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Multi-Terminal Source Coding: Can Zero-rate Encoders Enlarge the Rate Region?

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Abstract—Consider the multi-terminal source coding (MSC) problem wherein l discrete memoryless sources are compressed by l physically separate encoders, and a decoder is required to reconstruct the sources within certain distortion constraints. This paper focuses on the following question: Does the removal of a zero-rate link change the rate region? Though the question seems simple, its complication lies in the limiting nature of the rate region definition. Although intuition suggests that the answer should be no, resolving this question appears to be difficult for at least three reasons: (1) there is no known single-letter characterization of the MSC rate region; (2) there is no known elementary argument for rate-transfer from a zero-rate encoder to others; and (3) there is no known exponentially strong converse, whose existence would otherwise answer the question. In this paper, we answer the question for a number of special cases of the general MSC problem. Our proof techniques use a “super-code” style analysis along with new results from the helper problem. We note, however, that these techniques appear to fall short of answering the question in general.

I. INTRODUCTION

One of the primary goals of information theory is the explicit characterization of rate regions for transmitting data over a network meeting certain requirements. The requirements are either lossless reconstructions of sources or lossy reconstructions certified by a prescribed distortion measure [1]. Such network rate regions are usually defined using sequences of block codes [2] and have a form as follows.

$$\mathcal{R}(\mathcal{D}) = \bigcap_{\varepsilon > 0} \bigcup_{n \in \mathbb{N}} \mathcal{R}(\mathcal{D}, n, \varepsilon) = \lim_{\varepsilon \downarrow 0} \left(\bigcup_{n \in \mathbb{N}} \mathcal{R}(\mathcal{D}, n, \varepsilon) \right). \quad (1)$$

Here, $\mathcal{R}(\mathcal{D})$ represents the rate region for demands \mathcal{D} , and $\mathcal{R}(\mathcal{D}, n, \varepsilon)$ represents the set of rates at which there exists a block code of length n meeting the demands within a failure probability of ε . While properties such as convexity and closedness of the rate regions are straightforward to verify [1], continuity of rate regions w.r.t. the source statistics and the demands is harder to establish. Gu *et al.* have established the continuity of rate regions w.r.t. demands and source distribution for general classes of network problems [3]–[5]. Note that when a single-letter characterization of the rate region of a problem is known, it is almost trivial to ascertain the verity of such properties. However, multi-terminal information theory is fraught with simple problems such as the partial side-information (PSI) problem [6], multiple descriptions (MD) problem [7], and the multi-terminal source coding (MSC) problem [8] that remain unsolved.

In this work, we focus on one question: *Is the rate region of a network with zero rate on a link, the same as that of the network with that link deleted?* Though the question

seems simple, its complication lies in the limiting nature of the definition of rate regions. When the sources in the network emit non-i.i.d. symbols, several examples can be designed to show that asymptotically zero-rate links can alter the region (see Example 1 of Sec. IV-B). However, when the sources emit i.i.d. symbols, and when the demands are lossy (within a required distortion) and/or lossless reconstructions, the answer to this question (in cases where it is known) has always affirmed that zero-rate links do not alter the region. In a majority of network cases where the answer is known, an explicit description of the rate region is also known. In some cases, even if the rate region is unknown, the existence of an exponentially strong converse suffices to answer this question [5], [9]. However, the existence of such suitably strong converses is a hard information-theoretic problem in itself. Here, we attempt to answer this question for the multi-terminal source coding problem that is formalized in Sec. III. Note that for this problem in its generality, neither is the rate region, nor is the existence of a strong converse known.

We have been able to use standard information-theoretic tools in a constructive fashion to show that under many settings of the MSC problem, the rate region with zero rate on a link is the same that when the link is absent. In specific, we establish that in both PSI and MSC problems with two discrete memoryless sources (DMSs), zero-rate links can be deleted without altering the rate region. However, for more than two correlated sources, this result is established only when a certain Markov property holds for the source joint distribution and for specific distortion requirements.

The remainder of the paper is organized as follows. Section II summarizes the notations employed throughout this paper. Section III presents the formulation of the PSI and MSC problems and various terminologies associated with the definition of the rate region. Section IV presents the results and proofs and Section VI concludes the paper.

