{U}_{1/j}(x) \right\}.
\]

Obviously, \( \{C_j\}_{j \in \mathbb{N}} \) covers \( L(f) \). For every \( j \in \mathbb{N} \) we choose a cover \( \{C_{j,k}\}_{k \in \mathbb{N}} \) of \( C_j \) with \( C_{j,k} \subset C_j \) and \( \text{diam} C_{j,k} < \frac{1}{j} \) for every \( k \in \mathbb{N} \). For every \( x, y \in C_{j,k} \), \( x \in C_j \) and \( y \in \mathcal{U}_{1/j}(x) \) and therefore \( d(f(x), f(y)) \leq j|x - y| \). Thus \( f|_{C_{j,k}} \) is \( j \)-Lipschitz, and therefore ii) holds. Clearly, if \( f \) is measurable, the sets \( C_j \) are \( \mathcal{H}^n \)-measurable and the sets \( C_{j,k} \) can be chosen to be \( \mathcal{H}^n \)-measurable as well.

For the proof of i) we consider one such set \( C_{j,k} \) and denote it by \( C \). By Proposition 3.20 we know that \( \text{md} (f|_{C})_x \) exists at almost every point \( x \in C \). We show that for every Lebesgue density point \( x \in C \) such that \( \text{md} (f|_{C})_x \) exists, \( \text{md} f_x \) exists as well and coincides with \( \text{md} (f|_{C})_x \). The statement i) then follows directly.

Let now \( x \in C \) a density point of \( C \) such that \( \sigma := \text{md} (f|_{C})_x \) exists. Since \( f|_{C} \) is \( j \)-Lipschitz, \( \sigma(v) \leq j|v| \) for all \( v \in \mathbb{R}^n \). Let now \( \varepsilon > 0 \). Choose a \( \delta > 0 \) such that

- For every \( v, w \in B_{\delta}(0) \) with \( x + v, x + w \in C \),

\[
|d(f(x + v), f(x + w)) - \sigma(v - w)| \leq \varepsilon(|v| + |w|)
\]
For every \( r \leq \delta \), \( B_r(x) \cap C \) is \( \varepsilon r \)-dense in \( B_r(x) \).

Let \( z, z' \in B_{\delta}(0) \). By the second condition on \( \delta \) we find \( v \in B_{|z|}(0) \) and \( v' \in B_{|z'|}(0) \) such that \( x + v, x + v' \in C \) and \( |z - v| \leq \varepsilon |z| \) and \( |z' - v'| \leq \varepsilon |z'| \).

We can estimate

\[
|d(f(x + z), f(x + z')) - \sigma(z - z')| \\
\leq |d(f(x + v), f(x + v')) - \sigma(v - v')| \\
+ d(f(x + v), f(x + z)) + d(f(x + v'), f(x + z')) + \sigma(v - z) + \sigma(v' - z') \\
\leq |d(f(x + v), f(x + v')) - \sigma(v - v')| + 2j |z - v| + 2j |z' - v'| \\
\leq \varepsilon (|v| + |v'|) + 2j \varepsilon (|z| + |z'|) \\
\leq (2j + 1) \varepsilon (|z| + |z'|).
\]

Thus \( f \) is metrically differentiable at \( x \) and \( \text{md} f_x = \text{md}(f|C)_x \).

With the help of this proposition, we will prove Corollary 4.15, a generalization of the coarea formula, Theorem 1.1.

### 3.5 Coarea factor

The coarea factor \( C_m(\sigma) \) is an essential piece in the statement of the coarea formula. The quantity \( C_m(\text{md} f_x) \) corresponds to the \textit{m dimensional Jacobian} Federer uses in the statement of the coarea formula with rectifiable image space [Fed2, 3.2.1]. It is also an analogue of Kirchheim’s definition of Jacobians for norms [Kir, Definition 5] if the seminorms have rank \( m < n \). Connected with the metric differential, in the form \( C_m(\text{md} f_x) \), it measures the local dilation of \( f \) at the point \( x \) with respect to \( \mathcal{H}^n \) in the preimage and \( \mathcal{H}^m \) in the image.

**Definition 3.25** (coarea factor). Let \( 1 \leq m \leq n \) be and let \( \sigma \) be a seminorm on \( \mathbb{R}^n \) with rank at most \( m \). Let \( K = \ker \sigma \) and let \( T \) be any complement of \( K \) in \( \mathbb{R}^n \). The \( m \)-coarea factor \( C_m(\sigma) \) is defined as follows:

If \( \text{rank} \sigma < m \), then we define \( C_m(\sigma) = 0 \). If \( \text{rank} \sigma = m \), then \( C_m(\sigma) \) is the number satisfying

\[
C_m(\sigma) \mathcal{H}^n_e(A) = \int_T \mathcal{H}^{n-m}_e((y + K) \cap A) \, d\mathcal{H}^m_\sigma(y)
\]

for all \( \mathcal{H}^n \)-measurable subsets \( A \) of \( \mathbb{R}^n \).
In this definition, we choose a complement $T$ of $\ker \sigma$. Regarding any other such complement $T'$, the map $T \to T', y \mapsto (y + K) \cap T'$ defines an isometry between $(T, \| \cdot \|_\sigma)$ and $(T', \| \cdot \|_\sigma)$ preserving the quantity $\mathcal{H}^{n-m}_e((y + K) \cap A)$. Thus the choice of $T$ is irrelevant.

Given a norm $\sigma$ on $T$, the measure $\mathcal{H}_e^m$ is a scalar multiple of $\mathcal{H}^m_e$ on $T$. By choosing $T$ to be the orthogonal complement of $K$, Fubini’s theorem tells us that the right hand side of the equality is a scalar multiple of $\mathcal{H}^n_e(A)$ where the scalar factor does not depend on $A$. It follows that $C_m$ is well defined.

Remark that for $m = n$ we have
\[
C_n(\sigma) \mathcal{H}_e^n(A) = \int_{\mathbb{R}^n} \mathcal{H}_e^0(\{y\} \cap A) \, d\mathcal{H}_e^n(y) = \mathcal{H}^n_e(A).
\] (3.2)

In this case the coarea factor is identical to the Jacobian of $\sigma$ defined in [Kir].

We study Lipschitz maps with $\mathcal{H}^m$-$\sigma$-finite image spaces. A consequence of this restriction is, as we will see, that the metric differential of those maps can have rank greater than $m$ only on a set of $\mathcal{H}^n$-measure zero. Therefore it is unnecessary to define the coarea factor for seminorms of rank greater than $m$. In Section 6.1 we investigate what happens if we omit the $\sigma$-finiteness condition of the image space.

In the definition of $C_m$ we can choose $T$ to be the orthogonal complement of $K$. Consider a subset $A = A_T \times I$ where $A_T \subset T$ is $\mathcal{H}^m$-measurable and $0 < \mathcal{H}^m(A_T) < \infty$ and $I \subset K$ is a $\mathcal{H}^{n-m}$-measurable set with $\mathcal{H}^{m-n}(I) = 1$. Then we can apply the definition of $C_m$ and get
\[
C_m(\sigma) = \frac{\int_{A_T} \mathcal{H}^{n-m}_e(I) \, d\mathcal{H}_e^m(y)}{\mathcal{H}_e^n(A)} = \frac{\mathcal{H}^m(\mathcal{H}^m_e(A_T))}{\mathcal{H}^m_e(A_T)},
\] (3.3)

We will use this identity later on.

In the case $m = n$ the coarea formula is identical to the area formula. In this setting, this theorem is proven by Kirchheim in [Kir, Theorem 7 and Corollary 8].

**Theorem 3.26** (area formula). Suppose $f : A \to Y$ is a Lipschitz map where $A \subset \mathbb{R}^n$ is a $\mathcal{H}^n$-measurable subset and $Y$ is an arbitrary metric space. Then
\[
\int_A C_n(\text{md } f_x) \, d\mathcal{H}^n(x) = \int_Y \mathcal{H}^0(f^{-1}(y) \cap A) \, d\mathcal{H}^n(y).
\]
If $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{H}^n$-integrable, then
\[
\int_A g(x) C_n(\text{md } f_x) \, d\mathcal{H}^n(x) = \int_Y \int_{f^{-1}(y)} g(x) \, d\mathcal{H}^0(x) \, d\mathcal{H}^n(y).
\]
Note that $\mathcal{H}^0$ is the counting measure, i.e.
$\mathcal{H}^0(f^{-1}(y) \cap A) = \#(f^{-1}(y) \cap A)$ and $\int_{f^{-1}(y)} g(x) \, d\mathcal{H}^0(x) = \sum_{x \in f^{-1}(y)} g(x)$.

For the metric differential and the coarea factor, the chain rule is valid.

**Lemma 3.27** (chain rule).

1) Suppose $A \subset \mathbb{R}^n$ is a measurable set and $h : A \to \mathbb{R}^n$ is a Lipschitz map. Let $Y$ be an arbitrary metric space and let $f : h(U) \to Y$ be a Lipschitz map. At every $x \in A$ where $dh_x$ and $md_{f(h(x))}$ exist, $md(f \circ h)_x(v)$ exists for every $v \in \mathbb{R}^n$.

2) Let $\sigma$ be a norm on $\mathbb{R}^n$ and $L$ be an automorphism on $\mathbb{R}^n$. Then

$$C_n(\sigma L) = \det L C_n(\sigma).$$

**Proof.**

1) 

$$\lim_{|u|+|v| \to 0} \frac{|d(f \circ h(x + u), f \circ h(x + v)) - md_x(dh_x(u - v))|}{|u| + |v|}$$

$$\leq \lim_{|u|+|v| \to 0} \frac{|d(f(h(x) + dh_x u), f(h(x) + dh_x v)) - md_x(dh_x u - dh_x v)|}{|u| + |v|}$$

$$+ \lim_{|u|+|v| \to 0} \text{Lip} f \frac{|h(x + u) - h(x + v)| - |h(x) + dh_x u - (h(x) + dh_x v)|}{|u| + |v|}$$

$$= 0 + \lim_{|u|+|v| \to 0} \text{Lip} f \frac{|h(x + u) - h(x + v)| - |dh_x(u - v)|}{|u| + |v|}$$

$$= 0$$

2) Let $A \subset \mathbb{R}^n$ be a $\mathcal{H}^n$-measurable subset with $0 < \mathcal{H}^n(A) < \infty$. By the identity (3.2) we get

$$C_n(\sigma L) = \frac{\mathcal{H}^n_{\sigma L}(A)}{\mathcal{H}^n_e(A)} = \frac{\mathcal{H}^n_{\sigma L}(LA)}{\mathcal{H}^n_e(LA)} = \frac{\mathcal{H}^n_e(LA) \mathcal{H}^n_{\sigma L}(LA)}{\mathcal{H}^n_e(LA) \mathcal{H}^n_e(A)} = C_n(\sigma) \det L.$$
3.5. **COAREA FACTOR**

There is one last property of the metric differential we will use in the proof of the coarea formula.

**Lemma 3.28.** Suppose $A \subset \mathbb{R}^n$ is a measurable set, $(X, d_X)$ and $(Y, d_Y)$ are metric spaces and $f : A \to X$ and $g : A \to Y$ are Lipschitz maps. We equip $X \times Y$ with the metric $d((x,y), (x',y')) = d_X(x, x') + d_Y(y, y')$. Then at every $x \in A$ where $md f_x$ and $md g_x$ exist, the map

$$(f, g) : A \to X \times Y \quad x \mapsto (f(x), g(x))$$

is metrically differentiable and

$$md (f, g)_x = md f_x + md g_x,$$

in particular

$$\ker md (f, g)_x \cap \ker md g_x.$$

**Proof.**

$$\lim_{|u|+|v| \to 0} \frac{|d_{X \times Y}((f, g)(x + u), (f, g)(x + v)) - (md f_x(u - v) + md g_x(u - v))|}{|u| + |v|} \leq \lim_{|u|+|v| \to 0} \frac{|d_X(f(x + u), f(x + v)) - md f_x(u - v)|}{|u| + |v|}
+ \lim_{|u|+|v| \to 0} \frac{|d_Y(g(x + u), g(x + v)) - md g_x(u - v)|}{|u| + |v|}
= 0.$$  

The second statement follows immediately.
Chapter 4

Proof of the coarea formula

In this chapter we prove Theorem 1.1, the coarea formula. For a better overview we recall its setting: Given is a map \( f : \mathbb{R}^n \rightarrow Y \), where \( Y \) is an \( \mathcal{H}^m \)-\( \sigma \)-finite metric space, and an \( \mathcal{H}^n \)-measurable subset \( A \subset \mathbb{R}^n \). Our first goal is to prove (1.1).

To prove the coarea formula, we will perform the following steps. Up to a set of measure 0, we will divide \( A \) into sets \( A^i, 0 \leq i \leq n \), such that for every \( x \in A^i \), \( \text{md}_f x \) has rank \( i \). We denote the set \( \bigcup_{1 \leq i < m} A^i \) by \( A' \) and the set \( \bigcup_{m < i \leq n} A^i \) by \( A'' \). We will prove the coarea formula separately for \( A^m \) and \( A' \), and we show that \( \mathcal{H}^n(A'') = 0 \). This demonstrates that there is no need to define the coarea factor on \( A'' \).

The main task will be to handle \( A^m \). To do this, we choose a \( \lambda > 1 \) and divide \( A^m \) into countably many pieces, such that we can control \( \text{md}_f x \) on each piece in terms of \( \lambda \). With this information, we can prove the coarea formula on each of these pieces up to a factor of powers of \( \lambda \), and conclude the coarea formula on \( A^m \) up to this factor. Finally we let \( \lambda \) converge towards 1 what leads to the exact formula.

4.1 Preparation

Since we will prove the statement by dividing \( A \) into countably many measurable sets up to a set of measure zero, we need the following lemmata.

Lemma 4.1. Let \( A_1, A_2, \ldots \) be a disjoint sequence of measurable subsets of
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A. If (1.1) holds for every \( A_i, i \geq 1 \), then it also holds for \( \bigcup_{i=1}^{\infty} A_i \).

Proof.

\[
\int_A C_m(mdx) \, d\mathcal{H}^n(x) = \sum_{i=1}^{\infty} \int_{A_i} C_m(mdx) \, d\mathcal{H}^n(x)
\]

and

\[
\int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap \bigcup_{i=1}^{\infty} A_i) \, d\mathcal{H}^m(y) = \sum_{i=1}^{\infty} \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i) \, d\mathcal{H}^m(y)
\]

\[
= \sum_{i=1}^{\infty} \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i) \, d\mathcal{H}^m(y).
\]

\[\square\]

**Lemma 4.2.** Let \( A_0 \subset A \) with \( \mathcal{H}^n(A_0) = 0 \). Then (1.1) holds for \( A_0 \).

Proof. Obviously, \( \int_{A_0} C_m(mdx) \, d\mathcal{H}^n(x) = 0 \). From Theorem 2.4 it follows that

\[
\int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A_0) \, d\mathcal{H}^m(y) \leq (\text{Lip}f)^n \frac{\alpha_{n-m} \alpha_m}{\alpha_n} \mathcal{H}^n(A_0) = 0.
\]

\[\square\]

At the end of the proof we will need a slightly different version of Lemma 4.1. Since it is proven identically, we omit the proof here.

**Lemma 4.3.** Let \( A_1, A_2, \ldots \) be a disjoint sequence of measurable subsets of \( A \) and suppose \( \lambda > 1 \). If

\[
\lambda^{-1} \int_{A_i} C_m(mdx) \, d\mathcal{H}^n(x) \leq \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i) \, d\mathcal{H}^m(y)
\]

\[
\leq \lambda \int_{A_i} C_m(mdx) \, d\mathcal{H}^n(x)
\]

holds for every \( i \geq 1 \), then it also holds if \( A_i \) is replaced by \( \bigcup_{i=1}^{\infty} A_i \).
4.2 Subdivision

Since we demand \( Y \) to be \( \mathcal{H}^{m}\sigma\)-finite, we can assume \( Y \) to be \( \mathcal{H}^{m}\)-finite. Indeed, since \( Y \) is \( \mathcal{H}^{m}\sigma\)-finite we can cover it by a sequence of subsets \( \{ \tilde{Y}_i \}_{i \geq 1} \) with \( \mathcal{H}^{m}(\tilde{Y}_i) < \infty \) for all \( i \geq 1 \). Since the Hausdorff measure is Borel regular, we find a sequence of Borel subsets \( \{ Y_i \}_{i \geq 1} \) with \( \tilde{Y}_i \subset Y_i \) and \( \mathcal{H}^{m}(Y_i) = \mathcal{H}^{m}(Y_i) \). The subsets \( A_i = f^{-1}(Y_i) \cap A \) are therefore measurable for all \( i \geq 1 \). Thus Lemma 4.1 allows us to restrict us to the disjoint measurable sets \( A_i \setminus \bigcup_{k=1}^{i-1} A_k \). With the same lemma we can assume \( A \) to be bounded, and particularly \( \mathcal{H}^{n}\)-finite.

We need the following lemmata when subdividing the set \( A \).

**Lemma 4.4.** Let \( Y \) be a \( \mathcal{H}^{k}\sigma\)-finite metric space and let \( n \in \mathbb{N} \). We build the product space \( \mathbb{R}^{n} \times Y \) and equip it with the \( l^{p} \)-metric for \( p \in [1, \infty] \). Then there exists \( t > 0 \) such that for every \( V \subset \mathbb{R}^{n} \times Y \)

\[
\mathcal{H}^{n+k}(V) \leq t \cdot (\mathcal{H}^{n} \times \mathcal{H}^{k})(V).
\]

In particular, \( \mathcal{H}^{n+k} \ll (\mathcal{H}^{n} \times \mathcal{H}^{k}) \) on \( \mathbb{R}^{n} \times Y \).

**Proof.** From [Fed2, 2.10.45] for \( p = 2 \) follows that there exists a \( t > 0 \) such that \( \mathcal{H}^{n+k}(A \times B) \leq t \cdot \mathcal{H}^{n}(A)\mathcal{H}^{k}(B) \) for all \( A \subset \mathbb{R}^{n} \) and \( B \subset Y \). By definition we have

\[
(\mathcal{H}^{n} \times \mathcal{H}^{k})(V) = \inf \sum_{i} \mathcal{H}^{n}(A_{i})\mathcal{H}^{k}(B_{i})
\]

where the infimum is taken over all countable covers of \( V \) by sets \( A_{i} \times B_{i} \). For any such cover we have

\[
\mathcal{H}^{n+k}(V) \leq \sum_{i} \mathcal{H}^{n+k}(A_{i} \times B_{i}) \leq t \sum_{i} \mathcal{H}^{n}(A_{i})\mathcal{H}^{k}(B_{i}).
\]

It follows that \( \mathcal{H}^{n+k}(V) \leq t \cdot (\mathcal{H}^{n} \times \mathcal{H}^{k})(V) \). Since \( \text{id} : (\mathbb{R}^{n} \times Y, l^{2}) \to (\mathbb{R}^{n} \times Y, l^{p}) \) is \( \sqrt{2}\)-bi-Lipschitz for every \( p \in [1, \infty] \), the statement holds for all \( p \in [1, \infty] \).

**Lemma 4.5.** Let \( Y \) be an arbitrary metric space and let \( n \geq m \geq 1 \). Let \( I \subset Y \) and \( J \subset \mathbb{R}^{n-m} \) be such that \( \mathcal{H}^{m}(I) < \infty \) and \( \mathcal{H}^{n-m}(J) < \infty \). Let \( E \subset I \times J \) be an \( \mathcal{H}^{n}\)-measurable subset of \( Y \times \mathbb{R}^{n-m} \).

i) There exists a sequence \( \{ z_{i} \}_{i \in \mathbb{N}} \) in \( \mathbb{R}^{n-m} \) such that

\[
\mathcal{H}^{n}\left( E \setminus \left( p_{Y}(E \cap (Y \times \bigcup_{i \in \mathbb{N}} \{ z_{i} \})) \times \mathbb{R}^{n-m} \right) \right) = 0, \quad (4.1)
\]
where \( p_Y \) denotes the canonical projection \( Y \times \mathbb{R}^{n-m} \to Y \).

ii) There exists a Borel subset \( E_0 \subset E \) with \( \mathcal{H}^n(E_0) = \mathcal{H}^n(E) \), a sequence \( \{z_i\}_{i \in \mathbb{N}} \) in \( \mathbb{R}^{n-m} \) and a Borel partition \( \{G_i\}_{i \in \mathbb{N}} \) of \( E_0 \) such that

\[
p_Y(G_i) = p_Y(G_i \cap (Y \times \{z_i\}))
\]

for every \( i \in \mathbb{N} \).

Proof. By the previous lemma, every \( (\mathcal{H}^m \times \mathcal{H}^{n-m})\)-zeroset is also an \( \mathcal{H}^n \)-zeroset in \( Y \times \mathbb{R}^{n-m} \). Thus it suffices to construct a sequence fulfilling (4.1) with \( \mathcal{H}^n \) replaced by \( \mathcal{H}^m \times \mathcal{H}^{n-m} \). Let \( E_0 \) be a Borel set with \( E_0 \subset E \) and \( \mathcal{H}^n(E_0) = \mathcal{H}^n(E) \). A sequence fulfilling (4.1) with \( E_0 \) in place of \( E \) also fulfills (4.1) as it stands.

