Doctoral Thesis

Shear band propagation in soils and dynamics of tsunamigenic landslides

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SHEAR BAND PROPAGATION IN SOILS
AND DYNAMICS OF TSUNAMIGENIC LANDSLIDES

A dissertation submitted to

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for the degree of

Doctor of Sciences

presented by

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2009
Acknowledgment

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Abstract

Tsunami waves may be generated as a result of submerged landslides. Conventional geotechnical methods to model and calculate the wave height tend to underestimate the real height of the tsunami wave. A mechanism based on the phenomenon of shear band propagation along the failure surface of the landslide has been proposed to overcome this limitation. This mechanism is based on the energy balance (FMEB) approach from fracture mechanics and provides an opportunity to include an initial landslide velocity which could explain larger tsunami waves. One of the main advantages of this approach is that it allows a distinction to be made between progressive and catastrophic shear band propagation. The goal of this thesis is to validate the FMEB approach with physical tests and to develop an analytical solution of the mechanism of tsunamigenic landslides which takes into account dynamic effects during the failure process.

The rate of progressive shear band propagation is evaluated using two different test setups; the well known trapdoor test and a novel test setup, the shear-blade test, which allows curved shaped shear band propagation to be studied in granular materials. Analytical solutions based on the FMEB and the limiting equilibrium approaches are presented for both models for both cohesive and frictional-dilative materials. Comparison and validation of the analytical solutions with both experimental data and numerical calculations using FLAC, reveal that the FMEB is an appropriate tool for the modelling of shear band propagation in soils.

Within the framework of tsunamigenic landslides, the velocity of the catastrophic shear band propagation is a crucial parameter. In order to measure this velocity a series of preliminary biaxial tests under low confining stresses has been performed and the shear band propagation have been filmed using a high speed camera. As expected, the velocities of the shear band propagation appeared to be lower than the shear wave velocity.

Finally, a numerical solution and a closed form approximation for the calculation of the velocity of the shear band and of the initial velocity of
submerged landslides have been derived. The solution includes inertia effects and the viscous water resistance but neglects propagation and reflection of compaction waves (i.e. P-waves) in the sliding layer. When this method is applied to recent and historic landslides, it effectively produces the initial landslide velocities. These velocities are of great importance because they significantly affect the tsunami wave height predictions.
Zusammenfassung


Im Hinblick auf tsunamiauslösende Erdrutsche ist die Geschwindigkeit der katastrophalen Scherbandausbreitung von entscheidender Bedeutung. Um diese Geschwindigkeit zu messen sind in einem Biaxialtest erste Versuche durchgeführt und die Scherbandausbreitung mithilfe einer Hochgeschwindigkeitskamera gefilmt worden.
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Notation and Abbreviations

\( C_d \) [-] global hydrodynamic drag coefficient
\( D \) [kPa] stiffness parameter
\( D_t \) [J] energy dissipated along the shear band
\( D_r \) [%] relative density
\( D_o \) [J] energy dissipated in the process zone
\( E \) [kPa] Young’s modulus
\( E_{\text{ep}} \) [kPa] elasto-plastic modulus
\( F_{Fr} \) [-] Froude number
\( G \) [kPa] shear modulus
\( H \) [m] average water depth
\( I^* \); \( J \) [J/m\(^2\)] J-Integral (fracture mechanics)
\( K_o \) [-] coefficient of earth pressure at rest
\( K_b \) [-] coefficient accounting for the lack of constraint
\( K_d \) [-] coefficient for the reduction of dilation angle
\( L \) [m] total height of the trapdoor-test sample
\( M \) [kPa] constraint modulus
\( R \) [m] radius of the shear blade
\( T_i \) [-] traction vector
\( U \) [kPa] strain energy function
\( W \) [J] work, strain energy density
\( W_a \) [J] available energy
\( W_e \) [J] external work
\( W_i \) [J] internally stored / released work
\( a \) [-] regression parameter
\( b \) [m] width of the trapdoor
[-] regression parameter
\( c_0 \) [m/s] linear long-wave velocity
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d)</td>
<td>[mm]</td>
<td>shear band width</td>
</tr>
<tr>
<td>(d_{so})</td>
<td>[mm]</td>
<td>mean grain diameter</td>
</tr>
<tr>
<td>(g)</td>
<td>[m/s²]</td>
<td>gravity acceleration</td>
</tr>
<tr>
<td>(h)</td>
<td>[m]</td>
<td>landslide thickness</td>
</tr>
<tr>
<td>(l)</td>
<td>[m]</td>
<td>length of the shear band</td>
</tr>
<tr>
<td>(l_{cr})</td>
<td>[m]</td>
<td>critical length of the shear band</td>
</tr>
<tr>
<td>(i)</td>
<td>[m]</td>
<td>shear band propagation velocity</td>
</tr>
<tr>
<td>(n)</td>
<td>[-]</td>
<td>normal vector</td>
</tr>
<tr>
<td>(p_0)</td>
<td>[kPa]</td>
<td>pressure at the trapdoor or shear blade</td>
</tr>
<tr>
<td></td>
<td></td>
<td>initial total subhorizontal stress</td>
</tr>
<tr>
<td>(p_a)</td>
<td>[kPa]</td>
<td>active pressure</td>
</tr>
<tr>
<td>(p_L)</td>
<td>[kPa]</td>
<td>geostatic stress at depth (L)</td>
</tr>
<tr>
<td>(r)</td>
<td>[-]</td>
<td>ratio between peak and residual shear stresses</td>
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<td>(s_u)</td>
<td>[kPa]</td>
<td>undrained shear strength</td>
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<td>[s]</td>
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<tr>
<td>(v)</td>
<td>[m/s]</td>
<td>velocity of the sliding layer</td>
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<tr>
<td>(v_{max})</td>
<td>[m/s]</td>
<td>maximum slide velocity</td>
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<td>(v_s)</td>
<td>[m/s]</td>
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<td>(\Delta)</td>
<td>[-]</td>
<td>increment</td>
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<td>(\Gamma)</td>
<td>[-]</td>
<td>any curve surrounding a notch tip</td>
</tr>
<tr>
<td>(\Gamma_i)</td>
<td>[-]</td>
<td>surface along the notch tip</td>
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<tr>
<td>(\alpha_0)</td>
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<td>corrected rotation of the shear blade</td>
</tr>
<tr>
<td>(\gamma)</td>
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<td>unit weight of soil</td>
</tr>
<tr>
<td>(\gamma')</td>
<td>[kN/m³]</td>
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<td>(\gamma_w)</td>
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<td>(\delta)</td>
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<td>(\delta_m)</td>
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<td>relative linearized shear displacement before residual state</td>
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<tr>
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<td>Description</td>
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<td>-------------</td>
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<td>$\delta_r$</td>
<td>[m]</td>
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</tr>
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</tr>
<tr>
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<td>[-]</td>
<td>maximal compressive strain due to dilation</td>
</tr>
<tr>
<td>$\eta$</td>
<td>[rad]</td>
<td>angle between the shear tip the rotation axis and the initial position of the blade.</td>
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<td>[Pa s / m]</td>
<td>viscosity</td>
</tr>
<tr>
<td>$\nu$</td>
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<td>Poisson's ratio</td>
</tr>
<tr>
<td>$\rho$</td>
<td>[t/m$^3$]</td>
<td>density</td>
</tr>
<tr>
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<td>[kPa]</td>
<td>shear stress</td>
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<td>[kPa]</td>
<td>mobilized shear stress due to gravity (landslide)</td>
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<td>[kPa]</td>
<td>residual value of shear strength</td>
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<td>friction angle at constant volume</td>
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<tr>
<td>$\omega$</td>
<td>[m]</td>
<td>length of fracture process zone (softening zone)</td>
</tr>
<tr>
<td>$\bar{x}$</td>
<td>[-]</td>
<td>bar → normalized</td>
</tr>
<tr>
<td>$r, \theta, z$</td>
<td>[m] [rad] [m]</td>
<td>axis in cylindrical coordinate system</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>[m] [m]</td>
<td>axis in Cartesian coordinate system</td>
</tr>
</tbody>
</table>

FMEB fracture mechanics energy balance
LE limiting equilibrium
1 Introduction

1.1 Rationale and Objectives

Tsunamis are gigantic water waves, usually generated as a result of submarine earthquakes. These waves may cause tremendous damage, such as the disastrous event of December 26, 2004 in the Indian Ocean (Figure 1).

![Image](image1.jpg)

(a)

![Image](image2.jpg)

(b)

Figure 1. Aerial views of the shoreline at Banda Aceh, Indonesia (a) before and (b) after the 2004 tsunami (DigitalGlobe).
If an earthquake is strong enough, a tsunami may be generated directly as a result of the relative displacement of earth plates due to normal faulting. This mechanism has been most probably the main reason for the December 2004 disaster.

Nevertheless, there are seismic events which cause waves that are far larger than what can be expected from the motion of the plates. Some of these waves can be explained by the failure of submerged landslides (e.g., Wright & Rathje, 2003). Several cases of submerged landslides and their resulting tsunami wave have been studied; for instance the historic Storegga Slide off the coast of Norway (Bugge et al., 1988), the 1998 Aitape, Papua New Guinea tsunami (Tappin et al., 1999) as well as the relatively small-sized event in Chrütztrichter-basin of Lake Lucerne in the year 1601 (Schnellmann, 2004; Strasser, 2006).

The typical characteristics of submerged landslides which have been observed are their large length to thickness ratio, very mild inclination and high maximum landslide velocities.

Applying conventional limiting equilibrium theory to tsunami generating (i.e. tsunamigenic) landslides cannot fully explain the amount of energy released and the corresponding observed tsunami wave height; i.e. existing models underestimate the tsunami wave height. This is mainly because the limiting equilibrium approach does not treat the slope failure as a dynamic process and, therefore, does not allow the initial landslide velocity at the onset of the post-failure process to be taken into account. However, taking this velocity into account has been shown to be essential in order to explain observed tsunami wave heights as it allows treatment as a dynamic process.

Introduction of the mechanism of shear band propagation along the failure surface provides an attempt to explain this phenomenon (Puzrin & Germanovich, 2005). The fundamental idea behind this mechanism is that a shear band grows from an initial length and propagates along the potential failure surface until it reaches the full failure length. During this propagation, the material above the shear bands starts moving downwards and, as a consequence, the slope gets an initial landslide velocity (Figure 2) which leads
to a larger release of energy at the onset of the post failure landslide and so may explain the larger tsunami wave height.

Figure 2. Concept of the mechanism of shear band propagation along a potential failure surface leading to initial landslide velocity.

The mechanism of shear band propagation is based on the energy balance approach from fracture mechanics and was already postulated by Palmer & Rice in 1973.

One of the main advantages is that it allows distinguishing exactly between progressive (i.e. stable) and catastrophic (i.e. unstable) shear band propagation (Puzrin & Germanovich, 2003) in the sense that the shear band is driven by either external additional work or by internally released energy.

The goal of this thesis is to improve the understanding of the mechanisms of tsunamigenic landslides within this framework and to develop reliable computational and analytical models of the shear band propagation phenomenon. This would allow quantitative assessment of the initial landslide velocity and subsequently application of the model to real landslides.

More specifically, the objectives of this research are:

- Validation of the energy balance approach to shear band propagation in two-dimensional problems;

- Application of the energy balance approach to a solution of the dynamic boundary value problem: the propagation of the shear band in an infinite slope;
- Application of the obtained solution to real cases.

### 1.2 Structure of this Thesis

This thesis is structured as follows:

Following the introduction of the topic, an overview of the research area and the state of the art is provided (*chapter 2*).

In order to legitimize the application of the energy balance approach from fracture mechanics to the shear band propagation mechanism (and consequently to the mechanism of tsunamigenic landslides) a validation of the approach is required. Therefore, progressive shear band propagation has been studied by means of two different test setups; the trapdoor- and a novel shear blade test, which are described in chapters 3 and 4 respectively. Modelling the rate of the progressive shear band propagation in soils is the main focus of both chapters. Therefore analytical models of the progressive shear band propagation in trapdoor- and shear blade tests are derived and validated against physical and numerical test results.

Under certain conditions, particularly in case of the tsunamigenic landslides, the propagation mode of a shear band changes from progressive to catastrophic mode. In terms of the dynamics of tsunamigenic landslides, the velocity of the catastrophic shear band propagation becomes the crucial parameter for the determination of the initial landslide velocity and thus of the tsunami wave height. Hence, measurement of the velocity of the catastrophic shear band propagation allows understanding for a proper assessment and validation of the mechanism. For that reason, biaxial tests have been performed and recorded using a high-speed camera for the detection of the propagation velocity. The explanation of this test procedure, method of interpretation and results are presented and discussed in *chapter 5*.

In *chapter 6* a dynamic solution of the shear band propagation in submerged landslides is presented and is then applied in *chapter 7* to historic and recent
submerged landslides. The solution allows assessment of the initial landslide velocity of tsunamigenic landslides.

Chapter 8 summarises the main findings of this thesis and provides ideas for further investigation of this topic.
2 Overview and State of the Art

2.1 Tsunamis and their Sources

Tsunami waves are generated by a rapid displacement or motion of large volumes of water. The main sources of such events are normal faulting of earth plates during submarine earthquakes, volcanic eruptions or subaerial and submerged landslides. The submerged landslide mechanisms represent the main focus of this thesis. These mechanisms are described in more detail below.

2.1.1 Normal Faulting of Earth Plates

If a submarine earthquake is strong enough, a tsunami is generated directly as a result of the relative displacement of earth plates due to normal faulting. One plate moves up or down relative to its neighbouring plate as shown in Figure 3. If this relative displacement occurs under water a tsunami wave is triggered. Tsunami waves generated directly as a result of normal faulting can cause devastation on both local and remote coasts.

Figure 3. Normal faulting mechanism (from Bolt, 1993).

Motivated by the 2004 event, tsunami generation due to normal faulting of earth plates has been studied and analysed extensively in recent years (e.g., Lay et al., 2005; Ioualalen et al., 2007). But already before, events such as the
1896 Sanriku earthquake or the 1946 Aleutian Islands earthquakes (e.g., Kanamori, 1972) attracted the interest of researchers.

2.1.2 Volcanic Eruptions

Mechanisms of tsunami generation due to volcanic eruptions have been discussed intensely during the past few decades. Perhaps the most famous case is the 1883 Krakatau eruption in Sunda Strait between Java and Sumatra (e.g., Nomanbhoy & Satake, 1995), where due to an explosion, the northern part of Krakatau Island disappeared and a tsunami wave was triggered causing more than 34000 fatalities (Simkin & Fiske, 1983). Many different hypothesis explaining the cause of this event have been proposed, such as lateral blasts (Camus & Vincent 1983), large-scale collapse of the northern part (e.g., Self & Ramino, 1981), instability of pyroclastic flow deposits (e.g., Latter 1985; Verbeek, 1885) or submarine explosion (e.g., Yokoyama, 1981). Nevertheless the mechanism remains only partially understood.

2.1.3 Subaerial Landslides

There are few documented cases of large subaerial landslides falling into water and causing large localized impulse waves. One of the most spectacular examples was the Lituya Bay landslide on July 10, 1958 in Alaska. The landslide, triggered by an earthquake, fell into a narrow fjord and caused a wave that reached a maximum height of 520 m. However, when the wave reached the open ocean, its amplitude diminished quickly (Miller, 1960). The Vaiont reservoir disaster in 1963 (Müller, 1964) represents another event of this nature.

The mechanism of impulse waves generated by subaerial landslides has been extensively studied experimentally at the laboratory of hydraulics at ETH Zurich (VAW). As a result, empirical formulas for the main water wave characteristics as a function of the geometry of the used test setup have been proposed and improved by several authors (Fritz et al., 2003; Zweifel, 2004; Heller, 2007).
2.1.4 Submarine Landslides and Slumps

Already in the 1930’s researchers came to the conclusion that submarine landslides are to be considered as a significant cause of tsunamis (Gutenberg, 1939). Studies of subaqueous sedimentation have shown that submerged landslides are a common process not only at the seafloor (e.g., Bugge, 1988; Tappin, 1999; Greene, 2006) but also in lakes (Schnellmann, 2004; Strasser 2006). Since the early 1980’s various national and international projects have been related to the study of submarine mass movements and brought major advances in the understanding of this field. Due to the development of techniques for surveying of the seafloor using high-resolution multibeam echo-sounding and piston coring, the quality of bathymetric maps (in particular the description of morphology and the mapping of deposit forms from submarine mass movements) has been improved drastically (Locat & Lee, 2002). This has been demonstrated in case of the 1998 Aitape (Papua New Guinea) tsunami, where a new mapping technique revealed that the tsunami was most likely triggered by a sediment slump process (Tappin et al., 1999; Davies et al., 2003; Tappin et al. 2008). One of the major differences between slides and slups is that slumps have circular shaped failure surfaces and their length to thickness ratio is smaller.

One of the most famous historic events is the Storegga Slide tsunami off the coast of Norway (Jansen et al. 1987; Bugge et al., 1988; Dawson et al., 1988), which is described in more detail in chapter 7.

Many more cases have been reported, evaluated and modelled, such as the slides along the Costa Rica convergent margin (von Huene et al., 2004), around the Hawaiian Islands (e.g., Moore et al. 1994), in the eastern Mediterranean (e.g., Salamon et al., 2007) and the Gaviota and Goleta slides in Santa Barbara Basin, California (Greene et al., 2006) (Figure 4). For further information on the Goleta slide see and Chapter 7.

The run-up height of tsunami, caused by a landslide may be evaluated in several ways. One of these is from tsunami deposits in coastal areas affected by the wave (Bondevik et al., 2005). The other possible methods are a field
survey directly after the tsunami (e.g., Lynett et al., 2003) or by direct observation of the wave height (Cysat, 1601).

Figure 4. Detailed map of seafloor morphology of the Santa Barbara Basin. Computer generated hill shaded bathymetric image based on multibeam bathymetry (Greene et al. 2006).

2.2 Mechanics of Tsunamigenic Landslides

2.2.1 Failure Stages

In general, the following three stages in the evolution of the landslide-generated tsunami can be distinguished:

- stage I: triggering of slope failure
- stage II: post-failure landslide
- stage III: tsunami generation and propagation

Slope failure (stage I) occurs primarily on open continental-margin slopes and in the active river deltas in under- and normally consolidated sandy and clayey sediments. More detailed information on the mechanisms triggering slope failure is given in the next section.
At the onset of stage II, when the slide breaks out, the moving landslide generates the tsunami wave. Subsequently to the sliding process some landslides mobilize into flows and turbidity currents, whereas others remain slides or slumps with limited deformations and displacements (Locat & Lee, 2002).

During stage III, the water wave propagates through the water towards the shore. Commonly, three different boundary value problems for the analysis of the tsunami wave are distinguished:

- wave generation at the source
- wave propagation in the open water
- wave run-up along the continental slope into the shore

Researchers have been studying the process of tsunami generation by submarine landslides both analytically and numerically (Harbitz, 1992; Pelinovsky & Poplavsky, 1996; Ward, 2001; Tinti et al., 1997; Satake, 2001; Murty, 2003; Dutykh & Dias, 2009). While stages II and III of the evolution process are usually viewed as dynamic processes, conventional analysis of the slope failure (stage I) focuses commonly on the final limiting equilibrium state, which is a static condition.

In order to explain higher maximum velocities, models accounting for the initial acceleration from earthquakes have been developed (e.g., Harbitz, 1992). However it is known that this is not always legitimate. For instance, in the Aitape 1998 event, the submerged landslide occurred some 10 - 15 min after the earthquake (Davies et al., 2003). Consequently, earthquake acceleration cannot always explain an initial landslide velocity. The proposed mechanism of shear band propagation, which has the advantage of being independent of earthquake acceleration, provides an opportunity to explain this problem, while avoiding the introduction of earthquake acceleration.

### 2.2.2 Triggering Mechanisms of Submarine Landslides

Numerous mechanisms have been suggested to possibly contribute to the initiation of submarine landslides (Bardet et al, 2003; Wright & Rathije, 2003;
Locat & Lee, 2002). The authors distinguish between direct, indirect and combined triggering mechanisms.

**Direct mechanisms**

*Acceleration-induced sliding:* Due to an earthquake, soil slopes are subjected to horizontal and vertical accelerations. This may result in a reduction of the safety factor and initiate movement. If the acceleration is large enough, or if the time period is long enough, this may lead to a global failure of the slope. Most of the tools used for the analysis are quasi-static (e.g., Newmark, 1956), decoupled / coupled Newmark-type sliding analyses (e.g., Rathje & Bray, 2000) and nonlinear dynamic finite element analysis (e.g., Prevost et al., 1985). In spite of all analysis techniques it is unusual that seismic accelerations are high enough to cause large slope failure along the entire slide surface; they are more likely to serve as initiators, in combination with other mechanisms.

*Liquefaction-induced sliding:* Cyclic shearing caused by shear waves from earthquakes provokes the loss of shear strength. This occurs particularly in saturate deposits of loose cohesionless silts and sands due to generation of excess pore water pressure along the sliding surface (Morgenstern, 1967).

*Fault rupture-induced sliding:* Normal faulting may provoke surface profile changes. This may create unstable slopes which slip over the fault downwards (oversteepening of slopes).

**Indirect mechanisms**

*Water wave-induced sliding:* Large ocean waves generate changes in stress on the seafloor. This effect is also known as *storm wave loading*. The stress changes produce disturbing moments, tending to shear. A limiting equilibrium model has been proposed to predict slope instability due to ocean waves (Henkel, 1970).

*Rapid drawdown:* When a tsunami wave approaches a shoreline, the water level immediately drops and therefore removes the resisting water pressure. The immediate loss of water pressure may cause slope failures.
Shear strength and delayed failure mechanisms: Due to cyclic shear-loading during an earthquake, excess pore water pressure is accumulated along localized zones and, therefore, the effective stresses and shear strength are reduced. Still, the excess pore water pressure distribution is not necessarily constant along a sliding surface. Dense zones tend to dilate and water pressure drops, while loose zones tend to compact and produce excess pore water pressure. A conceptual framework taking time-dependency of the slope stability into account has been presented (Wright & Rathje, 2003).

Gas hydrate dissociation: In some cases, deposits of methane hydrate (i.e. form the decay of organic matter) are suspected to have caused the loss of shear strength in the soil by producing excess pore pressures (e.g., Nisbet & Piper, 1998; Xu & Germanovich, 2006).

All these models of the mechanism of landslide instability assume that the failure of a slope emerges along the entire failure surface simultaneously. For large landslides, where the length of the failure surface can be several kilometres long, this assumption does not seem to be realistic. Applying the mechanism of catastrophic shear band propagation along the failure surface, a much shorter initial weak failure zone is needed for the slope to reach failure state (Puzrin et al., 2004).

2.3 Progressive and Catastrophic Failure in Soils

Progressive and catastrophic failure in soils with strain-softening behaviour is commonly associated with the formation of shear bands along the failure surface. In material with strain-softening behaviour under shear loading conditions, most of the deformation is concentrated in narrow zones; i.e. shear bands, which lie between regions with relatively small shear deformation. Inside these shear bands, shear resistance drops from peak to residual strength.

Application of the energy balance approach allows distinguishing between progressive and catastrophic failure of soils. In the case of progressive failure, propagation of the shear band is a stable process, which means that
additional external energy to the system is required for the further propagation of the shear band. Catastrophic failure, on the other hand, is an unstable process and the shear band propagates without additional externally applied energy.

Strain localization and shear band formation can be observed in particulate materials exhibiting strain softening behaviour. These are overconsolidated clays and dense sands under drained shear conditions (e.g., Cuckson, 1977; Scarpelli, 1981) and under- or normally consolidated clays (e.g., Diaz-Rodriguez, et al., 1992; Puzrin et al. 1995, 1997; Watabe et al., 2002; Zumsteg et al., 2009) and loose sands under undrained shear conditions (e.g., Méghachou, 1993; Norris, 1999).

2.3.1 Progressive Failure in Slopes

In the literature, the term progressive is mainly associated with any growth of a slip surface. Although this terminology will be used here as well, in general the terminology progressive will be used for the stable shear band propagation, whereas the term catastrophic will be used when the propagation becomes unstable.

The phenomenon of progressive failure in soils has already been studied in the 1940’s (Terzaghi & Peck, 1948; Taylor, 1948), particularly for progressive failure in brittle soils, such as overconsolidated clays, and clay slopes (Skempton, 1964). Later it has been found that progressive failure may also be considered in normally consolidated sediments (e.g., Bernander et al., 1989). The importance of different values for peak and residual strength, particularly in clays, has been emphasised by Bjerrum (1967). In general, these models treated the phenomenon of progressive failure with respect to the shear resistance of soils assuming that the shear band appears at once along the entire failure surface.

It has been found that limiting equilibrium analysis cannot explain the mechanism of tsunamigenic landslides to the full extent. In particular, Scandinavian researchers have been focusing on the progressive failure extensively, mainly due to numerous large landslides that occurred in south-
western Sweden in normally consolidated sensitive clays and in Norwegian quick clay. Analytical models for stability calculations which account for progressive failure in soils considering the displacement-controlled shear softening behaviour of such soils have been presented (Bernander & Olofsson, 1981; Benander et al. 1989; Bernander, 2000). Simultaneously, steadily improving numerical codes have been developed and used for quasi-static stability calculations of progressive slides in sediments (e.g., Hoeg, 1972; Potts et al., 1990; Wiberg et al., 1990; Jostad & Andersen, 2004; Thakur, 2007).

2.3.2 Strain Localization and Shear Band Propagation

Strain localization in overconsolidated clays has been investigated experimentally using triaxial tests (e.g., Atkinson & Richardson, 1987; Asaoka et al., 1999) and studied with respect to the shear zone formation and propagation using a direct shear apparatus (Cuckson, 1977). It has been found that propagation of long shear bands could be treated as a ductile fracture mechanism. A first analytical model to this problem has been presented (Cuckson, 1977).

Following Cuckson’s investigation in overconsolidated clays, shear band propagation has also been studied in dense sand using direct shear apparatus (Scarpelli, 1981).

Experimental studies on strain localization in fine grained materials have been performed using biaxial tests (e.g., Mooney, et al., 1997; Desrues & Viggiani, 2004), hollow cylinder tests (Saada et al., 1999) and trapdoor tests (Vardoulakis, et al., 1981; Graf, 1984; Papamichos, et al., 2001). The performed studies mainly focused on the determination of the condition for strain localization; i.e. the shear band initiation, determination of the thickness and the orientation of the shear band.

Normalization of the thickness of a shear band with the mean grain size of the material has been found to be a convenient method to compare different materials. Therefore, the shear band width is defined as a multiple of the mean grain size of the material. It has been confirmed by several authors that
normalized shear band thickness decreases as grain size increases and as relative density decreases (Alshibli & Sture, 1999; Vardoulakis et al., 1981; Mühlhaus & Vardoulakis, 1987). The dependency observed by Alshibli & Sture (1999) for fine sands is shown in Figure 5. In clays, the thickness has been observed to be up to 200 times larger than the mean grain diameter.

![Figure 5. Shear band thickness (normalized) against the mean grain size from experimental tests (after Alshibli & Sture, 1999).](image)

A mathematical condition for the localized deformation based on bifurcation theory has been derived by Rudnicki & Rice (1975) and an analytical model for the determination of the thickness has been developed using Cosserat continuum theory (e.g. Gudehus & Nübel, 2004).

Bifurcation theory represents an alternative, mathematical approach for the definition of localization and formation of shear bands. Solutions are in general obtained by combining the constitutive relationship and equilibrium and compatibility equations on assumed localized shear bands and are solved as eigenvalue problem. The advantage of this approach is that it accounts for elasto-plastic soil behaviour and provides automatically the shear band orientation and deformation jump. However, in this approach the shear band
appears instantaneously along the entire failure surface and cannot be treated as a propagation process.

A fundamental review on the models based on bifurcation analysis is given by Vardoulakis & Sulem (2006).

Progressive Shear Band Propagation in Trapdoor Tests

In most conventional shear devices, such as triaxial- or biaxial- shear test devices, shear band formation always occurs in catastrophic mode. The trapdoor test is one of the few devices which allows investigation of progressive shear band propagation. Vardoulakis et al. (1981) provided a statical approach for the trapdoor problem based upon model test kinematics. Later on, constitutive models and finite element numerical calculations of the trapdoor problem were developed (e.g., De Borst & Vermeer, 1984; Koutsabeloulis & Griffiths, 1989; Smith, 1998; van Langen & Vermeer, 1991). In 1993 Tanaka & Sakai carried out numerical calculations to investigate scale effects of trapdoor problems with granular materials. Nevertheless, the rate of progressive shear band propagation has neither been studied nor modelled.

2.3.3 Fracture Mechanics Energy Balance Approach

In the classic paper of Palmer & Rice (1973) an approach for analysis of the growth of localized shear bands in the progressive failure of overconsolidated clay has been proposed. There, conditions for the growth of the shear band along a predefined slip surface have been derived in order to explain the progressive failure in overconsolidated clays, based on fracture mechanics energy balance principles (FMEB). The approach assumes a gradual decay of the shear strength from peak to residual value within the process zone at the shear band tip (Figure 7). The criteria for the shear band propagation were derived using the J-integral (Rice, 1968,a; 1968,b).