II. NOTATIONS

Throughout the paper, the following notations are employed. For $n_1, n_2 \in \mathbb{N}$, $n_1 \leq n_2$, $[n_1] \triangleq \{1, \dots, n_1\}$ and $[n_1 \sim n_2] \triangleq \{n_1, \dots, n_2\}$. $\mathbf{0}_k$ represents the $1 \times k$ all-zero vector. Uppercase letters (e.g., X, Y) are reserved for random variables (RVs) and the respective script versions (e.g., \mathcal{X}, \mathcal{Y}) correspond to their alphabets. The realizations of RVs are usually denoted by lowercase letter (e.g., x, y). Subscripts are used for components of vectors, i.e., $x_{[n]}$ denotes a vector of length n and x_i represents the i^{th} component of $x_{[n]}$. We let $\mathcal{S}_n^\varepsilon(P)$ to denote the set of all ε -strongly P -typical sequences

of length n [1]. When the underlying probability distribution is clear, H and I refers to the entropy and mutual information functionals. The Hamming distortion measure on a set \mathcal{X} is denoted by $\partial_H^{\mathcal{X}}$, and lastly, \mathbb{E} denotes the expectation operator.

III. PROBLEM DEFINITION

Given a DMS emitting $(X_i^{(1)}, \dots, X_i^{(l)})_{i \in \mathbb{N}}$ in an i.i.d. fashion with each symbol l -tuple having a joint distribution $p_{X^{(1)} \dots X^{(l)}}$, the *multi-terminal source coding (MSC) problem* aims to identify rates at which encoders have to separately encode sequences $\{x_i^{(k)}\}_{i \in \mathbb{N}}$, $k \in [l]$, using l encoders so that l suitably distorted reconstructions can be constructed at the joint decoder (see Fig. 1).

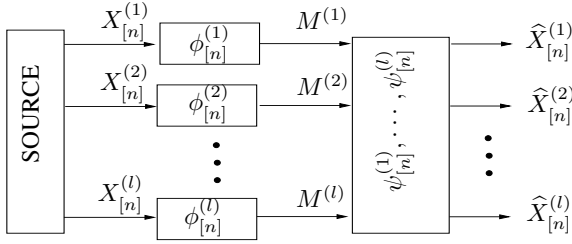


Fig. 1. The multi-terminal source coding problem

For $k \in [l]$, the reconstruction $(\hat{X}_i^{(k)})_{i \in \mathbb{N}}$ is a sequence of elements from the reconstruction alphabet $\mathcal{X}^{(k)}$ and the acceptability of the reconstruction is evaluated by a distortion criterion using the distortion measure $\partial^{(k)} : \mathcal{X}^{(k)} \times \mathcal{X}^{(k)} \rightarrow \mathbb{R}^+$. A rate-distortion pair $(\mathbf{R}, \mathbf{\Delta}) \triangleq (R_1, \dots, R_l, \Delta_1, \dots, \Delta_l)$ is said to be achievable if for each $\varepsilon > 0$, there exists an ε -achievable block code $(\phi_{[n]}^{(1)}, \dots, \phi_{[n]}^{(l)}, \psi_{[n]}^{(1)}, \dots, \psi_{[n]}^{(l)})$. That is, $\forall \varepsilon > 0, \exists n \in \mathbb{N}$, s. t. $\forall k \in [l]$, there exist encoders $\phi_{[n]}^{(k)} : \mathcal{X}_{[n]}^{(k)} \rightarrow \mathcal{M}^{(k)}$ and decoders $\psi_{[n]}^{(k)} : \mathcal{M}^{(1)} \times \dots \times \mathcal{M}^{(l)} \rightarrow \mathcal{X}_{[n]}^{(k)}$ satisfying:

- A1. $\frac{1}{n} \sum_{i=1}^n \mathbb{E} \partial^{(k)}(X_i^{(k)}, \hat{X}_i^{(k)}) \leq \Delta_k + \varepsilon$, where $\hat{X}_{[n]}^{(k)} \triangleq \psi_{[n]}^{(k)}(\phi_{[n]}^{(1)}(X_{[n]}^{(1)}), \dots, \phi_{[n]}^{(l)}(X_{[n]}^{(l)}))$, and
- A2. $|\mathcal{M}^{(k)}| \leq 2^{n(R_k + \varepsilon)}$.