We construct the sequence inductively. We define \( F_0 := p_Y(E_0) \) and

\[
q_0 : J \to \mathbb{R}, \quad q_0(z) = \mathcal{H}^m(E_0 \cap (Y \times \{z\})).
\]

By Fubini’s Theorem we have

\[
\mathcal{H}^m \times \mathcal{H}^{n-m}(E_0) = \int_J q_0(z) \, d\mathcal{H}^{m-n}(z).
\]

If \( \mathcal{H}^m \times \mathcal{H}^{n-m}(E_0) = 0 \), then (4.1) is fulfilled. Otherwise we can choose a \( z_1 \in \mathbb{R}^{n-m} \) with \( q_0(z_1) > 0 \). Inductively we define now

\[
F_i := F_{i-1} \setminus p_Y(E_0 \cap (Y \times \{z_i\})), \quad E_i := E_{i-1} \cap (F_i \times \mathbb{R}^{n-m}).
\]
Remark that all the $F_i$ are Borel sets, since $E_0 \cap (Y \times \{z_i\})$ is Borel. If $\mathcal{H}^m \times \mathcal{H}^{n-m}(E_i) = 0$, we are done. Otherwise, we define the function $q_i(z) = \mathcal{H}^m(E_i \cap (Y \times \{z\}))$ and choose a $z_{i+1}$ with $q_i(z_{i+1}) > 0$.

For every such sequence $\{z_i\}_{i \in \mathbb{N}}$ we get a sequence of sets $F_0 \supset F_1 \supset F_2 \ldots$. We define $F_\infty := \bigcap_{i \in \mathbb{N}} F_i$. Note that $F_\infty$ depends on the set $\{z_i\}_{i \in \mathbb{N}}$, but not on the order of the sequence, since

$$F_\infty = F_0 \setminus p_Y(E \cap (Y \times \bigcup_{i \in \mathbb{N}} \{z_i\})).$$

We also define $E_\infty := E \cap (F_\infty \times \mathbb{R}^{n-m})$. If for any such sequence $\mathcal{H}^m \times \mathcal{H}^{n-m}(E_\infty) = 0$ then we are done. So we assume now, that for every possible sequence $\{z_i\}_{i \in \mathbb{N}}$, $\mathcal{H}^m \times \mathcal{H}^{n-m}(E_\infty) > 0$.

By the continuity of the measure $\mathcal{H}^m(F_\infty) = \lim_{i \to \infty} \mathcal{H}^m(F_i)$ holds. Set

$$c := \inf \lim_{i \to \infty} \mathcal{H}^m(F_i) = \inf \mathcal{H}^m(F_\infty)$$

where the infimum is taken over all possible sequences. By taking the union of a converging sequence (of sequences), we see that the infimum is achieved. Denote this sequence now by $\{z_i\}_{i \in \mathbb{N}}$ and the respective sets by $F_i$ and $E_i$.

We claim that $c$ has to be zero. Indeed, if $c > 0$ we can argue as follows. Like above, we define

$$q_\infty : J \to \mathbb{R}, \quad q_\infty(z) = \mathcal{H}^m(E_\infty \cap (Y \times \{z\})).$$

Again we find a $\bar{z} \in \mathbb{R}^{n-m}$ with $q_\infty(\bar{z}) > 0$. Consider now the sequence $\bar{z}, z_1, z_2, \ldots$ and the respective set $\bar{F}_\infty$. With the formula above for $F_\infty$ we get

$$\bar{F}_\infty = F_\infty \setminus p_Y(E \cap (Y \times \bar{z})).$$

It follows that

$$\mathcal{H}^m(\bar{F}_\infty) = \mathcal{H}^m(F_\infty) - q_\infty(\bar{z}) < \mathcal{H}^m(F_\infty),$$

which contradicts the minimality assumption of the sequence $\{z_i\}_{i \in \mathbb{N}}$. Thus $\mathcal{H}^m(F_\infty) = 0$ and by Fubini’s Theorem $\mathcal{H}^m \times \mathcal{H}^{n-m}(E_\infty) = 0$, in contradiction to the above assumption. This completes the proof of 1.

Set $G_i := E_0 \cap ((F_{i-1} \setminus F_i) \times \mathbb{R}^{n-m})$ for all $i \in \mathbb{N}$. Since $F_i, F_{i-1}$ and $E_0$ are Borel sets, so is $G_i$. Together with the sequence $\{z_i\}_{i \in \mathbb{N}}$ the condition in ii) is fulfilled by construction. \qed
Let $\lambda > 1$. Now we start with the division process.

By Lemma 3.23 we find a finite set of $(n - m)$-dimensional subspaces $K_i \subset \mathbb{R}^n, 1 \leq i \leq N$ and a Borel partition $\{A_i\}_{1 \leq i \leq N}$ of $A^{(m)}$ such that the orthogonal projections

$$\ker \text{md}_x f \to K_i, \quad K_i \to \ker \text{md}_x f,$$

$$(\ker \text{md}_x f)^\perp \to K_i^\perp, \quad K_i^\perp \to (\ker \text{md}_x f)^\perp$$

are all $\lambda$-bi-Lipschitz. For every $1 \leq i \leq N$, we denote $K_i^\perp$ by $W_i$. For every subspace $V \subset \mathbb{R}^n$, we denote by $p_V$ the orthogonal projection $\mathbb{R}^n \to V$.

For every such subset we define the following map

$$u_i : A_i \to Y \times K_i,$$

$$x \mapsto (f(x), p_K(x)).$$

Where we equip $Y \times K_i$ with the metric $d((y, z), (y', z')) = d_Y(y, y') + |z - z'|$. One easily sees that at every point $x \in A_i$, $\text{md} (p_K)_x$ exists and it's kernel is $W_i$. By Lemma 3.28 we conclude that $\text{md} (u_i)_x$ exists for every $x \in A_i$ and since

$$\ker \text{md} (u_i)_x = \ker \text{md}_x f \cap \ker \text{md} (p_K)_x = K_i \cap W_i = \{0\}$$

it has rank $n$.

By Lemma 3.22 we find a measurable partition $(B_j)_{j \in \mathbb{N}}$ of $A_i$ and a sequence of norms $\sigma_j$ such that $u_i : B_j \subset \mathbb{R}^n_{\sigma_j} \to Y \times K_i$ is $\lambda$-bi-Lipschitz and $\text{md} (u_i)_x(v)$ is $\lambda$-close to $\sigma_j(v)$ for every $x \in B_j$. The union of the partitions of the $A_i$ gives us a partition of $A^{(m)}$, which we denote again by $\{A_i\}_{i \geq 1}$.

**Remark 4.6.** Consider a $y \in Y$ and one such set $A_i$. Then

$$f^{-1}(y) \cap A_i = u_i^{-1}((\{y\} \times K_i) \cap u_i(A_i)).$$

Since $u_i$ is bi-Lipschitz, and $((\{y\} \times K_i) \cap u_i(A_i))$ can be identified with a subset in $\mathbb{R}^{n-m}$, $f^{-1}(y) \cap A_i$ can be covered by a Lipschitz image of a subset of $\mathbb{R}^{n-m}$. We will need this fact in the proof of Theorem 4.16.

We can apply Lemma 4.5 on $E = u_i(A_i) \subset Y \times K_i$, $I = Y$ and $J = p_K(A_i)$, where we identify $K_i$ with $\mathbb{R}^{n-m}$. We assumed $A_i$ to be bounded, what implies $\mathcal{H}^{n-m}(p_K(A_i)) < \infty$, and we assumed $\mathcal{H}^n(Y) < \infty$ as well. The lemma gives us a disjoint countable family of sets $\{G_j\}_{j \in \mathbb{N}}$ covering $\mathcal{H}^n$-almost all of $u_i(A_i)$.
and a sequence \( \{z_j\}_{j \in \mathbb{N}} \) in \( K_i \), such that \( p_Y(G_j) = p_Y(G_j \cap (Y \times \{z_j\})) \). Let \( B_j = u_i^{-1}(G_j) \). Then we have
\[
f(B_j) = f(u_i^{-1}(G_j)) = p_Y(G_j) = p_Y(G_j \cap (Y \times \{z_j\})) = f(B_j \cap (z_j + W_i)).
\]
Since \( u_i \) is bi-Lipschitz, \( \{B_j\}_{j \in \mathbb{N}} \) covers almost all of \( A_i \). Again, we consider the union of these partitions and denote it by \( \{A_i\}_{i \in \mathbb{N}} \).

Summarizing, we have a measurable partition \( \{A_i\}_{i \in \mathbb{N}} \) of the set \( A^{(m)} \), a sequence of \( n - m \)-dimensional subspaces \( K_i \subset \mathbb{R}^n \) and their orthogonal complements \( W_i \), a sequence of \( \lambda \)-bi-Lipschitz maps \( u_i = (f, p_{K_i}) : A_i \to Y \times K_i \), a sequence of norms \( \sigma_i \) and points \( z_i \in K_i \) such that for every \( i \in \mathbb{N} \) the following properties hold.

- For every \( x \in A_i \), the orthogonal projections
  \[
  \ker \text{md} f_x \to K_i, K_i \to \ker \text{md} f_x, (\ker \text{md} f_x) \perp \to W_i, W_i \to (\ker \text{md} f_x) \perp
  \]
  are all \( \lambda \)-bi-Lipschitz.
- For every \( x \in A_i \), \( \text{md}(u_i)_x \) is \( \lambda \)-close to \( \sigma_i \).
- \( f(A_i) = f(A_i \cap (z_i + W_i)) \).

**Remark 4.7.** Regarding this last property, we can consider \( A_i \cap (z_i + W_i) \) as a subset of \( \mathbb{R}^m \). Thus \( f(A_i) \) is an \( m \)-rectifiable set in \( Y \), and so is \( f(A^{(m)} \setminus A_0) = f(\bigcup_{i \in \mathbb{N}} A_i) \), where \( A_0 = A \setminus \bigcup_{i \in \mathbb{N}} A_i \) is a set of measure zero. We will see that only \( A^{(m)} \setminus A_0 \) contributes to the coarea formula. For all other sets \( A^{(k)}, k \neq m \) as well as for the zero set \( A_0 \) or the zero set where \( \text{md} f_x \) is not defined, both sides of the coarea formula equal zero.

One might think that we can therefore restrict ourselves to rectifiable target spaces. This is not the case: If \( A \) is one of the sets listed above, \( f(A) \) itself can be of positive measure, despite the fact that
\[
\int_Y \mathcal{H}^{n-m}(f^{-1} \cap A) \, d\mathcal{H}^m = 0.
\]
This is illustrated in Example 6.2 where the image of these sets cover the whole target space and \( A^{(m)} = \emptyset \).

## 4.3 Properties

In this section, we will discuss some properties of the sets \( A_i \) resulting from the division process, and we will show that the function \( y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap A^{(m)}) \) is \( \mathcal{H}^m \)-measurable.
Lemma 4.8. Suppose \( i \in \mathbb{N} \). Then for all \( x \in A_i \)

\[
\lambda^{-2m} C_m(\sigma_i|W_i) \leq C_m(\text{md } f_x) \leq \lambda^{2m} C_m(\sigma_i|W_i)
\]

holds.

Proof. Let \( T \) be the orthogonal complement of \( \ker(\text{md } f_x) \), and \( p \) the orthogonal projection \( W_i \to T \), which is \( \lambda \)-bi-Lipschitz. Using formula (3.3), we know that for any \( \mathcal{H}^m \)-finite measurable set \( D \subset W \) we have

\[
C_m(\text{md } f_x) = \frac{\mathcal{H}^m_{\text{md } f_x}(p(D))}{\mathcal{H}^m_e(p(D))} = \frac{\mathcal{H}^m_{\text{md } f_x}(D)}{\mathcal{H}^m_e(p(D))} \leq \lambda^m \frac{\mathcal{H}^m_{\sigma_i|W_i}(D)}{\mathcal{H}^m_e(D)} = \lambda^{2m} C_m(\sigma_i|W_i)
\]

and the inequality in the other direction is proved analogously.

Lemma 4.9. For every \( y \in Y \) and every \( i \in \mathbb{N} \)

\[
\lambda^{-(n-m)} \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i) \leq \mathcal{H}^{n-m}(p_{K_i}(f^{-1}(y) \cap A_i)) \leq \lambda^{n-m} \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i)
\]

holds.

Proof. Denote \( f^{-1}(y) \cap A_i \) by \( M \). Assume \( M \neq \emptyset \), else the statement is trivial. Define \( p = p_{K_i}|_M \). Each point in \( u_i(M) \) is of the form \((y, p(x))\) for an \( x \in M \). Thus \( u_i(M) \) is isometric to \( p(M) \), and since \( u_i \) is \( \lambda \)-bi-Lipschitz, \( p \) is \( \lambda \)-bi-Lipschitz as well and in particular bijective. For a \( c > 0 \), \(|v| \leq c \sigma(v)\) for all \( v \in \mathbb{R}^n \). We have

\[
\mathcal{H}^{n-m}_e(M) \leq c^{n-m} \mathcal{H}^{n-m}_e(M) \leq c^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}_e(p(M)).
\]

Thus \( M \) has finite \( \mathcal{H}^{n-m}_e \)-measure. The above estimate holds as well for all measurable subsets of \( M \).

For every \( x \in M \), we have

\[
\lim_{\substack{|v| + |w| \to 0 \\ x + v, x + w \in M}} \frac{\text{md } f_x(v - w)}{|v| + |w|} = 0.
\]
Let $\sigma_x$ be the seminorm with $\ker \sigma_x = \ker \text{md} f_x$ and $\sigma_x|_W$ is the Euclidean metric. Since $\text{md} f_x|_W$ has full rank, there is a $c > 0$ such that $\sigma_x(v) \leq c \cdot \text{md} f_x(v)$ for all $v \in \mathbb{R}^n$. It follows that for every $x \in M$, we have

$$\lim_{|v| + |w| \to 0} \frac{\sigma_x(v - w)}{|v| + |w|} \leq c \cdot \lim_{|v| + |w| \to 0} \frac{\text{md} f_x(v - w)}{|v| + |w|} = 0$$

We can rewrite this as

$$\lim_{r \searrow 0} q_x(r) = 0$$

for all $x \in M$, where

$$q_x(r) = \sup_{v, w \in B_r(0)} \frac{\sigma_x(v - w)}{|v| + |w|}.$$ 

Let now $\varepsilon > 0$. Choose an $r_\varepsilon > 0$ small enough, such that

$$\mathcal{H}^{n-m}\{x \in M : q_x(r_\varepsilon) < \varepsilon\} \geq (1 - \varepsilon)\mathcal{H}^{n-m}(M).$$

Let $M_\varepsilon = \{x \in M : q_x(r_\varepsilon) < \varepsilon\}$. Let now $\delta > 0$ with $\delta < r_\varepsilon / \lambda$. Choose a $\delta$-cover $C$ of $p(M_\varepsilon)$, such that

$$\sum_{C \in \mathcal{C}} \alpha_{n-m} \left( \frac{\text{diam} C}{2} \right)^{n-m} \leq \mathcal{H}_\delta^{n-m}(p(M_\varepsilon)) + \delta.$$ 

Since $p$ is bijective, $\{p^{-1}(C)\}_{C \in \mathcal{C}}$ covers $M_\varepsilon$. Let $C \in \mathcal{C}$, $x \in p^{-1}(C)$ and $v \in \mathbb{R}^n$ such that $x + v \in p^{-1}(C)$. It follows that $|v| \leq \lambda \delta < r_\varepsilon$ and by the choice of $M_\varepsilon$ we know that $\frac{\sigma_x(v)}{|v|} < \varepsilon$. Let $v'$ be the image of $v$ under the projection in direction $W$ onto $\ker \sigma_x$. We have

$$|v - v'| = \sigma_x(v) < \varepsilon |v|.$$ 

It follows that

$$|v'| \geq |v| - |v - v'| = |v| - \sigma_x(v) > (1 - \varepsilon)|v|,$$

and since $p_K : \ker \sigma_x \to K_i$ is $\lambda$-bi-Lipschitz, we have

$$|v| < \frac{1}{1 - \varepsilon} |v'| \leq \frac{1}{1 - \varepsilon} \lambda |p(v')| = \frac{1}{1 - \varepsilon} \lambda |p(v)|. \quad (4.2)$$
It follows that for all $C \in \mathcal{C}$, $\text{diam} \ p^{-1}(C) \leq \frac{1}{1-\varepsilon} \lambda \text{diam} \ C$. In the construction of the Hausdorff measure, this leads to

$$
\mathcal{H}^n_{n-m}(M_{\varepsilon}) \leq \left(\frac{1}{1-\varepsilon}\right)^{n-m} \lambda^{n-m} (\mathcal{H}^n_{n-m}(p(M_{\varepsilon})) + \delta).
$$

Letting $\delta \searrow 0$, this gives

$$
\mathcal{H}^{n-m}(M_{\varepsilon}) \leq \left(\frac{1}{1-\varepsilon}\right)^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}(p(M_{\varepsilon})).
$$

We can conclude that

$$
\mathcal{H}^{n-m}(M) = \mathcal{H}^{n-m}(M_{\varepsilon}) + \mathcal{H}^{n-m}(M \setminus M_{\varepsilon})
$$

$$
\leq \left(\frac{1}{1-\varepsilon}\right)^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}(p(M_{\varepsilon})) + \varepsilon \mathcal{H}^{n-m}(M)
$$

$$
\leq \left(\frac{1}{1-\varepsilon}\right)^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}(p(M)) + \varepsilon c^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}(p(M))
$$

$$
= \left(\frac{1}{1-\varepsilon}\right)^{n-m} + \varepsilon c^{n-m}\right) \lambda^{n-m} \mathcal{H}^{n-m}(p(M)).
$$

Letting $\varepsilon \searrow 0$ gives the desired inequality. Obviously the analog inequality in the other direction, $\mathcal{H}^{n-m}(p(M)) \leq \lambda^{n-m} \mathcal{H}^{n-m}(M)$, holds since $p$ is 1-Lipschitz.

**Lemma 4.10.** The function

$$
g^{(m)} : Y \rightarrow \mathbb{R}
$$

$$
y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap A^{(m)})
$$

is $\mathcal{H}^m$-measurable.

**Proof.** First we choose a $\lambda > 1$. By the above construction we have the disjoint sequence of measurable subsets $\{A_i\}_{i \in \mathbb{N}}$ covering almost all of $A^{(m)}$. Let $A_0$ denote the zeroset $A^{(m)} \setminus \bigcup_{i \in \mathbb{N}} A_i$. Then we can define for every $i \in \mathbb{N} \cup \{0\}$ the function

$$
g_i : Y \rightarrow \mathbb{R}
$$

$$
y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i)$$
Remark that $g^{(m)}(y) = \sum_{i \in \mathbb{N} \cup \{0\}} g_i(y)$. By Lemma 4.2 we know that $g_0(y) = 0$ for $\mathcal{H}^m$-almost every $y \in Y$. Thus
\[
g^{(m)}(y) = \sum_{i \in \mathbb{N}} g_i(y)
\]
for $\mathcal{H}^m$-almost every $y \in Y$.

By applying Lemma 4.5 during the subdivision of $A$, we get Borel sets $G_i \subset Y \times \mathbb{R}^n$ with $u_i(A_i) = G_i$. We can thus define the Borel measurable function
\[
\tilde{g}_i : Y \rightarrow \mathbb{R}
\]
\[
y \mapsto \mathcal{H}^{n-m}(G_i \cap \{y\} \times \mathbb{R}^{n-m})
\]
By construction $p_{K_i}(f^{-1}(y) \cap A_i)$ and $G_i \cap \{y\} \times \mathbb{R}^{n-m}$ are isometric and hence have the same $\mathcal{H}^{n-m}$-measure. Lemma 4.9, which compares $\mathcal{H}^{n-m}(p_{K_i}(f^{-1}(y) \cap A_i))$ with $\mathcal{H}^{n-m}(f^{-1}(y) \cap A_i)$, thus leads to
\[
\lambda^{-(n-m)} \tilde{g}_i(y) \leq g_i(y) \leq \lambda^{n-m} \tilde{g}_i(y)
\]
for every $y \in Y$. It follows that for the Borel measurable function $g_\lambda := \sum_{i \in \mathbb{N}} \tilde{g}_i$ we have
\[
\lambda^{-(n-m)} g_\lambda(y) \leq g^{(m)}(y) \leq \lambda^{n-m} g_\lambda(y)
\]
for $\mathcal{H}^m$-almost every $y \in Y$. Remark that $g_\lambda$ depends on the partition of $A^{(m)}$, which again depends on $\lambda$, hence the index $\lambda$. Define $\lambda_i = 1 + 1/i$. It follows that the Borel measurable function
\[
\sup_{i \in \mathbb{N}} \lambda_i^{-(n-m)} g_\lambda_i(y)
\]
agrees $\mathcal{H}^m$-almost everywhere with $g^{(m)}(y)$. Hence $g^{(m)}$ is $\mathcal{H}^m$-measurable.

Of course in the end we need the function
\[
g : Y \rightarrow \mathbb{R}, \quad y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap A)
\]
to be measurable. We will prove this at the beginning of Section 4.7.
CHAPTER 4. PROOF OF THE COAREA FORMULA

4.4 Calculation

We perform the actual calculation of the coarea formula on each $A_i$ separately. To simplify notation, we denote it by $A$. We also denote $u_i$ by $u$, $\sigma_i$ by $\sigma$, $K_i$ by $K$, $W_i$ by $W$, $z_i$ by $z_0$ and $z_i + K_i^{+}$ by $W_0$.