The J-Integral

Consider a crack in a flat surface under plane strain conditions (see Figure 6.)
The J-Integral is used to calculate the strain energy release rate per unit fracture area in a linear or nonlinear elastic material as shown in Figure 6. (Rice, 1968a). It is defined in general form as

$$J = \int_{\Gamma} \left( W \, dy - T \frac{\partial u}{\partial x} \, ds \right)$$  \hspace{1cm} (2.1)

where $\Gamma$ is a curve surrounding the notch tip, $W$ denotes the strain energy density

$$W = \int_{\Gamma} \sigma_{ij} \, d\varepsilon_{ij}$$ \hspace{1cm} (2.2)

where $\varepsilon$ is the infinitesimal strain tensor. The traction vector $T$ is defined according to the outward normal along $\Gamma$, $T_i = \sigma_j n_j$, $u$ is the displacement vector and $ds$ is an element of arc length along $\Gamma$.

![Figure 6. Flat surfaced crack in plane-strain conditions. $\Gamma$ is any curve surrounding the crack tip; $\Gamma_i$ denotes the curved crack tip.](image)

In the fundamental paper, Rice (1968a) delivered the proof that the J-integral is independent of the path and therefore can also be written as

$$J = \int_{\Gamma_i} W \, dy$$ \hspace{1cm} (2.3)

so that $J$ is a measured strain at the crack tip.
The decay due to the softening of the material under shear conditions as proposed by Palmer & Rice (1973), is then defined as

\[ J - \tau_r, \delta_r = \int (\tau - \tau_r) d\delta \]  

(2.4)

where \( \tau_r \) denotes the residual shear stress of the material and therefore the integral denotes the shaded area in Figure 7.

Figure 7. Relationship between the shear resistance and the relative displacement along the shear band.

The concept of the J-Integral and linear elastic fracture mechanics has been widely applied to the study of factors governing the rate of propagation (e.g., Rice, 1973; Rice & Cleary, 1976; Rice & Simons, 1976) and of the failure criteria of rock structures in compressive and tensile stress fields (e.g., Germanovich & Cherepanov, 1995; Scavia, 1995; Grekov & Germanovich, 1998; Dyskin et al., 2000).

Puzrin & Germanovich (2005) applied the approach from Palmer & Rice to the shear band propagation of soils in an infinite slope and to the catastrophic shear band propagation in submerged slopes that are built of normally consolidated clays (Puzrin, et al., 2004). The derivation of the shear band propagation criterion in elasto-plastic materials is less convenient because the J-Integral is in general path dependent. However, the assumption that the process zone is much smaller than the length and height of the problem \( l >> h >> \omega \) still provides a way of solving it. In general form, the energy
balance criterion for the shear band propagation can then be written as the inequality

$$\Delta W_e \geq \Delta W_i + \Delta D_i + \Delta D_o$$  \hspace{1cm} (2.5)$$

where the term on the left side of the inequality denotes the external work applied to the system, the first term on the right side denotes the internally stored or released energy due to change in normal stress and the two last terms are the plastic work dissipated along the shear band due to the residual strength and due to softening along the process zone, respectively. The condition for the shear band propagation requires that the energy surplus produced in the body by incremental propagation of the shear band should exceed the energy required for this incremental propagation.

It has been shown (Puzrin, et al., 2004) that already a small weak zone may generate a full-scale landslide. Although this model is one-dimensional, its experimental validation and numerical simulations are difficult due to the catastrophic nature of the shear band propagation, which is a dynamic problem. Here, an attempt is provided to close the gap between the model and real soil behaviour by comparison of analytical models, based on the fracture mechanics energy balance approach, with numerical and experimental results.
3 Progressive Shear Band Propagation in Trapdoor Tests

3.1 Introduction

Progressive failure in soils has been commonly associated with the phenomenon of the shear band propagation. This is a challenging topic both in terms of understanding and modelling. Discontinuities, moving boundaries, scale effect are just some factors complicating the analysis. It is not surprising, therefore, that numerical simulations of the phenomenon of the shear band propagation bear a number of intrinsic problems. The major problem is mesh-dependency: the shear band width appears to be at least of the size of the mesh element. Higher order numerical formulations or introduction of interfaces along the shear band propagation path claim to solve the mesh dependency problem. In reality, however, they rely on certain “internal length” or “material length”, which is just another way to sneak the width of the shear band in. And this still does not guarantee that the propagation rate is modelled correctly.

Another problem of these simulations is that they make it difficult to understand, what are the true mechanical and physical mechanisms behind the shear band propagation (and not just mathematical and numerical conditions necessary for it to take place).

An alternative to the numerical modelling could be a simple analytical approach, based on clear mechanical principles and, naturally, free of the mesh-dependency problems. It could facilitate a better understanding of the phenomenon and serve as benchmarks for numerical simulations. In particular, Puzrin & Germanovich (2005) proposed to extend the fracture mechanics energy balance approach of Palmer & Rice (1973) to model catastrophic shear band propagation in an infinite slope in granular materials with zero shear strength.

In general, everything based on the energy balance makes an impression of a fundamental approach and it is relatively easy to forget that there are many rather restrictive assumptions involved. Needless to say, the ultimate proof is
always the experimental validation. Unfortunately, for the shear band propagation, physical modelling is also a challenge: unstable catastrophic propagation takes place very fast, while the stable progressive propagation has been observed only in very few experimental setups. One of these setups is a trapdoor test. Though the typical trapdoor test results have been extensively presented in the literature (e.g., Vardoulakis, et al., 1981; Graf, 1984; Papamichos, et al., 2001), not many successful attempts have been undertaken in modelling the rate of the shear band propagation in these tests.

The purpose of this chapter is to model the rate of the shear band propagation in trapdoor tests using the energy balance approach for cohesive and frictional-dilatant material and to validate the analytical model with results from experiments and numerical simulations.

### 3.2 Physical Trapdoor Tests

#### 3.2.1 Concept

In a trapdoor test, a narrow glass container is filled with soil in thin horizontal layers with coloured boundaries. At the bottom of the container, a trapdoor can be displaced in a controlled manner, upwards or downwards, causing passive (Figure 8a) and active (Figure 8b) failure, respectively. The failure is progressive and expresses itself in a growth of the shear bands, which can be observed when the shear band crosses a soil layer boundary. Proper modelling of the rate of the shear band propagation in this test is crucial both for the better understanding of the phenomenon and for applications in geotechnical and geohazards engineering. Specifically, could it be explained: why does an incremental displacement of the trapdoor $\Delta \delta_0$ cause a particular increment of the shear band length $\Delta l >> \Delta \delta_0$?
Figure 8. Trapdoor Test: (a) passive mode; (b) passive mode.

3.2.2 Test Setup

The trapdoor device consists of a box made of a stainless steel frame and the two main side walls made of glass of a thickness of 10 mm. The box has inside dimensions of $960 \times 150 \times 1000$ mm $(w \times t \times h)$ and the trapdoor in the middle of the bottom has an area of $200 \times 150$ mm $(b \times t)$. The trapdoor itself can be moved up- and downwards perpendicular to the bottom plate (Figure 9). The velocity of the trapdoor is controlled by a step-motor. All tests have been performed at a constant velocity of the trapdoor of 0.1 mm/sec, i.e. displacement controlled.
3.2.3 Material Properties

*Grain size distribution*

Poorly graded silica sand was used for the tests.
The grain size distribution of the material was analysed using a laser scattering particle size distribution analyser (LA-950). The measurable size ranges from 0.010 to 3000.0 μm. The grading curve of the used material is shown in Figure 10. The median grain diameter is equal to $d_{50} = 0.28$ mm.

**Material Stiffness**

The constrained modulus $M$ has been obtained from three oedometer-tests (Figure 11). From these tests, upper and lower bounds of the loading and unloading curves have been calculated. In analogy to the empirical correlation from Hardin & Drnevich (1972), for the loading and unloading curves, regression between the constrained modulus $M$ and the vertical stress $\sigma_v$ has been found to be of the form

$$M = a \cdot (\sigma_v)^b$$  \hspace{1cm} (3.1)

In Table 1, parameters are shown for the loading and unloading curves for both upper and lower bound values.

![Figure 11](image-url)  
*Figure 11. Plot of the constrained modulus $M$ against the vertical mean stress from oedometer tests. Black diamonds and lines denote test results and regression curves in loading, grey dots and lines in unloading, respectively.*
Table 1. Parameters of regression curves from oedometer test results.

<table>
<thead>
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<th>curve</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>loading curve - upper bound</td>
<td>253.42</td>
<td>0.87</td>
</tr>
<tr>
<td>loading curve - lower bound</td>
<td>113.1</td>
<td>0.97</td>
</tr>
<tr>
<td>unloading curve - upper bound</td>
<td>527.2</td>
<td>1</td>
</tr>
<tr>
<td>unloading curve - lower bound</td>
<td>378.75</td>
<td>1</td>
</tr>
</tbody>
</table>

The Poisson’s ratio has been chosen to be $\nu = 0.3$. This results in an earth pressure coefficient at rest of $K'_0 = \nu/(1 - \nu) = 0.43$.

**Shear Strength**

Triaxial tests have been performed at constant radial stresses of 50 and 100 kPa by Nater (2006). Figure 12 shows the friction angle against the maximum axial stresses.

From the triaxial test results the peak and residual friction angles (at constant volume) have been determined as $\phi'_p = 37^\circ$ and $\phi'_{cv} = 30^\circ$, respectively.

![Figure 12. Peak and residual friction angles from triaxial test results (after Nater, 2006).](image-url)
The friction between the glass boundary of the trapdoor and the sand has been found experimentally from sand on a glass plate with increasing inclination and is about $\phi' = 18^\circ$.

### 3.2.4 Experimental Program and Test Interpretation

Experimental program consisted of six trapdoor tests: three in the passive and three in the active mode. During the filling of the box, the sand was compacted to the relative density of $D_r = 80 - 90 \%$, resulting in the total unit weight of $\gamma = 17 \text{ kN/m}^3$. Horizontal marker lines were embedded every 3 cm at the glass boundary, allowing for the shear band evolution to be observed.

Tests were performed at 20 cm, 40 cm, and 60 cm sand heights in both, passive and active modes. The trapdoor velocity was 0.1 mm/s and the shear band propagation was recorded using a digital camera. In the simulation only the initial shear bands were modelled. In Figure 14 and Figure 15, the initial shear bands and their inclination to the vertical axis; i.e. the dilation angle are shown for all performed tests at the end-stage of the initial shear band. With an overlay of the initial picture, this allows for easier interpretation.

![Figure 13. Indicator of passed shear band tip.](image)

The trapdoor tests have been interpreted as follows. According to Vardoulakis et al. (1981), the residual friction can be mobilized after the relative displacement reaches a half of the thickness of the shear band. Therefore, a
discontinuity of this kind in a marker line indicated arrival of the shear band tip (Figure 13).

The fact that the real behaviour of the material above the shear band tip is elastic, while in the analytical model it is assumed to be rigid, has been taken care of by measuring the mean vertical displacement both at the trapdoor and at the height of the tip. At the tip, due to the curved shape of the marker line the displacement has been measured twice; in the middle and in the vicinity of the shear band in order to obtain an average displacement at the height of the shear band tip. The difference between the displacement at the tip and at the trapdoor \( \delta_0 = (\delta_t - \delta) / L \), in normalized form, is comparable to the normalized displacement due to stress changes in the elastic material within this zone; equal to the normalized displacement in the analytical model (see below).

### 3.2.5 Test Results

The shear band propagation rate was evaluated by plotting the position of the initial shear band tip \( \bar{l} = l / L \) (normalized with the total initial sand height in the corresponding test) against the normalized elastic displacement \( \bar{\delta}_0 \) measured from the moment of the shear band initiation. The results are plotted as a shaded area due to two reasons: First when the shear band is detected at a certain height, the tip will be somewhere between this and the next higher marker line. Second, there are two initial shear bands propagating, one from each edge of the trapdoor. The shaded area is therefore the envelope of these two shear bands. Results from physical tests are shown for both the active mode and passive mode. In Figure 16, the final stage of the first appeared shear bands are shown for each test.
Figure 14. Length and orientation of initial shear bands in active mode trapdoor tests at the end of the propagation.

Figure 15. Length and orientation of initial shear bands in passive mode trapdoor tests at the end of the propagation.
Figure 16. Shear band propagation in active (a) and passive (b) mode trapdoor tests. The normalized height of the shear band tip is plotted against the normalized displacement.
3.3 **Analytical Modelling**

3.3.1 **Formulation of the Problem**

At the bottom of a box of the height $L$, a trapdoor of the width $b$ is moved a distance $\delta_0$ downwards (active mode, see Figure 17) or upwards (passive mode), respectively. In this plain strain model, the shear bands propagate from the corners of the trapdoor at the inclination $\psi$ to the vertical axis $x$. The two vertical boundaries are sufficiently far not to influence the propagation.

![Figure 17. Analytical model of the shear band propagation in active mode.](image)

In a displacement controlled test, each value of the trapdoor displacement $\delta_0$ corresponds to a certain value of the average pressure $p_0$ acting on the trapdoor, and to a certain length of the shear band, with its vertical component defined as $l$. Our task is to find the dependency $l = l(\delta_0)$. Unlike the mentioned applications of the energy balance approach to the shear band propagation in soils (Palmer & Rice, 1973; Puzrin & Germanovich, 2005), the stresses normal to the shear band are not constant along the shear band length, which results in significantly different solutions for cohesive and frictional materials. Therefore, these cases have to be treated separately. Another major complication is due to the dilatant behaviour of densely packed frictional material. This results in different solutions for dilatant and non-dilatant frictional materials, which again have to be treated separately. A solution for frictional material without dilation has already been presented (Saurer & Puzrin, 2007). In this chapter, solutions for cohesive and frictional-
dilatant material are derived. However, before considering these particular cases, it is worth to summarize some assumptions common to all of them.

### 3.3.2 Assumptions

The following simplifications have been made for the plane strain deformation trapdoor problem:

- the shear band is assumed to be straight, and not slightly curved as in experiments;

- the shear bands propagate in vertical direction;

- the vertical normal stress \( \sigma_x(x) \) is assumed to be uniformly distributed along any horizontal line \( x = \text{const} \) between the shear bands, with its value being equal to the average vertical effective stress acting on this line;

- apart from the two small end zones \( \omega \), the shear resistance \( \tau \) along the shear band is equal to its residual value \( \tau_r \). At the tip of the shear band as well as at any point outside the band, the maximal shear resistance is equal to its peak value \( \tau_p \). Within the end zones; i.e. the process zones \( \omega \), the shear resistance \( \tau \) decreases as a function of the relative displacement \( \delta \) from its peak \( \tau_p \) to its residual \( \tau_r \) value (at the relative displacement \( \delta_r \)) (Figure 18);

- the soil between shear bands is linear elastic with constrained modulus \( M \);

- the soil outside and above the shear bands is rigid plastic;

- the initial geostatic stress is:

\[
\sigma_{so} = \gamma(L-x) = p_L(1-\bar{x})
\]  

(3.2)

where \( \bar{x} = x/L \), \( p_L = \gamma L \); \( \gamma \) is the unit volume weight of soil;
- the initial stress at the shear band tip $\bar{x} = \bar{t}$ is therefore:

$$\sigma_{\tau 0} = p_L \left(1 - \bar{t}\right)$$

(3.3)

- the friction between the glass walls and the soil is taken into account with $\tau_s$.

![Figure 18. Relationship between the shear resistance and the relative displacement along the shear band.](image)

### 3.3.3 Cohesive Material

First, the simpler case, the shear band propagation in non-dilatant cohesive material is going to be considered. The simplifications following from this assumption are:

- the shear stresses along the shear band and between the glass and the material do not depend on the normal stress magnitude;

- the shear bands propagate in the vertical direction.

In this derivation an active failure mode will be considered, with the results for the passive mode being given for the final results, only.
Stresses and Strains

Normal vertical Stresses

In the active case, the trapdoor is moving downwards and the shear stresses acting on the sliding body between the shear bands are directed upwards. The equilibrium of forces acting in vertical direction on an soil element (Figure 19) of infinitesimal thickness \( dx \) can be written as

\[
\gamma + \frac{d\sigma_x}{dx} - \frac{2 \cdot \tau_r}{b} - \frac{2 \cdot \tau_s}{t} = 0
\]  

which after integration produces a linear distribution of the normal vertical stress

\[
\int \frac{d\sigma_x}{dx} \, dx = \sigma_x(x) = \left( \gamma - \frac{2 \cdot \tau_r}{b} - \frac{2 \cdot \tau_s}{t} \right) x + c
\]  

Figure 19. Stresses acting on a soil element between the shear bands.

Using the boundary condition of constant stress at the trapdoor \( \sigma_x(x = 0) = p_0 \) gives \( c = p_0 \) and therefore we get the equation of the vertical stress distribution

\[
\sigma_x(x) = p_0 - \left( \gamma - \frac{2 \cdot \tau_r}{b} - \frac{2 \cdot \tau_s}{t} \right) x
\]  

or in the normalized form

\[
\sigma_x(x) = p_0 - p_L (1 - K_e) \bar{x}
\]
where
\[ K_c = 2 \left( \frac{\tau_c}{b} + \frac{\tau_s}{t} \right) \frac{L}{P_L} \] \quad \bar{x} = \frac{x}{L} \quad P_L = \gamma L \tag{3.8}

and as an expression for the stress at the tip of the shear band \( \tilde{I} \) we obtain
\[ \sigma_i = \sigma_i(\tilde{I}) = p_0 - P_L \left( 1 - K_c \right) \tilde{I} \tag{3.9} \]

**Strains and displacements**

Adopting the zero strain state at the initial geostatic stresses and considering the material outside the shear bands as rigid, the stress-strain behaviour of the material between the shear bands is described by the linear relationship

\[ \Delta \varepsilon_x = \frac{\Delta \sigma_x}{M} = \frac{\sigma_x - \sigma_{x0}}{M} \tag{3.10} \]

where \( M \) is the constrained modulus of soil:

\[ M = \frac{(1-\nu)}{(1+\nu)(1-2\nu)} E \tag{3.11} \]

where \( E \) is the Young’s modulus, \( \nu \) is the Poisson’s ratio. Note, that because the stress in the active mode is decreasing, the strain increment is negative.

The vertical displacement of soil between the shear bands can be calculated in general form as the integral of change in strain over the height

\[ \delta(x) = \int \Delta \varepsilon_x(x) dx + c \tag{3.12} \]

Using the boundary condition \( \delta(I) = \delta_r \), using Eq. (3.10), this can be then expressed as

\[ \delta(x) - \delta_r = \frac{1}{M} \int \left( \sigma_x(x) - \sigma_{x0}(x) \right) dx \tag{3.13} \]

Using Eqs. (3.2) and (3.7) and integrating gives (in the normalized form):

\[ \bar{\delta}(\bar{x}) - \bar{\delta}_r = \frac{1}{M} \left( p_L \left( \tilde{I} - \bar{x} \right) - p_0 \left( \tilde{I} - \bar{x} \right) - \frac{P_L K_c}{2} \left( \tilde{I}^2 - \bar{x}^2 \right) \right) \tag{3.14} \]

where
\[ \bar{\delta}(\bar{x}) = \delta(x)/L \quad \bar{\delta}_r = \delta_r/L \quad \tilde{I} = I/L \tag{3.15} \]
and for the displacement at the trapdoor

\[
\delta(0) - \delta_c = \delta_0(\bar{I}) = \frac{\bar{I}}{M} \left( p_L - p_0 - \frac{\bar{I}}{2} p_L K_c \right)
\]  

(3.16)

This is the displacement of the trapdoor as a function of the shear band length and of the pressure at the trapdoor.

**Energy Balance Propagation Criterion**

**Incremental propagation**

Consider an incremental propagation of the shear band with length \( l \) by an increment \( \Delta l \). Additional displacement of the trapdoor (and the entire soil body between the shear bands) \( \Delta \delta_0 \), caused by this propagation, is equal to the strain at the shear band tip multiplied by the length increment (in consequence to Eq. (3.10)):

\[
\Delta \delta_0 = \Delta \varepsilon(l) \cdot \Delta l = \frac{\sigma_x(l) - \sigma_{x0}(l)}{M} \cdot \Delta l = \frac{\sigma_f - \sigma_{f0}}{M} \cdot \Delta l
\]  

(3.17)

Substitution of Eqs. (3.3) and (3.9) into Eq. (3.17) gives

\[
\Delta \delta_0 = \Delta \varepsilon(l) \cdot \Delta l = \frac{\Delta l}{M} \left( p_L - p_0 - p_L K_c \bar{I} \right)
\]  

(3.18)

This relationship can be also obtained by differentiating Eq. (3.16) with respect to \( l \).

**Work Components**

The energy balance criterion requires that the surplus of external work and energy released in the body due to an incremental propagation of the shear band should exceed the elastic and plastic work required for this incremental propagation. Mathematically this can be expressed as the following inequality:

\[
\Delta W_e \geq \Delta W_i + \Delta D_f + \Delta D_s + \Delta D_o
\]  

(3.19)
where the energy balance is considered between the following five components:

- external work $\Delta W_e$ done by the gravitational and external forces on the movement of the body between the shear bands and the trapdoor;
- internal elastic energy $\Delta W_i$ stored or released due to the changes of stresses in the soil body between the bands;
- plastic work $\Delta D_l$ dissipated due to residual friction along the entire length of the shear bands $l$ within the shear band;
- plastic work $\Delta D_s$ dissipated due to friction along the length of the shear bands $l$ between the material and the glass walls;
- plastic work $\Delta D_\omega$ dissipated in the process zone $\omega$ due to the friction above residual.

As a result of the incremental shear band propagation, the stresses change only in the propagation zone between $l$ and $l + \Delta l$. Therefore the soil between the trapdoor $(x = 0)$ and the tip $(x = l)$ moves downwards as a rigid body by distance $\Delta \delta_0$ given by Eq. (3.17). The combined external work done on this displacement by gravity and by the pressure $p_0$ acting on the trapdoor can be expressed using Eq. (3.18):

$$\Delta W_e = p_0 \cdot b \cdot t \cdot \Delta \delta_0 - \gamma \cdot l \cdot b \cdot t \cdot \Delta \delta_0 = (p_0 - p_\perp \cdot \bar{I}) \cdot b \cdot t \cdot \frac{\Delta l}{M} \cdot (\sigma_I - \sigma_{\perp 0}) \quad (3.20)$$

Because, the stresses and strains change only in the propagation zone between $l$ and $l + \Delta l$, the change in the internal energy also occurs only in this area:

$$\Delta W_i = \int_{l + \Delta l}^{l + \Delta l + 1} \int_0^{\sigma} \sigma_y d(\Delta \varepsilon_y) \cdot b \cdot t = \Delta l \cdot b \cdot t \cdot \int_{\sigma_{\perp 0}}^{\sigma} \sigma_y d\left(\frac{\sigma_y - \sigma_{\perp 0}}{M}\right) = \Delta l \cdot b \cdot t \cdot \frac{\sigma_I^2 - \sigma_{\perp 0}^2}{2M} \quad (3.21)$$

The plastic work dissipated along the two shear bands is done by the residual strength $\tau_r$ on the relative incremental displacement $\Delta \delta_0$. Similarly the plastic work dissipated between the soil and the glass wall is taken into account by $\tau_s$. The two addends are given by:
\[ \Delta D_j = 2t \int_0^l \tau_j dx \cdot |\Delta \delta_j| = 2 \cdot t \cdot l \cdot \tau_j \cdot |\Delta \delta_j| \quad (3.22) \]

\[ \Delta D_s = 2b \int_0^l \tau_s dx \cdot |\Delta \delta_s| = 2 \cdot b \cdot l \cdot \tau_s \cdot |\Delta \delta_s| \quad (3.23) \]

Summation of Eq. (3.22) and (3.23) gives, using the expressions from Eqs. (3.8) and (3.17)

\[ \Delta D_j + \Delta D_s = 2 \cdot t \cdot b \cdot l \left( \frac{\tau_j}{b} + \frac{\tau_s}{l} \right) |\Delta \delta_j| = -t \cdot b \cdot p_i K \cdot \frac{Nl}{M} (\sigma_i - \sigma_{i0}) \quad (3.24) \]

and after substitution of Eq. (3.9) it follows

\[ \Delta D_j + \Delta D_s = t \cdot b \cdot (p_0 - p_i \bar{I} - \sigma_i) \cdot \frac{Nl}{M} (\sigma_i - \sigma_{i0}) \quad (3.25) \]

Finally, the plastic work due to material softening in the shear band process zone is given by the shadowed area in Figure 18. As this zone propagates distance \( \Delta l \) into the material, the plastic work dissipated in the process zone due to the friction above residual is given by:

\[ \Delta D_w = 2 \cdot t \cdot \Delta l \int_0^\delta \left( \tau - \tau_r \right) \delta \delta = \tau_r (r-1) \cdot \delta_m \cdot L \cdot t \cdot \Delta l \quad (3.26) \]

using

\[ r = \frac{\tau_p}{\tau_r}, \quad \delta_m = \frac{2}{\tau_r - \tau_p} \left( \tau - \tau_r \right) \delta \delta \]

\[ \delta_m = \left( \frac{2}{\tau_r - \tau_p} \right) (\tau - \tau_r) \delta \delta \quad (3.27) \]

**The Energy Balance Criterion**

Substitution of Eqs. (3.20) - (3.26) into the criterion from Eq. (3.19) gives the inequality

\[ \left( p_0 - p_i \bar{I} \right) \cdot b \cdot l \cdot \frac{Nl}{M} (\sigma_i - \sigma_{i0}) \geq \Delta l \cdot b \cdot t \cdot \frac{\sigma_i^2 - \sigma_{i0}^2}{2M} + t \cdot b \cdot \left( p_0 - p_i \bar{I} - \sigma_i \right) \cdot \frac{Nl}{M} (\sigma_i - \sigma_{i0}) \quad (3.28) \]

+ \tau_r (r-1) \cdot \delta_m \cdot L \cdot t \cdot \Delta l
which can be reduced to

\[
(\sigma_i - \sigma_{i0})^2 \geq \frac{\tau_c}{b}(r-1)L M \delta_m
\]  

(3.29)

When resolving this inequality, recall that in the active mode the normal stresses are decreasing, i.e. \( \sigma_i \leq \sigma_{i0} \). This leaves us with the following solution:

\[
\sigma_i \leq \sigma_{i0} - p_L \sqrt{\bar{p}_c}
\]

(3.30)

where \( \bar{p}_c = \frac{\tau_c L}{p_L b} (r-1) M \delta_m \); \( \bar{M} = \frac{M}{p_L} \)

representing the energy balance shear band propagation criterion.

**Progressive and Catastrophic Failure in a Stress Controlled Test**

Physical meaning of the energy balance criterion is better understood after the substitution of Eqs. (3.3) and (3.9) into Eq. (3.30), which gives the condition for the shear band propagation in a stress controlled trapdoor test:

\[
(p_0 - p_{cr}) - \frac{\tau_c L}{p_L b} (r-1) M \delta_m \leq 0
\]

(3.31a)

\[
(p_0 + p_{cr}) + \frac{\tau_c L}{p_L b} (r-1) M \delta_m \geq 0
\]

(3.31b)

Where Eq. (3.31a) correspond to the active, Eq. (3.31b) to the passive test results.

This condition can be interpreted as follows. If, in active mode, the pressure at the trapdoor is larger than critical, \( p_0 \geq p_{cr}(l) \), the shear band of the length \( l \) will not grow. When the pressure at the trapdoor drops below critical, \( p_0 < p_{cr}(l) \) the shear band will grow by an increment \( \Delta l \), and the critical pressure will decrease until the two pressures become equal: \( p_0 = p_{cr}(l + \Delta l) \). This describes a progressive (stable) propagation of the shear band, which in principle, continues until a critical length \( l_{cr}^{eb} \) is reached, such that

\[
\frac{dp_{cr}(l)}{dl} \bigg|_{l_{cr}^{eb}} = 0
\]

(3.32)
At this length, increase in shear band length would not lead to the decrease in the critical pressure anymore, and if the trapdoor pressure remains constant, the propagation of the shear band becomes catastrophic (unstable). It can only return to progressive propagation, if the pressure is reduced. In the case of Eq. (3.31), however, condition (3.32) is never satisfied, and the shear band propagation is always progressive, until another condition – limiting equilibrium becomes more critical (see section further below).

**Progressive and Catastrophic Failure in a Displacement Controlled Test**

In reality, the trapdoor tests are normally displacement controlled. Because during progressive propagation the critical pressure always stays equal to the trapdoor pressure \( p_0 = p_c \), substitution of Eq. (3.31) into the expression for the displacement at the trapdoor (3.16) gives the condition for the progressive shear band propagation in a displacement controlled mode:

\[
\delta(0) - \delta_c = \delta_0(\bar{l}) = \frac{l}{M} \left( \sqrt{\bar{p}_c + \frac{l K_c}{2}} \right)
\]  

(3.33)

Interpretation of this condition is similar to that of the stress controlled test. For any shear band length \( l \) there is a critical displacement \( \delta_c(l) \), such that if the trapdoor displacement \( \delta_0 \) is smaller than critical, \( \delta_0 \leq \delta_c(l) \), the shear band of the length \( l \) will not grow. When the trapdoor displacement \( \delta_0 \) increases above the critical, \( \delta_0 > \delta_c(l) \) the shear band will grow by an increment \( \Delta l \), and the critical displacement will increase until the two displacements become equal: \( \delta_0 = \delta_c(l + \Delta l) \). This describes a progressive (stable) propagation of the shear band, which, in principle, continues until a critical length \( l_{cr}^{ph} \) is reached, such that

\[
\left. \frac{\partial \delta_c(l)}{\partial l} \right|_{l = l_{cr}^{ph}} = 0
\]

(3.34)

At this length, increase in shear band length would not lead to the increase in the critical displacement anymore, and even if the trapdoor displacement remains constant, the propagation of the shear band will become catastrophic (unstable). In the case of Eq. (3.33), however, condition (3.34) cannot be satisfied for a positive length of the shear band. Therefore, the shear band
propagation is again always progressive, until another condition – limiting equilibrium becomes more critical.