Given $\mathbf{\Delta} \geq \mathbf{0}$, we say a rate vector \mathbf{R} is achievable if $(\mathbf{R}, \mathbf{\Delta})$ is achievable in the aforementioned sense, and denote $\mathcal{R}^{\text{MSC}}(\mathbf{\Delta})[p_{X^{(1)} \dots X^{(l)}}]$ to be the set of achievable rate vectors. This set, known as the rate region, is convex and closed [1]. For each distortion measure, we let $\partial_{\max}^{(k)} \triangleq \min_{\hat{x} \in \mathcal{X}^{(k)}} \sum_{x \in \mathcal{X}^{(k)}} p_{X^{(k)}}(x) \partial^{(k)}(x, \hat{x})$. Note that when $\Delta_k \geq \partial_{\max}^{(k)}$, the k^{th} encoder can even operate at zero rate. However, any message from this encoder can help decoders to obtain less-distorted reconstructions of other sources. Given distortions $\mathbf{\Delta}$, we set $\mathcal{H}(\mathbf{\Delta}) \triangleq \{k \in [l] : \Delta_k \geq \partial_{\max}^{(k)}\}$ to be the set of *helper* sources.

As a special case, the MSC problem with $l = 2$ and $\Delta_2 \geq \partial_{\max}^{(2)}$ is called as the *partial side-information (PSI) problem*. In this case, the rate region is independent of the actual value of Δ_2 and is denoted by $\mathcal{R}^{\text{PSI}}(\Delta_1)[p_{X^{(1)} X^{(2)}}]$. Lastly, when clear, we drop the reference to the underlying source distribution in rate region notations.

IV. THE RESULTS

In this section, we present the results and proofs. First, the invariance of the rate region under the deletion of zero-rate links is established for the PSI problem. The invariance is then proved for the MSC problem with two sources followed by a direct extension to multiple sources. Although the invariance result for the MSC problem subsumes that of the PSI problem, the proof techniques for the two cases are very different. While the proof for the MSC problem exploits the knowledge of the rate region for the common helper problem (See Appendix A), that of the PSI problem is self-contained and constructive in nature. Finally, the invariance for the MSC problem when $l > 2$ is established for a class of sources that have certain Markovian property.

A. The Partial Side-information Problem

Theorem 1: Let $R_{X^{(1)}}$ be the rate-distortion function for a DMS with distribution $p_{X^{(1)}}$ under the distortion measure $\partial^{(1)}$. Then,

$$\inf \{R : (R, 0) \in \mathcal{R}^{\text{PSI}}(\Delta_1)[p_{X^{(1)} X^{(2)}}]\} = R_{X^{(1)}}(\Delta_1) \quad (2)$$

Proof: Since $R \geq R_{X^{(1)}}(\Delta_1) \Rightarrow (R, 0) \in \mathcal{R}^{\text{PSI}}(\Delta_1)$, we only need to show the reverse implication. Let $\varepsilon > 0$ and $(R, 0) \in \mathcal{R}^{\text{PSI}}(\Delta_1)$. Let $(\phi_{[n]}^{(1)}, \phi_{[n]}^{(2)}, \psi_{[n]}^{(1)}, \psi_{[n]}^{(2)})$ be an ε -achievable code for this rate-distortion tuple. Set $U \triangleq \phi_{[n]}^{(1)}(X_{[n]}^{(1)})$ and $V \triangleq \phi_{[n]}^{(2)}(X_{[n]}^{(2)})$ and let \mathcal{U}, \mathcal{V} be their alphabets, respectively. Notating $q_n \triangleq p_{UX_{[n]}^{(1)}V}$, we have

$$\sum_{\substack{u \in \mathcal{U}, v \in \mathcal{V} \\ x_{[n]} \in \mathcal{X}_{[n]}^{(1)}}} \frac{q_n(u, x_{[n]}, v)}{n} \sum_{j=1}^n \partial^{(1)}((\psi_{[n]}^{(1)}(u, v))_j, x_j) \leq \Delta_1 + \varepsilon.$$

Now, choose $m \in \mathbb{N}$ and a code C_m of $2^{m(I(UX_{[n]}^{(1)}; V) + \varepsilon)}$ codewords from $\mathcal{V}_{[m]}$ (with each component of each codeword selected independently and identically using P_V) such that

B1. $\Pr[(X_{[n]})_{[m]} \in \mathcal{S}_m^\varepsilon(P_{X_{[n]}^{(1)}})] \geq 1 - \varepsilon$.