We will now define a map $h : A \rightarrow \mathbb{R}^n$ where we some kind of straighten the preimages $f^{-1}(y)$ of points in $Y$. For the following, we view $\mathbb{R}^n$ as $K + W$. Recall that $W_0 = W + z_0$ for some $z_0 \in K$. The facts that $f(A) = f(A \cap W_0)$ and that $f$ is injective on $A \cap W_0$ allows us to define the map. Define $h : A \rightarrow \mathbb{R}^n$ with

$$h(z + w) = z + w_{z+w}$$

where $w_{z+w}$ is the unique point in $W$ such that $f(z + w) = f(z_0 + w_{z+w})$. We also define a norm $\tau(z + w) = |z| + \sigma(w)$. Remark that $h|_{W_0} = \text{id}|_{W_0}$ and $\sigma|_{W} = \tau|_{W}$. We know that $u : A_\sigma \rightarrow Y \times K$ is $\lambda$-bi-Lipschitz, where $A_\sigma$ denotes the set $A$ endowed with the metric induced by $\sigma$. Thus we have

$$\sigma((z + w) - (z' + w')) \leq \lambda \, d(u(z + w), u(z' + w'))$$

$$= \lambda \, d(f(z + w), f(z' + w')) + |z - z'|$$

$$\leq \lambda^2 \, \sigma((z_0 + w_{z+w}) - (z_0 + w_{z'+w'})) + |z - z'|$$

$$= \lambda^2 \, \sigma(w_{z+w} - w_{z'+w'}) + |z - z'|$$

$$\leq \lambda^2 \, \tau((z + w_{z+w}) - (z' + w_{z'+w'}))$$

$$= \lambda^2 \, \tau(h(z + w) - h(z' + w')).$$

Analogously we show that $\sigma((z + w) - (z' + w')) \geq \lambda^{-2} \, \tau(h(z + w) - h(z' + w'))$ and it follows that $h : A_\sigma \rightarrow \mathbb{R}^n_\tau$ is $\lambda^2$-bi-Lipschitz. This leads to the following statement.

Lemma 4.11.

$$\lambda^{-2m} \mathcal{H}^n_e(h(A)) \leq \mathcal{H}^n_e(A) \leq \lambda^{2m} \mathcal{H}^n_e(h(A)).$$
Proof.

\[ H_c^n(h(A)) = \int_K H^m((z + W) \cap h(A)) \, dH^{n-m}(z) \]
\[ = \int_K C_m(\tau|W)^{-1} H^m_{\tau}((z + W) \cap h(A)) \, dH^{n-m}(z) \]
\[ \leq \lambda^2 \int_K C_m(\sigma|W)^{-1} H^m_{\sigma}((z + W) \cap A) \, dH^{n-m}(z) \]
\[ = \lambda^2 \int_K H^m((z + W) \cap A) \, dH^{n-m}(z) \]
\[ = \lambda^2 \lambda H^m(A). \]

Analogously we show the other direction. \[ \square \]

We denote \( f|_{W_0} \) by \( f_0 \). Note that

\[ h(f^{-1}(y) \cap A) = p_{W_0}^{-1}(f_0^{-1}(y)) \cap h(A). \]

By Lemma 4.9 we have

\[ \int_Y H^{n-m}(f^{-1}(y) \cap A) \, dH^m(y) \]
\[ \leq \lambda^{n-m} \int_Y H^{n-m}(p_K(f^{-1}(y) \cap A)) \, dH^m(y) \]
\[ = \lambda^{n-m} \int_Y H^{n-m}(h(f^{-1}(y) \cap A)) \, dH^m(y) \]
\[ = \lambda^{n-m} \int_Y H^{n-m}(p_{W_0}^{-1}(f_0^{-1}(y)) \cap h(A)) \, dH^m(y) \]
\[ = \lambda^{n-m} \int_{W_0} H^{n-m}(p_{W_0}^{-1}(x) \cap h(A)) \, C_m(\text{md} f_0x) \, dH^m_c(x) \]

where we used Theorem 3.26, the area formula, in the last equality.

By applying Lemma 4.8, Fubini’s theorem, Lemma 4.11 and Lemma 4.8
again, we get
\[ \lambda^{n-m} \int_{W_0} \mathcal{H}^{n-m}(p_{W_0}^{-1}(x) \cap h(A)) \ C_m(\text{md } f_0 x) \ d\mathcal{H}_e^m(x) \]
\[ \leq \lambda^{n+m} \int_{W_0} \mathcal{H}^{n-m}(p_{W_0}^{-1}(x) \cap h(A)) \ C_m(\sigma|W) \ d\mathcal{H}_e^m(x) \]
\[ = \lambda^{n+m} C_m(\sigma|W) \mathcal{H}^n(h(A)) \]
\[ \leq \lambda^{n+3m} C_m(\sigma|W) \mathcal{H}^n(A) \]
\[ \leq \lambda^{n+5m} \int_A C_m(\text{md } f_x) \ d\mathcal{H}^n(x). \]
Putting this together and re-introducing the index \( i \), we proved for every \( \{A_i\}, i \geq 1 \),
\[ \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i) \ d\mathcal{H}^m(y) \leq \lambda^{n+5m} \int_{A_i} C_m(\text{md } f_x) \ d\mathcal{H}^n(x). \]
Analogously, we obtain the estimate
\[ \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A_i) \ d\mathcal{H}^m(y) \geq \lambda^{-(n+5m)} \int_{A_i} C_m(\text{md } f_x) \ d\mathcal{H}^n(x). \]
Lemma 4.3 implies that the two estimates above hold for \( A^{(m)} \) as well. And this is true for every \( \lambda > 1 \). Hence
\[ \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A^{(m)}) \ d\mathcal{H}^m(y) = \int_{A^{(m)}} C_m(\text{md } f_x) \ d\mathcal{H}^n(x). \quad (4.3) \]

### 4.5 Lower rank

Let \( A' = \bigcup_{k=0}^{m-1} A^{(k)} \). By definition, \( C_m(\text{md } f_x) = 0 \) for every \( x \in A' \). So we want to show that
\[ \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A') \ d\mathcal{H}^m(y) = 0. \]
Again we can assume \( A' \) to be bounded.

We fix an \( x_0 \in A' \). The goal is now to construct a sequence \( \{r_i\}_{i \geq 1} \) such that for every \( \delta > 0 \)
\[ r_i^{-n} \mathcal{H}_\delta^m(f(B_{r_i}(x_0))) \mathcal{H}_\infty^{n-m}(B_{r_i}(x_0)) \to 0 \]
for $i \to \infty$. Let $\sigma_0 := \mathrm{md} f_{x_0}$.

For every $r > 0$, define the metric space $Y_r$ as $(Y, r^{-1}d_Y)$ and the map $\varphi_r : Y \to Y_r, y \mapsto y$. Furthermore, we define the map

$$f_r : \mathbb{R}^n \to Y_r, \quad f_r(x) = \varphi_r f(x_0 + rx).$$

**Remark 4.12.** For every set $M \in \mathbb{R}^n$ and every $r > 0$ the following identity holds:

$$\mathcal{H}^m_\delta(f(x_0 + M)) = r^m \mathcal{H}^m_{\delta/r}(f_r(1_r M)).$$

**Proof.**

$$r^m \mathcal{H}^m_{\delta/r}(f_r(1_r M)) = r^m \mathcal{H}^m_{\delta/r}(\varphi_r f(x_0 + M)) = \mathcal{H}^m_\delta(f(x_0 + M)).$$

We choose a zero sequence $\{\varepsilon_i\}_{i \geq 1}$. Let $0 < r_i < 1$, such that

$$d(f(x_0 + v), f(x_0 + w)) - \sigma_0(v - w) \leq \varepsilon_i(|v| + |w|)$$

for all $v, w$ in $B_{r_i}$. We define $E_i = f_{r_i}(B_1)$.

**Lemma 4.13.** Let $\delta \in (0, \infty]$. Then

$$\mathcal{H}^m_\delta(E_i) \to 0$$

for $i \to \infty$.

**Proof.** Let $E$ be the metric space $(B_1(0), \sigma_0)$ where we identify points with zero distance. Let $\varepsilon > 0$. Since rank $\sigma_0 < m$ we find a cover $C$ of $E$ by $N$ sets of diameter less than $d < \delta/3$ such that $Nd^m < \varepsilon$. For all $i$ large enough, we have $\varepsilon_i < d$. Then, for $v, w \in C \subset B_1(0), d_{E_i}(f_{r_i}(v), f_{r_i}(w)) < \text{diam} C + 2\varepsilon_i < 3d < \delta$. Thus we get a $\delta$-cover $C' = \{f_{r_i}(C)\}_{C \in C}$ of $E_i$ with

$$\sum_{C' \in C'} (\text{diam} C')^m < N^3 \text{d}^m < 3^m \varepsilon.$$

Therefore we have $\mathcal{H}^m_\delta(E_i) < 3^m \varepsilon$ for every $i$ large enough, which implies the desired statement. \qed
For every $\delta \in (0, \infty]$ we have

$$r_i^{-n} \mathcal{H}_{\delta}^m(f(B_{r_i}(x_0))) \mathcal{H}_{\infty}^{n-m}(B_{r_i}(x_0)) = r_i^{-m} \mathcal{H}_{\delta}^m(f(B_{r_i}(x_0))) \mathcal{H}_{\infty}^{n-m}(B_1(x_0))$$

$$= \mathcal{H}_{\delta/r_i}(f(B_1)) \mathcal{H}_{\infty}^{n-m}(B_1(x_0))$$

$$\leq \mathcal{H}_{\delta}^m(f(B_1)) \mathcal{H}_{\infty}^{n-m}(B_1(x_0))$$

$$\leq \alpha_{n-m} \mathcal{H}_{\delta}^m(E_i)$$

and with Lemma 4.13 we conclude $r_i^{-n} \mathcal{H}_{\delta}^m(f(B_{r_i}(x_0))) \mathcal{H}_{\infty}^{n-m}(B_{r_i}(x_0)) \to 0$ for $i \to \infty$.

Let now $\varepsilon > 0$. By Besicovitch’s Theorem, we can cover $A'$ with closed balls $B_{\rho_i}(x_i)$ with $x_i \in A'$ and $\rho_i < 1$, such that

$$\mathcal{H}_{\delta}^m(f(B_{\rho_i}(x_i))) \mathcal{H}_{\infty}^{n-m}(B_{\rho_i}(x_i)) \leq \rho_i^n \varepsilon$$

and

$$\sum_i \mathcal{H}^n(B_{\rho_i}(x_i)) \leq N_n \mathcal{H}^n(\tilde{A}')$$

where $\tilde{A}'$ is the set $\{x \in \mathbb{R}^n : d(x, A') \leq 1\}$ and $N_n$ is a constant, only depending on $n$. Since $A'$ is bounded, $\mathcal{H}^n(\tilde{A}')$ is finite.

With this covering of $A'$ we construct an upper estimate for $\int \mathcal{H}_{\infty}^{n-m}(f^{-1}(y) \cap A') \, d\mathcal{H}_{\delta}^m$. From

$$f^{-1}(y) \cap A' \subset \bigcup_{\{i : y \in f(B_{\rho_i}(x_i))\}} (B_{\rho_i}(x_i))$$

we conclude for every $y \in Y$

$$\mathcal{H}_{\infty}^{n-m}(f^{-1}(y) \cap A') \leq \mathcal{H}_{\infty}^{n-m}\left(\bigcup_{\{i : y \in f(B_{\rho_i}(x_i))\}} (B_{\rho_i}(x_i))\right)$$

$$\leq \sum_i \mathcal{H}_{\infty}^{n-m}(B_{\rho_i}(x_i)) \chi_{f(B_{\rho_i}(x_i))}(y).$$
It follows that
\[
\int^{*} H_{\infty}^{n-m}(f^{-1}(y) \cap A') \, dH_\delta^m \leq \sum_i H_\delta^m(f(B_{\rho_i}(x_i)))H_\infty^{n-m}(B_{\rho_i}(x_i)) \\
\leq \sum_i \rho_i^n \varepsilon \\
= \varepsilon/\alpha_n \sum_i H^m(B_{\rho_i}(x_i)) \\
\leq \varepsilon N_n H^n(\tilde{A}')/\alpha_n.
\]

This holds for every \(\varepsilon > 0\) and therefore \(\int^{*} H_{\infty}^{n-m}(A' \cap f^{-1}y) \, dH_\delta^m = 0\). This holds for every \(\delta \in (0, \infty]\). By Proposition 2.3 we know
\[
\int^{*} H_{\infty}^{n-m}(A' \cap f^{-1}y) \, dH^m(y) = \lim_{\delta \downarrow 0} \int^{*} H_{\infty}^{n-m}(A' \cap f^{-1}y) \, dH_\delta^m = 0.
\]

This implies that
\[
H_{\infty}^{n-m}(A' \cap f^{-1}y) = 0 \quad \text{for } H^m - \text{a.e. } y \in Y.
\]

By Lemma 2.6 this is equivalent to
\[
H_{\infty}^{n-m}(A' \cap f^{-1}y) = 0 \quad \text{for } H^m - \text{a.e. } y \in Y. \tag{4.4}
\]

We conclude that for \(A'\) both sides of the coarea formula equal 0.

### 4.6 Higher rank

We will verify that \(H^n(A') = H^n(\bigcup_{m<k \leq n} A^{(k)}) = 0\). Then, by Lemma 4.2, the coarea formula (1.1) is valid on \(A''\).

Assume that \(H^n(A^{(k)}) > 0\) for some \(k \in \{m+1, \ldots, n\}\). Choose \(\lambda > 1\). We perform the same two steps of the division process we did in section 4.2.

Let \(A_0\) be one of the sets resulting from this subdivision with \(H^n(A_0) > 0\). We get a norm \(\sigma\) on \(\mathbb{R}^n\), orthogonal subspaces \(K ((n-k)\)-dimensional) and \(W (k\)-dimensional) in \(\mathbb{R}^n\) and a map
\[
u : A_0 \rightarrow Y \times K \\
x \mapsto (f(x), p_K(x))\]
such that $u$ is $\lambda$-bi-Lipschitz with respect to $\sigma$ (on $\mathbb{R}^n$) and the metric

$$d((y, z), (y', z')) = d_Y(y, y') + |z - z'|$$

(on $Y \times K$). From the definition of $u$ follows for every $z \in K$ that

$$u(A_0 \cap (z + W)) = u(A_0) \cap (Y \times \{z\}).$$

We chose $A_0$ to have positive $\mathcal{H}^n$ measure. By Fubini’s Theorem we can choose a $z \in K$ such that $\mathcal{H}^k(A_0 \cap (z + W)) > 0$.

Then we can conclude:

$$\mathcal{H}^k(Y) \geq \mathcal{H}^k(u(A_0) \cap (Y \times \{z\}))$$

$$= \mathcal{H}^k(u(A_0 \cap (z + W)))$$

$$\geq \lambda^{-k} \mathcal{H}^k(A_0 \cap (z + W))$$

$$> 0.$$

Since $k > m$, it follows, that $Y$ is not $\mathcal{H}^m$-$\sigma$-finite. This contradicts the conditions. Therefore we conclude

$$\mathcal{H}^n(A'') = 0. \quad (4.5)$$

This result reconfirms that we did not need to define the coarea factor $C_m$ for seminorms with rank greater than $m$, if we restrict $Y$ to be $\mathcal{H}^m$-$\sigma$-finite.

### 4.7 Conclusion

As promised earlier, first we will prove the following lemma.

**Lemma 4.14.** The function

$$g : Y \rightarrow \mathbb{R}$$

$$y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap A)$$

is $\mathcal{H}^m$-measurable.

**Proof.** Let $g'$, $g''$ and $g_0$ be the functions $Y \rightarrow \mathbb{R}$ defined by

$$g'(y) := \mathcal{H}^{n-m}(f^{-1}(y) \cap A')$$

$$g''(y) := \mathcal{H}^{n-m}(f^{-1}(y) \cap A'')$$

$$g_0(y) := \mathcal{H}^{n-m}(f^{-1}(y) \cap (A \setminus (A' \cup A'' \cup A'))).$$
4.7. CONCLUSION

By (4.4) we know that $g'(y) = 0$ for almost every $y \in Y$. By (4.5) we know that $\mathcal{H}^n(A') = 0$ and by Proposition 3.2 we know $\mathcal{H}^n(A \setminus (A' \cup A'' \cup A^{(m)})) = 0$. Thus Lemma 4.2 shows that $g''(y) = 0$ and $g_0(y) = 0$ for almost every $y \in Y$ as well. Hence these three functions are $\mathcal{H}^m$-measurable. By Lemma 4.10 $g^{(m)}$ is $\mathcal{H}^m$-measurable as well and since
\[
g = g_0 + g' + g^{(m)} + g''
\]
so is $g$.

Proof of Theorem 1.1. By (4.3), (4.4) and (4.5) we know that (1.1) holds on $A^{(m)}, A'$ and $A''$. Together with $\mathcal{H}^n(A \setminus (A' \cup A^{(m)} \cup A'')) = 0$ Lemma 4.2 implies that (1.1) holds on $(A \setminus (A' \cup A^{(m)} \cup A''))$ too. Finally, by Lemma 4.1, the first statement (1.1) follows.

The generalization (1.2) follows from (1.1) by standard approximation procedures.

With the metric version of Stepanov’s theorem we can extend Theorem 1.1 to a larger class of maps.

**Corollary 4.15.** Suppose $f : \mathbb{R}^n \to Y$ is a measurable map where $Y$ is an $\mathcal{H}^m$-\(\sigma\)-finite metric space where $n \geq m \geq 1$. Define the set
\[
L(f) := \left\{ x : \limsup_{y \to x} \frac{d(f(y), f(x))}{|x - y|} < \infty \right\} \subset \mathbb{R}^n.
\]

If $A \subset L(f)$ is $\mathcal{H}^n$-measurable, then
\[
\int_A C_m(m f_x) \, d\mathcal{H}^n(x) = \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap A) \, d\mathcal{H}^m(y) \quad (4.6)
\]

and, given an $L^1$-integrable function $g : \mathbb{R}^n \to \mathbb{R}$,
\[
\int_A g(x) C_m(m f_x) \, d\mathcal{H}^n(x) = \int_Y \int_{f^{-1}(y) \cap A} g(x) \, d\mathcal{H}^{n-m} \, d\mathcal{H}^m(y). \quad (4.7)
\]

Proof. By Proposition 3.24 ii) we find a countable, measurable partition \(\{C_i\}_{i \in \mathbb{N}}\) of $L(f)$ such that $f|_{C_i}$ is Lipschitz. By Proposition 3.24 i) $f$ is metrically differentiable at $\mathcal{H}^n$-almost every point in $A$. We thus can prove (4.6) and (4.7) on each $C_i$ separately by applying Theorem 1.1.
Theorem 4.16 (rectifiable level sets). Let $Y$ be an $H^m$-$\sigma$-finite metric space and let $A \subset \mathbb{R}^n$ be a measurable set. Let $f : A \rightarrow Y$ be a Lipschitz map. Then for $H^m$-almost every $y \in Y$, $f^{-1}(y)$ is an $H^{n-m}$-rectifiable set.

Proof. As described in the proof above, we have a measurable decomposition $A = A' \cup A^{(m)} \cup A_0$ where $H^n(A_0) = 0$ (and $A'' \subset A_0$). By (4.4) we know that for $H^m$-almost every $y \in Y$ we have $H^{n-m}(f^{-1}(y) \cap A') = 0$ and in particular $\cap f^{-1}(y) \cap A'$ is $H^{n-m}$-rectifiable. By Lemma 4.2, the same holds for $A_0$.

By Remark 4.6, we know that there is a countable partition $\{A_i\}_{i \geq 1}$ of $A^{(m)}$ such that for every $i$, $f^{-1}(y) \cap A_i$ can be covered by a Lipschitz image of a subset of $\mathbb{R}^{n-m}$. Therefore, $f^{-1}(y) \cap A^{(m)}$ can be covered by countably many Lipschitz images of subsets of $\mathbb{R}^{n-m}$ and is therefore $H^{n-m}$-rectifiable.

We conclude that

$$f^{-1}(y) = (f^{-1}(y) \cap A') \cup (f^{-1}(y) \cap A^{(m)}) \cup (f^{-1}(y) \cap A_0)$$

is $H^{n-m}$-rectifiable for $H^m$-almost every $y \in Y$. \hfill \square
Chapter 5

Coarea formula on rectifiable sets

In this section we will extend the domain of $f$ from $\mathbb{R}^n$ to an $\mathcal{H}^n$-rectifiable space $X$. As in the Euclidean case we consider a Lipschitz map $f : X \rightarrow Y$, where $Y$ is an $\mathcal{H}^m$-$\sigma$-finite metric space. Our goal is to define a coarea factor $C_m(f, x)$ at almost every point $x \in X$ with respect to $f$ such that the coarea formula

$$\int_E C_m(f, x) \, d\mathcal{H}^n(x) = \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap E) \, d\mathcal{H}^m(y)$$

holds for every measurable subset $E \subset X$. To do this, we will fix a parametrization of $X$ by Lipschitz maps $\alpha_i$ on $U_i \subset \mathbb{R}^n$ and define the coarea factor $C_m(f, x)$ for $x \in \alpha_i(U_i)$. We will show that this factor is almost everywhere independent of the chosen parametrization.

5.1 Rectifiable sets

**Definition 5.1** (rectifiable set, rectifiable space). Suppose $X$ is a metric space and $E \subset X$.

1) The set $E$ is called $n$-rectifiable if there are countable families of $\mathcal{H}^n$-measurable sets $U_i \subset \mathbb{R}^n$ and of Lipschitz maps $f_i : U_i \rightarrow X$, such that $E \subset \bigcup_i f_i(U_i)$.  