**Limiting Equilibrium Propagation**

In the process of the shear band propagation, the soil above the shear band tips is considered to be rigid plastic. The growth of the shear bands will be governed by the energy balance until the soil column above them reaches the state of the limiting equilibrium. This occurs when the weight of the soil column can no more be supported by the pressure $\sigma_l$ at its bottom and the shear stresses $\tau_p$ and $\tau_s$ at its sides (Figure 20). After that moment the propagation is governed by the limiting equilibrium.

$$\int \sigma_l dx - \int \tau_p dx - \int \tau_s dx \leq 0$$

In normalized form, using Eq. (3.3) and Eq. (3.8), this gives the criterion for the shear band propagation:

$$\sigma_l \leq \sigma_{i0} - p_l \mathcal{K}_e n_c (1 - \mathcal{I})$$

where $n_c = \frac{\mathcal{K}_{c,LE}}{\bar{K}_c}$; $\mathcal{K}_{c,LE} = 2 \left( \frac{\tau_p + \tau_s}{b} \right) \frac{L}{p_L}$; $\bar{K}_c = 2 \left( \frac{\tau_p + \tau_s}{b} \right) \frac{L}{p_L}$

**The Limiting Equilibrium Criterion**

The limiting equilibrium of forces acting on the soil column above the tips of the shear bands (Figure 20) is given by

$$b \cdot t \cdot \sigma_l \leq (L - l) \cdot b \cdot t \cdot \gamma - 2 \cdot t \cdot \int l \tau_p dx - 2 \cdot t \cdot \int s \tau_s dx$$

(Figure 20. Stresses acting on the remaining soil element above the shear band tips in active mode.)
Progressive and Catastrophic Failure in a Stress Controlled Test

Substitution of (3.3) and (3.9) into (3.36) gives the condition for the shear band propagation in a stress controlled test:

\[ p_0 \leq p_{cr} = p_L \left(1 - K_c \left(\hat{l} - n_c \hat{l} + n_c\right)\right) \]  
\[ \text{(3.38a)} \]

\[ p_0 \geq p_{cr} = p_L \left(1 + K_c \left(\hat{l} - n_c \hat{l} + n_c\right)\right) \]  
\[ \text{(3.38b)} \]

This describes a stable propagation of the shear band, i.e., the progressive failure in active Eq. (3.38a) and passive mode Eq. (3.38b). This stable limiting equilibrium driven propagation will start at the certain initial length \( l_{0e} \), at which the conditions (3.31) and (3.38) are for the first time satisfied simultaneously:

\[ \delta_{le} = \frac{1}{n_c K_c} \]  
\[ \text{(3.39)} \]

where

\[ p_c = \frac{\tau_c L}{p_L b (r-1) M \delta_n} \quad M = \frac{M}{p_L} \]

This progressive failure will take place up to a point when the shear band reaches a positive critical length \( l_{cr} \) such that

\[ \frac{\partial p_{cr} (l)}{\partial l} \bigg|_{l=l_{cr}} = 0 \]  
\[ \text{(3.40)} \]

Progressive and Catastrophic Failure in a Displacement Controlled Test

With \( p_0 = p_{cr} \) and the substitution of (3.38) into the equation of the displacement of the trapdoor (3.16), gives

\[ \delta_0 (\hat{l}) \geq \delta_{cr} (\hat{l}) = \frac{\hat{l}}{M K_c} \left(n_c - n_c \hat{l} + \frac{\hat{l}}{2}\right) \]  
\[ \text{(3.41)} \]

This describes a progressive (stable) propagation of the shear band, which, starts at the length \( l_{0e} \), given in Eq. (3.39), and continues until a critical length \( l_{cr} \) is reached.
\[ \tilde{I}_{cr} = -\frac{n_c}{2n_c - 1} \]  

which satisfies the condition:

\[ \frac{\partial \delta_{cr} (l)}{\partial l} \bigg|_{l=\Delta l} = 0 \]  

Therefore, starting from

\[ \delta_{cr} (\tilde{I}_{cr}) = \frac{K_c}{2M} \frac{n_c^2}{2n_c - 1} \]  

the propagation becomes unstable, i.e. catastrophic.

### 3.3.4 Validation of Energy Terms

Since the analytical solutions incorporate various simplifications which may affect the solutions significantly, the question arises how much these simplifications cost us in terms of energy as compared to the exact solution. Therefore, in order to validate the models and provide a proper assessment, a comparison of the obtained solution with the exact solution is required.

Specifically, the following simplifications are of major concern:

- The soil beyond the area enveloped by the shear bands is assumed to be rigid. What is the effect of neglecting the elastically stored work due to this rigidity in comparison with the total internally stored / released energy?

- We assume that the stresses are distributed constant along a horizontal cross section. How much does this assumption affect the simplified solution in comparison with the exact solution?

**Method**

The validation of the analytical solution was performed as follows: The energies available for the propagation of the shear band after a propagation of predefined length \( \Delta l \) and under predefined applied pressures at the trapdoor
were calculated. The pressure at the trapdoor was varied within a range in the vicinity of the critical pressure calculated from the analytical solution $p_{cr}$. The resulting available energy $W_a$ is equal to the sum of the applied, stored and dissipated energies due to an incremental propagation of the shear band $\Delta l$ in the model are calculated analytically with

$$W_a = \Delta W_s + \Delta W_f + \Delta D_n$$  \hspace{1cm} (3.45)

In this context it is assumed that the numerical model is a close approximation to the exact solution, therefore the energies are comparable to those from the exact solution.

The cohesive material model is investigated for three different cases in active mode, listed in Table 2.

**Table 2. Cases studied with respect to the energy terms.**

<table>
<thead>
<tr>
<th>case</th>
<th>test height $L$ [m]</th>
<th>shear band length $l$ [m]</th>
<th>normalized $\bar{l} = l / L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.4</td>
<td>0.13</td>
<td>0.325</td>
</tr>
<tr>
<td>B</td>
<td>0.6</td>
<td>0.13</td>
<td>0.216</td>
</tr>
<tr>
<td>C</td>
<td>0.6</td>
<td>0.26</td>
<td>0.433</td>
</tr>
</tbody>
</table>

For each of these geometries, the available energy after a prolongation of the shear band by $\Delta l$ (of 20 mm) was calculated for both the analytical and the numerical model.

The numerical model used is shown in Figure 21. After calculation of the initial state stresses due to the gravitational force, the fixation below the trapdoor has been replaced by a pressure $p_0$ and along the shear band, shear stress has been applied.

After the calculation, the total energy of stage 1 (shear band length $= l$) was subtracted from the total energy of stage 2 (shear band length $= l + \Delta l$), resulting in the change of total energy due to the incremental prolongation of the shear band.
Energies from the analytical models are calculated from Eqs. (3.20) - (3.22).

In the numerical model, the external energy for each step is calculated with

\[
\Delta W_e = \sum_{\text{grid}} (\Delta y \cdot \gamma \cdot t \cdot A_{\text{grid}}) - p_0 \cdot t \cdot b / 2 \cdot \Delta \delta_{\text{average}}
\]  

(3.46)

where \( A_{\text{grid}} \) is the area of the element, \( \Delta y \) is the difference of the displacement of each element between the stages in vertical direction. In the first term on the right side of Eq. (3.46), the energy due to the gravity load and displacement of the material within the trapdoor test is calculated by summation over the elements, in the second term, the change of energy due to the displacement and the pressure working against the displacement is calculated.

The change of the internal energy is calculated with

\[
\Delta W_i = W_i(t + \Delta t) - W_i(t)
\]

(3.47)

where

\[
W_i = \frac{1}{2} \sum \left[ (\sigma_{xx} e_{xx} + 2\sigma_{xy} e_{xy} + \sigma_{yy} e_{yy}) A_{\text{grid}} \right]
\]

(3.48)
is the internal energy for one stage.

The dissipated energy along the shear band is calculated by

$$\Delta D_i = (l + \Delta l) \tau_i \delta_{i+1,av} - l \tau_i \delta_{i,av} \quad (3.49)$$

where $\delta_{i,av}$ is the average displacement along the shear band.

**Results and Discussion**

Results from the validation process are shown below. The plots show the resulting available energy as a function of the applied pressure at the trapdoor. The vertical dashed line represents the critical pressure calculated from Eq. (3.31) and the corresponding value of the energy needed to overcome the softening within the process zone. The results illustrate that the energies agree within reasonable boundaries to the exact results.

Although there is a tendency that in the beginning of the propagation, the FMEB approach tends to overestimate the available energy (Figure 22b), with increasing length of the shear band, the energies are in a close range to the numerical solutions (Figure 22c).
Figure 22. Comparison of the energy terms for the three cases from Table 2; (a) case A; (b) case B (c) case C. The lines denote the solution from energy balance criterion, dots denote the results from FLAC.
Recalling that the analytical solution involves a number of rather restrictive assumptions, the fact that the solutions lie within a comparable range delivers a first validation of the applicability of the energy balance approach from fracture mechanics.

### 3.3.5 Frictional-Dilatant Material

As a next step, the trapdoor model with frictional-dilatant material for the active case is considered. A solution for the shear band propagation in frictional material has been published (Saurer & Puzrin, 2007), neglecting the effect of dilation and the shear resistance between the glass and the soil. Here, these additional effects are considered. For simplicity, in the following derivations again the active mode is considered. Results for the passive mode are presented in the appendix. Until now, the shear resistance along the shear band was not a function of the vertical stresses and was constant along the entire propagation path. In case of frictional material this is no longer the case. Dilation is taken into account by linear volume increase in the shear band until a certain relative displacement $\delta$ is reached. This allows taking into account both shear resistance along the glass and dilation at the same time. According to Taylor (1948), followed by Skempton & Bishop (1950) and Houlsby (1991), the dependency between friction angle and dilation can be written in the form (Figure 23a)

\[
\tan \phi' = \tan \phi'_v + K_d \tan \psi
\]

where $\psi$ is the dilation and $\phi'_v$, residual friction angle, respectively and the factor $K_d$ lies in the range of $1 \geq K_d > 0$. Because of the assumption that the maximum volume is reached after a relative displacement of $\delta$, within the shear band, an expression for the dilation angle $\psi$ (Figure 23b) can be written as

\[
\tan \psi = (1 - \delta/\delta) \tan \psi_{\text{max}}
\]

for $\delta < \delta$. 

---

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Due to dilation the vertical shear band expands. The expansion of the shear band can be calculated by integration of the linearly decreasing dilation angle over the relative shear displacement $\delta$.

$$u = \int_0^\delta \left(1 - \frac{\delta}{\delta_r}\right) \tan \psi_{\text{max}} \, d\delta = \left(\delta - \frac{\delta^2}{2\delta_r}\right) \tan \psi_{\text{max}}$$  \hspace{1cm} (3.52)

At the end of the dilation process, i.e. after the relative displacement of $\delta_r$, the total displacement in y-direction at the shear band is then

$$u = \frac{\delta_r}{2} \tan \psi_{\text{max}}$$  \hspace{1cm} (3.53)

For simplicity it is assumed that this expansion causes a homogeneous strain in y-direction $\varepsilon_y = 2u/b$ within the elastic sector. In reality, the total expansion of the shear band results in compression both in- and outside the sector, and will cause a smaller horizontal strain. This effect of the lack of constraint is taken care of by a factor $K_b \leq 1$:

$$\varepsilon_y = \frac{\delta_r}{b} K_b \tan \psi_{\text{max}} = \text{const.}$$  \hspace{1cm} (3.54)

From the linear elasticity equations

$$\Delta \varepsilon_z = \frac{1}{E} \left(\Delta \sigma_z - \nu (\Delta \sigma_y + \Delta \sigma_z)\right)$$

$$\Delta \varepsilon_y = \frac{1}{E} \left(\Delta \sigma_y - \nu (\Delta \sigma_z + \Delta \sigma_y)\right)$$

$$\Delta \varepsilon_x = \frac{1}{E} \left(\Delta \sigma_z - \nu (\Delta \sigma_z + \Delta \sigma_y)\right)$$  \hspace{1cm} (3.55)

where $\Delta \sigma_i = \sigma_i - \sigma_{i,0}$ is the difference between the actual and the initial stress.

With the relation from the boundary conditions $\varepsilon_z = 0$ and the expression from Eq. (3.54) follows

$$\Delta \sigma_y = M (\Delta \varepsilon_y + K_0 \Delta \varepsilon_z); \quad \Delta \sigma_z = M (\varepsilon_z + K_0 \Delta \varepsilon_z) = K_0 \Delta \sigma_z + M \left(1 - K_0^2\right) \varepsilon_y$$  \hspace{1cm} (3.56)

From the constitutive equation in Eq. (3.56) follows the expression for the change of strain in vertical direction at the tip as a function of change in vertical stress and dilation:
\[ \Delta \varepsilon_i = \frac{1}{M} (\sigma_i - \sigma_{i0}) - K_0 \varepsilon_y \]  

(3.57)

For the calculation of the peak resistance \( \varepsilon_y = MIN = 0 \) is used and can be written as

\[ \tau_p = K_0 \sigma_x (\tan \varphi_{cv} + K_d \tan \psi_{\text{max}}) \]  

(3.58)

while for residual shear resistance, where \( \varepsilon_y = MAX = \frac{\delta_r}{b} K_0 \tan \psi \) is used follows

\[ \tau_r = (K_0 \sigma_x + M (1 - K_0) \varepsilon_y) \tan \phi'_{cv} \]  

(3.59)

Friction along the glass boundary can be expressed via

\[ \tau_s = (K_0 \sigma_x + MK_0 (1 - K_0) \varepsilon_y) \tan \phi'_{sv} \]  

(3.60)

Figure 23. Dependency between shear strength and dilation angle from the relative displacement within the shear band tip.
**3 Progressive Shear Band Propagation in Trapdoor Tests**

**Stresses and Strains**

Stress distribution and deformations within the elastic part are derived in analogy to the cohesive material case by calculating the equilibrium of the vertical forces acting on a horizontal elementary slice within the elastic part resulting in the differential equation

\[
\frac{d\sigma_x}{dx} = \frac{2 \cdot \tau_\epsilon}{b} + \frac{2 \cdot \tau_\phi}{t} - \gamma
\]

(3.61)

Normalisation and substitution of Eqs. (3.59) and (3.60) into the differential equation results in

\[
\frac{d\sigma_x}{dx} = \sigma_x \overline{K}_f + p_f - p_L
\]

(3.62)

where

\[
p_L = \gamma L
\]

\[
\overline{K}_f = 2LK_0 \left(\frac{\tan \phi'_{\text{crit}}}{b} + \frac{\tan \phi'_{\text{int}}}{t} \right)
\]

(3.63)

\[
p_f = 2LM \left(1 - K_0\right) \frac{\delta_r}{b} K_h \tan \psi_{\text{max}} \left(1 + K_0\right) \frac{\tan \phi_r}{b} + K_0 \frac{\tan \phi'_{\text{int}}}{t}
\]

The solution of this differential equation has been found to be of the form

\[
\sigma_x = c_1 e^{c_2 x} + c_3
\]

(3.64)

Using the boundary condition \(\sigma_x(x = 0) = p_0\), the vertical stress in normalized form is

\[
\sigma_x(\overline{x}) = \left( p_0 - \frac{p_L - p_f}{\overline{K}_f} \right) e^{\overline{x}/\gamma} + \frac{p_L - p_f}{\overline{K}_f}
\]

(3.65)

Substitution of (3.65) and (3.3) into (3.13) gives the displacement at the height of the trapdoor as a function of the stress at the trapdoor and the length of the shear band:

\[
\overline{\delta}_0 = \frac{1}{MK_f} \left( \left( p_0 - \frac{p_L - p_f}{\overline{K}_f} \right) e^{\overline{\epsilon}_f'/\gamma} - 1 \right) + \left( p_L - p_f \right) \overline{I} - \frac{p_L}{2M} \overline{I} (2 - \overline{I} - K_0 \overline{\epsilon}_f \overline{I})
\]

(3.66)
Energy Balance Propagation Criterion

Incremental propagation

Similar to the cohesive material case, an incremental propagation of the shear band is considered, which results in an increase of its length $\Delta l$, at the constant normal stress at the blade $p_0$. This causes additional displacement of the trapdoor $\Delta \delta_0$, which is equal to the strain at the shear band tip multiplied by this incremental propagation. This gives using Eq. (3.57)

$$\Delta \delta_0 = \Delta \epsilon \Delta l = \left( \frac{\sigma_i - \sigma_{i0}}{M} - K_0 \overline{\epsilon}_y \right) \Delta l$$  \hspace{1cm} (3.67)

where $\sigma_i$ and $\sigma_{i0}$ denote the actual and initial vertical stress at the shear band tip.

Work Components

Energy terms in the energy balance criterion Eq. (3.19) are derived in analogy to the cohesive material.

The external work component is

$$\Delta W_e = (p_0 - p_t \cdot I) \cdot b \cdot t \cdot \left( \frac{\sigma_i - \sigma_{i0}}{M} - K_0 \overline{\epsilon}_y \right) \Delta l$$  \hspace{1cm} (3.68)

Calculation of the internally stored energy in the elastic part is more complex than in the case of the cohesive material, because due to dilation the stresses perform work not only on vertical but also on horizontal strains in $y$-direction. Therefore, the incremental elastic work should be calculated as an integral of the specific strain energy function $U$ over the area of the elastic wedge:

$$\Delta W_i = \Delta l \cdot b \cdot t \cdot \int dU = \Delta l \cdot b \cdot t \cdot U(\epsilon_i, \overline{\epsilon}_y)$$  \hspace{1cm} (3.69)

To define this specific strain energy $U$, which is a unique function of the strain state, note that it serves as a potential for stresses producing the linear elastic law (3.56)
Equations (3.70) are satisfied when the strain energy function is given by the following expression

\[
U = \int dU = \frac{M}{2} (\Delta \epsilon_x)^2 + MK_0 \epsilon_y \Delta \epsilon_x + \sigma_{x0} \Delta \epsilon_x + \frac{M}{2} \epsilon_y^2 + \sigma_{y0} \epsilon_y
\]

which can be simplified using Eq. (3.57) to

\[
U = \frac{\sigma_i^2 - \sigma_{i0}^2}{2M} + \frac{M}{2} \left(1 - K_0^2\right) \epsilon_y^2
\]

Substitution of Eq. (3.72) into (3.69) gives the expression for the internally stored elastic energy component:

\[
\Delta W_i = \Delta l \cdot b \cdot l \cdot \left(\frac{\sigma_i^2 - \sigma_{i0}^2}{2M} + \frac{M}{2} \left(1 - K_0^2\right) \epsilon_y^2 \right)
\]

Dissipated energy along the shear bands and along the glass is given by

\[
\Delta D_i + \Delta D_o = 2 \int_0^l (\tau, t + \tau_s b) dx \cdot |\Delta \delta_y|
\]

Substitution of the equilibrium from Eq. (3.61) into this expression, using Eq. (3.67) gives

\[
\Delta D_i + \Delta D_o = tb \left(p_0 - \sigma_i - p_i \tilde{t}\right) \Delta l \left(\frac{\sigma_i - \sigma_{i0}}{M} - K_0 \epsilon_y \right)
\]

Finally, dissipated energy within the process zone at the shear band tip is defined as

\[
\Delta D_w = 2 \Delta l \int_0^{\delta} (\tau - \tau_s) b d\delta
\]

The shear strength in the strain softening branch is given by

\[
\tau = \sigma_{y0} \tan \varphi'
\]
which in combination with Eqs. (3.50), (3.51) and (3.58) gives

\[ \tau - \tau_r = \sigma_y K_d \tan \psi = \sigma_y K_d (1 - \delta / \delta_r) \tan \psi_{\text{max}} \]  

(3.78)

Normal horizontal stresses are obtained from the second expression in Eq. (3.56)

\[ \sigma_y = K_0 \sigma_x + M \left( 1 - K_0^2 \right) \frac{2u}{b} \]  

(3.79)

which after substitution of Eq. (3.52) and applying the correction factor \( K_b \) gives

\[ \sigma_y = K_0 \sigma_x + M \left( 1 - K_0^2 \right) \frac{2}{b} \left( \delta - \frac{\delta^2}{2 \delta_r} \right) \tan \psi_{\text{max}} \]  

(3.80)

The dissipated work in the process zone can be calculated by substitution of Eq. (3.80) and (3.78) into (3.76)

\[ \Delta D_{\omega} = 2 \Delta t \int_0^\delta \left( K_0 \sigma_x + M \left( 1 - K_0^2 \right) \frac{2}{b} \left( \delta - \frac{\delta^2}{2 \delta_r} \right) \tan \psi_{\text{max}} \right) \left( 1 - \frac{\delta}{\delta_r} \right) \tan \psi_{\text{max}} d\delta \]  

(3.81)

and integrated to produce in simplified form

\[ \Delta D_{\omega} = \Delta t b \frac{K_d}{K_b} \left( K_0 \sigma_y + \frac{M}{2} \varepsilon_y \left( 1 - K_0^2 \right) \right) \]  

(3.82)

**Progressive and Catastrophic Failure in a Stress Controlled Test**

Substitution of the work increments (3.68), (3.73), (3.75) and (3.82) into the energy balance criterion

\[ \Delta W_c \geq \Delta W_i + \Delta D_t + \Delta D_s + \Delta D_{\omega} \]  

(3.83)

gives the criterion for the shear band propagation

\[ \sigma_i \leq \sigma_{i0} + \bar{\varepsilon}_y K_0 M \left( 1 + \frac{K_d}{K_b} \right) \]  

\[ - \sqrt{2 \sigma_{i0} K_0 M \bar{\varepsilon}_y \left( 1 + \frac{K_d}{K_b} \right) + M^2 \bar{\varepsilon}_y^2 \left( 1 + \frac{K_d}{K_b} \right) \left( 1 + K_0^2 \left( 1 + \frac{K_d}{K_b} \right) \right)} \]  

(3.84)
Substitution into Eq. (3.65) gives with \( \sigma_{t_0} = p_L (1 - \tilde{l}) \) and \( \tilde{M} = M / p_L \):

\[
p_f = p_f / p_L \quad \text{the solution for the stress controlled test.}
\]

\[
\frac{p_0}{p_L} \leq \left[ (1 - \tilde{l}) - \sqrt{2(1 - \tilde{l}) K_o M \bar{\varepsilon}_y \left( 1 + \frac{K_d}{K_b} \right) + \tilde{M}^2 \bar{\varepsilon}_y^2 \left( 1 + \frac{K_d}{K_b} \left( 1 + \frac{K_o}{K_b} \right) \right) \right]
\]

\[
+ \bar{\varepsilon}_y K_o \tilde{M} \left( 1 + \frac{K_d}{K_b} \right) - \frac{1}{\bar{K}_f} + \frac{p_f}{\bar{K}_f} \right] e^{-\bar{\varepsilon}_y \tilde{l}} - \frac{p_f}{\bar{K}_f} + \frac{1}{\bar{K}_f}
\]

where \( \bar{K}_f = 2 L K_0 \left( \tan \frac{\phi'_c}{b} + \frac{\tan \phi'_d}{t} \right) \), \( \tilde{M} = \frac{M}{p_L} \)

and \( \tilde{p}_f = 2 L \tilde{\bar{\varepsilon}}_y (1 - K_o) \left( 1 + K_o \right) \frac{\tan \phi'_c}{b} + K_0 \frac{\tan \phi'_d}{t} \)

As long as the criterion from Eq. (3.32) is not valid, the propagation will propagate progressive, else catastrophic.

**Progressive and Catastrophic Failure in a Displacement Controlled Test**

With \( p_0 = p_{c_r} \) and the substitution of (3.85) into (3.66) gives the equation for the propagation in the displacement controlled test:

\[
\delta(0) = \frac{1}{MK_f} \left[ \left( (1 - \tilde{l}) + \bar{\varepsilon}_y K_o \tilde{M} \left( 1 + \frac{K_d}{K_b} \right) - \frac{1}{\bar{K}_f} + \tilde{p}_f \right) \right]
\]

\[
- \sqrt{2(1 - \tilde{l}) K_o \tilde{\bar{\varepsilon}}_y \left( 1 + \frac{K_d}{K_b} \right) + \tilde{M}^2 \tilde{\bar{\varepsilon}}_y^2 \left( 1 + \frac{K_d}{K_b} \left( 1 + \frac{K_o}{K_b} \right) \right) \}} \times \left[ 1 - e^{-\bar{\varepsilon}_y \tilde{l}} \right] + (1 - \tilde{p}_f \tilde{\bar{\varepsilon}}_y) \tilde{l} - \tilde{\bar{\varepsilon}}_y \tilde{l} / M - \tilde{l} / 2 - \bar{\varepsilon}_y \tilde{l}
\]

Again, when the shear band reaches the critical length the energy balance driven propagation will change from progressive to catastrophic mode.
**Limiting Equilibrium Propagation Criterion**

From the *limiting equilibrium approach* the condition for the progressive propagation in a stress controlled tests is

$$\frac{P_0}{P_L} \leq \left(1 - \bar{I}\right) - K_0\left(\frac{L}{b} (\tan \varphi'_{cv} + K_d \tan \psi_{max}) + \frac{L}{t} \tan \varphi'_{s}\right)(1 - \bar{I})^2 - \frac{1 - \bar{P}_{fs}}{K_{fs}} e^{-\pi_{f_s}^{\bar{t}}} + 1 - \bar{P}_{fs}$$

(3.87)

and in the displacement controlled test

$$\bar{\delta}_0(\bar{I}) = \frac{1}{MK_{fs}} \left[\left(1 - \bar{I}\right) - K_0\left(\frac{L}{b} (\tan \varphi'_{cv} + K_d \tan \psi_{max}) + \frac{L}{t} \tan \varphi'_{s}\right)(1 - \bar{I})^2 - \frac{1 - \bar{P}_{fs}}{K_{fs}}\right] - \frac{1}{M} \left(\bar{I} - \bar{I}^2 / 2\right)$$

(3.88)

Summarized solutions for both active and passive mode trapdoor tests are provided in the appendix.
3.4 Sensitivity Analysis

A sensitivity study has been performed in order to understand parameter dependency of the solution. Sensitivity has been checked for two parameters: the stiffness parameter $M$ and the relative displacement $\delta$, taking the assumption of linear decrease of strength $\delta_n = \delta$ into account.

![Graph showing sensitivity analysis](image)

(a) active mode

(b) passive mode

Figure 24. Sensitivity of the constrained modulus in the model with frictional material.
In Figure 24, the shape of the curves with frictional material is plotted for constrained modulus $M$ of 250, 500 and 1000 kPa for both active and passive mode for the test height of $L = 0.4$ m. With increasing stiffness, the rate of propagation increases.

The chain dotted line refers to the intersection between the two criteria; in the lower part the fracture mechanics energy balance; in the upper part the limiting equilibrium criterion drives the shear band propagation.

A parameter which is difficult to determine is the relative displacement needed to reach friction at constant volume at the shear band tip (Figure 25). Since it has been assumed that this parameter is equal to half of to the shear band width, which then again is a multiple of the mean grain diameter, here the dependency on the shear band width is shown in comparison with a multiple of the mean grain diameter of the used material. Parameters have been chosen $L = 0.6$ m; $M = 250$ kPa and all other parameters similar as defined previously.

Two tendencies can be observed. With increasing shear band width, the shear band propagation becomes slower and the intersection between energy balance and limiting equilibrium criteria decreases.

Figure 25. Sensitivity on the parameter $\delta_i$. 

\[ \delta_i = 10d_{50} \]
\[ \delta_i = 15d_{50} \]
\[ \delta_i = 20d_{50} \]
\[ \delta_i = 30d_{50} \]
3.5 Comparison with Experimental Data

Comparison between experiments and analytical curves is performed using two different kinds of plots: in the first set of plots (Figure 27a and b), the propagation rates of both the analytical solutions and the experiments are shown. The analytical curves are fitted to the experimental data set using method of least squared errors and varying the constrained modulus $M$. From the shape of the curves a statement can be made with respect to the progressive and catastrophic propagation mode. In the second plot (Figure 28), the stiffness calculated from the regression analysis is compared to the stiffness obtained from oedometer tests and plotted against the range of stress states reached at the shear band tip during the propagation according the FMEB approach. With the exception of the constrained modulus $M$, all the parameters have been calculated as of the material properties section and are summarized in Table 3. The parameter $K_b$, taking into account the lack of constraint in the experiments, has been validated using the numerical code FLAC. Comparison between two tests – one without dilatancy and one with dilatancy-softening – has been performed to calculate this parameter. Results are shown in Figure 26. For the active mode, this parameter has been chosen to be about $K_b = 0.4$, for the passive mode $K_b = 0.1$. The parameter $K_d = 0.7$ has been found to be appropriate for both test modes. The shear band width has been chosen to be of the size of 20 times the mean grain diameter.

Comparison of the shape of the propagation rate (Figure 27) shows that with increasing sample heights the quality of agreement between experimental and analytical curves increases. However, the lowest test height (particularly A20), shows a rather poor agreement with respect to the shape of the curves. The main reason for this could be found from the increasing accuracy of the interpretation of experimental results with increasing test height since larger elastic deformations can be measured and therefore with a higher precision. In the passive case, this effect has a rather large influence because in these tests, elastic deformations are difficult to measure. Still a range of propagation rate has been obtained and represents qualitatively well the expected behaviour of the used soil. Note that for the lower test heights, the limiting equilibrium has been found to be the main driving criterion, while for the larger
tests the energy balance criterion drives the shear band in the initial phase of propagation.