B2. $\Pr\left[\left\{ (u_{[m]}, (x_{[n]})_{[m]}) \in \mathcal{U}_{[m]} \times (\mathcal{X}_{[n]})_{[m]} : \exists \mathbf{c} \in C_m \text{ s.t. } (u_{[m]}, (x_{[n]})_{[m]}, \mathbf{c}) \in \mathcal{S}_m^\varepsilon(q_n) \right\} \geq 1 - \varepsilon\right]$.

Consider the scheme where the $X^{(2)}$ encoder sends a constant message and $X^{(1)}$ encoder sends the index of a $\mathbf{v} \in C_m$ that is jointly typical with $(U_{[m]}, (X_{[n]}^{(1)})_{[m]})$ in addition to $U_{[m]} \triangleq (\phi_{[n]}^{(1)}(X_{[n]}^{(1)}), \dots, \phi_{[n]}^{(1)}(X_{[n-m-n+1 \sim nm]}^{(1)}))$. Note that by B2, for almost all source realizations, at least one such \mathbf{v} exists. If no such typical vector exists, the first codeword is transmitted by default. This scheme can be effected with a rate

$$\tilde{R}_1 = R_1 + \frac{1}{n} I(UX_{[n]}^{(1)}; V) + \varepsilon \stackrel{(a)}{\leq} R_1 + 2\varepsilon, \quad (3)$$

where (a) follows because the data processing inequality for $UX_{[n]}^{(1)} \oplus X_{[n]}^{(2)} \oplus V$ ensures $I(UX_{[n]}^{(1)}; V) \leq I(X_{[n]}^{(2)}; V) \leq \log_2 |\mathcal{V}| \leq n\varepsilon$. The decoder uses the indices to generate $\hat{X}_{[l+1 \sim (l+1)n]}^{(1)} = \psi_{[n]}^{(1)}(U_{l+1}, V_{l+1})$, $l = 0, \dots, m-1$. By construction, we guarantee an average distortion of no more

than $(\Delta_1 + \varepsilon)(1 + \varepsilon)(1 - 2\varepsilon) + 2\varepsilon\partial_{\max}^{(1)} < \Delta_1 + \varepsilon(1 + 2\partial_{\max}^{(1)})$. Since ε is arbitrary, we see that if $(R_1, 0) \in \mathcal{R}^{\text{PSI}}(\Delta_1)$, then the rate R_1 suffices to construct a rate-distortion code for $p_{X^{(1)}}$ meeting a distortion of Δ_1 under $\partial^{(1)}$. ■

B. Multi-terminal Source Coding Problem with Two Sources

We first present an example that shows that the i.i.d. nature of the source is important for zero-rate links to play an insignificant role in shaping rate regions.

Example 1: Consider the MSC problem for the $\{X_i^{(1)}, X_i^{(2)}\}_{i \in \mathbb{N}}$, where $\{X_i^{(1)}\}$ are i.i.d. with each index $X_i^{(1)}$ having a distribution $p_{X^{(1)}}$, and $X_i^{(2)} = (X_i^{(1)}, A)$, where A is a discrete RV over \mathcal{A} with distribution p_A that: (1) is statistically independent of each $X_{[s]}^{(1)}$ for $s \in \mathbb{N}$, and (2) meets $H(A) > 0$. Then, for $\partial^{(1)} = \partial_H^{X^{(1)}}$ and $\partial^{(2)} = \partial_H^{X^{(2)}}$ and $\Delta = \mathbf{0}_2$, $(H(X), 0)$ is achievable, since one can use a good compression scheme for the $X^{(1)}$ side and convey A using $\lceil \log_2 |\mathcal{A}| \rceil$ bits from the $X^{(2)}$ side. However, by deleting the link from the $X^{(2)}$ encoder, one cannot reconstruct the $X^{(2)}$ with zero distortion.

The following result shows such an event cannot occur for i.i.d. sources.

Theorem 2: Let \mathcal{R}^{H} , \mathcal{R}^{PRD} denote the rate region for the common helper problem and the rate-distortion function for the partially-blind rate-distortion problem (see Appendix A), respectively. Then, the following are equivalent.