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ii) The set $E$ is called $\mathcal{H}^n$-rectifiable if there is an $n$-rectifiable set $E' \subset X$ such that $\mathcal{H}^n(E \setminus E') = 0$.

For an $\mathcal{H}^n$-rectifiable set $E$, if $E' = \bigcup_i f_i(U_i)$ the sequence of pairs of maps and domains $\{(f_i, U_i)\}_{i \geq 1}$ is called a parametrization of $E$.

iii) The space $X$ is called $\mathcal{H}^n$-rectifiable if it is an $\mathcal{H}^n$-rectifiable subset of itself.

Often rectifiable sets are defined without the measurability condition on the sets $U_i$. Since we want to integrate over the space $X$ or the set $E$ itself, as well as over the sets $U_i$, we include this condition. Otherwise, any non-$\mathcal{H}^n$-measurable subset of $\mathbb{R}^n$ would be a $\mathcal{H}^n$-rectifiable space, which would not serve our purpose.

The following lemma allows us to specify the parametrization, i.e. the choice of the maps $\alpha_i$ and the sets $U_i$.

**Lemma 5.2.** Let $X$ be an $\mathcal{H}^n$-rectifiable space and suppose $\lambda > 1$. Then there exists a parametrization $\{(U_i, \alpha_i)\}_{i \in \mathbb{N}}$ and a sequence of norms $\{\tau_i\}_{i \in \mathbb{N}}$ on $\mathbb{R}^n$ such that

i) for all $i \in \mathbb{N}$, $\alpha_i$ is $\lambda$-bi-Lipschitz,

ii) for almost every $x \in U_i$, $\text{md} \alpha_ix$ is $\lambda$-close to $\tau_i$,

iii) the $\alpha_i(U_i)$ are pairwise disjoint subsets in $X$.

Additionally, suppose $E \subset X$ is an $\mathcal{H}^n$-measurable subset. Then

$$\alpha_i^{-1}(E \cap \alpha_i(U_i)) \subset U_i$$

is measurable.

**Proof.** Let $\{\beta_i, V_i\}_{i \geq 1}$ be a parametrization of $X$. We can extend the $\beta_i$ to Lipschitz maps $\tilde{\beta}_i : \mathbb{R}^n \to \bar{X}$ where $X$ is embedded via the Kuratowski embedding in $\bar{X} = l^\infty(X)$. For a given $\lambda > 1$ we can apply Lemma 3.22 on every $\tilde{\beta}_i$. For every $i \in \mathbb{N}$ we get Borel a partition $\{B_{ij}\}_{j \geq 1}$ and a sequence of norms $\tau_{ij}$ such that $\tilde{\beta}_i$ is $\lambda$-bi-Lipschitz on $B_{ij}$ with respect to $\tau_{ij}$ and $\text{md} \tilde{\beta}_ix$ is $\lambda$-close to $\tau_{ij}$.

We know that $\tilde{\beta}_i : B_{ij} \cap V_i \to X \subset \bar{X}$. Since $B_{ij} \cap V_i$ is measurable, we can exhaust it by compact sets $C_{ijk}$. By renaming, we get a family of maps $\tilde{\beta}_i : \mathbb{R}^n \to \bar{X}$, compact sets $C_i$ with $\tilde{\beta}_i(C_i) \subset X \subset \bar{X}$ and norms
5.2. Coarea factor on rectifiable sets

We are now able to define the coarea factor of $f$.

**Definition 5.3** (coarea factor). Suppose $X$ is an $\mathcal{H}^n$-rectifiable space, $Y$ is an $\mathcal{H}^m$-$\sigma$-finite metric space where $n \geq m \geq 1$ and $f : X \to Y$ is a Lipschitz map. Suppose $\{(U_i, \alpha_i)\}_{i \in \mathbb{N}}$ is a disjoint bi-Lipschitz parametrization of $X$. Let $x = \alpha_i(x) \in \alpha_i(U_i)$ be such that $\text{md} \circ \alpha_i \neq \emptyset$ exists and $\text{md} \alpha_i$ exists and is a norm. The $m$-coarea factor of $f$ at $x$ is defined as

$$C_m(f, x) = \frac{C_m(\text{md} \circ \alpha_i)_{\text{md}} C_{n-m}(\text{md} \alpha_i \mid \ker (\text{md} \circ \alpha_i))}{C_n(\text{md} \alpha_i)}.$$ 

This definition depends on the choice of the parametrization $\{(U_i, \alpha_i)\}_{i \geq 1}$. However, the coarea factor is almost everywhere independent of this choice as we see in the following proposition.

**Proposition 5.4.** Let $X, Y, f$ be as in Definition 5.3. Then the coarea factor $C_m(f, x)$ exists for $\mathcal{H}^n$-almost every $x \in X$. It does almost everywhere not depend on the parametrization of $X$, i.e. given two disjoint measurable bi-Lipschitz parametrizations $\{(U_i, \alpha_i)\}_{i \geq 1}$ and $\{(V_j, \beta_j)\}_{j \geq 1}$ with respective coarea factors $C_m(f, x)$ and $C'_m(f, x)$, then $C_m(f, x) = C'_m(f, x)$ for almost every $x \in X$.

**Proof.** First we show the uniqueness of the coarea factor. Let $\{(U_i, \alpha_i)\}_{i \in \mathbb{N}}$ and $\{(V_j, \beta_j)\}_{j \in \mathbb{N}}$ be two disjoint measurable parametrizations of $X$. Let $U_{ij} = \alpha_i^{-1}(U_i) \cap \beta_j(V_j)$ and $V_{ji} = \beta_j^{-1}(\alpha_i(U_i) \cap \beta_j(V_j))$. Since $\alpha_i$ and $\beta_j$ are bi-Lipschitz on $U_{ij}$ and $V_{ji}$, so is $\beta_j^{-1} \alpha_i : U_{ij} \to V_{ji}$. Then almost every point $x \in X$ has the property that it can be written as $x = \alpha_i(x_{\alpha}) = \beta_i(x_{\beta})$.

$\tau_i$ such that i) and ii) are fulfilled by $\tilde{\beta}_i$ and $C_i$. By construction we have $\mathcal{H}^n(X \setminus \bigcup_i \tilde{\beta}_i(C_i)) = 0$.

For every $i \in \mathbb{N}$ the image $\tilde{\beta}_i(C_i)$ is compact in $\bar{X}$ and therefore we can define Borel sets

$$A_i = \tilde{\beta}_i(C_i) \setminus \bigcup_{j<i} \tilde{\beta}_j(C_j).$$

The preimages $U_i = \tilde{\beta}_i^{-1}(A_i)$ are therefore measurable. Remark that by construction $U_i$ is a subset of $V_j$ for some $j \in \mathbb{N}$. Let $\alpha_i = \beta_j|_{U_i} = \tilde{\beta}_j|_{U_i}$. Now the parametrization $\{(U_i, \alpha_i)\}_{i \in \mathbb{N}}$ fulfills all the requested properties.  

for some Lebesgue density points $x_\alpha \in U_{ij}$ and $x_\beta \in V_{ji}$ and that $\beta_j^{-1} \alpha_i$ is differentiable at $x_\alpha$. Consider now one such $x \in X$ and let $h = \beta_j^{-1} \alpha_i$, i.e. $\alpha_i = \beta_j \circ h$. Remark that $dh_{x_\alpha}$ is regular, since $h : \mathbb{R}^n \to \mathbb{R}^n$ is bi-Lipschitz. For simplicity we omit the indices of $\alpha_i$ and $\beta_j$. By the uniqueness of the metric differential at density points, we have $md \alpha_{x_\alpha} = md (\beta \circ h)_{x_\alpha}$ and $md (f \circ \alpha)_{x_\alpha} = md (f \circ \beta \circ h)_{x_\alpha}$. With the chain rule for the metric differential (Lemma 3.27) we conclude

$$md \alpha_{x_\alpha} = md \beta_{x_\beta} \cdot dh_{x_\alpha}$$

and

$$md (f \circ \alpha)_{x_\alpha} = md (f \circ \beta)_{x_\beta} \cdot dh_{x_\alpha}.$$  

Obviously, if we rotate the coordinates of $U_{ij}$ or $V_{ji}$, the coarea factors do not change. Therefore we can assume $md \alpha$ and $md \beta$ to have the same kernel $K$. It follows $dh_{x_\alpha}(K) = K$. We calculate the coarea factors with the help of identity (3.3). Let $A_T \subset T := K\perp$ be measurable, finite and positive with respect to $\mathcal{H}^{n-m}$. Let $p_T$ denote the orthogonal projection onto $T$. Remark that $p_T : T' \to T$ is an isometry with respect to $\mathcal{H}^{m}_{md \beta_{x_\beta}}$ for every subspace $T'$ transversal to $K$. Just for this calculations, we use $\sigma = md (f \circ \beta)_{x_\beta}$ and $L = dh_{x_\alpha}$. By Lemma 3.27 we get

$$C_n(md \alpha_{x_\alpha}) = C_n(md \beta_{x_\beta} L) = C_n(md \beta_{x_\beta}) det L$$

and since $L(K) = K$, we can apply the lemma on $K$ as well and get

$$C_{n-m}(md \alpha_{x_\alpha} |_K) = C_{n-m}(md \beta_{x_\beta} L|_K) = C_{n-m}(md \beta_{x_\beta} |_K) det(L|_K).$$
For the third factor, we calculate directly

\[
\mathbf{C}_m(\text{md} (f \circ \alpha)_{x_\alpha}) = \mathbf{C}_m(\text{md} (f \circ \beta)_{x_\beta} \cdot dh_{x_\alpha}) \\
= \mathbf{C}_m(\sigma L) \\
= \frac{\mathcal{H}_\sigma^m(A_T)}{\mathcal{H}_e^m(A_T)} \\
= \frac{\mathcal{H}_\sigma^m(LA_T)}{\mathcal{H}_e^m(A_T)} \\
= \frac{\mathcal{H}_\sigma^m(p_T LA_T)}{\mathcal{H}_e^m(A_T)} \\
= \frac{\mathcal{H}_\sigma^m(p_T LA_T)}{\mathcal{H}_e^m(A_T)} \cdot \det(L) \\
= \mathbf{C}_m(\text{md} (f \circ \beta)_{x_\beta}) \frac{\mathcal{H}_e^m(p_T LA_T)}{\mathcal{H}_e^m(A_T)}.
\]

Putting everything together leads to

\[
\mathbf{C}_m(f, x) = \frac{\mathbf{C}_m(\text{md} (f \circ \alpha)_{x_\alpha}) \mathbf{C}_{n-m}(\text{md} \alpha_{x_\alpha} | \ker \text{md} (f \circ \alpha)_{x_\alpha})}{\mathbf{C}_n(\text{md} \alpha_{x_\alpha})} \\
= \frac{\mathbf{C}_m(\text{md} (f \circ \beta)_{x_\beta}) \mathbf{C}_{n-m}(\text{md} \beta_{x_\beta} | \ker \text{md} (f \circ \beta)_{x_\beta})}{\mathbf{C}_n(\text{md} \beta_{x_\beta})} \\
= \frac{\mathcal{H}_\sigma^m(p_T LA_T)}{\mathcal{H}_e^m(A_T)} \det(L | K) \\
\cdot \frac{\det(L)}{\det L} \\
= \mathbf{C}_m'(f, x) \cdot 1.
\]

Where we use that \(\frac{\mathcal{H}_\sigma^m(p_T LA_T)}{\mathcal{H}_e^m(A_T)} \det(L | K) = \det L\), which is a simple task in linear algebra.

It remains to show that for every disjoint bi-Lipschitz parametrization, the coarea factor exists almost everywhere. Let \(\{(U_i, \alpha_i)\}_{i \geq 1}\) be such a parametrization. By Lemma 5.2, we can choose a \(\lambda > 1\) and we find another representation \(\{(V_i, \beta_i)\}_{i \geq 1}\) of \(X\), such that for every \(i \geq 1\), \(\text{md} (\beta_i)_{x_\beta}\) is \(\lambda\)-close to a norm \(\tau_i\). Thus \(\text{md} (\beta_i)_{x_\beta}\) is a norm as well for every \(\bar{x} \in V_i\). It follows that the coarea factor, defined with the representation \(\{(V_i, \beta_i)\}_{i \geq 1}\), exists for almost every \(x \in X\). And by the uniqueness of the coarea factor the same holds if we define it with the original representation \(\{(U_i, \alpha_i)\}_{i \geq 1}\). Therefore \(\mathbf{C}_m(f, x)\) is defined for almost every \(x \in X\). □
5.3 Statement

Now we are able to state the coarea formula on rectifiable sets.

**Theorem 5.5.** Let $X$ be an $\mathcal{H}^n$-rectifiable metric space. Suppose $n \geq m \geq 1$ and suppose $Y$ is an $\mathcal{H}^m$-$\sigma$-finite metric space. Suppose $f : X \to Y$ is a Lipschitz map and $E \subset X$ is an $\mathcal{H}^n$-measurable subset. Then

$$\int_E C_m(f, x) \, d\mathcal{H}^n(x) = \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap E) \, d\mathcal{H}^m(y).$$

(5.1)

Suppose $g : X \to \mathbb{R}$ is an $\mathcal{H}^n$-integrable function. Then

$$\int_E g(x) C_m(f, x) \, d\mathcal{H}^n(x) = \int_Y \int_{f^{-1}(y) \cap E} g(x) \, d\mathcal{H}^{n-m} \, d\mathcal{H}^m(y).$$

(5.2)

Choose a $\lambda > 1$. As in the Euclidean case we will prove the coarea formula up to a factor of powers of $\lambda$. First, we choose a suitable parametrization of $E$ and then we proceed as well by dividing the set $E$ into countably many pieces. After this subdivision we perform the calculation on the preimages of the charts which then is very similar to the calculation in the Euclidean case. To work this way, we need the Lemmata 4.1, 4.2 and 4.3 to hold on rectifiable sets as well. The following three lemmata are the respective generalizations. We omit their proofs since they are identical to those in the Euclidean case.

**Lemma 5.6.** Let $E_1, E_2, \ldots$ be a disjoint sequence of measurable subsets of $E$. If (5.1) holds for every $E_i, i \in \mathbb{N}$ then it also holds for $\bigcup_{i=1}^{\infty} E_i$.

**Lemma 5.7.** Let $E_0 \subset E$ be with $\mathcal{H}^n(E_0) = 0$. Then (5.1) holds for $E_0$.

**Lemma 5.8.** Let $E_1, E_2, \ldots$ be a disjoint sequence of measurable subsets of $E$ and suppose $\lambda > 1$. If for every $i \in \mathbb{N}$

$$\lambda^{-1} \int_{E_i} C_m(f, x) \, d\mathcal{H}^n(x) \leq \int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap E_i) \, d\mathcal{H}^m(y)$$

$$\leq \lambda \int_{E_i} C_m(f, x) \, d\mathcal{H}^n(x)$$

then it also holds for $\bigcup_{i=1}^{\infty} E_i$.

5.4 Subdivision

We start the subdivision process by applying Lemma 5.2. Thus there exists a measurable, disjoint representation $\{(U_i, \alpha_i)\}_{i \geq 1}$ of $E$ and a norms $\{\tau_i\}_{i \in \mathbb{N}}$
on $\mathbb{R}^n$ such that for every $i \in \mathbb{N}$, $\alpha_i$ is $\lambda$-bi-Lipschitz and, for almost every $x \in U_i$, $\text{md} \alpha_i x$ is $\lambda$-close to $\tau_i$. We will prove the coarea formula up to a factor of powers of $\lambda$ for each $\alpha_i(U_i)$ separately. So we fix one $i \in \mathbb{N}$ and for simplicity we denote now $U_i$ by $U$, $\alpha_i$ by $\alpha$, $\tau_i$ by $\tau$ and $f \circ \alpha_i$ by $\bar{f}$.

Now we have a situation very similar to the Euclidean case. There is a Lipschitz map $\bar{f}$ defined on a measurable subset of $\mathbb{R}^n$ with values in $Y$. The problem remains to transform the calculation in a way such that in the end we get an integral on $\alpha(U)$ instead of on $U$. For this reason, we chose the representation in a way that we can control $\alpha$ and its metric differential by the norm $\tau$.

By Theorem 3.2 we know that $\text{md} \bar{f}_x$ exists for almost every $x \in U$. As in the Euclidean case we divide the subset $\{x \in U : \text{md} \bar{f}_x \text{ exists}\}$ into the three sets

$$A^{(m)} = \{x \in U : \text{rank} \text{md} \bar{f}_x = m\},$$
$$A' = \{x \in U : \text{rank} \text{md} \bar{f}_x < m\},$$
$$A'' = \{x \in U : \text{rank} \text{md} \bar{f}_x > m\}.$$

The sets $A'$ and $A''$ can be handled exactly as in the Euclidean case. The proof in Section 4.6 applied on the Lipschitz function $\bar{f}$ shows that $\mathcal{H}^n(A'') = 0$. Since $\alpha$ is Lipschitz we get

$$\mathcal{H}^n(\alpha(A'')) = 0,$$

and by Lemma 5.6 we know that (5.1) holds on $A''$.

By Definition 3.25 we get $C_m(\text{md} \bar{f}_x) = 0$ and by Definition 5.3 we get $C_m(f, \alpha(x)) = 0$ for every $x \in A'$. This leads to

$$\int_{\alpha(A')} C_m(f, x) \, d\mathcal{H}^n(x) = 0.$$

Using $C_m(\text{md} \bar{f}_x) = 0$, the coarea formula on Euclidean spaces for $\bar{f}$ and the
Lipschitz property of \( \alpha \) we get
\[
\int_{A'} C_m(\text{md } \bar{f}_x) \, d\mathcal{H}^n(x) = 0
\]
\[
\Rightarrow \int_Y H^{n-m}(\bar{f}^{-1}(y) \cap A') \, d\mathcal{H}^m(y) = 0
\]
\[
\Rightarrow \int_Y H^{n-m}(\alpha(\bar{f}^{-1}(y) \cap A')) \, d\mathcal{H}^m(y) = 0
\]
\[
\Rightarrow \int_Y H^{n-m}(f^{-1}(y) \cap \alpha(A')) \, d\mathcal{H}^m(y) = 0.
\]

Thus (5.1) holds on \( \alpha(A') \).

This leaves us with \( A^{(m)} \). The main idea of the proof is the same as in the Euclidean case. Since we calculate everything on \( U \), unsurprisingly there are some more steps in the calculation estimating the distortion by the map \( \alpha \). However, also the calculations in \( U \) itself cannot be performed equally.

First, we want to divide \( A^{(m)} \) into subsets such that for every subset \( A_i \) we have an \((n-m)\)-dimensional subspace \( K_i \subset \mathbb{R}^n \) with the property that the orthogonal projections
\[
\ker \text{md } \bar{f}_x \to K_i, K_i \to \ker \text{md } \bar{f}_x, (\ker \text{md } \bar{f}_x)^\perp \to K_i^\perp, K_i^\perp \to (\ker \text{md } \bar{f}_x)^\perp
\]
are all \( \lambda \)-bi-Lipschitz for every \( x \in A_i \). But we need this property to hold with respect to \( \tau \), not to the Euclidean norm. We need thus the following lemma.

**Lemma 5.9.** Let \( \lambda > 1 \) and let \( \tau \) be a norm on \( \mathbb{R}^n \). Then there exists a \( \mu > 1 \) such that for every two subspaces \( K \) and \( K' \) the following holds: If the orthogonal projection \( p : K \to K' \) is \( \mu \)-bi-Lipschitz with respect to the Euclidean metric, then it is \( \lambda \)-bi-Lipschitz with respect to \( \tau \).

**Proof.** Let \( K \) and \( K' \) be two \( m \)-dimensional subspaces and let \( \lambda > 1 \) be given. Let \( l, L \in \mathbb{R} \) such that \( l|v| \leq \tau(v) \leq L|v| \) for all \( v \in \mathbb{R} \). Assume that \( p : K \to K' \) is \( \mu \)-bi-Lipschitz with respect to the Euclidean metric for a \( \mu > 1 \). Then for every \( v \in K \) the following holds
\[
|v - p(v)| = \sqrt{|v|^2 - |p(v)|^2} = |p(v)| \sqrt{|v|^2 - |p(v)|^2} - 1 \leq |p(v)| \sqrt{\mu^2 - 1}
\]
\[
|v - p(v)| = \sqrt{|v|^2 - |p(v)|^2} = |v| \sqrt{1 - \frac{|p(v)|^2}{|v|^2}} \leq |v| \sqrt{1 - \frac{1}{\mu^2}}
\]
5.4. SUBDIVISION

and we can estimate

$$\frac{\tau(v)}{\tau(p(v))} \leq 1 + \frac{L}{l} \frac{|v - p(v)|}{|p(v)|} \leq 1 + \frac{L}{l} \sqrt{\mu^2 - 1}$$

$$\frac{\tau(p(v))}{\tau(v)} \leq 1 + \frac{L}{l} \frac{|v - p(v)|}{|v|} \leq 1 + \frac{L}{l} \sqrt{1 - \frac{1}{\mu^2}}$$

Both estimates converge to one as $\mu \to 1$. Thus, by choosing $\mu$ small enough, we can ensure that every such $p$ is $\lambda$-bi-Lipschitz with respect to $\tau$.

Let now $\mu = \mu(\lambda, \tau) > 1$ be as in the lemma above. By Lemma 3.28 we find a measurable partition $\{A_i\}_{1 \leq i \leq N}$ and $(m - n)$-dimensional subspaces $\{K_i\}_{1 \leq i \leq N}$ such that for every $x \in A_i$ the four projections above are $\mu$-bi-Lipschitz with respect to the Euclidean metric, and hence $\lambda$-bi-Lipschitz with respect to $\tau$.