![Diagram](image)

**Figure 26.** Parameter $K_h$ plotted against the normalized height in the test; numerical results (dots); chosen parameter for the analytical curves (lines)

**Table 3. Parameters of analytical solutions.**

<table>
<thead>
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<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
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<tbody>
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<td>[m]</td>
</tr>
<tr>
<td>$t$</td>
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<td>[m]</td>
</tr>
<tr>
<td>$b$</td>
<td>0.2</td>
<td>[m]</td>
</tr>
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</tr>
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<td>$K_d$</td>
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</tr>
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</table>
Figure 27. Results of the shear band propagation along the normalized height of the test against the normalized elastic displacement; (a) active mode; (b) passive mode results. Experimental (shaded) and analytical results (lines).
In Figure 28, the constrained moduli from the analytical model show a high agreement with the regression curves from the oedometer tests. The plot shows that first of all the constrained moduli of all tests are within the expected range and second, the constrained moduli of the active mode are
rather at the higher bound, which is logical since this mode is more likely to a
unloading test and therefore the material is expected to behave stiffer. However, the stiffness is not comparable to the unloading curve because of
the different kinematics as opposed to unloading in oedometer tests.

3.6 Analytical versus Numerical Solutions

3.6.1 Method

Due to their reasonable numerical stability, certain finite difference methods
have proven to be more suitable for numerical analysis of the post-failure
strain-softening behaviour of soils. One of these methods, FLAC (Fast
Lagrangian Analysis of Continua), was used for numerical simulations of the
trapdoor tests with frictional-dilatant material.

Because the main focus was put on the rate of the shear band propagation it
was important to make sure that the shear band follows the experimentally
observed path. Therefore, the mesh was generated such that the shear band
was pushed for propagating along the predefined path and to stay within the
band of a one element width, i.e. within its experimentally observed width.
This allows for the negative effects of mesh dependency to be eliminated.

An example of the mesh during the simulation is given in Figure 29. The soil
was modelled as a continuum using Mohr-Coulomb model with strain-
softening along the shear band. The mesh size along the shear band was
chosen to be identical to the width of the shear band thickness. The residual
friction was assumed to be reached at the relative displacement 2.5 mm,
corresponding to the shear strain of $\epsilon = 0.25$, plain strain conditions have
been applied.

In the analytical solution, the displacement outside the propagation zone is
assumed to be zero. In the experiment and numerical analysis they are non-
zero, because the material outside the zone bounded by the band is not rigid.
Therefore, for the proper comparison of the rates of the shear band
propagation between the numerical and analytical models, the shear blade
displacement has been corrected by the deformations ahead of the shear band propagation zone.

Figure 29. FLAC mesh of passive mode trapdoor test (L=40cm) after a displacement at the trapdoor of 6 mm. Decreasing shear resistance from 37° to 30° is observed along the shear band. Contour interval is 1°.

Identical parameters have been defined for both the analytical and numerical models. For the comparison of results, the previously used concept has been applied and is shown in Figure 26 and Figure 28. Analytical curves of the propagation rate have been fitted to the experiments ignoring the friction between the glass and the soil by variation of the constraint modulus. Then again the stiffness parameter in the numerical simulations has been adjusted such that the propagation rates correspond to those from analytical and therefore experimental results (Figure 30). The constraint moduli for both active and passive mode are plotted against the normal stress at the initiation of the shear band at the symmetry axis of the test and are shown in Figure 31.
3.6.2 Comparison

Identical rates of shear band propagation ignoring the shear resistance along the boundaries can only be obtained with much lower stiffness parameters. While in active mode (analytical $\phi'_s = 0^\circ$), the range of stiffness is still at the lower bound of the regression curves from oedometer tests, in passive mode it is already beyond the expected range. Results obtained from numerical simulation are even further away from the range. Numerical results of the lowest test heights (A20; P20) have a weak correlation with the analytical curves and the corresponding constraint moduli are extremely small. For higher tests, however, the results are less dramatic with respect to the shape of the propagation rates, but still the stiffness of the material is underestimated.

It has been shown that the shear band propagation can be modelled numerically, taking the mesh-dependency problem into account. With this particular 2D numerical code it is not possible, however, to include the friction along the sides, whereas in the analytical solution this can be introduced straightforward. Therefore and because the stiffness of the material has been underestimated, a benchmark model is essential to obtain correct rates of propagation of the shear band in a boundary value problem.
Figure 30. Best fit of shear band propagation rates from analytical solution (continuous line) and numerical simulations (dashed line). A and P denote active and passive mode. The number denotes the height of the test in [cm].
Figure 31. Constraint modulus $M$ against the vertical stress at the shear band tip. Comparison of results with best fit of the analytical solution with and without friction along the glass and of the numerical solution. Black lines denote the boundaries from oedometer loading test.
3.7 Conclusions

The progressive rate of shear band propagation has been investigated experimentally in the trapdoor test in active and passive mode for three different test heights. Three different analytical approaches have been used to get an insight to the mechanism. Analytical models have been obtained using two propagation criteria; the fracture mechanics energy balance and the limiting equilibrium criteria.

In the first analytical model cohesive material has been taken into account. This model has been used to validate the energy balance approach using FLAC. Results show that the energy balance criterion produces results that are in a reasonable range to the approximated exact energies obtained by the numerical model.

Furthermore an analytical model has been obtained for frictional-dilatant material including the friction between the glass and the soil. The analytical models have been used for the comparison with results from physical test data. Comparison of the constrained modulus with the stresses at the tip of the shear band has turned out to be the most convenient because it allows assessing the analytical models in a single 2D plot. From the obtained results the following statements can be made:

- The energy balance approach provides both qualitatively and quantitatively reasonable results of the shear band propagation in trapdoor tests.

- For a proper comparison of the results, the friction at the boundaries should not be neglected.

- The analytical model does not depend on the thickness of the shear band and, therefore, allow for such problems as mesh-dependency or internal length to be avoided.

- Validation of the Energy Balance Approach has shown that it can be used for the modelling of shear band propagation in frictional materials.
4 Progressive Shear Band Propagation in Shear Blade Tests

4.1 Introduction

In the previous section and in most of the existing experimental and numerical studies, the propagation of planar shaped shear bands has been investigated using trapdoor-, biaxial- or triaxial-test setups (e.g., Vardoulakis et al., 1981; de Borst & Vermeer, 1984; Tanaka & Sakai, 1993, Saurer & Puzrin, 2007). These results are useful to understand the shear band propagation in slides.

However there are tsunami events, such as the Papua New Guinea Tsunami 1998 (Tappin et al, 1999, Synolakis et al., 2002), where slumps with curve shaped sliding surfaces have been found to be the main source of the tsunami wave. In order to simulate and study the propagation of cylinder shaped shear bands in soils, a novel test, the shear blade test has been developed.

The larger scope of this study is the experimental validation of the energy balance approach to the cylinder-shaped shear band propagation in soils. In particular, the purpose of this chapter is to apply the energy balance approach to the problem of a shear blade test and obtain analytical solutions for cohesive and frictional-dilatant material, with their initial validation against some experimental and numerical results.

4.2 Physical Tests

With the purpose of investigating the rate of shear band propagation in frictional material, a novel test device, the shear blade apparatus, has been developed. The device allows for a shear band of approximately semi-circular shape to propagate progressively (in a stable manner), simulating the shape of a sliding surface in slumps.
4.2.1 Test Setup

The device consists of a box of the 120 × 120 mm plan area and a height of 25 mm. A two-winged 80 mm long and 25 mm high shear blade is located in the centre of the box (Figure 32). The rotational velocity of the blade is controlled by a step-motor while the torque is measured continuously by a static torque sensor installed between the step-motor and the box.

![Figure 32. The Shear Blade device: (a) photograph; (b) conceptual setup](image)

The bottom plate of the box is made out of transparent acrylic glass (Figure 32). The pressure on the top plate can be varied and is controlled by a pneumatic cylinder above the plate with an electronic pressure controller. For the observation of the surface of the material, tests are recorded using a progressive scan CCD camera with a maximal resolution of 1392 × 1040 pixels at the rate of 2 full frames per second.

4.2.2 Material Properties

The experimental program consisted of three shear blade tests using dry silty sand with a mean grain size of $d_{50} = 0.06$ mm (Figure 33).

Standard-oedometer tests up to a vertical load of 200 kPa have been performed. In Figure 34, the constrained modulus $M$ for loading is plotted against the vertical load for stresses above 25 kPa. In addition, upper and lower bounds for these values have been generated using a regression curve.
of the form $M = a \cdot (\sigma_v)^b$. The upper bound is given by $a = 189.73$ and $b = 0.67$, for the lower bound $a = 166.3$ and $b = 0.62$.

Figure 33. Grain size distribution of material.

Figure 34. Constrained modulus $M$ against vertical load $\sigma_v$ from three oedometer tests on silty sand and bounding regression curves.
From direct shear tests and by measurement of the angle of repose, friction angles at peak and constant volume have been determined as $\phi'_p = 45^\circ$ and $\phi'_v = 39^\circ$, the friction between the soil and the boundaries has been determined as $\phi'_s = 30^\circ$, respectively.

### 4.2.3 Sample Preparation and Test Procedure

The sample preparation was performed with the shear box facing upwards. The box was filled with the soil compacted up to the relative density of $D_r = \text{80-90}\%$. Radial marker lines were added onto the surface at the intervals of $\pi/8$ [rad] (Figure 35a) and the box was covered with the glass. After having applied a constant pressure of 15 kPa, the apparatus was carefully turned upside down.

After these preparation steps the shear blade in the sample was rotated at a constant velocity of 0.2 rpm and the surface of the sample was filmed by the camera.

![Figure 35. Shear band propagation in experimental test: (a) blade at initial position; (b) blade at $\alpha = 0.26$ [rad].](image)
4.2.4 Test Interpretation

The tests were interpreted as follows. After Vardoulakis et al. (1981) the shear band tip represents a point where the shear strength in the band has dropped to the residual level. This is assumed to take place when the relative shear displacement has reached half of the shear band width (which is about 20 mean grain sizes, i.e. 1.2 mm). After a certain rotation of the blade, on the passive side (in front of the blade), a circular shear band starts to propagate from the tips of the blade crossing one marker line after another (Figure 35b). The progress of this shear band has been recorded and plotted as a function of the shear blade rotation angle, corrected by the rotation of the material at the tip of the shear band. On the active side of the blade, i.e. behind the shear blade, a diffusive active failure zone has been detected.

Results from the three experiments are plotted in Figure 36. The angle between the blade and the shear band tip $\eta$ is plotted against the corrected rotation of the blade $\alpha_0$.

![Figure 36. Experimental results from shear blade tests.](image-url)
4.3 Analytical Modelling – Cohesive Material

For simplicity (and for completeness) we begin with an analytical solution for cohesive material. This implies that the shear strength of soil is independent of normal stresses and drops from its peak value $\tau_p$ to residual shear strength $\tau_r$ solely as a function of the relative displacement (Figure 37a). The shear strength of $\tau_r$ at the top and bottom boundary plates is also independent of normal stresses.

4.3.1 Geometry and Assumptions

The real test setup suggests plane stress boundary conditions, but with a uniform out of plane displacement which is more suitable for the plane strain constraint. Therefore, for the analytical problem both the plane stress and plain strain solutions have been obtained. The following assumptions have been made:

- the shear band is assumed to be perfectly cylinder shaped;
- in the modeled test we control the displacement of the blade $\alpha_0$ and try to predict the corresponding angular length $\eta$ of the shear band propagating in front of the blade in the material (Figure 37b);
- behind the blade, the active failure shear zone develops with a moving boundary $\eta_{LE}$;
- the soil inside the sector bounded by the blade wing and the shear band is linear elasto-plastic with loading modulus $E$ and Poisson’s ratio $\nu$;
- the soil outside this sector is rigid-plastic;
- apart from the small end zone $\omega$, the shear resistance $\tau$ along the shear band is equal to its residual value $\tau_r$. At the tip of the shear band as well as at any point outside the band, the maximal shear resistance is equal to its peak value $\tau_p$. Within the end zones, called the process zones $\omega$, the shear resistance decreases as a function of the relative...
displacement $\delta$ from its peak $\tau_p$ to its residual $\tau_r$ value (at the relative displacement $\delta_r$) (Figure 37a);

- the circumferential normal stress $\sigma_\theta$ is assumed to be uniformly distributed along any radial line $\theta = \text{const}$, with its value being equal to the average vertical effective stress acting on this line;

- the shear strength at the boundary plates is $\tau_s$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure37.png}
\caption{The simplified model of the shear blade test: (a) shear strength along the band; (b) geometry.}
\end{figure}

The initial pressure in the out of plane direction is $\sigma_{z,0} = p_v$. This generates a uniform stress state within the $r - \theta$ – plane given by

$$\sigma_{r,0} = \sigma_{\theta,0} = \sigma_{z,0} K_0$$

(4.1)

where $K_0 = \nu/(1 - \nu)$ is the earth pressure coefficient at rest.
4.3.2 Cohesive Material - Plane Stress

Solutions for both, plane stress and plane strain conditions are presented in this paper. In this section, solution for plane stress conditions is derived. Please note that due to the rotational symmetry of the problem, only half of the model is considered for the derivations.

**Stresses and Strains**

The plane stress condition implies that the stress in the out of plane direction does not change

\[ \sigma_z = \sigma_{z,0} = \text{const.} , \quad \Delta \sigma_z = 0 \]  

(4.2)

Furthermore, because the average radial strain remains equal to zero due to the confinement in the radial direction, radial strains caused by the rotation of the blade are neglected, so that from the linear elastic relationship it follows:

\[ \varepsilon_r = \frac{1}{E} (\sigma_r - \nu(\sigma_\theta + \sigma_z)) = 0 \]  

(4.3)

where \( \sigma_\theta = \sigma_{\theta,0} + \Delta \sigma_{\theta} \) and \( \sigma_r = \sigma_{r,0} + \Delta \sigma_r \) are the circumferential and radial normal stresses, respectively, given by the initial value plus the increment due to rotation. From Eq. (4.3) it follows that

\[ \sigma_r = \nu \sigma_\theta + \nu \sigma_{z,0} \]  

(4.4)

the circumferential and radial stress changes are related by

\[ \Delta \sigma_r = \nu \Delta \sigma_\theta \]  

(4.5)

Therefore, using Eqs. (4.2) and (4.5), the incremental elastic stress-strain relationship in circumferential direction is given by

\[ \Delta \varepsilon_\theta = \frac{1}{E}(\Delta \sigma_\theta - \nu(\Delta \sigma_r + \Delta \sigma_z)) = \frac{1}{D}(\sigma_\theta - \sigma_{\theta,0}) \]  

(4.6)

where the stiffness parameter \( D \) is defined as \( D = E/(1-\nu^2) \) for plane stress conditions.
The equilibrium of the momentum around the centre of an elementary wedge with opening angle $d\theta$ within the elastic part of the model (Figure 38) can be written as

$$\sigma_\theta \frac{R^2}{2} = (\sigma_\theta + d\sigma_\theta) \frac{R^2}{2} + \tau_r \cdot R^2 \cdot d\theta + \frac{2}{3} \frac{R^3}{t} \tau_s \cdot d\theta$$

(4.7)

which can be simplified to

$$\frac{d\sigma_\theta}{d\theta} = -2 \left( \tau_r + \frac{2}{3} \frac{R}{t} \tau_s \right)$$

(4.8)

Integration of (4.8) using the boundary condition $\sigma_\theta(0) = p_0$ results in the expression for the circumferential stress as a function of angle $\theta$, normal stress at the blade $p_0$ and the shear stresses at the boundaries of the elastic wedge

$$\sigma_\theta(\theta) = p_0 - 2 \left( \tau_r + \frac{2}{3} \frac{R}{t} \tau_s \right) \cdot \theta$$

(4.9)

![Figure 38. Equilibrium of momentum of an elementary elastic wedge.](image)

The circumferential displacement (i.e., rotation $\alpha(\theta)$) in the elastic part can be calculated by integration of the strains over the arc-length of the corresponding sector. Using the assumption that the process zone is small compared to the total length of the shear band, we obtain the boundary condition $\alpha(\eta + \omega) = \alpha(\eta) = 0$. The rotation $\alpha(\theta)$ is calculated using Eqs. (4.6), (4.9) and (4.1):

$$\alpha = \int \Delta e_\theta d\theta = \frac{1}{D_\theta} \int_0^\eta (\sigma_\theta - \sigma_{\theta,0}) d\theta$$

$$= \frac{1}{D} \left( p_0 - K_\theta \sigma_{\theta,0} (\eta - \theta) - \left( \tau_r + \frac{2}{3} \frac{R}{t} \tau_s \right) \cdot (\eta^2 - \theta^2) \right)$$

(4.10)
Using \( \theta = 0 \), gives the rotation of the blade as a function of the stress at the blade \( p_0 \) and the length of the shear band \( \eta \).

\[
\alpha(\theta = 0) = \alpha_0 = \frac{\eta}{D} \left( p_0 - K_0 \sigma_{z,0} - \left( \tau_r + \frac{2R}{3t} \tau_z \right) \cdot \eta \right)
\]  

(4.11)

**Energy Balance Propagation Criterion**

**Incremental Propagation**

Consider an incremental prolongation of the shear band, which results in an increase of its angular length \( \Delta \eta \), at the constant normal stress at the blade \( p_0 \). This causes additional rotation of the shear blade \( \Delta \alpha_0 \), which is equal to the strain at the shear band tip multiplied by this incremental propagation. This gives, using Eq. (4.6)

\[
\Delta \alpha(\theta = 0) = \Delta \alpha_0 = \Delta \varepsilon_0(\eta) \cdot \Delta \eta = \frac{1}{D} \left( \sigma^\eta_{o} - \sigma^\eta_{o,0} \right) \cdot \Delta \eta
\]  

(4.12)

where \( \sigma^\eta_{o} \) denotes tangential stress at the shear band tip \( \theta = \eta \).

**Work Components**

Initial propagation of the shear bands is modelled using the fracture mechanics energy balance approach. This approach sets a condition for an incremental propagation: the incremental sum of work of external forces and released internal energy should be higher than the incremental sum of the elastically stored energy and the dissipated plastic work in the shear band, its process zone and along the boundaries.

Mathematically this can be expressed as the following inequality:

\[
\Delta W_e \geq \Delta W_i + \Delta D_i + \Delta D_s + \Delta D_{\omega}
\]  

(4.13)

where the energy balance has been considered between the following five components:

- external work \( \Delta W_e \) done by the external force acting on the shear blade on the displacement of the shear blade;
4 Progressive Shear Band Propagation in Shear Blade Tests

- internal elastic energy $\Delta W_i$ stored or released due to the changes of stresses in the soil wedge inside the shear band;

- plastic work $\Delta D_i$ dissipated due to residual friction within the shear band along the entire length of the shear band $l$;

- plastic work $\Delta D_s$ dissipated due to friction between the soil wedge and the boundary plates;

- plastic work $\Delta D_\omega$ dissipated in the process zone $\omega$ of the shear band due to the friction above residual.

Note that due to the assumption that for an incremental shear band propagation the stresses change only in the propagation zone between $\eta$ and $\eta + \Delta \eta$, the soil between the blade ($\theta = 0$) and the tip ($\theta = \eta$) rotates as a rigid body by angle $\Delta \alpha_0$.

The first term in Eq. (4.13), the external work, using Eq. (4.12), is given by the expression

$$\Delta W_e = p_0 \cdot \frac{R^2}{2} \cdot \Delta \alpha_0 = p_0 \cdot \frac{R^2}{2} \frac{t}{D} (\sigma_0^\eta - \sigma_0^{\theta,0}) \Delta \eta \quad (4.14)$$

which is the momentum due to the pressure applied to the shear blade multiplied by its incremental rotation.

Because stresses and strains change only in the propagation zone between $\eta$ and $\eta + \Delta \eta$, change of internal elastic energy is limited to this area. Using Eq. (4.6) follows

$$\Delta W_i = t \int_\eta^{\eta + \Delta \eta} \int_0^{\Delta \epsilon} \int_0^{\Delta \sigma} \sigma_0 \, d\epsilon \, dr \, d\theta$$

$$= \frac{t}{D} \cdot \frac{R^2}{4} \left( \sigma_0^2 - \sigma_0^{\theta,0} \right) \Delta \eta \quad (4.15)$$

The sum of plastic work dissipated along the shear blade and along the boundary plates can be calculated as
\[ \Delta D_i + \Delta D_o = t \int_0^{\Delta \alpha_i} \int_0^{\Delta \alpha_o} \tau_i R d \alpha \] 
\[ + 2 \int_0^{\Delta \alpha_i} \int_0^{\Delta \alpha_o} \tau_i R d \alpha \frac{d r d \theta}{\eta R} \] 
\[ = \frac{R^2 t}{2 D} \left( p_0 - \sigma^{\eta}_{\theta} \right) \left( \sigma^\eta_{\theta} - \sigma^\eta_{\theta,0} \right) \cdot \Delta \eta \] 
\[ (4.16) \]

where the second equality is derived using equilibrium from Eq. (4.9).

Finally, the plastic work dissipated in the process zone (Figure 37a) due to the friction above residual is

\[ \Delta D_o = \int_0^{\Delta \alpha_i} \int_0^{\Delta \alpha_o} \tau_i \left( \tau - \tau \right) d \delta \frac{R d \eta}{\eta} \] 
\[ = \frac{(r_i - 1) \tau_r - \Delta \alpha_r \cdot R^2 \cdot t \Delta \eta}{2} \] 
\[ (4.17) \]

where \( r_i = \tau_i / \tau_r \); \( \alpha_r = \delta_r / R \) and assuming linear decrease of the shear strength \( \delta_r = \delta_m \).

**Energy Balance Criterion**

Substitution of Eqs. (4.14) - (4.17) into Eq. (4.13) gives after some manipulation the inequality

\[ \left( \sigma^\eta_{\theta} - \sigma^\eta_{\theta,0} \right)^2 \geq 2(r_i - 1) \tau_r D \alpha_r \] 
\[ (4.18) \]

which represents the shear band propagation criterion.

Substitution of Eq. (4.9) into Eq. (4.18) gives the condition for the shear band propagation in a stress controlled test.

\[ p_0 \geq p_{cr} = \sigma_{z,0} K_0 + \sqrt{2(r_i - 1) \tau_r D \alpha_r} + 2 \left( \tau_r + \frac{2 R}{3} \tau_i \right) \frac{\tau}{\eta} \] 
\[ (4.19) \]

Then, after substitution of Eq. (4.19), using \( p_0 = p_{cr} \) into the equation of the rotation of the blade (Eq. (4.11)), the condition for the displacement controlled test is given by:

\[ \alpha_0 = \frac{p_0}{D} \left( \sqrt{2(r_i - 1) \tau_i D \alpha_r} + \left( \tau_r + \frac{2 R}{3} \tau_i \right) \eta \right) \] 
\[ (4.20) \]
This relationship between the shear band length and the shear blade rotation can be written in a more compact way

\[
\alpha_0 = \frac{\eta}{D} \left( \sqrt{\bar{p}_c \eta K_c} + \frac{\eta K_c}{2} \right)
\]  

(4.21)

where \( \bar{D} = \frac{D}{\sigma_{z,0}}; \ \bar{p}_c = 2(\tau_s - 1)\bar{r}_s \bar{D} \alpha_s; \ \bar{K}_c = 2 \left( \tau_p + \frac{2 R}{3 L_s} \right); \ \bar{r}_s = \frac{\tau_p}{\sigma_{z,0}} \) and \( \bar{r}_s = \frac{\tau_s}{\sigma_{z,0}} \).

The energy balance criterion controls the shear band propagation up to the state when the remaining sector between the tip of the shear band \( \eta \) and the current boundary of the active failure shear zone \( \eta_{LE} \) reaches the state of the limiting equilibrium. After that, further progressive and catastrophic propagation is driven by the limiting equilibrium criterion.

**Limiting Equilibrium Approach**

\[ \text{Figure 39. Geometry of the shear blade test during limiting equilibrium condition.} \]

The failure will take place when the remaining sector between the tip of the shear band \( \eta \) and the current boundary of the active failure shear zone \( \eta_{LE} \) reaches the state of the limiting equilibrium. This will happen when the shear stresses along the shear band path between the current tip of the shear band \( \eta \) and the current boundary of the active failure shear zone \( \eta_{LE} \) in Figure 39 reach the peak strength of the material. The stresses acting at the sector
boundaries are: pressure $\sigma^2_\theta$ at the boundary with the elastic region; the active earth pressure $p_a = \sigma_{z,0} K_a$ at the boundary with the active failure zone; the peak strength of soil $\tau_p$ along the shear band path; and the shear strength $\tau_s$ and at the boundaries between the soil and the top and bottom plates. The limiting condition for the momentum equilibrium can therefore be written as

$$\left(\sigma^2_\theta - \sigma_{z,0} K_a\right) \frac{t R^2}{2} \geq \tau_p (\eta_{LE} - \eta) t R^2 + \tau_s (\eta_{LE} - \eta) \frac{2}{3} R^3$$

(4.22)

where $K_a = \left(1 - \sin \phi' \right) / \left(1 + \sin \phi' \right)$ denotes the active earth pressure coefficient.

Using Eq. (4.9) and defining the ratio $m_c = \frac{K_{c,LE}}{K_c}$, where $K_{c,LE} = 2 \left( \tau_p + \frac{2}{3} \frac{R}{t} \tau_s \right)$ and $K_c = 2 \left( \tau_s + \frac{2}{3} \frac{R}{t} \tau_s \right)$ we obtain the relation

$$\frac{p_0}{\sigma_{z,0}} \geq (\eta_{LE} - \eta) m_c K_c + K_a + K_c \cdot \eta$$

(4.23)

which represents the criterion for the stress controlled test. Substitution of Eq. (4.23) into Eq. (4.11) gives the solution for the displacement controlled test:

$$\alpha_0 = \frac{\eta}{D} \left( m_c (\eta_{LE} - \eta) + \frac{\eta}{2} K_c + (K_a - K_a) \right)$$

(4.24)

### 4.3.3 Cohesive Material - Plane Strain

In plane strain conditions, strains in the out-of-plane direction do not change. Within the linear elasticity, this can be written as

$$\varepsilon_z = \frac{1}{E} (\sigma_z - \nu (\sigma_\theta + \sigma_r)) = 0$$

(4.25)

Neglecting radial strains leads to the expression

$$\varepsilon_r = \frac{1}{E} (\sigma_r - \nu (\sigma_\theta + \sigma_z)) = 0$$

(4.26)
From Eqs. (4.25) and (4.26) gives an expression for the stiffness parameter $D$ in Eq. (4.6), for plane strain conditions:

$$\Delta e_\theta = \frac{1}{D}(\Delta \sigma_\theta), \quad D = \frac{1-\nu}{(1+\nu)(1-2\nu)}E = M$$

(4.27)

i.e., the stiffness $D$ in plane strain conditions is equal to the constrained modulus $M$. Apart from this difference, all the derivations and final remain expressions are identical to those for the plane stress conditions in cohesive material.

**4.4 Analytical Modelling – Frictional-Dilatant Material**

With the principle approaches to the shear band propagation in the shear blade test being established in Section 4.3 for a simpler case a cohesive material, it is now possible to extend this method to a more general case of the frictional material, which would allow for comparison with experimental results from Section 4.2. The main complications compared to the cohesive material are:

- dependency of the peak and residual shear strength on the normal stresses;

- dilation of the soil in the shear band.

**4.4.1 Shear Strength and Dilation**

Densely packed frictional material experiences dilation when sheared. According to Taylor (1948), the dependency between friction angle and dilation (Figure 40a) can be calculated from

$$\tan \phi' = \tan \phi'_{cv} + K_d \tan \psi$$

(4.28)

where $\psi$ is the dilation and $\phi'_{cv}$ residual friction angle at constant volume, respectively and the factor $K_d$ lies in the range of $1 \geq K_d > 0$. Because of the assumption that the maximum volume is reached after a relative displacement
of \( \delta_c \) within the shear band we can write a linear expression for the dilation angle \( \psi \)

\[
\tan \psi = (1 - \delta_c/\delta) \tan \psi_{\text{max}}
\]

(4.29)

for \( \delta < \delta_c \) (see Figure 40b).

![Figure 40. Dependency between shear strength and dilation angle from the relative displacement within the shear band tip.](image)

The peak and residual shear strength in the shear band depend on the radial stress \( \sigma_r \) and are calculated using Eqs. (4.28) and (4.29) for \( \delta = 0 \) and \( \delta = \delta_c \), respectively:

\[
\tau_p = \sigma_r \left( \tan \varphi'_{cv} + K_d \tan \psi_{\text{max}} \right) \quad , \quad \tau_r = \sigma_r \tan \varphi'_{cv}
\]

(4.30)

Due to dilation the cylinder shaped shear band expands. The expansion of the shear band can be calculated by integration of the assumed linearly decreasing dilation angle over the relative shear displacement \( \delta \).
At the end of the dilation process, i.e. after the relative displacement of $\delta_r$, the total radial displacement at the shear band is then

$$u = \frac{\delta_r}{2} \tan \psi_{\text{max}} \tag{4.32}$$

For simplicity it is assumed that this expansion causes a homogeneous radial strain $\bar{\varepsilon}_r = u/R$ within the elastic sector bounded by the radius $R$. In reality, the total expansion of the shear band results in compression both in- and outside the sector, and will cause a smaller radial strain. This effect of the lack of constraint is taken care of by a factor $K_b \leq 1$:

$$\bar{\varepsilon}_r = \frac{\delta_r}{2R} K_b \tan \psi_{\text{max}} \tag{4.33}$$

### 4.4.2 Frictional-Dilatant Material - Plane Stress

#### Stresses and Strains

Accounting for the initial stress state from Eq. (4.1) and considering that under plane stress conditions, stresses in the out-of-plane direction do not change $\Delta \sigma_z = 0$, linear elastic constitutive equations

$$\Delta \varepsilon_\theta = \frac{1}{E} (\Delta \sigma_\theta - \nu(\Delta \sigma_r + \Delta \sigma_z)) \tag{4.34}$$

$$\Delta \varepsilon_r = \frac{1}{E} (\Delta \sigma_r - \nu(\Delta \sigma_\theta + \Delta \sigma_z)) \tag{4.35}$$

$$\Delta \varepsilon_z = \frac{1}{E} (\Delta \sigma_z - \nu(\Delta \sigma_\theta + \Delta \sigma_r)) \tag{4.36}$$

can be rewritten as

$$\sigma_\theta = D(\Delta \varepsilon_\theta + \nu \bar{\varepsilon}_r) + K_0 \sigma_{z,0} \tag{4.37}$$

$$\sigma_r = D(\nu \Delta \varepsilon_\theta + \bar{\varepsilon}_r) + K_0 \sigma_{z,0} \tag{4.38}$$
with the stiffness parameter defined as $D = \frac{E}{1 - v^2}$, and the homogeneous radial strain caused by dilation $\Delta \varepsilon_r = \varepsilon_r$.