- C1. $(R, 0) \in \mathcal{R}^{\text{H}}(\Delta_1, \Delta_2)[p_{X^{(1)}X^{(2)}}]$.
- C2. $(R, 0) \in \mathcal{R}^{\text{MSC}}(\Delta_1, \Delta_2)[p_{X^{(1)}X^{(2)}}]$.
- C3. $R \geq \mathcal{R}^{\text{PRD}}(\Delta_1, \Delta_2)[p_{X^{(1)}X^{(2)}}]$.

Proof: It is straightforward to see that C3 \Rightarrow C2 and C2 \Rightarrow C1. To show C1 \Rightarrow C3, let $(R, 0) \in \mathcal{R}^{\text{H}}(\Delta_1, \Delta_2)$. Then, from the rate region for the common helper problem (Appendix A), we have $p_{U^*V^*X^{(1)}X^{(2)}} \in \mathcal{P}^{\text{H}}(\Delta_1, \Delta_2)$, such that $R_2 = I(X^{(1)}X^{(2)}; V^*|U^*) = 0$. This functional being zero in conjunction with the chain $U^* \circ X^{(1)} \circ X^{(2)}$ establish $V^* \circ U^* \circ X^{(1)} \circ X^{(2)}$. Now, for $j = 1, 2$, let f_j denote functions that map $\mathcal{U}^* \times \mathcal{V}^*$ to the respective reconstruction alphabets $\mathcal{X}^{(j)}$, such that $f_j(U^*, V^*)$ meets required distortion constraint of Δ_j under $\partial^{(j)}$. Define for $j = 1, 2$, functions $h_j: \mathcal{U}^* \rightarrow \mathcal{V}^*$, $\tilde{f}_j: \mathcal{U}^* \rightarrow \mathcal{X}^{(j)}$ by

$$h_j(u) \triangleq \arg \min_{v \in \mathcal{V}^*} \sum_{x \in \mathcal{X}^{(j)}} p_{X^{(j)}|U^*}(x|u) \partial^{(j)}(x, f_j(u, v)). \quad (4)$$

$$\tilde{f}_j(u) \triangleq f_j(u, h_j(u)). \quad (5)$$

Observe that by construction, for $j = 1, 2$,

$$\mathbb{E} \partial^{(j)}(X^{(j)}, \tilde{f}_j(U^*)) \leq \mathbb{E} \partial_x(X^{(j)}, f_j(U^*, V^*)) \leq \Delta_j. \quad (6)$$

Thus, there exists a distribution $p_{U^*X^{(1)}X^{(2)}}$ with (1) $|\mathcal{U}^*| \leq |\mathcal{X}^{(1)}| |\mathcal{X}^{(2)}| + 4$; (2) $U^* \circ X^{(1)} \circ X^{(2)}$; and (3) functions \tilde{f}_j that provide reconstructions $\hat{X}^{(j)}$ from U^* meeting the required distortions. Therefore, $p_{U^*X^{(1)}X^{(2)}} \in \mathcal{P}^{\text{PRD}}(\Delta_1, \Delta_2)$ (possibly after altering the definition of \mathcal{R}^{PRD} to include auxiliary RVs with alphabet sizes up to $|\mathcal{X}^{(1)}| |\mathcal{X}^{(2)}| + 4$, which does not alter the PRD rate region). Therefore, we have

$$(R, 0) \in \mathcal{R}^{\text{H}}(\Delta_1, \Delta_2) \Rightarrow R \geq \mathcal{R}^{\text{PRD}}(\Delta_1, \Delta_2). \quad \blacksquare$$