Here we proceed as in the Euclidean case. On each $A_i$ we define the map

$$u_i : A_i \to Y \times \mathbb{R}^{n-m}$$

$$x \mapsto (\bar{f}(x), p_{K_i}(x))$$

and we endow $Y \times \mathbb{R}^{n-m}$ with the metric defined by $d((y, z), (y', z')) = d_Y(y, y') + \tau(z - z')$. Exactly as in the Euclidean case we use Lemmata 3.22 and 4.5 to get a countable measurable refinement of the partition $\{A_i\}_{1 \leq i \leq N}$ which we denote again by $\{A_i\}_{i \in \mathbb{N}}$. Additionally we get a sequence of $(n - m)$-dimensional subspaces $K_i$ with orthogonal complements $W_i$, a sequence of maps $u_i : A_i \to Y \times K_i$, a sequence of norms $\sigma_i$ and a sequence of points $z_i \in K_i$ such that the following properties hold for every $i \in \mathbb{N}$:

- $u_i : A_i \to Y \times K_i$ is $\lambda$-bi-Lipschitz with respect to $\sigma_i$ on $A_i$.
- For every $x \in A_i$, the orthogonal projections
  
  $$\ker \text{md} \bar{f}_x \to K_i, K_i \to \ker \text{md} \bar{f}_x,$$
  $$\ker \text{md} \bar{f}_x \to K_i^\perp, K_i^\perp \to \ker \text{md} \bar{f}_x \perp$$

  are $\lambda$-bi-Lipschitz with respect to the norm $\tau$.
- For every $x \in A_i$, $\text{md}(u_i)_x$ is $\lambda$-close to $\sigma_i$.
- $\bar{f}(A_i) = \bar{f}(A_i \cup (z_i + W_i))$. 


CHAPTER 5. COAREA FORMULA ON RECTIFIABLE SETS

For the calculations we need the following adaptation of Lemma 4.9:

**Lemma 5.10.** For every \( y \in Y \) and \( i \in \mathbb{N} \)

\[
\lambda^{-(n-m)}H^{n-m}_r(f^{-1}(y) \cap A_i) \leq H^{n-m}_r(p_{K_i}(f^{-1}(y) \cap A_i)) \\
\leq \lambda^{n-m}H^{n-m}_r(f^{-1}(y) \cap A_i)
\]

holds.

**Proof.** Denote \((f^{-1}(y) \cap A_i)\) by \(M\) and assume it is not empty. Let \( p = p_{K_i}|_M \). Each point in \( u_i(M) \) is of the form \((y, p(x))\) for an \( x \in M \). Thus \( u_i(M) \) is isometric to \( p(M) \) endowed with \( \tau \), and since \( u_i \) is \( \lambda\)-bi-Lipschitz, \( p \) is \( \lambda\)-bi-Lipschitz as well and in particular bijective. For a \( c > 0 \), \( \tau(v) \leq c\sigma(v) \) for all \( v \in \mathbb{R}^n \). We have

\[
H^{n-m}_r(M) \leq c^{n-m}H^{n-m}_\sigma(M) \leq c^{n-m} \lambda^{n-m}H^{n-m}_r(p_{K}(M)).
\]

Thus \( M \) has finite \( H^{n-m}_r \)-measure. The above estimate holds as well for all measurable subsets of \( M \).

For every \( x \in M \), we have

\[
\lim_{|v|+|w| \to 0, x+v,x+w \in M} \frac{\text{md} \bar{f}_x(v-w)}{\tau(v) + \tau(w)} \leq \lim_{|v|+|w| \to 0, x+v,x+w \in M} \frac{\text{md} \bar{f}_x(v-w)}{l(|v|+|w|)} = 0.
\]

Where \( l > 0 \) such that \( \tau(v) \geq l|v| \) for all \( v \in \mathbb{R} \).

Let \( \sigma_x \) be the seminorm with \( \ker \sigma_x = \ker \text{md} \bar{f}_x \) and \( \sigma_x|_W = \tau|_W \). Since \( \text{md} \bar{f}_x|_W \) has full rank, there is a \( c > 0 \) such that \( \sigma_x(v) \leq c \cdot \text{md} \bar{f}_x(v) \) for all \( v \in \mathbb{R}^n \). It follows, that for every \( x \in M \), we have

\[
\lim_{|v|+|w| \to 0, x+v,x+w \in M} \frac{\sigma_x(v-w)}{\tau(v) + \tau(w)} = 0.
\]

We can rewrite this as

\[
\lim_{r \searrow 0} q_x(r) = 0
\]

for all \( x \in M \), where

\[
q_x(r) = \sup_{v,w \in B_r(0)} \frac{\sigma_x(v-w)}{\tau(v) + \tau(w)}.
\]
Let now $\varepsilon > 0$. Choose a $r_\varepsilon > 0$ small enough, such that

$$\mathcal{H}^{n-m}\{x \in M : q_x(r_\varepsilon) < \varepsilon\} \geq (1 - \varepsilon)\mathcal{H}^{n-m}(M).$$

Let $M_\varepsilon = \{x \in M : q_x(r_\varepsilon) < \varepsilon\}$. Let now $\delta > 0$ with $\delta < r_\varepsilon/\lambda$. Choose a $\delta$-cover (with respect to $\tau$) $C$ of $p(M_\varepsilon)$, such that

$$\sum_{C \in C} \alpha_{n-m}2^{-(n-m)}(\text{diam } C)^{n-m} \leq \mathcal{H}^{n-m}_\delta(p(M_\varepsilon)) + \delta.$$ 

Since $p$ is bijective, $\{p^{-1}(C)\}_{C \in C}$ covers $M_\varepsilon$. Let $C \in C$, $x \in p^{-1}(C)$ and $v \in \mathbb{R}^n$ such that $x + v \in p^{-1}(C)$. It follows that $\tau(v) \leq \lambda \delta < r_\varepsilon$ and by the choice of $M_\varepsilon$ we know $\frac{x(v)}{\tau(v)} < \varepsilon$. Let $v'$ be the image of $v$ under the projection in direction $W$ onto $\ker \sigma_x$. Since $v - v' \in W$, we have

$$\tau(v - v') = \sigma_x(v) < \varepsilon \tau(v).$$

It follows that

$$\tau(v') \geq \tau(v) - \tau(v - v') = \tau(v) - \sigma_x(v) > (1 - \varepsilon)\tau(v)$$

and since $p_{K_i} : \ker \sigma_x \to K_i$ is $\lambda$-bi-Lipschitz, we have

$$\tau(v) < \frac{1}{1 - \varepsilon} \tau(v') \leq \frac{1}{1 - \varepsilon} \lambda \tau(p(v')) = \frac{1}{1 - \varepsilon} \lambda \tau(p(v)).$$

It follows that for all $C \in C$, $\text{diam}_\tau p^{-1}(C) \leq \frac{1}{1 - \varepsilon} \lambda \text{diam}_\tau C$. In the construction of the Hausdorff measure, this leads to

$$\mathcal{H}^{n-m}_{\tau,\delta}(M_\varepsilon) \leq \left(\frac{1}{1 - \varepsilon}\right)^{n-m} \lambda^{n-m} \left(\mathcal{H}^{n-m}_{\tau,\delta}(p(M_\varepsilon)) + \delta\right).$$

Letting $\delta \searrow 0$, this leads to

$$\mathcal{H}^{n-m}_\tau(M_\varepsilon) \leq \left(\frac{1}{1 - \varepsilon}\right)^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}_\tau(p(M_\varepsilon)).$$

We conclude

$$\mathcal{H}^{n-m}_\tau(M) = \mathcal{H}^{n-m}_\tau(M_\varepsilon) + \mathcal{H}^{n-m}_\tau(M \setminus M_\varepsilon)$$

$$\leq \left(\frac{1}{1 - \varepsilon}\right)^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}_\tau(p(M_\varepsilon)) + \varepsilon \mathcal{H}^{n-m}_\tau(M)$$

$$\leq \left(\frac{1}{1 - \varepsilon}\right)^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}_\tau(p(M)) + \varepsilon c^{n-m} \lambda^{n-m} \mathcal{H}^{n-m}_\tau(p(M))$$

$$= \left(\frac{1}{1 - \varepsilon}\right)^{n-m} + \varepsilon c^{n-m}\lambda^{n-m} \mathcal{H}^{n-m}_\tau(p(M)).$$
Letting \( \varepsilon \to 0 \) gives
\[
\mathcal{H}^{n-m}_\tau(M) \leq \lambda^{n-m} \mathcal{H}^{n-m}_\tau(p(M)).
\]

Unlike in the Euclidean case, \( p \) does not have to be 1-Lipschitz with respect to \( \tau \). However, the other inequality \( \mathcal{H}^{n-m}_\tau(p(M)) \leq \lambda^{n-m} \mathcal{H}^{n-m}_\tau(M) \) can be proven with the same techniques used in the proof above.

Note that Lemma 4.8 holds as well for \( \bar{f} \), as it does for \( f \) in the Euclidean case. Because the metric differential is independent of the norm on the domain the two cases are identical. However, additionally we need a slight adaptation of this lemma as well.

**Lemma 5.11.**
\[
\lambda^{-2(n-m)} C_{n-m}(\tau|K) \leq C_{n-m}(\tau|_{\text{ker md } \bar{f}_x}) \leq \lambda^{2(n-m)} C_{n-m}(\tau|K),
\]

**Proof.** Let \( D \subset \ker \text{md } \bar{f}_x \) be a measurable set with positive finite measure. Then by formula (3.3) we get
\[
C_{n-m}(\tau|_{\text{ker md } \bar{f}_x}) = \frac{\mathcal{H}^{n-m}_\tau(D)}{\mathcal{H}^{n-m}_e(D)} \leq \lambda^{2(n-m)} \frac{\mathcal{H}^{n-m}_\tau(p_K(D))}{\mathcal{H}^{n-m}_e(p_K(D))} = \lambda^{2(n-m)} C_{n-m}(\tau|K).
\]

And the other inequality is proven analogously.

Before we start calculating the integral of the function \( y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap E) \), we show that this function is actually measurable.

**Lemma 5.12.** The function
\[
g : Y \to \mathbb{R}, \quad y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap E)
\]
is \( \mathcal{H}^m \)-measurable.

**Proof.** First we choose a \( \lambda > 1 \). Let \( \{(U_i, \alpha_i)\}_{i \in \mathbb{N}} \) the respective parametrization of \( E \) with norms \( \{\tau_i\}_{i \in \mathbb{N}} \) we can find by Lemma 5.2, and let \( A'_i, A''_i \) and
\{A_{ij}\}_{j \in \mathbb{N}} \text{ the covering of } U_i \text{ obtained by the above process. We remark that for the sets } A'_i \text{ and } A''_i, \text{ where } \text{md } \bar{f}_x \text{ has rank lower and larger than } m \text{ respectively,}

\mathcal{H}^{n-m}(f^{-1}(y) \cap A'_i) = \mathcal{H}^{n-m}(f^{-1}(y) \cap A''_i) = 0

\text{for almost every } y \in Y \text{ as proven above. We estimate for almost every } y \in Y, \text{ with the help of Lemmata 4.9 and 5.10,}

\mathcal{H}^{n-m}(f^{-1}(y) \cap E) = \mathcal{H}^{n-m}(f^{-1}(y) \cap \bigcup_{i,j \in \mathbb{N}} \alpha_i(A_{ij}))

\leq \lambda^{n-m} \sum_{i,j \in \mathbb{N}} \mathcal{H}_{\tau_i}^{n-m}(f^{-1}(y) \cap A_{ij})

\leq \lambda^{2(n-m)} \sum_{i,j \in \mathbb{N}} \mathcal{H}_{\tau_i}^{n-m}(p|K_i(f^{-1}(y) \cap A_{ij}))

= \lambda^{2(n-m)} \sum_{i,j \in \mathbb{N}} C_{n-m}(\tau_i|K_i) \mathcal{H}_e^{n-m}(p|K_i(f^{-1}(y) \cap A_{ij}))

\leq \lambda^{3(n-m)} \sum_{i,j \in \mathbb{N}} C_{n-m}(\tau_i|K_i) \mathcal{H}_e^{n-m}(f^{-1}(y) \cap A_{ij})

\text{and analogously}

\mathcal{H}^{n-m}(f^{-1}(y) \cap E) \geq \lambda^{-3(n-m)} \sum_{i,j \in \mathbb{N}} C_{n-m}(\tau_i|K_i) \mathcal{H}_e^{n-m}(f^{-1}(y) \cap A_{ij}).

\text{Denote the function } \sum_{i,j \in \mathbb{N}} C_{n-m}(\tau_i|K_i) \mathcal{H}_e^{n-m}(f^{-1}(y) \cap A_{ij}) \text{ by } g_\lambda(y). \text{ By Lemma 4.14, } g_\lambda(y) \text{ is } \mathcal{H}^m \text{-measurable for every } \lambda > 1.

\text{Choose a sequence } \lambda_j = 1 + \frac{1}{j}. \text{ Since}

\mathcal{H}^{n-m}(f^{-1}(y) \cap E) = \lim_{j \to \infty} g_{\lambda_j}(y)

\text{for almost every } y \in Y, \text{ the function } y \mapsto \mathcal{H}^{n-m}(f^{-1}(y) \cap E) \text{ is } \mathcal{H}^m \text{-measurable as well.} \quad \square

5.5 Calculation

\text{To do the calculation we fix again one such subset } A_i \text{ and denote it by } A. \text{ Denote also } \alpha(A) \text{ by } E, \sigma_i \text{ by } \sigma, K_i \text{ by } K, K_i^\perp \text{ by } W, z_i \text{ by } z_0 \text{ and } z + W_i \text{ by } W_0.
As in the Euclidean case we define the map \( h : A \to \mathbb{R}^n \) by
\[
h(z + w) = z + w_{z+w}
\]
where \( w_{z+w} \) is the unique point in \( W \) satisfying \( \bar{f}(z + w) = \bar{f}(z_0 + w_{z+w}) \). We also need the following adaption of Lemma 4.11

**Lemma 5.13.**
\[
\lambda^{-2m} \mathcal{H}^n_\tau(h(A)) \leq \mathcal{H}^n_\tau(A) \leq \lambda^{2m} \mathcal{H}^n_\tau(h(A)).
\]

**Proof.** With Lemma 4.11 we get
\[
\lambda^{-2m} \mathcal{H}^n_\tau(h(A)) = \lambda^{-2m} C_n(\tau) \mathcal{H}^n_e(h(A)) \leq C_n \mathcal{H}^n_e(A) = \mathcal{H}^n_\tau(A)
\]
and the analogue for the other inequality. \( \square \)

We denote \( f|_{W_0} \) by \( \bar{f}_0 \). Now we can start the calculation.

By the bi-Lipschitz property of \( \alpha \) and Lemma 5.10 we have
\[
\int_Y \mathcal{H}^{n-m}(f^{-1}(y) \cap E) \, d\mathcal{H}^m(y)
\]
\[
\leq \lambda^n \int_Y \mathcal{H}^{n-m}(\bar{f}^{-1}(y) \cap A) \, d\mathcal{H}^m(y)
\]
\[
\leq \lambda^{2n-m} \int_Y \mathcal{H}^{n-m}(p_K(\bar{f}^{-1}(y) \cap A)) \, d\mathcal{H}^m(y)
\]
\[
= \lambda^{2n-m} \int_Y \mathcal{H}^{n-m}(h(\bar{f}^{-1}(y)) \cap h(A)) \, d\mathcal{H}^m(y)
\]
\[
= \lambda^{2n-m} \int_Y \mathcal{H}^{n-m}(p_{W_0}(\bar{f}^{-1}(y)) \cap h(A)) \, d\mathcal{H}^m(y)
\]
\[
= \lambda^{2n-m} \int_{W_0} \mathcal{H}^{n-m}_e(p_{W_0}^{-1}(x) \cap h(A)) \, C_{n-m}(\tau|_K) C_m(m \, d\bar{f}_0) \, d\mathcal{H}^m_e(x)
\]
where we used Theorem 3.26, the area formula, twice to obtain the last equality.

By applying Lemma 4.8, Fubini's Theorem, Lemma 5.13, Lemma 4.8 again,
5.5. **CALCULATION**

Lemma 5.11 and the area formula, we get

\[
\lambda^{2n-m} \int_{W_0} \mathcal{H}^{n-m}_e \left( \frac{p^{-1}}{W_0} (x) \cap h(A) \right) C_{n-m}(\tau|K) C_m(\sigma|W) d\mathcal{H}^m_e (x) \\
\leq \lambda^{2n+m} \int_{W_0} \mathcal{H}^{n-m}_e \left( \frac{p^{-1}}{W_0} (x) \cap h(A) \right) C_{n-m}(\tau|K) C_m(\sigma|W) d\mathcal{H}^m_e (x) \\
= \lambda^{2n+m} C_{n-m}(\tau|K) C_m(\sigma|W) \mathcal{H}^n (h(A)) \\
\leq \lambda^{2n+3m} C_{n-m}(\tau|K) C_m(\sigma|W) \mathcal{H}^n (A) \\
\leq \lambda^{2n+5m} \int_A C_{n-m}(\tau|K) C_m(\text{md} \bar{f}_x) d\mathcal{H}^n (x) \\
\leq \lambda^{4n+3m} \int_A C_{n-m}(\tau|\ker \text{md} \bar{f}_x) C_m(\text{md} \bar{f}_x) d\mathcal{H}^n (x) \\
\leq \lambda^{4n+3m} \int_E \frac{C_{n-m}(\tau|\ker \text{md} \bar{f}_x) C_m(\text{md} \bar{f}_x)}{C_n(\text{md} \alpha_x)} d\mathcal{H}^n (x).
\]

So we proved for every \( E_i = \alpha(A_i) \)

\[
\int_Y \mathcal{H}^{n-m} (f^{-1}(y) \cap E_i) d\mathcal{H}^m (y) \leq \lambda^{4n+3m} \int_{E_i} C_m (f, x) d\mathcal{H}^n (x) \tag{5.3}
\]

and analogously we get

\[
\int_Y \mathcal{H}^{n-m} (f^{-1}(y) \cap E_i) d\mathcal{H}^m (y) \geq \lambda^{-(4n+3m)} \int_{E_i} C_m (f, x) d\mathcal{H}^n (x). \tag{5.4}
\]

Now recall the whole problem. We started with a rectifiable space \( X \) such that it suffices the conditions in Theorem 5.5 and a measurable subset \( E \subset X \). For every \( \lambda \) we constructed a measurable parametrization \( \{ (U_i, \alpha_i) \}_{i \in \mathbb{N}} \). Since the parametrization depends on \( \lambda \), we first have to put the pieces together and prove (5.3) and (5.4) on the whole set \( E \). Not till then we can let \( \lambda \) converge to one.

Every \( U_i \) we splitted into \( A_i^{(m)} \), \( A_i' \), \( A_i'' \) and a set of \( \mathcal{H}^n \)-measure zero \( A_{0,i} \). We showed that (5.1) holds on \( \alpha_i (A_i'), \alpha_i (A_i'') \) and by Lemma 5.7 it also holds on \( \alpha_i (A_{0,i}) \). Up to a set of measure zero we then subdivided \( A_i^{(m)} \) into subsets \( A_{ij} \) and proved (5.3) and (5.4) on \( \alpha_i (A_{ij}) \).
On every set where (5.1) holds, obviously (5.3) and (5.4) hold as well. Define $E_0$ to be the zero set $E \setminus \bigcup_{i=1}^{\infty} \alpha_i(U_i)$. Lemma 5.8 implies then, that

$$E = E_0 \cup \bigcup_{i=1}^{\infty} \left( \alpha_i(A_i^{(m)}) \cup \alpha_i(A_i') \cup \alpha_i(A_{0,i}) \cup \bigcup_{j=i}^{\infty} \alpha_i(A_{ij}) \right)$$

satisfies (5.3) and (5.4) as well. This holds for every $\lambda > 1$, and hence (5.1) holds on $E$. Here as well, the generalization (5.2) follows by standard approximation procedures from (5.1). The proof of Theorem 5.5 is complete.

As in the Euclidean case, we can prove a theorem about rectifiable level sets.

**Theorem 5.14** (rectifiable level sets). Let $X$ be an $\mathcal{H}^n$-rectifiable metric space and let $Y$ be an $\mathcal{H}^m$-$\sigma$-finite metric space. Suppose $f : X \rightarrow Y$ is a Lipschitz map. Then for $\mathcal{H}^m$-almost every $y \in Y$, $f^{-1}(y)$ is an $\mathcal{H}^{n-m}$-rectifiable set.

**Proof.** Fix any $\lambda > 1$. Then Lemma 5.2 provides a parametrization $\{(U_i, \alpha_i)\}_{i \in \mathbb{N}}$ and a sequence of norms $\{\tau_i\}_{i \in \mathbb{N}}$ on $\mathbb{R}^n$, such that the maps $\alpha_i$ are $\lambda$-bi-Lipschitz with respect to $\tau_i$. Since different norms on $\mathbb{R}^n$ are equivalent, the maps $\alpha_i$ are Lipschitz with respect to the Euclidean norm.

By Theorem 4.16 the set $\alpha^{-1}(f^{-1}(y)) \subset U_i$ is $\mathcal{H}^{n-m}$-rectifiable for every $i \in \mathbb{N}$; and since rectifiability is preserved under Lipschitz maps, the subset $f^{-1}(y) \cap \alpha_i(U_i) \subset X$ is $\mathcal{H}^{n-m}$-rectifiable as well.