Radial normal stress on the shear band will change depending on the level of dilation and can be calculated using linear elastic relationship

$$\varepsilon_r = \frac{1}{E} \left( \sigma_r - v(\sigma_\theta + \sigma_z) \right)$$  \hspace{1cm} (4.39)

For peak and residual shear strength radial strains $\varepsilon_r = 0$ and $\varepsilon_r = \varepsilon_r = \frac{\delta}{2R} K_h \tan \psi_{max}$ are considered, respectively, so that from Eqs. (4.30) and (4.39):

$$\tau_p = \sigma_r (\tan \phi'_{cv} + K_d \tan \psi_{max}) = v(\sigma_\theta + \sigma_z) (\tan \phi'_{cv} + K_d \tan \psi_{max})$$  \hspace{1cm} (4.40)

$$\tau_r = \sigma_r \tan \phi'_{cv} = (v(\sigma_\theta + \sigma_z) + E\varepsilon_r) \tan \phi'_{cv}$$  \hspace{1cm} (4.41)

In plane stress conditions, because $\Delta \sigma_z = 0$, shear resistance along the boundaries between the top and bottom plates is constant:

$$\tau_s = \sigma_z \tan \phi'_s = \sigma_{z,0} \tan \phi'_s$$  \hspace{1cm} (4.42)

Stress distribution and deformations within the elastic part are derived in analogy to the cohesive material case (Eq. (4.8)) by calculating the equilibrium of moments acting on an elementary wedge, resulting in

$$\frac{d\sigma_\theta}{d\theta} = -2\tau_r - \frac{4}{3} \tau_s$$  \hspace{1cm} (4.43)

Substitution of Eqs. (4.41) and (4.42) into this expression results in the differential equation for the circumferential normal stress

$$\frac{d\sigma_\theta}{d\theta} = -2v(\sigma_\theta + \sigma_{z,0}) + E\varepsilon_r) \tan \phi'_{cv} - \frac{4}{3} \sigma_{z,0} \tan \phi'_s$$  \hspace{1cm} (4.44)

Rewriting this expression using

$$a = 2v \tan \phi'_{cv} \quad \text{and} \quad b = 2 \left( \frac{E\varepsilon_r \tan \phi'_{cv} + \sigma_{z,0}}{v \tan \phi'_{cv} + \frac{2}{3} \frac{R}{t} \tan \phi'_s} \right)$$  \hspace{1cm} (4.45)
gives

\[
\frac{d\sigma_\theta}{d\theta} = -a\sigma_\theta - b \tag{4.46}
\]

The solution of this differential equation is of the form

\[
\sigma_\theta = c_1 e^{-a\theta} - \frac{b}{a} \tag{4.47}
\]

Using the boundary condition on the shear blade \(\sigma_\theta(\theta = 0) = p_0\), it follows that \(c_1 = p_0 + \frac{b}{a}\) and therefore

\[
\sigma_\theta = \left(p_0 + \frac{b}{a}\right) e^{-a\theta} - \frac{b}{a} \tag{4.48}
\]

From the constitutive equations in Eqs. (4.37) and (4.38) follows the expression of circumferential strains as a function of the change in circumferential stress and dilation:

\[
\Delta \varepsilon_\theta = \frac{1}{D} \left(\sigma_\theta - \sigma_{\theta,0}\right) - v\bar{\varepsilon}_r \tag{4.49}
\]

Integration over the elastic area yields the rotation as a function of the pressure along the blade and of the shear band tip, respectively: The rotation \(\alpha(\theta)\) is calculated by integration of these strains similar to Eq. (4.10). For \(\theta = 0\), the rotation of the blade as a function of the stress at the blade \(p_0\) and the length of the shear band \(\eta\) can be written as:

\[
\alpha_0 = \int_0^\eta \Delta \varepsilon_\theta d\theta = \int_0^\eta \left(\frac{1}{D} \left(\sigma_\theta - K_\theta \sigma_{z,0}\right) - v\bar{\varepsilon}_r\right) d\theta
\]

\[
= \frac{1}{D} \int_0^\eta \left(p_0 e^{-a\theta} + \frac{b}{a} e^{-a\theta} - \frac{b}{a} - K_\theta \sigma_{z,0} - Dv\bar{\varepsilon}_r\right) d\theta
\]

\[
= \frac{1}{D} \left(\frac{1}{a} \left(p_0 + \frac{b}{a}\right) \left(1 - e^{-a\eta}\right) - \eta \left(\frac{b}{a} + K_\theta \sigma_{z,0} + Dv\bar{\varepsilon}_r\right)\right) \tag{4.50}
\]
**Energy Balance Criterion**

**Incremental Propagation**

Similar to the cohesive material case, an incremental propagation of the shear band is considered, which results in an increase of its angular length \( \Delta \eta \), at the constant normal stress at the blade \( p_0 \). This causes additional rotation of the shear blade \( \Delta \alpha_0 \), which is equal to the strain at the shear band tip multiplied by this incremental propagation. This gives

\[
\Delta \alpha_0 = \Delta \epsilon_\theta \Delta \eta = \frac{1}{D} \left( \frac{1}{R^2} \left( \sigma^\theta_\theta - \sigma^\theta_{\theta,0} \right) - \nu \epsilon_r \right) \Delta \eta
\]

(4.51)

where \( \sigma^\theta_\theta \) denotes tangential stress at the shear band tip \( \theta = \eta \).

**Work Components**

Energy terms in the Energy Balance Criterion Eq. (4.13) are derived in analogy to the cohesive material.

The external work component is

\[
\Delta W_e = p_0 \frac{R^2}{2} t \Delta \alpha_0 = p_0 \frac{R^2}{2} t \left( \frac{1}{D} \left( \sigma^\theta_\theta - \sigma^\theta_{\theta,0} \right) - \nu \epsilon_r \right) \Delta \eta
\]

(4.52)

Calculation of the internally stored energy in the elastic wedge is more complex than in the case of the cohesive material, because due to dilation the stresses perform work not only on circumferential but also on radial strains. Therefore, the incremental elastic work should be calculated as an integral of the specific strain energy function \( U \) over the area of the elastic wedge:

\[
\Delta W_i = \frac{R^2 t}{2} \Delta \eta \int_0^\epsilon \frac{U(r, \theta, z)}{\epsilon_r} \, dU = \frac{R^2 t}{2} \Delta \eta U(\Delta \epsilon_\theta, \epsilon_r, \epsilon_z)
\]

(4.53)

To define this specific strain energy \( U \), which is a unique function of the strain state, it should be noted that it serves as a potential for stresses producing the linear elastic law (Eq. (4.37) and (4.38))

\[
\sigma_\theta = \frac{\partial U}{\partial \epsilon_\theta} = D \Delta \epsilon_\theta + D \nu \epsilon_r + K_0 \sigma_{r,0}
\]

(4.54)
\[ \sigma_r = \frac{\partial U}{\partial \epsilon_r} = D \nu \Delta \epsilon_\theta + \epsilon_r + K_0 \sigma_{z,0} \quad (4.55) \]

\[ \sigma_z = \frac{\partial U}{\partial \epsilon_z} = \sigma_{z,0} \quad (4.56) \]

Eqs. (4.54) - (4.56) are satisfied when the strain energy function is given by the following expression

\[ U = \frac{D}{2} \left( (\Delta \epsilon_\theta^0)^2 + 2 \nu \epsilon_r \Delta \epsilon_\theta^0 + \epsilon_r^2 \right) + \left( \Delta \epsilon_\theta^0 + \epsilon_r \right) K_0 \sigma_{z,0} + \epsilon_z \sigma_{z,0} \quad (4.57) \]

Substitution of Eq. (4.57) into (4.53) gives the expression for the internally stored elastic energy component:

\[ \Delta W_i = \frac{R^2 t}{2} \Delta \eta \left[ \frac{D}{2} \left( (\Delta \epsilon_\theta^0)^2 + 2 \nu \epsilon_r \Delta \epsilon_\theta^0 + \epsilon_r^2 \right) + \left( \Delta \epsilon_\theta^0 + \epsilon_r \right) K_0 \sigma_{z,0} + \epsilon_z \sigma_{z,0} \right] \quad (4.58) \]

which, by using expressions from Eq. (4.49)

\[ \Delta \epsilon_\theta^0 = \left( \frac{1}{D} \left( \sigma_\theta^0 - \sigma_{z,0} K_0 \right) - \nu \epsilon_r \right) \quad (4.59) \]

and the expression for \( \epsilon_z \) derived from Eqs. (4.35) and (4.36)

\[ \epsilon_z = \left( \frac{1}{D} K_0 \left( K_0 \sigma_{z,0} - \sigma_\theta^0 \right) - \nu \epsilon_r \right) \quad (4.60) \]

can be transferred written as

\[ \Delta W_i = \frac{R^2 t}{2} \Delta \eta \left[ \frac{D}{2} \left( \frac{1}{D} \left( \sigma_\theta^0 - \sigma_{z,0} K_0 \right) - \nu \epsilon_r \right)^2 + 2 \nu \epsilon_r \left( \frac{1}{D} \left( \sigma_\theta^0 - \sigma_{z,0} K_0 \right) - \nu \epsilon_r \right) + \epsilon_r^2 \right] \]

\[ + \left( \frac{1}{D} \left( \sigma_\theta^0 - \sigma_{z,0} K_0 \right) - \nu \epsilon_r \right) + \epsilon_r \right) K_0 \sigma_{z,0} + \frac{1}{D} K_0 \left( K_0 \sigma_{z,0} - \sigma_\theta^0 \right) - \nu \epsilon_r \right) \sigma_{z,0} \]

and further simplified to

\[ \Delta W_i = \frac{R^2 t}{4} \Delta \eta \left( \frac{1}{D} \left( \sigma_\theta^0 - \sigma_{z,0} K_0 \right)^2 + D \epsilon_r^2 (1 - \nu^2) \right) \quad (4.61) \]
Dissipated work along the shear band and at the boundary plates is given by

\[
\Delta D_i + \Delta D_s = t \int_0^\eta \int_0^{\Delta \alpha_0} \tau_r R \, d\theta + \frac{2}{3} \int_0^\eta \int_0^{\eta \Delta \alpha_0} R \, d\alpha d\theta
\]

\[
= tR^2 \int_0^\eta \int_0^{\eta \Delta \alpha_0} \left( \tau_r + \frac{2R}{3t} \tau_s \right) d\alpha d\theta
\]

(4.62)

Substitution of the equilibrium from Eq. (4.43) into this expression gives

\[
\Delta D_i + \Delta D_s = -\frac{tR^2}{2} \int_0^\eta \int_0^{\eta \Delta \alpha_0} \frac{d\sigma_\theta}{d\theta} d\alpha d\theta = t\Delta \alpha_0 \frac{R^2}{2} \left( p_0 - \sigma_0^\theta \right)
\]

(4.63)

which, after substitution of Eq. (4.51), can be written as

\[
\Delta D_i + \Delta D_s = \left( \frac{1}{D} (\sigma_\theta - \sigma_{\theta,0}) - \nu \varepsilon_i \right) \Delta \eta \frac{R^2t}{2} \left( p_0 - \sigma_0^\theta \right)
\]

(4.64)

Finally, the dissipated energy at the shear band tip (process zone) is defined as

\[
\Delta D_\omega = R \Delta \eta \int_0^{\delta_s} (\tau - \tau_r) d\delta
\]

(4.65)

The shear strength in the strain softening branch is given by

\[
\tau = \sigma_r \tan \phi'
\]

(4.66)

which in combination with Eqs. (4.28) - (4.30) gives

\[
\tau - \tau_r = \sigma_r K_d \tan \psi = \sigma_r K_d \left( 1 - \delta / \delta_r \right) \tan \psi_{\text{max}}
\]

(4.67)

Normal radial stresses are obtained from Eq. (4.39)

\[
\sigma_r = \nu (\sigma_\theta + \sigma_{z,0}) + E \varepsilon_r = \nu (\sigma_\theta + \sigma_{z,0}) + E \frac{u}{R}
\]

(4.68)

which after substitution of Eq. (4.31) and applying correction \( K_b \) gives

\[
\sigma_r = \nu (\sigma_\theta + \sigma_{z,0}) + E \frac{K_b}{R} \left( \delta - \frac{\delta^2}{2\delta_r} \right) \tan \psi_{\text{max}}
\]

(4.69)
The dissipated work in the process zone can be calculated by substituting Eqs. (4.67) and (4.69) into (4.65)

\[
\Delta D_w = R \Delta \eta K_d \tan \psi_{\text{max}} \left( \int_0^\delta \left( \nu (\sigma_0 + \sigma_{z,0}) + E \frac{K_d}{R} \left( \delta - \frac{\delta^2}{2\delta_f} \right) \tan \psi_{\text{max}} \right) \left( 1 - \frac{\delta}{\delta_f} \right) d\delta \right)
\]

and integrating the resulting expression to produce

\[
\Delta D_w = R \Delta \eta K_d \tan \psi_{\text{max}} \left( \frac{\nu \sigma_0 \delta_f}{2} + \frac{\nu \sigma_{z,0} \delta_f}{2} + \frac{\delta_f^2}{8R} E K_b \tan \psi_{\text{max}} \right)
\]

which can be rewritten as

\[
\Delta D_w = R^2 \Delta \eta \frac{K_d}{K_b} \left( \nu \sigma_0 \varepsilon_f + \nu \sigma_{z,0} \varepsilon_f + \varepsilon_r^2 \frac{D}{2} (1 - \nu^2) \right) \tag{4.70}
\]

**Energy Balance Criterion**

Substitution of the work increments (4.52), (4.61), (4.64), (4.70) into the energy balance criterion (Eq. (4.13)) gives

\[
p_0 \left( \frac{1 - \nu^2}{E} (\sigma_0 - \sigma_{0,0}) - \nu \varepsilon_f \right) \geq \frac{1}{2} \left( \frac{1}{D} (\sigma_0 - \sigma_{0,0})^2 K_0 + D \varepsilon_r^2 (1 - \nu^2) \right) + \\
+ \left( \frac{1}{D} (\sigma_0 - \sigma_{0,0}) - \nu \varepsilon_f \right) \left( p_0 - \sigma_0^2 \right) + \frac{2 K_d}{K_b} \left( \nu \sigma_0 \varepsilon_f + \nu \sigma_{z,0} \varepsilon_f + \varepsilon_r^2 \frac{D(1 - \nu^2)}{2} \right) \tag{4.71}
\]

which can be reduced to the following shear band propagation criterion

\[
\sigma_0^2 \geq 2 D \sigma_0^2 \varepsilon_f \left( 1 + \frac{2 K_d}{K_b} \right) + \sigma_{z,0}^2 K_0^2 + \frac{4 K_d}{K_b} D \nu \sigma_{z,0} \varepsilon_f + \varepsilon_r^2 D^2 (1 - \nu^2) \left( 1 + \frac{2 K_d}{K_b} \right)
\]

and after solving this quadratic inequality:

\[
\sigma_0^2 \geq D \nu \varepsilon_f \left( 1 + \frac{2 K_d}{K_b} \right) + \frac{1}{2} \left( 2 D \nu \varepsilon_f \left( 1 + \frac{2 K_d}{K_b} \right) \right)^2 + \\
+ 4 \left( \sigma_{z,0}^2 K_0^2 + \frac{4 K_d}{K_b} D \nu \sigma_{z,0} \varepsilon_f + \varepsilon_r^2 D^2 (1 - \nu^2) \left( 1 + \frac{2 K_d}{K_b} \right) \right) \tag{4.72}
\]
written in normalized form this gives

\[ \frac{\sigma''_o}{\sigma_{z,0}} \geq D v \bar{\varepsilon}_d \left( 1 + \frac{2K_d}{K_b} \right) + \left[ K_0^2 + \frac{4K_d}{K_b} D v \bar{\varepsilon}_d \right. \]

\[ + \bar{\varepsilon}_r^2 D^2 \left( 1 + \frac{2K_d}{K_b} \right) + \nu^2 \left( 1 + \frac{2K_d}{K_b} \right) - 1 \left( 1 + \frac{2K_d}{K_b} \right) \right]^{\frac{1}{2}} \]

\[ (4.73) \]

where \( \bar{D} = D/\sigma_{z,0} \).

From Eq. (4.48), which in the normalized form becomes

\[ \frac{\sigma''_o}{\sigma_{z,0}} = \left( \frac{p_0}{\sigma_{z,0}} + \bar{K}_f \right) e^{-\eta} - \bar{K}_f, \text{ where } \bar{K}_f = \frac{b}{a \sigma_{z,0}} \]

(4.74)

it follows that

\[ \frac{p_0}{\sigma_{z,0}} \geq \left[ \bar{K}_f + D v \bar{\varepsilon}_d \left( 1 + \frac{2K_d}{K_b} \right) + \left[ K_0^2 + \frac{4K_d}{K_b} D v \bar{\varepsilon}_d \right. \]

\[ + \bar{\varepsilon}_r^2 D^2 \left( 1 + \frac{2K_d}{K_b} \right) + \nu^2 \left( 1 + \frac{2K_d}{K_b} \right) - 1 \left( 1 + \frac{2K_d}{K_b} \right) \right]^{\frac{1}{2}} e^{-\eta} - \bar{K}_f \]

\[ (4.75) \]

This is the condition for the stress controlled test. Its substitution into Eq. (4.50) gives the condition for the displacement controlled test, i.e., the relationship between the shear band length and the shear blade rotation:

\[ \alpha_o = \frac{1}{aD} \left[ \bar{K}_f + D v \bar{\varepsilon}_d \left( 1 + \frac{2K_d}{K_b} \right) + \left[ K_0^2 + \frac{4K_d}{K_b} D v \bar{\varepsilon}_d \right. \]

\[ + \bar{\varepsilon}_r^2 D^2 \left( 1 + \frac{2K_d}{K_b} \right) + \nu^2 \left( 1 + \frac{2K_d}{K_b} \right) - 1 \left( 1 + \frac{2K_d}{K_b} \right) \right]^{\frac{1}{2}} e^{-\eta} - 1 \]

\[ (4.76) \]

\[ - \frac{\eta}{D} \left( \bar{K}_f + K_0 + D v \bar{\varepsilon}_d \right) \]
4 Progressive Shear Band Propagation in Shear Blade Tests

where

\[
\overline{K}_f = \frac{b}{a \sigma_{z,0}}
\]

\[
a = 2 \nu \tan \phi'_{cv}
\]

\[
b = 2 \left( E \varepsilon_y \tan \phi'_{cv} + \sigma_{z,0} \left( \nu \tan \phi'_{cv} + \frac{2}{3} \frac{R}{t} \tan \phi'_{s} \right) \right)
\]

**Limiting Equilibrium Criterion**

The limiting equilibrium condition from Eq. (4.22) holds for the frictional case as well, with a difference that the peak shear strength and the strength on the boundary plate contacts are defined by Eqs. (4.40) and (4.42), respectively. After the corresponding substitutions, the condition for the shear band propagation according to the limiting equilibrium can be written as

\[
\sigma_\theta \geq \frac{2 \sigma_{z,0} \left( \eta_{LE} - \eta \left( \nu \tan \phi'_{cv} + \nu \tan \psi_{\text{max}} + \frac{2}{3} \frac{R}{t} \tan \phi'_{s} \right) + K_a \right)}{(1 - 2 \nu (\tan \phi'_{cv} + \tan \psi_{\text{max}})(\eta_{LE} - \eta))} (4.78)
\]

which can be resolved with respect to the pressure applied by the shear blade using Eq. (4.48)

\[
\frac{p_0}{\sigma_{z,0}} \geq \frac{2 \left( \eta_{LE} - \eta \left( \nu \tan \phi'_{cv} + \nu \tan \psi_{\text{max}} + \frac{2}{3} \frac{R}{t} \tan \phi'_{s} \right) + K_a \right)}{(1 - 2 \nu (\tan \phi'_{cv} + \tan \psi_{\text{max}})(\eta_{LE} - \eta))} e^{\nu \eta K_a} - K_f (4.79)
\]

Substitution of the above inequality into Eq. (4.50) results in the expression for the shear band propagation for the displacement controlled test:

\[
\alpha_0 = \frac{1}{D a} \frac{2 \left( \eta_{LE} - \eta \left( \nu \tan \phi'_{cv} + \nu \tan \psi_{\text{max}} + \frac{2}{3} \frac{R}{t} \tan \phi'_{s} \right) + K_a \right)}{(1 - 2 \nu (\tan \phi'_{cv} + \tan \psi_{\text{max}})(\eta_{LE} - \eta))} \left( e^{\nu \eta} - 1 \right) + \frac{K_f}{\overline{K}_f} (4.80)
\]

\[-\frac{\eta}{D} (K_f + K_0 + D \nu \varepsilon_x)\]

where \( K_f = \frac{b}{a \sigma_{z,0}} \) from Eq. (4.77).
4.4.3 Frictional-Dilatant Material - Plane Strain

**Stresses and Strains**

Accounting for initial stress state from Eq. (4.1) and including that in plane strain conditions, strains in the out of plane direction do not change, linear elastic equations can be written as

\[
\Delta \varepsilon_\theta = \frac{\Delta \sigma_\theta}{E} - \frac{\nu}{E} (\Delta \sigma_r + \Delta \sigma_z) \quad (4.81)
\]

\[
\Delta \varepsilon_r = \frac{\Delta \sigma_r}{E} - \frac{\nu}{E} (\Delta \sigma_\theta + \Delta \sigma_z) \quad (4.82)
\]

\[
\Delta \varepsilon_z = \frac{\Delta \sigma_z}{E} - \frac{\nu}{E} (\Delta \sigma_\theta + \Delta \sigma_r) = 0; \quad \Delta \sigma_z = \nu (\Delta \sigma_\theta + \Delta \sigma_r) \quad (4.83)
\]

Substitution of the latter expression in Eq. (4.83) into (4.82) and (4.81) gives, after completion of the dilation process; i.e. \( \Delta \varepsilon_r = \bar{\varepsilon}_r \), using the stiffness as defined in the second expression in Eq. (4.27) the expressions for tangential strains

\[
\Delta \varepsilon_\theta = \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)E} \Delta \sigma_\theta - \frac{\nu}{(1 - \nu)} \bar{\varepsilon}_r = \frac{1}{D} (\sigma_\theta - K_0 \sigma_{z,0}) - K_0 \bar{\varepsilon}_r \quad (4.84)
\]

Note that the stiffness in plane strain is defined as \( D = \frac{(1 - \nu)}{(1 + \nu)(1 - 2\nu)} E = M \)

For circumferential stresses we get therefore

\[
\sigma_\theta = (\Delta \varepsilon_\theta + K_0 \bar{\varepsilon}_r) D + K_0 \sigma_{z,0} \quad (4.85)
\]

Similarly for radial stresses we obtain

\[
\sigma_r = D (\bar{\varepsilon}_r + K_0 \Delta \varepsilon_\theta) + K_0 \sigma_{z,0} \quad (4.86)
\]

Combination of Eq. (4.86) with (4.84) gives an expression for the radial stress as a function of tangential stress

\[
\sigma_r = (\sigma_\theta + \sigma_{z,0}) (1 - K_0) K_0 + (1 - K_0^2) D \bar{\varepsilon}_r \quad (4.87)
\]

Radial normal stress on the shear band will change depending on the level of dilation and can be calculated using linear elastic relationship from Eq. (4.39)
For peak and residual shear strength $\Delta \varepsilon_r = 0$ and $\Delta \varepsilon_r = \bar{\varepsilon}_r = \frac{\delta_r}{2R} K_h \tan \psi_{\text{max}}$, respectively, so that from Eqs. (4.30) and (4.87) it follows:

$$
\tau_p = \sigma_r (\tan \varphi'_{cv} + K_d \tan \psi_{\text{max}}) = \left(\sigma_\theta + \sigma_{z,0} (1 - K_0)\right) K_0 \left(\tan \varphi'_{cv} + K_d \tan \psi_{\text{max}}\right)$$

(4.88)

$$
\tau_r = \sigma_r \tan \varphi'_{cv} = \left[\left(\sigma_\theta + \sigma_{z,0} (1 - K_0)\right) K_0 + \left(1 - K_0^2\right) D \bar{\varepsilon}_r\right] \tan \varphi'_{cv}$$

(4.89)

Shear stress along the boundary can be expressed as $\tau_s = \sigma_z \tan \varphi_s$, and can be written as a function of tangential stress

$$
\tau_s = \left((\sigma_\theta - K_0 \sigma_{z,0}) K_0 + D K_0 \bar{\varepsilon}_r (1 - K_0) + \sigma_{z,0}\right) \tan \varphi'_{s}$$

(4.90)

Stress distribution and deformations within the elastic part are derived in analogy to the cohesive material case (Eq. (4.8) and (4.43)) by calculating the equilibrium of moments acting on an elementary wedge, resulting in

$$
\frac{d\sigma_\theta}{d\theta} = -2 \left(\sigma_\theta + \sigma_{z,0} (1 - K_0)\right) K_0 + \left(1 - K_0^2\right) D \bar{\varepsilon}_r \tan \varphi'_{cv}$$

$$- \frac{4R}{3t} \left((\sigma_\theta - K_0 \sigma_{z,0}) K_0 + D K_0 \bar{\varepsilon}_r (1 - K_0) + \sigma_{z,0}\right) \tan \varphi'_{s}$$

(4.91)

In analogy to Eq. (4.46) with

$$
a = 2K_0 \left(\tan \varphi'_{cv} + \frac{2R}{3t} \tan \varphi'_{s}\right)$$

$$b = 2(1 - K_0) \left[\sigma_{z,0} K_0 \tan \varphi'_{cv} + \frac{2R}{3t} (1 + K_0) \tan \varphi'_{s}\right]$$

(4.92)

$$+ D \bar{\varepsilon}_r \left((1 + K_0) \tan \varphi'_{cv} + \frac{2R}{3t} K_0 \tan \varphi'_{s}\right)$$

The solution of this differential equation is of the form shown in Eq. (4.47) and its solution can be written as Eq. (4.48); i.e. we get the identical expression for the tangential stress.

Integration of the expression for the circumferential strains (Eq. (4.84)) over the elastic area yields the rotation as a function of the pressure along the blade and of the shear band tip, respectively: The rotation $\alpha(\theta)$ is calculated by integration of these strains similar to Eq. (4.10). For $\theta = 0$, we get the rotation of the blade as a function of the stress at the blade $p_0$ and the length of the shear band $\eta$:
\[
\alpha_0 = \int_0^\theta \Delta \varepsilon_\theta d\theta = \int_0^\theta \left( \frac{\sigma_\theta - \sigma_{\theta,0}}{D} - K_0 \varepsilon_r \right) d\theta \\
= \frac{1}{D} \left( p_0 + \frac{b}{a} \right) \left( 1 - e^{-a\eta} \right) - \eta \left( \frac{b}{a} + K_0 \sigma_{\theta,0} + D K_0 \varepsilon_r \right) \\
= \frac{1}{D} \left( p_0 + \frac{b}{a} \right) \left( 1 - e^{-a\eta} \right) - \eta \left( \frac{b}{a} + K_0 \sigma_{\theta,0} + D K_0 \varepsilon_r \right) 
\]

(4.93)

**Energy Balance Criterion**

**Incremental Propagation**

Similar to the cohesive material case, an incremental propagation of the shear band is considered, which results in an increase of its angular length \( \Delta \eta \), at the constant normal stress at the blade \( p_0 \). This causes additional rotation of the shear blade \( \Delta \alpha_0 \), which is equal to the strain at the shear band tip multiplied by this incremental propagation. This gives

\[
\Delta \alpha_0 = \varepsilon_\theta \Delta \eta = \left( \frac{1}{D} \left( \sigma_\theta^0 - \sigma_{\theta,0} \right) - K_0 \varepsilon_r \right) \Delta \eta 
\]

(4.94)

**Work Components**

Energy terms in the Energy Balance Criterion Eq. (4.13) are derived in analogy to the cohesive material.