At this point, we would like to remark that the inner bound $\mathcal{R}_{in}^{\text{MSC}}(\Delta_1, \Delta_2)$ by Berger and Tung [10] and the outer bound $\mathcal{R}_{out}^{\text{MSC}}(\Delta_1, \Delta_2)$ obtained from traditional converse techniques (that replaces the chain $U \circ X^{(1)} \circ X^{(2)} \circ V$ in the inner bound with $U \circ X^{(1)} \circ X^{(2)}$ and $X^{(1)} \circ X^{(2)} \circ V$) also agree on the $R_2 = 0$ plane. That is,

$$(R, 0) \in \mathcal{R}_{in}^{\text{MSC}}(\Delta_1, \Delta_2) \Leftrightarrow (R, 0) \in \mathcal{R}_{out}^{\text{MSC}}(\Delta_1, \Delta_2) \quad (7)$$

$$\Leftrightarrow R \geq \mathcal{R}^{\text{PRD}}(\Delta_1, \Delta_2), \quad (8)$$

thereby providing an alternate proof of the invariance result for the MSC problem when $l = 2$. Further, Theorem 2 can be extended for the $l > 2$ setting to show that zero-rate encoders cannot help when there is only one link carrying positive rate.

Theorem 3: For $l > 2$ and $\Delta \geq 0$

$$(R, \mathbf{0}_{l-1}) \in \mathcal{R}^{\text{MSC}}(\Delta)[p_{X^{(1)} \dots X^{(l)}}] \Leftrightarrow R \geq \mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)} \dots X^{(l)}}].$$

Proof: Note that

$$(R, \mathbf{0}_{l-1}) \in \mathcal{R}^{\text{MSC}}(\Delta)[p_{X^{(1)} \dots X^{(l)}}] \Rightarrow (R, 0) \in \mathcal{R}^{\text{MSC}}(\Delta)[p_{X^{(1)}Y}],$$

where $Y = (X^{(2)} \dots X^{(l)})$. Notice here that the distortions $\partial^{(k)}$ for $k > 1$ can be equivalently seen as distortion measures for the Y -source. However, from Theorem 2, we notice that

$$(R, 0) \in \mathcal{R}^{\text{MSC}}(\Delta)[p_{X^{(1)}Y}] \Rightarrow R \geq \mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)}Y}]. \quad (9)$$

However, since $\mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)}Y}] = \mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)} \dots X^{(l)}}]$, the proof is complete because $R \geq \mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)} \dots X^{(l)}}]$ is achievable for the MSC problem. ■

C. Multi-terminal Source Coding Problem for $l > 2$ sources

Here, we show that for a class of sources and under certain distortions Δ , the MSC rate region with zero rates on certain links is the same as that of the MSC problem with the same constraints and with the zero-rate links deleted.

Theorem 4: Suppose $\exists S \subset [l]$, $i \in [l] \setminus S$, such that $X^{(S)} \circ X^{(i)} \circ X^{((S \cup \{i\})^c)}$. Additionally, if $S \subseteq \mathcal{H}(\Delta)$, then all rate vectors in $\mathcal{R}^{\text{MSC}}(\Delta) \cap \{R_j = 0 : j \in S\}$ are achievable even if the encoders encoding $X^{(j)}$, $j \in S$ send a constant message.

Proof: Since the proof is a simple multi-source adaptation of that of Theorem 1 that establishes a rate-transfer argument, we present only an outline of the proof. Given an ε -achievable code C with l encoders, construct a block supercode C' with a bigger block length, wherein the encoders corresponding to the indices of S transmit constant messages, and the i^{th} encoder constructs a codebook that will transmit along with its usual message, additional message that corresponds to a typical realization of the messages that would be originally sent over the $|S|$ zero-rate links, i.e., from the encoders encoding $X^{(j)}$, $j \in S$. In doing so, the rate from the encoders encoding $\{X_j : j \in S\}$ is transferred to that of i . Note that this additional rate incurred is bounded above by $|S|\varepsilon$. The proof is complete by noting that ε is arbitrary. ■

Note that the above result is different from that of $l = 2$ case, since for $l > 2$, the rate regions of suitable sub-networks may be unknown. For example, consider the MSC problem

for a DMS with distribution $p_{X_1 X_2 X_3}$ s. t. $X_1 \oplus X_2 \oplus X_3$. Theorem 4 guarantees

$$\mathcal{R}^{\text{MSC}}(\Delta_1, \Delta_2, \partial_{\max}^{(3)}) \cap \{R_3 = 0\} \cong \mathcal{R}^{\text{PSI}}(\Delta_1, \Delta_2)[p_{X^{(1)} X^{(2)}}],$$

$$\mathcal{R}^{\text{MSC}}(\partial_{\max}^{(1)}, \Delta_2, \Delta_3) \cap \{R_1 = 0\} \cong \mathcal{R}^{\text{PSI}}(\Delta_2, \Delta_3)[p_{X^{(2)} X^{(3)}}],$$

where \cong signifies that the right-hand region is the appropriate projection of the one on the left. Note that this result is previously unknown, since the rate region for the PSI problem remains open. Additionally, $\forall \Delta \geq \mathbf{0}_3$, Theorem 3 guarantees

$$(0, R_2, 0) \in \mathcal{R}^{\text{MSC}}(\Delta) \Rightarrow R_2 \geq \mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)} X^{(2)} X^{(3)}}].$$