Since the set $X_0 = X \setminus \bigcup \alpha_i(U_i)$ has $\mathcal{H}^n$ measure zero, we have

$$\mathcal{H}^{n-m}(f^{-1}(y) \cap X_0) = 0$$

for $\mathcal{H}^m$-almost every $y \in Y$. Thus we can conclude that

$$f^{-1}(y) = (f^{-1}(y) \cap X_0) \cup \bigcup_{i \in \mathbb{N}} (f^{-1}(y) \cap \alpha_i(U_i))$$

is $\mathcal{H}^{n-m}$-rectifiable for $\mathcal{H}^m$-almost every $y \in Y$. \qed
Chapter 6

Examples

In this chapter we will construct two examples. In the first one we observe what happens if we omit the restriction of $\sigma$-finiteness to the target space. We show that the coarea formula cannot be extended to arbitrary, not $\sigma$-finite metric spaces. We will construct the example with $n = 2$ and $m = 1$ such that the image is not $H^1$-$\sigma$-finite. However, we ensure that at almost every point $x$ in the domain we have $\text{rank}_m f_x = 1$; and thus the coarea factor is still well defined. We can calculate both sides of the coarea formula though, and we see that they do not coincide.

In a second step we modify this example in such a way that the target space is 1-Hausdorff-dimensional. The coarea formula is still well defined and still fails. Thus in Theorem 1.1 we cannot replace the $H^m$-$\sigma$-finiteness condition by the weaker condition of $m$-Hausdorff-dimensionality.

The second example has a purely unrectifiable set of positive measure as target space. It shows that there are examples one cannot calculate with the coarea formula for rectifiable target spaces.

6.1 Omitting the $\sigma$-finiteness

Example 6.1. Let $C \subset I = [0, 1]$ be a cantor set with $H^1$-measure $\frac{1}{2}$. To construct this set we use the standard construction for the cantor set, while we choose the width of the open segments we remove at step $n$, such that their union has measure $2^{-n-1}$, starting with step 1. This leaves us with a set of measure $\frac{1}{2}$. After the $n^{th}$ step, each connected component of $C_n$ has measure
less than $2^{-n}$. We conclude that each connected component of $C$ consists of a single point. Let

$$
\gamma : \mathbb{R} \to \mathbb{R}, \quad \gamma(t) = \mathcal{H}^1(I \cap C^c \cap (-\infty, t]).
$$

Obviously, $\gamma$ is nondecreasing. For $s \leq t \in \mathbb{R}$ we have

$$
\gamma(t) - \gamma(s) = \mathcal{H}^1(I \cap C^c \cap (s,t]) \leq \mathcal{H}^1((s,t]) = t - s.
$$

Thus $\gamma$ is $1$-Lipschitz. By Rademacher’s theorem we know that $\dot{\gamma}(t)$ exists almost everywhere. By the density theorem for the Lebesgue measure, we know that almost every $t \in C$ is a density point of $C$, i.e.

$$
\lim_{\varepsilon \to 0} \frac{\mathcal{H}^1((t-\varepsilon,t+\varepsilon) \cap C)}{2\varepsilon} = 1.
$$

Thus almost every $t \in C$ is a density point where $\dot{\gamma}$ exists. For such $t$ we have

$$
\dot{\gamma}(t) = \frac{\dot{\gamma}(t)^+ + \dot{\gamma}(t)^-}{2} = \lim_{\varepsilon \to 0} \frac{\gamma(t + \varepsilon) - \gamma(t) + \gamma(t) - \gamma(t - \varepsilon)}{2\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathcal{H}^1((t - \varepsilon,t + \varepsilon) \cap C^c)}{2\varepsilon} = \lim_{\varepsilon \to 0} 1 - \frac{\mathcal{H}^1((t - \varepsilon,t + \varepsilon) \cap C)}{2\varepsilon} = 0.
$$

Thus $\dot{\gamma}(t) = 0$ for almost every $t \in C$. Suppose now $s,t \in C$ with $s < t$. Since the connected components of $C$ consist only of single points and by construction of $C$, there exists an open set $(u,v) \subset C^c$ with $s < u < v < t$, which obviously has positive measure. It follows that $\gamma(s) \neq \gamma(t)$. We conclude that $\gamma : C \to \gamma(C)$ is bijective. Apply now the coarea formula for $n = 2, m = 1, A = C \times I$ on the following map:

$$
f : \mathbb{R}^2 \to \mathbb{R}^2, f(t,y) = (\gamma(t),y).
$$

We can calculate the metric differential and get for almost every $x = (t,y) \in A$

$$
\text{md} f_x = \sigma
$$

where $\sigma(v_1,v_2) = |v_2|$ for every $(v_1,v_2) \in \mathbb{R}^2$. We can see easily that $C_1(\sigma) = 1$. Writing down the coarea formula we get for the left hand side

$$
\int_A C_1(\text{md} f_x) \, d\mathcal{H}^2(x) = \int_A 1 \, d\mathcal{H}^2 = \mathcal{H}^2(C \times I) = 1/2
$$
and for the right hand side

$$\int_{\mathbb{R}^2} \mathcal{H}^1(f^{-1}(y) \cap A) \, d\mathcal{H}^1(y) = \int_{f(A)} \mathcal{H}^1(\{\text{pt}\}) \, d\mathcal{H}^1(y) = \int_{f(A)} 0 \, d\mathcal{H}^1(y) = 0$$

since $f : A \to f(A)$ is bijective. We can see that the coarea formula does not hold in this example, even though $\text{md } f_x$ has rank 1 for $\mathcal{H}^2$-almost every $x \in A$. In Theorem 1.1 it is therefore not possible to omit the $\sigma$-finiteness condition on the target space $Y$.

We will now modify this example such that the target space is 1-Hausdorff-dimensional, while the coarea formula still fails. To do this we specify our Cantor set $C$.

At first we define a different Cantor set $D$. We start with $D_0 := [0, \frac{1}{2}]$. For $n \geq 1$ we define the sets $D_n$ iteratively in the following way: The set $D_n$ is defined as the remaining set, when we remove from each connected component of $D_{n-1}$ a centered open interval of a certain length $l_n > 0$, and we define $D := \bigcap_{n \geq 0} D_n$. We choose the lengths $\{l_n\}_{n \geq 1}$ such that $D_n$ has positive measure for $n \geq 1$ and the Hausdorff dimension of $D$ equals zero. It is well known that such a choice is possible.

Now we define $C$. Put $C_0 := I$. For $n \geq 1$ we define the sets $C_n$ to be the remaining set, when we remove from each connected component of $C_{n-1}$ a centered open interval of length $l_n$ as chosen above, and again we define $C := \bigcap_{n \geq 0} C_n$. From the construction of $D$ it follows that

$$\sum_{n=1}^{\infty} 2^{n-1} l_n = \frac{1}{2}$$

and therefore $\mathcal{H}^1(C) = \frac{1}{2}$.

There is a natural bijection $\beta$ between the connected components of $C$ and $D$. Since these components are single points, $\beta$ is a map $\beta : C \to D$. We can characterize this bijection by the condition that if $t$ lies in the $m$th connected component of $C_n$, $\beta(t)$ has to lie in the $m$th connected component of $D_n$.

Regarding the constructions of $C, D$ and $\beta$, we see that for $t \in C$

$$\gamma(t) = \sum_{(a,b) \in J_t} b - a = \beta(t)$$

where $J_t$ is the set of all open segments $(a, b)$ fulfilling $a, b \in C, (a, b) \cap C = \emptyset$ and $b \leq t$. 
Therefore we have \( \gamma(C) = \beta(C) = D \). In the example above this guarantees that \( f(A) \subset D \times I \).

We can therefore restrict ourselves to the target space \( D \times I \). This does not affect any of the calculations. By construction, \( D \) has Hausdorff dimension zero. By Lemma 4.4 it follows that the Hausdorff dimension of \( D \times I \) is less than or equal to one, and since \( D \times I \) contains an isometric copy of \( I \) it is greater than or equal to one. Thus \( D \times I \) is 1-Hausdorff-dimensional. Remark that since \( D \) is uncountable, \( D \times I \) consists of uncountably many copies of \( I \) and is therefore not \( \mathcal{H}^1 \)-\( \sigma \)-finite.

So we constructed an example where the target space is not \( \mathcal{H}^m \)-\( \sigma \)-finite any more, but it is \( m \)-Hausdorff-dimensional, and the coarea formula fails. This shows that we cannot relax the condition in Theorem 1.1 from \( \mathcal{H}^m \)-\( \sigma \)-finiteness to \( m \)-Hausdorff-dimensionality.

### 6.2 Metric trees

The second example will be the one we referred to in Remark 4.7. We will construct a space \( Y \) with Hausdorff dimension 2 and positive \( \mathcal{H}^2 \)-measure, together with a Lipschitz map \( f : X \to Y \) where \( X \) is an open set in \( \mathbb{R}^3 \), and the set \( A^{(2)} \subset X \), i.e. the set where \( \text{md} f_x \) exists and has rank 2, is empty.

In this example, \( Y \) will be a totally 2-unrectifiable space. It does not possess any 2-rectifiable subsets of positive \( \mathcal{H}^2 \)-measure. Therefore, in this example, we cannot apply Federer’s coarea formula [Fed2, 3.2.12]. However, \( Y \) itself has positive and finite \( \mathcal{H}^2 \)-measure.

In preparation for this example, we investigate the metric differential of Lipschitz maps \( f : A \to Y \) where \( A \subset \mathbb{R}^n \) and \( Y \) is a metric tree. We will use the following notation: For any two points \( y, y' \) in a metric space \( Y \), \( [y, y'] \subset Y \) denotes a geodesic segment joining \( y \) and \( y' \).

**Definition 6.1** (metric tree). Let \( Y \) be a geodesic metric space, i.e. a metric space such that the metric is intrinsic and for every two points \( y, y' \) there exists a geodesic joining \( y \) and \( y' \). \( Y \) is called a metric tree if for every three points \( y_1, y_2, y_3 \in Y \) and any geodesics joining them, \( [y_1, y_2] \) is contained in \( [y_1, y_3] \cup [y_2, y_3] \).

Metric trees have several nice properties:

**Lemma 6.2.** Suppose \( Y \) is a metric tree.

i) Then \( Y \) is uniquely geodesic.
ii) Every continuum (compact connected set) connecting two points $y, y' \in Y$ contains the geodesic $[y, y']$.

iii) If $f : S^1 \rightarrow Y$ is continuous, then there exist $x, x' \in S^1$ such that $f(x) = f(x')$ and $d_{S^1}(x, x') \geq \pi/2$, where $d_{S^1}$ denotes the intrinsic metric on $S^1$.

Proof. i): Assume $y_1, y_2 \in Y$ such that there exist two different geodesics $\gamma, \gamma'$ joining $y_1$ and $y_2$. Then without loss of generality, there exists a $y_3 \in \gamma \setminus \gamma'$. We see that $y_3$ divides $\gamma$ into two geodesic segments $\gamma_1$ and $\gamma_2$ which join $y_3$ with $y_1$ and $y_2$ respectively, and we have $\gamma = \gamma_1 \cup \gamma_2$. The geodesic triangle $y_1, y_2, y_3$ with edges $\gamma_1, \gamma_2, \gamma'$ must fulfill $\gamma_1 \cup \gamma_2 \subset \gamma'$, thus $\gamma \subset \gamma'$. Since $\gamma'$ is closed and $y_3 \notin \gamma'$, we have $d(y_3, \gamma') > 0$. This implies that the length of $\gamma$ is strictly larger than the length of $\gamma'$, contradicting to $\gamma$ being a geodesic.

ii): Suppose $y_1, y_2 \in Y$ and $\gamma$ is a continuum connecting $y_1$ and $y_2$. Assume that there exists a point $y_3 \in [y_1, y_2]$ such that $y_3 \notin \gamma$. Since $\gamma$ is closed, $d(y_3, \gamma) > 0$ and there exists an $\varepsilon > 0$ such that $B_{2\varepsilon}(y_3) \cap \gamma = \emptyset$.

Suppose $y, y' \in \gamma$ with $d(y, y') \leq \varepsilon$. Considering the geodesic triangle with vertices $y, y', y_3$, we see that $[y, y']$ has length at most $\varepsilon$, whereas the other two sides have length larger than $2\varepsilon$. It follows, that

$$[y, y_3] \cap B_\varepsilon(y_3) = [y', y_3] \cap B_\varepsilon(y_3).$$

Since $\gamma$ is compact and connected, we have $[y_1, y_3] \cap B_\varepsilon(y_3) = [y_2, y_3] \cap B_\varepsilon(y_3)$, which is in contradiction to $y_3$ lying on the geodesic $[y_1, y_2]$.

iii): Assume $f : S^1 \rightarrow X$ is a continuous map such that every two points $x, x' \in S^1$ with $d_{S^1}(x, x') \geq \pi/2$ are mapped to two distinct points. Consider the four points $a = (1, 0), b = (0, 1), c = (-1, 0), d = (0, -1)$ in $S^1$. By assumption they are mapped to pairwise different points. Denote by $\langle ab \rangle$ the shortest arc connecting $a$ and $b$ in $S^1$. By assumption we have $f(\langle ab \rangle) \cap f(\langle cd \rangle) = \emptyset$. Thus there exists a point $y \in [f(a), f(d)]$ such that $y \notin f(\langle ab \rangle) \cup f(\langle cd \rangle)$.

Consider the continuum

$$f(\langle ab \rangle) \cup [f(b), f(c)] \cup f(\langle cd \rangle).$$

This continuum connects $f(a)$ and $f(d)$, and by ii) it contains $[f(a), f(d)]$ and therefore $y$ as well. Since $y \notin f(\langle ab \rangle) \cup f(\langle cd \rangle)$ we have $y \in [f(b), f(c)]$. Again by ii) it follows that $y \in f(\langle ad \rangle) \cap f(\langle bc \rangle)$. This contradicts the assumption. \qed

In [Lan, Theorem 1.2], U. Lang proved an extension theorem which can be applied to metric trees:
Proposition 6.3. Let $X$ be a metric space and $Y$ a complete metric tree. Then every $\lambda$-Lipschitz map $f : S \to Y$ defined on a subset of $X$ possesses a $\lambda$-Lipschitz extension $\bar{f} : X \to Y$.

Definition 6.4 (purely unrectifiable). Given $m \in \mathbb{N}$, a metric space $Y$ is called purely $m$-unrectifiable if every $m$-rectifiable subset $E \subset Y$ has $\mathcal{H}^m$-measure zero.

Lemma 6.5. Every metric tree is purely 2-unrectifiable.

Proof. Suppose $Y$ is a metric tree and $f : A \to Y$ is a Lipschitz map defined on a subset $A$ of $\mathbb{R}^2$. We will show that for every such $f$ and $A$, $\mathcal{H}^2(f(A)) = 0$, which implies the statement.

By Proposition 6.3 there exists an extension $\bar{f} : \mathbb{R}^2 \to Y$ of $f$. Let $x \in A$ be a density point of $A$ where $\text{md } \bar{f}_x$ exists. Then, by Lemma 3.19, $\text{md } f_x$ exists as well and equals $\text{md } \bar{f}_x$.

Assume now that there exist $f$ and $A$ as above with $\mathcal{H}^2(f(A)) > 0$. Then, by Theorem 3.26, there exists an $x \in A$ which is a density point of $A$ such that $\text{rank } \text{md } \bar{f}_x = 2$. Let $\delta > 0$ be such that $\text{md } \bar{f}_x(v) > 2\delta$ for $|v| \geq 1$. We can find an $r > 0$ such that for every $v, w \in B_r(0)$ we have

$$d(\bar{f}(x + v), \bar{f}(x + w)) \geq \text{md } \bar{f}_x(v - w) - \delta(|v| + |w|).$$

Define $g : S^1 \to Y$ by $g(z) := \bar{f}(x + rz)$. By Lemma 6.2, iii) we find $z, z' \in S^1$ with $g(z) = g(z')$ and $|z - z'| \geq 1$. Using the formula above we get a contradiction by

$$0 = d(g(z), g(z')) = d(\bar{f}(x + rz), \bar{f}(x + rz'))$$
$$\geq \text{md } \bar{f}_x(r(z - z')) - \delta r(|z| + |z'|)$$
$$= r(\text{md } \bar{f}_x(z - z') - 2\delta)$$
$$> 0.$$ 

Example 6.2. First we construct a fractal set in $\mathbb{R}^3$ in the following way. Let $W_0 := [0, 1]^3$ be the unit cube in $\mathbb{R}^3$. Let $D$ be the set of nine points consisting of the corners and the center of the cube $[0, \frac{2}{3}]^3$. For $i \geq 1$ we define the sets $W_i \subset \mathbb{R}^3$ inductively by

$$W_i := D + \frac{1}{3}W_{i-1}.$$
Finally we define \( W := \bigcap_{i=1}^{\infty} W_i \).

By a result of Falconer [Fal, Theorem 8.6], \( W \) has Hausdorff dimension 2 and \( 0 < \mathcal{H}^2(W) < \infty \).

In the following arguments we will utilize the notion of triadic cubes. A triadic cube is of the form

\[
(n_13^{-n}, n_23^{-n}, n_33^{-n}) + [0, 3^{-n}]^3 \subset \mathbb{R}^3
\]

where \( n, n_1, n_2, n_3 \in \mathbb{Z} \).

We constructed the subset \( W \subset \mathbb{R}^3 \). First we notice that \( W \) is a metric tree, i.e. all geodesic triangles in \( W \) are tripods. We claim that the intrinsic metric \( d_I \) in \( W \) is bi-Lipschitz equivalent to the Euclidean distance \( d_e(x,y) = |x-y| \).

Indeed, suppose \( x \neq y \in W \). Let \( \gamma \) be a geodesic in \( W \) from \( x \) to \( y \). Let \( C \) be a largest triadic cube with \( x, y \neq C \) and \( \gamma \cap \bar{C} \neq \emptyset \), and denote its side length by \( d \). It follows that \( |x-y| \geq d \). If the geodesic \( \gamma \) passes through the interior of more than six triadic cubes with side length \( d \), it must pass through a triadic cube with side length \( 3d \) not containing \( x \) and \( y \). Thus \( \gamma \) passes at most through the interior of six triadic cubes of side length \( d \), and it follows that \( d_I(x,y) \leq 6\sqrt{3}d \leq 6\sqrt{3}|x-y| \). Obviously, \( |x-y| \leq d_I(x,y) \). We get that \( d_e \) and \( d_I \) are \( \lambda \)-bi-Lipschitz equivalent, where \( \lambda := 6\sqrt{3} \).

Set \( X := (-1,2)^3 \) and \( Y := (W,d_I) \). We consider now the identity map \( f : W \to Y \). By the remarks above \( f \) is \( \lambda \)-Lipschitz. By Proposition 6.3 there is a \( \lambda \)-Lipschitz extension \( \tilde{f} : X \to Y \).

Suppose \( x \in X \) where \( \text{md} \, \tilde{f}_x \) exists. Exactly as in the proof of Lemma 6.5 we see that \( \text{md} \, \tilde{f}_x \) has rank less than two. Thus, wherever \( C_2(\text{md} \, \tilde{f}_x) \) exists, it equals zero. And we conclude with Theorem 1.1

\[
\int_Y \mathcal{H}^1(\tilde{f}^{-1}(y)) \, d\mathcal{H}^2(y) = \int_X C_2(\text{md} \, \tilde{f}_x) \, d\mathcal{H}^3(x) = 0.
\]
Chapter 7

Coarea inequality

In this chapter we provide a proof for the coarea inequality (Theorem 2.4) and study its equality cases in the setting where $X$ is an $\mathcal{H}^n$-rectifiable metric space and $Y$ is a $\mathcal{H}^m$-$\sigma$-finite metric space. The first section containing the proof is self-contained.

7.1 Proof

For convenience we state the coarea inequality again.

**Theorem 7.1 (Coarea inequality).** If $f : X \to Y$ is a Lipschitz map of metric spaces, $0 \leq m \leq n$, then

$$\int_Y \mathcal{H}^{n-m}(f^{-1}(y)) \, d\mathcal{H}^m(y) \leq (\text{Lip } f)^m \frac{\alpha_n}{\alpha_n} \mathcal{H}^n(X).$$

(F.1)

Federer proved the coarea inequality for Lipschitz maps between metric spaces, fulfilling some additional conditions [Fed2, 2.10.25]. There he states that they might be superfluous. Shortly after, R. O. Davies writes in [Dav] the following: “H. Federer tells me that this work answers a question raised in *Geometric measure theory*, (...) consequently the supplementary conditions (...) are superfluous.” We provide a proof of the coarea inequality without the extra conditions. It mainly follows Federer’s proof and makes use of Davies’ result.

First, we introduce some notation and definitions.

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**Definition 7.2 (fine covering).** Suppose $X$ is a metric space and $A \subset X$. A collection $\mathcal{F}$ of subsets of $X$ is called a fine covering of $A$ if for every $x \in A$ and every $\varepsilon > 0$ there exists an $C \in \mathcal{F}$ such that $x \in C$ and $\operatorname{diam} C \leq \varepsilon$.