The external work component is

\[
\Delta W_e = p_0 \frac{R^2 t}{2} \Delta \alpha_0 = \frac{R^2 t}{2} p_0 \left( \frac{\sigma_\theta^0 - \sigma_{\theta,0}}{D} - K_0 \varepsilon_r \right) \Delta \eta 
\]

(4.95)

Calculation of the internally stored energy in the elastic wedge is performed in analogy to the plane stress case (Eq. (4.53) as an integral of the specific strain energy function \( U \) over the area of the elastic wedge. Recall that in plane strain condition the out-of-plane direction is zero and therefore

\[
\Delta W_i = \frac{R^2 t}{2} \Delta \eta \int dU = \frac{R^2 t}{2} \Delta \eta U(\varepsilon_\eta, \varepsilon_r) 
\]

(4.96)
To define this specific strain energy $U$, which is a unique function of the strain state, note that it serves as a potential for stresses producing the linear elastic law (Eqs. (4.85) and (4.86)).

$$\sigma_\theta = \frac{\partial U}{\partial \varepsilon_\theta} = D\Delta\varepsilon_\theta + DK_0\varepsilon_r + K_0\sigma_{\varepsilon,0} \quad (4.97)$$

$$\sigma_r = \frac{\partial U}{\partial \varepsilon_r} = D\varepsilon_r + DK_0\Delta\varepsilon_\theta + K_0\sigma_{\varepsilon,0} \quad (4.98)$$

This results in the expression for the stain energy function

$$U = \frac{D}{2}\left(\varepsilon_r^2 + 2K_0\Delta\varepsilon_\theta\varepsilon_r + (\Delta\varepsilon_\theta)^2\right) + K_0\sigma_{\varepsilon,0}(\Delta\varepsilon_\theta + \varepsilon_r) \quad (4.99)$$

Substitution of Eq. (4.99) into (4.96) gives the expression for the internally stored elastic energy component

$$\Delta W_i = \frac{R^2t}{2}\Delta\eta\left(\frac{D}{2}\left(\varepsilon_r^2 + 2K_0\Delta\varepsilon_\theta\varepsilon_r + (\Delta\varepsilon_\theta)^2\right) + K_0\sigma_{\varepsilon,0}(\Delta\varepsilon_\theta + \varepsilon_r)\right) \quad (4.100)$$

which, by using expression $\varepsilon_\theta^0 = \frac{\sigma_\theta^0 - \sigma_{\varepsilon,0} - K_0\varepsilon_r}{D}$, can be transferred written as

$$\Delta W_i = \frac{R^2t}{2}\Delta\eta\left(\frac{D}{2}\varepsilon_r^2\left(1 - K_0^2\right) + \frac{\sigma_\theta^0 - K_0\sigma_{\varepsilon,0}^2}{2D} + \sigma_{\varepsilon,0}\varepsilon_rK_0\left(1 - K_0\right)\right) \quad (4.101)$$

Dissipated work along the shear band and at the boundary plates is given by

$$\Delta D_j + \Delta D_z = t\int_0^{\Delta\alpha_0} \int_0^{\Delta\alpha_0} \int_0^R \tau_R d\alpha R d\theta + 2\int_0^{\Delta\alpha_0} \int_0^R \int_0^R \tau_R d\alpha r dr d\alpha d\theta \quad (4.102)$$

Substitution of the equilibrium Eq. (4.43) into this expression gives

$$\Delta D_j + \Delta D_z = -\frac{tR^2}{2} \int_0^{\Delta\alpha_0} \int_0^{\Delta\alpha_0} \frac{d\sigma_\theta}{d\theta} d\alpha d\theta = t\Delta\alpha_0 R^2 \left(p_0 - \sigma_\theta^0\right) \quad (4.103)$$

which, after substitution of Eq. (4.94), can be written as
Finally, the dissipated energy at the shear band tip (process zone) is defined as

$$\Delta D_{ao} = R\Delta \eta_0 \int_0^\delta \tau \, d\delta$$  \hspace{1cm} (4.105)$$

The shear strength in the strain softening branch is given by

$$\tau = \sigma_r \tan \phi'$$  \hspace{1cm} (4.106)$$

which in combination with Eqs. (4.28) - (4.30) gives

$$\tau - \tau_r = \sigma_r K_d \tan \psi = \sigma_r K_d \left(1 - \delta/\delta_r\right) \tan \psi_{max}$$  \hspace{1cm} (4.107)$$

Normal radial stresses are obtained from Eq. (4.39)

$$\sigma_r = \nu \left(\sigma_\theta + \sigma_{z,0}\right) + E \varepsilon_r = \nu \left(\sigma_\theta + \sigma_{z,0}\right) + E \frac{u}{R}$$  \hspace{1cm} (4.108)$$

which after substitution of (4.31) and applying correction $K_b$ gives

$$\sigma_r = \nu \left(\sigma_\theta + \sigma_{z,0}\right) + E \frac{K_b}{R} \left(\delta - \frac{\delta^2}{2\delta_r}\right) \tan \psi_{max}$$  \hspace{1cm} (4.109)$$

The dissipated work in the process zone can be calculated by substituting Eqs. (4.107) and (4.109) into Eq. (4.105):

$$\Delta D_{ao} = R\Delta \eta K_d \tan \psi_{max} \int_0^\delta \nu \left(\sigma_\theta + \sigma_{z,0}\right) + E \frac{K_b}{R} \left(\delta - \frac{\delta^2}{2\delta_r}\right) \tan \psi_{max} \left(1 - \frac{\delta}{\delta_r}\right) d\delta$$

and integrating the resulting expression to produce

$$\Delta D_{ao} = R\Delta \eta K_d \tan \psi_{max} \left(\frac{v\sigma_\theta \delta_r}{2} + \frac{v\sigma_{z,0} \delta_r}{2} + \frac{\delta_r^2}{8R} E K_b \tan \psi_{max}\right)$$

which can be rewritten as

$$\Delta D_{ao} = R^2 \Delta \eta K_d \frac{K_d}{K_b} \left(\sigma_\theta^\eta + \sigma_{z,0} (1 - K_0) \varepsilon_r K_0 + \varepsilon_r^2 \frac{D}{2} (1 - K_0^2)\right)$$  \hspace{1cm} (4.110)$$
Energy Balance Criterion

Substitution of the work increments Eqs. (4.95), (4.101), (4.104) and (4.110) into the energy balance criterion in Eq. (4.13) gives

\[ p_0 \left( \frac{\sigma_0^n - \sigma_0^{n,0}}{D} - K_0 \varepsilon_r \right) \geq \left( \frac{D}{2} \varepsilon_r^2 (1 - K_0^2) + \frac{\sigma_0^{n,2} - K_0^2 \sigma_{z,0}^2 + \sigma_{z,0} \varepsilon_r K_0 (1 - K_0)}{2D} \right) \]

\[ + \left( \frac{\sigma_0^n - \sigma_0^{n,0}}{D} - K_0 \varepsilon_r \right) \left( p_0 - \sigma_0^n \right) + 2 \frac{K_d}{K_b} \left( \sigma_0^n + \sigma_{z,0} (1 - K_0) \varepsilon_r K_0 + \frac{D}{2} \varepsilon_r^2 (1 - K_0^2) \right) \]

which can be reduced to the following shear band propagation criterion

\[ (\sigma_0^n - K_0 \sigma_{z,0})^2 \geq 2 \sigma_0^n DK_0 \varepsilon_r \left( 1 + 2 \frac{K_d}{K_b} \right) \]

\[ + \left( 2 \sigma_{z,0} \varepsilon_r DK_0 + D^2 \varepsilon_r^2 (1 + K_0) \right) \left( 1 - K_0 \right) \left( 1 + 2 \frac{K_d}{K_b} \right) \]

(4.111)

and after solving this quadratic inequality:

\[ \sigma_0^n \geq \sigma_{z,0} K_0 + \varepsilon_r DK_0 \left( 1 + 2 \frac{K_d}{K_b} \right) \]

\[ + \sqrt{\varepsilon_r D \left( 1 + 2 \frac{K_d}{K_b} \right) \left( \varepsilon_r D \left( 1 + 2 \frac{K_d}{K_b} K_0^2 \right) + 2K_0 \sigma_{z,0} \right)} \]

(4.112)

or in normalized form it can be written as

\[ \frac{\sigma_0^n}{\sigma_{z,0}} \geq K_0 + \varepsilon_r \overline{D} K_0 \left( 1 + 2 \frac{K_d}{K_b} \right) \]

\[ + \sqrt{\varepsilon_r \overline{D} \left( 1 + 2 \frac{K_d}{K_b} \right) \left( \varepsilon_r \overline{D} \left( 1 + 2 \frac{K_d}{K_b} K_0^2 \right) + 2K_0 \right)} \]

(4.113)

where \( \overline{D} = \frac{D}{\sigma_{z,0}} \).

Substituted into Eq. (4.48) it follows
\[
\frac{P_0}{\sigma_{z,0}} \geq \left[ \bar{\xi} \bar{D} K_0 \left( 1 + 2 \frac{K_d}{K_b} \right) + \sqrt{\bar{\xi} \bar{D} \left( 1 + 2 \frac{K_d}{K_b} \right) \left( 1 + 2 \frac{K_d}{K_b} \right) + 2K_0} \right] e^{\eta \epsilon} - \bar{K}_f \] (4.114)

This is the condition for the stress controlled test. Its substitution into Eq. (4.93) gives the condition for the displacement controlled test, i.e., the relationship between the shear band length and the shear blade rotation:

\[
\alpha_0 = \frac{1}{a \bar{D}} \left[ \bar{K}_f + K_0 \left( 1 + \bar{D} \bar{\xi}_r \left( 1 + 2 \frac{K_d}{K_b} \right) \right) + \sqrt{\bar{\xi}_r \bar{D} \left( 1 + 2 \frac{K_d}{K_b} \right) \bar{\xi}_r \bar{D} \left( 1 + 2 \frac{K_d}{K_b} \right) + 2K_0} \right] \left( e^{\eta \epsilon} - 1 \right) (4.115)
\]

where

\[
\bar{D} = \frac{D}{\sigma_{z,0}} ; \quad \bar{K}_f = \frac{b}{a \sigma_{z,0}} ; \quad \bar{\xi}_r = \frac{\delta K_b}{2R} \tan \psi_{\text{max}}
\]

\[
a = 2K_0 \left( \tan \phi'_{cv} + \frac{2}{3} t \tan \phi'_{s} \right)
\]

\[
b = 2(1 - K_0) \left( K_0 \tan \phi'_{cv} + \frac{2}{3} t (1 + K_0) \tan \phi'_{s} \right) + D \bar{\xi}_r \left( 1 + K_0 \right) \tan \phi'_{cv} + \frac{2}{3} K_0 \tan \phi'_{s} \right)
\]

**Limiting Equilibrium Criterion**

The limiting equilibrium condition can be found in analogy to the plane stress case and can be written in normalized form

\[
\frac{\sigma_\theta}{\sigma_{z,0}} \geq \frac{2(\eta_{LE} - \eta)(1 - K_0) \left( K_0 (\tan \phi'_{cv} + K_d \tan \psi_{\text{max}} ) + \frac{2}{3} t \tan \phi'_{s} (1 + K_0) + K_d \right)}{1 - 2K_0(\eta_{LE} - \eta) \left( \tan \phi'_{cv} + K_d \tan \psi_{\text{max}} + \frac{2}{3} t \tan \phi'_{s} \right)} (4.116)
\]

Substitution of this expression into Eq. (4.48) and then into Eq. (4.93) gives the analytical conditions for the shear band propagation according the limiting
equilibrium approach for the stress and displacement controlled test, respectively.

### 4.5 Parametric Study

A sensitivity study has been performed in order to understand parameter dependency of the solution. Sensitivity has been checked for two parameters: the stiffness parameter $M$ and the relative displacement $\delta_r$, assuming linear decrease of strength $\delta_m = \delta_r$.

![Figure 41. Sensitivity of the constrained modulus in the model with frictional material.](image)

In Figure 41 the shape of the curves with frictional material is plotted for constrained modulus $M$ of 250, 500 and 1000 kPa. With increasing stiffness, the rate of propagation increases. In this study the active failure zone has been neglected. Therefore the shear band is purely driven by the energy balance criterion.
Another parameter which is difficult to choose is the relative displacement $\delta_r$ needed to reach friction at constant volume at the shear band tip. It has been assumed that this parameter is equal to half of the shear band width. Correspondingly, Figure 42 shows dependency of the rate of the shear band propagation on the relative displacement $\delta_r$ as a multiple of the mean grain diameter. With increasing relative displacement, the rate of the shear band propagation slows down. This is because more energy is dissipated in the process zone of the shear band. The sensitivity of the shear band propagation rate is, however, rather low.
4.6 Comparison of Analytical Curves with Experimental Data

Comparison between experiments and analytical curves is performed using two plots: in the first plot (Figure 43), the propagation rates of both the analytical solutions and the experiments are shown. The analytical curves are fitted to the experimental data set using method of least squared errors and changing the stiffness parameter $D$ for both the fastest and the slowest propagation path. Then, in a second plot (Figure 44), the stiffness $D$ obtained from the above regression analysis is compared to the experimental stiffness curves obtained from oedometer tests.

Because in oedometer tests, stresses are much higher than stresses applied in the shear blade tests, regression curves have been calculated in order to be able to compare the values.

Table 4. Input parameters used in the analytical model.

<table>
<thead>
<tr>
<th>Geometry</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>0.04 [m]</td>
</tr>
<tr>
<td>$t$</td>
<td>0.025 [m]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Material properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_p'$</td>
<td>45.0 [°]</td>
</tr>
<tr>
<td>$\phi_{cv}'$</td>
<td>39.0 [°]</td>
</tr>
<tr>
<td>$\phi_s'$</td>
<td>30.0 [°]</td>
</tr>
<tr>
<td>$d_{s0}$</td>
<td>0.058 [mm]</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.3 [-]</td>
</tr>
<tr>
<td>$K_d$</td>
<td>1.0 [-]</td>
</tr>
<tr>
<td>$K_b$</td>
<td>0.5 [-]</td>
</tr>
</tbody>
</table>

Coefficient $K_b$ defining the compression strain cause by dilation within shear band in the sector inside the shear band has been adopted as 0.5. In order to validate this parameter, a numerical simulation has been performed to investigate the displacement inside and outside the shear band due to a pressure increase along the shear band: It has been found that $K_b = 0.5$ is an appropriate assumption. Remaining parameters used for the analytical solution are given in Table 4.
Figure 43. Experimental and analytical results of the shear band propagation in shear blade tests: (a) plane stress; (b) plane strain.
From the plot shown in Figure 44 it can be concluded that correct rates of propagation of the shear band are predicted using the stiffness from the lower range obtained in the oedometer tests.

The area between the corners A-B-E-D has been obtained including regression analysis of the angle of the active failure zone $\eta_{LE}$, which has been found to be between 1.57 and 1.7 [rad]. If the active failure zone is ignored results between points C and F are obtained.

Figure 44. Constrained moduli plotted versus the stress acting on the failure plane (quad) in comparison with the regression curves and results from oedometer tests (curves and dots).
4.7 Validation of Analytical Solution with Numerical Analysis

A preliminary attempt has been also made to model the shear blade tests numerically. Due to their reasonable numerical stability, certain finite difference methods have proven to be more suitable for numerical analysis of the post-failure strain-softening behaviour of soils. One of these methods, FLAC (Fast Lagrangian Analysis of Continua), was used for numerical simulations of the shear blade tests.

Because we are focusing on the rate of the shear band propagation it was important to make sure that the shear band follows the experimentally observed path. Therefore, the mesh was generated in such a way, that the shear band was forced to propagate in the similar direction as in the experiments and to stay within the band of a one element width, i.e. within its experimentally observed width. This allows for the negative effects of mesh dependency to be eliminated.

![FLAC mesh at initial state](image)

**Figure 45.** FLAC mesh at initial state. Dark grey coloured material is elastic, along the predefined shear band path it is elasto-plastic with softening in friction and dilation (light grey).
The mesh generated for the simulation is given in Figure 45. The soil was modelled as a continuum using Mohr-Coulomb model with strain-softening along the shear band and elastic material beyond the shear band. The mesh size along the shear band was chosen to be identical to the width of the shear band thickness of twenty times the mean grain diameter in the tests (1.2 mm). The residual friction was assumed to be reached at the relative displacement $\delta_r = 0.6\,\text{mm}$, corresponding to the shear strain of $\varepsilon_s = 0.25$, plain strain conditions have been applied. The rest of the parameter values were adopted from the analytical model. The friction between the glass and the soil was ignored.

In the analytical solution, the displacement outside the propagation zone is assumed to be zero. In the experiment and numerical analysis they are non-zero, because the material outside the zone bounded by the band is not rigid. Therefore, for the proper comparison of the rates of the shear band propagation between the numerical and analytical models, the shear blade displacement has been corrected by the deformations ahead of the shear band propagation zone.

Identical parameters have been defined for both the analytical and numerical models. For the comparison of results, we make use of the concept used previously and shown in Figure 43 and Figure 44. Analytical curves of the propagation rate have been fitted to the experiments ignoring the friction between the glass and the soil by variation of the constraint modulus. Then again the stiffness parameter in the numerical simulations has been adjusted such that the propagation rates are comparable with those from analytical and experimental results (Figure 46). The constraint moduli are plotted against the normal stress at the initiation of the shear band at the symmetry axis of the test and are shown in Figure 47.
Figure 46. Regression of numerical and analytical results against experimental data.

Figure 47. Constraint modulus against the stress at the shear band tip. Comparison of the range of analytical and numerical results in comparison with the range from oedometer results (black lines).
For this simplified numerical model for both, plane strain and plane stress conditions the shear band propagated but, as expected, at much higher forces than in the experiments.

4.8 Conclusions

A novel test device for studying the progressive shear band propagation in cylinder shaped shear bands has been presented. An analytical model based on the fracture mechanics energy balance and the limiting equilibrium approaches has been derived for both, plane stress and plane strain conditions. Solutions are provided for cohesive and frictional-dilatant materials. Analytical solutions have been validated and checked against numerical simulation. Comparison of experiments with the analytical solution provides an assessment of the solution. Results show that the method provides both qualitatively and quantitatively reasonable results. This study confirms that the energy balance approach from fracture mechanics is an appropriate means for the modelling of shear band propagation in cohesive and frictional-dilatant soils.
5 Catastrophic Shear Band Propagation in Biaxial Tests

5.1 Introduction

The velocity of the shear band propagation is a non-conventional soil parameter. However, it is essential for the assessment of the tsunamigenic landslides. In the previous chapters, the rate of the shear band propagation has been evaluated for the progressive case. In this chapter, an attempt is provided to catch the process of the catastrophic shear band propagation and to measure the velocity of the shear band propagation in dry silty sand using a high-speed camera.

A biaxial test setup has been chosen to observe and measure this process. The main advantage of biaxial test setups is that the shear band grows along a plane which is, due to the boundary conditions, always normal to the plane of observation. Compared with the triaxial test, in a biaxial test setup it is sufficient to record the sample during the test from one direction, using one camera only.

Figure 48. Concept of biaxial test: initial sample (dashed line) and localized shear band (grey).
Displacement controlled biaxial compression tests have been performed. Initially, a uniform pressure $\sigma_z = \sigma_x = \text{const.}$ is applied on a sample under plane strain conditions ($\varepsilon_y = 0$). During the test itself, the sample is sheared by applying a vertical velocity $\dot{z} = \text{const.} > 0$, while the stress in horizontal direction is kept constant ($\sigma_x = \text{const.}$). In particulate material exhibiting strain-softening behaviour, such as dense granular material, this provokes the propagation of a shear band throughout the sample (Figure 48).

Shear band formation and dependency on boundary conditions in biaxial tests have been studied by several authors, analytical models based on bifurcation theory (Vardoulakis et al. 1978; Vardoulakis, 1980) and on theoretical considerations of the limitations from classic soil mechanic solutions (Vermeer, 1990) have been developed and discussed.

Here a novel biaxial test has been developed. Although the test has been designed as simple as possible, it incorporates the following advantages:

- It allows testing without application of a rubber sleeve around the sample. Therefore the particles can be observed directly and, in consequence, detection of the shear band propagation throughout the sample is permitted.

- Very low horizontal stresses can be applied. This is an advantage because the shear band propagation velocity is expected to be dependent on the confining stress level.
5.2 Physical Tests

5.2.1 Test Setup

Figure 49. Biaxial Test Setup.
The test setup has been designed for a sample of dimensions 80 × 240 × 86 mm (b × h × t). The biaxial test setup consists of a box with inside dimension 180 × 300 × 80 mm. The frame is made out of 18 mm thick phenolic coated multilayer timber plate. The rear and front walls are made out of 10 mm acrylic glass to provide observation of the shear band throughout the sample. The glass plates are fixed to the wood frame by screws to facilitate the sample preparation (see below). In horizontal direction, the sample is supported by 1.5 mm thick steel plates. Horizontal stress $\sigma_x$ is applied by two pneumatic standard cylinders (DSN-16-50-P, Festo Ltd.) of 16 mm in diameter on each side of the sample. A simple cylindrical roller bearing prevents the steel plate’s friction at the bottom during shearing of the sample. Cylinder pressure is controlled by a pneumatic pressure controller (Proportion-Air Inc.). The maximum horizontal pressure depends on the pneumatic pressure available and the area of the sample. With the chosen setup, a maximum horizontal stress of 4 kPa can be applied. The vertical displacement was applied from the bottom of the test upwards using a hydraulic press (walter+bai Ltd.) with a maximum displacement rate of 10 mm/sec up to 50 mm displacement. To prevent the sample from eccentric application of displacement in vertical direction, the wooden plunger above the sample is lead by two fixation blocks (see Figure 49).

During the test, the sample surface was filmed using a high-speed camera built by AOS industries (X-Motion) (Videal, 2009). The movie was saved in avi (audio video interlace) movie format with a resolution of 800 × 1024 pixels at a frame rate of 250 fps (full frames), and shutter time of 100 $\mu$s, resulting in a maximum recording time of 10 sec, limited by the buffer memory of 2.6 GB.

### 5.2.2 Sample Preparation and Test Procedure

Dry silty-sand was used for the tests. The material was filled and compacted with the box lying on one of the glass side walls. Marker lines out of graphite powder were placed upon the panel fixed to the frame and the box installed upon the hydraulic jack.  

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Tests were performed at horizontal stresses of 1 and 4 kPa and vertical velocities of 5 and 10 mm/s. In total, 9 tests have been performed; an overview is given in Table 5.

Table 5. Biaxial test program.

<table>
<thead>
<tr>
<th>test no.</th>
<th>horizontal stress $\sigma_3$ [kPa]</th>
<th>vertical velocity [mm/sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>13-1 to 13-4</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>14-1 to 14-4</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>15-1 to 15-4</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

5.2.3 Post-processing of Data

For the post-processing, the avi-files have been decompiled to full frames and saved in png (portable network graphics) file format. This format has been chosen because it allows lossless data compression. The frames have been cut to the size according to the area of interest of ~330 x 910 pixels.

In order to be able to analyse the frames with respect to the shear band propagation velocity, the frames have been filtered and modified using MATLAB’s built in image processing toolbox.

In order to see a displacement field of the test, from each frame $f_n$, where $n$ denotes the frame number, the frame $f_{n-10}$ has been subtracted. After this, the resulting image has been filtered using $wiener2$, a 2-D adaptive noise-removal filter. In the next step, edges in the images have been detected using the Sobel method function. This function returns edges at those points where the gradient in the image is at a local maximum. After this, the complementary image has been saved as a new file.
5.3 Results of Physical Tests

As a consequence of the post-processing, pure white; i.e. no visible marker lines denote zero displacement, whereas black lines denote displacement. Since the velocity is applied in fact at the bottom of the sample, pure compression would result in an image with increasing intensity and distance of black lines from the top towards the lower end of the image.

Although the images show qualitatively the propagation of the shear band, quantitative interpretation of results is still a challenging topic. Here, an attempt is made to calculate the catastrophic shear band propagation velocity from two qualitatively good test results (test no 15-1; 13-2). In this context quality is assessed high, when the shear band is clearly detectable along a path and the test is symmetric; i.e. the vertical plates move parallel to each other.

5.3.1 Test 13-2

From Test No. 13-2, which has been performed under a horizontal stress of 1 kPa, characteristic frames are shown in Figure 50.

As can be seen from Figure 50, the shear band propagates between the frame 020 and frame 040 over the length of approximately 8 cm. This results in a shear band propagation velocity of

\[
\dot{i} = \frac{\Delta l}{\Delta t} = \frac{0.08}{20/250} = 1.0 \text{ m/s}
\]  \hspace{1cm} (5.1)

which represents a lower bound.

Comparing the frames in Figure 51, it can be found that the shear band propagates between frame 026 and 034 over the length of about 8 cm, which gives another attempt to obtain a propagation velocity:

\[
\dot{i} = \frac{\Delta l}{\Delta t} = \frac{0.08}{8/250} = 2.5 \text{ m/s}
\]  \hspace{1cm} (5.2)
Figure 50. (a)-(f) frames 000 - 050 of Test No. 13-2.
5.3.2 Test 15-1

Test 15-1 has been performed under a horizontal stress of 4 kPa. In Figure 52, a series of images, during which the propagation of the shear band through the sample takes place is presented. Here, only every twentieth processed image is plotted.

Results show, that the shear band propagation clearly takes place between frame 020, where no shear band can be detected and frame 060, where the shear band has propagated through the sample over a length of about 200 mm. This represents, for these boundary conditions, a first lower bound of the propagation velocity. Taking into account the frame rate of 250 fps leads to the propagation velocity of 1.25 m/s.

Having a closer look at the individual frames between frame 047 and frame 054 (Figure 53), careful attention leads to the thesis that the shear band propagates between frame 049 and frame 052 over the length of ~11 cm, where the wedge in the lower right part brightens up due to the propagation of the shear band. This would result in a shear band propagation velocity of

\[
\dot{i} = \frac{\Delta l}{\Delta t} = \frac{0.11}{3/250} = 9.31 \text{ m/s}
\]  

(5.3)
Figure 52. Processed images of Test 15-1, in (a) – (f) each twentieth frame is plotted.
5.4 An Upper Bound - Shear Wave Velocity

An upper bound of the shear band propagation velocity can be derived from dynamic fracture mechanics concepts. The motion of any crack leads to a sudden unloading in the surface traversed by the crack. This will emit stress waves, which may return upon reflection at the surface to influence the further motion of the crack. But during the dynamic event the total energy balance must be satisfied (e.g., Hellan, 1984).

Therefore, the shear wave velocity represents an upper bound of the shear band propagation velocity; \[ \dot{i} \leq v_s \]. As a first approximation, the shear wave velocity can be calculated from

\[
v_s = \frac{G}{\sqrt{\rho}} \tag{5.4}
\]

where \( G \) is the shear modulus and \( \rho \) is the density of the material, which is about \( \rho = 1700 \text{ kg/m}^3 \). From oedometer test results, the constraint modulus of the material can be calculated as a function of the vertical stress with \( M = 190 \cdot (\sigma_y)^{0.67} \). Assuming the Poisson’s ratio of \( \nu = 0.3 \), the shear modulus can be calculated using

\[
G = \frac{E}{2(1+\nu)} \text{ where } E = M \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \tag{5.5}
\]
resulting in the formulation

\[ G = \frac{M \left(1 - 2\nu\right)}{2 \left(1 - \nu\right)} \]  \hspace{1cm} (5.6)

Here, for consistency, the larger stress is calculated, which is the vertical stress in the used biaxial test to determine the constraint modulus of the sample: Taking into account the residual friction angle of \( \phicv = 38^\circ \) of the material results in a maximum vertical stress of about \( \sigma_v = \sigma_h / \tan(38^\circ) = 1.23 \) kPa or 4.9 kPa for the horizontal stress \( \sigma_h \) of 1 and 4 kPa, respectively.  
This gives, using \( M = 190 \cdot (\sigma_v)^{0.67} \), a constraint modulus of \( M \) of 218 kPa and 554.0 kPa, respectively.

Therefore, an upper bound of the shear wave velocity is obtained by

\[ v_s = \sqrt{\frac{M \left(1 - 2\nu\right)}{2 \rho \left(1 - \nu\right)}} \]  \hspace{1cm} (5.7)

and is about \( v_s = 6.06 \) m/s for the tests with a horizontal stress of 1 kPa and 9.64 m/s for 4 kPa, respectively.

### 5.5 Comparison

For comparison, velocities of the shear band propagation from the test results and upper bound are plotted against the calculated vertical stress at failure (Figure 54). The plot shows that all results obtained lie within a realistic range below the upper bound which is represented by the shear wave velocity represented by the black line.

Note that the upper bound increases very fast at low stresses, reaching velocity of 10 m/s already at a vertical stress of 6 kPa and becomes less steep with increasing stresses.
5.6 Conclusions

Measuring the catastrophic shear band propagation velocity is not easy. A novel biaxial test setup that allows application of very low confining stresses has been developed. One of the main findings of the performed tests is that they show clearly that the shear band does not appear instantaneously but propagates throughout the test specimen.

Quantitative interpretation of results with respect to the velocity of the shear band, however, is more difficult because of the lack of explicitly definable objective indicators. Still results indicate the tendencies that the shear band propagation velocities for the performed tests are in the range of 1.0 - 10 m/sec and that they are smaller than the corresponding shear wave velocities.