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VI. CONCLUSIONS

The invariance of the MSC rate region under the deletion of zero-rate links was studied. Though the question of invariance remains open in general, it was shown that the rate region remains unaltered if zero-rate links are deleted from the PSI and MSC problems with two correlated DMSs. When more than two correlated DMSs are present, it was established that the deletion of zero-rate links from some helper encoders do not alter the MSC rate region provided the source distribution has a certain Markov structure.

APPENDIX A

ALLIED PROBLEMS AND THEIR RATE REGIONS

Problem 1: (The partially-blind rate-distortion problem) Given a discrete source emitting $(X_i^{(1)}, \dots, X_i^{(l)})_{i \in \mathbb{N}}$ in an i.i.d. fashion with each symbol l -tuple having the joint distribution $p_{X^{(1)} \dots X^{(l)}}$. The problem aims to identify the rates at which the $X^{(1)}$ sequence can be encoded so that suitably “noisy” reconstruction $(\widehat{X}_i^{(k)})_{i \in \mathbb{N}}$ for each $k \in [l]$ is constructed by the block decoder. The acceptability of the reconstructions are determined by distortion criteria using distortion measures $\partial^{(k)} : \mathcal{X}^{(k)} \times \widehat{\mathcal{X}}^{(k)} \rightarrow \mathbb{R}^+$. A pair (R_1, Δ) is said to be achievable if for each $\varepsilon > 0$, $\exists n \in \mathbb{N}$, $\phi_{[n]}^{(1)} : \mathcal{X}_{[n]}^{(1)} \rightarrow \mathcal{M}^{(1)}$ and $\psi_{[n]} : \mathcal{M}^{(1)} \rightarrow \widehat{\mathcal{X}}_{[n]}^{(1)} \times \dots \times \widehat{\mathcal{X}}_{[n]}^{(l)}$ s. t.:

$$\text{D1. } \frac{1}{n} \sum_{i=1}^n \mathbb{E} \partial^{(k)}(X_i^{(k)}, \widehat{X}_i^{(k)}) \leq \Delta_k + \varepsilon, \forall k \in [l], \text{ and}$$

$$\text{D2. } |\mathcal{M}^{(1)}| \leq 2^{n(R_1 + \varepsilon)}.$$

The infimum of achievable rates $\mathcal{R}^{\text{PRD}}(\Delta)[p_{X^{(1)} \dots X^{(l)}}]$ can be shown to be as follows.

$$\mathcal{R}^{\text{PRD}}(\Delta) = \inf_{p_{U X^{(1)} \dots X^{(l)}} \in \mathcal{P}^{\text{PRD}}(\Delta)} I(X^{(1)}; U),$$

where $\mathcal{P}^{\text{PRD}}(\Delta)$ is the set of distributions $p_{U X^{(1)} \dots X^{(l)}}$ s. t.:

$$\text{E1. } U \oplus X^{(1)} \oplus (X^{(2)} \dots X^{(l)}), |\mathcal{U}| \leq |\mathcal{X}^{(1)}| + l, \text{ and}$$

$$\text{E2. } \forall k \in [l], \exists f^{(k)} : \mathcal{U} \rightarrow \widehat{\mathcal{X}}^{(k)}, \mathbb{E} \partial^{(k)}(X^{(k)}, f^{(k)}(U)) \leq \Delta_k.$$