**Definition 7.3.** Suppose $\eta$ is a real-valued function defined on a subfamily of the closed subsets of a metric space $X$, and $x \in X$. Then we define

$$\limsup_{S \to x} \eta(S) := \lim_{\delta \searrow 0} \sup \{ \eta(S) : \eta(S) \text{ is defined}, x \in S, \operatorname{diam} S \leq \delta \}.$$ 

For a nonnegative integer $n$ and a subset $S \subset X$ we set

$$\zeta^n(S) := \alpha_n \left( \frac{\operatorname{diam} S}{2} \right)^n.$$ 

**Definition 7.4 (weighted integral).** Let $X$ be a metric space, $f : X \to [0, \infty]$ and $\delta \in (0, \infty]$. We define the weighted integral of $f$ as

$$\int_X f \, d\mathcal{H}_\delta^n := \inf \sum_{C \in \mathcal{C}} u(C) \mathcal{H}_\delta^n(C) \quad (7.2)$$

where the infimum is taken over all countable families $\mathcal{C}$ of subsets of $X$ and $[0, \infty]$-valued functions $u$ on $\mathcal{C}$ such that

$$\sum_{C \in \mathcal{C}} u(C) \chi_C(x) \geq f(x) \quad \forall \ x \in X. \quad (7.3)$$

Alternatively we can define the weighted integral by

$$\int_X f \, d\mathcal{H}_\delta^n := \inf \sum_{C \in \mathcal{C}} u(C) \zeta^n(C)$$

and take the infimum over all countable families $\mathcal{C}$ of subsets $C$ with $\operatorname{diam} C \leq \delta$ and functions $u : \mathcal{C} \to [0, \infty]$ satisfying (7.3). This is how Federer defines the weighted integral in [Fed2, 2.10.24]. The two definitions are equivalent, which follows immediately from the constructions of $\mathcal{H}_\delta^n$ and $\mathcal{H}_\delta^n$.

The following theorem is the result Federer is referring to. It is proved in [Dav, Theorem 8, Example 1].

**Theorem 7.5 (Davies’ theorem).** Suppose $X$ is a metric space, $n \in \mathbb{N}$ and $\delta > 0$. Furthermore suppose $A_1 \subset A_2 \subset A_3 \ldots$ is an increasing sequence of subsets of $X$ and $A = \bigcup_{i \in \mathbb{N}} A_i$. Then

$$\mathcal{H}_\delta^n(A) = \lim_{i \to \infty} \mathcal{H}_\delta^n(A_i).$$
Vitali’s theorem is well known. A proof of a more general statement can be found in [Fed2, 2.8.4 – 2.8.6].

**Theorem 7.6** (Vitali’s covering theorem). Suppose $X$ is a metric space and $\mathcal{F}$ is a family of closed subsets of $X$. For every $S \in \mathcal{F}$ we define

$$\hat{S} := \bigcup \{ C \in \mathcal{F} : C \cap S \neq \emptyset, \text{diam } C \leq 2 \cdot \text{diam } S \}.$$ 

Then there exists a disjoint subfamily $\mathcal{G} \subset \mathcal{F}$ such that

$$\bigcup_{S \in \mathcal{F}} S \subset \bigcup_{S \in \mathcal{G}} \hat{S}.$$ 

If $A \subset X$ and $\mathcal{F}$ is a fine covering of $A$, then we find such a subfamily $\mathcal{G}$ with the additional property that for every finite $\mathcal{G}' \subset \mathcal{G}$,

$$A \subset \bigcup_{S \in \mathcal{G}'} S \cup \bigcup_{S \in \mathcal{G} \setminus \mathcal{G}'} \hat{S}.$$ 

We remark that we always have $\text{diam } \hat{S} \leq 5 \text{diam } S$ and thus $\zeta^n(\hat{S}) \leq 5^n \zeta^n(S)$ for every $S \in \mathcal{F}$.

To prove the coarea inequality, the main problem lies in proving the subsequent proposition. Note that in the definition of the Hausdorff measure, as well as in the definition of the outer measures $H^n_\delta$ or the weighted integral, we can restrict ourselves to the case where the respective coverings consist of closed sets, without affecting the definitions. This holds since $\zeta^n(S) = \zeta^n(\text{cl}(S))$ for every subset $S \subset X$.

**Proposition 7.7.** Let $X$ be a metric space and suppose $f : X \to [0, \infty]$. Then

$$\lim_{\delta \searrow 0} \int_X f \, dH^n_\delta = \int_X f \, dH^n.$$ 

Cf. [Fed2, 2.10.24].

To prove this proposition we need some lemmata.

**Lemma 7.8.** Suppose $\mu$ is a Borel measure on $X$, $V$ is an open subset of $X$, $B \subset V$, $t > 0$ and

$$\limsup_{S \to x} \frac{\mu(S)}{\zeta^n(S)} > t$$

whenever $x \in B$, then $\mu(V) \geq t \mathcal{H}^n(B)$.  

Proof. For \( \mu(V) = \infty \) the statement is trivial. Thus we assume \( \mu(V) < \infty \).

We observe that for each \( \delta > 0 \) the family
\[
\{ S \subset X : S \text{ closed}, \mu(S) > t \zeta^n(S), B_{2 \text{diam } S}(S) \subset V, \text{diam } S < \delta/5 \}
\]
is a fine covering of \( B \). By Theorem 7.6 we find a disjoint subfamily of closed sets \( G \) such that
\[
B \subset \bigcup_{S \in G'} S \cup \bigcup_{S \in \hat{G} \setminus G'} \hat{S}
\]
for any finite \( G' \subset G \). Let now \( \varepsilon > 0 \), and choose a finite \( G' \subset G \) such that
\[
\sum_{S \in \hat{G} \setminus G'} \mu(S) \leq \varepsilon.
\]
Then we can calculate
\[
\mathcal{H}^n_\delta(B) \leq \sum_{S \in G'} \zeta^n(S) + \sum_{S \in \hat{G} \setminus G'} \zeta^n(\hat{S}) \\
\leq \sum_{S \in G'} \frac{1}{t} \mu(S) + \sum_{S \in \hat{G} \setminus G'} \frac{5^n}{t} \mu(S) \\
\leq \frac{1}{t}(\mu(V) + 5^n \varepsilon).
\]
We conclude \( \mu(V) \geq t \mathcal{H}^n_\delta(B) \) for every \( \delta > 0 \), and hence \( \mu(V) \geq t \mathcal{H}^n(B) \).

Lemma 7.9. Suppose \( A \subset X \) is a Borel set of finite \( \mathcal{H}^n \)-measure. Then
\[
\limsup_{S \to x} \frac{\mathcal{H}^n(A \cap S)}{\zeta^n(S)} \leq 1
\]
for \( \mathcal{H}^n \)-almost every \( x \in X \).

Proof. Define
\[
B_m := \left\{ x : \limsup_{S \to x} \frac{\mathcal{H}^n(A \cap S)}{\zeta^n(S)} > 1 + \frac{1}{m} \right\}.
\]
Using the Borel regularity of the measure \( \mathcal{H}^n | A \) and applying Lemma 7.8 with \( \mu = \mathcal{H}^n | A \), \( B = B_m \) and \( t = 1 + \frac{1}{m} \) we get
\[
\infty > \mathcal{H}^n(A \cap B_m) = \inf \{ \mathcal{H}^n(A \cap V) : V \text{ open, } B_m \subset V \} \geq (1 + \frac{1}{m}) \mathcal{H}^n(B_m).
\]
We conclude \( \mathcal{H}^n(B_m) = 0 \) for every \( m \in \mathbb{N} \) what implies the desired statement.
7.1. PROOF

Proof of Proposition 7.7. The inequality
\[ \lim_{\delta \searrow 0} \int_X f \, d\mathcal{H}^n_\delta \leq \int_X f \, d\mathcal{H}^n \]
holds by construction. We have to prove the other direction.

We define \( A := \{ x : f(x) > 0 \} \). At first, we consider the case where \( A \) is \( \mathcal{H}^n\)-\( \sigma \)-finite.

We will now construct an increasing sequence of Borel sets \( B_j \) and positive numbers \( \varepsilon_j \), such that
\[ \mathcal{H}^n(B_j \cap S) \leq (1 + \frac{1}{j})\zeta^n(S) \quad (7.4) \]
for all closed sets \( S \) with \( \text{diam } S \leq \varepsilon_j \), and such that
\[ \mathcal{H}^n\left(A \setminus \bigcup_{j \in \mathbb{N}} B_j\right) = 0. \quad (7.5) \]

By the Borel regularity of \( \mathcal{H}^n \) and the \( \mathcal{H}^n\)-\( \sigma \)-finiteness of \( A \), we can choose an increasing sequence of Borel sets \( \tilde{B}_j \) with finite \( \mathcal{H}^n \)-measure satisfying (7.5). For every \( \varepsilon > 0 \) we set
\[ \tilde{B}_{j,\varepsilon} := \{ x \in \tilde{B}_j : \mathcal{H}^n(\tilde{B}_j \cap S) \leq (1 + \frac{1}{j})\zeta^n(S) \quad \forall \ x \in S \text{ closed, diam } S \leq \varepsilon \}. \]

By Lemma 7.9 we know that
\[ \mathcal{H}^n\left(\tilde{B}_j \setminus \bigcup_{\varepsilon > 0} \tilde{B}_{j,\varepsilon}\right) = 0. \]

The goal is now to construct a sufficiently large Borel subset of \( \tilde{B}_j \) contained in some \( \tilde{B}_{j,\varepsilon} \). Since \( \tilde{B}_{j,\varepsilon} \) is not necessarily Borel, we have to do some work. We choose \( \varepsilon_j > 0 \) small enough such that
\[ \mathcal{H}^n(A \cap (\tilde{B}_j \setminus \tilde{B}_{j,2\varepsilon_j})) \leq \frac{1}{j}. \]

Suppose \( x \in \text{cl}(\tilde{B}_{j,2\varepsilon_j}) \) and \( S \) is a closed set with \( x \in S \) and \( 0 < \text{diam } S \leq \varepsilon_j \). Then we can choose a sequence \( \{x_i\}_{i \in \mathbb{N}} \subset \tilde{B}_{j,2\varepsilon_j} \) converging to \( x \). For \( i \) large enough, the sets \( S_i := S \cup \{x_i\} \) satisfy the condition \( \mathcal{H}^n(\tilde{B}_j \cap S_i) \leq (1 + \frac{1}{j})\zeta^n(S_i) \). We can calculate
\[ \frac{\mathcal{H}^n(\tilde{B}_j \cap S)}{\zeta^n(S)} = \lim_{i \to \infty} \frac{\mathcal{H}^n(\tilde{B}_j \cap S)}{\zeta^n(S_i)} = \lim_{i \to \infty} \frac{\mathcal{H}^n(\tilde{B}_j \cap S_i)}{\zeta^n(S_i)} \leq 1 + \frac{1}{j}. \]
Thus \( x \in \tilde{B}_{j,\varepsilon_j} \) and therefore \( \text{cl}(\tilde{B}_{j,2\varepsilon_j}) \subset \tilde{B}_{j,\varepsilon_j} \). So the Borel sets \( B_j := \tilde{B}_j \cap \text{cl}(\tilde{B}_{j,2\varepsilon_j}) \) fulfill the conditions (7.4) and (7.5).

We continue by choosing countable families \( C_j \) of closed sets with diameter less than \( \varepsilon_j \) and \([0, \infty]\)-valued functions \( u_j \) on \( C_j \) such that

\[
 g_j(x) = \sum_{S \in C_j} u_j(S) \chi_S \geq f(x)
\]

for every \( x \in X \) and

\[
 \lim_{j \to \infty} \sum_{S \in C_j} u_j(S) \zeta^n(S) = \lim_{\delta \searrow 0} \int_X f \, dH^n_{\delta}.
\]

Additionally, we define \( h_j := g_j \cdot \chi_{B_j} \). We then can calculate

\[
 \int_X f \, dH^n \leq \int_X \liminf_{j \to \infty} h_j \, dH^n
\]

\[
 \leq \liminf_{j \to \infty} \int_X h_j \, dH^n
\]

\[
 = \liminf_{j \to \infty} \sum_{S \in C_j} u_j(S) H^n(B_j \cap S)
\]

\[
 \leq \liminf_{j \to \infty} (1 + \frac{1}{j}) \sum_{S \in C_j} u_j(S) \zeta^n(S)
\]

\[
 = \lim_{\delta \searrow 0} \int_X f \, dH^n_{\delta}.
\]

This proves the statement for the case where \( A \) is \( H^n \)-\( \sigma \)-finite.

It remains to prove the case where \( A \) is not \( H^n \)-\( \sigma \)-finite. We show that in this case \( \lim_{\delta \searrow 0} \int_X f \, dH^n_{\delta} = \infty \), which implies the statement.

We deduce this by showing that

\[
 H^n(\{ x : f(x) > t \}) \leq t^{-1} 5^n \lim_{\delta \searrow 0} \int_X f \, dH^n_{\delta}
\]

(7.6)

for every \( 0 < t < \infty \). By the non-\( H^n \)-\( \sigma \)-finiteness of \( A \), the left hand side has then to be infinite for some \( t > 0 \), which implies the right hand side to be infinite as well. It follows \( \lim_{\delta \searrow 0} \int_X f \, dH^n_{\delta} = \infty \).

We can reduce (7.6) to the fact that for every countable family of closed \( \delta \)-subsets \( C \), every \([0, \infty]\)-valued function \( u \) on \( C \) and every \( 0 < t < \infty \) we have

\[
 H^n_{\delta_0} \left( \left\{ x : \sum_{x \in S \in C} u(S) > t \right\} \right) \leq t^{-1} 5^n \sum_{C} u(S) \zeta^n(S).
\]

(7.7)
This being so, since for every pair \((C, u)\) fulfilling \(\sum_{x \in S \in C} u(S) > f(x)\), we then have

\[
\mathcal{H}^n_{5\delta} \left( \{ x : f(x) > t \} \right) \leq \mathcal{H}^n_{5\delta} \left( \left\{ \sum_{x \in S \in C} u(S) > t \right\} \right) \leq t^{-15n} \sum_C u(S)\zeta^n(S).
\]

Taking the infimum over all such pairs \((C, u)\) we get

\[
\mathcal{H}^n_{5\delta} \left( \left\{ x : f(x) > t \right\} \right) \leq t^{-15n} \int_X f \, d\mathcal{H}^n_{5\delta}
\]

and by letting \(\delta \searrow 0\) we get (7.6).

We are thus left to show (7.7). It suffices to consider the case where \(u\) takes values in \(\mathbb{R}_{\geq 0}\). This is so, since for any pair \((C, u)\) with \(u : C \to [0, \infty]\) we can replace \(u\) with the function \(u'(S) = \min \{u(S), t\}\); and if \((C, u')\) satisfies (7.7), then so does \((C, u)\).

By Davies’ theorem (Theorem 7.5), it suffices to prove (7.7) for finite families \(C\). This is so, since for a countable family \(C = \{S_i\}_{i \in \mathbb{N}}\) we obviously have

\[
\lim_{j \to \infty} \sum_{i=1}^{j} u(S_i)\zeta^n(S_i) = \sum_{S \in C} u(S)\zeta^n(S)
\]

and Davies’ theorem tells us that we also have

\[
\lim_{j \to \infty} \mathcal{H}^n_{5\delta} \left( \left\{ x : \sum_{x \in S \in \{S_1, \ldots, S_j\}} u(S) > t \right\} \right) = \mathcal{H}^n_{5\delta} \left( \left\{ x : \sum_{x \in S \in C} u(S) > t \right\} \right).
\]

We can further assume \(u\) to have rational values and conclude the statement for real values by approximation. Thus we are left with the following:

Suppose \(C\) is a finite set of closed sets \(S\) with \(\text{diam} S \leq \delta\) and \(u : C \to \mathbb{Q}_{\geq 0}\). Then

\[
\mathcal{H}^n_{5\delta} \left( \left\{ x : \sum_{x \in S \in C} u(S) > t \right\} \right) \leq t^{-15n} \sum_C u(S)\zeta^n(S).
\]

Let \(N\) be the largest common denominator of the \(u(S)\), and denote \(v(S) := N \cdot u(S)\) and \(k := \lceil Nt \rceil\). Then we can reduce the above inequality to

\[
\mathcal{H}^n_{5\delta} \left( \left\{ x : \sum_{x \in S \in C} v(S) \geq k \right\} \right) \leq k^{-15n} \sum_C v(S)\zeta^n(S).
\]

We will prove this inequality by finding \(k\) subfamilies of \(C\) of disjoint sets, such that every \(S \in C\) belongs to at most \(v(S)\) of these subfamilies.
We denote the set \( \{ x : \sum_{x \in S} v(S) \geq k \} \) again by \( A \). Inductively we will define functions \( v_0, v_1, \ldots, v_k \) from \( C \) to the nonnegative integers and subfamilies \( C_1, \ldots, C_k \) by starting with \( v_0 := v \). For every \( 1 \leq j \leq k \) we do the following:

By induction we know that \( A \subset \{ x : \sum_{x \in S} v_{j-1}(S) \geq k - j + 1 \} \). With Vitali’s theorem we can find a disjoint subfamily \( \hat{S} \subset C \cap \{ S : v_{j-1}(S) \geq 1 \} \) such that \( A \subset \bigcup_{S \in C} \hat{S} \). Then we define \( v_j(S) := \begin{cases} v_{j-1}(S) - 1 & \text{if } S \in C_j \\ v_{j-1}(S) & \text{else.} \end{cases} \)

Since we chose \( C_j \) to be disjoint, we have \( A \subset \{ x : \sum_{x \in S} v_j(S) \geq k - j \} \).

By this construction \( v_k(S) \geq 0 \) for every \( S \in C \), and therefore every such \( S \) is contained in at most \( v(S) \) of the \( C_j \). Then we find that

\[
kH^n_{\delta}(A) \leq \sum_{j=1}^{k} \sum_{S \in C_j} \zeta^n(\hat{S}) \leq 5^n \sum_{j=1}^{k} \sum_{S \in C_j} \zeta^n(S) \leq 5^n \sum_{S \in C} v(S) \zeta^n(S)
\]

and the proof is complete. \( \Box \)

The generalization from the \( \mathcal{H}^n\)-\( \sigma \)-finite case to the arbitrary case could also be proven using the existence of large, \( \mathcal{H}^n \) finite subsets. Assume that a certain metric space \( X \) possesses subsets of arbitrarily large, finite \( \mathcal{H}^n \)-measure. If \( A = \{ x : f(x) > 0 \} \) is not \( \mathcal{H}^n\)-\( \sigma \)-finite, then one of the sets \( A_k = \{ x : f(x) \geq \frac{1}{k} \} \) is \( \mathcal{H}^n \) infinite. We denote this \( k \) by \( k_0 \). For every \( \mathcal{H}^n \) finite subset \( A' \subset A_{k_0} \) we then have

\[
\lim_{\delta \searrow 0} \int_{X}^* f d\mathcal{H}^n_{\delta} \geq \lim_{\delta \searrow 0} \int_{A'}^* f d\mathcal{H}^n_{\delta} = \int_{A'}^* f d\mathcal{H}^n \geq \frac{1}{k} \mathcal{H}^n(A').
\]

As assumed above, \( \mathcal{H}^n(A') \) can be arbitrarily large, and \( \lim_{\delta \searrow 0} \int_{X}^* f d\mathcal{H}^n_{\delta} = \infty \) follows.

However, the existence of such sets of positive, finite \( \mathcal{H}^n \) measure is not trivial. In [DR], Davies and Rogers give an example of a space \( X \) and a
Hausdorff measure $\mathcal{H}^h$ such that $\mathcal{H}^h(X) = \infty$, and there exist no subsets of $X$ with finite positive $\mathcal{H}^h$ measure. In the construction of the measure $\mathcal{H}^h$, $h$ is a monotone function $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, and $h(\text{diam}(S))$ plays the role of $\zeta^n(S)$ in the construction of $\mathcal{H}^n$. In their example, $h(t)$ converges to zero for $t \to 0$ much faster than $t^n$.

In [How], Howroyd actually proves the existence of subsets with arbitrarily large, finite $\mathcal{H}^n$ measure for a large class of spaces, namely for analytic subsets of complete, separable metric spaces. These are the subsets with a representation $\bigcup_{i_1, i_2, \ldots} \bigcap_{r=1}^{\infty} F(i_1, \ldots, i_r)$, where $F(i_1, \ldots, i_r)$ is closed for each finite sequence of integers. In his proof he employs Theorem 7.5 by Davies. Therefore we provide the proof here using this theorem directly.

Having proved Proposition 7.7, the proof of the coarea inequality runs straightforward. The only non-standard technique we need is monotone convergence for arbitrary, in particular non-measurable, functions with respect to the upper integral. This holds as the following lemma ensures.

**Lemma 7.10.** Suppose $X$ is a metric space and $\mu$ is a Borel regular outer measure on $X$. Suppose $\{g_i\}_{i \in \mathbb{N}}$ is a non-decreasing sequence of $[0, \infty]$-valued, not necessarily measurable functions on $X$. Set $g := \lim_{i \to \infty} g_i$. Then

$$\int g \, d\mu = \lim_{i \to \infty} \int g_i \, d\mu.$$  

**Proof.** By $g \geq g_i$ for every $i \in \mathbb{N}$ we have $\int g \, d\mu \geq \lim_{i \to \infty} \int g_i \, d\mu$. We have to prove the other direction.