These preliminary tests represent a first step towards a more detailed investigation of the propagation velocity. Although results are promising, the dependency from soil parameters cannot be determined from this data set. Therefore as a next step, better quality of the test results could be obtained when the test setup is modified such that the vertical plates are prevented from inclination and by using a larger sample size.
6 A Dynamic Solution of the Shear Band Propagation in Submerged Landslides

6.1 Introduction

Understanding the mechanisms of tsunamis and their sources is a key task for the tsunami hazard assessment and mitigation. Recent devastating tsunami events, such as the 1998 Papua New Guinea tsunami and the event in 2004 in the Indian Ocean have aroused the public and scientific interest on an improved understanding of the triggering mechanisms and tsunami hazard assessment (Tappin et al., 1999; Liam Finn, 2003; Okal & Synolakis, 2003; Ioualalen et al., 2007). Although tsunamis often occur directly due to normal faulting of earth plates (Yamashita & Sato, 1974), it has been shown that submarine landslides, triggered by earthquakes, may also cause tsunami waves of a significant height. Overview on earthquake related triggering mechanisms of submarine and shoreline slope instabilities have been provided by several authors (Bardet et al., 2003; Wright & Rathije, 2003). The authors distinguish between direct, such as acceleration- or liquefaction-induced sliding, and indirect triggering mechanisms, such as a delayed failure mechanism due to excess pore water pressure. The general tendency, however, is to assume that the landslide fails simultaneously along the entire sliding surface, which can be tens and hundreds kilometres long. This assumption is also behind the fact that numerical simulations of landslide induced tsunamis (Harbitz, 1992; Murty, 2003; Bondevik et al., 2005) tend to underestimate the tsunami wave height.

These limitations can be overcome, if the landslide failure is considered as a dynamic process, and not as a static limiting equilibrium event. Such an approach has been proposed by Puzrin & Germanovich (2003; 2005), who suggested that an initial shear band emerges along a certain length of the potential failure surface. Within this shear band the shear strength drops due to the softening behavior of the material. Therefore, the soil above this
weakened zone starts moving downwards, causing the shear band to propagate further along the potential failure surface. This produces an initial landslide velocity already before the slide reached the state of the global limiting equilibrium; i.e. the post failure stage.

Analysis of the mechanism is based on the energy balance approach of Palmer & Rice (1973). For the shear band to propagate, the energy surplus produced in the body by an incremental propagation of the shear band should exceed the energy required for this propagation. The main advantage of this model is that it allows distinguishing between progressive and catastrophic shear band propagation and treats the shear band as a true physical process and not just as a mathematical bifurcation problem (Rudnicky & Rice, 1975). Applications of the energy balance approach to the phenomenon of progressive shear band propagation in trapdoor- and shear blade tests on sand and silt have been investigated experimentally, analytically and numerically (Saurer & Puzrin, 2007; 2008). It seems that the energy balance approach provides a reasonable quantitative description of the shear band propagation phenomena in granular materials.

Analysis of the catastrophic shear band propagation in an infinite submerged slope built of normally consolidated clays has shown that relatively short initial failure zones are sufficient to cause a full-scale landslide (Puzrin et al., 2004). An attempt to assess the initial landslide velocity at failure was also made Puzrin & Germanovich (2003), based on a quasi-static approach, neglecting the fundamental dynamic terms.

As will be shown below, this initial landslide velocity at failure plays an important role for the tsunami height assessment. Therefore, in spite of the complexity of the dynamic problem, it is worth exploring a possibility of producing a better estimate of this velocity. This chapter briefly outlines an attempt to provide an improved approximation of the true dynamic solution. In this simplified approach, the stress distribution in the sliding layer is calculated using inertia terms and the viscous resistance of the water, but excluding propagation and reflection of elastic waves. In spite of this simplification, the energy balance includes the kinetic energy of the moving landslide and leads
to a non-linear differential equation. While this equation can be solved numerically, for large lengths of the shear band the landslide velocity asymptotically approaches a closed form solution. This allows for estimation of the initial landslide velocity at the moment of failure.

6.2 Dynamic Shear Band Propagation in an Infinite Slope

6.2.1 Geometry and Soil Behaviour

Consider an infinite slope inclined by angle $\alpha$ to the horizontal with a discontinuity zone at the depth $h$ parallel to the slope (Figure 55).

![Figure 55. Propagation of the shear band in an infinite slope. Parameters $\tau_g$, $\tau_r$ are the same as defined in Palmer & Rice (1973).](image)

Starting from the initial weak zone of the length larger than critical $l > l_{cr}$ (Puzrin et al., 2004), a shear band propagates down the slope parallel to the surface. At the top of this zone, the soil fails in active failure with the active pressure $p_a$. It is assumed that the length of the discontinuity $l$ is sufficiently larger than its depth and the length of the process zone $\omega$: $l > h >> \omega$. Within this small process zone, the shear resistance $\tau$ gradually drops from the peak $\tau_p$ to the residual value $\tau_r$, as a function of the relative displacement (Figure 56). Within the rest of the shear band, the shear resistance is constant and equal to $\tau_r$. Outside the shear band and at the tip of the process zone, the shear resistance is equal to the peak value $\tau_p$. If the gravitational shear stress $\tau_g$ above the shear band exceeds the residual shear strength $\tau_r$, the soil above
the shear band starts moving downwards, driving the shear band to propagate along the slope, until it comes to the surface and the slope fails (Figure 55). We are interested in the velocity of the shear band propagation and of the landslide at the moment when the slide fails.

\[
\int_0^\delta (\tau - \tau_r) \, d\delta = \frac{1}{2} (\tau_p - \tau_r) \delta_m = I
\]

Figure 56. Strain softening behaviour in the shear band process zone.

Although the normal stress in the x-direction is a function of depth z, in this derivation only the average value of this stress across the sliding layer \( \sigma_s(x) \) is of interest. Before the shear band propagation, the average normal stress in the intact slope is \( \bar{\sigma}_x = p_0 \). As the shear band propagates, it starts growing. Linear elasto-plastic behavior in the sliding layer is assumed \( \bar{\sigma}_x = p_0 + \varepsilon_x / E_{ep} \), where \( E_{ep} \) is the elasto-plastic modulus of soil.

6.2.2 Equation of Motion

The main simplification of the proposed model is in considering the sliding layer in the dynamic case, when all the points above the band are moving with the same velocity \( \dot{v} \) and, therefore, acceleration \( \ddot{v} \). Then, at each moment of time, the layer above the shear band can be considered as rigid body (Figure 57) and from the equation of motion for this body we obtain the average normal stress above the tip of the shear band:

\[
h \bar{\sigma}_x(l) = h \bar{\sigma}_c = \left( \tau_g - \tau_r \right) l + p_u h - \rho h \dot{v} - \rho h \ddot{v} - \mu v l
\]
where $\rho$ is the density of the soil, $\mu$ is the coefficient of proportionality that scales with the viscosity of water, $\mu_w$. The fourth term on the right side reflects the fact that the mass of the moving body is increasing during the shear band propagation (the additional mass accelerates from zero velocity to $v$).

### 6.2.3 Energy Balance Approach

The energy balance criterion for an incremental dynamic propagation of the shear band can be expressed in the following equation:

$$\Delta W_e - \Delta W_i - \Delta D_l - \Delta D_\mu - \Delta K = \Delta D_\omega$$

where $W_e$ is the external work made by gravitational forces on downhill movements of the layer; $W_i$ denotes the internal work of the normal stress acting parallel to the slope surface on the change of strains in the layer; $D_l$ is the dissipated energy due to plastic work along the shear band; $D_\mu$ is the dissipated energy at the soil-water interface; $K$ is the kinetic energy and $D_\omega$ is the plastic work required to overcome the peak shear resistance at the tip of the band, i.e. the softening in the process zone, see Eqs. (6.5) – (6.11). This equation is the same as suggested by Palmer & Rice (1973) for the static case, except it includes the change $\Delta K$ of the kinetic energy and $\Delta D_\mu$ of the dissipated energy at the soil-water interface.

Incremental propagation of the shear band by (Figure 57) over the time increment $\Delta t$ produces displacement of the entire sliding layer, proportional to the strain $\varepsilon_i$ in the portion of the sliding layer above $\Delta l$:

$$\Delta \delta = \varepsilon_i \Delta l$$

(6.3)

Velocity of the sliding layer is then given by

$$v = \frac{\Delta \delta}{\Delta t} = \varepsilon_i \frac{\Delta l}{\Delta t} = \varepsilon_i \dot{l}$$

(6.4)

where $\dot{l} = \Delta l / \Delta t$ is the velocity of the shear band propagation.

The corresponding work increments are then given by the following expressions:
\[ \Delta W_c = \tau_s l \varepsilon_i \Delta l + p_a h \varepsilon_i \Delta l \]  

(6.5)
is the increment of the external work due to the work of gravity on the moving layer and of the active pressure at the failure scar on the displacements induced by the incremental shear band propagation;

\[ \Delta W_i = \Delta lh \int_0^{\varepsilon_i} \sigma_i d\varepsilon_i \]  

(6.6)
is the internal work of the normal stress acting parallel to the slope surface on the change of sub-horizontal strains in the layer above the shear band increment;

\[ \Delta D_i = \tau_s l \varepsilon_i \Delta l \]  

(6.7)
is the dissipated plastic work of the residual shear strength along the failure surface on the displacements induced by the incremental shear band propagation;

\[ \Delta D_{\omega} = \Delta l \int_0^{\delta} (\tau - \tau_r) d\delta = I \Delta l \]  

(6.8)
is the plastic work required to overcome the peak shear resistance at the tip of the band, i.e. the softening in the process zone, where

\[ I = \int_0^{\delta} (\tau - \tau_r) d\delta \]  

(6.9)

Eqs. (6.5) - (6.9) do not differ from those derived for the static analysis of the shear band propagation (Puzrin & Germanovich, 2005). The following two energy terms, however, are specific for the dynamic analysis. The first term

\[ \Delta D_{\mu} = \mu \nu l \varepsilon_i \Delta l \]  

(6.10)
is the dissipated energy due to the viscous resistance of water proportional to the velocity of the slide on the displacements induced by the incremental shear band propagation. Here \( \mu \) is the coefficient of proportionality that scales with the viscosity of water, \( \mu_w \). The second term
\[ \Delta K = v \Delta (mv) = v \Delta (\rho hl v) = \rho hl v \Delta v + \rho h v^2 \Delta l \]  \hspace{1cm} (6.11)

is the change in the kinetic energy of the sliding layer induced by the incremental shear band propagation. The change in kinetic energy is defined as a product of velocity and the increment of momentum, and for the bodies with the changing mass it has two components: due to increasing velocity and mass, respectively. The mass of the sliding layer increases proportionally to the increasing length of the shear band.

Substituting these equations into the energy balance and dividing each term by the time increment \( \Delta t \), after certain manipulations we obtain

\[ \left[ (\tau_g - \tau_r) + p_a h - \mu vl - \rho hl v - \rho hl v \right]_i - h \int_0^\varepsilon \sigma d \varepsilon = I \]  \hspace{1cm} (6.12)

where from Figure 56 (the dashed line corresponding to a linear approximation of the shear strength diagram):

\[ I = \frac{\delta}{2} (\tau - \tau_r) d \delta = \frac{1}{2} (\tau _p - \tau _r) \delta_m \]  \hspace{1cm} (6.13)

The term in the square brackets in equation (6.12) can be recognised from equation (6.1), leading to

\[ \bar{\sigma} \varepsilon_i - \int_0^{\varepsilon_i} \bar{\sigma} d \varepsilon = \frac{I}{h} \]  \hspace{1cm} (6.14)

Figure 57. Motion of the sliding layer.
6.2.4 Differential Equation of the Shear Band Propagation

The left side of the Eq. (6.14) is equal to the complimentary strain energy:

\[ \sigma_i \varepsilon_i - \int_0^{\varepsilon_i} \sigma_x d\varepsilon_x = \int \varepsilon_x d\sigma_x \]  

(6.15)

The average linear strain \( \varepsilon_x \) can be related to the average normal stress \( \sigma_x \) in the layer along the shear band:

\[ \varepsilon_x = \frac{\sigma_x - p_0}{E_{ep}} \]  

(6.16)

Eq. (6.16) can be then substituted into (6.15), integrated, and the result substituted into (6.14):

\[ \sigma_i \varepsilon_i - \int_0^{\varepsilon_i} \sigma_x d\varepsilon_x = \frac{(\sigma_i - p_0)^2}{2E_{ep}} = \frac{I}{h} \]  

(6.17)

This gives

\[ \sigma_i = p_0 + \sqrt{\frac{2E_{ep}}{h}} ; \quad \varepsilon_i = \frac{\sigma_i - p_0}{E_{ep}} = \sqrt{\frac{2I}{hE_{ep}}} \]  

(6.18)

i.e., the shear band propagates at the constant normal lateral stress in the sliding layer above the band tip.

Equation of motion (6.1) can be then rewritten as:

\[ \rho_h \dot{v} + \rho \dot{v} + \mu v - (\tau_g - \tau_r) (l - l_{cr}) = 0 \]  

(6.19)

where

\[ l_{cr} = \frac{h \sigma_i - p_0}{\tau_g - \tau_r} = \frac{\sqrt{2IE_{ep}h - (p_a - p_b)h}}{\tau_g - \tau_r} \]  

(6.20)

is the critical length of the initial shear band beyond which it starts propagating. Substitution of Eq. (6.4) into (6.19) gives the following non-linear second order differential equation:
(6.21)

\[(y + l_{cr})\ddot{y} + (y)^2 + a(y + l_{cr})\dot{y} - by = 0\]

where

\[y = l - l_{cr} \quad (6.22)\]

\[a = \frac{\mu}{\rho h} ; \quad b = \frac{(\tau_g - \tau_r)}{\varepsilon \rho h} = \frac{(\tau_g - \tau_r)}{\rho h} \sqrt{\frac{h\varepsilon_{ep}}{2l}} \quad (6.23)\]

with initial conditions:

\[y(0) = \dot{y}(0) = 0 \quad (6.24)\]

### 6.2.5 Simplified Solution for the Velocity of the Shear Band Propagation

Equation (6.21) can be solved numerically. However, for large lengths of the shear band \(y \gg l_{cr}\) and zero viscosity \((a = 0\) for sub-aerial slides), it can be simplified as

\[y\ddot{y} + (y)^2 - by = 0 \quad (6.25)\]

and solved with initial conditions (6.24) in the closed form:

\[y = \frac{b}{6} t^2 \quad (6.26)\]

Therefore,

\[l = y + l_{cr} = \frac{b}{6} t^2 + l_{cr} ; \quad \dot{l} = \ddot{y} = \frac{b}{3} t \quad (6.27)\]

and dependency of the landslide velocity on the shear band length is given by

\[v = \varepsilon\dot{l} = \varepsilon \frac{\sqrt{2b}}{3} \sqrt{l - l_{cr}} \quad (6.28)\]

Finally, substituting (6.18) and (6.23) into (6.28) results in

\[v = \sqrt{\frac{8l_{cr}}{9h\varepsilon_{ep}} \frac{\tau_g - \tau_r}{\rho h} \sqrt{l - l_{cr}}} \quad (6.29)\]
6.2.6 Limiting Condition for the Shear Band Propagation Velocity

Approximations (6.27) - (6.29) are, strictly speaking, only valid for sub-aerial landslides. For submarine landslides, viscosity cannot be neglected \((a \neq 0)\) and another approximation has to be made. Introducing dimensionless length and time

\[ Y = y/L_{cr} \quad ; \quad T = at \]  \hspace{1cm} (6.30)

into Eq. (6.21) and dividing both parts by \(Y + 1\), we obtain

\[ \ddot{Y} + \frac{(\dot{Y})^2}{Y + 1} + \dot{Y} - c \frac{Y}{Y + 1} = 0 ; \quad \text{where} \quad c = \frac{b}{a^2 L_{cr}} \]  \hspace{1cm} (6.31)

We are looking for a limiting condition for the shear band propagation velocity. If for large \(Y \gg 1\) velocity stabilizes, the second term in the above equation becomes small and the forth term approaches \(c\), leading to

\[ \ddot{Y} + \dot{Y} - c = 0 \]  \hspace{1cm} (6.32)

which can also be solved in closed form with initial conditions (6.24). We obtain \(\dot{Y} = c(1 - e^{-at})\) or

\[ \dot{Y} = \frac{b}{a} \left(1 - e^{-at}\right) \]  \hspace{1cm} (6.33)

which confirms that the shear band velocity cannot grow infinitely and stabilizes at larger \(y\), limiting the initial landslide velocity to the maximum value of

\[ v = \varepsilon_l \dot{y} = \varepsilon_l \frac{b}{a} \frac{\tau_s - \tau_c}{\mu} \]  \hspace{1cm} (6.34)

The above relationship is obtained by substituting \(a, b\) and \(\varepsilon_l\) from equations (6.18) and (6.23).

Note that the shear band propagation velocity is not limited by the elastic wave velocity and, therefore, theoretically, the shear band (or crack) growth can be supersonic (Freund, 1990) in our formulation. This situation arises because elastic wave propagations in the sliding body were ignored and one-
dimensional description of the deformations at the front, \( x = l \), of the thin sliding layer was employed. This is somewhat analogous to cases in fracture mechanics when the loading is applied directly at the crack tip and does not require energy delivered by the means of elastic waves (Winkler et al., 1970; Curran et al., 1970).

### 6.3 Parametric Study

Numerical solution of Eq. (6.21), and in particular its analytical approximation (6.29) provide an opportunity to study effects of the landslide geometry and soil properties on the initial post-failure landslide velocity, i.e., at the moment when the sliding layer separates from the underlying sediment. This is the final velocity with respect to the model developed in our work, but this velocity is considered as an initial velocity with respect to the consecutive processes such as possible generation of a tsunami or a deposit of turbidities following the landslide.

#### 6.3.1 Soil Properties

Coefficients of Eqs. (6.20), (6.21) and (6.29) include terms, which have to be expressed via the slide geometry and soil properties. Assumptions with respect to these relationships are summarized in this section.

The gravitational shear stress is calculated as in the static case (Palmer & Rice, 1973)

\[
\tau_g = \gamma \cdot h \cdot \sin \alpha
\]

(6.35)

where \( \gamma \) is the total unit weight of soil. In undrained saturated conditions, where \( \gamma' = \gamma - \gamma_w \) (\( \gamma_w \) denotes the unit weight of water), the peak undrained shear strength of clay can be estimated using the formulation

\[
\tau_p = s_u = k(\gamma - \gamma_w)h
\]

(6.36)

where \( k \) is a factor relation the undrained shear strength to the effective stress normal to the failure plane.
Along the sliding layer a static liquefaction is assumed, where the development of pore water pressures causes the residual shear strength to vanish:

\[ \tau_r = 0 \]  
\[ (6.37) \]

The tension crack at the top of the sliding layer is assumed to be filled with water, acting as active pressure onto the moving layer

\[ p_a = \gamma_v \frac{h}{2} \]  
\[ (6.38) \]

while the initial total sub-horizontal stress is

\[ p_0 = K_0 (\gamma - \gamma_v) \frac{h}{2} + \gamma_w \frac{h}{2} \]  
\[ (6.39) \]

where \( K_0 \) is the earth pressure coefficient at rest.

In sedimentary soils, the increase of stiffness with depth can be roughly assessed from the following empirical relationship:

\[ E_{\text{eq}}(h) = E_0 \sqrt{ \frac{h}{h_0} } \]  
\[ (6.40) \]

where \( E_0 \) is the deformation modulus of soil at the depth \( h_0 \).

Finally, substitution of Eq. (6.36) and (6.37) into the expression of specific work dissipated within the process zone (6.13) gives

\[ I = \frac{1}{2} k (\gamma - \gamma_v) h \delta_m \]  
\[ (6.41) \]

### 6.3.2 Velocity of the Shear Band Propagation

Now all the coefficients of the differential equation (6.21) and its solution (6.29) can be expressed via the slide geometry and soil parameters. Substitution of Eq. (6.35)-(6.41) into Eq. (6.20) gives a formula for the critical shear band length, such that if the initial weak zone exceeds this length, the propagation of the shear band becomes catastrophic:
which can be reduced to

\[ l_{cr} = \frac{E_0 \delta_m}{k(\gamma - \gamma_w)} \sqrt{\frac{h}{h_0}} + \frac{K_0 h}{2k} \quad (6.43) \]

The parameters in the differential equation (6.21) are then given by

\[ y = l - \frac{E_0 \delta_m}{k(\gamma - \gamma_w)} \sqrt{\frac{h}{h_0}} - \frac{K_0 h}{2k}; \quad a = \frac{\mu}{\rho h}; \quad b = g \sin \alpha \sqrt{\frac{E_0}{k(\gamma - \gamma_w) \delta_m}} \quad (6.44) \]

where \( g = \gamma / \rho \approx 9.81 \text{ m/s}^2 \approx 10 \text{ m/s}^2 \) is the gravity acceleration.

Substituting Eqs. (6.35) - (6.43) into Eq. (6.29), gives the post-failure landslide velocity as a function of true soil parameters and the landslide size:

\[ v = \sqrt{\frac{4k(\gamma - \gamma_w) \delta_m \gamma \sin \alpha}{9E_0 \cdot \sqrt{\frac{h}{h_0}} \rho \left( l - \frac{E_0 \delta_m}{k(\gamma - \gamma_w)} \sqrt{\frac{h}{h_0}} - \frac{K_0 h}{2k} \right)}} \quad (6.45) \]

### 6.3.3 Effects of the Viscosity of Water

In this section, the effect of the viscosity of water on slide velocity is analyzed. Because the analytical approximations (6.29) and (6.45) were achieved by neglecting the coefficient of proportionality \( \mu \) that scales with the viscosity of water, \( \eta \), finite differences numerical integration of equation (6.21) with initial conditions (6.24) and parameters (6.44) have been used and the results were compared with the analytical approximation (6.45). Two different landslide geometries have been considered: a shallow long slope (slope 1) and a deeper rather short slope (slope 2). Geometries are shown in Table 6.
Table 6. Geometric parameters of the two slopes.

<table>
<thead>
<tr>
<th>parameter</th>
<th>slope 1</th>
<th>slope 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope length L [km]</td>
<td>150</td>
<td>4.5</td>
</tr>
<tr>
<td>depth of failure surface h [m]</td>
<td>114</td>
<td>600</td>
</tr>
<tr>
<td>slope inclination α [°]</td>
<td>0.5</td>
<td>12</td>
</tr>
</tbody>
</table>

Soil parameters have been chosen for a normally consolidated clay: \( \gamma = 20kN/m^3; \ K_0 = 0.5; \ k = 0.25; \ \delta_m = 0.1m \). The stiffness curve is calibrated assuming the elasto-plastic stiffness in loading to be \( E_0 = 1000kPa \) at the depth of \( h_0 = 100m \).

The density of water is \( \gamma_w = 10kN/m^3 \) and its dynamic viscosity \( \mu_w = 0.001 Pa \cdot s/m \).

Results of the comparison are presented in Figure 58a,b, for the slopes 1 and 2, respectively. It follows, that for both geometries the difference between the numerical solution and its analytical approximation becomes negligibly small, when the shear band length reaches the failure length \( l = L \) and at this moment the sliding layer separates from the underlying sediment. In the vicinity of the critical length, the approximation limits the velocity from above, mainly due to the non-zero acceleration at \( l = l_{cr} \). Very soon, however, accelerations in both solutions become very close.

Exact determination of the coefficient of proportionality \( \mu \) requires a solution of the corresponding hydrodynamic problem, which is beyond the scope of this thesis. The parametric study of the range of these coefficients, however, indicates that for a rather broad range of coefficients \( \mu \) (namely, \( \mu = (10^0 - 10^2) \times \mu_w \) for slope 1 and \( \mu = (10^0 - 10^4) \times \mu_w \) for slope 2), the analytical formula (6.45) produces a reasonable approximation of the numerical solution (Figure 58).
As is seen, within the adopted assumptions, the water does not have a significant effect on the shear band.

Figure 58. Comparison of numerical and analytical solutions for two geometries: (a) slope 1; (b) slope 2.
Figure 59 shows the solution and its approximation in the vicinity of $l = l_{cr}$. It demonstrates why the approximation limits the velocity from above, namely due to the non-zero acceleration at $l = l_{cr}$. Very soon, however, accelerations in both solutions become very close. All obtained results are shown further below in the summary section.

### 6.3.4 Parametric and Sensitivity Study

Because the comparison between numerical solution and its analytical approximation has shown a good agreement at larger $l/L$ ratios, the sensitivity study on different geometric parameters can be performed using the analytical closed form solution. Substitution of the soil parameter values for normally consolidated clays presented in the previous section into Eqs. (6.44) and (6.45) gives formulas for the critical length

$$l_{cr} = \frac{\sqrt{h} + h/2}{4 \cdot \sin \alpha} \quad (6.46)$$

and the initial landslide velocity

$$v = \sqrt{\frac{\sin \alpha \cdot L - \frac{\sqrt{h^3}}{24} - \frac{1}{12}}{3 \cdot \sqrt{\frac{4}{h}}} \cdot \frac{\sqrt{h} + h/2}{4 \cdot \sin \alpha}} \quad (6.47)$$
which are the functions of the slide geometry only. Using Eq. (6.47), the effect of the landslide inclination $\alpha$ on the initial landslide velocity is investigated in Figure 60a for a constant thickness of the sliding layer $h = 100 m$. The effect of the thickness of the sliding layer on the initial landslide velocity is investigated in Figure 60b for a constant landslide inclination of $\alpha = 2^\circ$. Results are shown in a semi-logarithmic scale.

As is seen in Figure 60a, the increased landslide inclination leads (for the same length at failure) to a drastic increase in the initial velocity, mainly, because the critical length for a steeper slide is shorter and it has more time to accelerate.

From Figure 60b it can be concluded, that the increased landslide depth leads (for the same length at failure) to a decrease in the initial velocity, mainly,
because the critical length for a deeper slide is longer and it has less time to accelerate, even though its initial acceleration is higher.

6.4 Conclusions

A stable numerical solution and its closed form approximation have been obtained for the velocity of the submarine tsunamigenic landslides at failure. The landslide mechanism, based on the phenomenon of the dynamic shear band propagation, is analyzed using the energy balance approach. Inertia effects and viscous water resistance have been included into the analysis, while the propagation and reflection of the P-waves within the sliding layer have been neglected.

In contrast to the biaxial tests, where the shear band propagation velocity cannot exceed the shear wave velocity limited by the boundary value problem, here the shear band propagation velocity is not limited by the shear wave velocity and, for an infinite slope, this velocity can become rather large because the proposed formulation does not require energy delivered to the shear band tip. In the nature, however, this does not represent a problem because no infinite slopes exist and the shear band would sooner or later propagate to the surface, causing the slope failure at a finite velocity.
7 Application to Historic and Recent Tsunamigenic Landslides

The content of this chapter has been submitted to Granular Matter and is currently under review.

In the recent years, a number of attempts have been undertaken in the literature to simulate numerically tsunami events, triggered by submarine landslides (e.g., Murty, 2003; Harbitz, 2006). However, only few of them have been able to deliver an explanation for the trigger source to the full extent. Most of them underestimate the tsunami wave height, indicating a necessity to account for some kind of the initial post-failure landslide velocity. In this section an attempt is made:

- to validate the shear band propagation hypothesis as a possible source of this initial landslide velocity for a number of historic and recent tsunamigenic landslides;

- to assess qualitatively the effect of this initial velocity on the observed tsunami wave heights.

7.1 Storegga Slide

The Storegga slides, situated on the continental slope off the western coast of Norway, are among the largest and best-studied submarine landslides in the world (Figure 61) (Bugge et al., 1988; Haflidason et al., 2004; Kvalstad et al., 2005). It is recognized that three separate landslide events occurred in the area. The first slide occurred about 30’000-50’000 years before present involving a volume of 3880 km³, the average thickness of 114 m and a run-out distance of 350-400 km from the headwall (Bugge et al., 1988), during which the altitude of the centre of gravity decreased by roughly 1500 m. According to the geological data, this landslide generated a tsunami wave with a height of 10-12 m on the coast of Norway; on the Shetland Island it was higher than
20 m (Bondevik et al., 2005). The average water depth of the slide before the failure was about 1500 m.

Figure 61. Location of Storegga Slide (marked area) and the flow direction (arrow) off the coast of Norway (after Bugge et al., 1988).

The second slide occurred about 8'200 years ago. It has been found that the second Storegga slide consisted of one giant slide, with a volume of about 3100 km$^3$ and a runout of about 750 km, followed by a multitude of smaller events. The average thickness was about 144 m, and the height of a tsunami wave generated was of the same order of magnitude as for the first slide. The third event was limited to the upper part of the second slide scar and has not been considered here.

Numerous numerical simulations of the tsunami wave heights for the first and second Storegga slides (Harbitz, 1992; Bondevik et al., 2005; Grilli & Watts; 2005a,b; De Blasio et al., 2005) indicated that rather high maximum velocities
of up to 35-60m/s are required to get a correct correlation to the run-out
distances. Such high velocities are difficult to explain without accounting for
some kind of the initial post-failure landslide velocity.