Problem 2: (The common helper problem) Given a discrete source emitting $(X_i^{(1)}, X_i^{(2)})_{i \in \mathbb{N}}$ in an i.i.d. fashion with each pair having the joint distribution $p_{X^{(1)} X^{(2)}}$. The problem aims to identify the rates at which information must be sent by

encoders so that suitably “noisy” versions $(\widehat{X}_i^{(1)})_{i \in \mathbb{N}}$ and $(\widehat{X}_i^{(2)})_{i \in \mathbb{N}}$ are constructed by a joint block decoder. Here, the first encoder (the helper encoder) has access to the $X^{(1)}$ -sequence, whereas the second one has access to both $X^{(1)}$ - and $X^{(2)}$ -sequences. As before, the acceptability of reconstructions are evaluated by distortion criteria using distortion measures $\partial^{(1)} : \mathcal{X}^{(1)} \times \widehat{\mathcal{X}}^{(1)} \rightarrow \mathbb{R}^+$ and $\partial^{(2)} : \mathcal{X}^{(2)} \times \widehat{\mathcal{X}}^{(2)} \rightarrow \mathbb{R}^+$. A quadruplet $(R_1, R_2, \Delta_1, \Delta_2)$ is said to be achievable if for each $\varepsilon > 0$, $\exists n \in \mathbb{N}$, $\phi_{[n]}^{(1)} : \mathcal{X}_{[n]}^{(1)} \rightarrow \mathcal{M}^{(1)}$, $\phi_{[n]}^{(12)} : \mathcal{X}_{[n]}^{(1)} \times \mathcal{X}_{[n]}^{(2)} \rightarrow \mathcal{M}^{(12)}$, $\psi_{[n]}^{(1)} : \mathcal{M}^{(1)} \times \mathcal{M}^{(12)} \rightarrow \widehat{\mathcal{X}}_{[n]}^{(1)}$ and $\psi_{[n]}^{(2)} : \mathcal{M}^{(1)} \times \mathcal{M}^{(12)} \rightarrow \widehat{\mathcal{X}}_{[n]}^{(2)}$ s. t.

$$\text{F1. } \frac{1}{n} \sum_{i=1}^n \mathbb{E} \partial^{(j)}(X_i^{(j)}, \widehat{X}_i^{(j)}) \leq \Delta_j + \varepsilon, j = 1, 2, \text{ where}$$

$$\widehat{X}_{[n]}^{(j)} \triangleq \psi_{[n]}^{(j)}(\phi_{[n]}^{(1)}(X_{[n]}^{(1)}), \phi_{[n]}^{(12)}(X_{[n]}^{(1)}, X_{[n]}^{(2)})), \text{ and}$$

$$\text{F2. } |\mathcal{M}_t| \leq 2^{n(R_t + \varepsilon)}, t = 1, 12.$$

Even though the problem defines two separate encoders, allowing the $X^{(1)}$ encoder to send its encoded message to the $X^{(1)} X^{(2)}$ encoder does not alter the rate region. This setting is more readily seen as the common helper setup [11]. The set $\mathcal{R}^{\text{H}}(\Delta_1, \Delta_2)$ of achievable rates is given by

$$\mathcal{R}^{\text{H}}(\Delta) = \left\{ \begin{array}{l} R_1 \geq I(X^{(1)}; U) \\ R_{12} \geq I(X^{(1)} X^{(2)}; V|U) : p_{X^{(1)} X^{(2)} UV} \in \mathcal{P}^{\text{H}}(\Delta) \end{array} \right\},$$

where $\mathcal{P}^{\text{H}}(\Delta)$ is the set of distributions $P_{UV X^{(1)} X^{(2)}}$ s. t.:

$$\text{G1. } U \oplus X^{(1)} \oplus X^{(2)}, |\mathcal{U}| \leq |\mathcal{X}^{(1)}| |\mathcal{X}^{(2)}| + 4.$$

$$\text{G2. } |\mathcal{V}| \leq (|\mathcal{X}^{(1)}| |\mathcal{X}^{(2)}| + 2)^2 - 2.$$

$$\text{G3. } \exists f_1 : \mathcal{U} \times \mathcal{V} \rightarrow \widehat{\mathcal{X}}^{(1)}, \mathbb{E} \partial^{(1)}(X^{(1)}, f_1(U, V)) \leq \Delta_1.$$

$$\text{G4. } \exists f_2 : \mathcal{U} \times \mathcal{V} \rightarrow \widehat{\mathcal{X}}^{(2)}, \mathbb{E} \partial^{(2)}(X^{(2)}, f_2(U, V)) \leq \Delta_2.$$

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