For every $i \in \mathbb{N}$ we find a $\mu$-measurable function $\tilde{h}_i$ such that $\tilde{h}_i \geq g_i$ and

$$\int \tilde{h}_i \, d\mu \leq \int g_i \, d\mu + \frac{1}{i}.$$  

We define a non-decreasing sequence $\{h_i\}_{i \in \mathbb{N}}$ by $h_i := \inf_{j \geq i} \tilde{h}_j$. Then we have $g_i \leq g_j \leq h_j$ for all $j \geq i$ and therefore $h_i \geq g_i$. With $h_i \leq \tilde{h}_i$ it follows that

$$\int g_i \, d\mu \leq \int h_i \, d\mu \leq \int \tilde{h}_i \, d\mu \leq \int g_i \, d\mu + \frac{1}{i}$$  

and we conclude

$$\lim_{i \to \infty} \int h_i \, d\mu = \lim_{i \to \infty} \int g_i \, d\mu.$$  

We define $h := \lim_{i \to \infty} h_i$ and by $h_i \geq g_i$ we have $h \geq g$. By the monotone convergence theorem for measurable functions, we conclude
\[ \int h \, d\mu = \lim_{i \to \infty} \int h_i \, d\mu. \] It follows that
\[ \int^* g \, d\mu \leq \int h \, d\mu = \lim_{i \to \infty} \int h_i \, d\mu = \lim_{i \to \infty} \int^* g_i \, d\mu. \]

\[ \int^* Y \left( H_{n-\alpha}^{1/j}(f^{-1}(y)) \right) \, dH_{\alpha}^m(y) = \lim_{j \to \infty} \int^* Y \left( H_{n-\alpha}^{1/j}(f^{-1}(y)) \right) \, dH_{\alpha}^m(y) \leq (\text{Lip} f)^m \frac{\alpha_m \alpha_{n-\alpha}}{\alpha_m} \, H^n(X). \]

We see that the pair \((\{f(C)\}_{C \in \mathcal{C}}, \zeta^{n-m}(C))\) satisfies the condition in Definition 7.4. Applying Proposition 7.7, we have for every positive integer \(i\)
\[ \int^* Y \left( H_{n-\alpha}^{1/j}(f^{-1}(y)) \right) \, dH_{\alpha}^m(y) \leq (\text{Lip} f)^m \frac{\alpha_m \alpha_{n-\alpha}}{\alpha_m} \, H^n(X). \]
7.2 Equality cases

In this section we will examine the equality cases of the coarea inequality in the setting in which we have proven the coarea formula. Namely where $X$ is an $H^n$-rectifiable metric space and $Y$ is an $H^m$-$\sigma$-finite metric space.

By Lemma 5.12 the integrand on the left hand side in (7.1) is measurable, and therefore we can replace the upper integral by an integral.

For a seminorm $\sigma$ of rank $m$ on $\mathbb{R}^n$ and a measurable set $A \subset (\ker \sigma)^\perp$ with $0 < H^{n-m}(A) < \infty$ we will need the identity

$$C_m(\sigma) = \frac{\mathcal{H}_\sigma^{n-m}(A)}{\mathcal{H}^{n-m}(A)}$$

which we showed above.

We will need the following theorem from convex geometry [Sch, Theorem 6.1.1].

**Theorem 7.11** (Brunn-Minkowski). Suppose $K_0, K_1$ are convex, compact subsets of $\mathbb{R}^n$ and $0 \leq \lambda \leq 1$. Then

$$H^n((1-\lambda)K_0 + \lambda K_1)^{1/n} \geq (1-\lambda)H^n(K_0)^{1/n} + \lambda H^n(K_1)^{1/n}.$$ 

Equality holds if $K_0$ and $K_1$ either lie in parallel hyperplanes, if $K_1 = \mu K_0 + v$ for $v \in \mathbb{R}^n$ and $\mu > 0$ or if one of $K_0$ and $K_1$ is a point.

**Remark 7.12.** In the case where $H^n(K_0) = H^n(K_1) > 0$ the conclusion of the Brunn-Minkowski theorem simplifies to

$$H^n((1-\lambda)K_0 + \lambda K_1) \geq H^n(K_0)$$

with equality if and only if $K_0$ is a translate of $K_1$.

**Proposition 7.13.** Suppose $n > m > 0$, $X$ is an $H^n$-rectifiable space with a parametrization $(U_i, \alpha_i)_{i \in \mathbb{N}}$, $Y$ is an $H^m$-$\sigma$-finite metric space and $f : X \to Y$ is a 1-Lipschitz map. Suppose $X' \subset X$ is the set of all points $x \in X$ such that $C_m(f, x)$ exists. Then equality holds in (7.1) if and only if at almost every point $x \in X'$ the following property holds:

There exists a subspace $T \subset \mathbb{R}^n$ complementary to $\ker \sigma$ such that for every $u \in \ker \sigma$ and $v \in T$

$$\tau(u + v) = \max\{\tau(u), \sigma(v)\}$$

where $(U, \alpha)$ belongs to the parametrization of $X$ such that $\alpha(\bar{x}) = x$ for an $\bar{x} \in U$, $\tau = \text{md} \alpha_{\bar{x}}$ and $\sigma = \text{md}(f \circ \alpha)_{\bar{x}}$. 

Note that for a Lipschitz map \( f : X \to Y \) with arbitrary positive Lipschitz constant \( \text{Lip} f \) we can apply the above theorem by rescaling the metric of \( Y \) by the factor \( (\text{Lip} f)^{-1} \). The resulting condition on the metric differentials of the map with respect to the original metric is

\[
\tau(u + v) = \max\{\tau(u), (\text{Lip} f)^{-1}\sigma(v)\}
\]

where \( \sigma, \tau \) are as in Proposition 7.13.

Proof. In the following we denote the closed unit ball in a subspace \( V \subset \mathbb{R}^n \) with respect to a norm \( \eta \) by \( D^\eta_V \) and by \( D^\eta \) for \( V = \mathbb{R}^n \). Recall that \( \mathcal{H}^n \) denotes the Hausdorff measure with respect to the Euclidean norm.

With help of the coarea formula on rectifiable spaces, the coarea inequality (7.1) is equivalent to

\[
\int_X C_m(f, x) d\mathcal{H}^n(x) \leq \frac{\alpha_n - m\alpha_m}{\alpha_n} \mathcal{H}^n(X) \tag{7.9}
\]

This holds not only for \( X \), but for every measurable subset of \( X \), and \( C_m(f, x) \) is a measurable function on \( X \). Thus (7.9) is equivalent to

\[
C_m(f, x) \leq \frac{\alpha_n - m\alpha_m}{\alpha_n} \tag{7.10}
\]

holding for almost every \( x \in X \). Equality holds in (7.1) if and only if equality holds in (7.10) for almost every \( x \in X \).

We will thus investigate under which circumstances

\[
C_m(f, x) = \frac{\alpha_n - m\alpha_m}{\alpha_n} \tag{7.11}
\]

is fulfilled.

Assume now \( x \in X \) such that \( C_m(f, x) \) exists. Let \( (U, \alpha) \in (U_i, \alpha_i)_{i \in \mathbb{N}} \) be such that \( x \in \alpha(U) \) and put \( \bar{x} = \alpha^{-1}(x) \).

Further we denote the norm \( \text{md} \alpha_{\bar{x}} \) by \( \tau \), the seminorm \( \text{md}(f \circ \alpha)_{\bar{x}} \) by \( \sigma \) and \( \ker \sigma \) by \( K \). By applying the definition of the coarea factor and equation
(7.8) we can calculate
\[ C_m(f, x) = \frac{C_m(\sigma) \cdot C_{n-m}(\tau|_K)}{C_n\tau} \]
\[ = \frac{\mathcal{H}^n(D_{\sigma}^+) \cdot \mathcal{H}^{n-m}(D^K)}{\mathcal{H}^n(D_{\tau}) \cdot \mathcal{H}^{n-m}(D^K)} \]
\[ = \frac{\alpha_{n-m} \alpha_{m}}{\alpha_n} \cdot \frac{\mathcal{H}^n(D_{\tau})}{\mathcal{H}^n(D_{\sigma}^+) \cdot \mathcal{H}^{n-m}(D^K)}. \]

Thus (7.11) can be reformulated to
\[ \mathcal{H}^n(D_{\tau}) = \mathcal{H}^m(D_{\sigma}^+) \cdot \mathcal{H}^{n-m}(D^K). \]

Since \( f \) is 1-Lipschitz, we have \( \sigma(v) \leq \tau(v) \) for all \( v \in \mathbb{R}^n \). Thus by Lemma 7.14 equality holds in the coarea inequality if and only if the stated property is fulfilled.

\textbf{Lemma 7.14.} Suppose \( \tau \) is a norm on \( \mathbb{R}^n \) and \( \sigma \) is a seminorm on \( \mathbb{R}^n \) with kernel \( K \) such that \( \sigma(v) \leq \tau(v) \) for every \( v \in \mathbb{R}^n \). Let \( k := \dim K \). Then the following are equivalent.

\begin{enumerate}
  \item \[ \mathcal{H}^n(D_{\tau}) = \mathcal{H}^k(D_{\sigma}^+) \cdot \mathcal{H}^{n-k}(D_{\sigma}^+) \]
  \item There exists a complement \( T \) of \( K \) such that \( D_{\tau} = D^K_{\tau} + D^T_{\sigma} \).
  \item There exists a complement \( T \) of \( K \) such that for every \( u \in K, v \in T \):
  \[ \tau(u + v) = \max\{\tau(u), \sigma(v)\} \]
\end{enumerate}

\textbf{Proof.} Clearly \( ii) \Leftrightarrow iii) \) since they are just two formulations of the same condition. \( ii) \Rightarrow i) \) follows directly by integrating. Thus we are left to prove \( i) \Rightarrow ii) \).
For every $v \in \mathbb{R}^n$ we define $D_v := D_\tau \cap (v + K)$. Note that $D_0 = D^K_\tau$.

We will now use the convexity of $D_\tau$ to show that every $D_v$, that is not empty, is a translation of $D_0$, i.e. that the intersections of $D_\tau$ with subspaces parallel to $K$ only differ by a translation. The main idea to do this is the following: Any midpoint of a segment joining a point in $D_v$ and one in $D_{-v}$ lies in $K$ by construction. Since $D_\tau$ is convex, this midpoint also lies in $D_\tau$, and therefore it lies in $D_0 = D_\tau \cap K$. We observe that the set of these midpoints is exactly $D_v + D_{-v}/2$. With the help of the Brunn-Minkowski theorem we then can estimate $\mathcal{H}^k(D_v)$ by $\mathcal{H}^k(D_0)$. Together with the condition given in i) we show that they have to coincide.

Suppose $v \in \mathbb{R}^n$. Observe that by the symmetry of $D_\tau$ we have $\mathcal{H}^k(D_v) = \mathcal{H}^k(D_{-v})$. Suppose $w \in D_v$ and $w' \in D_{-v}$. Then $w = v + k$ and $w' = -v + k'$ for some $k, k' \in K$. Their midpoint $w + w' = v + w + k = k + k'$ lies in $K$. By the convexity of $D_\tau$ it also lies in $D_\tau$, and we conclude $D_v + D_{-v}/2 \subset D_0$.

Let $p$ be the orthogonal projection onto $K$. Regarding $p(D_v)$ and $p(D_{-v})$, we can see that $D_{\pm v}$ is a translate of $p(D_{\pm v})$, $\mathcal{H}^k(p(D_v)) = \mathcal{H}^k(p(D_{-v}))$ and that the sets $D_v + D_{-v}/2$ and $p(D_v) + p(D_{-v})/2$ coincide. Thus we can apply the Brunn-Minkowski theorem in $K$ in the form as in Remark 7.12, and get

$$\mathcal{H}^k(D_0) \geq \mathcal{H}^k\left(\frac{D_v + D_{-v}}{2}\right) = \mathcal{H}^k\left(\frac{p(D_v) + p(D_{-v})}{2}\right) \geq \mathcal{H}^k(p(D_v)) = \mathcal{H}^k(D_v). \quad (7.12)$$

Equality holds if and only if in the two performed estimations equality holds. In the first one this is the case if and only if the two convex sets $D_v + D_{-v}/2$ and $D_0$ coincide. In the second one, according to Brunn-Minkowski, if and only if $D_v$ and $D_{-v}$ (respectively $p(D_v)$ and $p(D_{-v})$) are translates of each other. If both these conditions are fulfilled, then the three sets $D_0$, $D_v$ and $D_{-v}$ are translates of each other and there exists a $\tilde{v} \in \mathbb{R}^n$ such that $D_v = D_0 + \tilde{v}$ and $D_{-v} = D_0 - \tilde{v}$.

We can now calculate $\mathcal{H}^n(D_\tau)$ by integrating $\mathcal{H}^k(D_v) = \mathcal{H}^k(D_\tau \cap (v + K))$ over $K^\perp$. We notice that for any $v \in K^\perp$ with $\sigma(v) > 1$ we have
Therefore equality holds in the second line and thus $H^v$ is empty for $v \in K^\perp \setminus D^K_{\sigma}$. Together with the assumption i), i.e. $\mathcal{H}^n(D_\tau) = \mathcal{H}^m(D^K_{\sigma}) \cdot \mathcal{H}^k(D^K_\tau)$, we get

$$\mathcal{H}^n(D_\tau) = \int_{D^K_{\sigma}} \mathcal{H}^k(D_v) \, d\mathcal{H}^m(v) \leq \int_{D^K_{\sigma}} \mathcal{H}^k(D_0) \, d\mathcal{H}^m(v) = \mathcal{H}^m(D^K_{\sigma}) \cdot \mathcal{H}^k(D_0) = \mathcal{H}^m(D^K_{\sigma}) \cdot \mathcal{H}^k(D^K_\tau) = \mathcal{H}^n(D_\tau).$$

Therefore equality holds in the second line and thus $\mathcal{H}^k(D_v) = \mathcal{H}^k(D_0)$ for almost every $v \in D^K_{\sigma}$. Since $D_\tau$ is closed and convex, $\mathcal{H}^k(D_v) = \mathcal{H}^k(D_0)$ holds for every $v \in D^K_{\sigma}$. For every $v \in D^K_{\sigma}$ we have therefore equality in (7.12), and we know that this implies that there is a $\tilde{v} \in \mathbb{R}^n$ such that $D_v = D_0 + \tilde{v}$ and $D_{-v} = D_0 - \tilde{v}$. This defines a map $\varphi : D^K_{\sigma} \to \mathbb{R}^n$ by $\varphi(v) = \tilde{v}$. Note that by construction $\varphi(v) \in v + K$. We will show now that $\varphi$ is the restriction of a linear map on $K^\perp$ to $D^K_{\sigma}$.

By the constructions above we have $\varphi(0) = 0$ and $\varphi(-v) = -\varphi(v)$. Suppose $v \in D^K_{\sigma}$ and $r \in (0, 1)$. Then we know that $D_v = D_0 + \varphi(v)$ and $D_{rv} = D_0 + \varphi(rv)$. We have $rD_v + (1-r)D_0 \subset (rv + K)$ and with the convexity of $D_\tau$ it follows that $rD_v + (1-r)D_0 \subset D_{rv}$. But both sets are translations of the compact set $D_0$ and therefore they must coincide. Thus we have

$$\varphi(rv) + D_0 = D_{rv} = rD_v + (1-r)D_0 = r(\varphi(v) + D_0) + (1-r)D_0 = r\varphi(v) + D_0$$

and it follows $\varphi(rv) = r\varphi(v)$.

Suppose $v, w \in D^K_{\sigma}$. With the same argument as above, we can show that $\frac{D_v + D_w}{2} = D_{\frac{v+w}{2}}$. If follows

$$\varphi\left(\frac{v+w}{2}\right) + D_0 = D_{\frac{v+w}{2}} = \frac{D_v + D_w}{2} = \frac{1}{2}(\varphi(v) + D_0) + \frac{1}{2}(\varphi(w) + D_0) = \frac{\varphi(v)}{2} + \frac{\varphi(w)}{2} + D_0$$

and we conclude $\varphi\left(\frac{v+w}{2}\right) = \frac{\varphi(v)}{2} + \frac{\varphi(w)}{2}$. Thus $\varphi$ is the restriction of a linear map on $K^\perp$, which we denote again by $\varphi$. 

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\[ \tau(v') \geq \sigma(v') = \sigma(v) > 1 \] for every $v' \in (v + K)$. Therefore $D_\tau \cap (v + K)$ is empty for $v \in K^\perp \setminus D^K_{\sigma}$ and it suffices to integrate over $D^K_{\sigma} \subset K^\perp$. For every $v \in D^K_{\sigma}$ we know that this implies that there is a $\tilde{v} \in \mathbb{R}^n$ such that $D_v = D_0 + \tilde{v}$ and $D_{-v} = D_0 - \tilde{v}$. This defines a map $\varphi : D^K_{\sigma} \to \mathbb{R}^n$ by $\varphi(v) = \tilde{v}$. Note that by construction $\varphi(v) \in v + K$. We will show now that $\varphi$ is the restriction of a linear map on $K^\perp$ to $D^K_{\sigma}$.

By the constructions above we have $\varphi(0) = 0$ and $\varphi(-v) = -\varphi(v)$. Suppose $v \in D^K_{\sigma}$ and $r \in (0, 1)$. Then we know that $D_v = D_0 + \varphi(v)$ and $D_{rv} = D_0 + \varphi(rv)$. We have $rD_v + (1-r)D_0 \subset (rv + K)$ and with the convexity of $D_\tau$ it follows that $rD_v + (1-r)D_0 \subset D_{rv}$. But both sets are translations of the compact set $D_0$ and therefore they must coincide. Thus we have

$$\varphi(rv) + D_0 = D_{rv} = rD_v + (1-r)D_0 = r(\varphi(v) + D_0) + (1-r)D_0 = r\varphi(v) + D_0$$

and it follows $\varphi(rv) = r\varphi(v)$.

Suppose $v, w \in D^K_{\sigma}$. With the same argument as above, we can show that $\frac{D_v + D_w}{2} = D_{\frac{v+w}{2}}$. If follows

$$\varphi\left(\frac{v+w}{2}\right) + D_0 = D_{\frac{v+w}{2}} = \frac{D_v + D_w}{2} = \frac{1}{2}(\varphi(v) + D_0) + \frac{1}{2}(\varphi(w) + D_0) = \frac{\varphi(v)}{2} + \frac{\varphi(w)}{2} + D_0$$

and we conclude $\varphi\left(\frac{v+w}{2}\right) = \frac{\varphi(v)}{2} + \frac{\varphi(w)}{2}$. Thus $\varphi$ is the restriction of a linear map on $K^\perp$, which we denote again by $\varphi$. 

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We now know for every $v \in K^\perp$ the intersection of $v + K$ with $D_\tau$. Since $\varphi(v) \in v + K$, $\varphi(K^\perp)$ is a complement of $K$ and since $K = \ker \sigma$, we have $\varphi(D^K_\sigma) = D^K_\sigma$. With this information we can construct $D_\tau$, getting

$$D_\tau = D_0 + \varphi(D^K_\sigma) = D^K_\tau + D^K_\sigma.$$

By setting $T = \varphi(K^\perp)$ the condition in ii) is fulfilled.

7.3 Examples

First we give an example showing that in the proof of the coarea inequality the functions $H^n(\delta^{-1}(f))$ and $H^m(f^{-1}(y))$ are not necessarily measurable.

**Example 7.1.** Suppose $I = [0, 1]$ and let $g : I \to \mathbb{R}_{\geq 0}$ be a non-measurable function. Then we define the set $A \subset I \times \mathbb{R}$ by

$$A = \{(x, y), 0 \leq y \leq g(x)\}$$

and the Lipschitz map $f : A \to I, (x, y) \to x$.

Regarding the functions $H^n(\delta^{-1}(x))$ and $H^m(\delta^{-1}(y))$ for some $\delta > 0$, we notice that they equal $g(x)$ and are therefore not measurable.

This shows that the functions $H^n(\delta^{-1}(y))$ and $H^m(f^{-1}(y))$, which appear in the statement and the proof of the coarea inequality, may indeed turn out to be non-measurable.

In the second example we show that a map from a Euclidean space to a metric space, which is non-degenerate, cannot be an equality case of the coarea inequality.

**Example 7.2.** Suppose $A \subset \mathbb{R}^n_\sigma$ is an $H^n$-measurable set, $Y$ is an $H^m$-finite metric space and

$$f : A \to Y.$$

We consider a non-degenerate situation in the sense that $0 < m < n$ and $H^m(f(A)) > 0$. Assume that for $f$ in the coarea inequality equality holds. What can we say about $\sigma$?

We can regard $\mathbb{R}^n_\sigma$ as a rectifiable space with the representation

$$(A, \alpha).$$
where \( A \subset \mathbb{R}^n \) and \( \alpha = \text{id} : \mathbb{R}^n \to \mathbb{R}^n \). We observe that \( \text{md} \alpha_x = \sigma \) at every density point of \( A \). We consider a \( x \in A \) such that \( \text{md} (f \circ \alpha)_x \) has rank \( m \) and denote \( \text{md} (f \circ \alpha)_x \) by \( \tau \). Thus, by Proposition 7.13, there exists a subspace \( T \) complementary to \( \ker \tau \) such that
\[
\sigma(u + v) = \max\{\sigma(u), \tau(v)\}
\]
for every \( u \in \ker \tau \) and \( v \in T \). We can define a seminorm \( \nu \) on \( \mathbb{R}^n \) to be the seminorm with kernel \( T \) and agreeing with \( \sigma \) on \( \ker \tau \). Then we can rewrite the above condition as
\[
\sigma(x) = \max\{\nu(x), \tau(x)\}.
\]
In the non-degenerate case the seminorms \( \nu \) and \( \tau \) have non-degenerate kernels which are complementary subspaces. Thus \( \mathbb{R}^n_\sigma \) is the direct sum of two normed vector spaces, endowed with the maximum norm. In particular \( \sigma \) cannot be the Euclidean norm.
Bibliography