### 7.1.1 Calculation of Initial Landslide Velocity

Here, calculation procedure is presented for the First Storegga Slide with the
average height of \( h = 114 \, m \), the average inclination \( \alpha = 0.5^\circ \) and the total unit
weight of the soil \( \gamma = 17 \, kN/m^3 \); i.e. \( \rho = 1.7 \, t/m^3 \). Therefore, the gravitational
shear stress is

\[
\tau_g = \gamma \cdot h \cdot \sin \alpha = 17 \cdot 114 \cdot \sin(0.5^\circ) = 16.9 \, kPa
\]  

(7.1)

The undrained shear strength for the calculation of the \( \tau_p \) in normally
consolidated clays may be approximated using the formula \( \tau_p = s_u = 1/4 \, \gamma' \, z \); assuming total static liquefaction: \( \tau_r = 0 \); \( \delta_m = 0.1 \, m \), so that

\[
I = \frac{1}{2} (\tau_p - \tau_r) \delta_m = 9.98 \, kN/m
\]  

(7.2)

The at rest earth pressure coefficient \( K_0 = 0.5 \). For the calculation of the earth
pressure at the top of the sliding layer a gap filled with water is assumed:

\[
p_a = \gamma_a z
\]  

(7.3)

The initial average stress in x-direction

\[
p_0 = K_0 \gamma' z + \gamma_a z
\]  

(7.4)

Therefore at the average height of \( z = h/2 = 57 \, m \):

\[
p_a - p_0 = -K_0 \gamma' z = -199.5 \, kPa
\]  

(7.5)

Including a secant Young’s modulus of \( E_{cp} = 1.0 \, MPa \); viscosity of water
\( \mu = 0.001 \, Pa \cdot s \cdot m^{-1} \) and substitution of these parameters into the Eqs. (6.20)
and (6.23) gives for the critical length

\[
l_{cr} = \frac{\sqrt{2IE_{cp} \gamma} - (p_a - p_0)h}{\tau_g - \tau_c} \approx 1434 \, m
\]  

(7.6)
and parameters

\[
a = \frac{\mu}{\rho h} = 5.16 \cdot 10^{-6} \text{ s}^{-1}
\]  \hspace{1cm} (7.7)

\[
b = \frac{(\tau_g - \tau_r)}{\rho h} \sqrt{\frac{h E_{rr}}{2 I}} = 6.60 \text{ ms}^{-2}
\]  \hspace{1cm} (7.8)

Eq. (6.31) can be now solved together with the boundary conditions (6.24) numerically. However, the dimensionless parameter \( c \) in this case appears to be very large:

\[
c = 1.43 \cdot 10^8
\]  \hspace{1cm} (7.9)

which could lead to numerical instabilities. In order to stabilize the solution, the initial condition has to be applied in the form:

\[
Y(0) = \varepsilon << 1
\]  \hspace{1cm} (7.10)

For the failure length \( l = 150 \) km, which was suggested by Harbitz (1992), numerical solution predicted the landslide velocity of \( v_0 = 10.66 \text{ m/s} \), while the analytical approximation was \( v_0 = 10.69 \text{ m/s} \).

### 7.2 Western Goleta Slide

Goleta Slide is situated near Coal Oil Point off the shelf edge in the Santa Barbara Basin in California (Figure 62). It consists of three major slides (i.e. lobes) situated beneath each other.

Intense detailed measurements using bathymetry, seismic reflection and remote operated vehicles (ROV) (Greene et al., 2006) allowed determination of the geometry of the slide. The compound has a total length of 14.6 km, a width of 10.5 km. Because it is unlikely that the entire slide failed at once, for an assessment of the presented theory only the western part is considered. The western slide had a length at failure of 6.33 km, average thickness of 39 m and is located at the average water depth of 455 m. Average inclination of the slide was 6.6°. From oxygen isotope stratigraphy the age of the slide has
been estimated to be about 5'500 years (Lee et al., 2004). Landslide analysis (Grilli & Watts, 2005; Watts, 2000) produced a maximum landslide velocity of 22.8 m/s and a simulated wave height of maximum 10 m (Greene et al., 2006).

Figure 62. Seafloor image of Goleta Slide in Santa Barbara Channel (Greene et al., 2006). Marked area: Western Goleta Slide.

### 7.3 Slides in Lake Lucerne

The Weggis and Chrütztrichter Slides located in Lake Lucerne, Switzerland (Figure 63 and Figure 64) occurred after an earthquake in the year 1601 and triggered a tsunami wave, with a height up to 4 m (Cysat, 1601). Field investigations (Schnellmann, 2004; Strasser, 2008) using bathymetric field mapping techniques and core drillings in the bed of the lake allowed for estimation of the volume (8.5\(\times\)10^6 m\(^3\) and 0.18\(\times\)10^6 m\(^3\)) and an average thickness (3.75 m and 5.5 m) of the two slides, respectively. The average run-out distance of the Weggis slide was in the range of 1500 m, loosing approximately 50 m in average altitude. Chrütztrichter slide had a run-out
distance of about 600-800 m. The average water depths at Weggis and Chrüztrichter slides are roughly 100 m and 90 m, respectively.

Figure 63. Overview map of Lake Lucerne and its surroundings (from Strasser, 2008).

Figure 64. Bathymetric map of a part of Lake Lucerne with outlines of the Weggis and Chrüztrichter Slides. Dark grey indicates erosional, light grey depositional zones. Contour interval is 10 m (after Schnellmann, 2004; Strasser, 2008).
7.4 Summary of Initial Velocity Calculations

The summary of the landslide parameters and the corresponding calculated initial velocity is shown in Table 7. Parameters for the Storegga slides have been found from Harbitz (1992), Bondevik et al. (2005) and Bugge et al. (1988). Parameters for the Goleta slide have been taken from Grilli & Watts (2005), Greene et al. (2006), Lee et al. (2004) and Watts (2000). For the Weggis and Chrütztrichter slides, the peak strength has been back-calculated from the static stability analysis.

Table 7. Summary of calculated initial slide velocities for investigated landslides.

<table>
<thead>
<tr>
<th>Landslide</th>
<th>$L$ [km]</th>
<th>$h$ [m]</th>
<th>$\alpha$ [°]</th>
<th>$\gamma$ [kN/m$^3$]</th>
<th>$\tau_p$ [kPa]</th>
<th>average water depth $H$ [m]</th>
<th>initial velocity [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Storegga 1</td>
<td>150</td>
<td>114</td>
<td>0.5</td>
<td>17</td>
<td>$0.25\sigma'_v$</td>
<td>1500</td>
<td>10.5</td>
</tr>
<tr>
<td>Storegga 2</td>
<td>150</td>
<td>144</td>
<td>0.5</td>
<td>17</td>
<td>$0.25\sigma'_v$</td>
<td>1500</td>
<td>10.2</td>
</tr>
<tr>
<td>Goleta</td>
<td>6.33</td>
<td>39</td>
<td>6.6</td>
<td>18</td>
<td>$0.25\sigma'_v$</td>
<td>455</td>
<td>9.3</td>
</tr>
<tr>
<td>Weggis</td>
<td>1</td>
<td>3.75</td>
<td>5-15</td>
<td>16</td>
<td>$0.3 - 0.7\sigma'_v$</td>
<td>100</td>
<td>4.2-8.9</td>
</tr>
<tr>
<td>Chrütztrichter</td>
<td>0.2</td>
<td>5.5</td>
<td>~12</td>
<td>16</td>
<td>$0.56\sigma'_v$</td>
<td>90</td>
<td>3.2</td>
</tr>
</tbody>
</table>

As is seen, the initial velocities for these very different landslides lie in a rather narrow range between 3.2 and 10.5 m/sec. These are non-negligible velocities, but are they sufficient to affect the tsunami wave height significantly? The quantitative answer to this question can be only given on the basis of numerical hydrodynamic analysis of individual cases. Some qualitative assessment, however, can be provided based on the consideration presented in the following section.
7.5 Dependency of Tsunami Wave Height on Landslide Velocity

Water wave amplitude above an underwater landslide scale with characteristics of the solid block motion (Watts, 2000). In general, wave height $\eta$ and landslide velocity $v_{\text{max}}$ are correlated via the Froude number (Harbitz et al., 2006). For tsunamigenic landslides this number relates the linear long-wave velocity $c_0$ at a water depth of $H$; $c_0 = \sqrt{gH}$ (where $g$ is the gravity acceleration) to the maximum landslide velocity $v_{\text{max}}$ and is defined as

$$Fr = \frac{v_{\text{max}}}{\sqrt{gH}}$$

(7.11)

Sub-critical, critical and super-critical landslide motions are defined as $Fr < 1$, $Fr = 1$ and $Fr > 1$, respectively. Critical landslide motion produces tsunami waves several times higher than the thickness of the landslide $h$ (Figure 65). For larger water depths, most of the landslides are going to be sub-critical, therefore, a substantial initial velocity of the landslide would result in significantly larger increase in the tsunami wave height $\Delta \eta$.

![Figure 65. Typical dependency of the tsunami height on the landslide velocity (after Ward, 2001).](image-url)
For the Storegga slide a maximum landslide velocity of about 60 m/s can be assumed to be an upper bound (de Blasio et al., 2005). Simulation of Greene et al. (2006) resulted in a maximum landslide velocity for the Goleta slide of 22.8 m/s.

In order to make an assessment of the maximum velocity of the slides in Lake Lucerne, simplified assessment of the maximum landslide velocity can be calculated according to Grilli & Watts (2005).

\[ v_{\text{max}} = \sqrt{gH} \sqrt{\frac{L \sin \alpha \pi (\rho - 1)}{2C_d \tan \alpha - H \tan \psi}} \]  

(7.12)

where \( H \) denotes the average water depth above the sliding body and \( C_d \) is the global hydrodynamic drag coefficient, \( L \) is the length of the slide, \( \rho \) is the material density and \( \psi \) is the friction angle below the slide.

In order to get an upper bound of the maximum landslide velocity Eq. (7.12) can be rewritten, assuming zero friction below the slide \( \psi = 0 \) and \( C_d = 1.0 \) to

\[ v_{\text{max}} = \sqrt{\frac{L \sin \alpha}{2} \pi (\gamma - \gamma_w)} \]  

(7.13)

From Eq. (7.13) we obtain for the Weggis slide a maximum velocity of 28.5 m/s and for Chrütztrichter 19.7 m/s.

Assuming an average water depth of the Storegga landslide of 1500 m results in a tsunami wave velocity \( c_0 = \sqrt{gH} \) of about 120 m/s; Goleta slide \( c_0 = 67 m/s \) for Weggis and Chrütztrichter \( c_0 = 32 m/s \) and \( c_0 = 30 m/s \), respectively.

The calculated Froude numbers for all the landslides are summarized in Table 8.

As is seen from Table 8, the Froude numbers for all the landslides are smaller than unity; that is their motion appears to be subcritical. The initial velocities (Table 7) represent 10-30% of the estimated maximum velocities. Accounting for these initial velocities will allow for increase in the Froude number, leading to increase in predicted tsunami wave height (Figure 65), possibly providing
explanation for the estimated wave heights in Table 8. This, however, will require a proper hydrodynamic analysis, in which the initial velocity from Table 7 can be used as an input parameter.

Table 8. Summary of calculated Froude numbers for investigated landslides.

<table>
<thead>
<tr>
<th>Landslide</th>
<th>Average water depth $H$ [m]</th>
<th>Wave velocity $u_f$ [m/s]</th>
<th>Max landslide velocity $u$ [m/s]</th>
<th>Froude number</th>
<th>Tsunami wave height $H_t$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Storegga 1</td>
<td>1500</td>
<td>122</td>
<td>60</td>
<td>0.5</td>
<td>10-12</td>
</tr>
<tr>
<td>Storegga 2</td>
<td>1500</td>
<td>122</td>
<td>60</td>
<td>0.5</td>
<td>10-12</td>
</tr>
<tr>
<td>Goleta</td>
<td>455</td>
<td>67</td>
<td>22.8</td>
<td>0.34</td>
<td>10</td>
</tr>
<tr>
<td>Weggis</td>
<td>100</td>
<td>32</td>
<td>28.5</td>
<td>0.89</td>
<td>4</td>
</tr>
<tr>
<td>Chrütztrichter</td>
<td>90</td>
<td>30</td>
<td>19.7</td>
<td>0.65</td>
<td>4</td>
</tr>
</tbody>
</table>

7.6 Conclusions

The analytical closed form approximation, derived in chapter 6, has been applied to a number of historic and recent landslides to calculate the initial landslide velocity.

For a number of different landslides these velocities appeared to be of the order of magnitude of 1-10 m/s. For subcritical slides this may affect significantly the tsunami wave heights and run-out distances.

The Storegga landslides are the longest landslides discovered so far, and their initial velocities in the order of 10 m/s are, probably, the largest initial velocities that could develop in realistic environment.
8 Summary and Outlook

8.1 Summary of the Main Findings

The first part of this thesis focuses on the validation of the energy balance approach to the shear band propagation in granular soils. The rate of progressive (i.e. stable) shear band propagation has been studied in a trapdoor test and in a novel shear blade test. In the trapdoor test, the shear band propagates along a planar surface, whereas the shear blade test allows curve shaped shear bands to be studied.

The analytical solution for the rate of progressive shear band propagation involves parameters which are difficult to determine, such as $K_b$ (a factor reflecting effect of dilation; of development of normal strains inside the shear band), and $K_d$ (a factor reflecting effect of dilation of the peak strength). Despite this, and the fact that the analytical solution includes several rather restrictive assumptions, comparison of the analytical solution with data from experiments and with numerical simulations, revealed that the energy balance approach is a useful tool for the simulation of the rate of progressive shear band propagation in soils. In fact for both the trapdoor test and the shear blade test, comparable rates of shear band propagation have been obtained when using materials of comparable stiffness.

Measurements of the catastrophic (i.e. unstable) shear band velocity in soils have been made using a biaxial test setup. Tests have been performed under very low confining stresses and videotaped using a high-speed camera. Preliminary measurements of the catastrophic shear band propagation velocity have been obtained. All measurements lie below the corresponding shear wave velocities, which represent an upper bound for the shear band propagation velocities. So far the obtained results represent a first step towards a more detailed investigation of the catastrophic shear band propagation velocity.

A simplified solution of the shear band propagation in submerged landslides has also been derived. In obtaining this simplified solution, the stress distribution in the sliding layer is calculated using inertia terms and the viscous
resistance of the water, but neglects propagation and reflection of elastic waves. Despite this simplification, the energy balance includes the kinetic energy of the moving landslide and leads to a non-linear differential equation, which can be solved numerically. For large shear band lengths the numerical solution asymptotically approaches a closed form solution, which allows for estimation of the initial landslide velocity at the moment of failure.

The analytical approximation has been applied to a number of historic and recent landslides. Initial landslide velocities appeared to be of a 1-10 m/s magnitude. For subcritical slides this may significantly affect the tsunami wave heights and run-out distances.

8.2 Outlook

Analytical solutions of the progressive shear band propagation have been derived for cohesive, frictional (Saurer & Puzrin, 2007; 2008) and frictional-dilatant materials. However, such solutions have been validated experimentally only for the frictional and frictional-dilatant materials. Validation of the analytical solution for cohesive materials would additionally support the applicability of the shear band propagation mechanism in cohesive soils.

Numerical simulation of the shear band propagation have been obtained, but, as expected, at much lower stiffness (in the trapdoor test) or at much higher stress levels (in the shear blade test) than in the experiment, respectively. These results by no means demonstrate that numerical modelling of the shear blade tests is impossible – it just shows that it is not trivial within commercially available software, which, in fact, is also true for many other cases of the shear band propagation. The author would like to hope that this thesis will encourage other and more sophisticated and successful attempts to model these tests numerically. Meanwhile, the analytical solutions proposed in this thesis could provide a reasonably good reference.

The measurement of the catastrophic shear band propagation in soils is a challenging topic and therefore provides an impetus for further research. Propagation of the shear band has been observed and measurements of the
first order of magnitude have been obtained. Further improvement of the biaxial test apparatus for tests under low confining stresses and the use of more elaborate image analysis tools, such as PIV, would allow for a more precise understanding and measurement of this process.

Both a numerical solution and an analytical approximation for initial landslide velocity and its influence on the tsunami wave height have been proposed and applied to numerous historic and recent landslides. For a proper validation of the proposed solution, application to further case histories is required. A database of tsunamigenic landslides is currently in preparation at Georgia Institute of Technology. This will offer a great opportunity to more accurately verify the presented solution.

Implementation of the solution into calculation models for the prediction of potential submarine landslides would allow for more accurate calculations of water wave amplitudes. Therefore, this could be a useful tool for the risk assessment of coastal and offshore-constructions. Furthermore, the presented solution could be applied to other research fields, such as snow avalanche hazard assessment and risk assessment of potential subaerial landslides and debris flows.

Progressive failure of soil has quite often been found as the reason for damages in geotechnical engineering. The analytical solution of the progressive shear band propagation and its application to geotechnical problems opens new avenues for both the risk assessment and the design of geotechnical structures. For instance, in analogy to the trapdoor test, the presented concept can be applied to safety calculations of potential sinkholes during the construction of shallow tunnels in both granular and cohesive soils. Likewise, in analogy to the shear blade test, it may also be applied to the bearing capacity calculations for progressive failure of shallow foundations.
References


References


Videal AG, X-Motion: Mobile high resolution camera system. Technical Specifications.


Appendices

Appendix A: Analytical Solutions of the Trapdoor Test

Upper signs correspond to active; lower signs to passive mode tests

Cohesive Material – FMEB-Approach

Stress controlled test

\[
p_0 \leq p_\sigma = p_L \left( 1 \mp \bar{K} \frac{\bar{I}}{p_c} \right)
\]

Displacement controlled test

\[
\delta_0 \bar{I} = \frac{\bar{I}}{M} \left( \sqrt{\frac{p_c}{2}} + \frac{\bar{K}}{p_c} \right)
\]

Cohesive Material – LE-Approach

Stress controlled test

\[
p_0 \geq p_\sigma = p_L \left( 1 \pm \bar{K} \bar{I} - n_\tau + n_c \right)
\]

Displacement controlled test

\[
\delta_0 \bar{I} = \frac{\bar{I}}{M} \bar{K} \left( n_c - n_\tau \bar{I} + \frac{i}{2} \right)
\]

where

\[
\begin{align*}
\bar{p}_c &= \frac{\tau_s}{b} \frac{L}{p_L} (r-1) \bar{M} \delta_m; \quad \bar{M} = \frac{M}{p_L}; \quad r = \frac{\tau_c}{\tau_s}; \quad \delta_m = \delta_m / L; \quad M = \frac{(1-v)}{(1+v)(1-2v)} E; \\
\bar{K}_s &= 2 \left( \frac{\tau_s}{b} + \frac{\tau_c}{t} \right) \frac{L}{p_L}; \quad \bar{K}_{s,LE} = 2 \left( \frac{\tau_s}{b} + \frac{\tau_c}{t} \right) \frac{L}{p_L}; \quad n_\tau = \frac{\bar{K}_{s,LE}}{\bar{K}_s}; \quad p_L = \gamma L.
\end{align*}
\]
Frictional-Dilatant Material – FMEB-Approach

Stress controlled test

\[
p_b \leq \left[ (1-\bar{I}) \mp \sqrt{2(1-\bar{I})K_0\bar{M}\bar{\epsilon}_y \left( 1 + \frac{K_d}{K_b} \right) + \bar{M}^2\bar{\epsilon}_y^2 \left( 1 + \frac{K_d}{K_b} \right) \left( 1 + K_0^2 \left( 1 + \frac{K_d}{K_b} \right) \right) } \right]
\]

\[+ \bar{\epsilon}_y K_0 \bar{M} \left[ 1 + \frac{K_d}{K_b} \mp \frac{1}{K_f} + \frac{\bar{p}_f}{K_f} \right] e^{i\pi \bar{y}_f} - \bar{\rho}_f + \frac{1}{K_f} \]

Displacement controlled test

\[
\bar{\delta}_0(\bar{I}) = \frac{1}{MK_f} \left[ (1-\bar{I}) + \bar{\epsilon}_y K_0 \bar{M} \left( 1 + \frac{K_d}{K_b} \right) \mp \frac{1}{K_f} + \frac{\bar{p}_f}{K_f} \right]
\]

\[+ \sqrt{2(1-\bar{I})K_0\bar{M}\bar{\epsilon}_y \left( 1 + \frac{K_d}{K_b} \right) + \bar{M}^2\bar{\epsilon}_y^2 \left( 1 + \frac{K_d}{K_b} \right) \left( 1 + K_0^2 \left( 1 + \frac{K_d}{K_b} \right) \right) } \times \]

\[\times (\pm 1 \mp e^{i\pi \bar{y}_f}) \bar{I} - \bar{I}\bar{p}_f - \frac{\bar{I}}{M} (1-\bar{I}/2) - K_0\bar{\epsilon}_y \bar{I} \]

Frictional-Dilatant Material – LE-Approach

Stress controlled test

\[
p_b \leq \left( (1-\bar{I}) \mp K_0 \left( \frac{L}{b} (\tan \phi_{cv} + K_d \tan \psi_{max}) + \frac{L}{t} \tan \phi_s \right) \left( 1-\bar{I} \right)^2 \mp \frac{1}{K_f} + \frac{\bar{p}_f}{K_f} \right) e^{i\pi \bar{y}_f} \]

\[\pm \frac{1}{K_f} - \frac{\bar{p}_f}{K_f} \]

Displacement controlled test

\[
\bar{\delta}_0(\bar{I}) = \frac{1}{MK_f} \left[ (1-\bar{I}) \mp K_0 \left( \frac{L}{b} (\tan \phi_{cv} + K_d \tan \psi_{max}) + \frac{L}{t} \tan \phi_s \right) \left( 1-\bar{I} \right)^2 \mp \frac{1}{K_f} + \frac{\bar{p}_f}{K_f} \right] \times \]

\[\times (\pm 1 \mp e^{i\pi \bar{y}_f}) \bar{I} - \bar{I}\bar{p}_f - \frac{\bar{I}}{M} (1-\bar{I}/2) - K_0\bar{\epsilon}_y \bar{I} \]

where

\[\bar{M} = \frac{M}{p_c}; \quad \bar{p}_f = 2L\bar{M}\bar{\epsilon}_y \left( 1 - K_0 \right) \left( 1 + K_0 \right) \frac{\tan \phi_{cv}}{b} + K_0 \frac{\tan \phi_s}{t}; \]

\[\bar{K}_f = 2LK_0 \left( \frac{\tan \phi_{cv}}{b} + \frac{\tan \phi_s}{t} \right); \quad \bar{\epsilon}_y = \frac{\delta}{b} K_b \tan \psi_{max} \]
Appendix B: Analytical Solutions of the Shear Blade Test

Cohesive Material – FMEB-Approach

Stress controlled test

\[
\frac{p_0}{\sigma_{z,0}} \geq K_0 + \sqrt{\bar{p}_c + K_c \cdot \eta}
\]

Displacement controlled test

\[
\alpha_0 = \frac{\eta}{D} \left( \sqrt{\bar{p}_c + \frac{\eta K_c}{2}} \right)
\]

Cohesive Material – LE-Approach

Stress controlled test

\[
\frac{p_0}{\sigma_{z,0}} \geq (\eta_{LE} - \eta)m_c \bar{K_c} + K_a + \bar{K_c} \cdot \eta
\]

Displacement controlled test

\[
\alpha_0 = \frac{\eta}{D} \left( m_c (\eta_{LE} - \eta) + \frac{\eta}{2} \bar{K_c} + \frac{K_a - K_0}{K_c} \right)
\]

where

\[
D = \frac{D}{\sigma_{z,0}}; \quad \bar{p}_c = 2(r_s - 1)\bar{\tau} D \alpha_r; \quad m_c = \frac{K_{c,LE}}{K_c}; \quad \bar{K}_c = 2\left( \frac{\tau_s}{\sigma_{z,0}} \right); \quad \bar{\tau} = \frac{\tau_s}{\sigma_{z,0}}; \quad \text{and} \quad \bar{\tau}_s = \frac{\tau_s}{\sigma_{z,0}}
\]

in plane stress

\[
D = \frac{E}{1 - \nu^2}
\]

in plane strain

\[
D = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} E = M
\]
Frictional-Dilatant Material – FMEB-Approach – Plane Stress

Stress controlled test

\[
\frac{p_0}{\sigma_{z,0}} \geq \left\{ \bar{K}_f + \overline{D} \bar{\varepsilon}_f \left[ 1 + \frac{2K_d}{K_b} \right] + \left[ K_0^2 + \frac{4K_d}{K_b} \overline{D} \bar{\varepsilon}_r + \frac{\bar{\varepsilon}_r^2 \overline{D}^2 \left[ 1 + \frac{2K_d}{K_b} \right]}{\left( 1 + \frac{2K_d}{K_b} \right)^2 - 1 \left( 1 + \frac{2K_d}{K_b} \right) \right]^{\frac{1}{2}} \right\} e^{\eta} - \bar{K}_f
\]

Displacement controlled test

\[
\alpha_0 = \frac{1}{a \overline{D}} \left\{ \bar{K}_f + \overline{D} \bar{\varepsilon}_f \left[ 1 + \frac{2K_d}{K_b} \right] + \left[ K_0^2 + \frac{4K_d}{K_b} \overline{D} \bar{\varepsilon}_r + \frac{\bar{\varepsilon}_r^2 \overline{D}^2 \left( 1 + \frac{2K_d}{K_b} \right) + \nu^2 \left( \frac{2K_d}{K_b} \left[ 1 + \frac{2K_d}{K_b} \right] \right) \right]^{\frac{1}{2}} \right\} e^{\eta} - 1
\]

\[
- \frac{\eta}{\overline{D}} (\bar{K}_f + K_0 + \overline{D} \bar{\varepsilon}_r)
\]

Frictional-Dilatant Material – LE-Approach – Plane Stress

Stress controlled test

\[
\frac{p_0}{\sigma_{z,0}} \geq \left\{ \frac{2(\eta_{LE} - \eta) \left( \nu \tan \varphi'_{cv} + \nu \tan \psi_{max} + \frac{2R}{3} \tan \varphi' s \right) + K_a}{(1 - 2\nu (\tan \varphi'_{cv} + \tan \psi_{max} (\eta_{LE} - \eta))) + \bar{K}_f} \right\} e^{\eta} - \bar{K}_f
\]

Displacement controlled test

\[
\alpha_0 = \frac{1}{a \overline{D}} \left\{ \frac{2(\eta_{LE} - \eta) \left( \nu \tan \varphi'_{cv} + \nu \tan \psi_{max} + \frac{2R}{3} \tan \varphi' s \right) + K_a}{(1 - 2\nu (\tan \varphi'_{cv} + \tan \psi_{max} (\eta_{LE} - \eta))) + \bar{K}_f} \right\} e^{\eta} - 1
\]

\[
- \frac{\eta}{\overline{D}} (\bar{K}_f + K_0 + \overline{D} \bar{\varepsilon}_r)
\]

where \( \bar{K}_f = \frac{b}{a \sigma_{z,0}} \); \( a = 2\nu \tan \varphi'_{cv} \); \( b = 2 \left( E \bar{\varepsilon}_r \tan \varphi'_{cv} + \sigma_{z,0} \nu \tan \varphi'_{cv} + \frac{2R}{3} \tan \varphi' s \right) \)

\[
K_a = (1 - \sin \varphi'_{cv})/(1 + \sin \varphi'_{cv})
\]
Frictional-Dilatant Material – FMEB-Approach – Plane Strain

Stress controlled test

\[
p_0 \geq \frac{p_0}{\sigma_{z,0}} \geq \left[ \varepsilon, D K_0 \left( 1 + 2 \frac{K_d}{K_b} \right) \right] \\
+ \left[ \varepsilon, D \left( 1 + 2 \frac{K_d}{K_b} \right) \left( \varepsilon, D \left( 1 + 2 \frac{K_d}{K_b} K_0^2 \right) + 2 K_f \right) + K_f + K_0 \right] e^{a \eta} - K_f
\]

Displacement controlled test

\[
\alpha = \frac{1}{aD} \left[ K_f + K_0 \left( 1 + \bar{D} \varepsilon \right) \left( 1 + 2 \frac{K_d}{K_b} \right) \right] \\
+ \left[ \varepsilon, D \left( 1 + 2 \frac{K_d}{K_b} \right) \left( \varepsilon, D \left( 1 + 2 \frac{K_d}{K_b} K_0^2 \right) + 2 K_f \right) \right] \left( e^{a \eta} - 1 \right)
\]

\[- \frac{\eta}{D} (K_f + K_0 (1 + \bar{D} \varepsilon))
\]

Frictional-Dilatant Material – LE-Approach – Plane Strain

Stress controlled test

\[
p_0 \geq \frac{p_0}{\sigma_{z,0}} \geq \left( 2(\eta_{LE} - \eta)(1 - K_0) \left( K_0 (\tan \phi' + K_d \tan \psi_{max}) + \frac{2 R}{3 t} \tan \phi' \left( 1 + K_0 \right) \right) + K_a \right) \\
\left( 1 - 2 K_0 (\eta_{LE} - \eta) \left( \tan \phi' + K_d \tan \psi_{max} + \frac{2 R}{3 t} \tan \phi' \right) \right) + \bar{K}_f + K_f \] \\
\left( e^{a \eta} - K_f \right)
\]

Displacement controlled test

\[
\alpha = \frac{1}{aD} \left( 1 - e^{-a \eta} \right) \times \\
\left( 2(\eta_{LE} - \eta)(1 - K_0) \left( K_0 (\tan \phi' + K_d \tan \psi_{max}) + \frac{2 R}{3 t} \tan \phi' \left( 1 + K_0 \right) \right) + K_a \right) \left( 1 - 2 K_0 (\eta_{LE} - \eta) \left( \tan \phi' + K_d \tan \psi_{max} + \frac{2 R}{3 t} \tan \phi' \right) \right) + \bar{K}_f + K_f \] \\
\left( e^{a \eta} - K_f \right)
\]

\[- \frac{\eta}{D} (K_f + K_0 + \bar{D} K_0 \varepsilon)
\]

\[
\bar{D} = \frac{D}{\sigma_{z,0}} ; \ K_f = \frac{b}{a \sigma_{z,0}} ; \ \varepsilon = \frac{\delta K_b}{2R} \tan \psi_{max} ; \ a = 2 K_0 \left( \tan \phi' + \frac{2 R}{3 t} \tan \phi' \right) ;
\]
\[ b = 2(1 - K_0) \left[ \sigma_{z,0} \left( K_0 \tan \varphi'_{cv} + \frac{2}{3} \frac{R}{l} (1 + K_0) \tan \varphi'_s \right) 
+ D E_r \left( (1 + K_0) \tan \varphi'_{cv} + \frac{2}{3} \frac{R}{l} K_0 \tan \varphi'_s \right) \right] \]